a primer on geometry optimization

CIX Summer School 2023 | U Michigan

Prof. Alexandra Ion

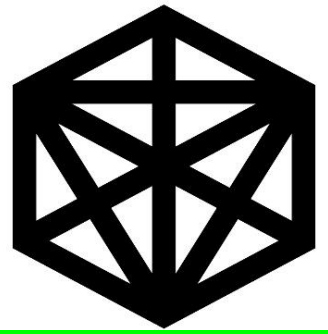

interactive structures lab

hi, I am alex.

Assistant Professor
at CMU HCII

Postdoc, 2020:
ETH Zurich
Olga Sorkine-Hornung, Graphics

PhD, 2019:
Hasso Plattner Institute
Patrick Baudisch, HCI



interactive structures lab

digital fabrication

image: formlabs.com

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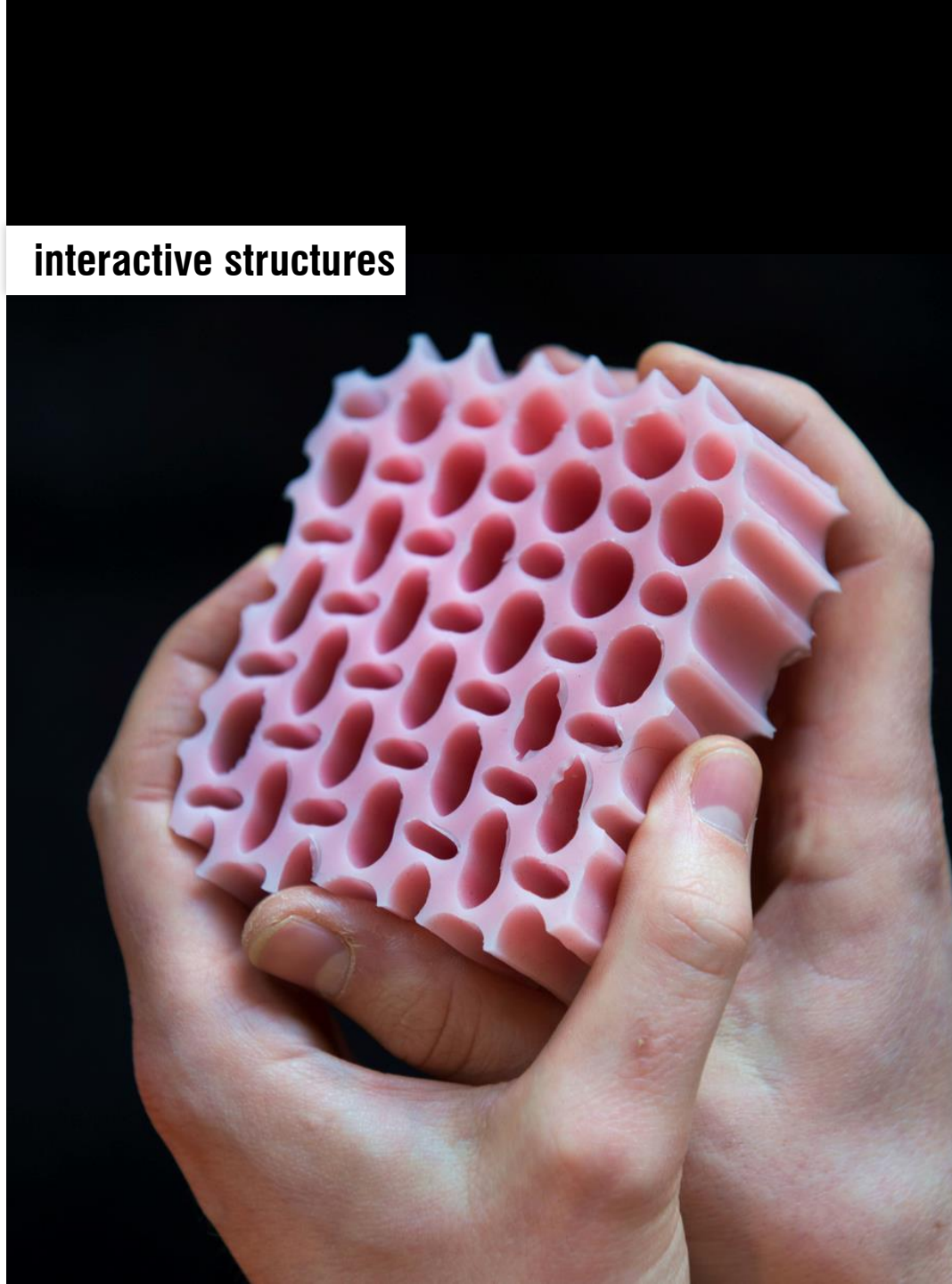
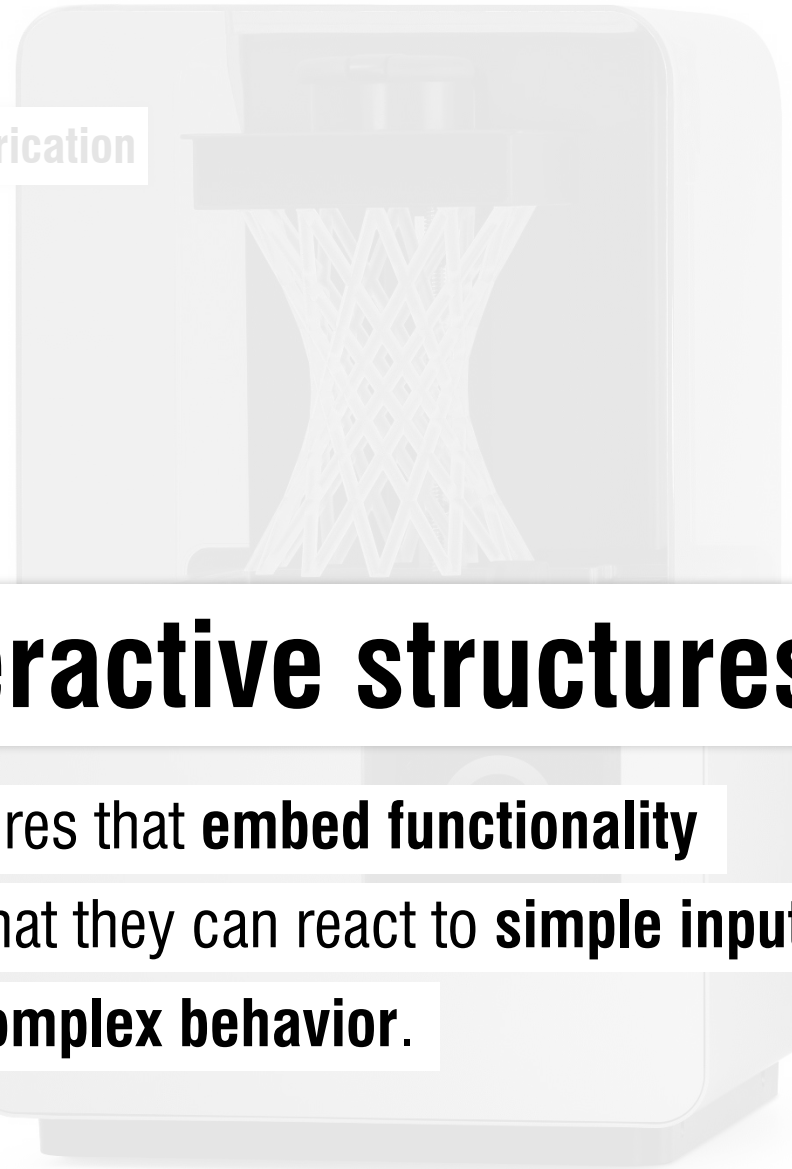
interactive structures



digital fabrication



interactive structures

structures that **embed functionality** such that they can react to **simple input** with **complex behavior**.

image: formlabs.com

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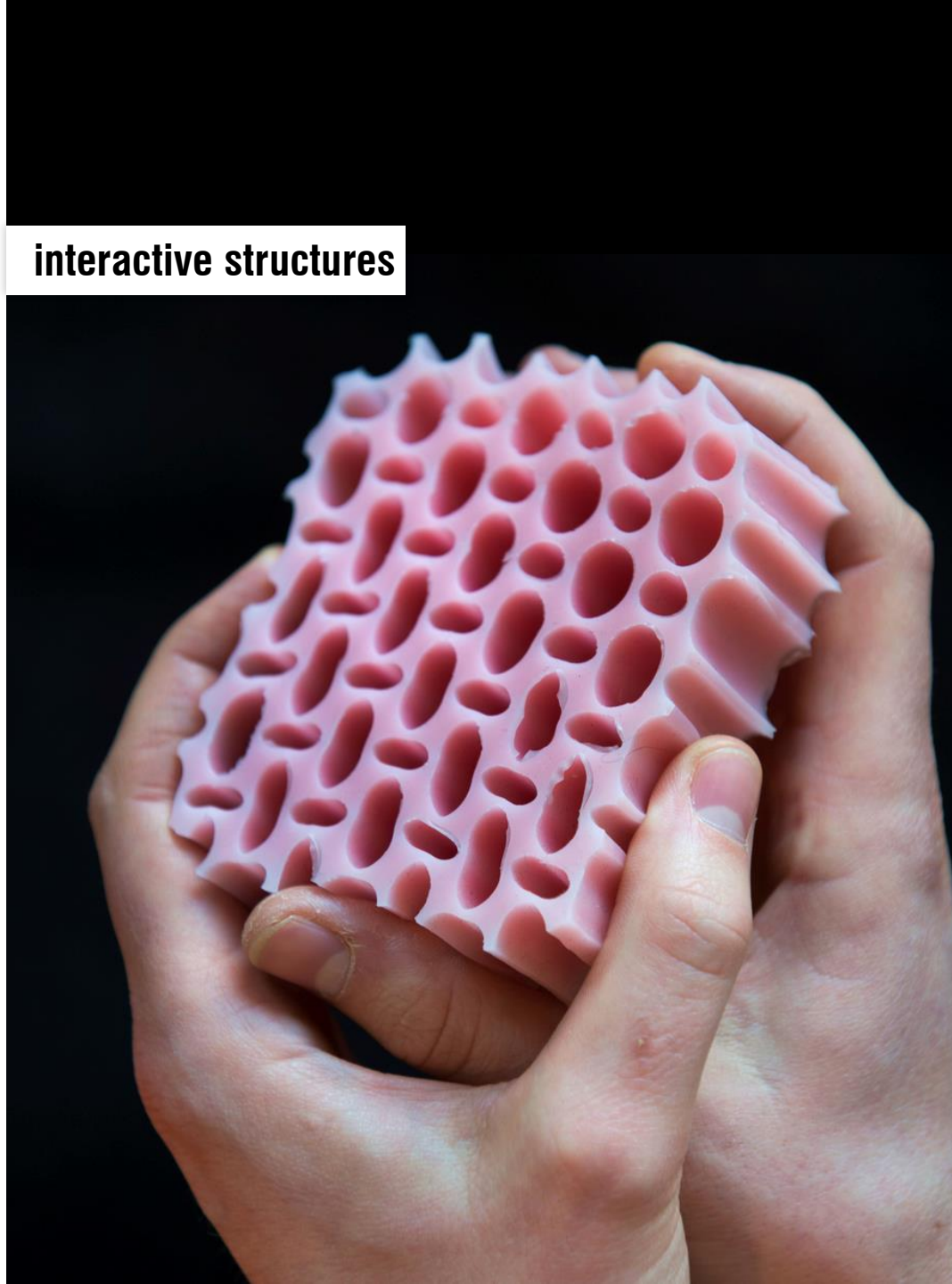
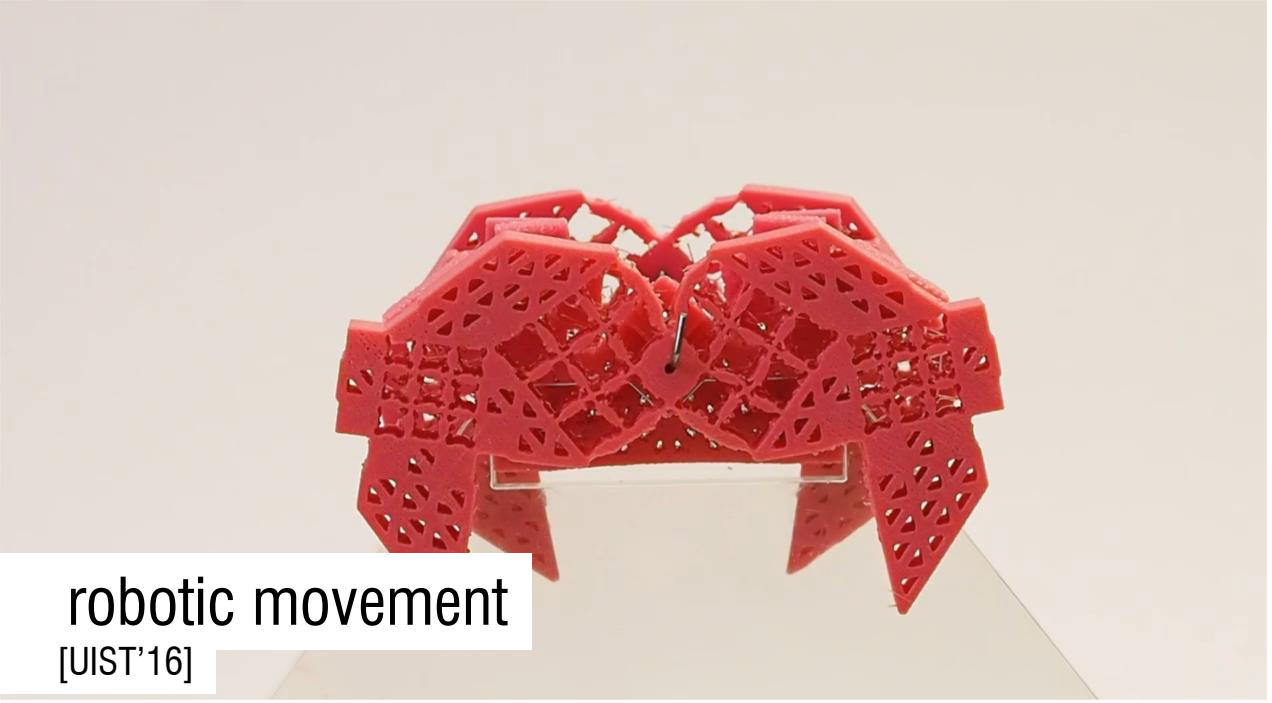
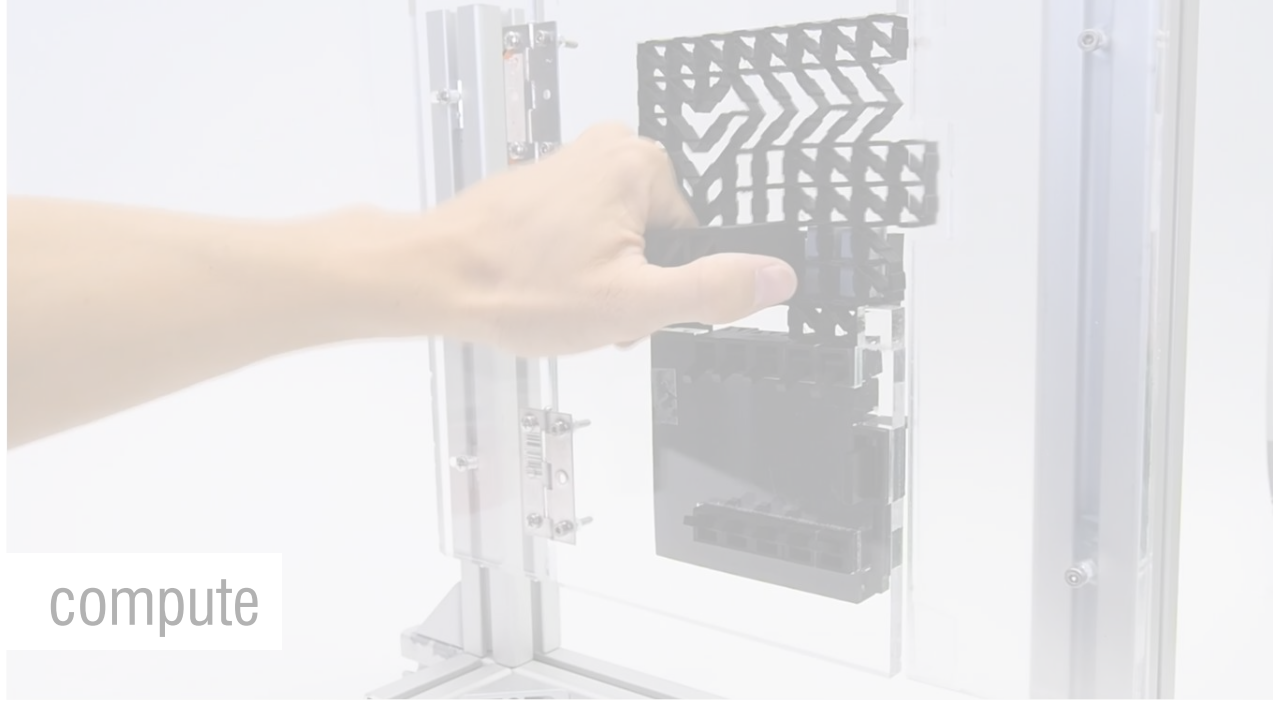
interactive structures

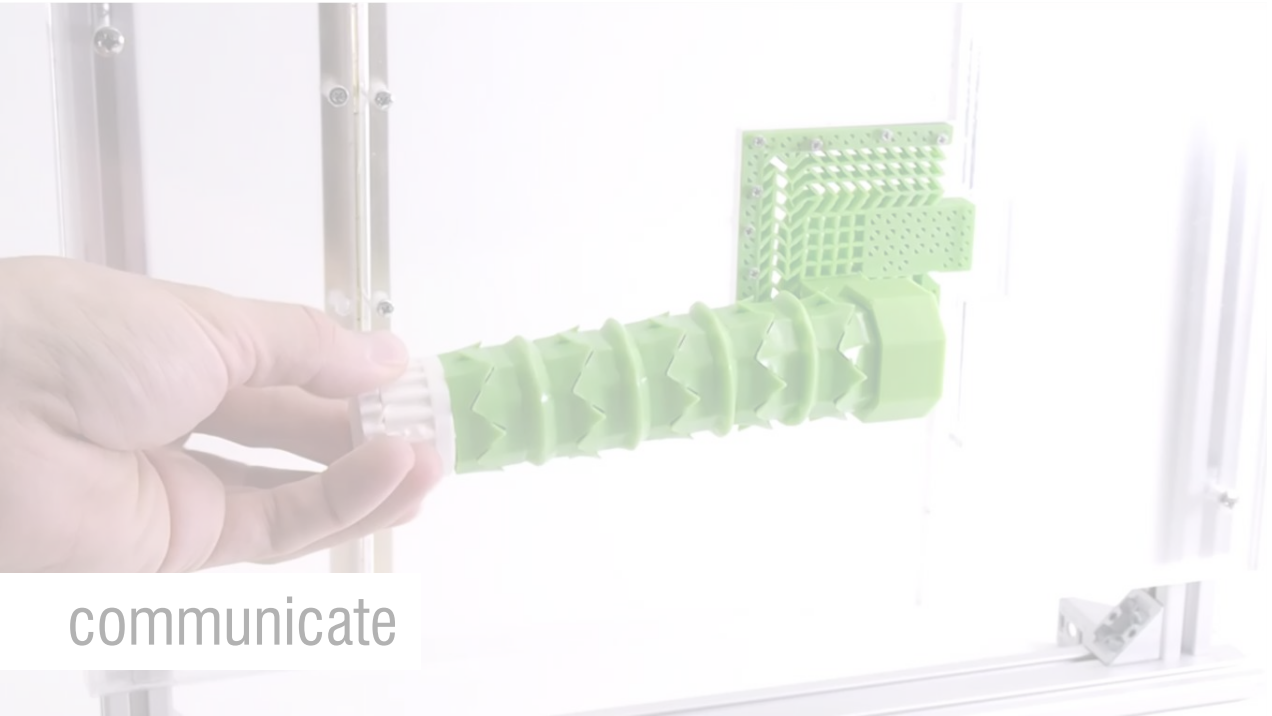




robotic movement
[UIST'16]



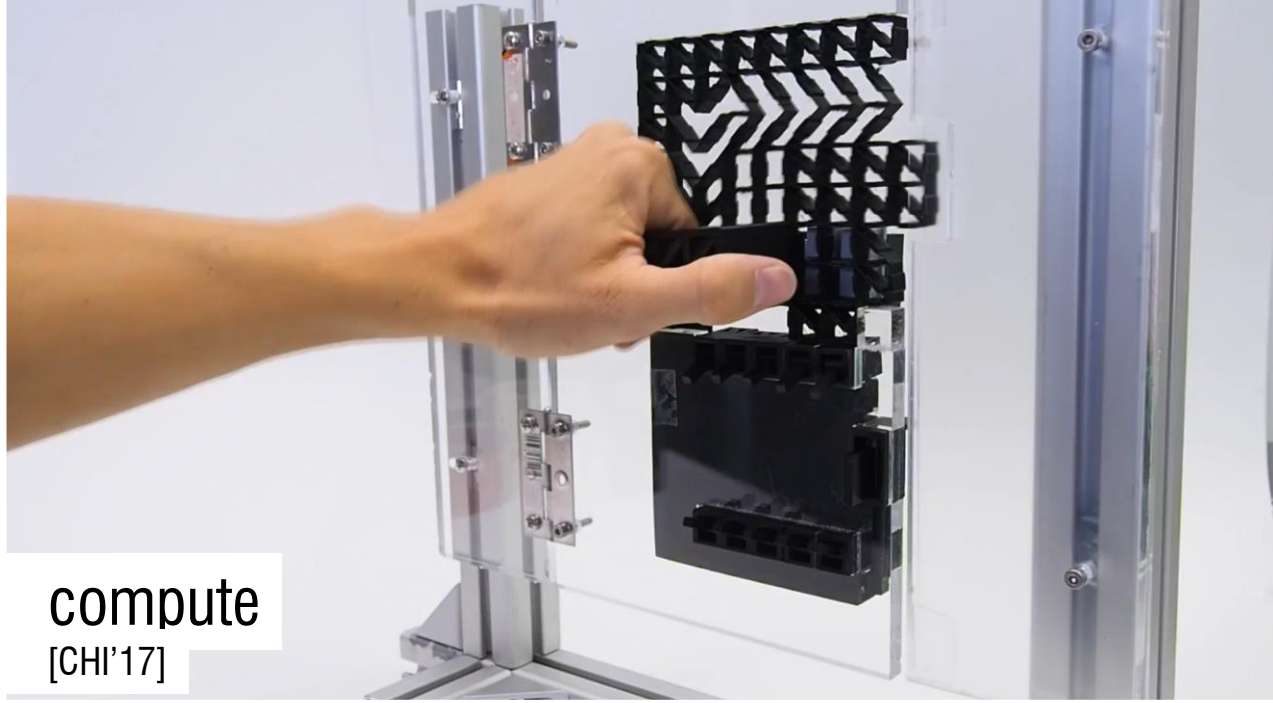
compute



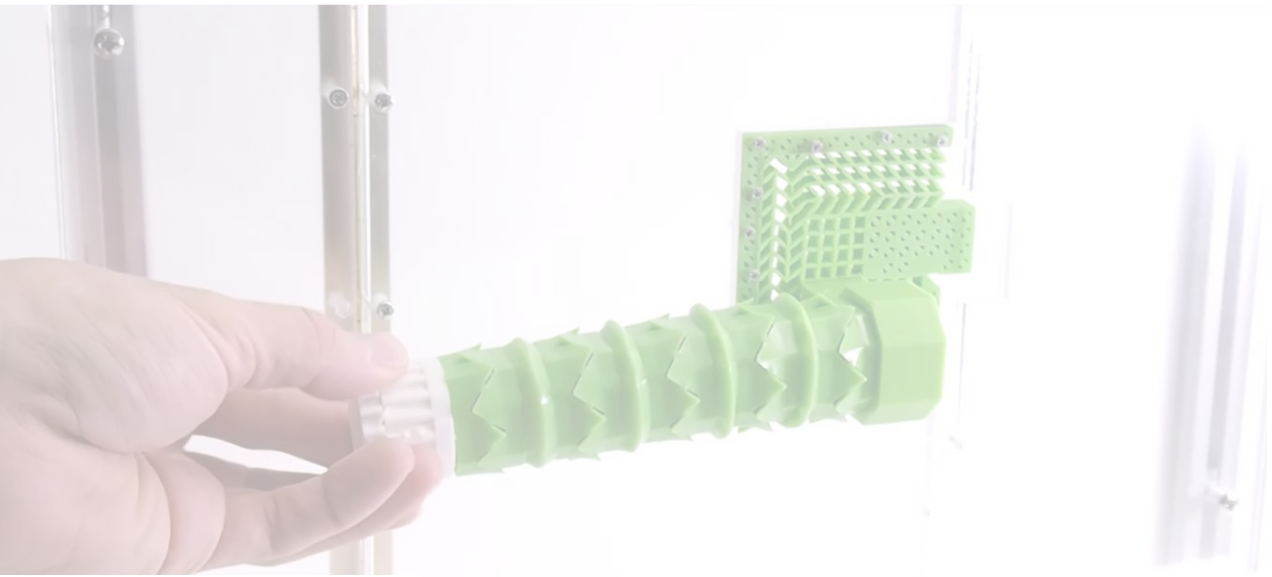
communicate

mass-fabricable

robotic movement



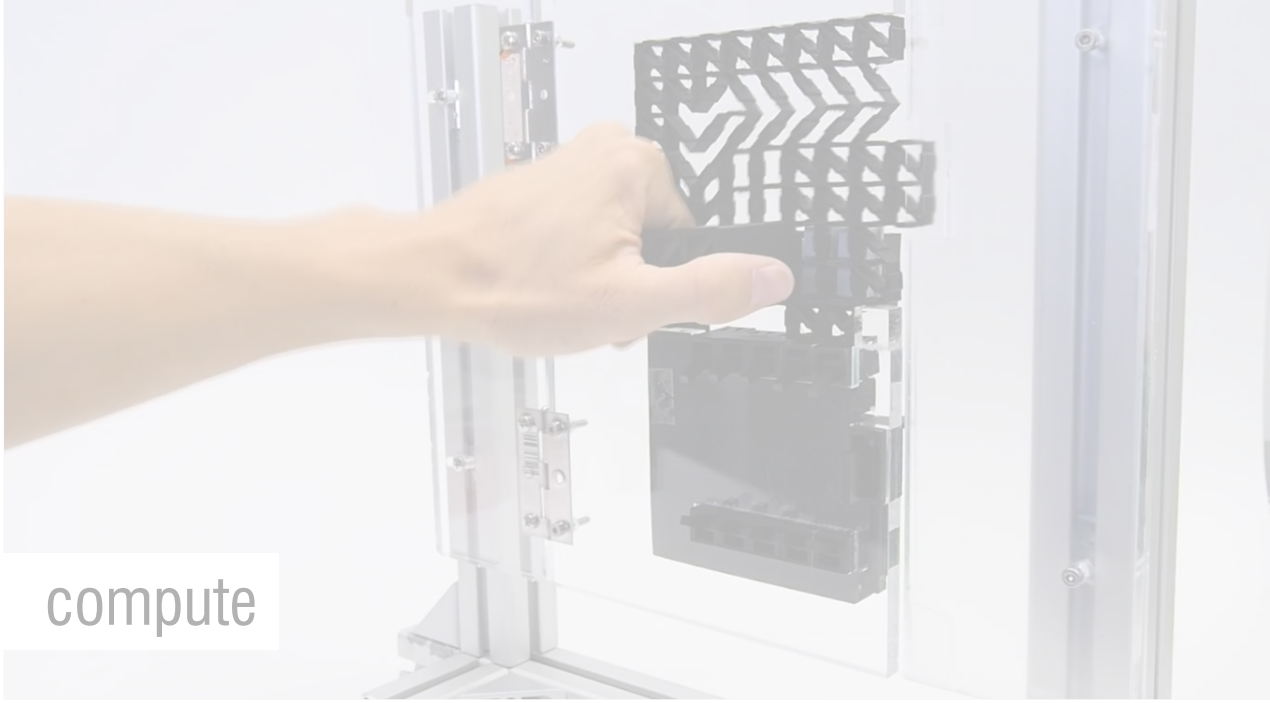
compute
[CHI'17]



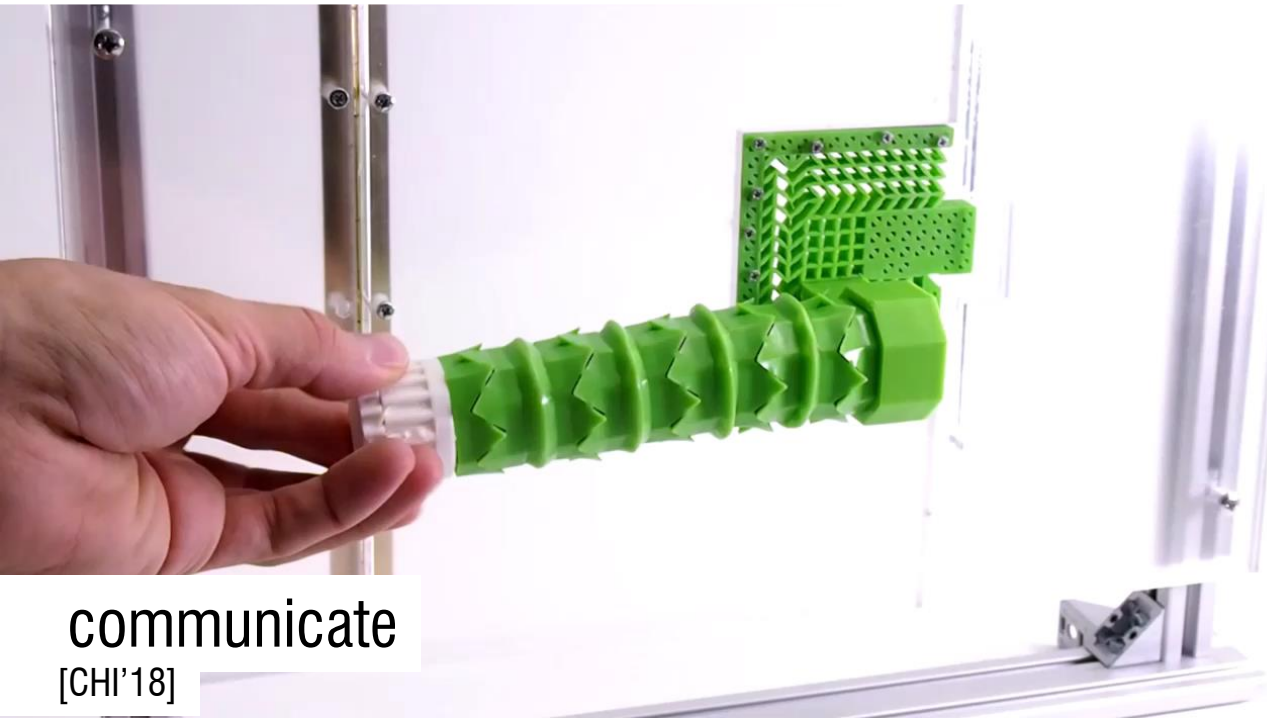
communicate

mass-fabricable

robotic movement



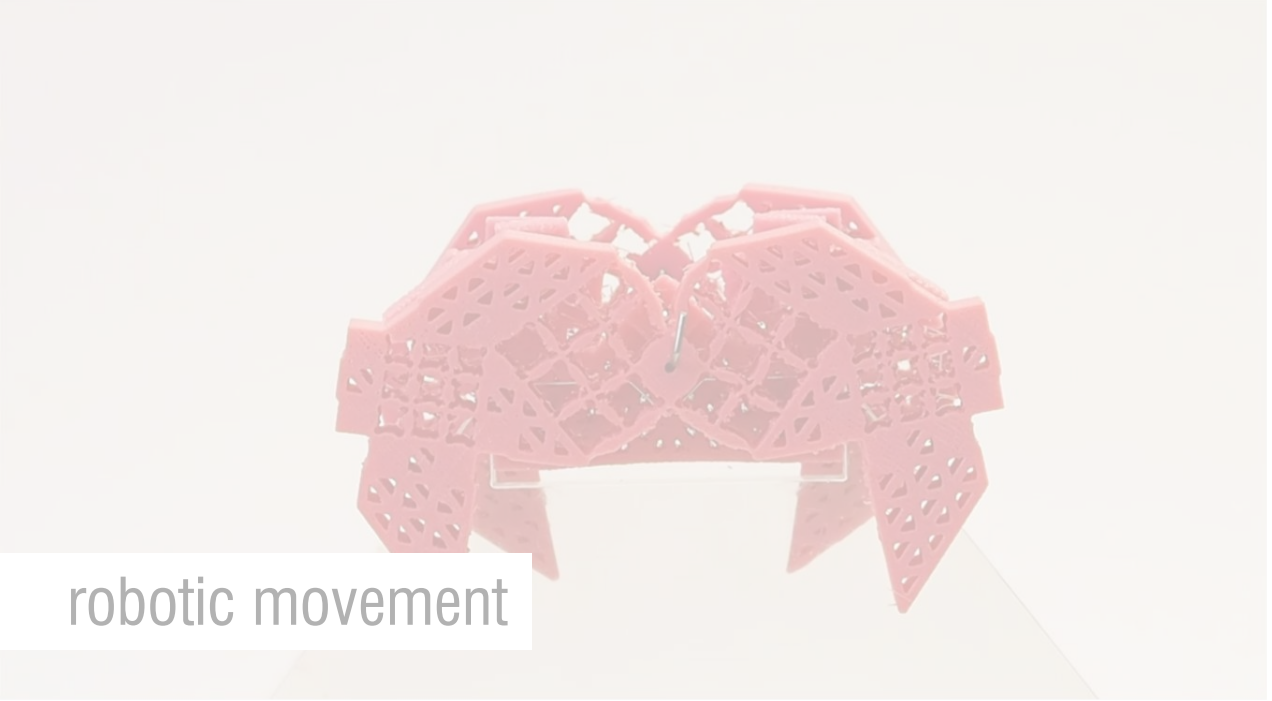
compute



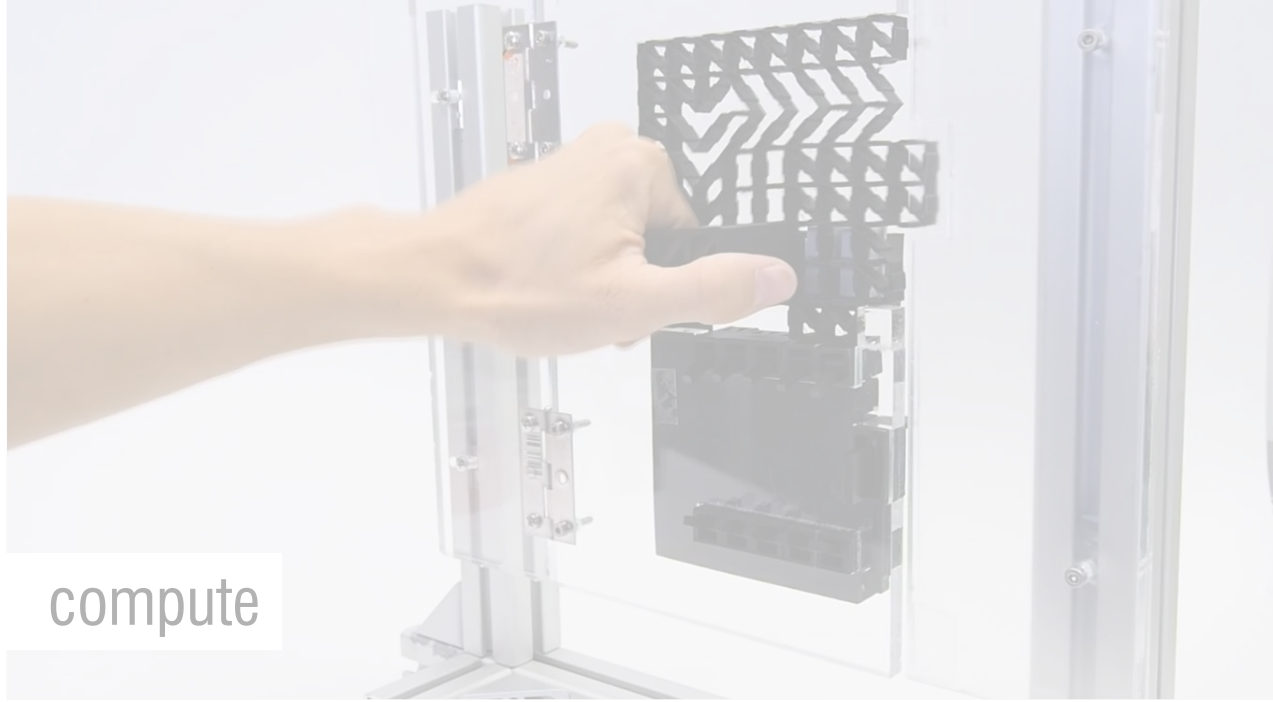
communicate

[CHI'18]

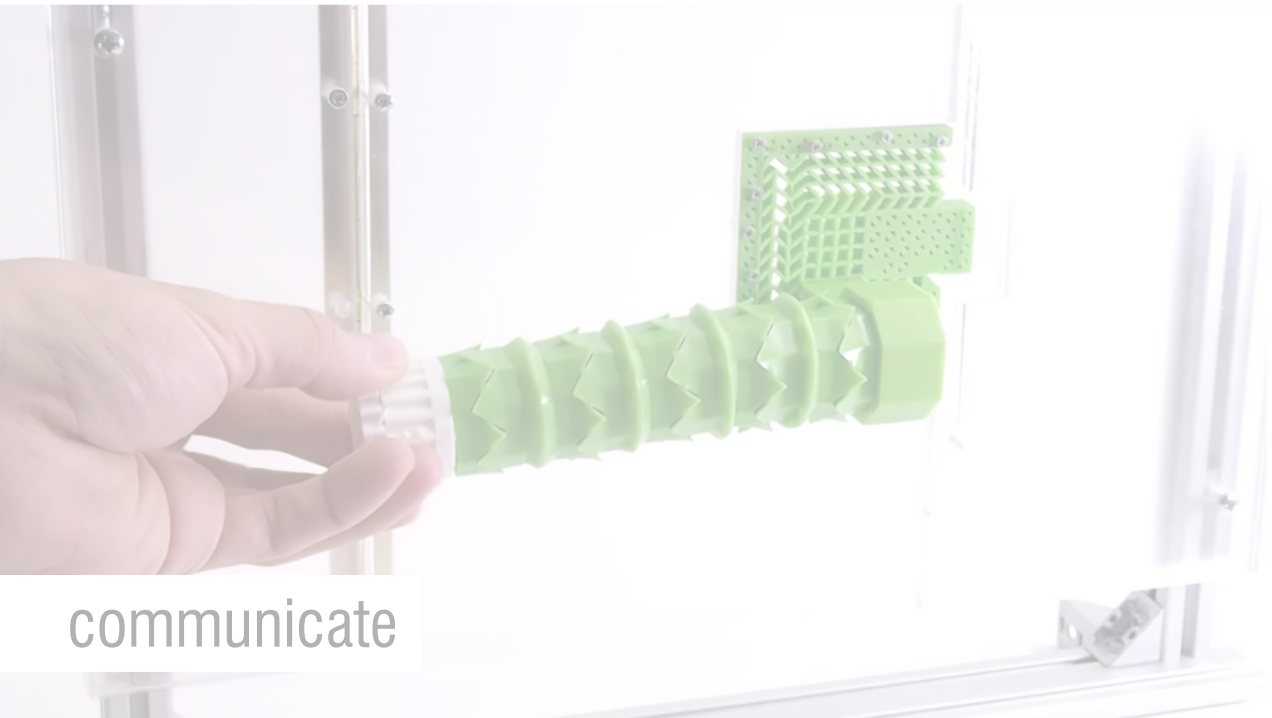
mass-fabricable



robotic movement



compute



communicate

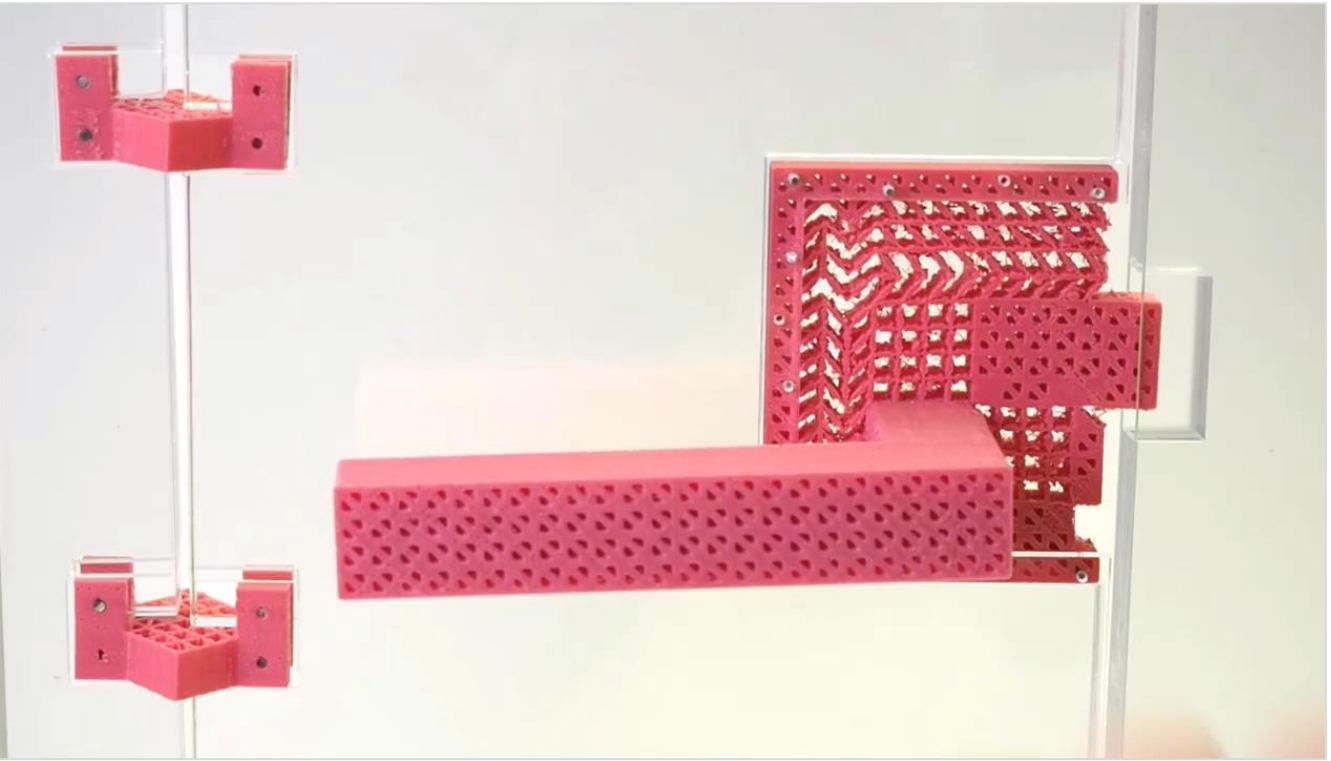
mass-fabricable

[CHI'21]

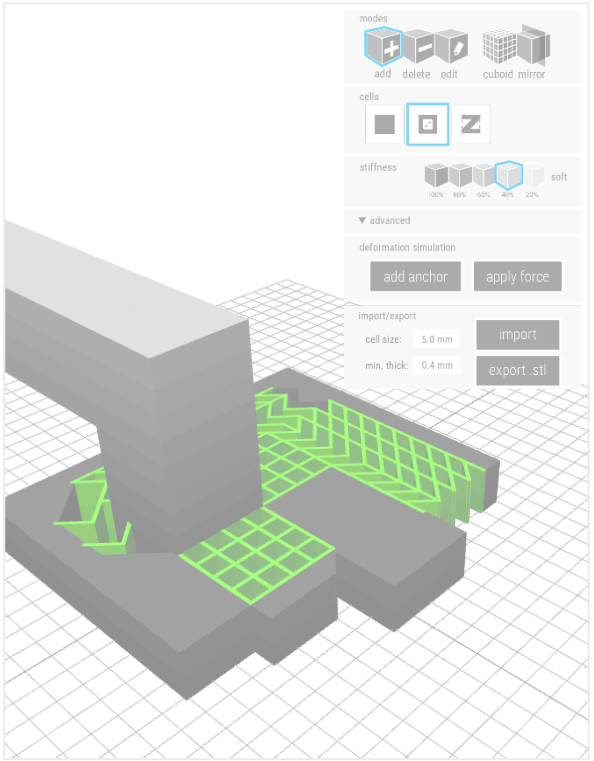
HCI + geometry + engineering + design

the work we do in the **lab**

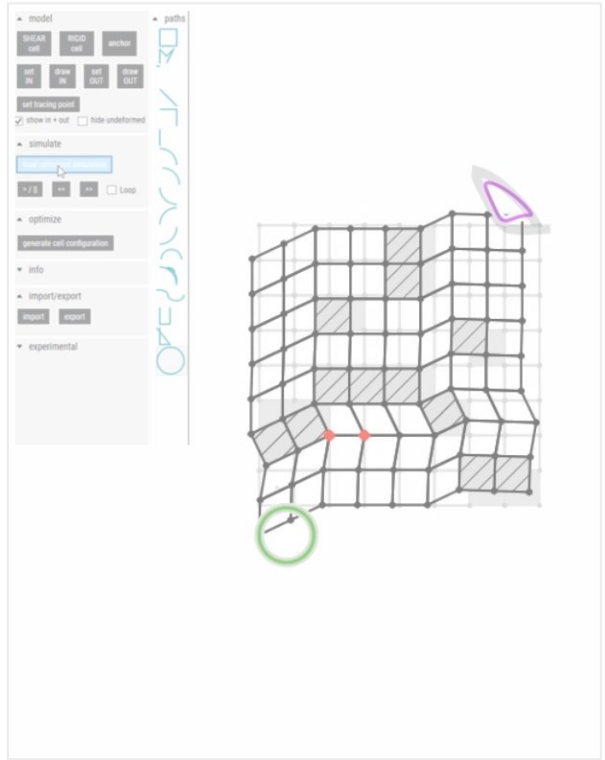
example: Metamaterial Mechanisms
[UIST'16, CHI'19]



structures



editor

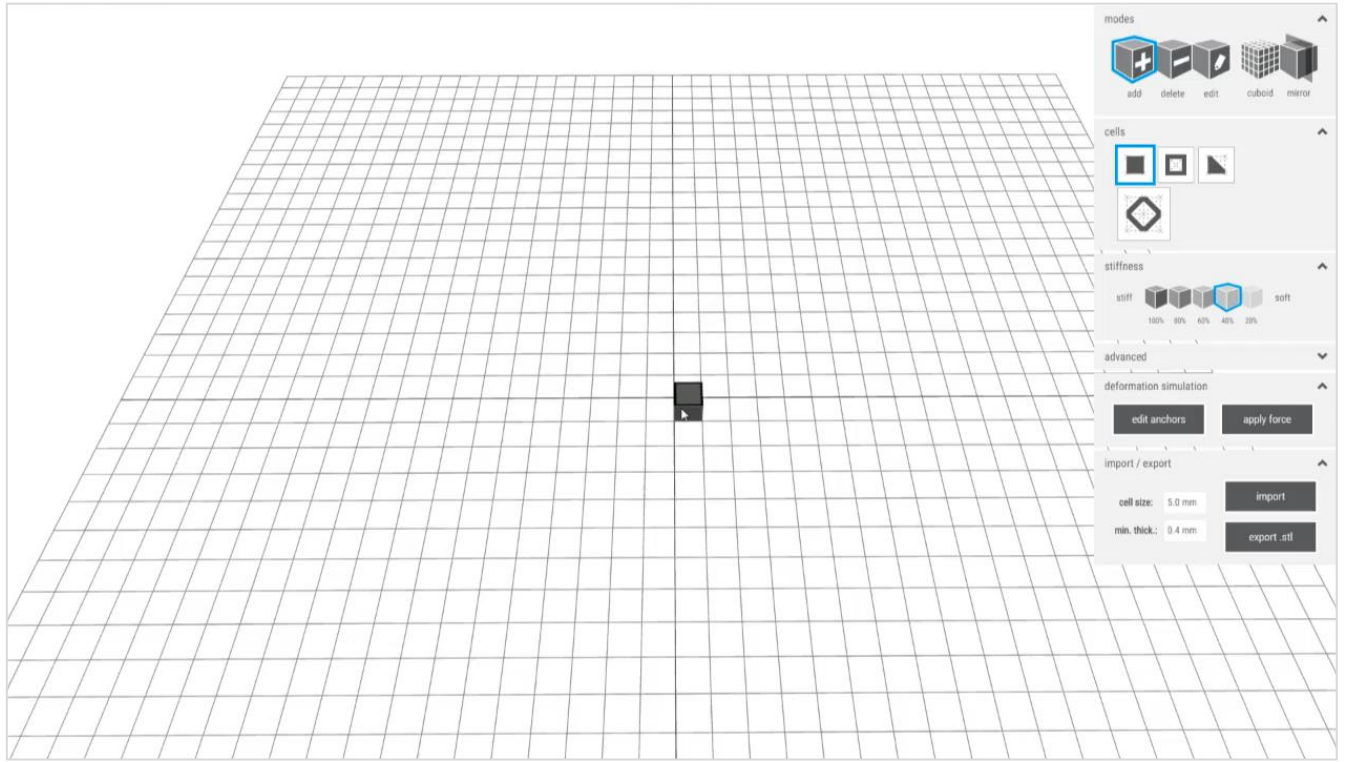


inverse design

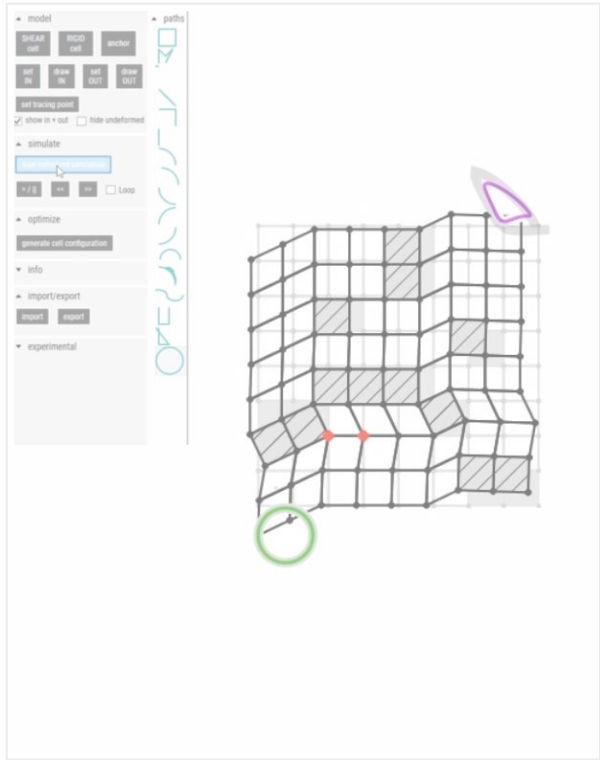
the work we do in the **lab**

example: Metamaterial Mechanisms
[UIST'16, CHI'19]

structures



editor

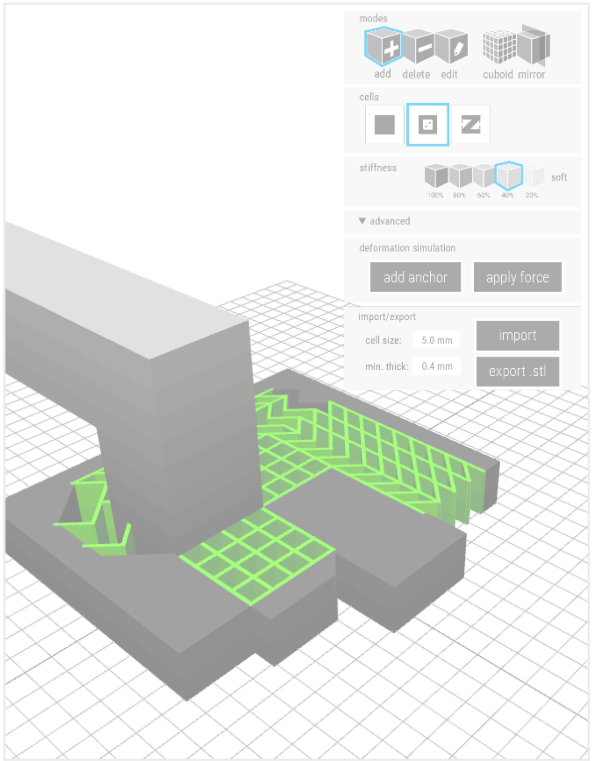


inverse design

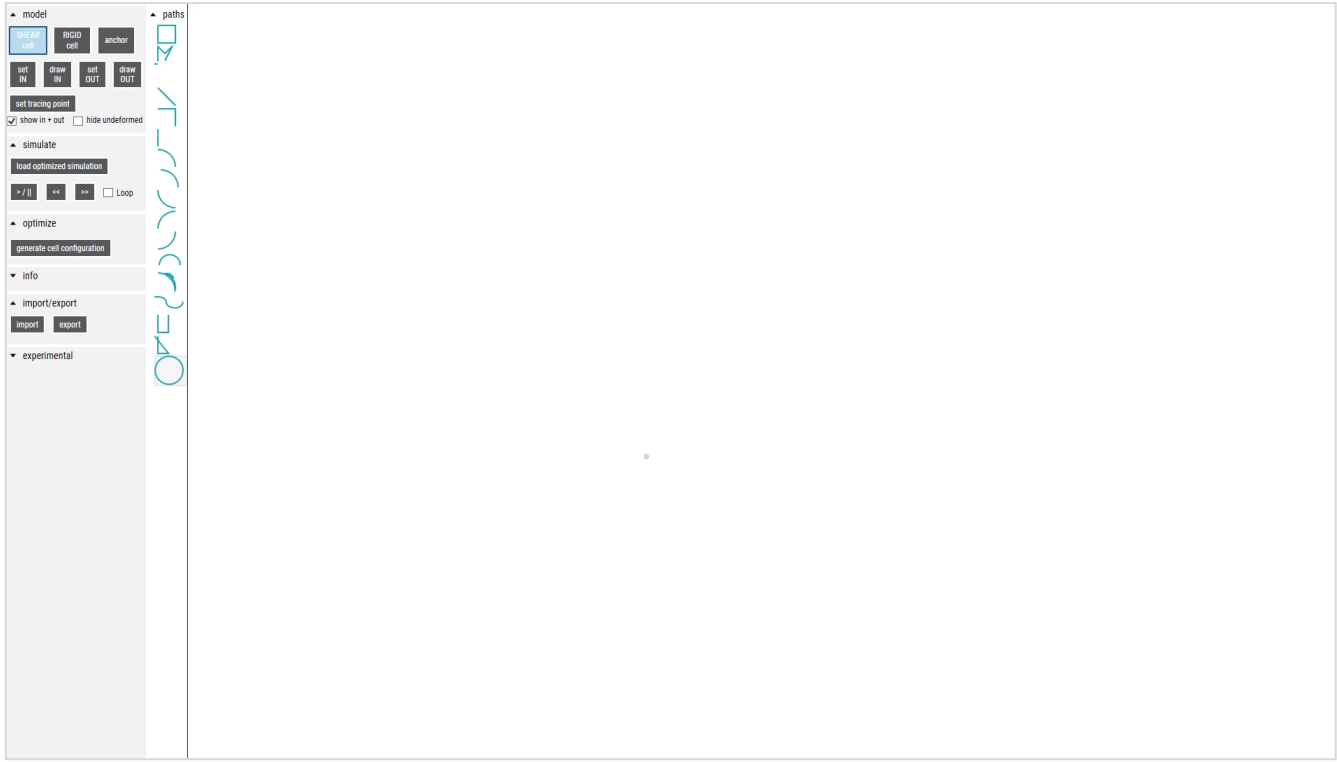
the work we do in the **lab**

example: Metamaterial Mechanisms
[UIST'16, CHI'19]

structures



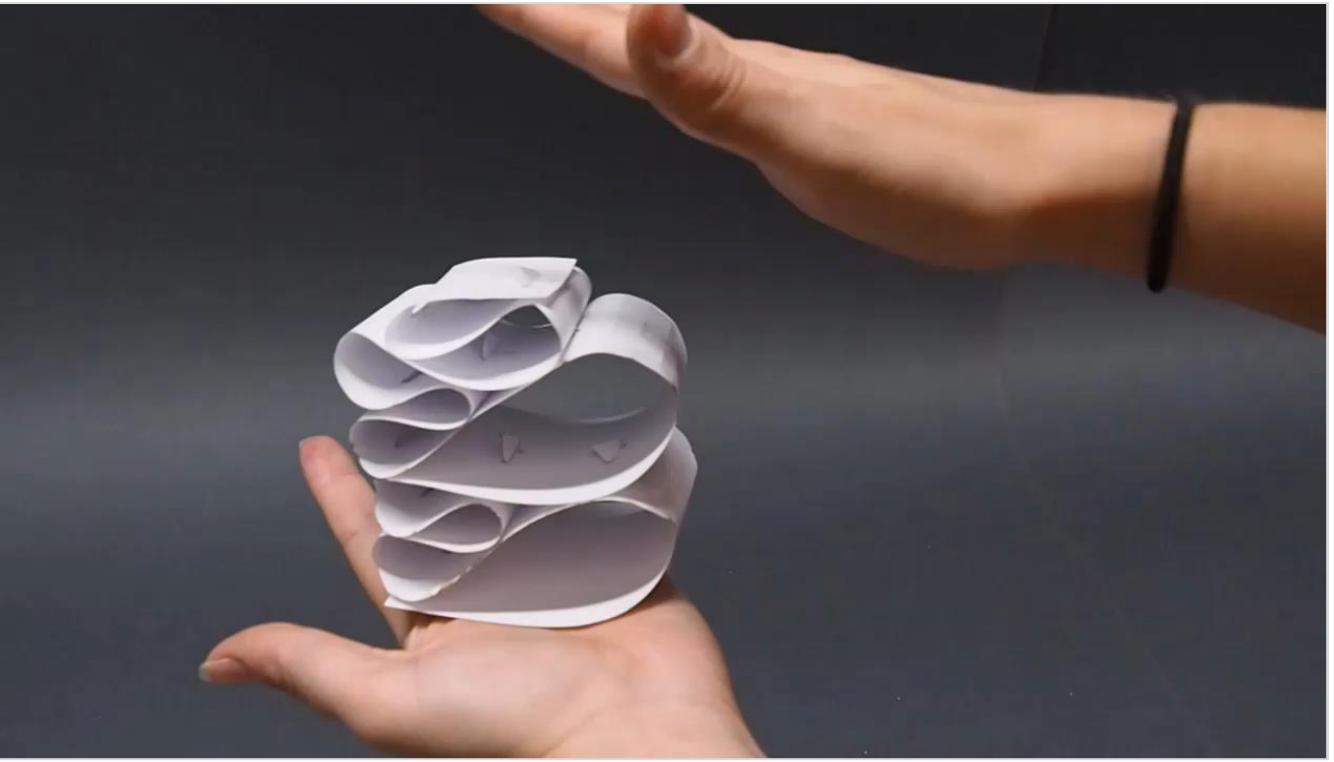
editor



inverse design
combinatorial optimization, meta-heuristics

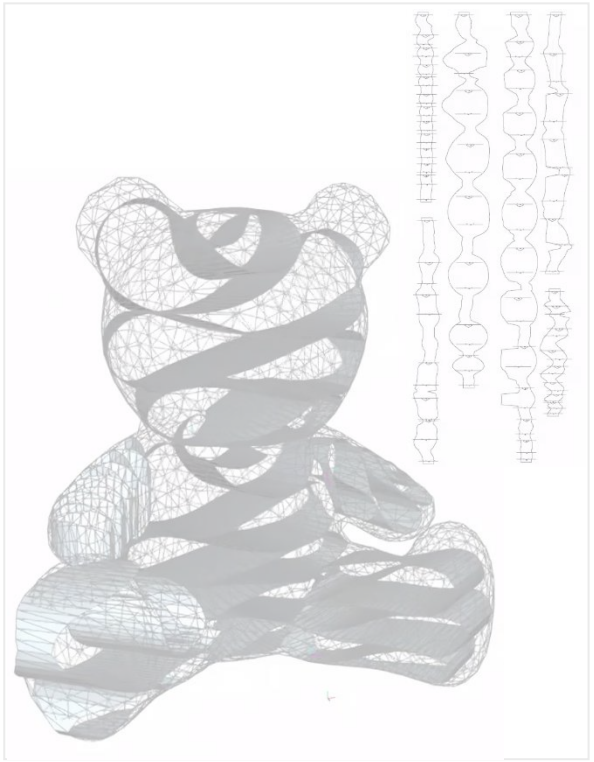
the work we do in the **lab**

example: Developable Metamaterials [CHI'21]



structures

developable, mass-manufacturable, easy to assemble



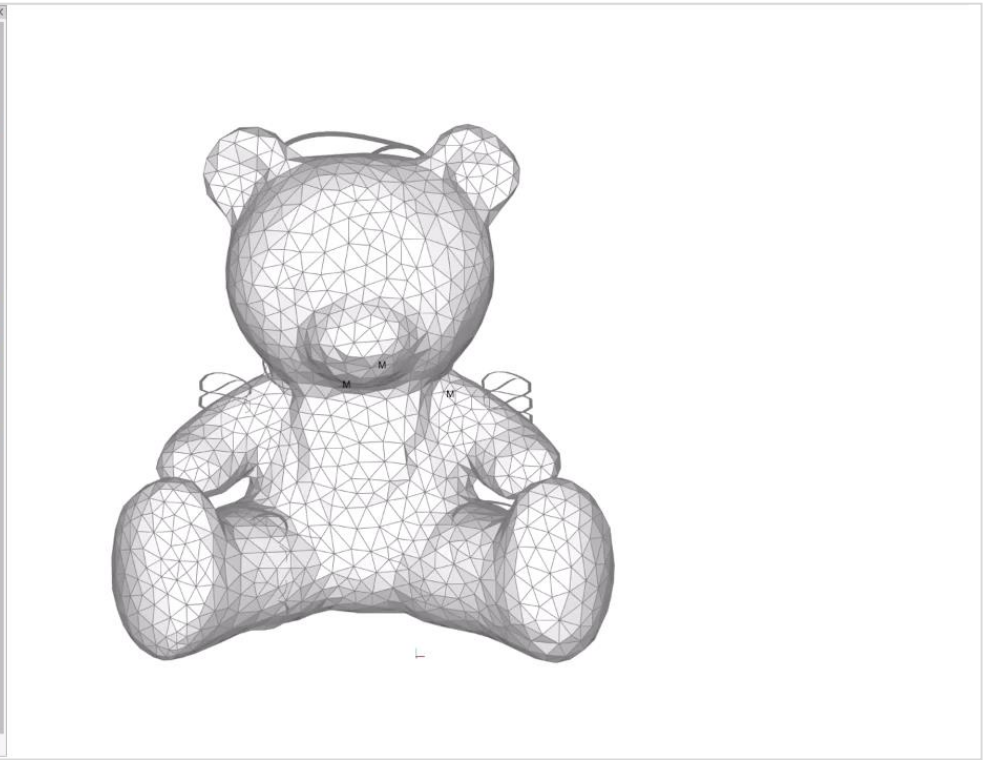
inverse design

impactful applications

the work we do in the **lab**

example: Developable Metamaterials [CHI'21]

structures



inverse design

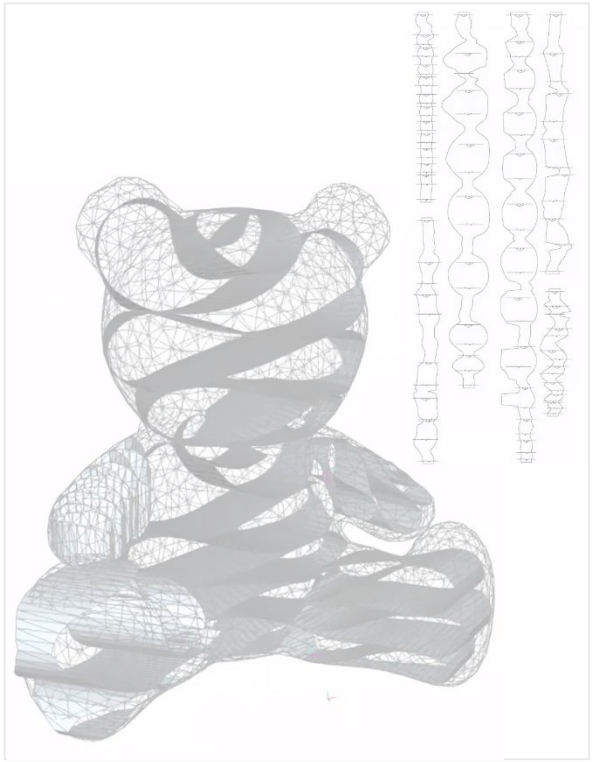
thin shells, physics simulation, shape optimization

impactful applications

the work we do in the **lab**

example: Developable Metamaterials [CHI'21]

structures

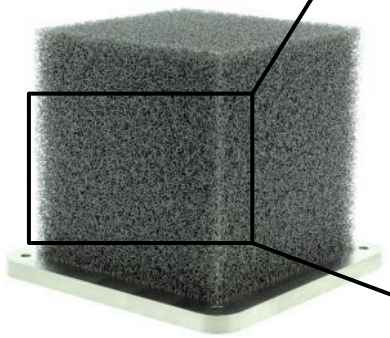


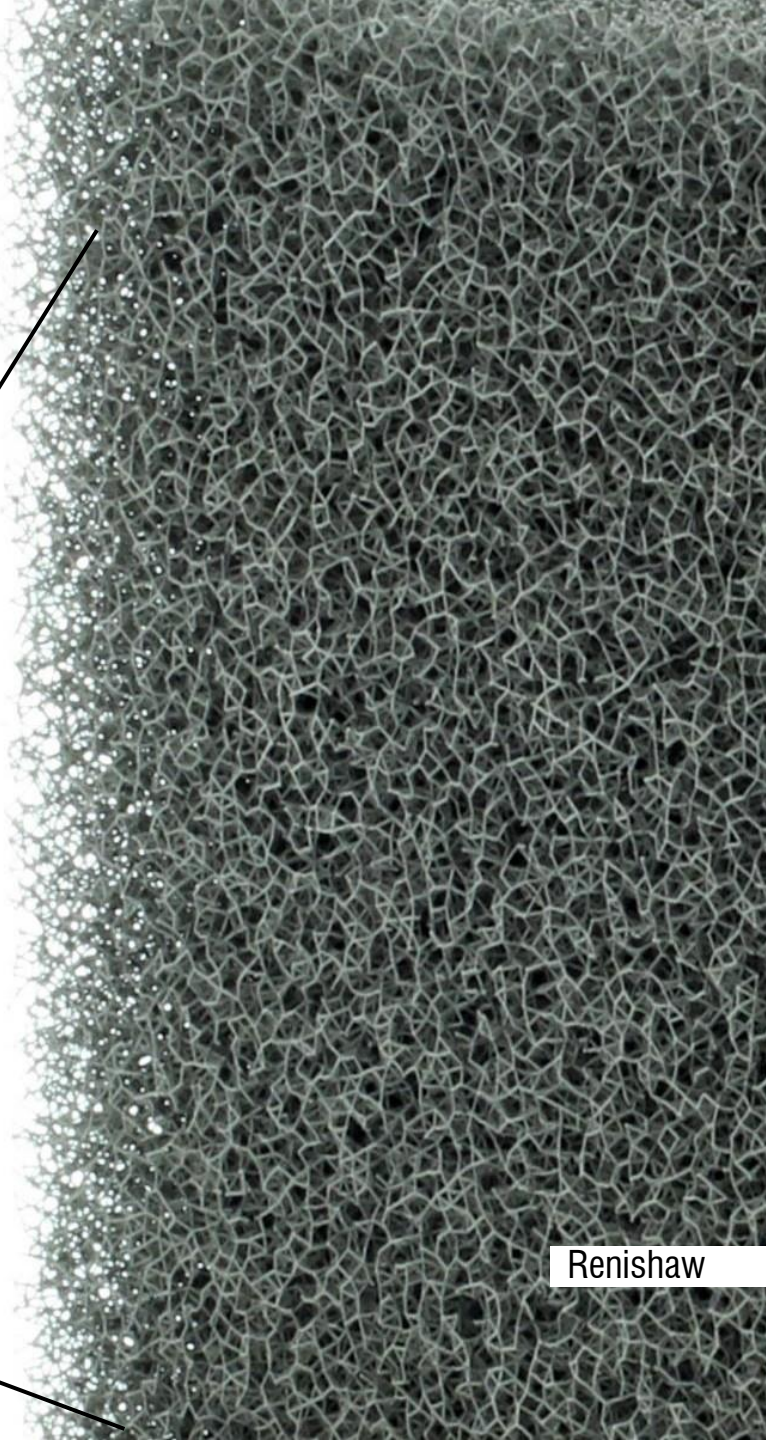
inverse design

impactful applications

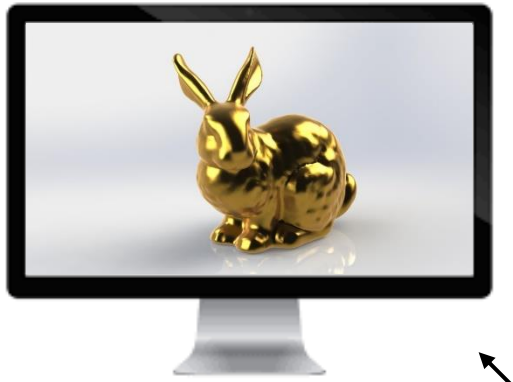
customizable, affordable, sensorized and actuated prostheses

Geometric complexity





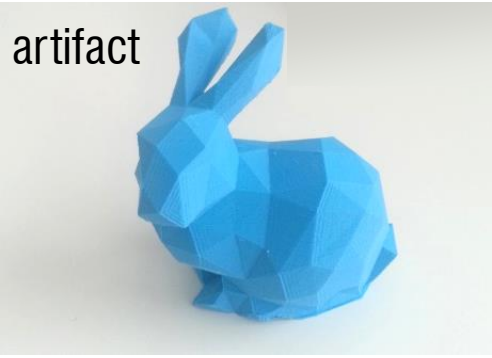
Renishaw

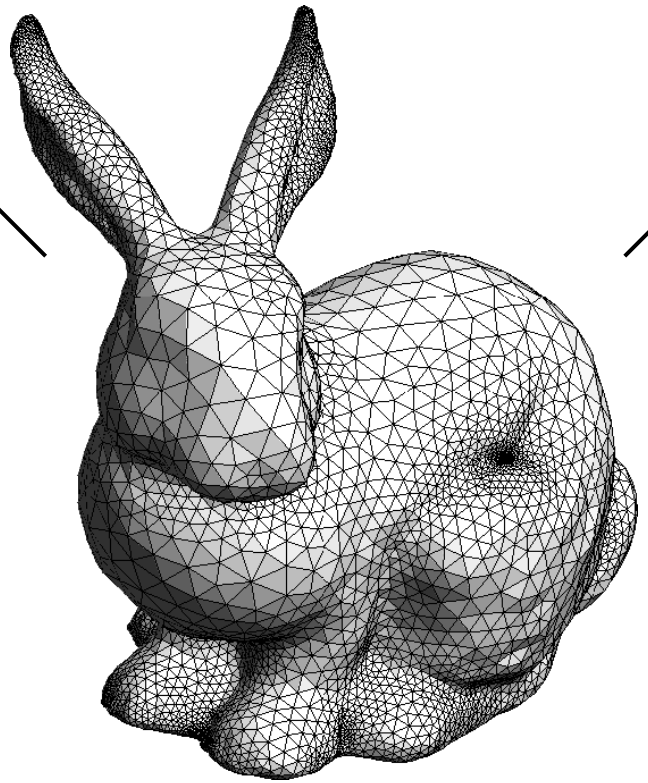


2D output

3D output

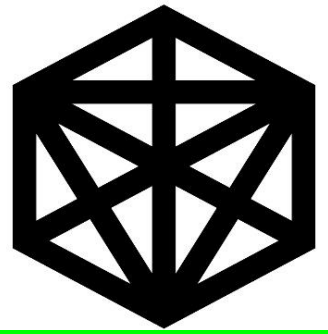
artifact





geometry

geometry



interactive structures lab

basics of geometry processing

Carnegie Mellon University

Minor updates in git, please pull quickly. No need to rebuild.

1.1 Introduction to libigl

1.2 DDG curves

1.3 Smoothing meshes

1.1 Introduction to libigl

1.2 DDG curves

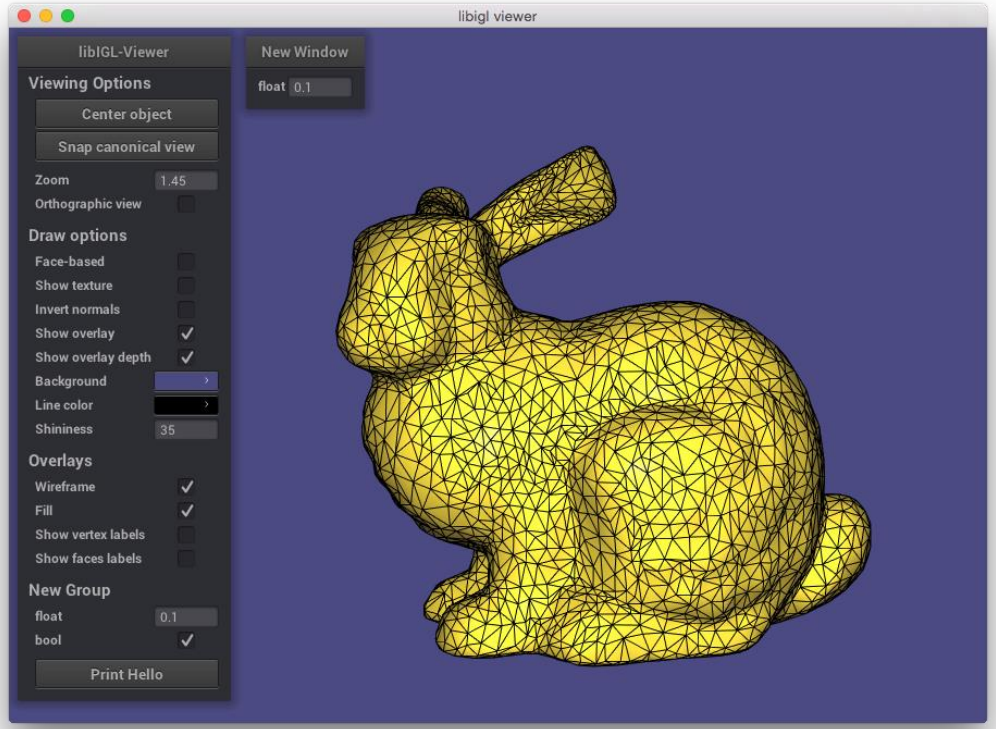
1.3 Smoothing meshes


libigl

libigl data structures & style

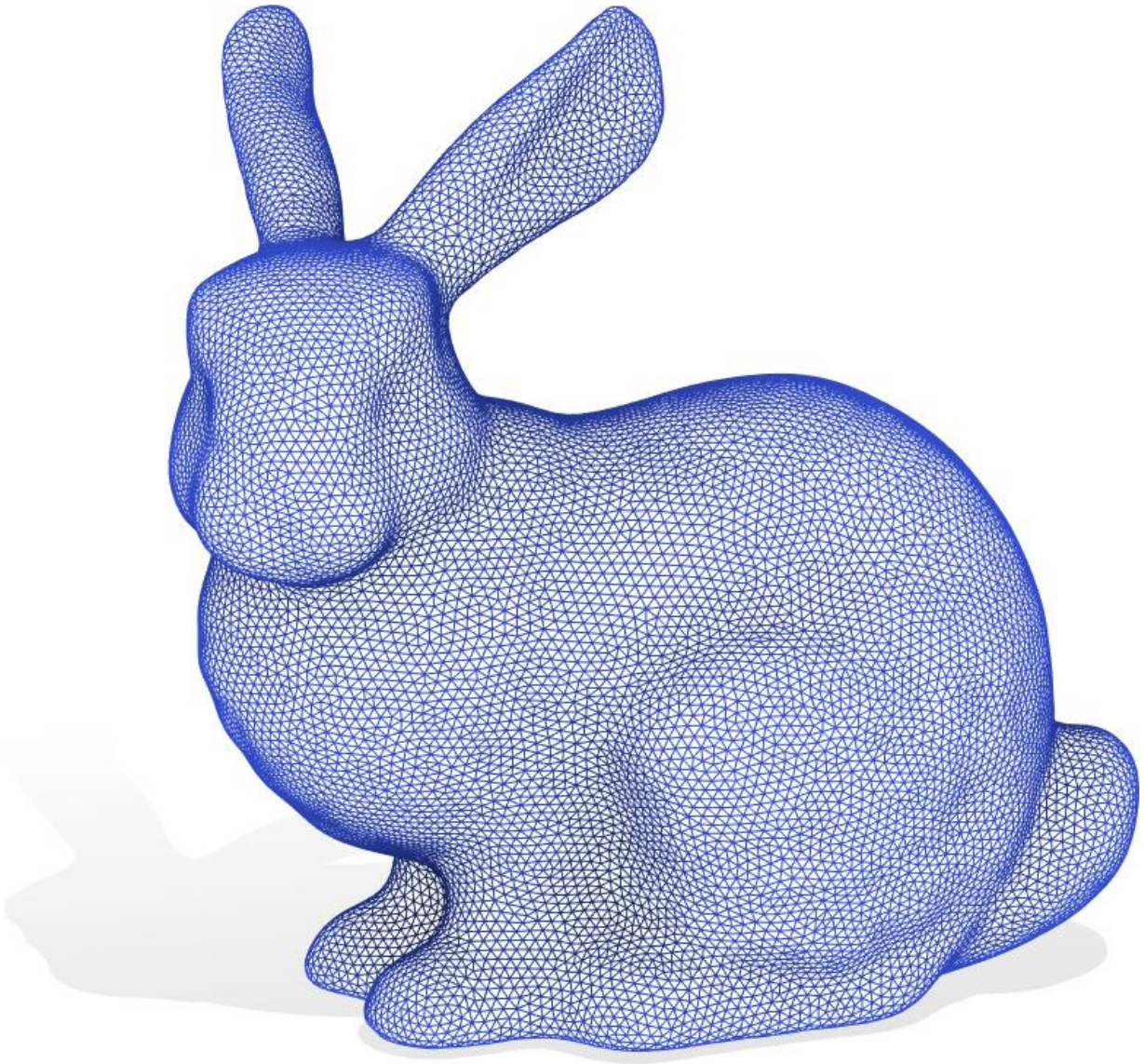
V

F



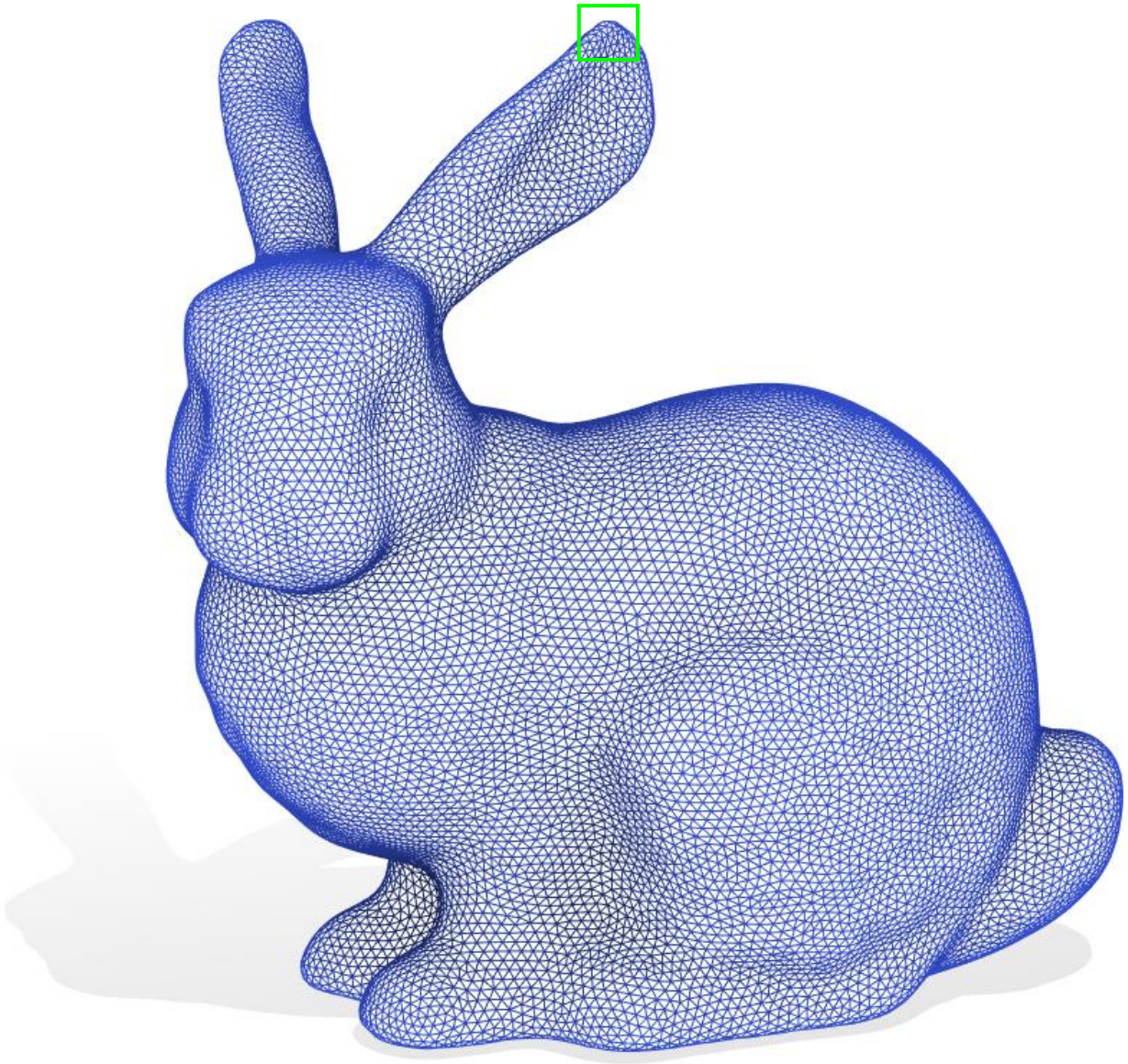
Jacobson & Panozzo, "libigl course"

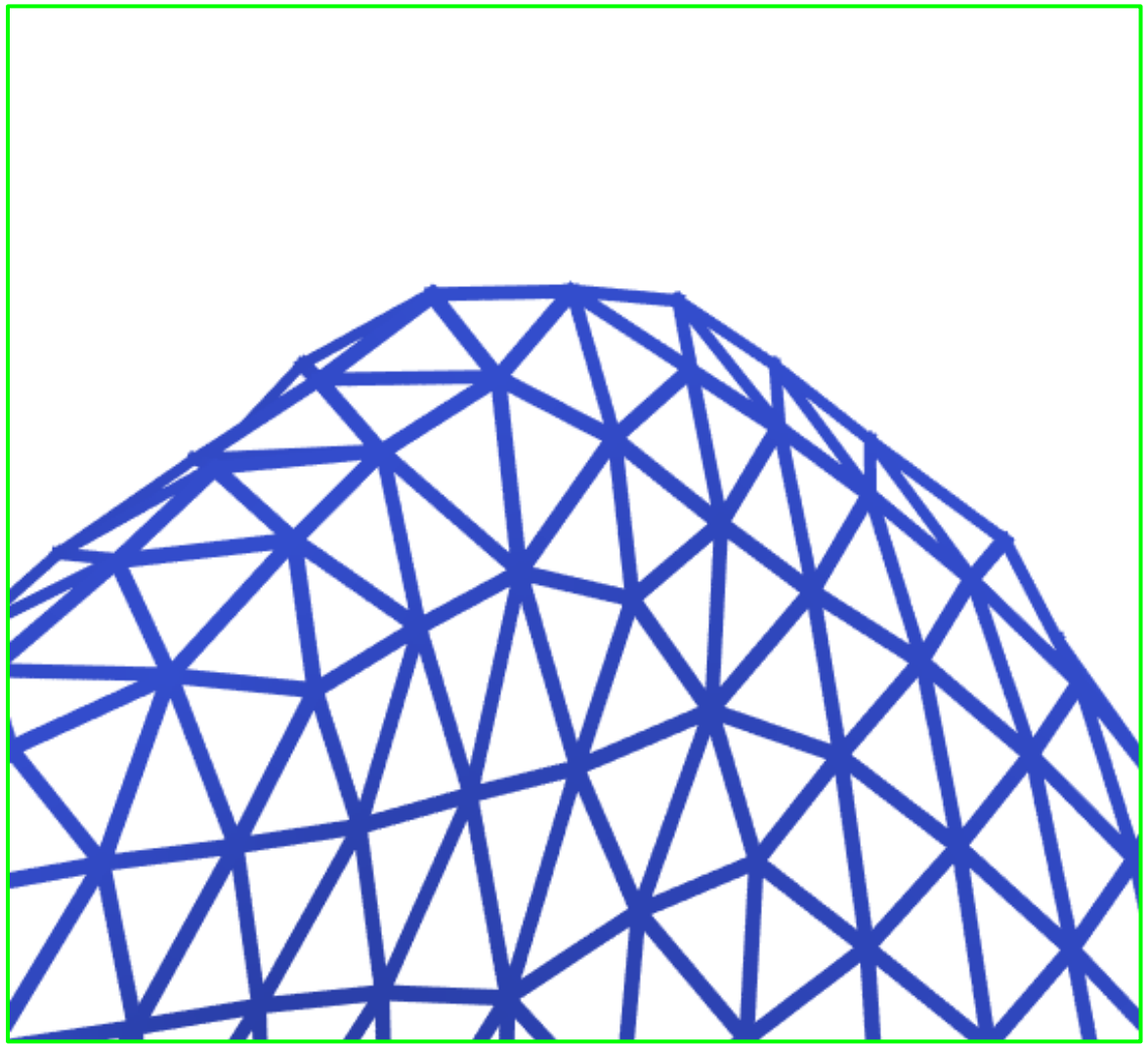
Jacobson & Panozzo, "libigl course"

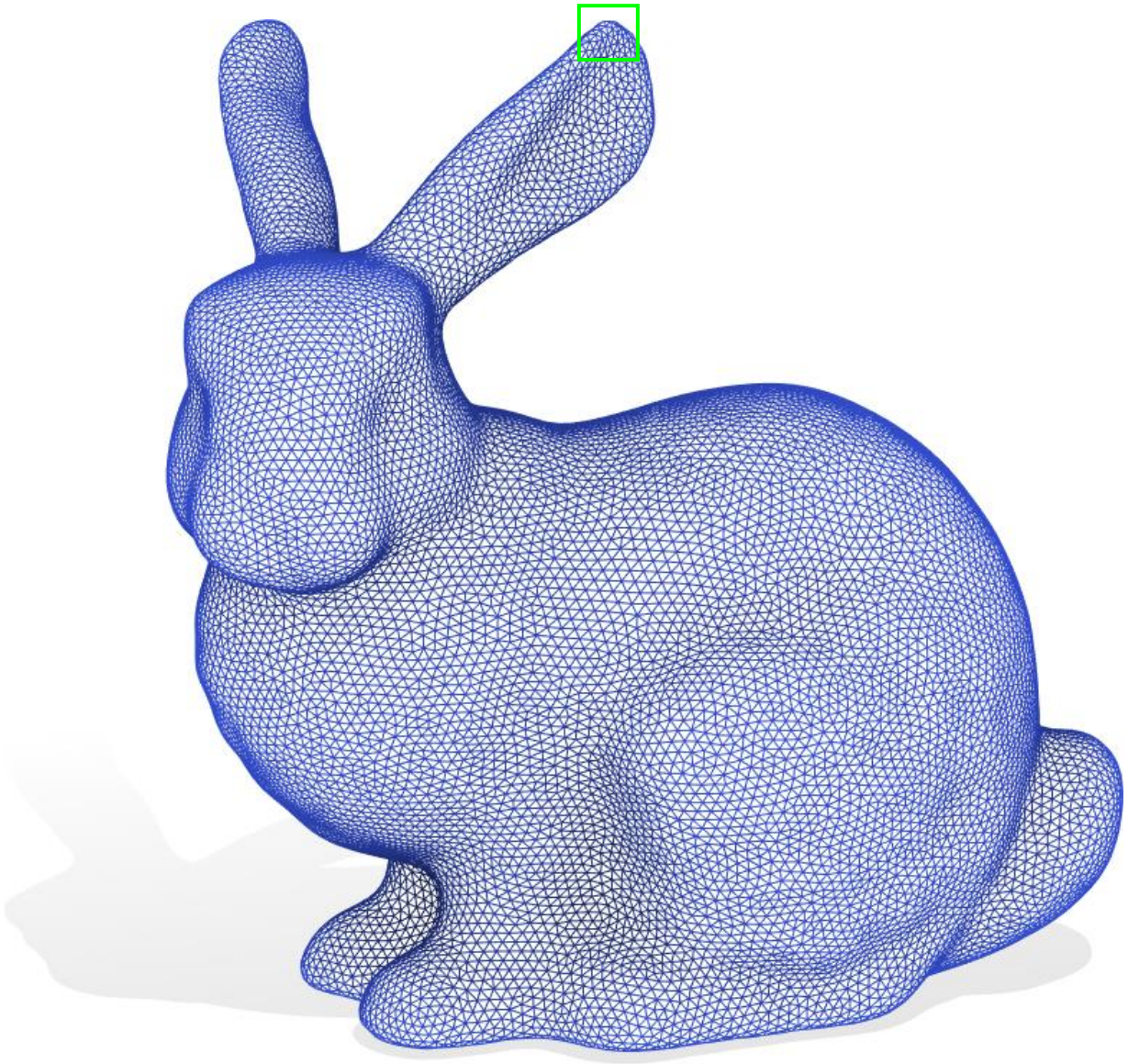


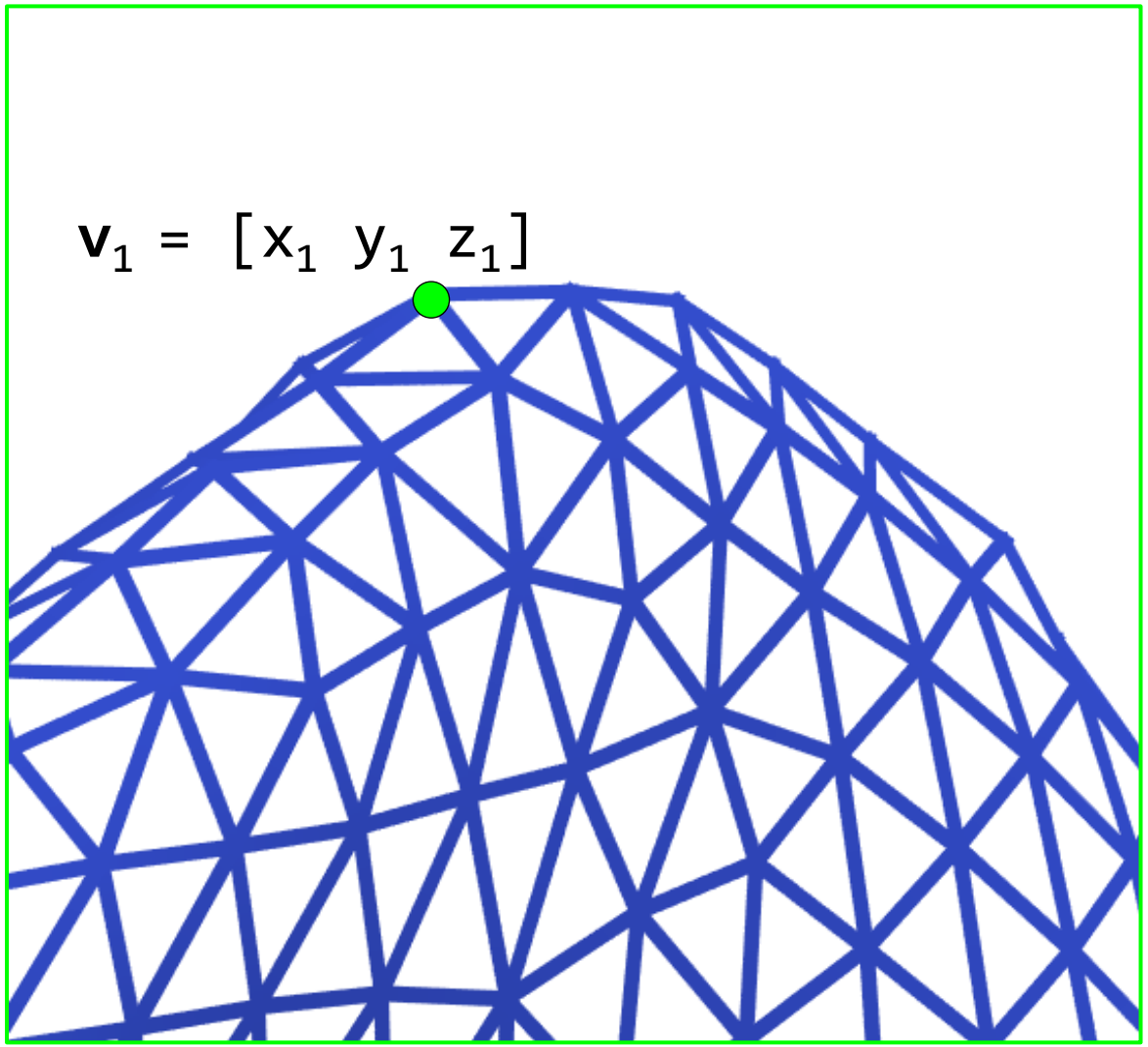
Triangle meshes discretize surfaces

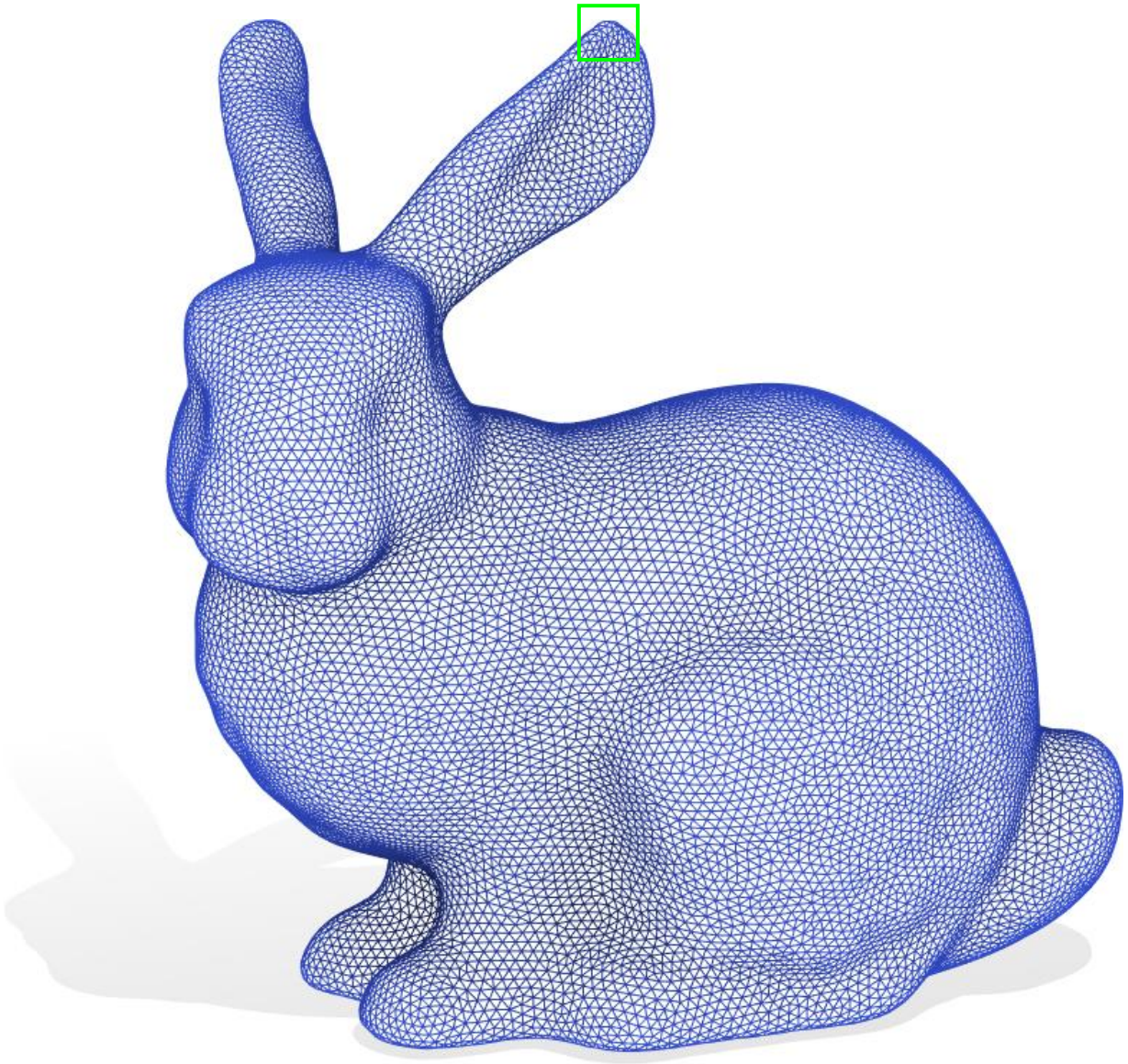
Jacobson & Panozzo, "libigl course"

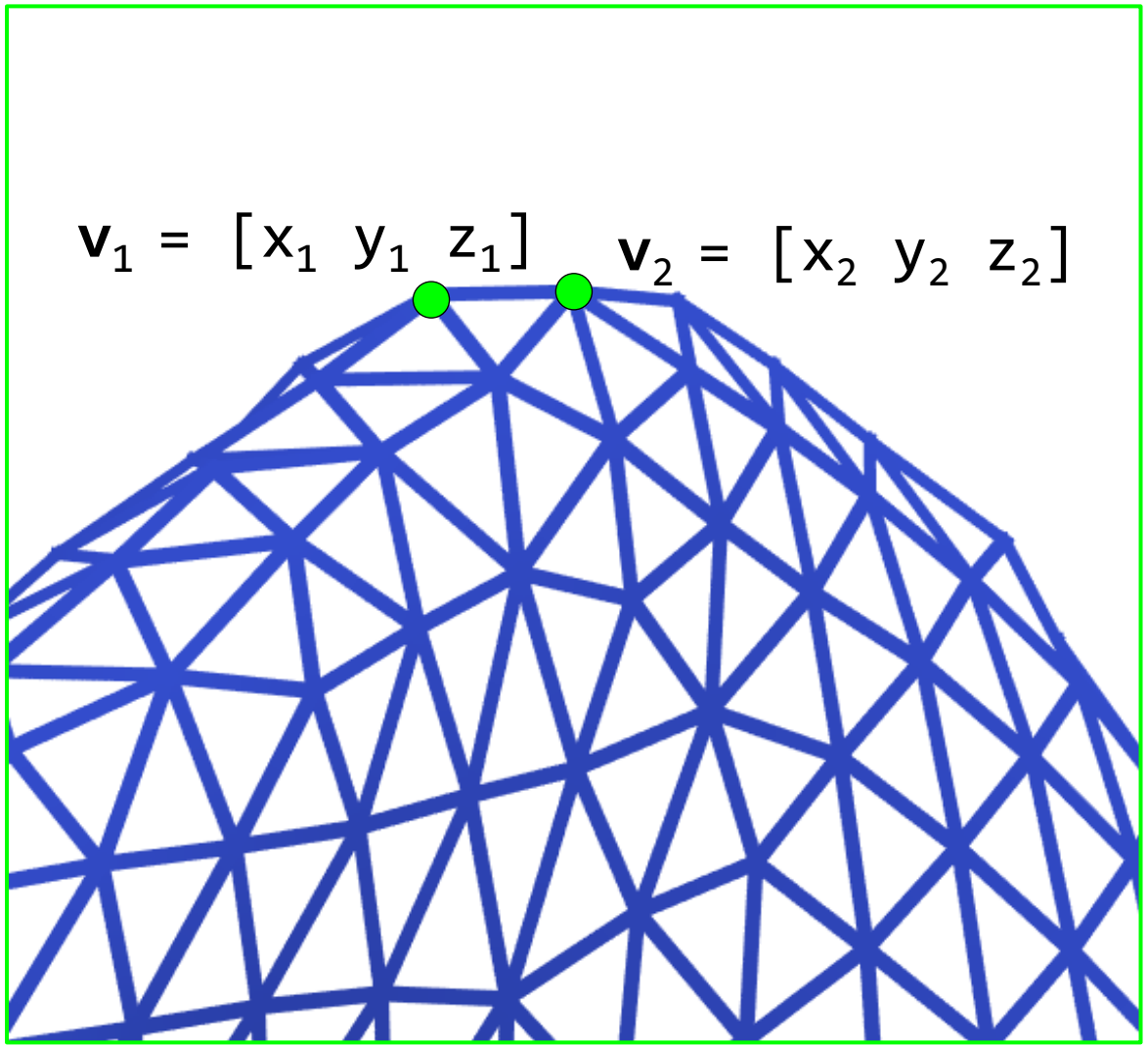


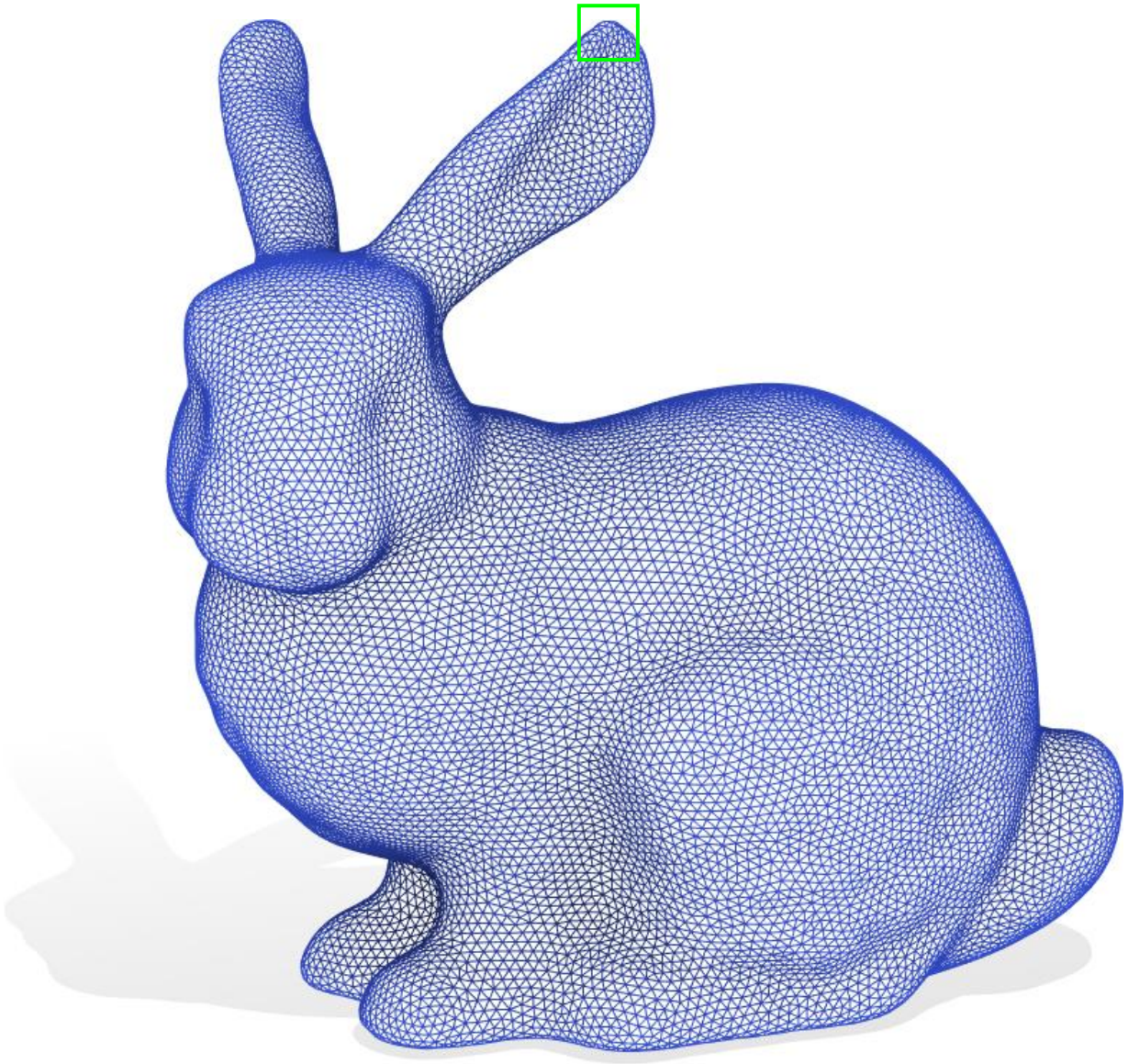


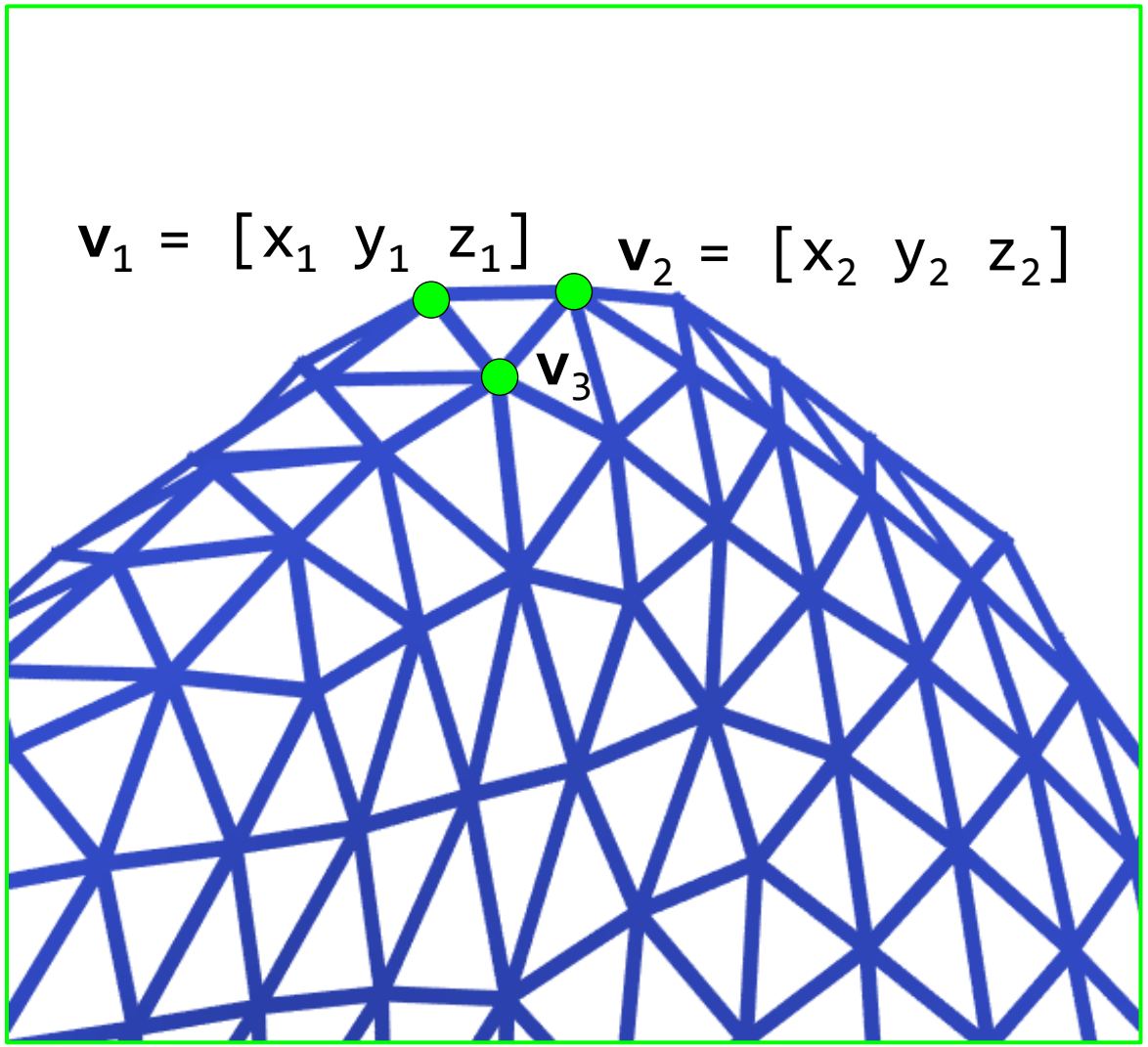


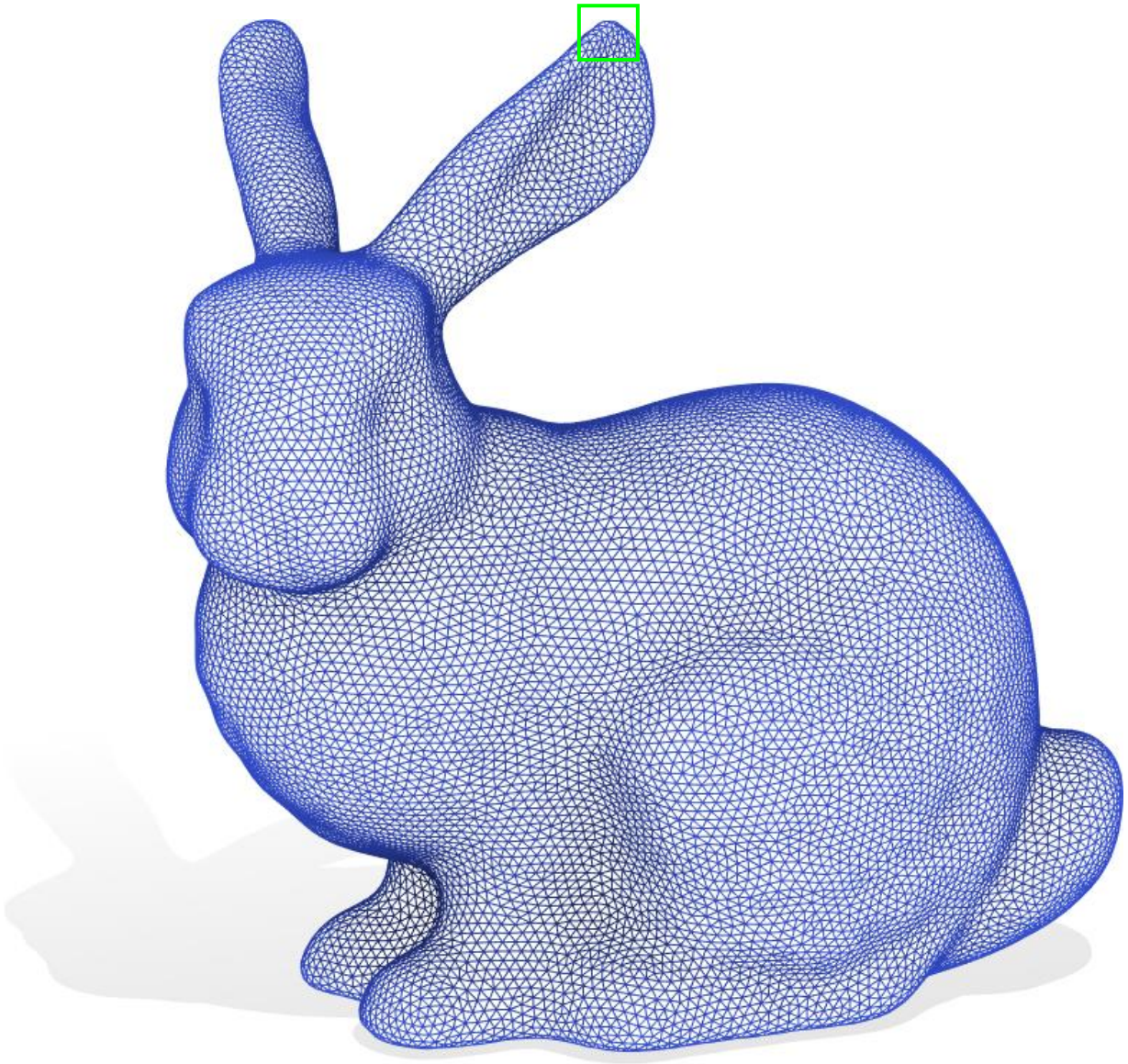

$\mathbf{v}_1 = [x_1 \ y_1 \ z_1]$

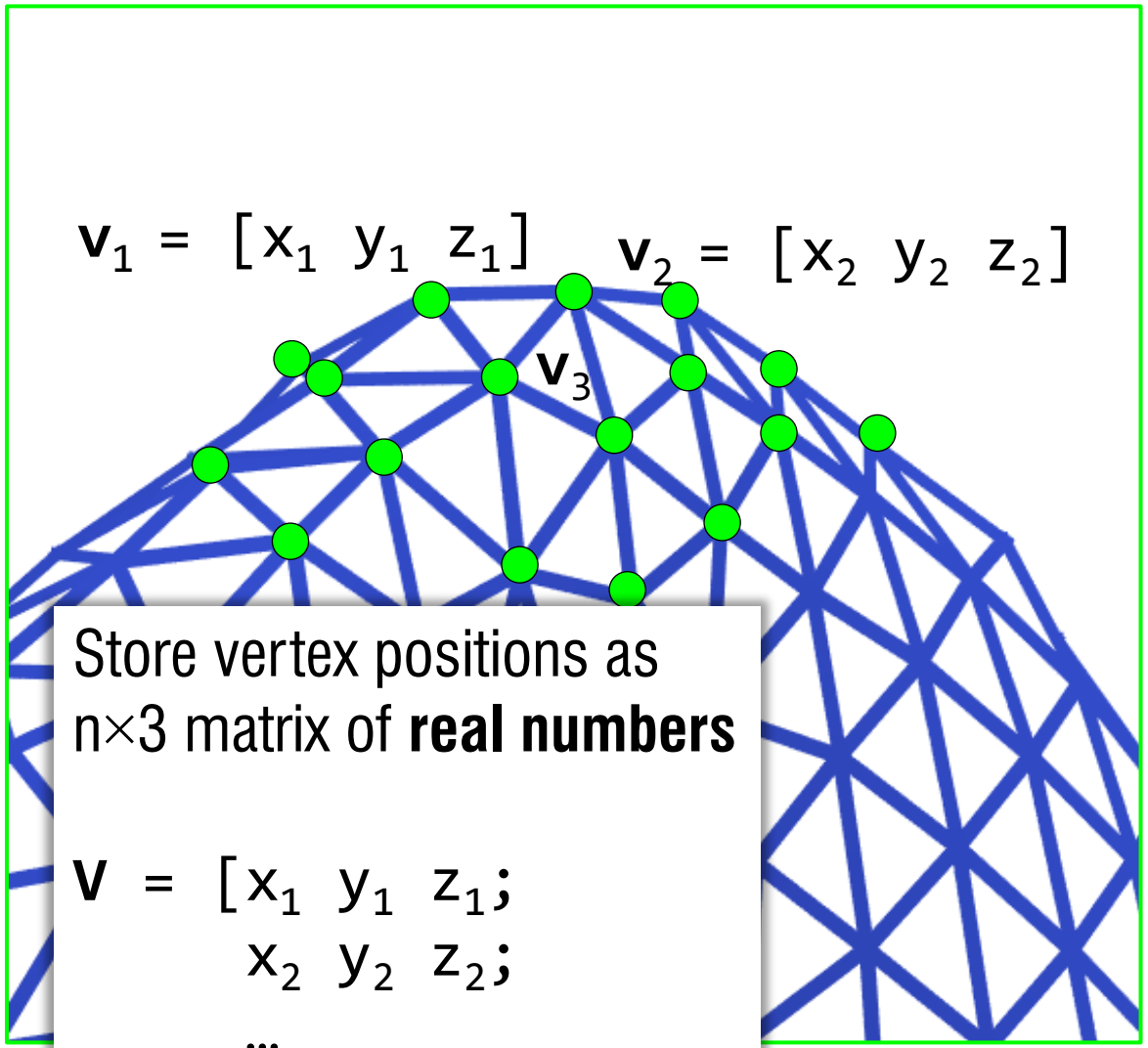



$\mathbf{v}_1 = [x_1\ y_1\ z_1]$
$\mathbf{v}_2 = [x_2\ y_2\ z_2]$



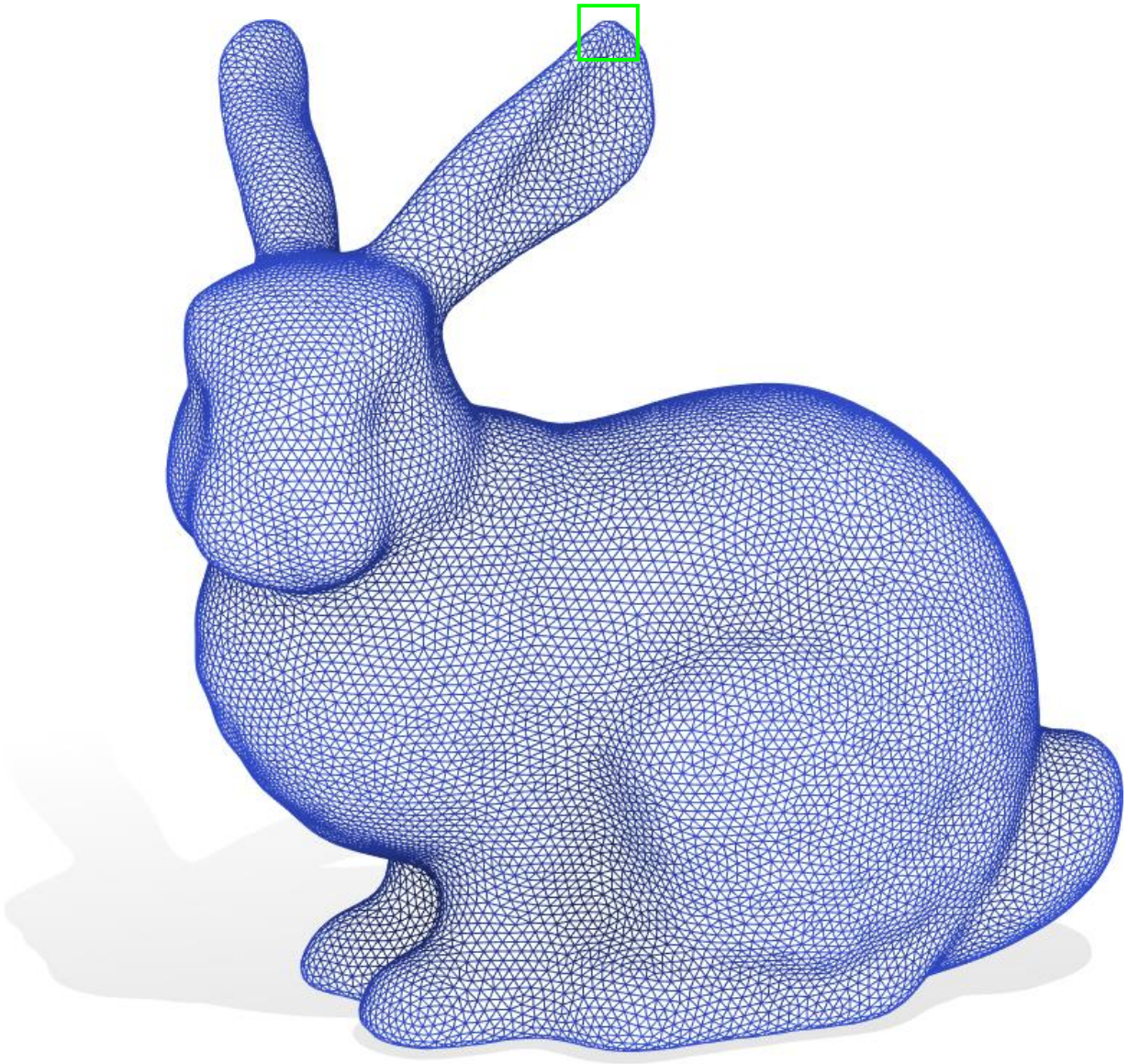

$\mathbf{v}_1 = [x_1\ y_1\ z_1]$
$\mathbf{v}_2 = [x_2\ y_2\ z_2]$
$\mathbf{v}_3$

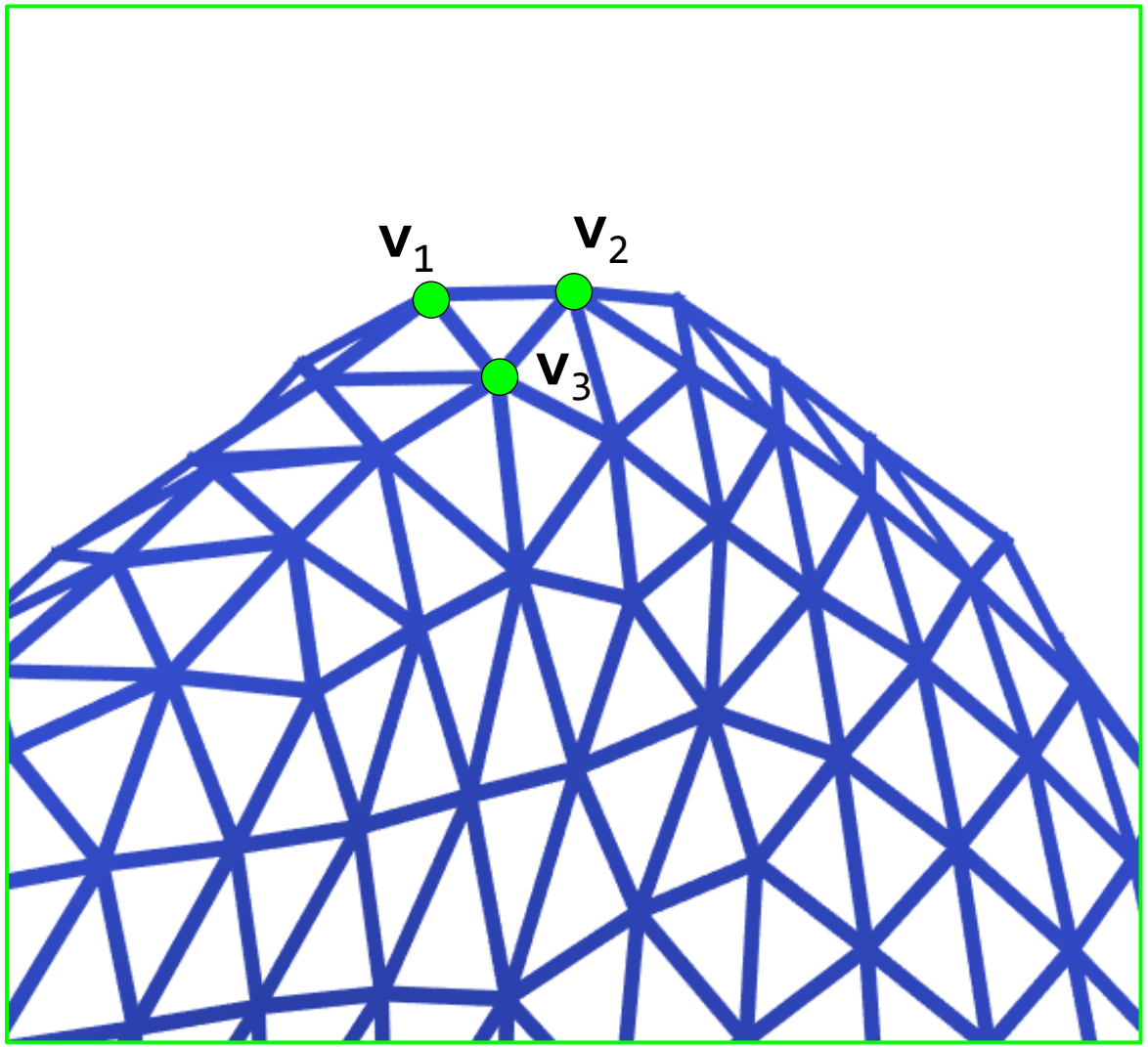


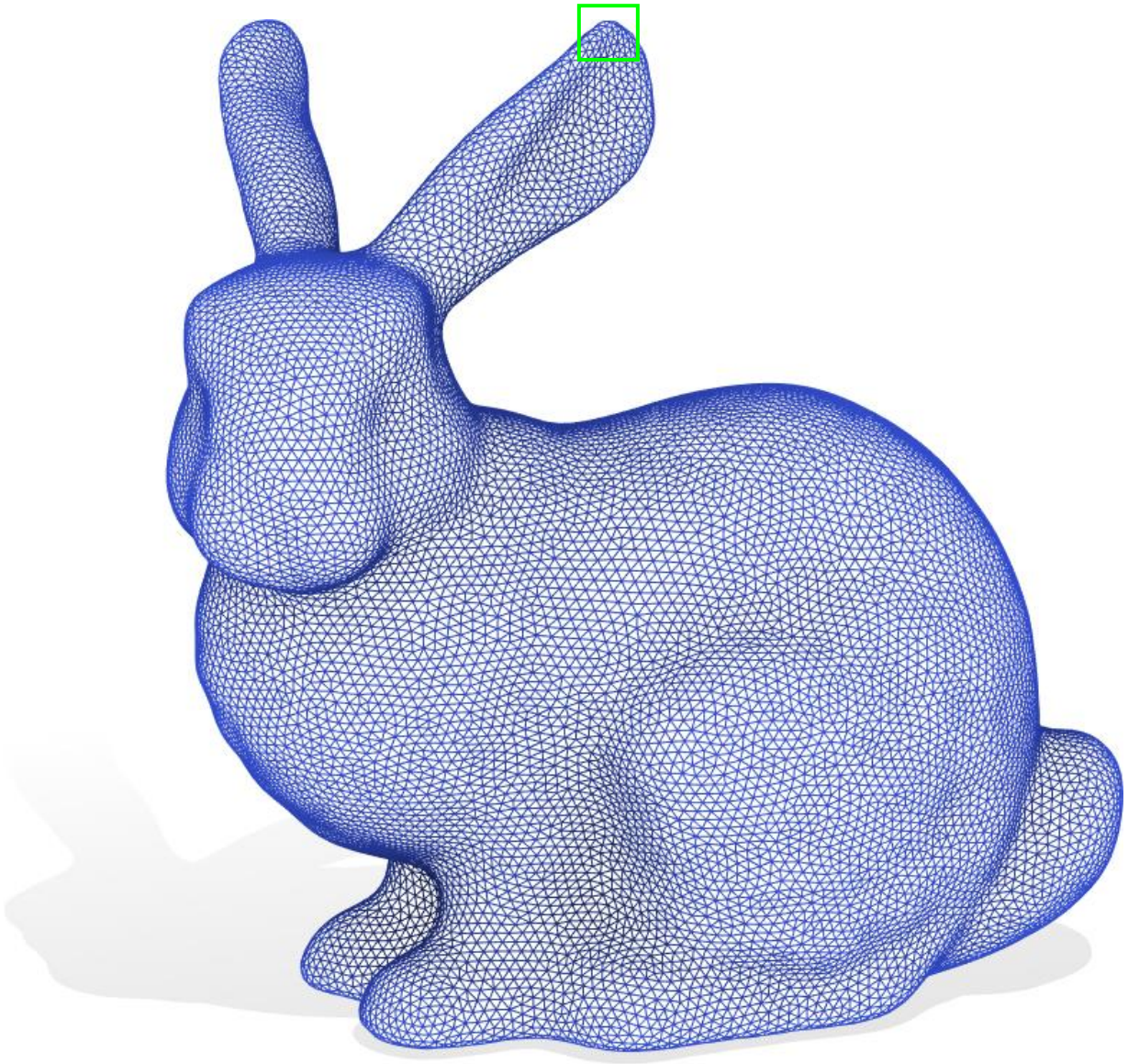


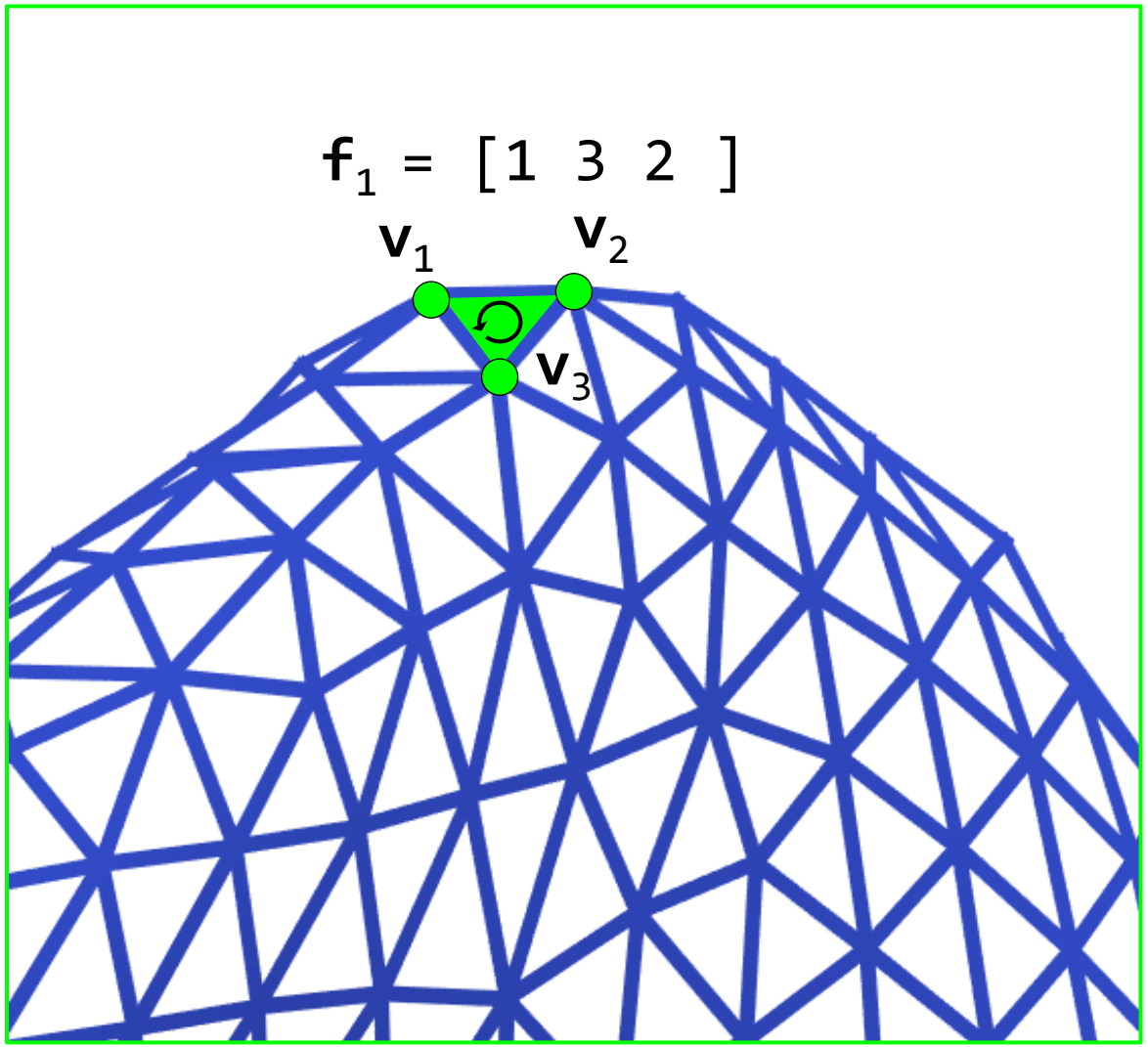
Store vertex positions as n×3 matrix of **real numbers**

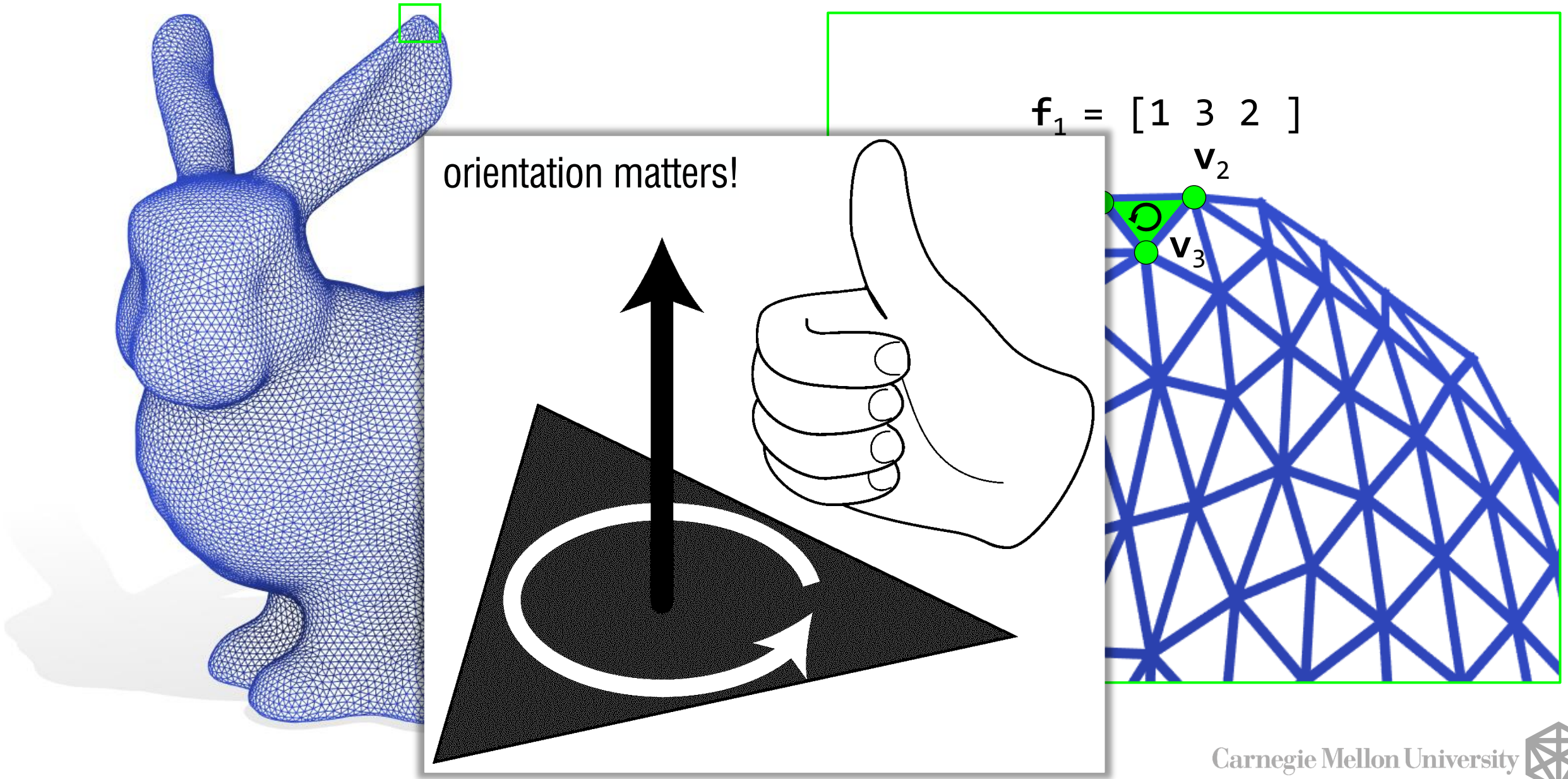
$$\mathbf{V} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \dots & & \\ x_n & y_n & z_n \end{bmatrix}$$

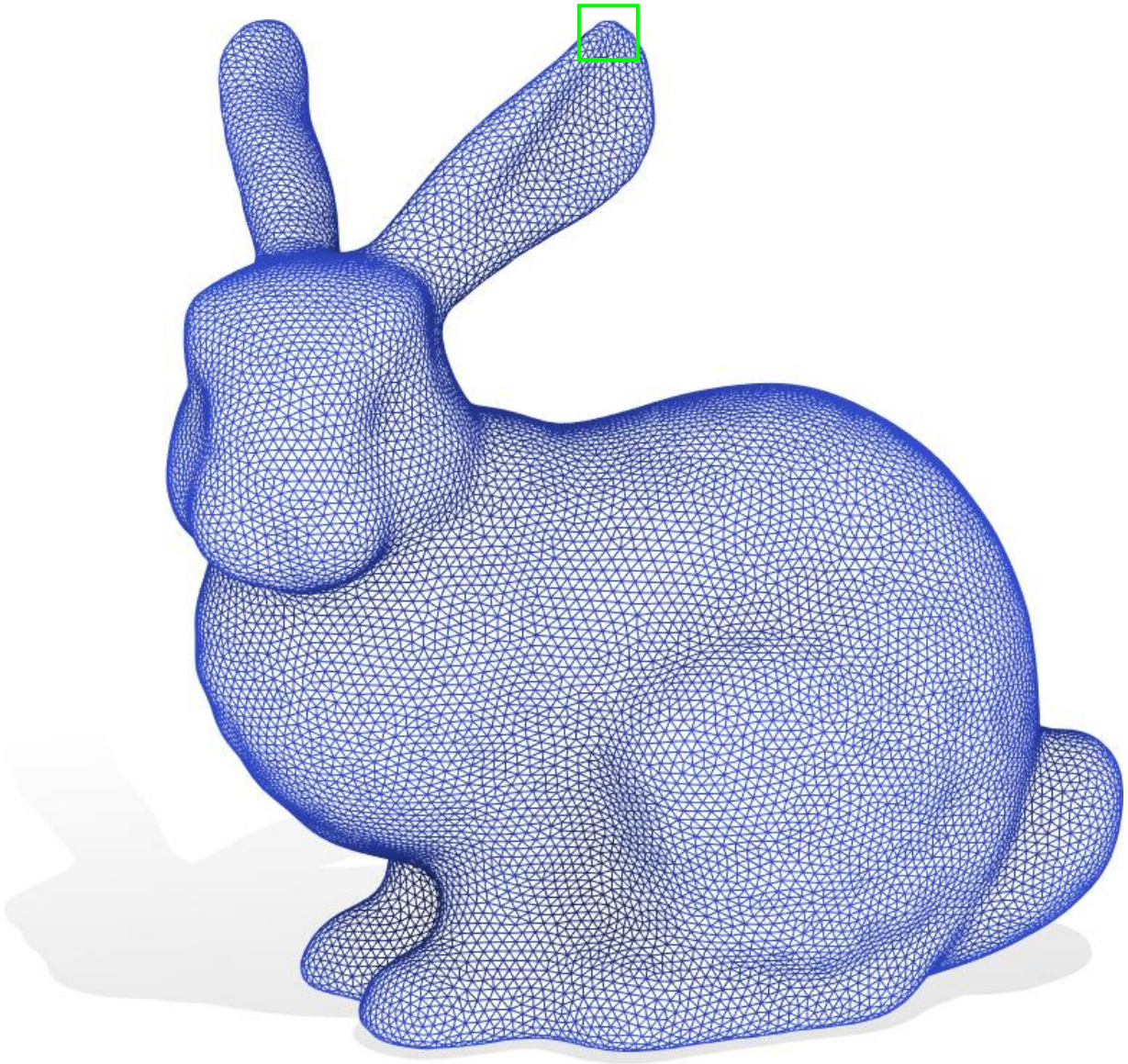


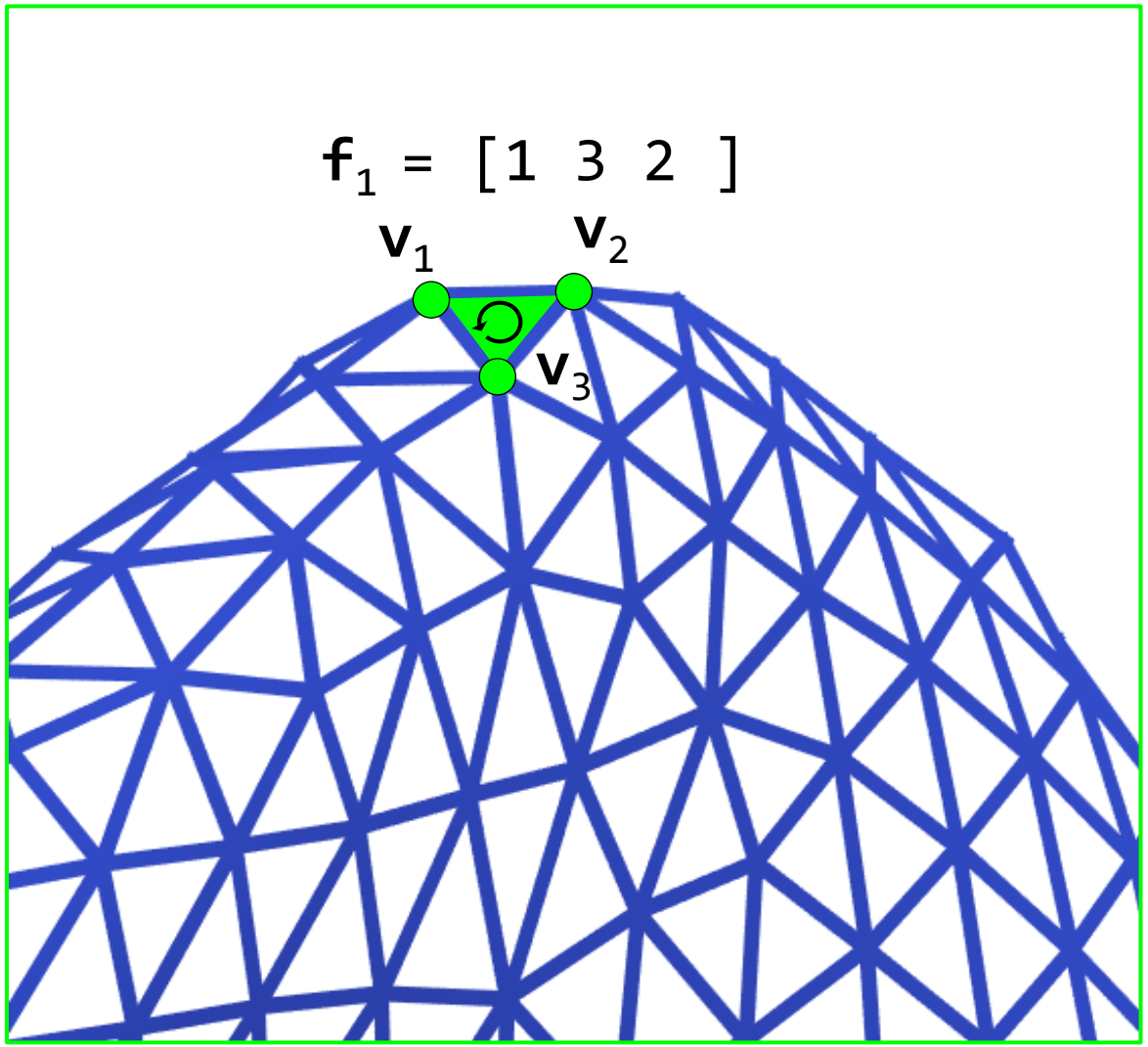

v_1
v_2
v_3

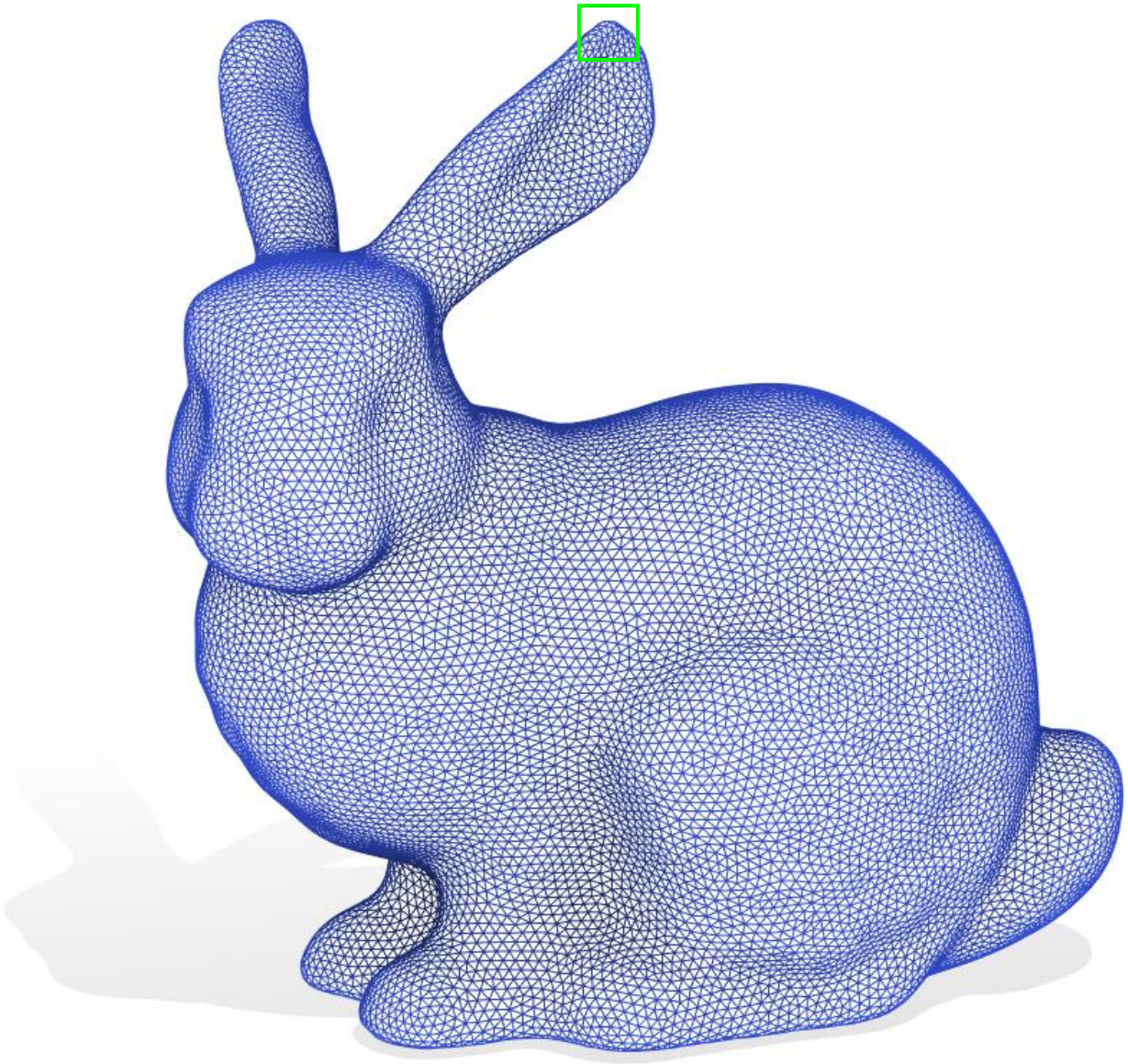


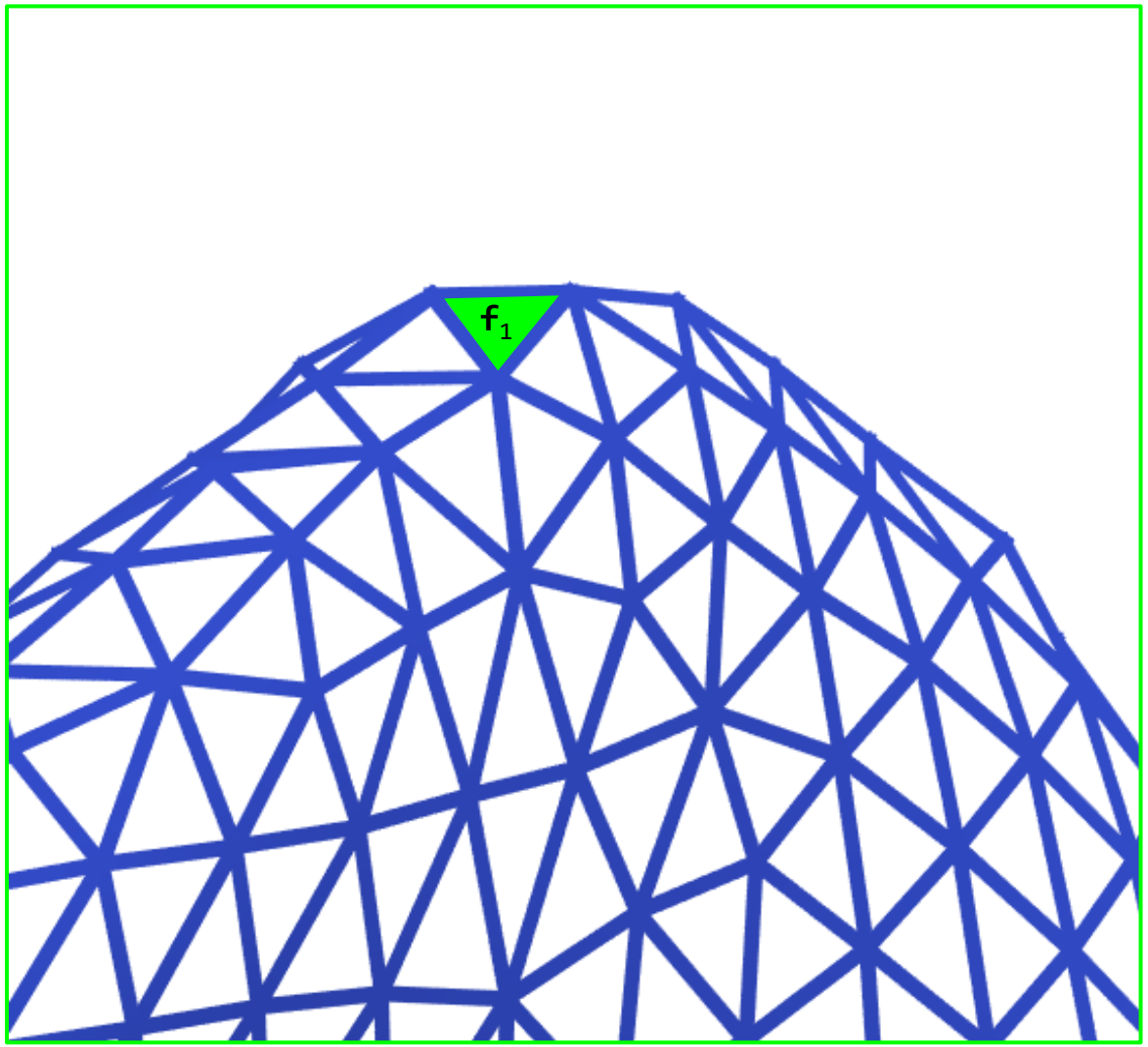

$f_1 = [1\ 3\ 2\ ]$
v_1
v_2
v_3

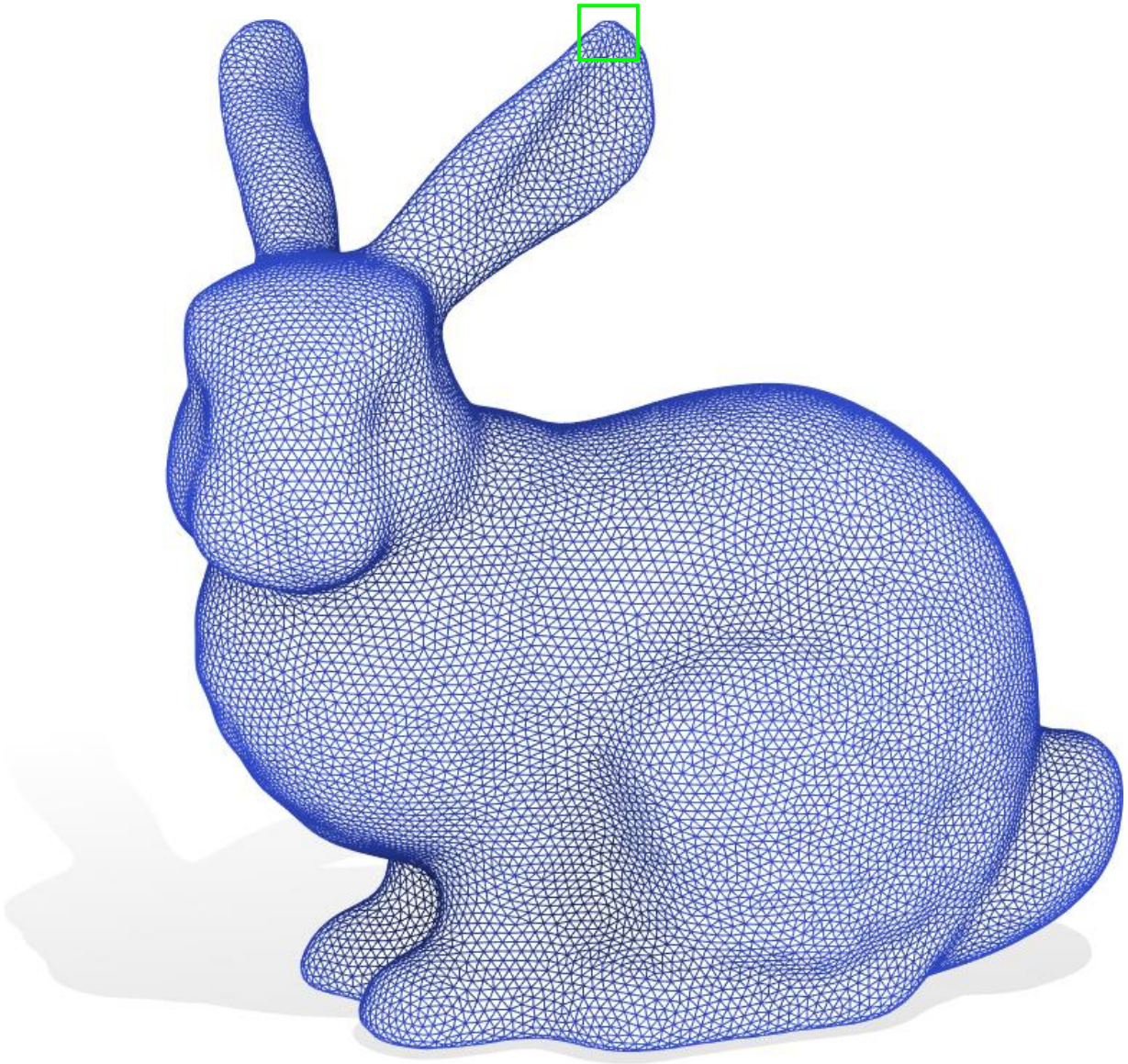

$\mathbf{f}_1 = [1\ 3\ 2\ ]$
$\mathbf{v}_2$
$\mathbf{v}_3$
orientation matters!

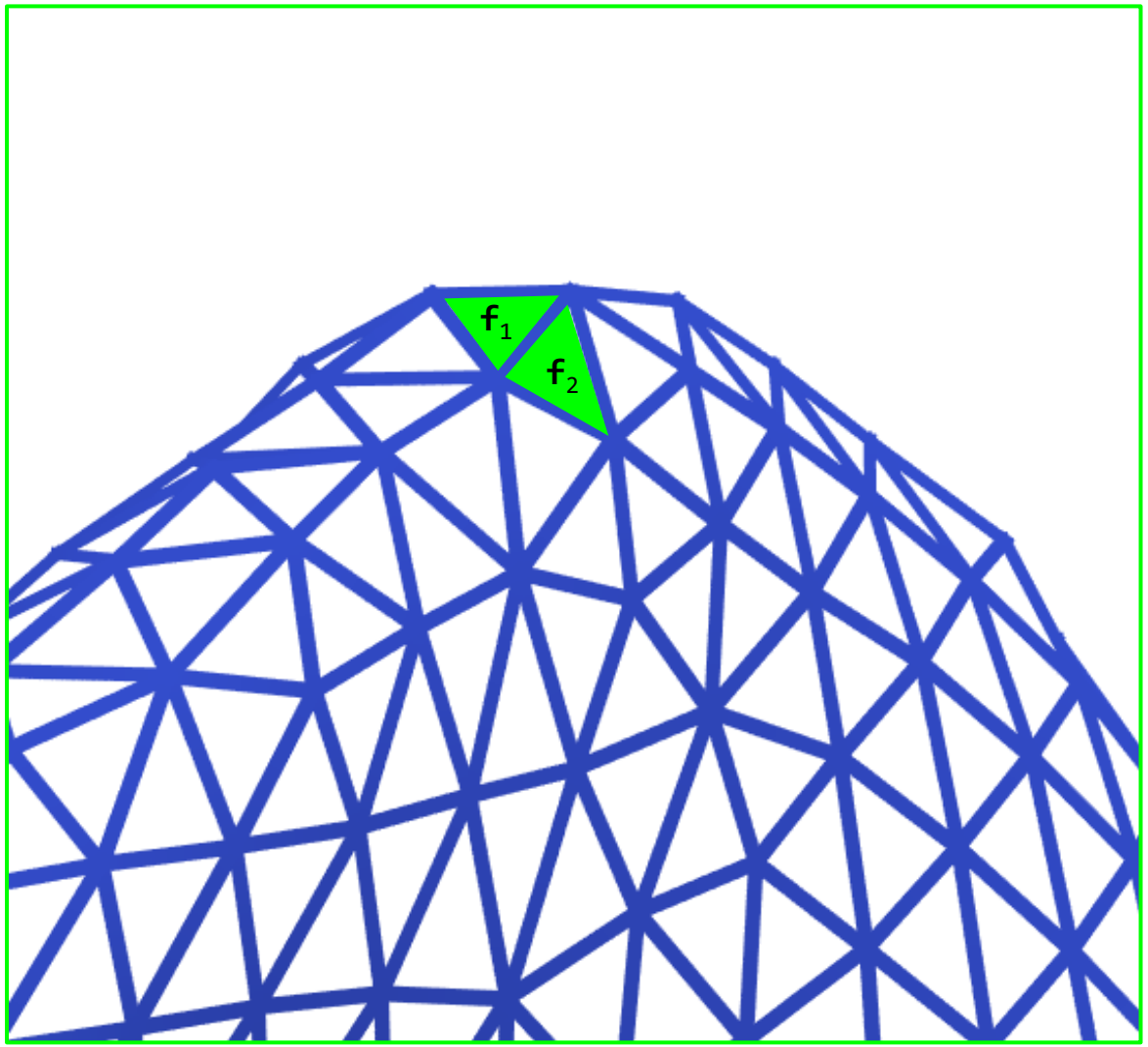


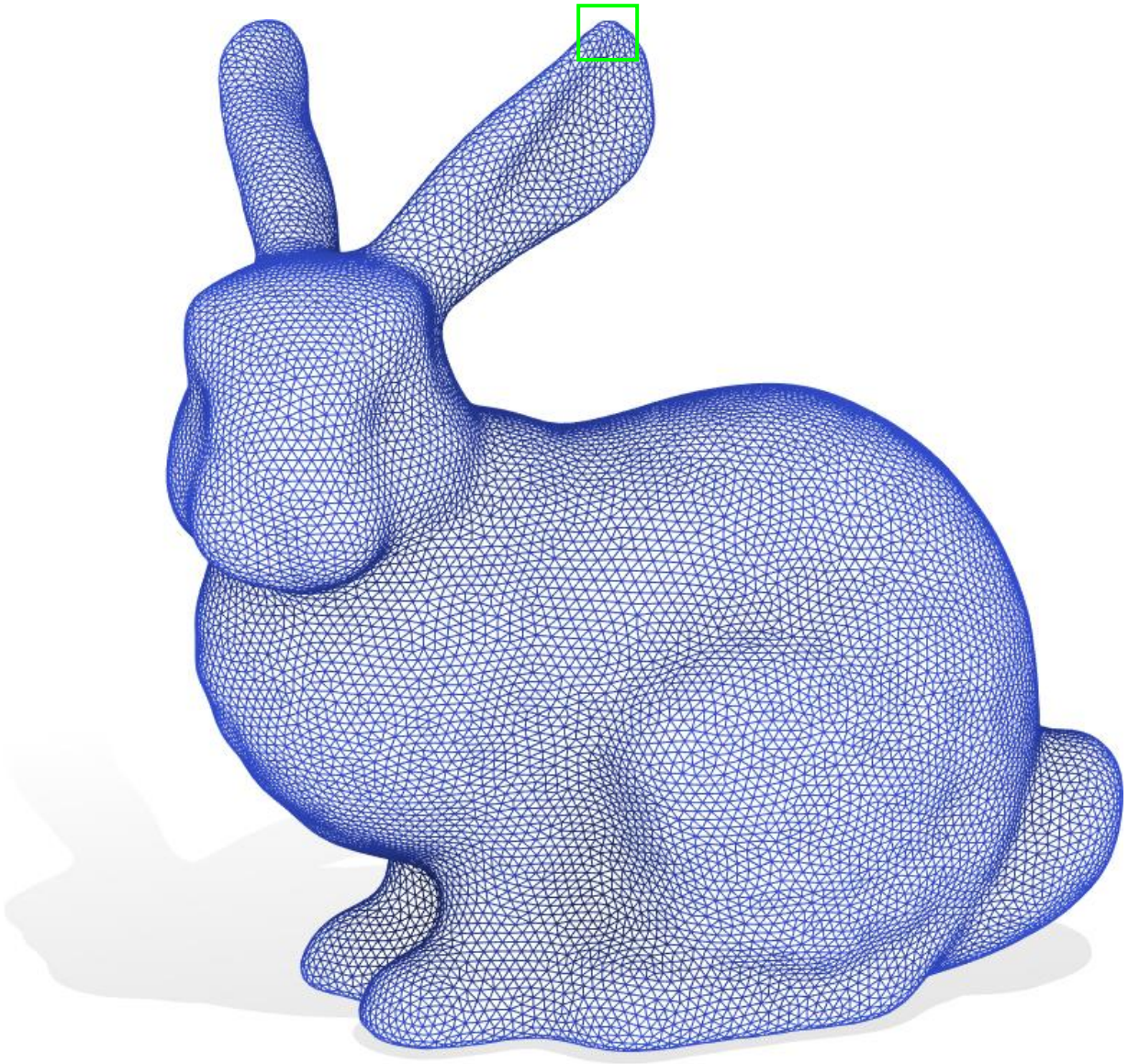

f_1 = [1 3 2]
v_1
v_2
v_3

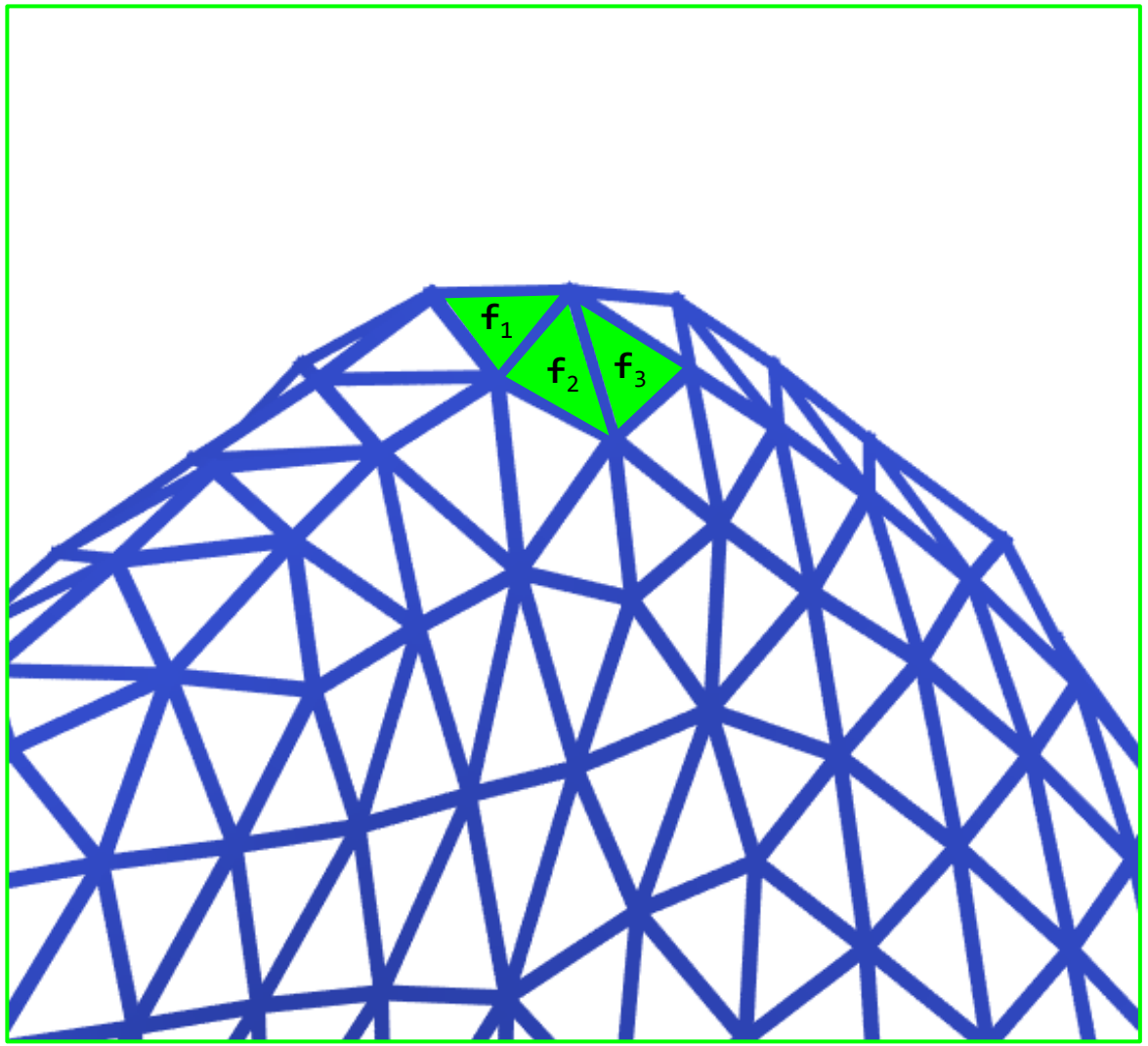


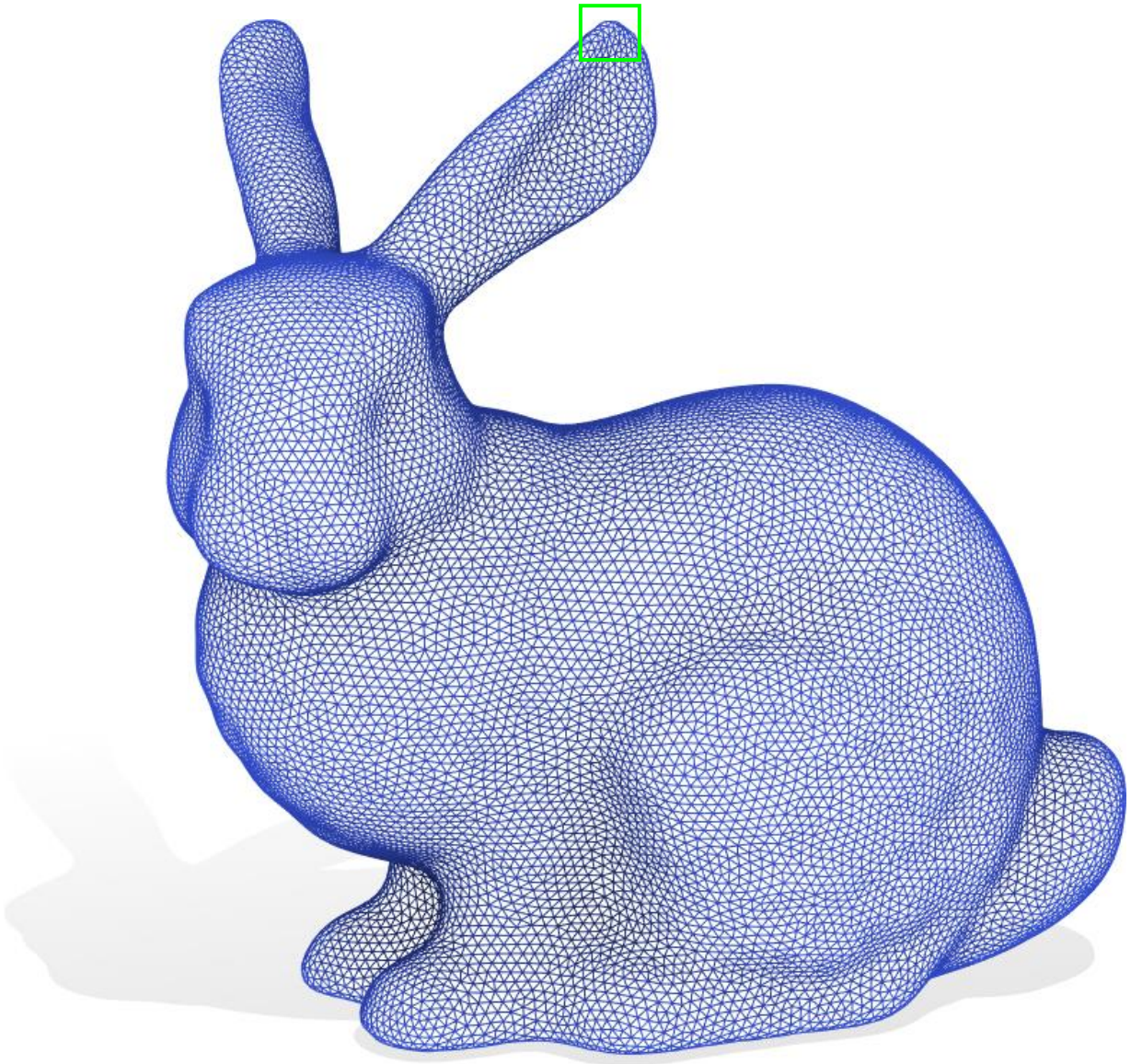

f_1

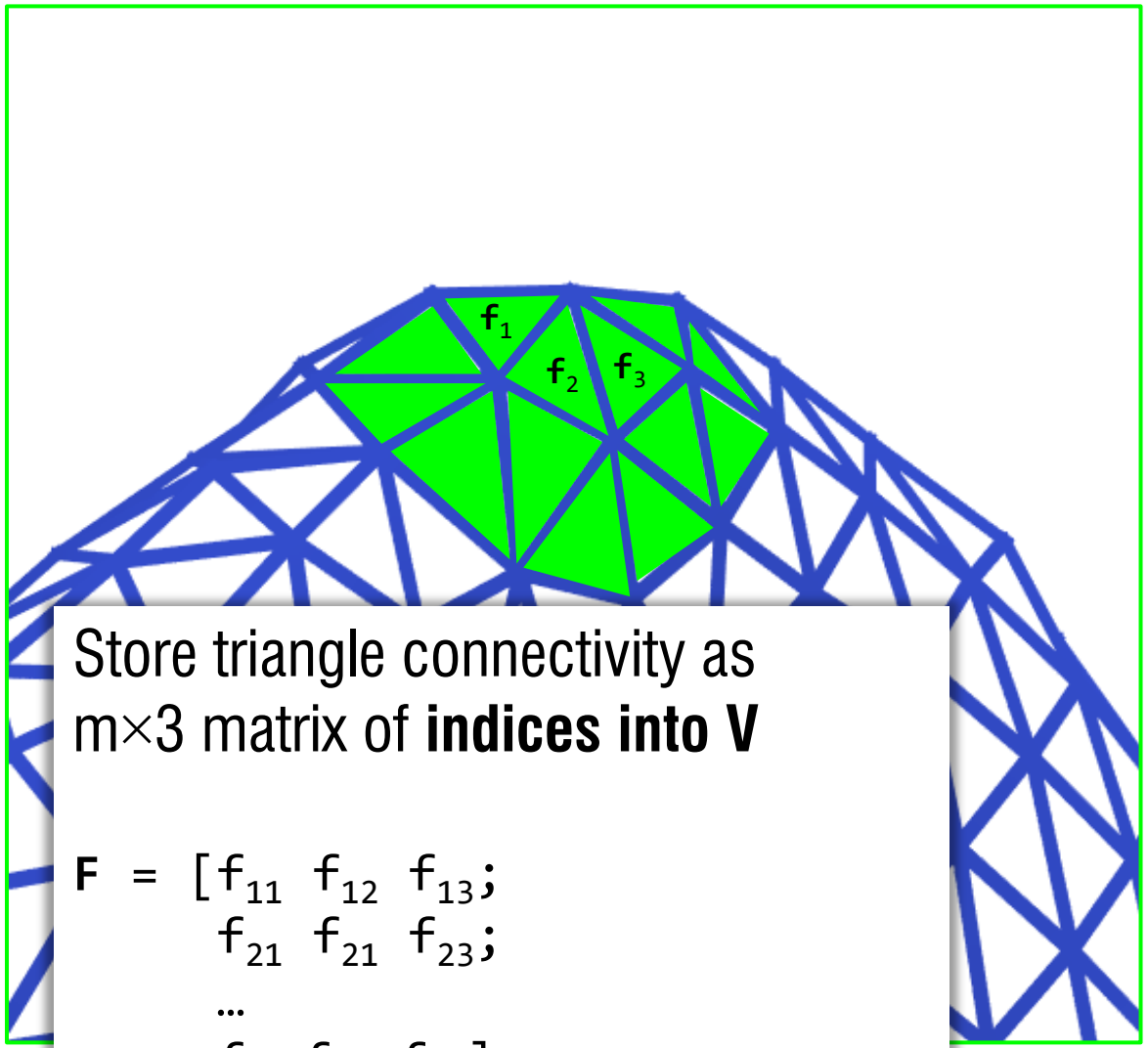



f_1
f_2



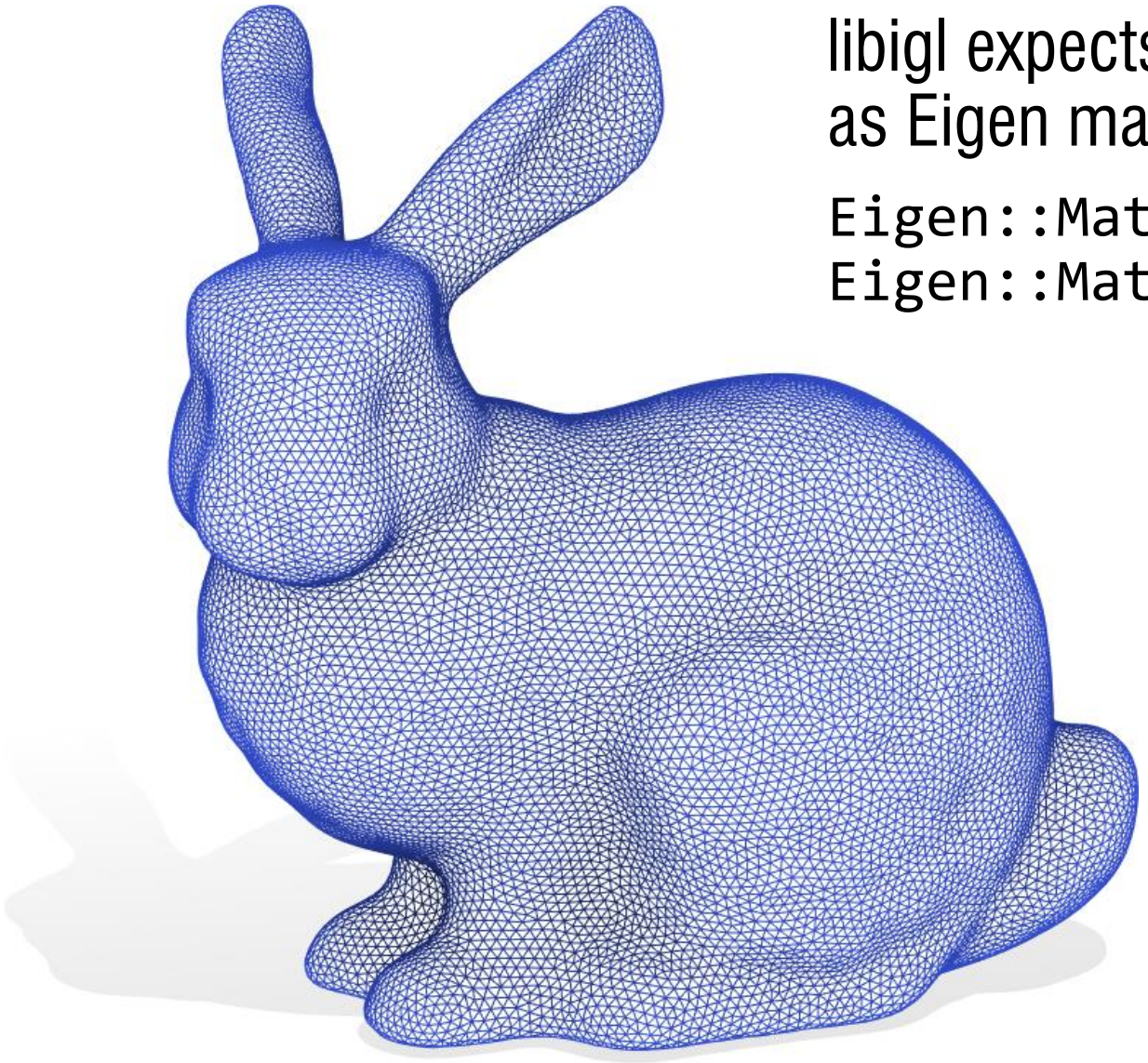

f_1
f_2
f_3





Store triangle connectivity as m×3 matrix of **indices into V**

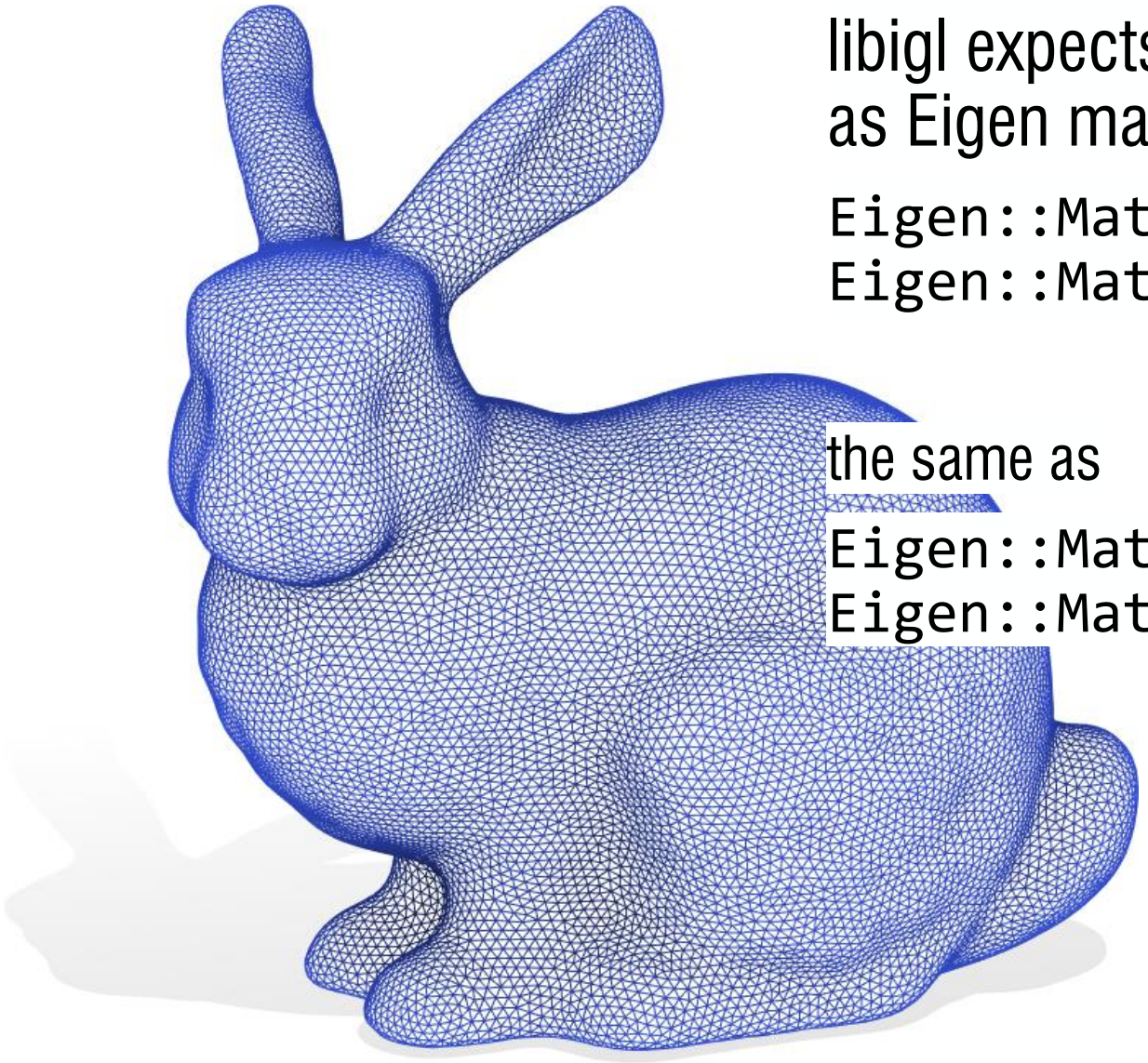
$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{21} & f_{23} \\ \dots & & \\ f_{n1} & f_{n2} & f_{n3} \end{bmatrix}$$



libigl expects vertex positions and triangle indices as Eigen matrices

```
Eigen::Matrix<double,Eigen::Dynamic,3> V;
Eigen::Matrix<int,Eigen::Dynamic,3> F;
```

Jacobson & Panozzo, "libigl course"



libigl expects vertex positions and triangle indices as Eigen matrices

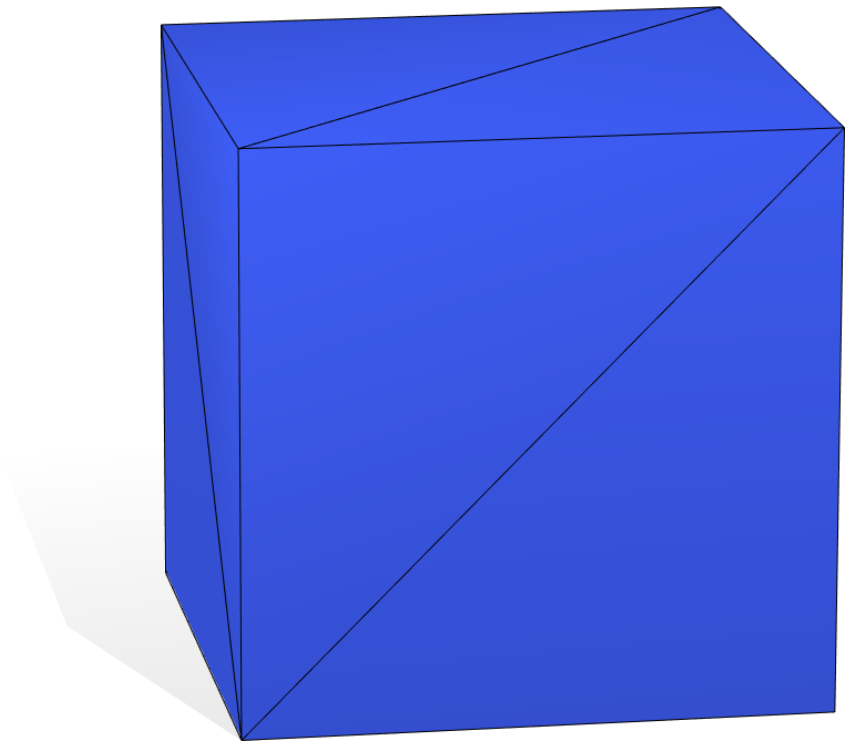
```
Eigen::Matrix<double,Eigen::Dynamic,3> V;
Eigen::Matrix<int,Eigen::Dynamic,3> F;
```

the same as

```
Eigen::MatrixXd V;
Eigen::MatrixXi F;
```

Jacobson & Panozzo, "libigl course"

libigl expects vertex positions and triangle indices as Eigen matrices

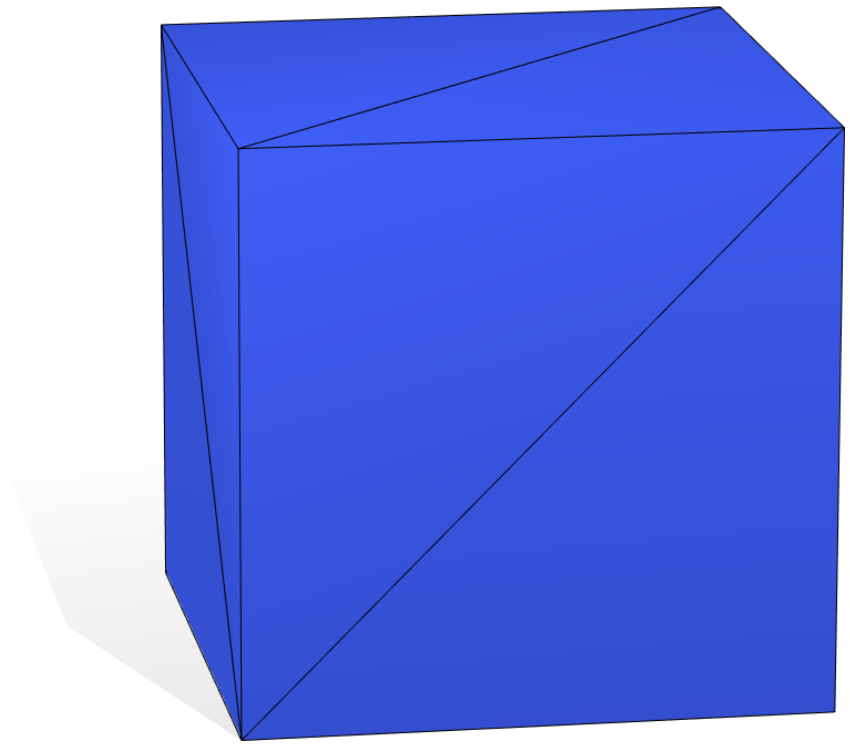


```
Eigen::MatrixXd V(8,3);
V<<
  0.0, 0.0, 0.0,
  0.0, 0.0, 1.0,
  0.0, 1.0, 0.0,
  0.0, 1.0, 1.0,
  1.0, 0.0, 0.0,
  1.0, 0.0, 1.0,
  1.0, 1.0, 0.0,
  1.0, 1.0, 1.0;
```

```
Eigen::MatrixXi F(12,3);
F<<
  0, 6, 4,
  0, 2, 6,
  0, 3, 2,
  0, 1, 3,
  2, 7, 6,
  2, 3, 7,
  4, 6, 7,
  4, 7, 5,
  0, 4, 5,
  0, 5, 1,
  1, 5, 7,
  1, 7, 3;
```

Jacobson & Panozzo, "libigl course"

libigl expects vertex positions and triangle indices as Eigen matrices



```
Eigen::MatrixXd V;
Eigen::MatrixXi F;
igl::read_triangle_mesh(“cube.obj”,V,F);
```

Jacobson & Panozzo, "libigl course"

libigl expects v ons and triangle indices as Eigen matri

```
Eigen::Mat
Eigen::Mat
igl::read_ (“cube.obj”,V,F);
```



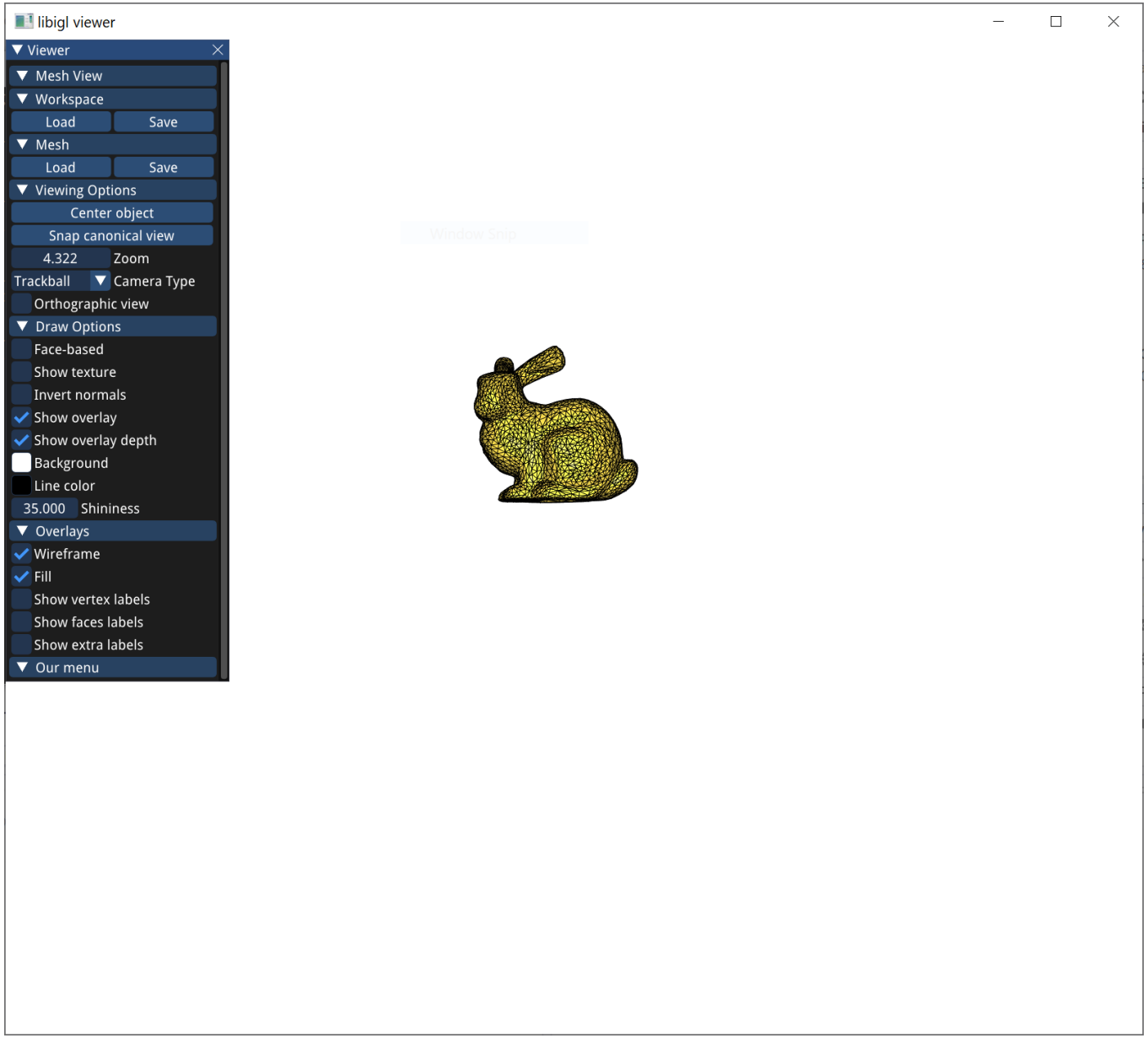
```
# cube.obj
v 0 0 0
v 0 0 1
v 0 1 0
v 0 1 1
v 1 0 0
v 1 0 1
v 1 1 0
v 1 1 1
f 1 7 5
f 1 3 7
f 1 4 3
f 1 2 4
f 3 8 7
f 3 4 8
f 5 7 8
f 5 8 6
f 1 5 6
f 1 6 2
f 2 6 8
f 2 8 4
```

Jacobson & Panozzo, "libigl course"

Let's go!

Run `1_intro_igl`

Solution is in `0_intro_igl__solution`



Quick refs

libigl: great tutorials, courses, etc

https://libigl.github.io/tutorial

https://libigl.github.io/tutorial/#other-matlab-style-functions

API: none really, but search in github (one-function one-file principle)

→ IGL, main contributors Alec Jacobson & Danielle Panozzo 👏

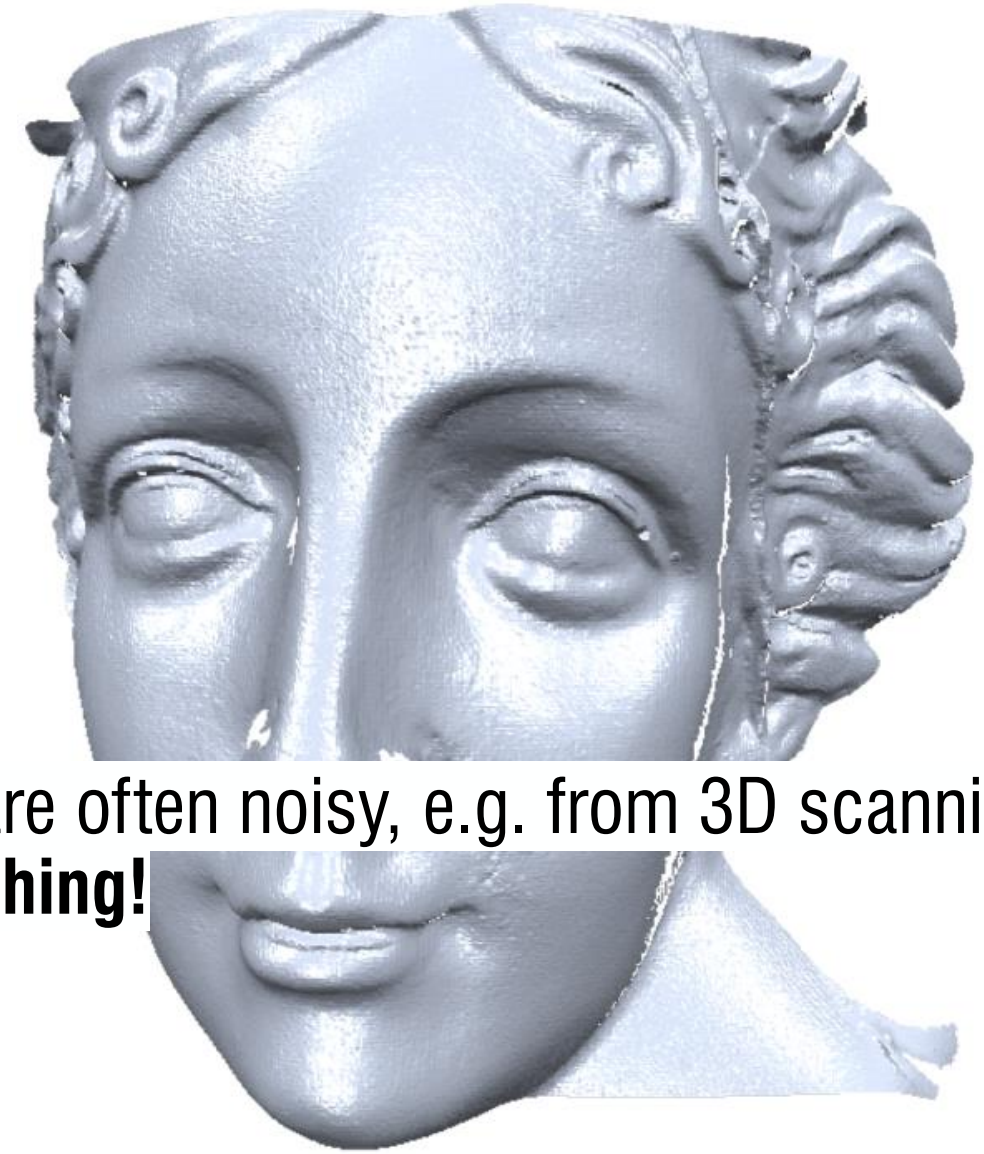
Eigen: https://eigen.tuxfamily.org/dox/group__QuickRefPage.html


10min **Break**

1.1 Introduction to libigl

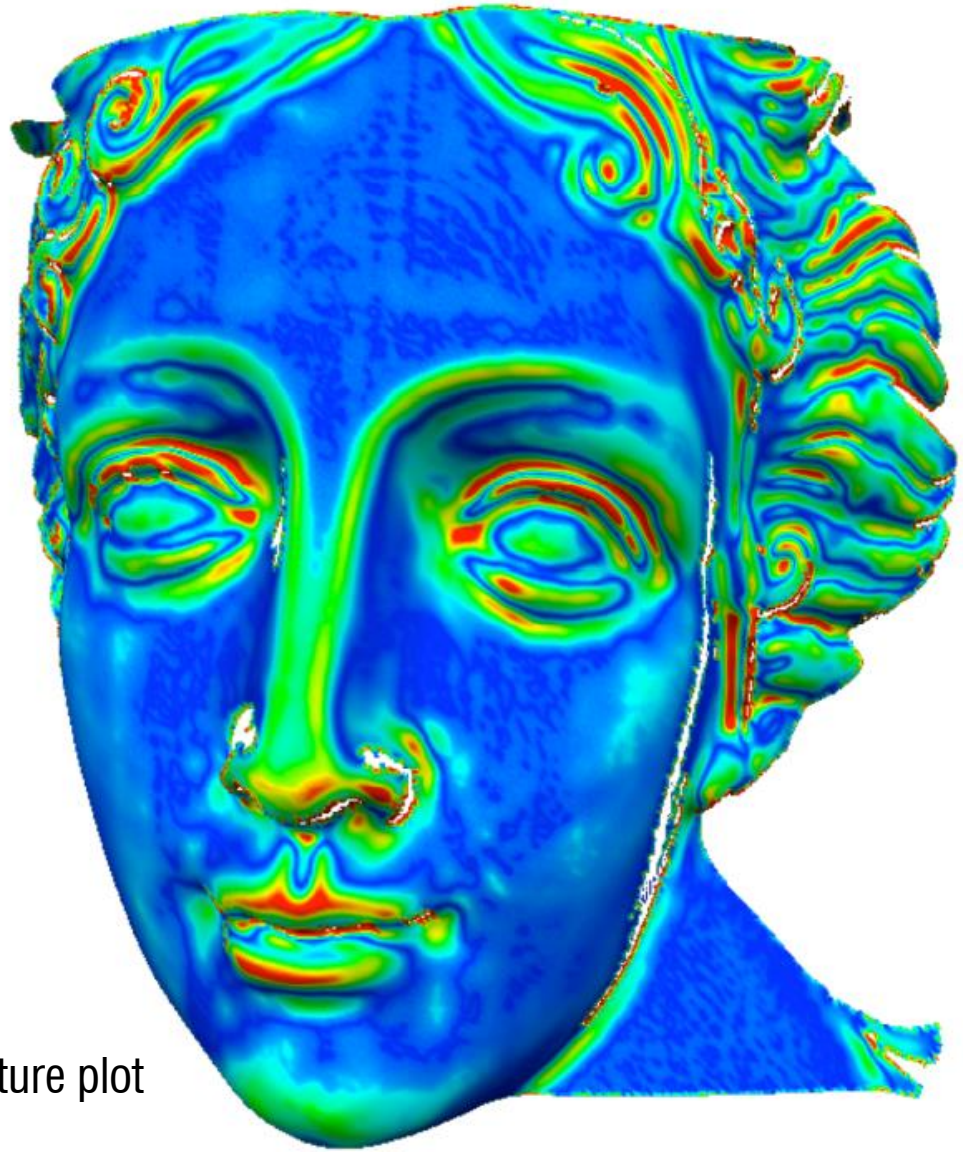
1.2 DDG curves

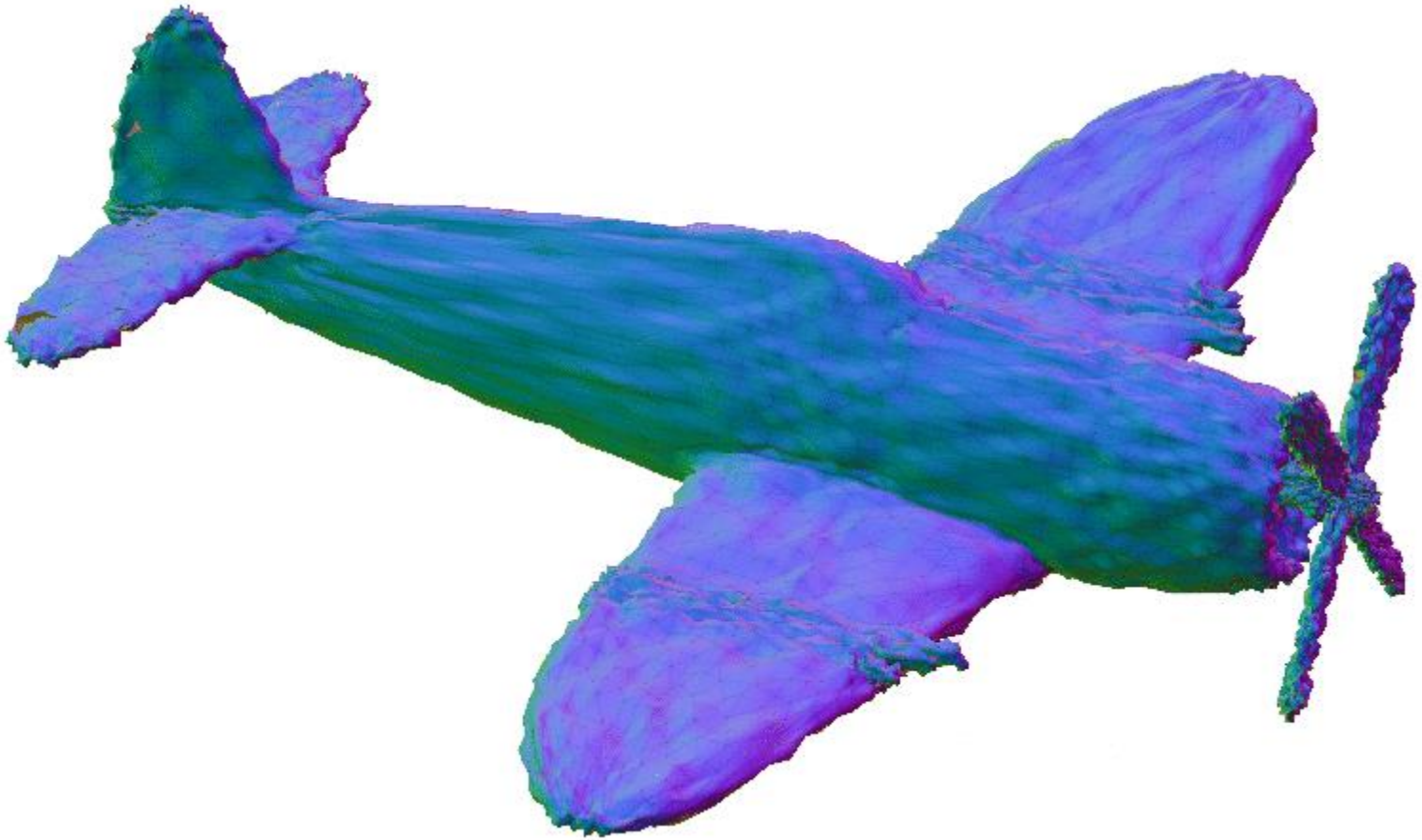
1.3 Smoothing meshes



meshes are often noisy, e.g. from 3D scanning
→ **smoothing!**

mean curvature plot





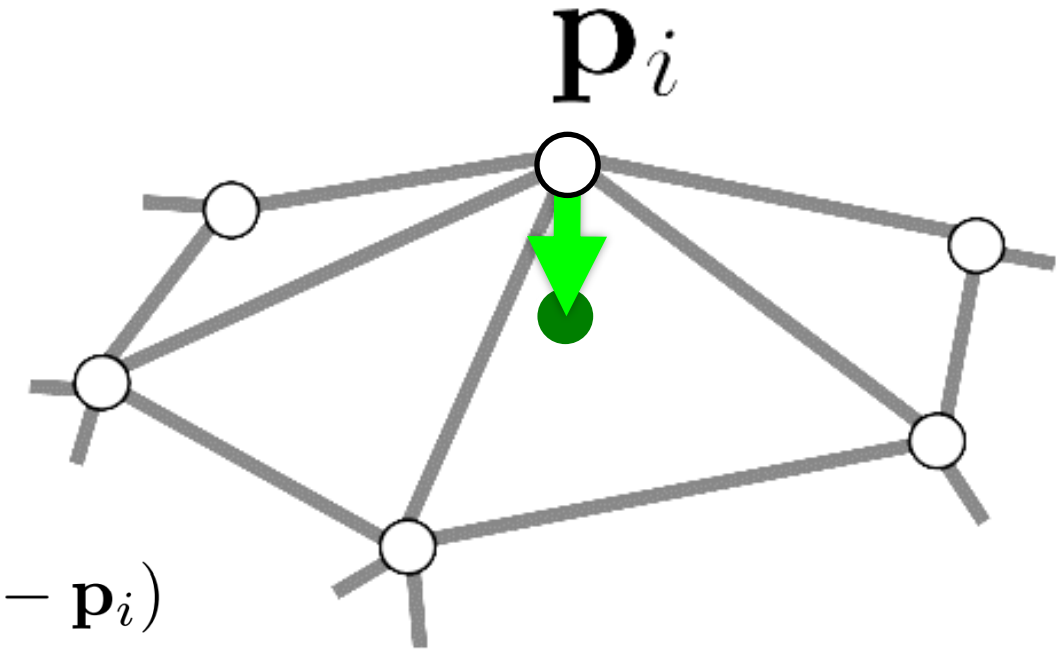
Alec Jacobson, "Geometry Processing Smoothing"

Mesh smoothing

High-pass filter: extracts local surface detail

Detail = ***smooth***(surface) – surface

Assumption: smoothing = averaging

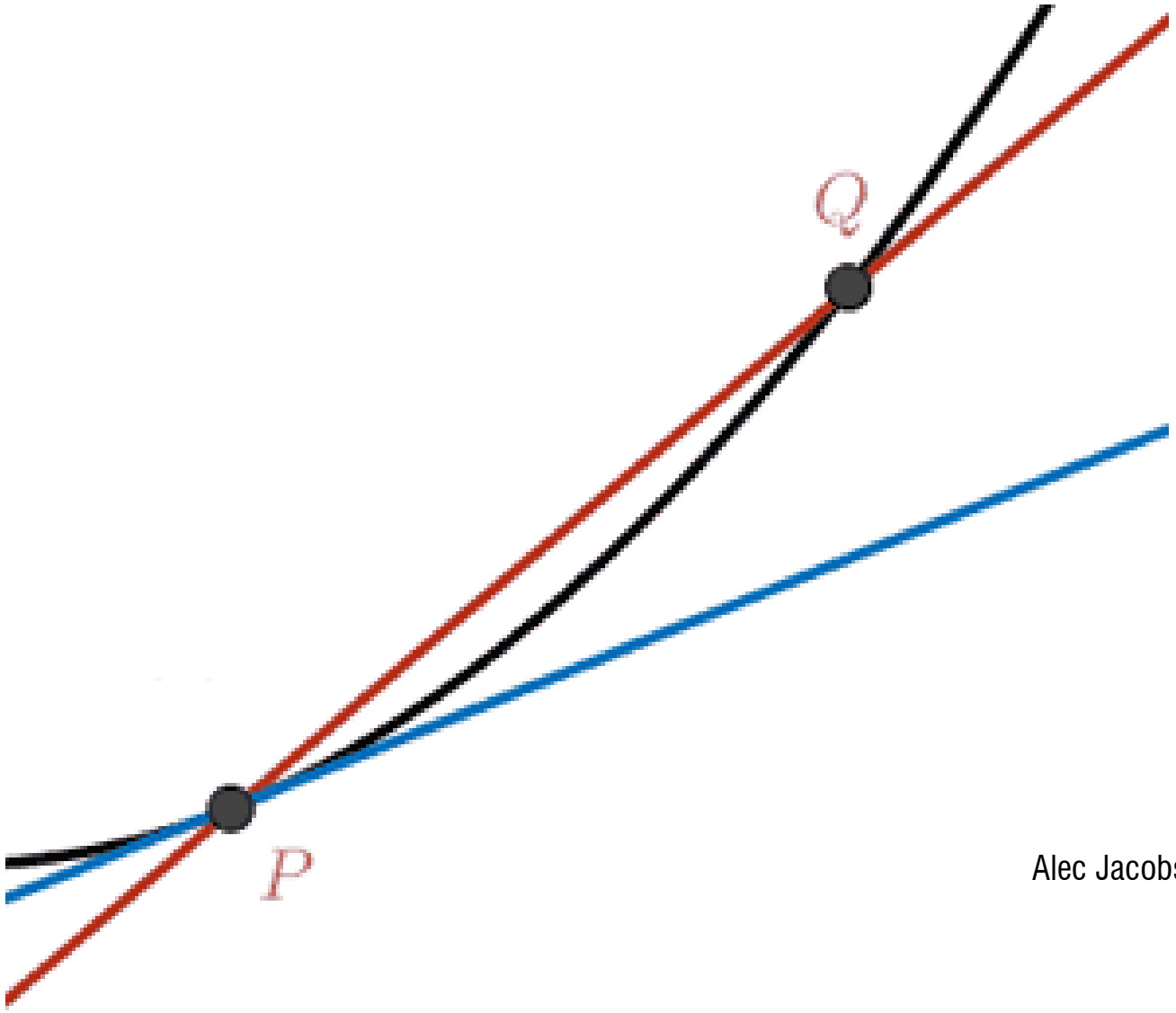


$$\delta_i = \frac{1}{W_i} \sum_{j \in \mathcal{N}(i)} w_{ij}(\mathbf{p}_j - \mathbf{p}_i)$$

Olga Sorkine-Hornung, ETH, Shape Modeling course

Let's start with curves

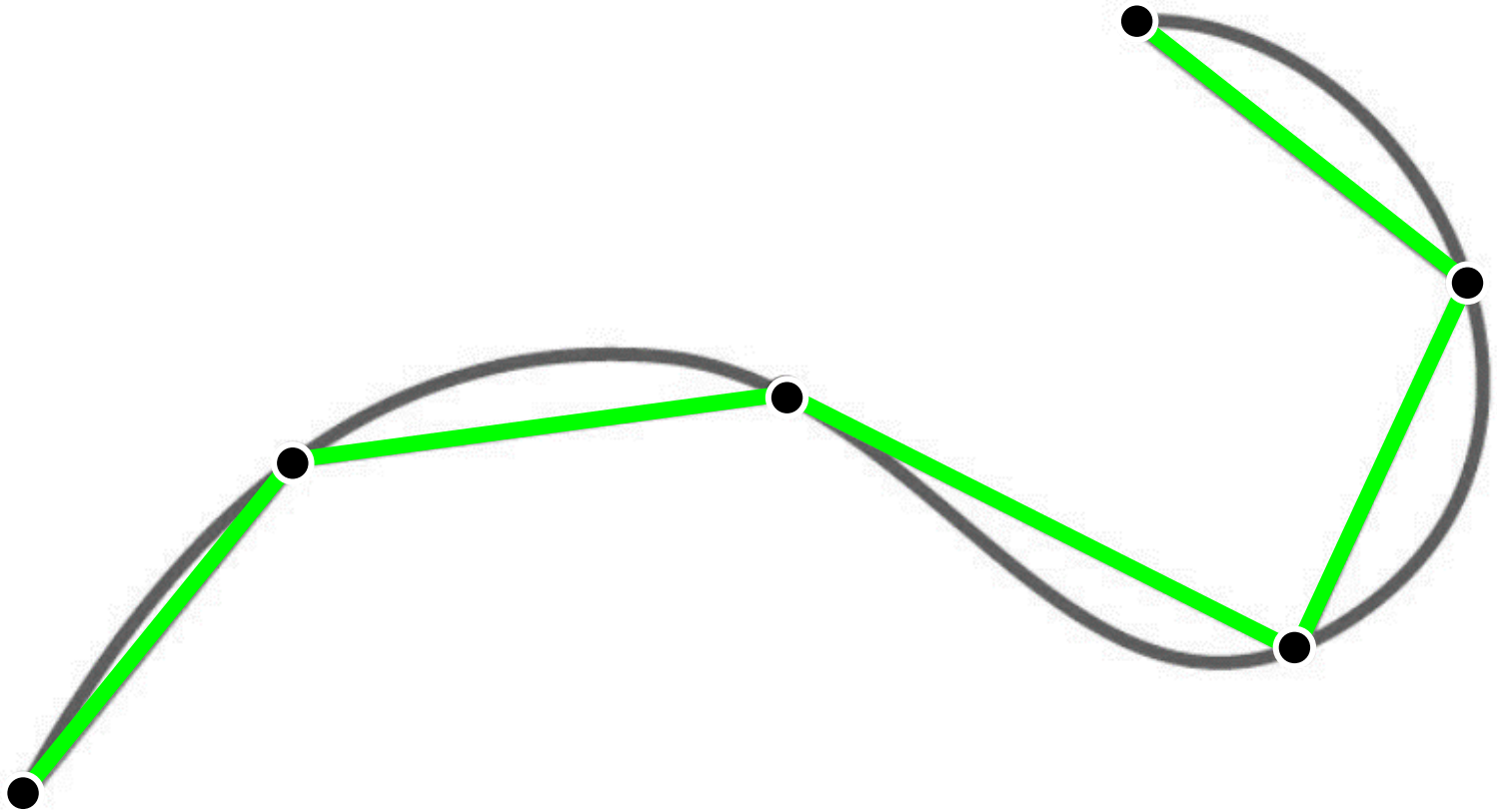
Differential Geometry



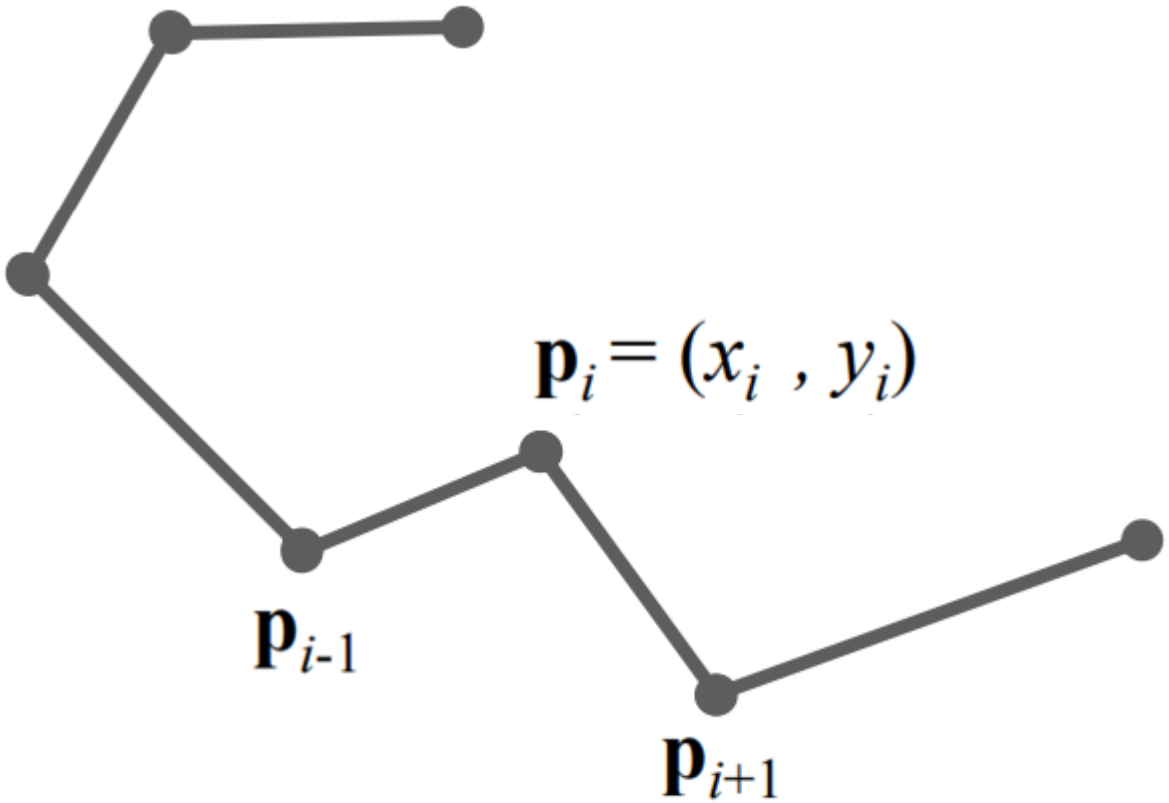
Alec Jacobson, "Geometry Processing - Curvature"

Discrete Differential Geometry

Piecewise-linear curve: finite number of vertices on the curve connected by lines

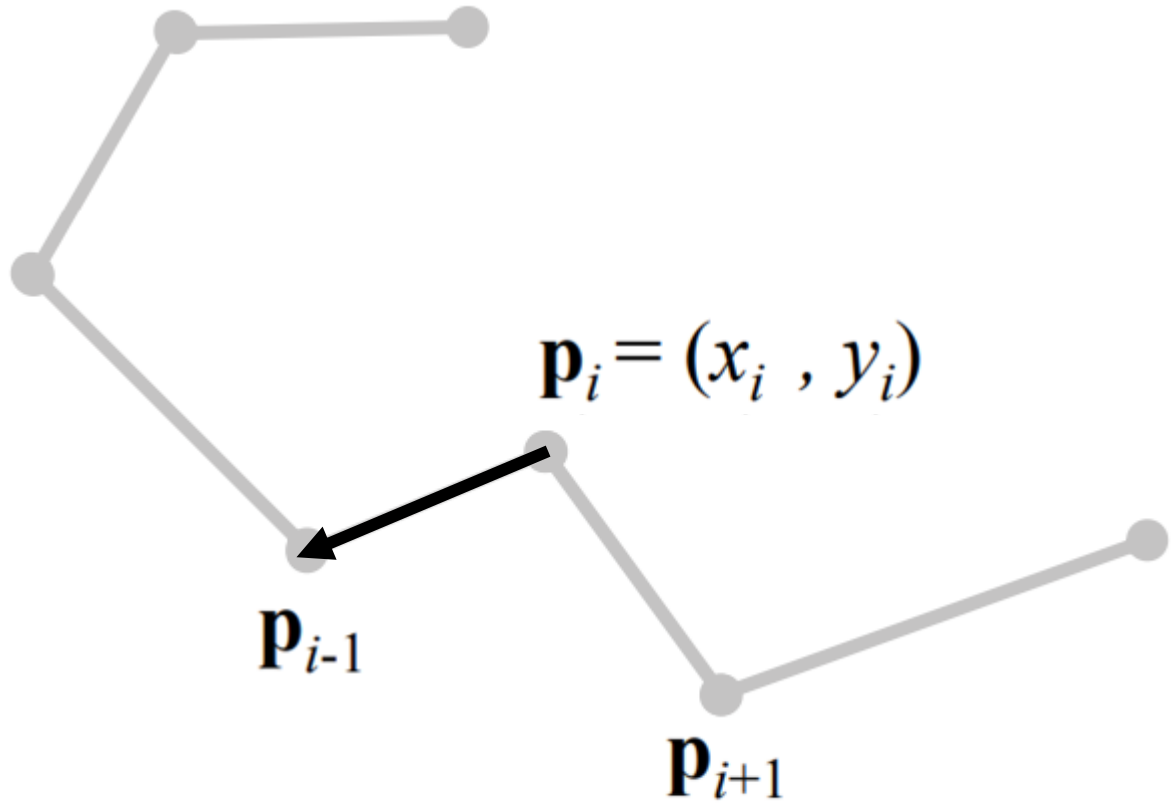


Laplacian smoothing
on a planar discrete curve



Laplacian smoothing

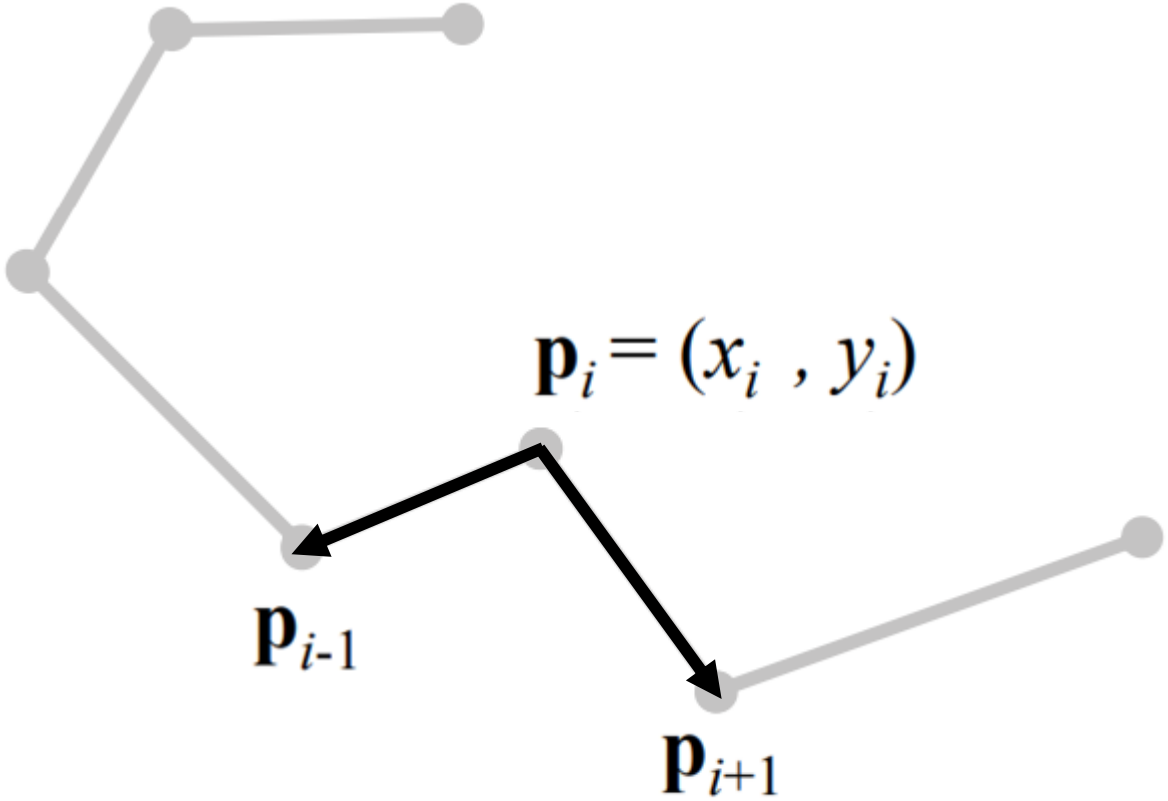
$$L(\mathbf{p}_i) = \frac{1}{2}(\mathbf{p}_{i-1} - \mathbf{p}_i) + \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_i)$$



Carnegie Mellon University

Laplacian smoothing

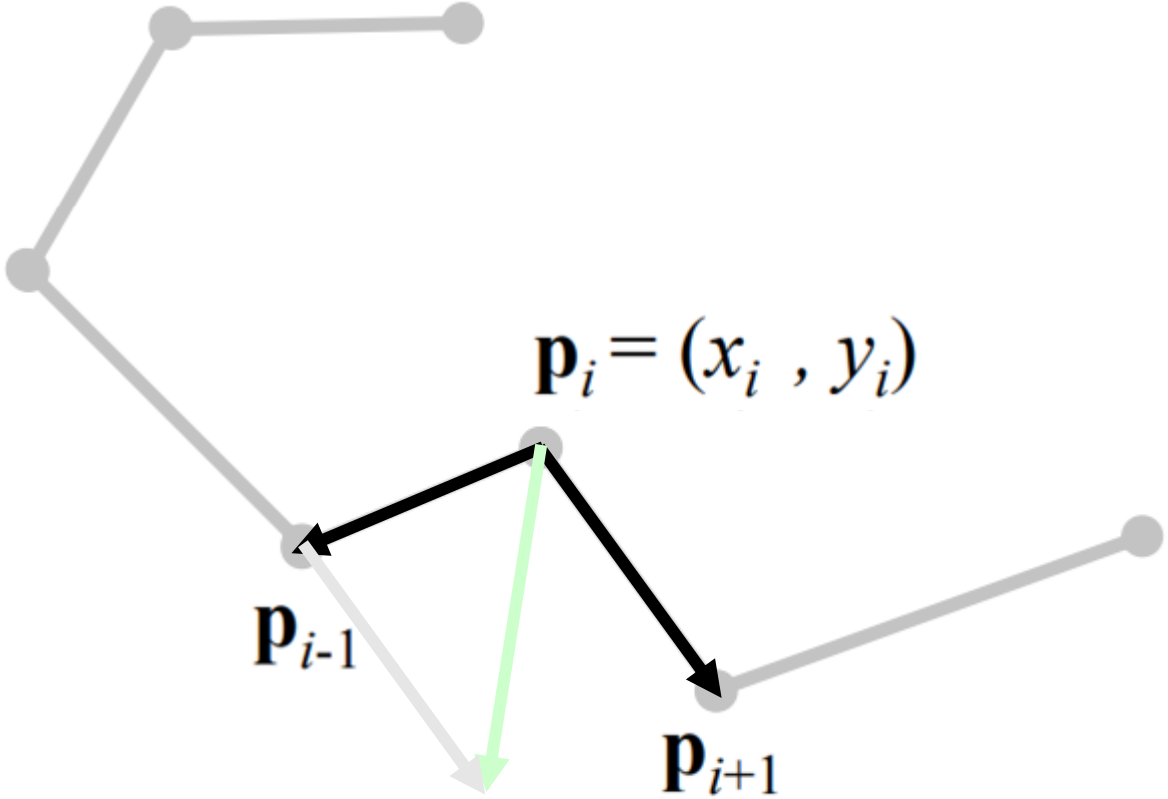
$$L(\mathbf{p}_i) = \frac{1}{2}(\mathbf{p}_{i-1} - \mathbf{p}_i) + \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_i)$$



Carnegie Mellon University

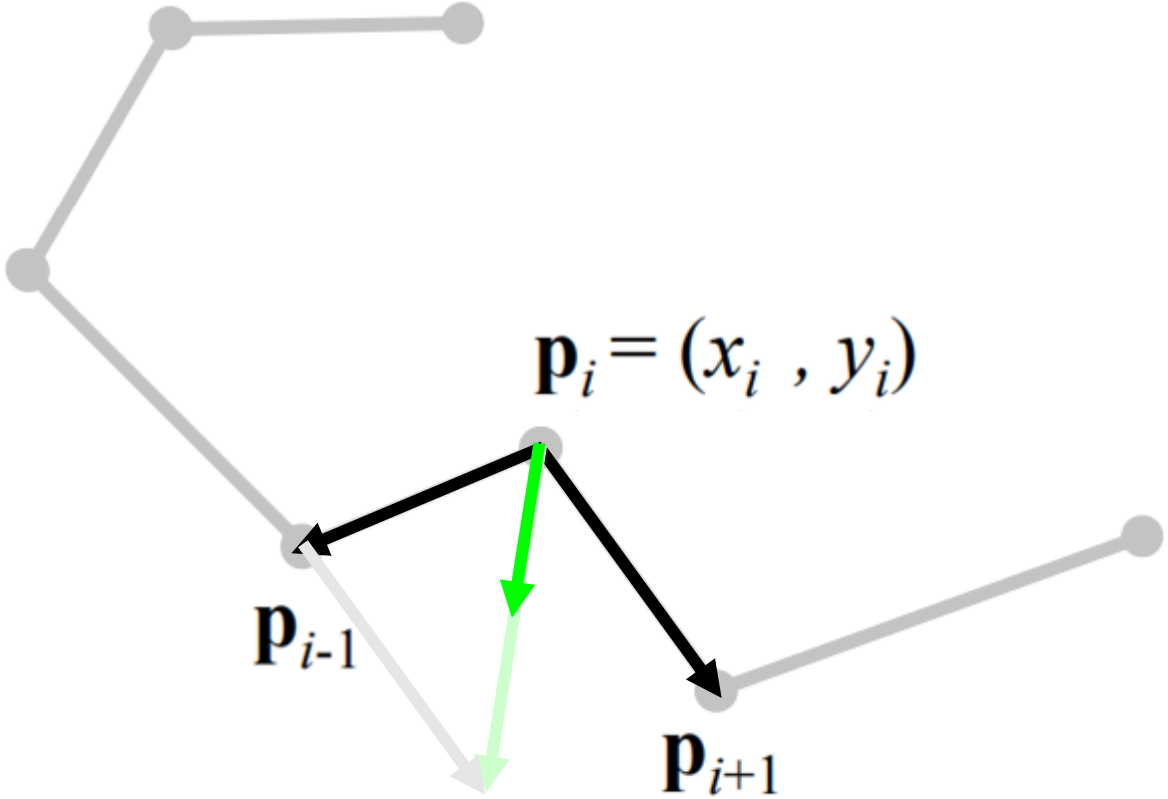
Laplacian smoothing

$$L(\mathbf{p}_i) = \frac{1}{2}(\mathbf{p}_{i-1} - \mathbf{p}_i) + \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_i)$$



Laplacian smoothing

$$L(\mathbf{p}_i) = \frac{1}{2}(\mathbf{p}_{i-1} - \mathbf{p}_i) + \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_i)$$

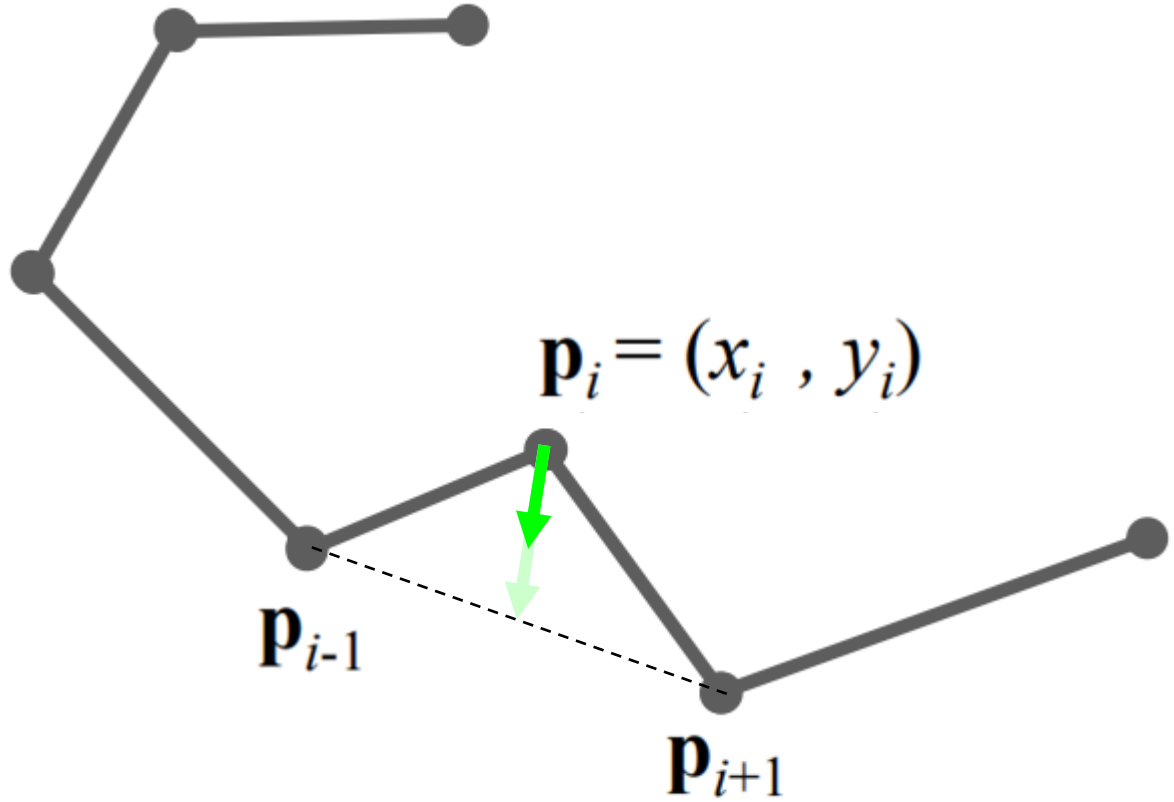


Laplacian smoothing

$$L(\mathbf{p}_i) = \frac{1}{2}(\mathbf{p}_{i-1} - \mathbf{p}_i) + \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_i)$$

$$\tilde{\mathbf{p}} = \mathbf{p} + \lambda L\mathbf{p}$$

move the vertex **stepwise** towards their smooth position



Laplacian smoothing

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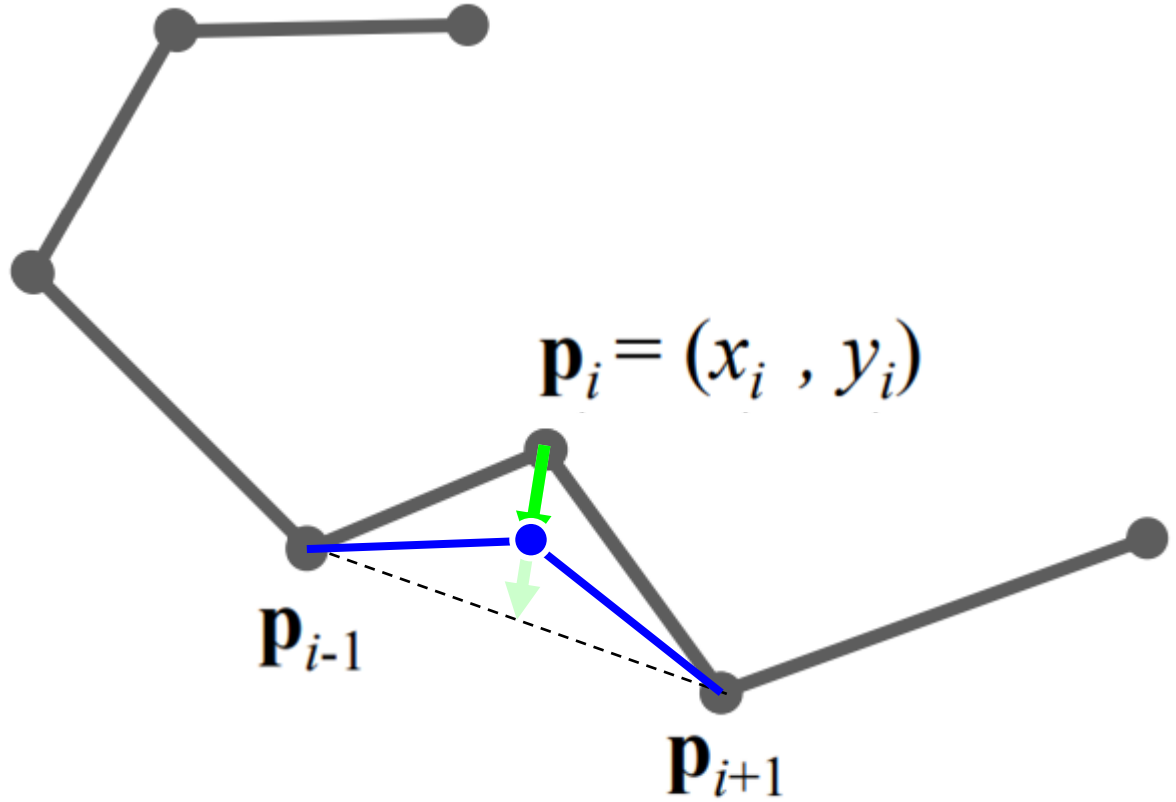

Laplacian smoothing

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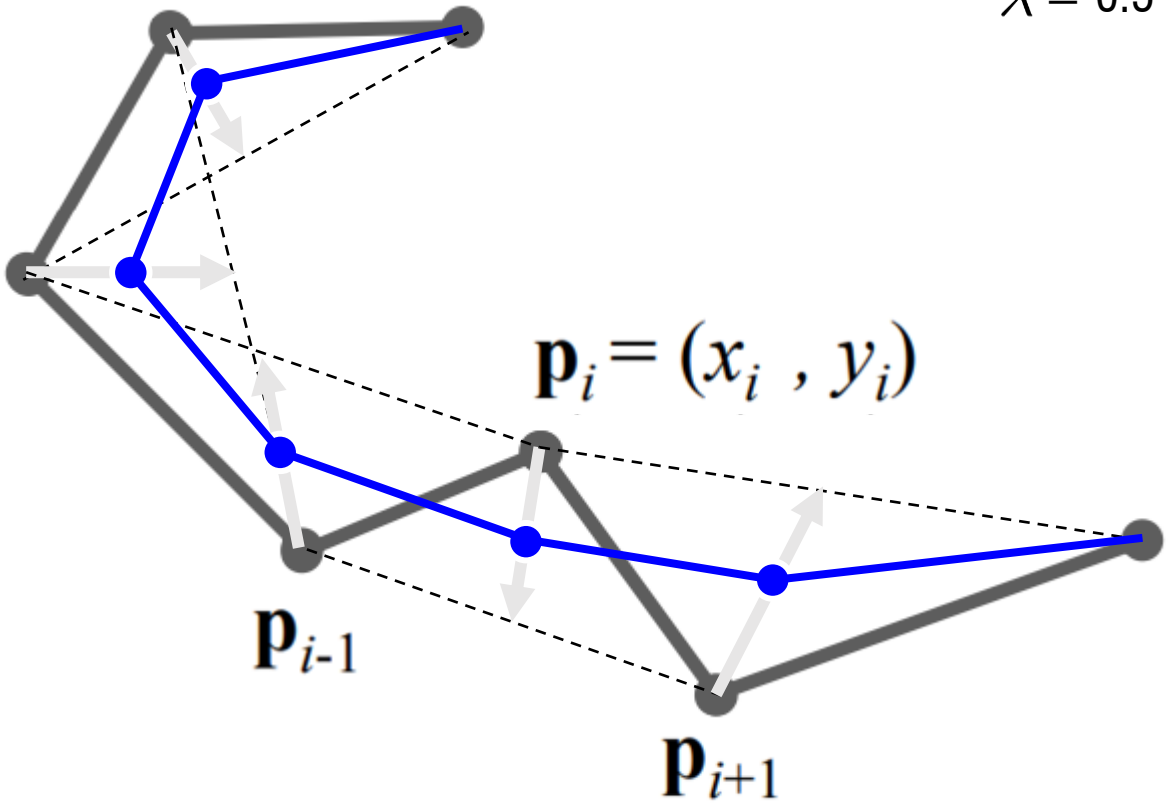


Any drawbacks?

<30sec brainstorming>

Let's see...

run `2_mesh_smoothing`
[key c]

Taubin smoothing

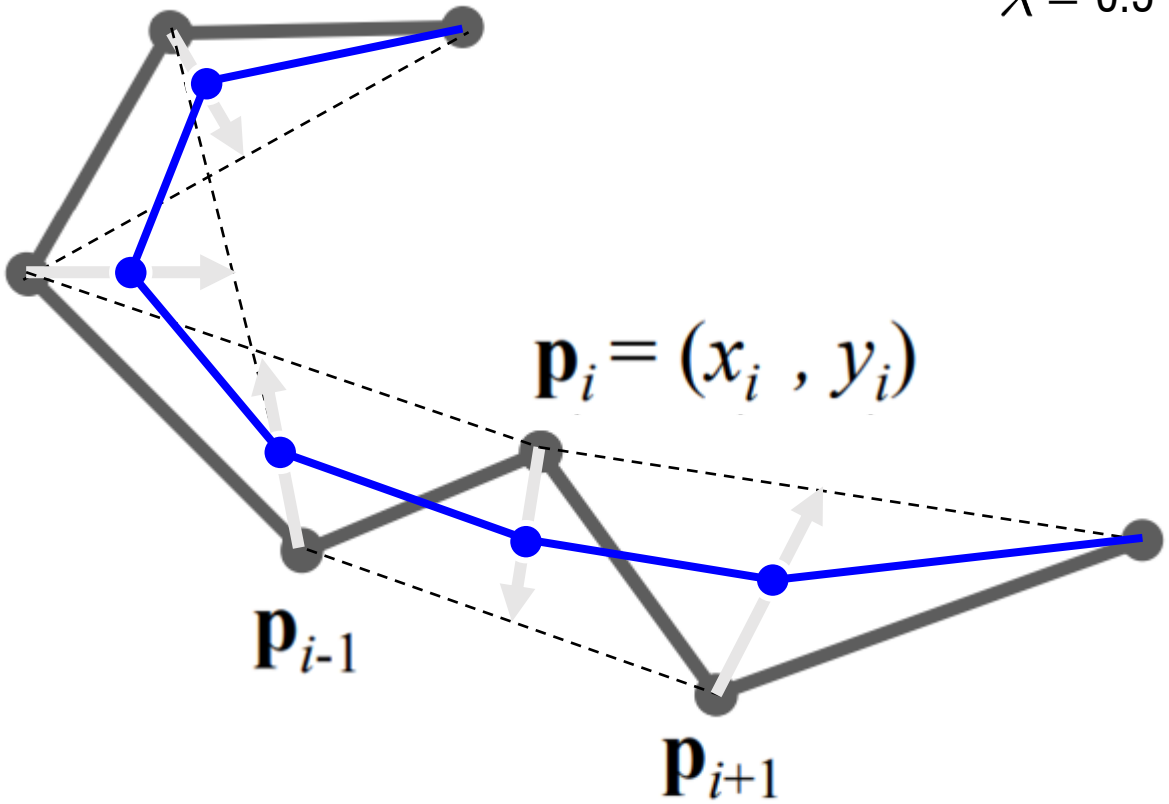
$$L(\mathbf{p}_i) = \frac{1}{2}(\mathbf{p}_{i-1} - \mathbf{p}_i) + \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_i)$$

$$\tilde{\mathbf{p}} = \mathbf{p} + \lambda L\mathbf{p}$$

$$\tilde{\mathbf{p}} = \mathbf{p} + \mu L\mathbf{p}$$

$\lambda > 0$ to smooth

$\mu < 0$ to **inflate**



Taubin smoothing

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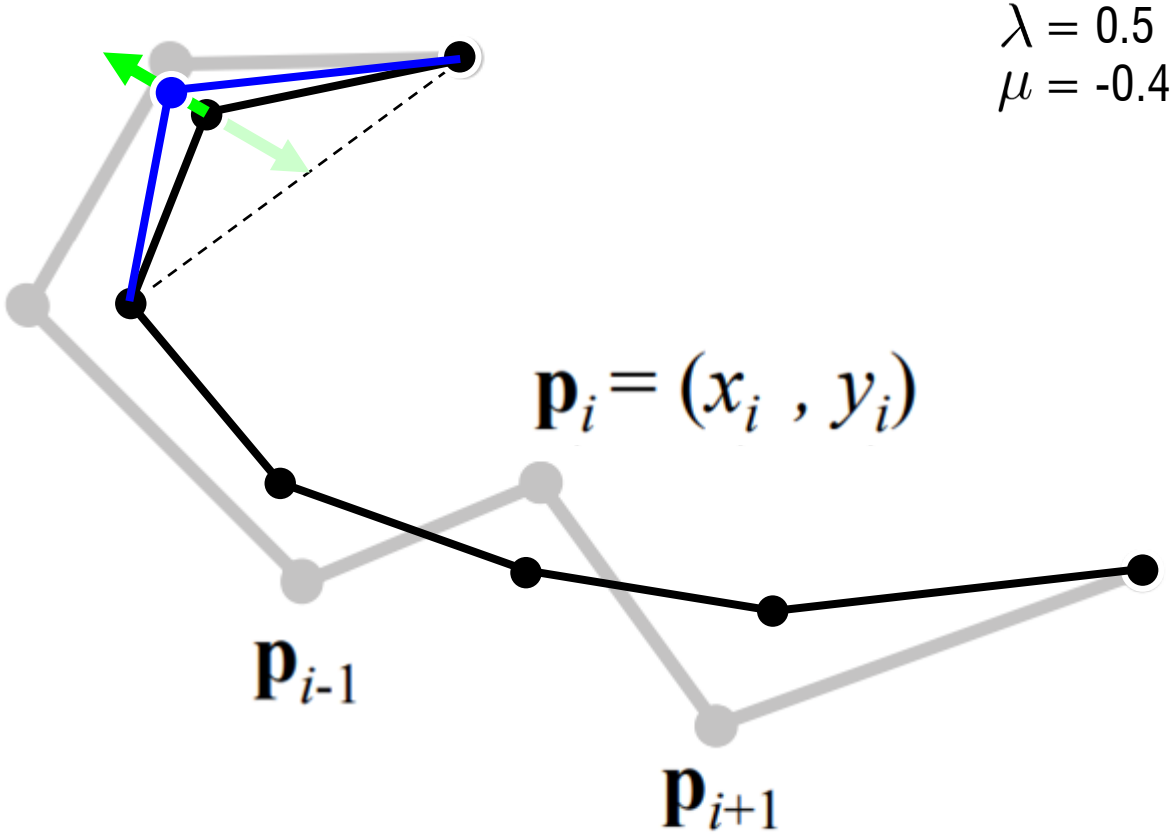

Taubin smoothing

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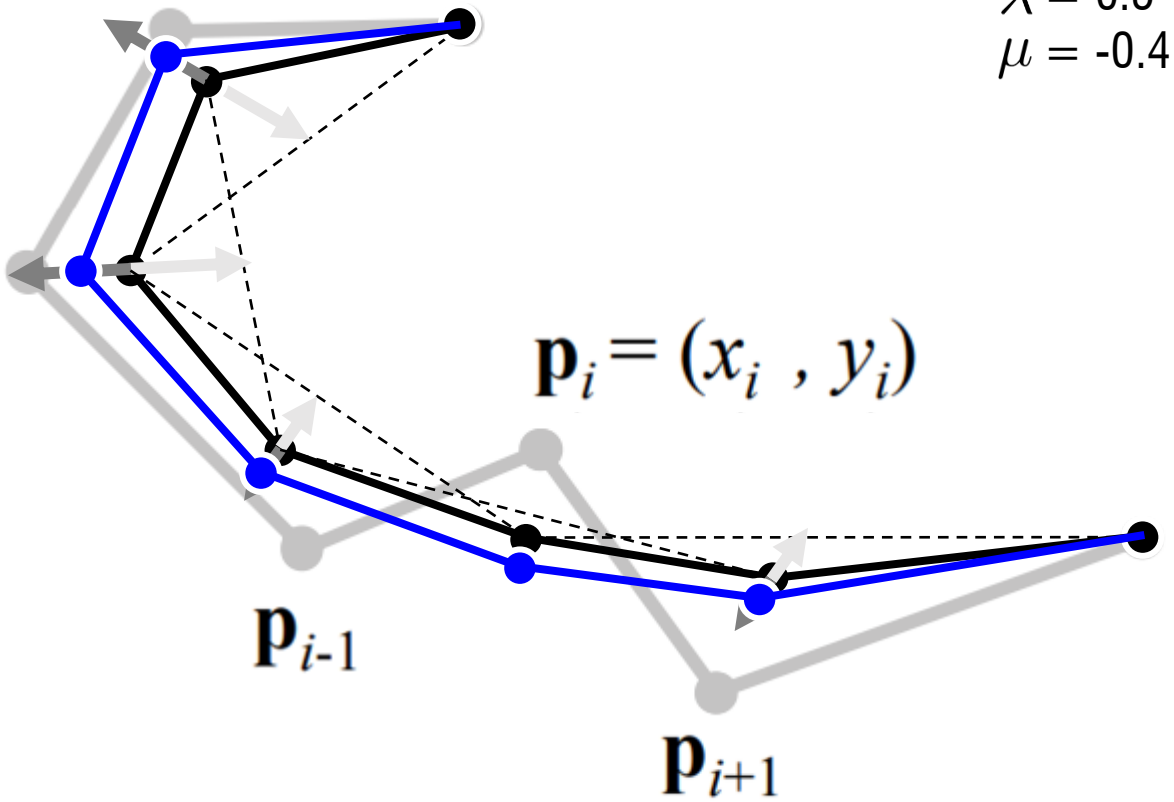


1.1 Introduction to libigl

1.2 DDG curves

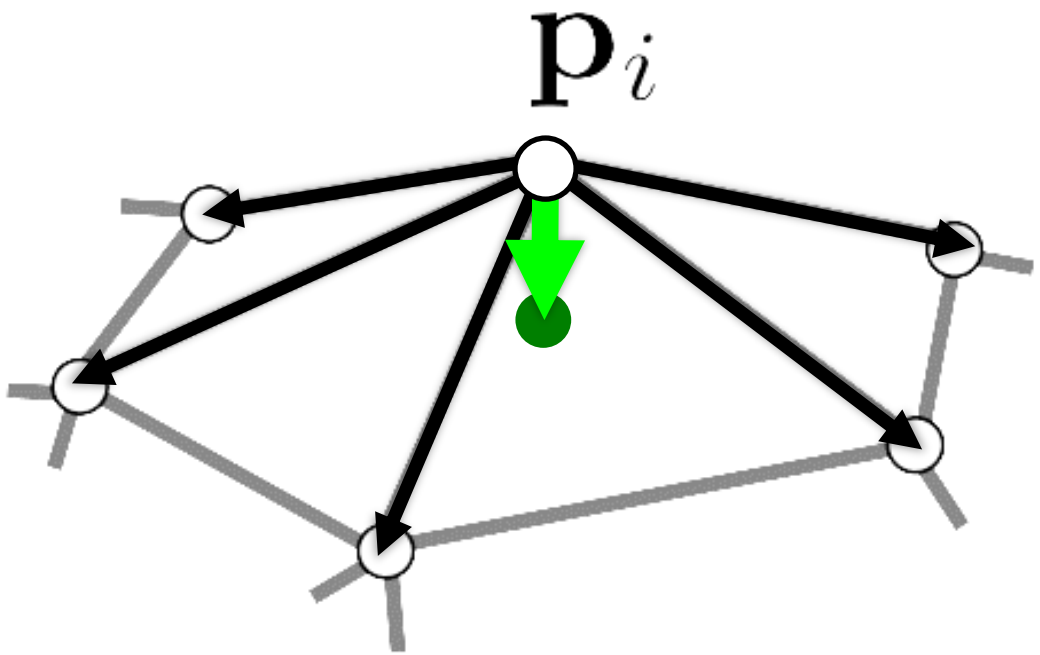
1.3 Smoothing meshes

Same idea as for curve smoothing

Assumption: smoothing = averaging

Uniform Laplacian or **Umbrella** operator

$$\delta_i = \frac{1}{W_i} \sum_{j \in \mathcal{N}(i)} w_{ij}(\mathbf{p}_j - \mathbf{p}_i)$$



Weighing schemes

$$\delta_i = \frac{1}{W_i} \sum_{j \in \mathcal{N}(i)} w_{ij}(\mathbf{p}_j - \mathbf{p}_i)$$

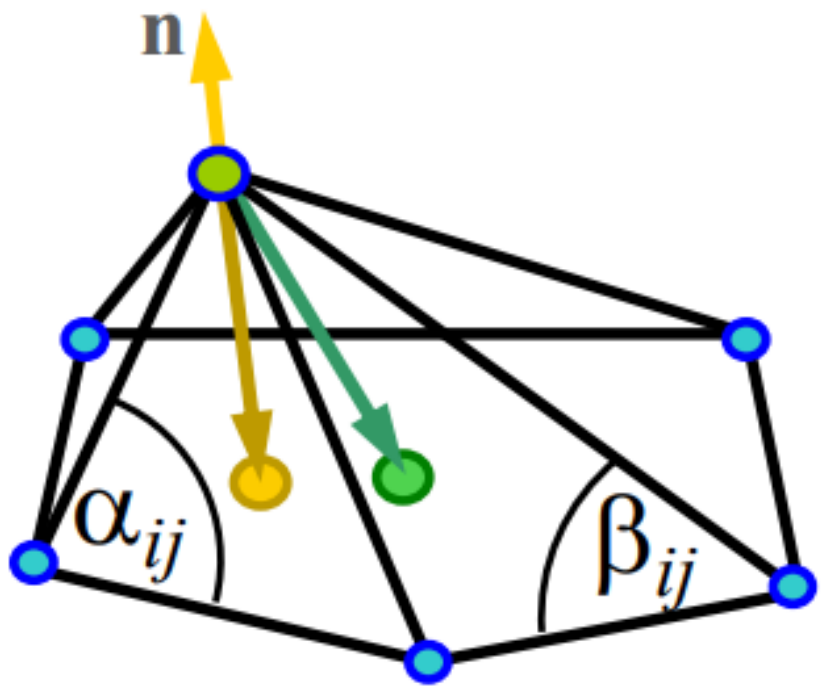
Ignoring the geometry:

δ_{uniform} : $W_i = 1,\ w_{ij} = 1/|N(i)|$

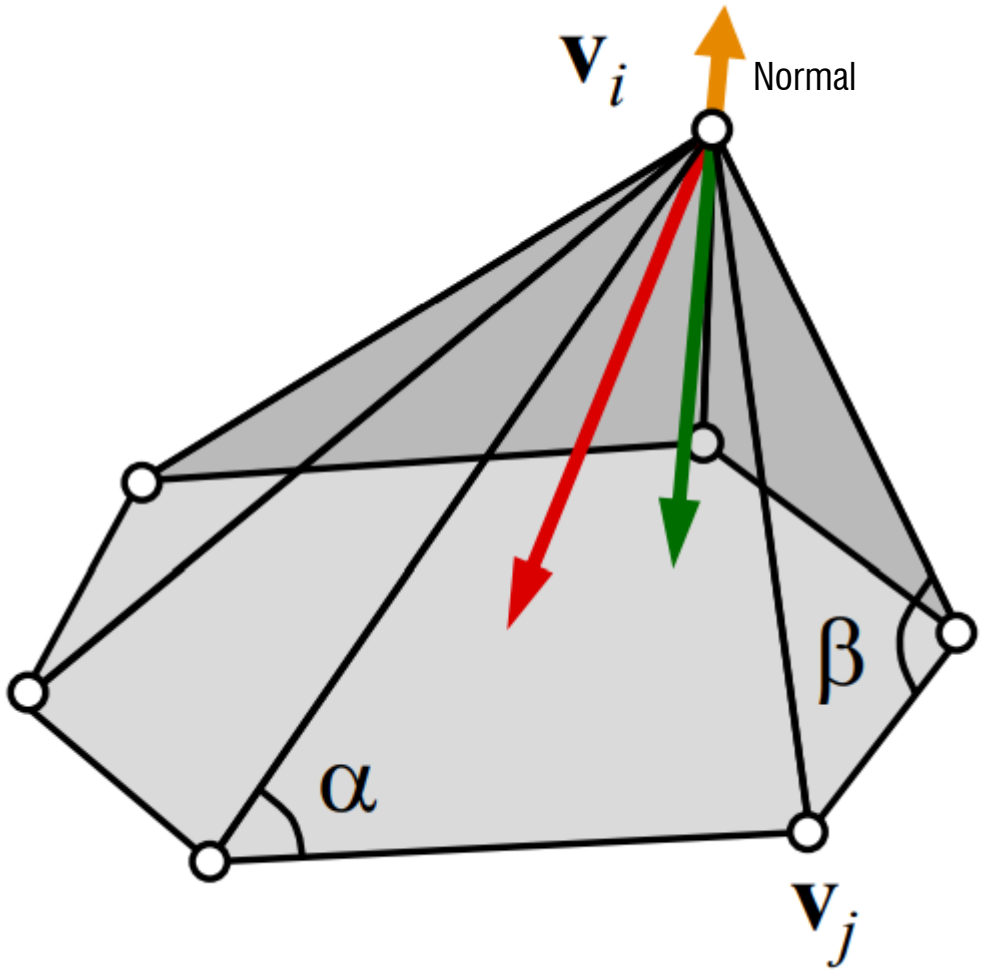
Integrate over Voronoi region of the vertex:

δ_{cotan} : $w_{ij} = 0.5(\cot \alpha_{ij} + \cot \beta_{ij})$

$W_i = A_i$



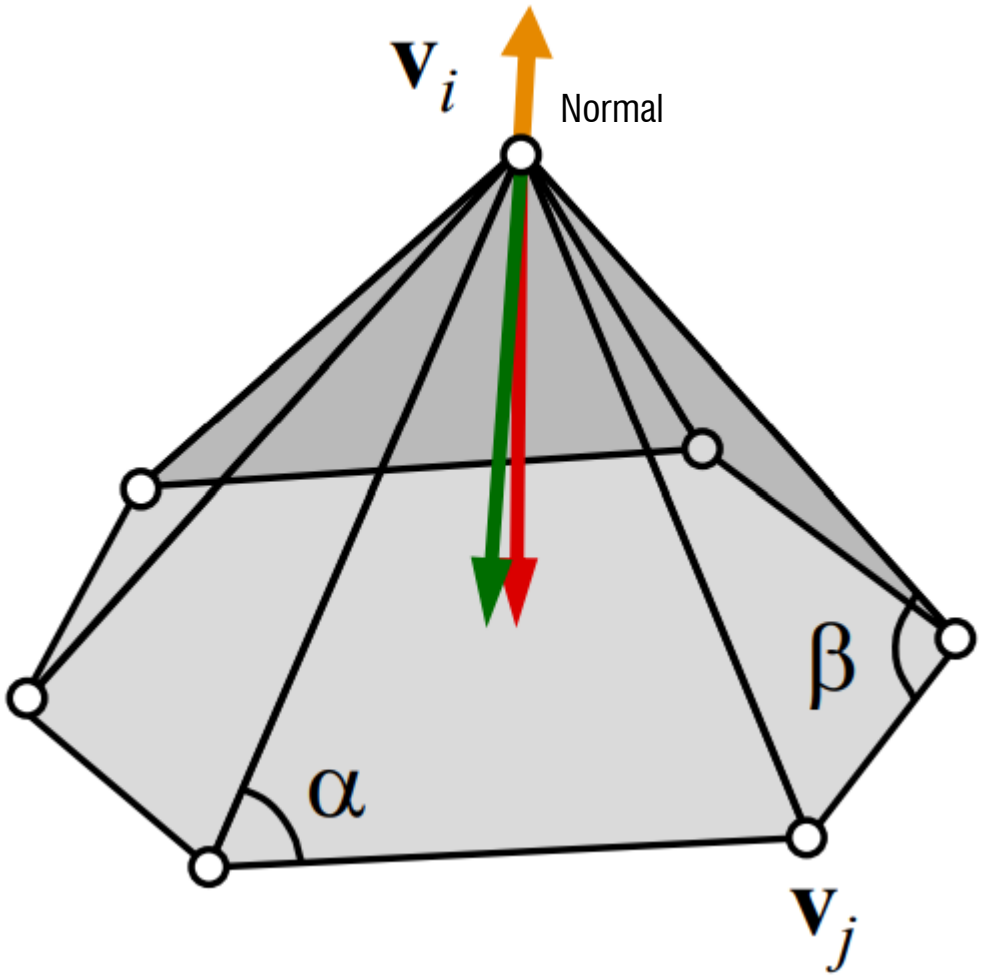
Olga Sorkine-Hornung, ETH, Shape Modeling course



Uniform Laplacian (Umbrella operator)

Cotangent Laplacian (Laplace-Beltrami)

For nearly equal edge lengths
Uniform ≈ Cotangent

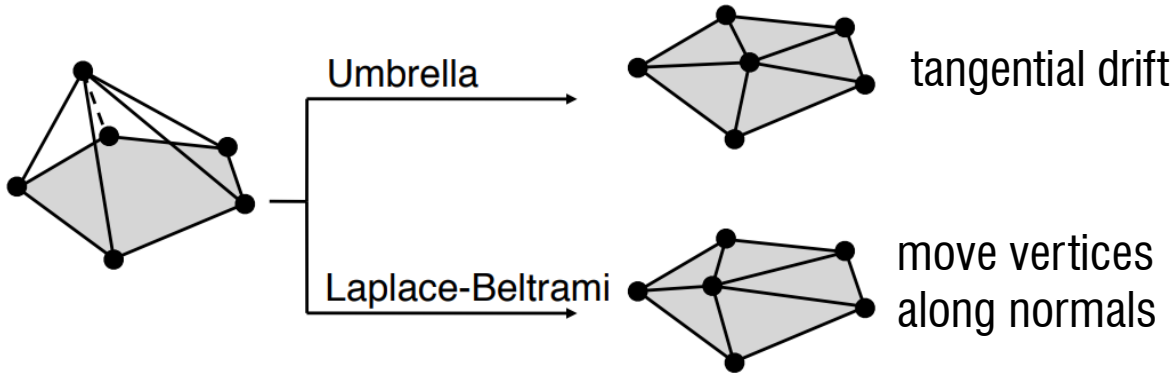


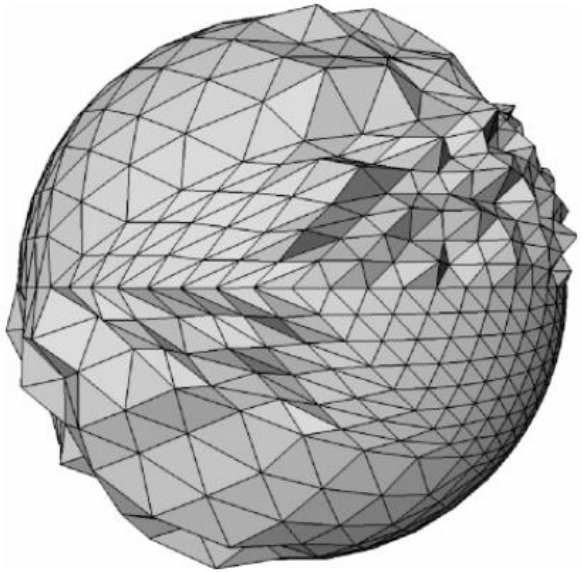
Uniform Laplacian (Umbrella operator)

Cotangent Laplacian (Laplace-Beltrami)

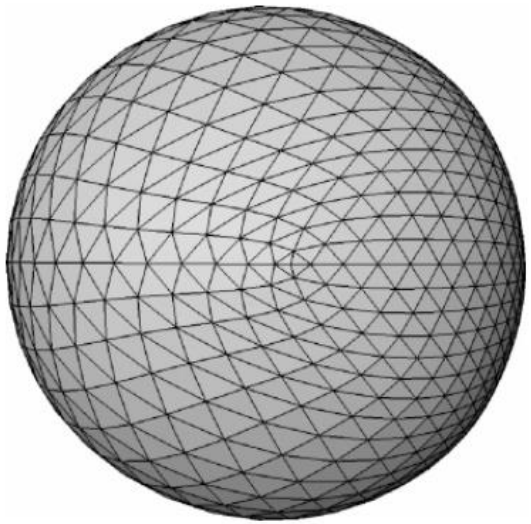
For nearly equal edge lengths
Uniform ≈ Cotangent

The Uniform Laplacian moves the geometry tangentially as well

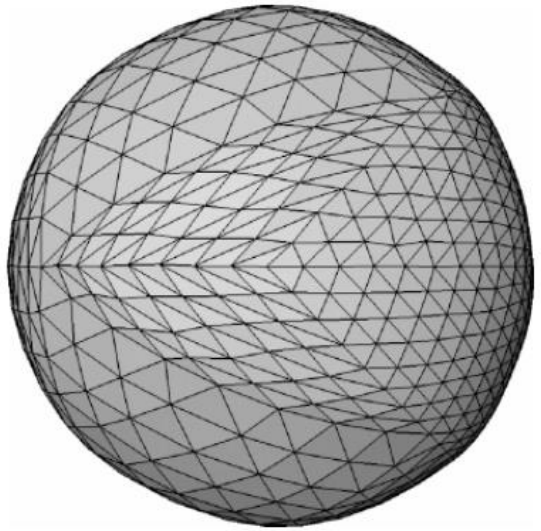




original

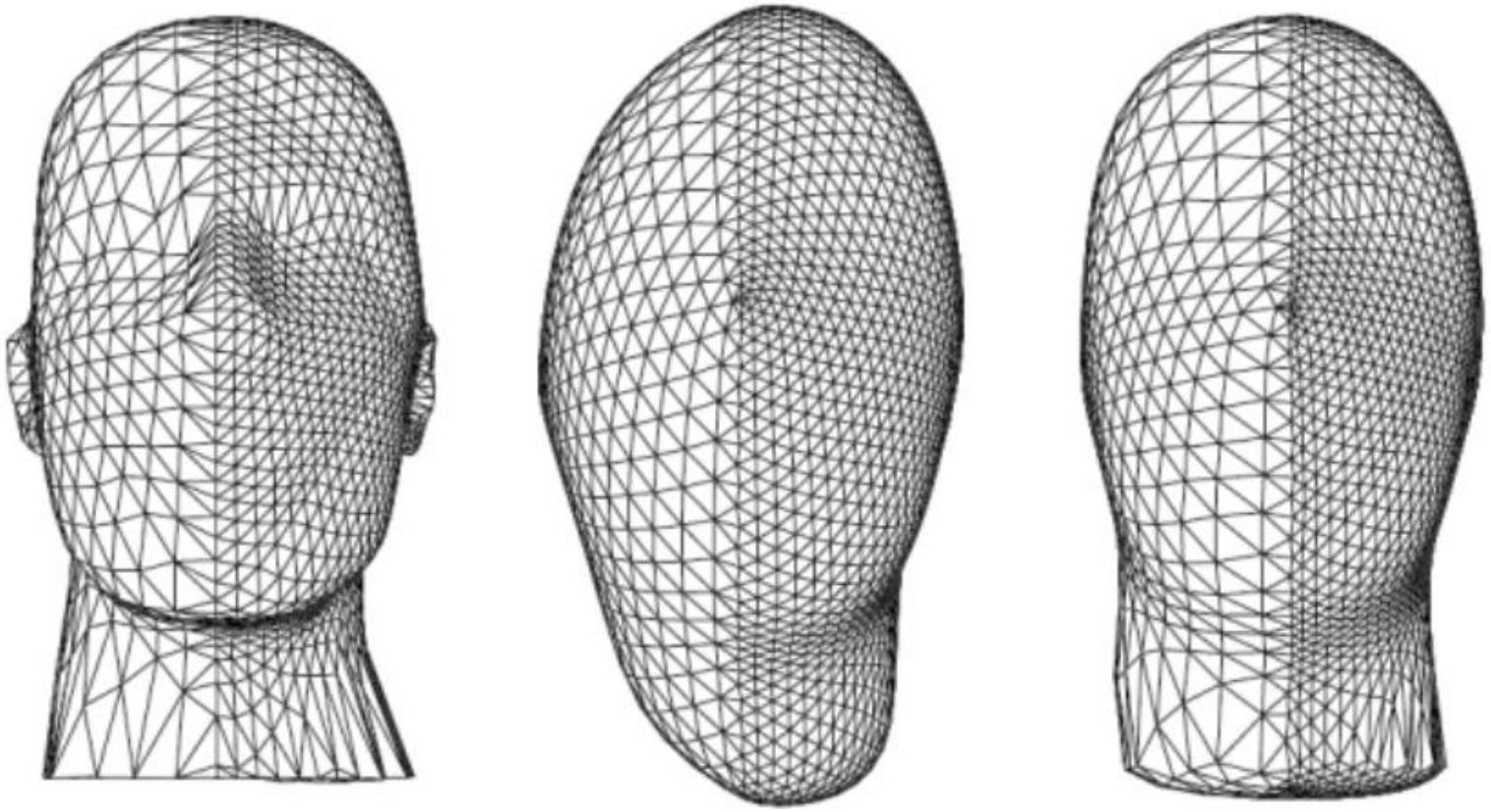


uniform



cotan

The Uniform Laplacian moves the geometry tangentially as well



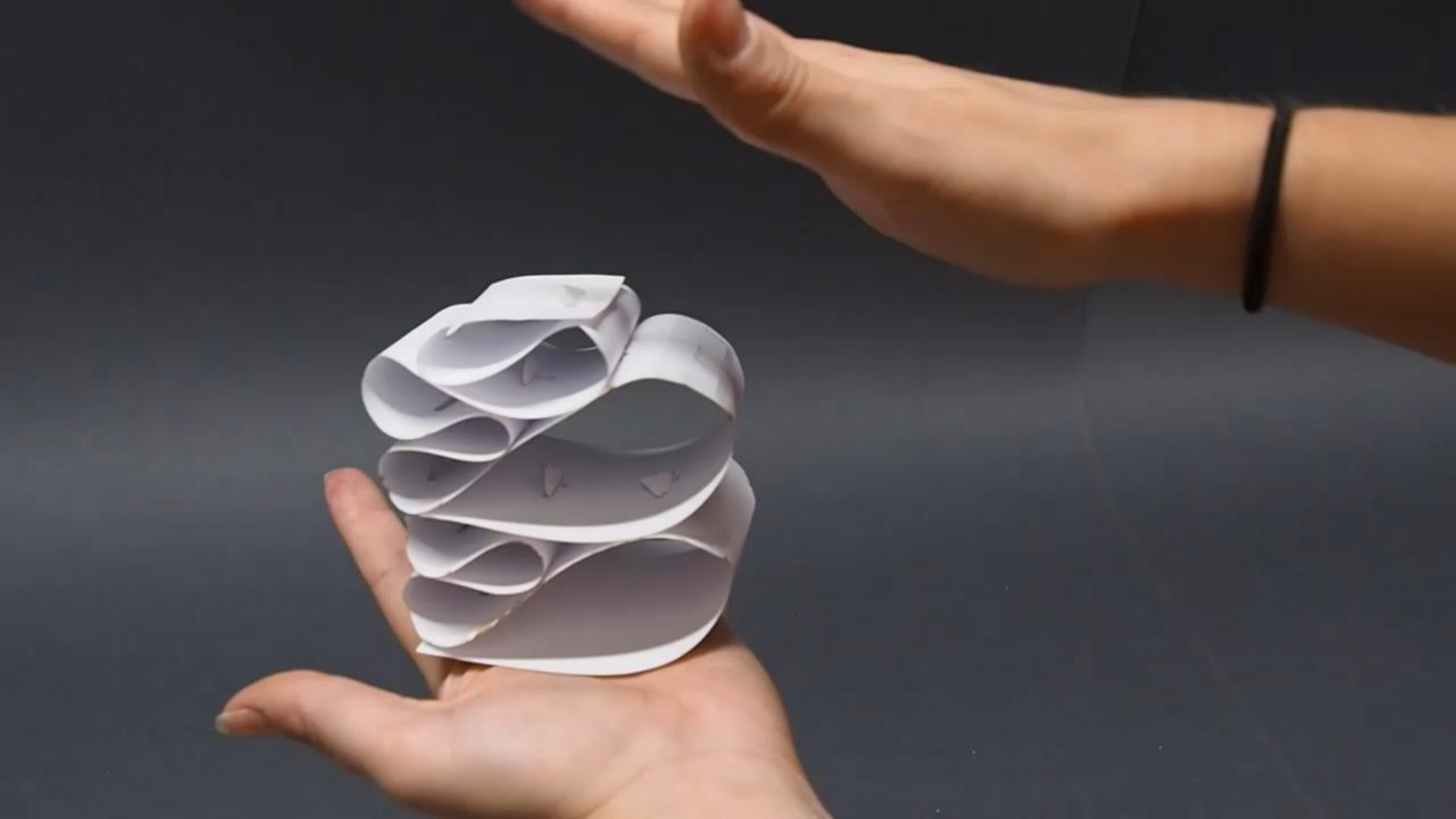
original uniform cotan

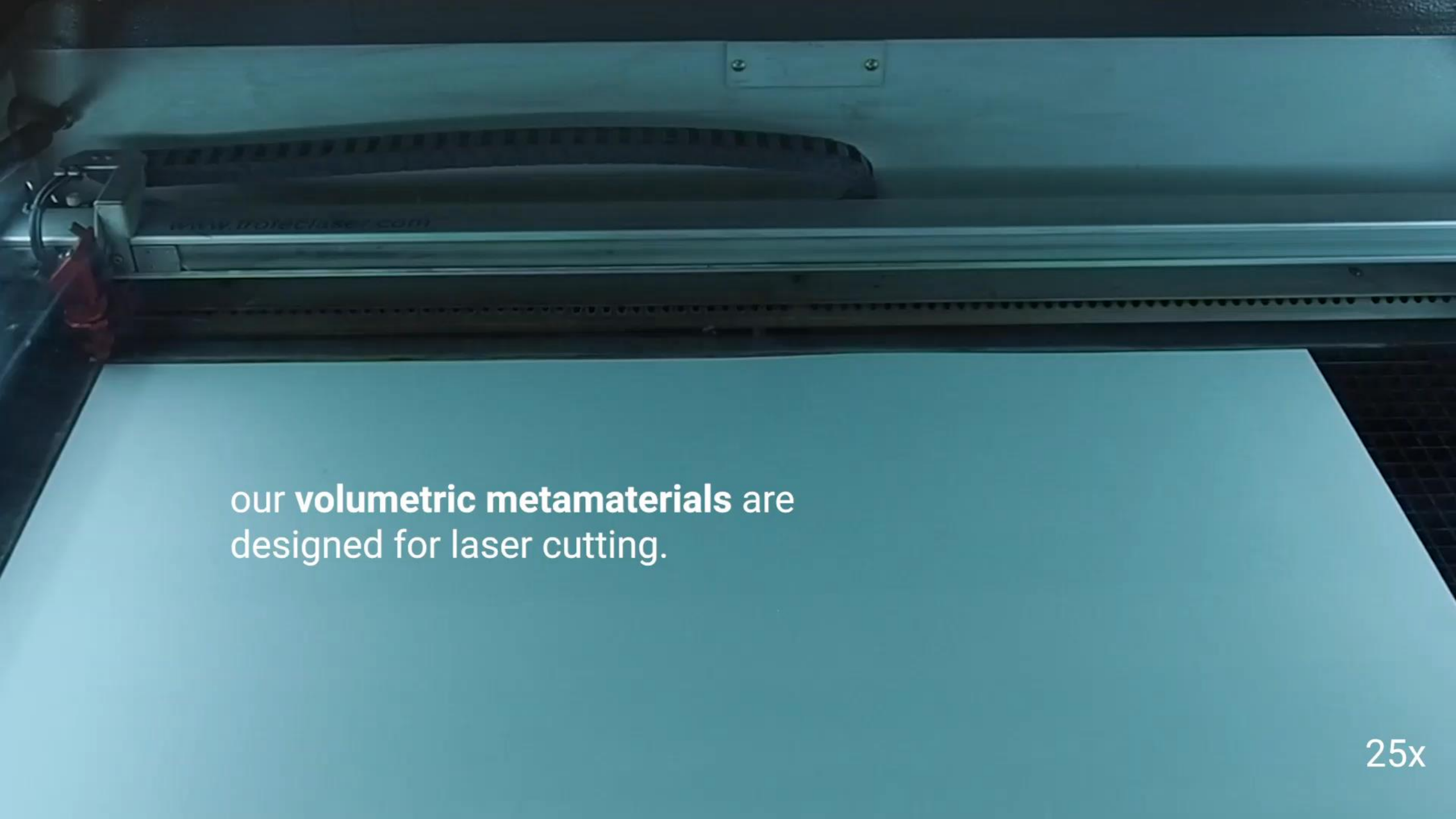
See an example in `2_mesh_smoothing` [key m]

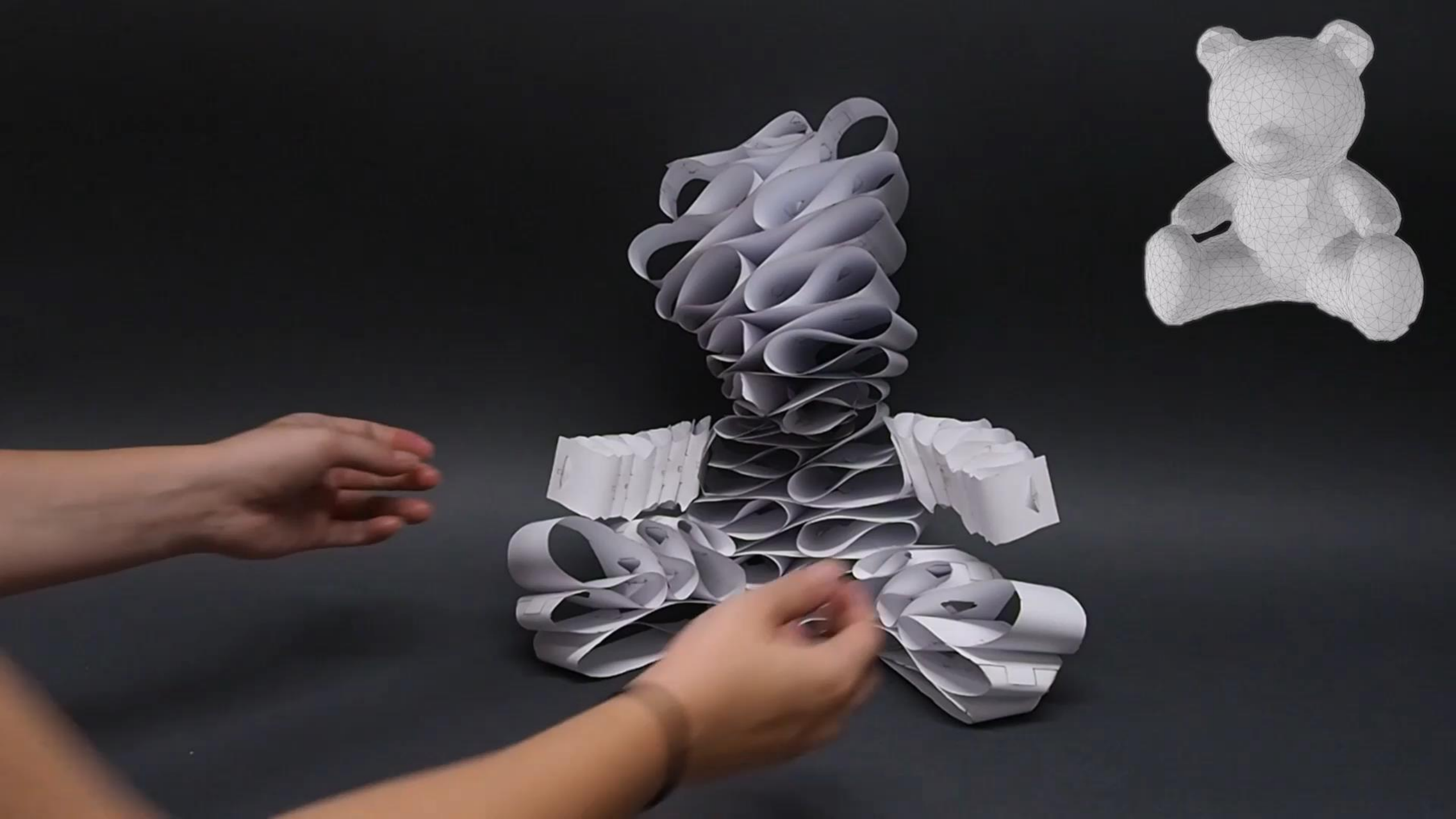
10min **Break**

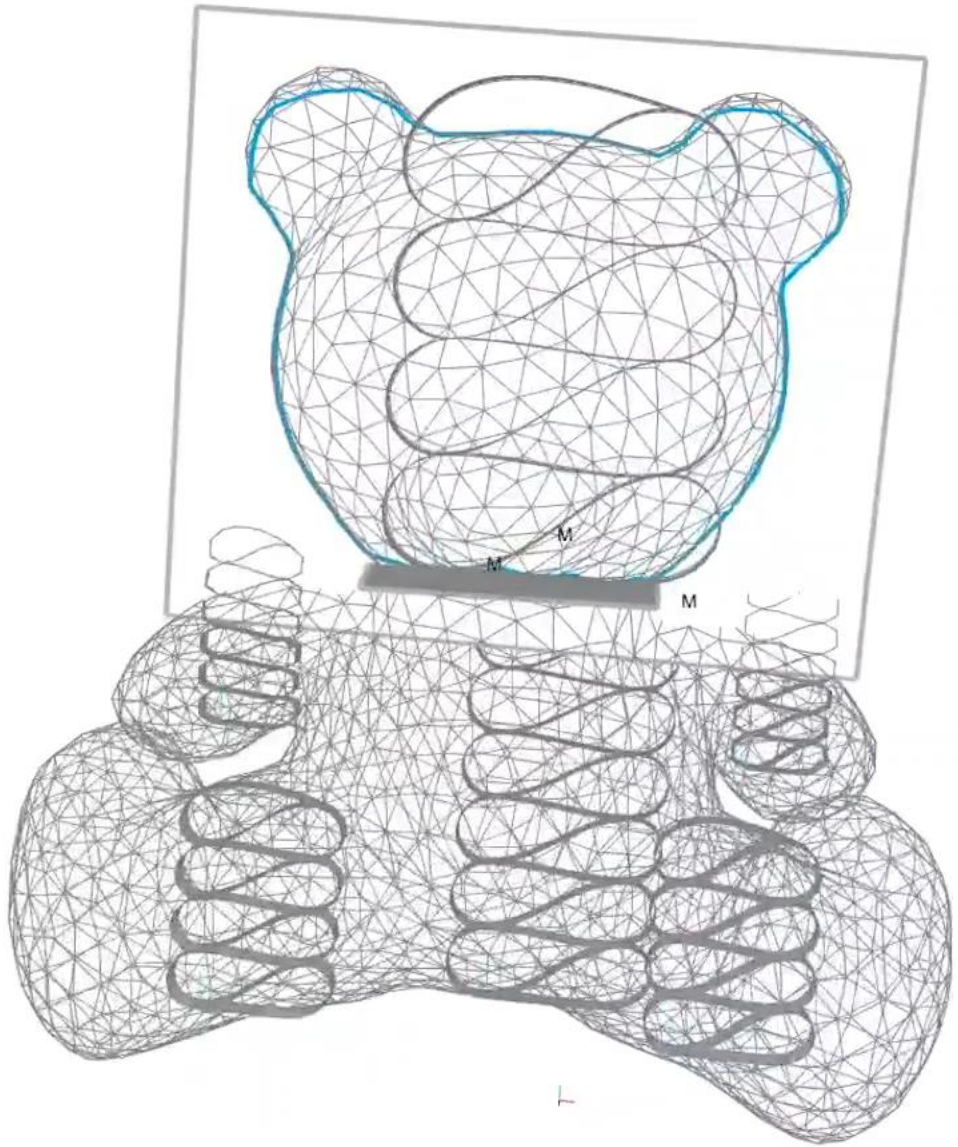
Ruffles

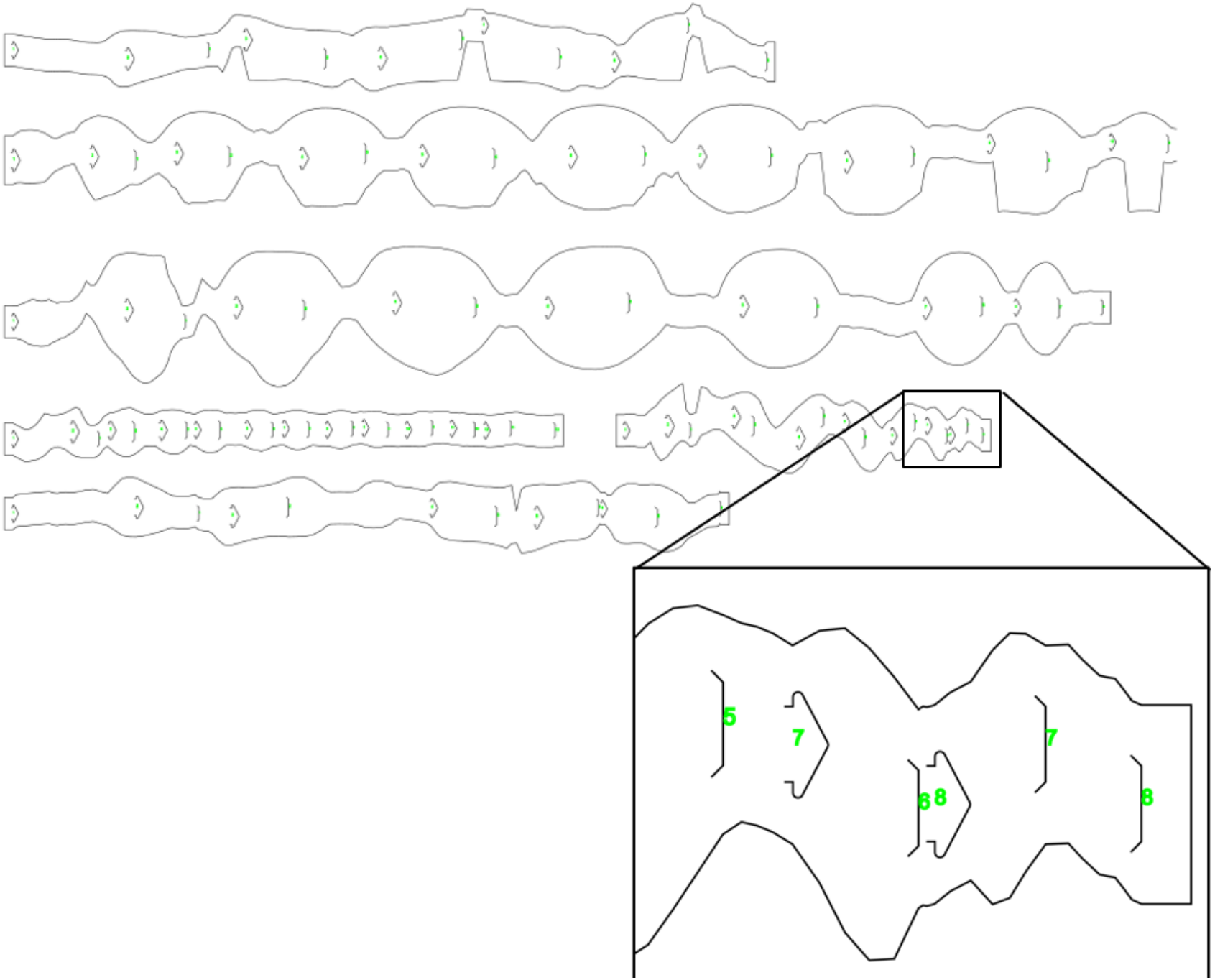
thin sheets, such as paper,
have a **very low bending stiffness**




our **volumetric metamaterials** are
designed for laser cutting.
25x

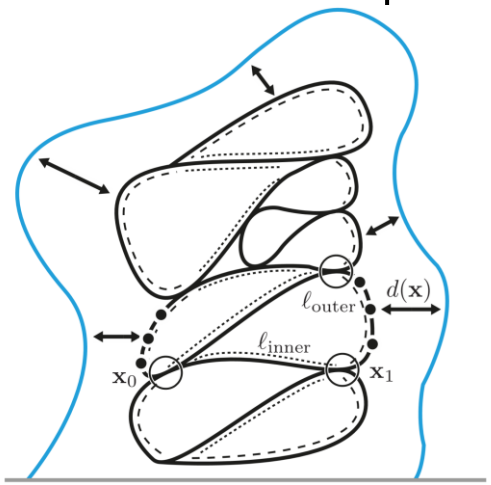



M
M
M


5
7
8
8
7
8

update shape

simulation



target shape

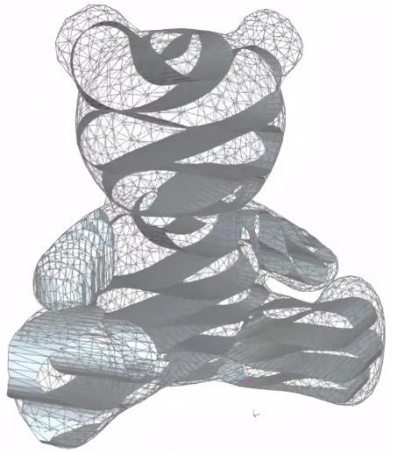
distance between target and clostest point

inner part of section

outer part of section

$$\ell'_{inner} = \alpha \cdot (d(\mathbf{x}_0) + \ell_{inner} + d(\mathbf{x}_1))$$

$$\ell'_{\text{outer}} = \ell_{\text{outer}} + \eta_{\text{outer}} \cdot \min_{\boldsymbol{x}_i \in \text{Section}} d(\mathbf{x}_i)$$



properties

1. smooth deformation
   based on Discrete Shells [Grispun et al. 2003]
2. smooth connections
3. no self-intersection
   extening AirMeshes [Müller et al. 2015]

error < ε ?

no

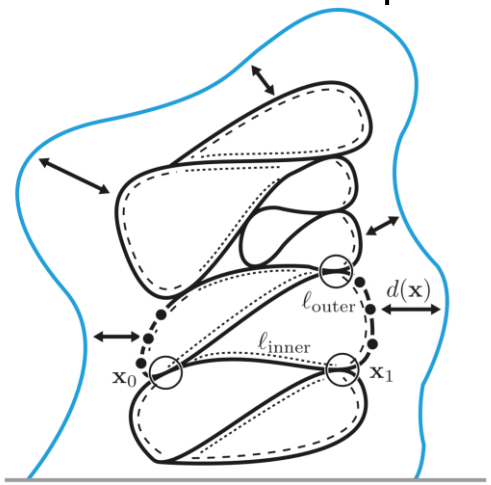
yes

final result

inner: simulation

update shape

simulation



target shape

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inner part of section

outer part of section

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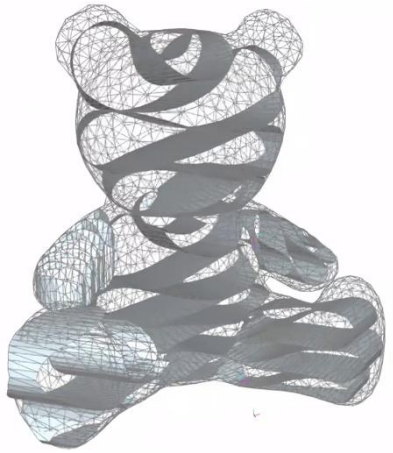

properties

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3. no self-intersection
   extening AirMeshes [Müller et al. 2015]

error < ε ?

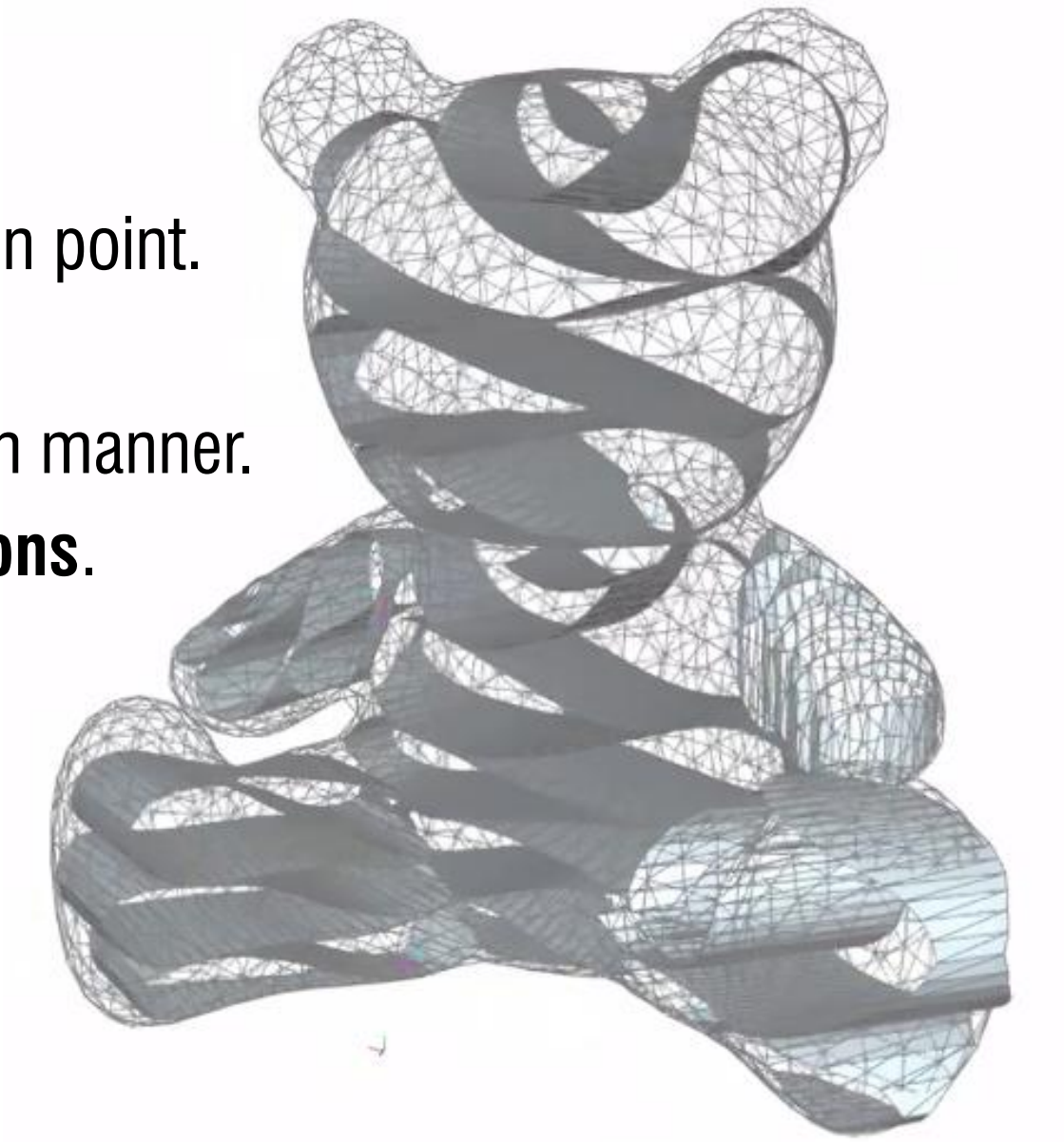
no

yes

final result

properties

1. The ruffle deforms **smoothly** at any given point.
2. The ruffle does **not stretch**, like paper.
3. The ruffle **connections** meet in a smooth manner.
4. The ruffle does not have **self-intersections**.



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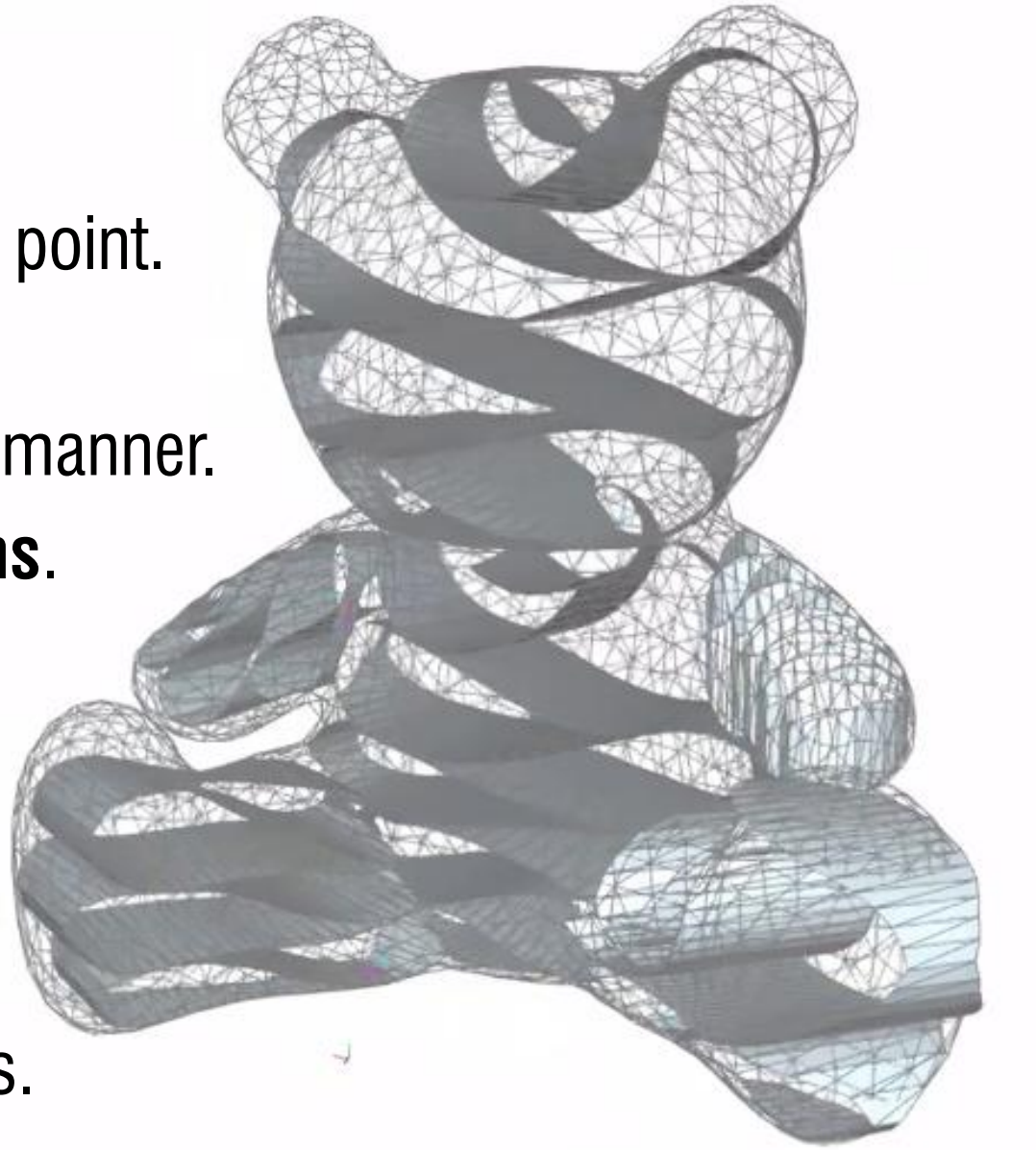
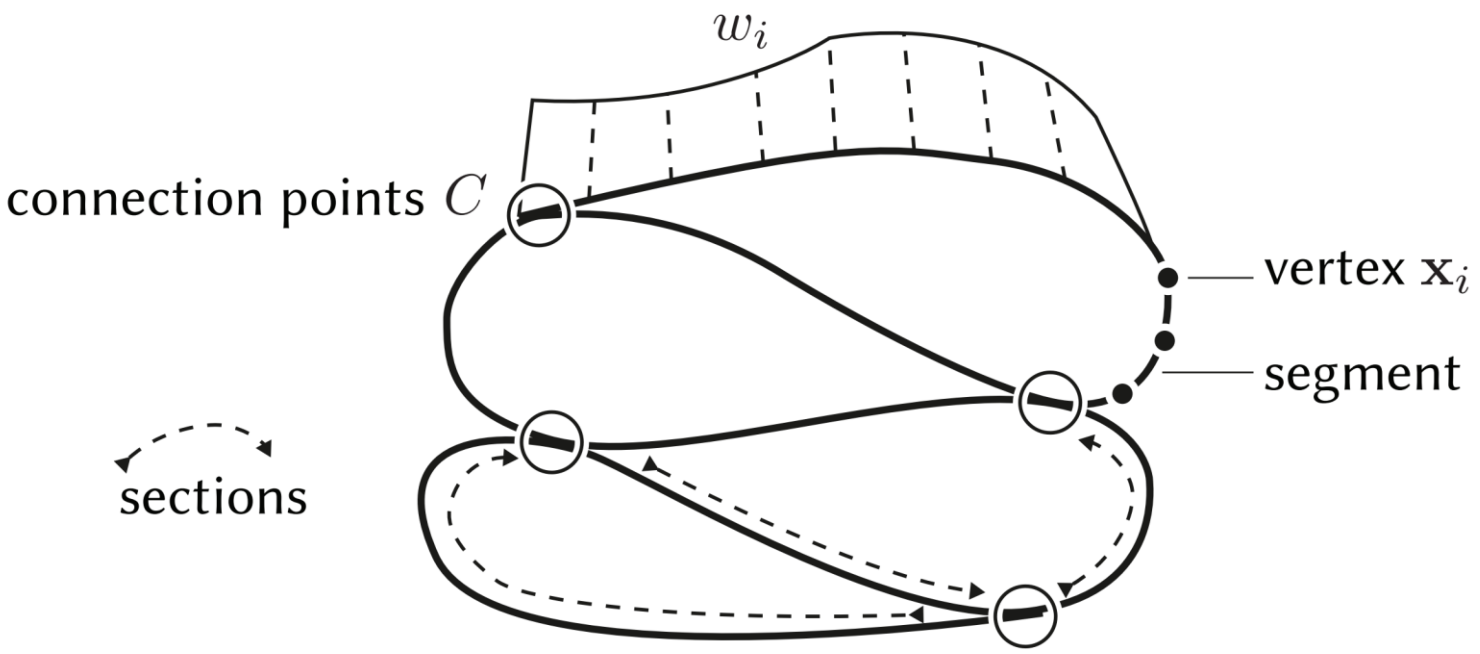


The simulation will be a minimizer

$$\mathbf{x}^* = \arg\min_{\mathbf{x}} E(\mathbf{x})$$

where the energy **E(x)** models the properties.

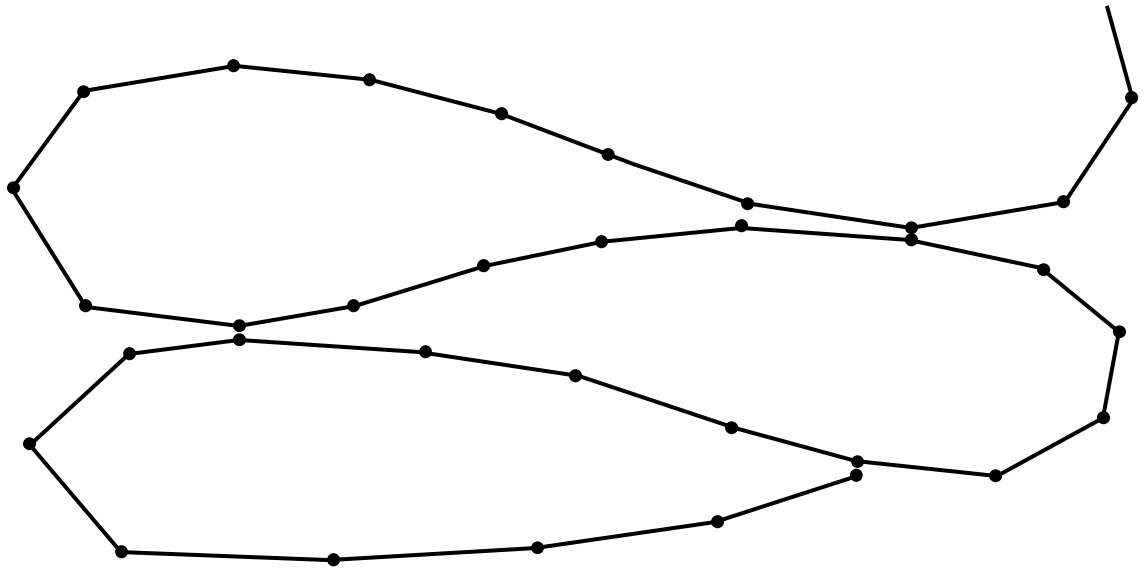
notation



Carnegie Mellon University

properties → energies

1. The ruffle deforms **smoothly** at any given point.

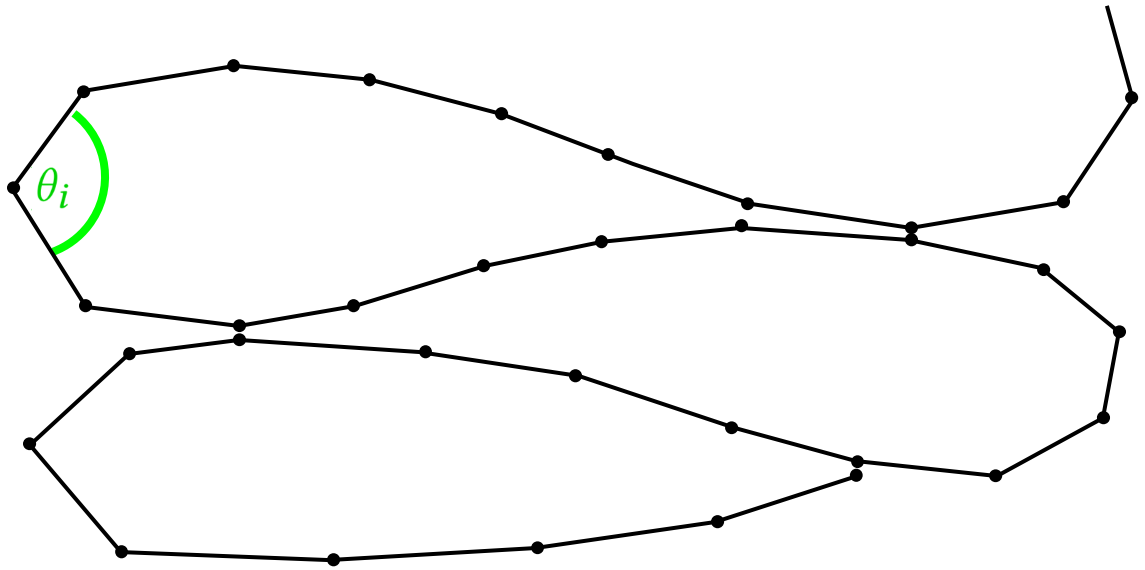


We want to get a small energy when the paper is at rest *(flat)*, and a higher number the more we deform it. How would you model this smoothness energy?

<30sec brainstorming>

properties ⟶ energies

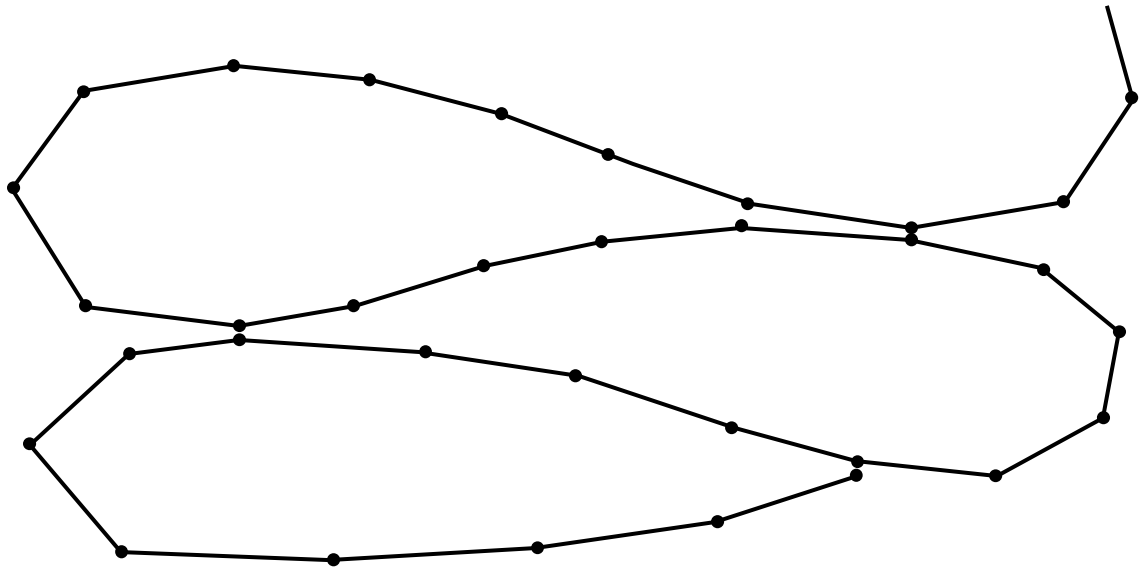
1. The ruffle deforms **smoothly** at any given point.



$$E_B(\mathbf{x}) = \sum_{i=2}^{N-1} (\theta_i - \pi)^2 w_i / \bar{\ell}_i$$

properties → energies

2. The ruffle does **not stretch**, like paper.

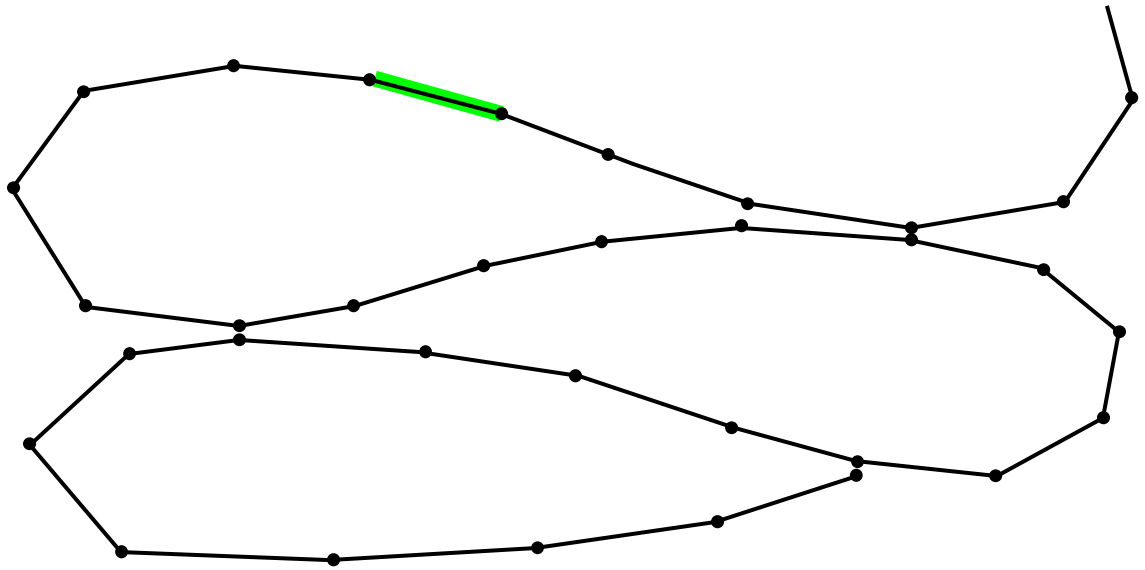


We want: small energy when the *lengths* stay the same, higher energy when deformed. How would you model this stretch energy?

<30sec brainstorming>

properties → energies

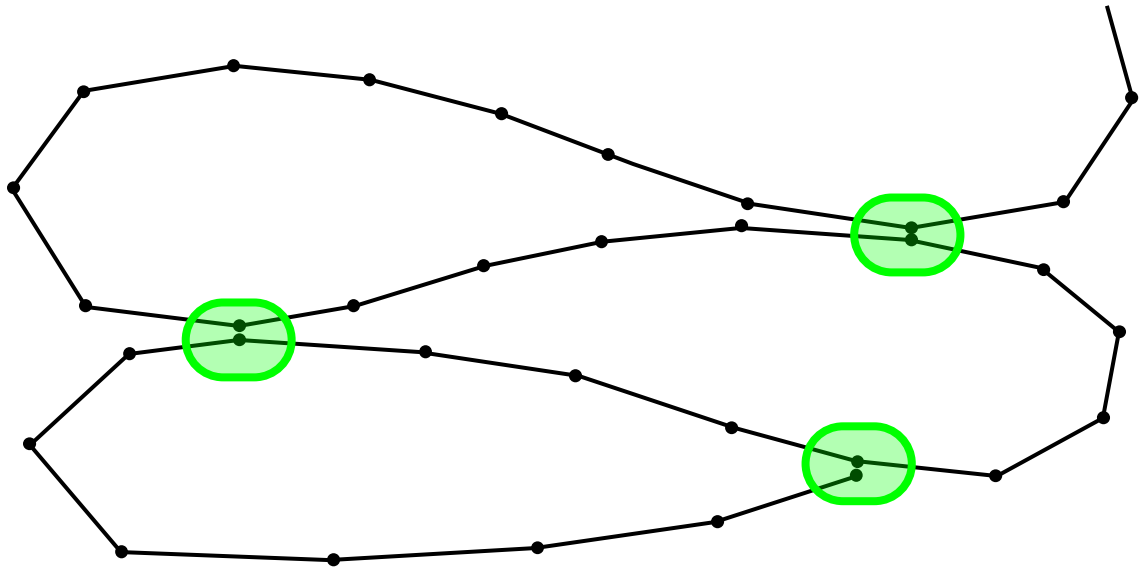
2. The ruffle does **not stretch**, like paper.



$$E_M(\mathbf{x}) = \sum_{i=1}^{N-1} \left(\|\mathbf{x}_{i+1} - \mathbf{x}_i\| - \ell_i \right)^2$$

properties → energies

3. The ruffle **connections** meet in a smooth manner.

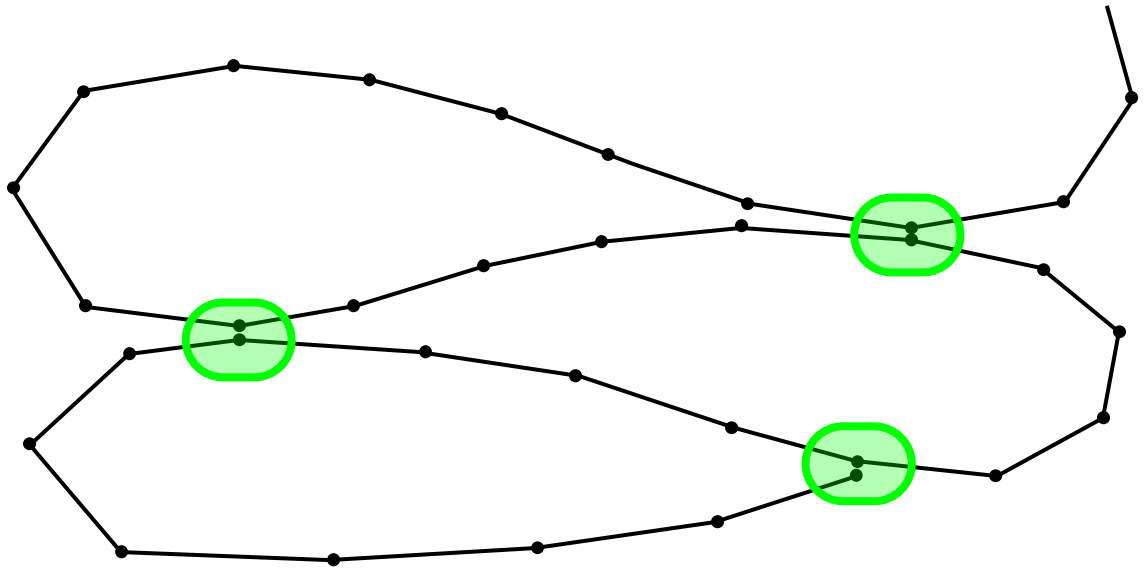


We want: small energy when the *tangents* are the same, higher energy when different. How would you model this energy?

<30sec brainstorming>

properties ⟶ energies

3. The ruffle **connections** meet in a smooth manner.

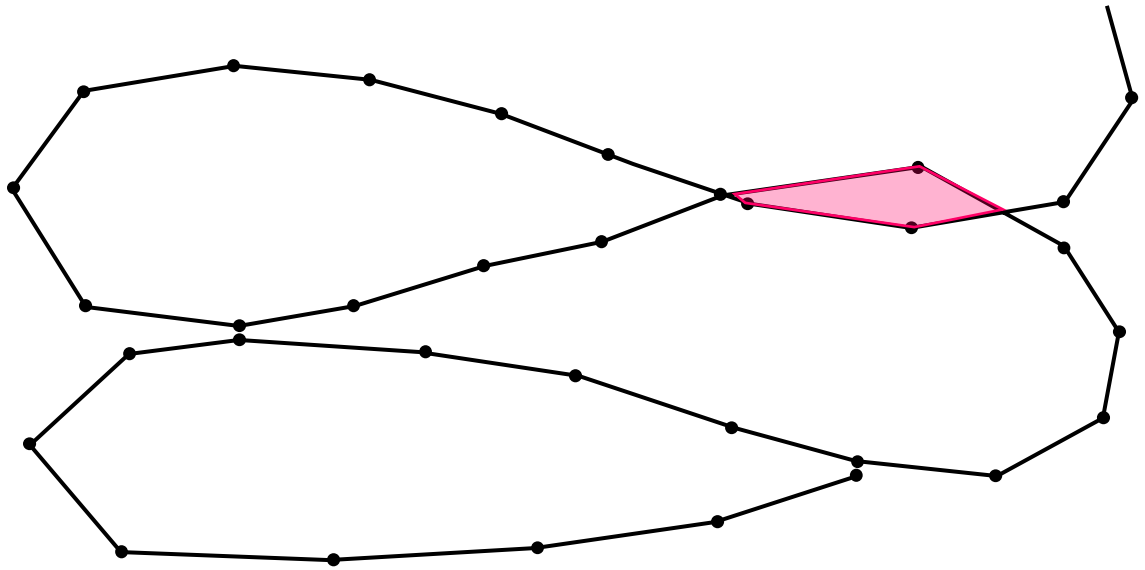


$$E_C(\mathbf{x}) = \sum_{(i,j)\in C} (\theta_{ij}^{+} - \pi)^2 w_i / \bar{\ell}_{ij}^{+} + (\theta_{ij}^{-} - \pi)^2 w_i / \bar{\ell}_{ij}^{-}$$

$$\theta_{ij}^{\pm} = \angle(\mathbf{x}_{i\pm1}, \mathbf{x}_i, \mathbf{x}_{j\pm1})$$

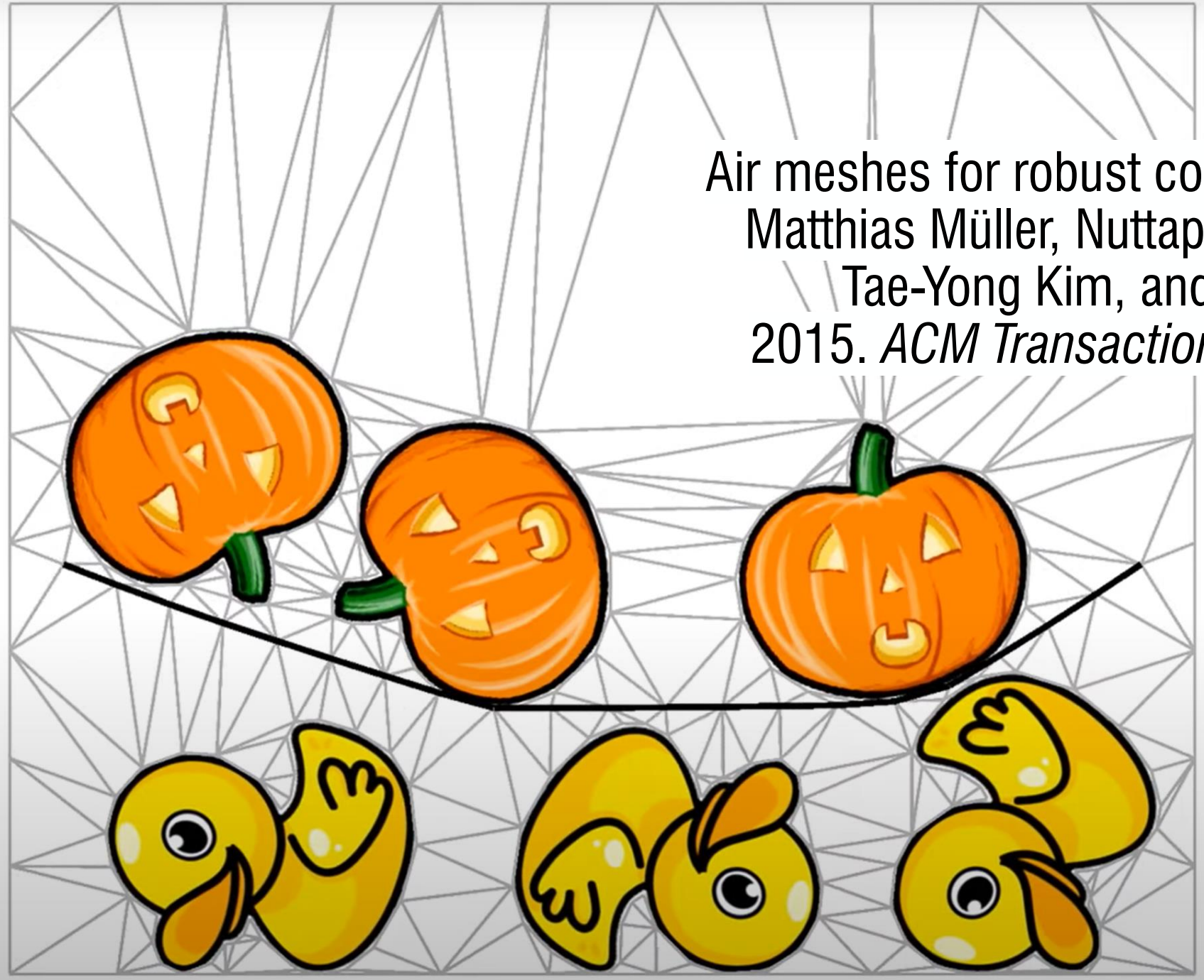
properties → energies

4. The ruffle does not have **self-intersections**.



We want: small energy at no intersection, high energy otherwise. This is a long-standing challenge in engineering.

We use...



Video

Air meshes for robust collision handling.
Matthias Müller, Nuttapong Chentanez,
Tae-Yong Kim, and Miles Macklin.
2015. *ACM Transactions on Graphics.*

Simulation

Putting it all together

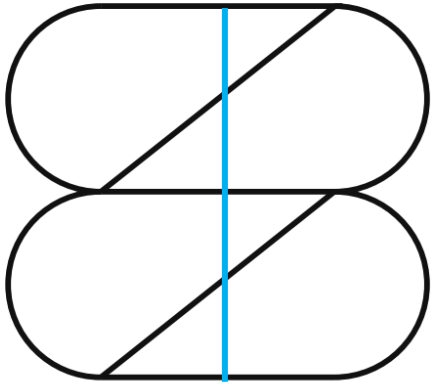
$$E(\mathbf{x}) = k_M E_M(\mathbf{x}) + k_B(E_B(\mathbf{x}) + E_C(\mathbf{x})) + k_{AM} E_{AM}(\mathbf{x})$$

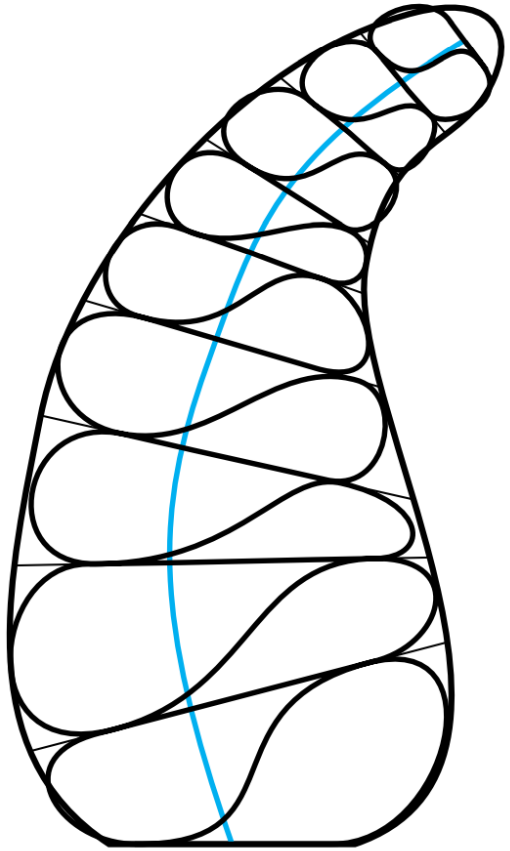
and solve for

$$\mathbf{x}^* = \arg\min_{\mathbf{x}} E(\mathbf{x})$$

Initial guess

Ruffle construction by *generator curve*



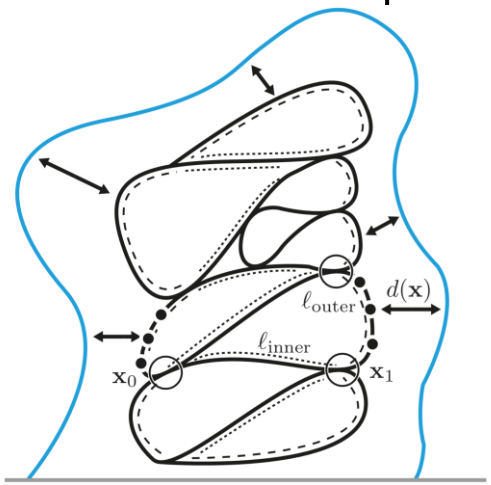


Let's see this in `3_ruffle_simulation`

outer: optimization

update shape

simulation



target shape

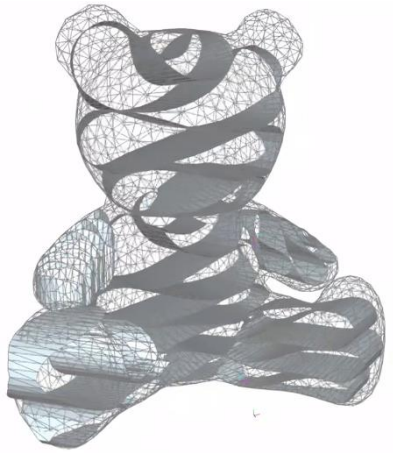
distance between target and clostest point

inner part of section

outer part of section

$$\ell'_{inner} = \alpha \cdot (d(\mathbf{x}_0) + \ell_{inner} + d(\mathbf{x}_1))$$

$$\ell'_{\text{outer}} = \ell_{\text{outer}} + \eta_{\text{outer}} \cdot \min_{\boldsymbol{x}_i \in \text{Section}} d(\mathbf{x}_i)$$



properties

1. smooth deformation
   based on Discrete Shells [Grispun et al. 2003]
2. smooth connections
3. no self-intersection
   extening AirMeshes [Müller et al. 2015]

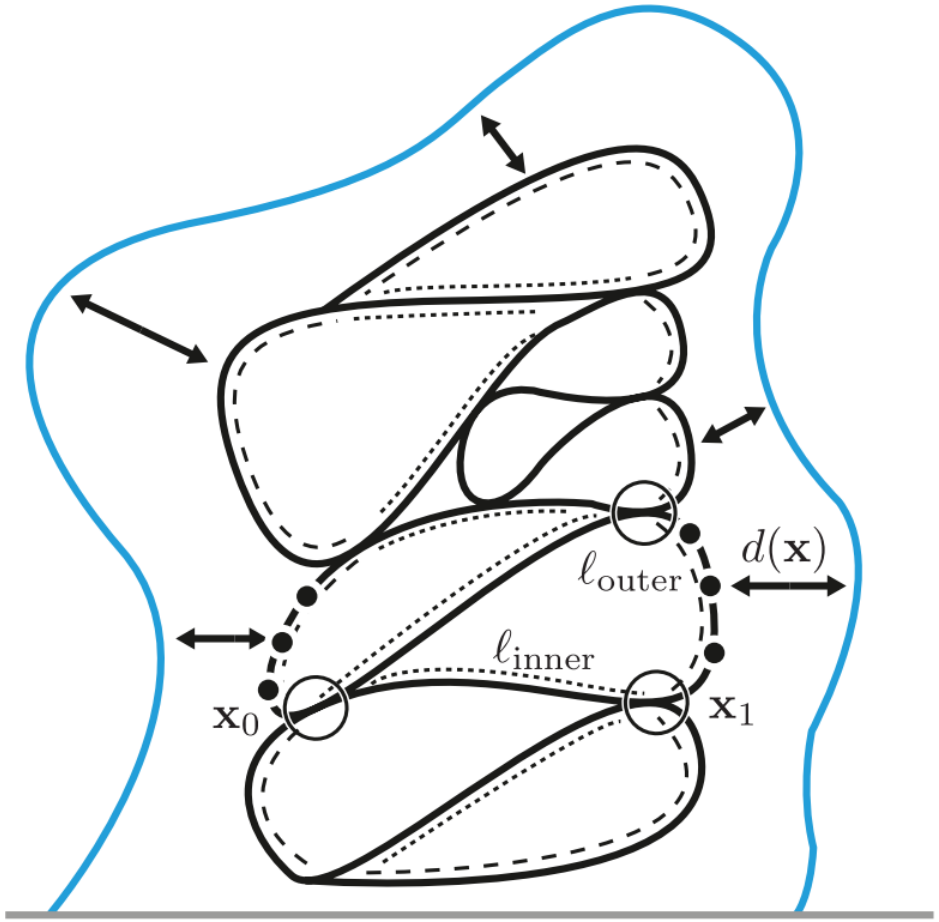
error < ε ?

no

yes

final result

shape approximation



target shape

distance between target and clostest point

inner part of section

outer part of section

$$\ell'_{inner} = \alpha \cdot (d(\mathbf{x}_0) + \ell_{inner} + d(\mathbf{x}_1))$$

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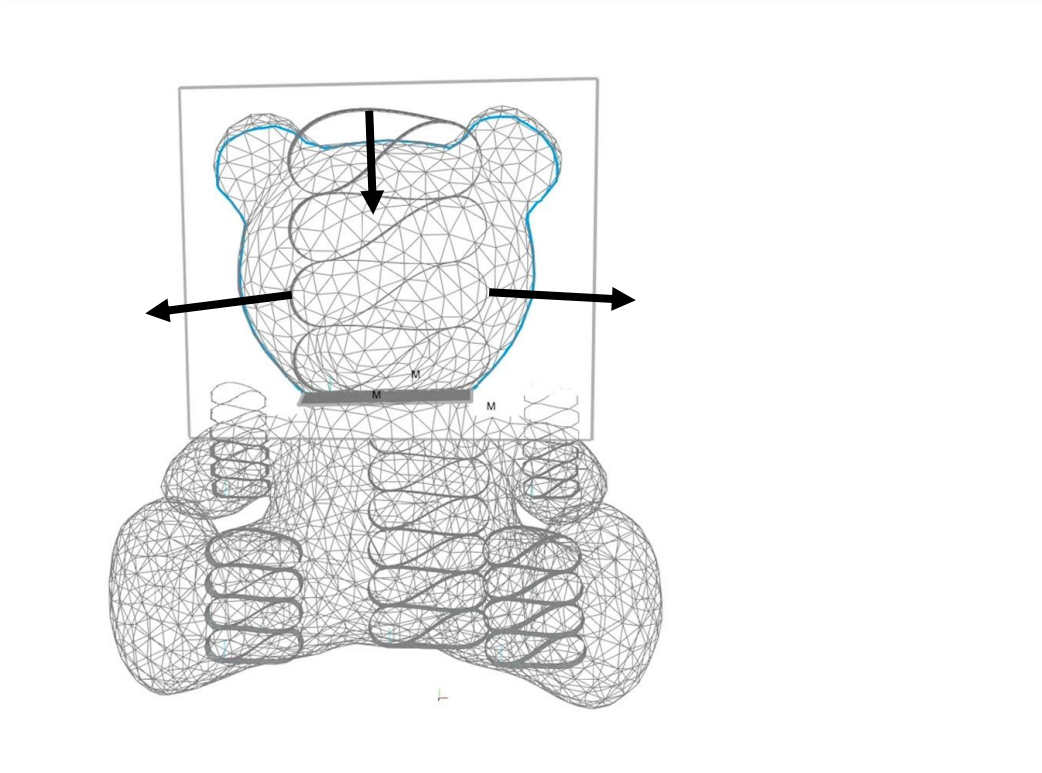
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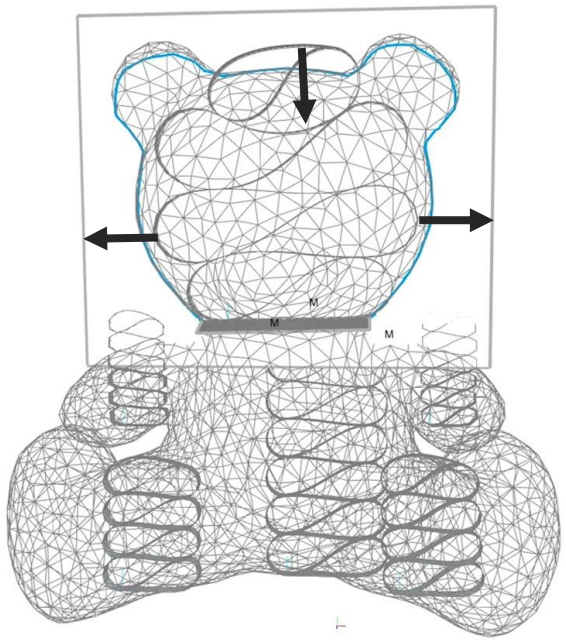
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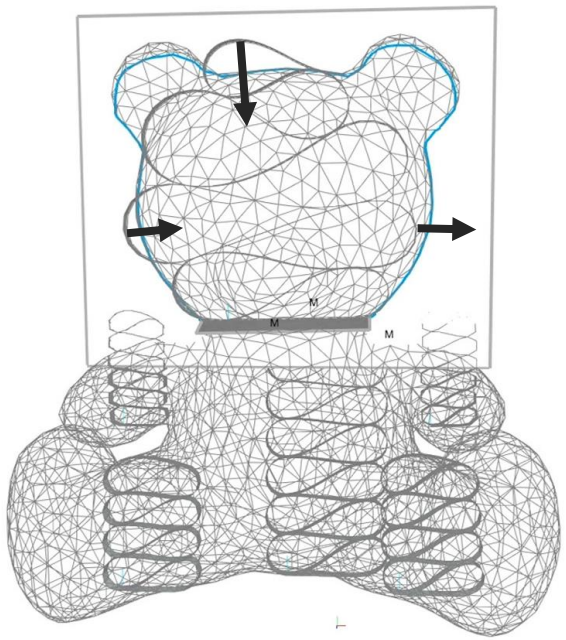
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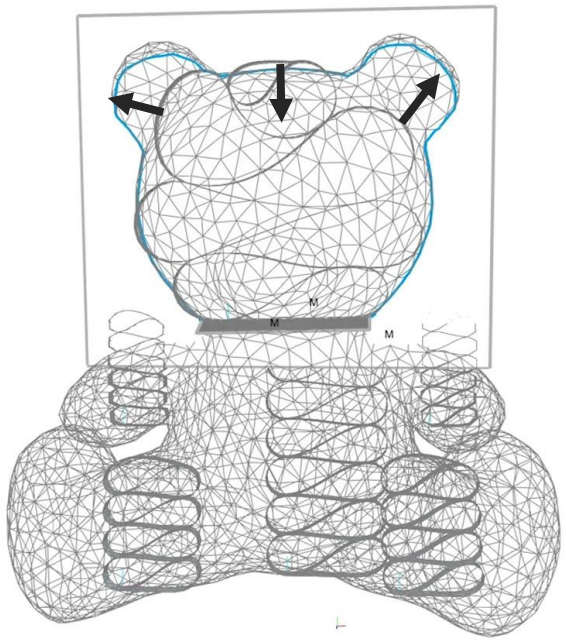
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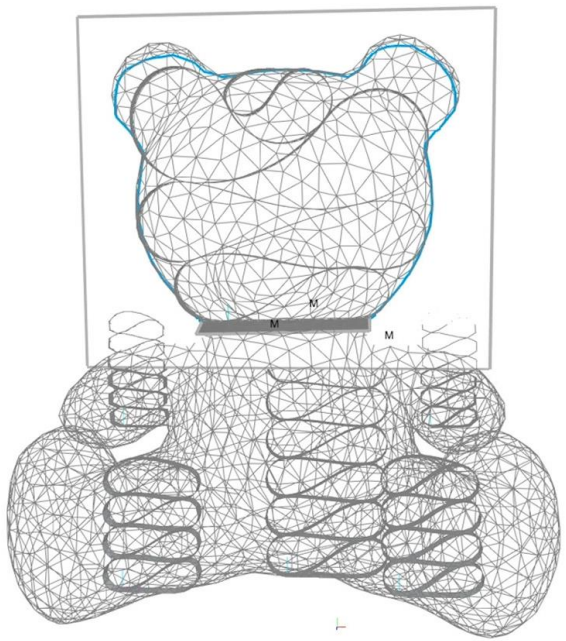
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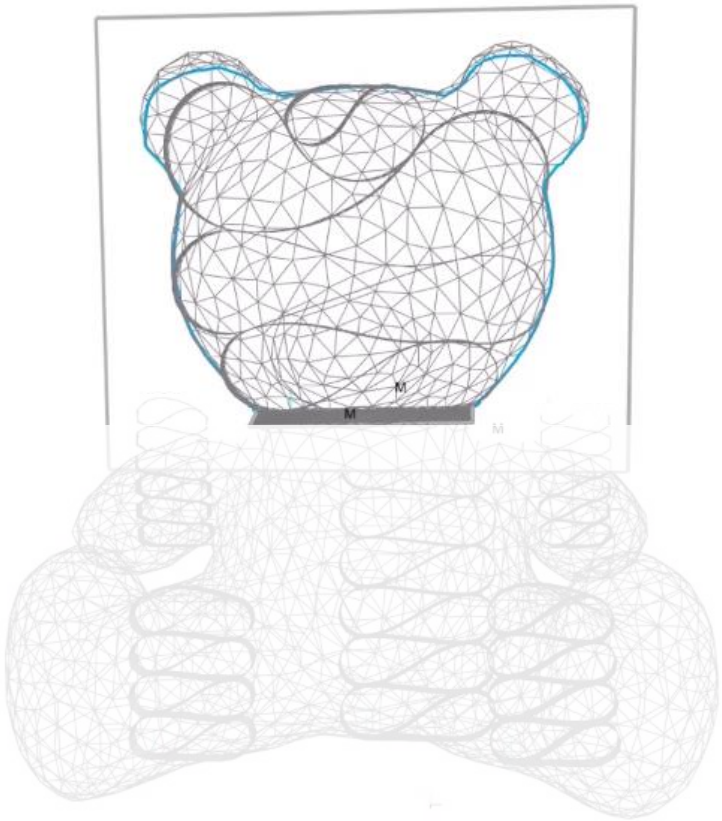
scope



current → **next**

optimize shape → optimize for **forces**

summary

Summary

Geometry is awesome

Generate & edit meshes, send to 3D printer → physical objects!

Generate complex materials

Useful tools & data

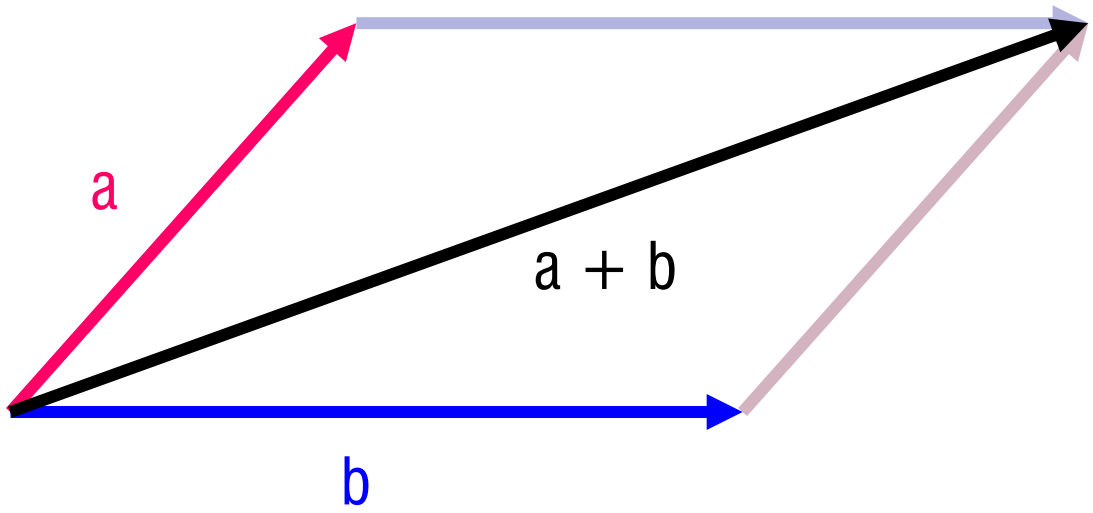
MeshLab

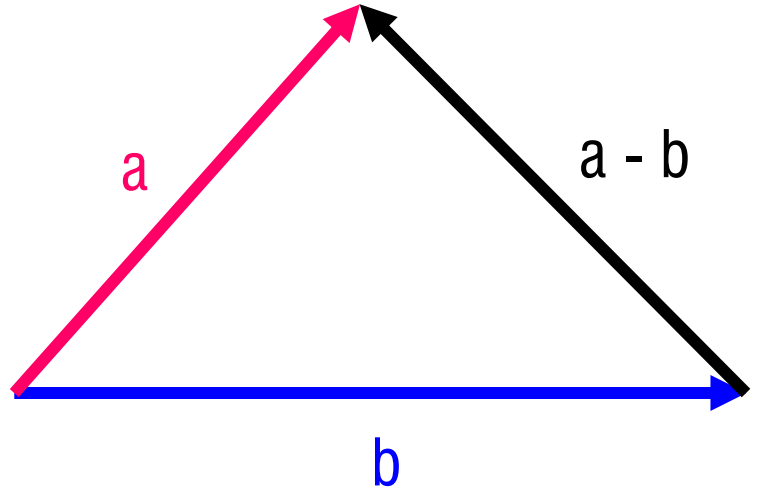
Open Flipper

Meshes: https://github.com/libigl/libigl-tutorial-data

end

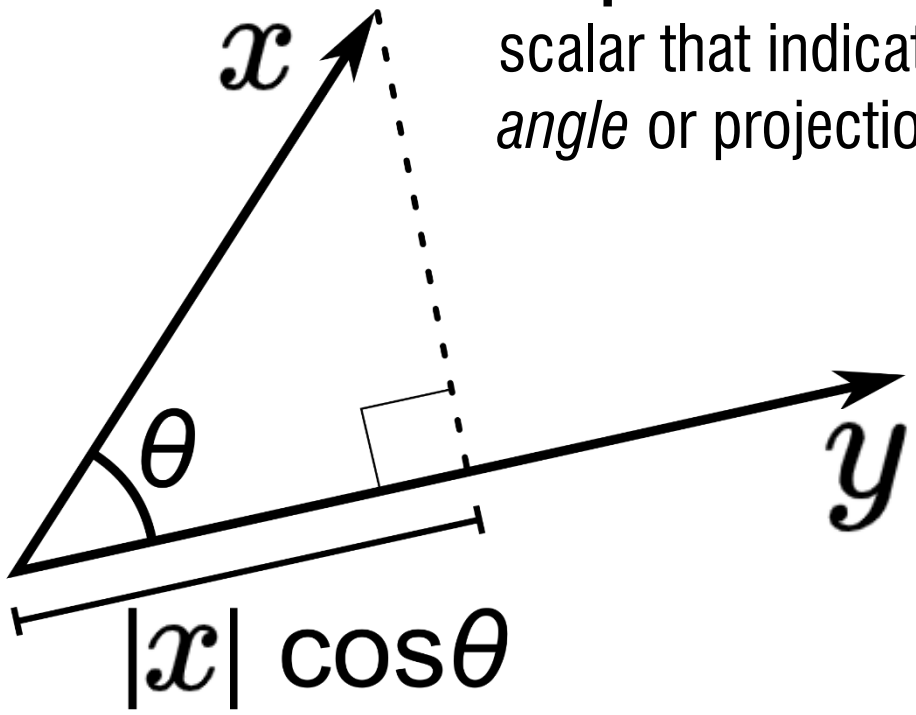
Vector cheat sheet





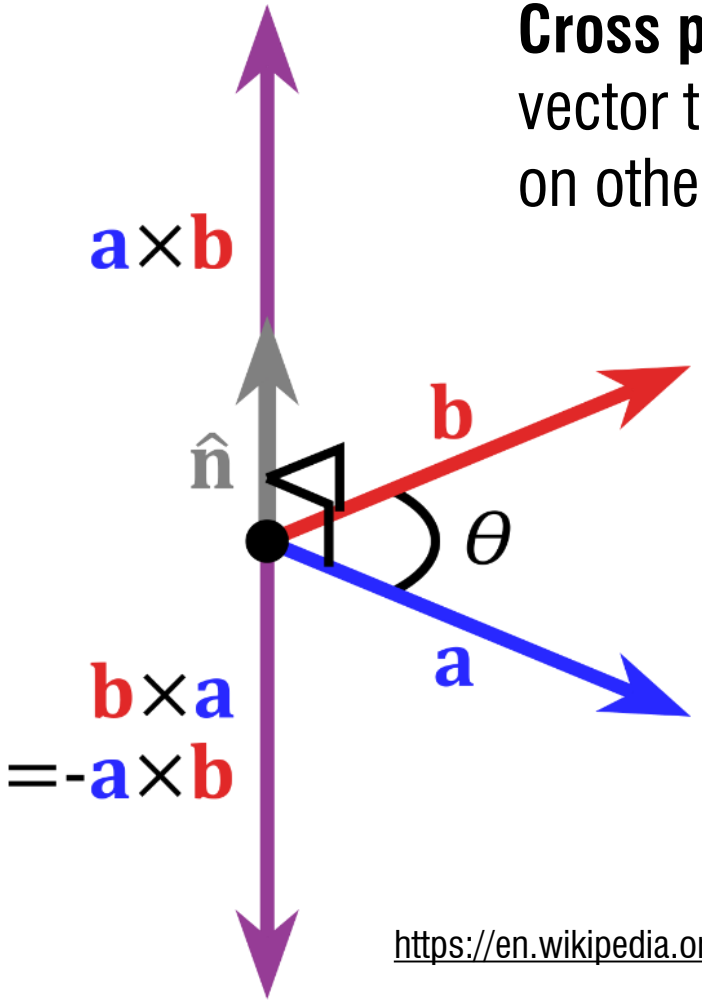
https://en.wikipedia.org/wiki/Euclidean_vector

Vector cheat sheet



Dot product:
scalar that indicates *angle* or projection

$$\angle(x, y) = \arccos \frac{\langle x, y \rangle}{\|x\| \, \|y\|}$$



Cross product:
vector that is normal on other two vectors

https://en.wikipedia.org/wiki/Cross_product