THE DISTRIBUTION OF STRESS IN THE NEIGHBOURHOOD OF A FLAT ELLIPTICAL CRACK IN AN ELASTIC SOLID

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1. *Introduction.* This paper is mainly concerned with the distribution of stress near a flat elliptical crack in a body of infinite extent under a uniform tension at infinity perpendicular to the plane of the crack. After an analytical solution of this problem was found the authors received a copy of a paper by Sadowsky and Sternberg (3) in which they solved the more general problem of the stress concentration around a tri-axial ellipsoidal cavity in an elastic body of infinite extent, the body at infinity being in a uniform state of stress whose principal axes are parallel to the axes of the cavity. The method of solution adopted by the present writers for the special case of the elliptical crack is somewhat different from the more general work of Sadowsky and Sternberg, and is, moreover, surprisingly simple, so it seems to be of value to present this solution here.

In addition, the problem of the indentation of a flat semi-infinite surface by a flat-ended elliptical cylindrical punch is also considered. This problem is, however, only partly solved.

2. *Fundamental equations.* If rectangular cartesian coordinates x, y, Z, with $z = x+iy$ the complex variable and $\bar{z} = x-iy$ its complex conjugate are used, it is known (1) that the elastic equations of equilibrium and the stress-strain relations for isotropic materials may be put in the form*

$$\left.\begin{aligned} D &= 4\frac{\partial F(z,\bar{z},Z)}{\partial \bar{z}}, \quad w = \frac{\partial G(z,\bar{z},Z)}{\partial Z},\\ \Theta &= -\frac{\mu'}{1-\eta}\frac{\partial^2}{\partial Z^2}(H-2\eta G),\\ \Phi &= 16\mu'\frac{\partial^2 F}{\partial \bar{z}^2}, \quad \Upsilon = 2\mu'\frac{\partial L}{\partial \bar{z}},\\ \widehat{zz} &= -\mu'\nabla_1^2 H = -\frac{\mu'}{1-\eta}\frac{\partial^2}{\partial Z^2}(\eta H-2G), \end{aligned}\right\} \tag{2·1}$$

where u, v, w are the components of displacements and where

$$D = u+iv, \quad \Theta = \widehat{xx}+\widehat{yy}, \quad \Phi = \widehat{xx}-\widehat{yy}+2i\widehat{xy}, \quad \Upsilon = \widehat{xz}+i\widehat{yz}, \quad = 0\,? \tag{2·2}$$

$$\left.\begin{aligned} 2(1-2\eta)\nabla^2 F &= (1-2\eta)\nabla^2 G = -\Delta,\\ \Delta = \nabla_1^2(F+\bar{F})+\frac{\partial^2 G}{\partial Z^2}&, \quad \nabla_1^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}},\\ H = F+\bar{F}+G, \quad L &= \frac{\partial}{\partial Z}(2F+G), \quad L+\bar{L} = 2\frac{\partial H}{\partial Z}. \end{aligned}\right\} \tag{2·3}$$

* The z in the notation for the stresses need not be confused with the complex variable.

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A bar placed over a quantity denotes the complex conjugate of that quantity. Poisson's ratio η is related to Lamé's constants λ', μ' by $\eta = \frac{1}{2}\lambda'/(\lambda'+\mu')$.

For the present work, functions F, G, H which satisfy (2·3) may be expressed in the form

$$\left.\begin{aligned} F &= 2\omega + 4(1-\eta)\phi + 2Z\frac{\partial\phi}{\partial Z}, \\ G &= 4\omega - 8(1-\eta)\phi + 4Z\frac{\partial\phi}{\partial Z}, \\ H &= 8\omega \qquad\qquad\qquad + 8Z\frac{\partial\phi}{\partial Z}, \end{aligned}\right\} \tag{2·4}$$

where ω, ϕ are real harmonic functions, i.e. they satisfy the equations

$$\nabla^2\omega = 0, \quad \nabla^2\phi = 0. \tag{2·5}$$

3. *Statement of the problem for an elliptical crack.* Suppose that the crack occupies the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \tag{3·1}$$

in the plane $Z = 0$. A uniform tension p at infinity acts parallel to the Z-axis. The resulting distribution of stress consists of the system

$$\widehat{zz} = p, \quad \Theta = \Phi = \Psi = 0, \tag{3·2}$$

together with a distribution of stress in the region $Z \geqslant 0$ which vanishes at infinity and which satisfies the conditions

$$\left.\begin{aligned} &\text{(i)}\ \ \Psi = 0 && (Z = 0), \\ &\text{(ii)}\ \ \widehat{zz} = -p && \left(\frac{x^2}{a^2}+\frac{y^2}{b^2} < 1,\ Z = 0\right), \\ &\qquad w = 0 && \left(\frac{x^2}{a^2}+\frac{y^2}{b^2} > 1,\ Z = 0\right). \end{aligned}\right\} \tag{3·3}$$

From (2·1), (2·3) and (2·4) it is seen that the condition (i) in (3·3) can be satisfied by taking

$$\omega + \phi = 0$$

everywhere. The displacements and stresses may then be reduced to the forms

$$\left.\begin{aligned} D &= 8\frac{\partial}{\partial \bar{z}}\left\{(1-2\eta)\phi + Z\frac{\partial\phi}{\partial Z}\right\}, \\ w &= -8(1-\eta)\frac{\partial\phi}{\partial Z} + 4Z\frac{\partial^2\phi}{\partial Z^2}, \\ \Theta &= -8\mu'\left\{(1+2\eta)\frac{\partial^2\phi}{\partial Z^2} + Z\frac{\partial^3\phi}{\partial Z^3}\right\}, \\ \Phi &= 32\mu'\frac{\partial^2}{\partial \bar{z}^2}\left\{(1-2\eta)\phi + Z\frac{\partial\phi}{\partial Z}\right\}, \\ \widehat{zz} &= -8\mu'\frac{\partial^2\phi}{\partial Z^2} + 8\mu' Z\frac{\partial^3\phi}{\partial Z^3}, \\ \Psi &= 16\mu' Z\frac{\partial^3\phi}{\partial \bar{z}\,\partial Z^2}. \end{aligned}\right\} \tag{3·4}$$

p

$\frac{\partial\phi}{\partial Z} = 0$ outside of ellipse

$\frac{\partial^2\phi}{\partial Z^2} = \frac{p}{8\mu'}$ in ellipse

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It will be noticed that the expressions (3·4) are valid for any problem in which condition (i) in (3·3) is satisfied, independently of (ii). On using (3·4) condition (ii) may now be put in the form

$$\left.\begin{aligned} 8\mu'\frac{\partial^2\phi}{\partial Z^2} &= p \quad \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}<1,\ Z=0\right),\\ \frac{\partial\phi}{\partial Z} &= 0 \quad \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}>1,\ Z=0\right).\end{aligned}\right\} \tag{3·5}$$

Since ϕ, and therefore $\partial\phi/\partial Z$, are harmonic functions, $\partial\phi/\partial Z$ is equivalent to the velocity potential due to a flat elliptical disk moving with uniform velocity $p/8\mu'$ perpendicular to its plane through an infinite incompressible fluid which is at rest at infinity*. The solution of this problem is well known (see (2)) and will be given in the next section in terms of ellipsoidal coordinates.

4. *Ellipsoidal coordinates.* The coordinates (x, y, Z) of any point may be expressed in terms of a triply orthogonal system (λ, μ, ν) in the form

λ, μ

Analyse dimensionelle?

$$\left.\begin{aligned} a^2(a^2-b^2)\,x^2 &= (a^2+\lambda)\,(a^2+\mu)\,(a^2+\nu),\\ b^2(b^2-a^2)\,y^2 &= (b^2+\lambda)\,(b^2+\mu)\,(b^2+\nu),\\ a^2b^2Z^2 &= \lambda\mu\nu,\end{aligned}\right\} \tag{4·1}$$

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where $$\infty > \lambda \geqslant 0 \geqslant \mu \geqslant -b^2 \geqslant \nu \geqslant -a^2. \tag{4·2}$$

In the plane $Z = 0$ the inside of the ellipse $x^2/a^2+y^2/b^2 = 1$ is given by $\lambda = 0$, and the outside by $\mu = 0$.

The following partial derivatives are needed, namely,

$$\frac{\partial\lambda}{\partial x} = \frac{x}{2(a^2+\lambda)\,h_1^2},\quad \frac{\partial\lambda}{\partial y} = \frac{y}{2(b^2+\lambda)\,h_1^2},\quad \frac{\partial\lambda}{\partial Z} = \frac{Z}{2\lambda h_1^2}, \tag{4·3}$$

where $$4h_1^2Q(\lambda) = (\lambda-\mu)\,(\lambda-\nu),\quad Q(\lambda) = \lambda(a^2+\lambda)\,(b^2+\lambda). \tag{4·4}$$

Alternative expressions for x, y, Z in terms of elliptic functions may also be obtained if necessary (see (4)).

On recalling the hydrodynamical analogy of the previous section, $\partial\phi/\partial Z$ may be put in the form

$$\frac{\partial\phi}{\partial Z} = AZ\int_\lambda^\infty \frac{ds}{s\sqrt{\{Q(s)\}}}, \tag{4·5}$$

where A is a constant, and hence

$$\phi = \frac{A}{2}\int_\lambda^\infty \left\{\frac{x^2}{a^2+s}+\frac{y^2}{b^2+s}+\frac{Z^2}{s}-1\right\}\frac{ds}{\sqrt{\{Q(s)\}}}. \tag{4·6}$$

The function ϕ in (4·6) is known to be harmonic, since, apart from a multiplying constant, it represents the gravitational potential at an external point of a uniform elliptical plate, and it can at once be verified that $\partial\phi/\partial Z$ is given by (4·5), since

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{Z^2}{\lambda}-1 = 0. \tag{4·7}$$

Except for the determination of the constant A the solution of the problem is now complete. Displacements and stresses in cartesian coordinates (x, y, Z) may be obtained

* S. G. Michlin (*App. Math. Mech.* (*Prikl. Mat. Mekh.*), 10 (1946), 304) has also reduced stress problems of the type considered in this paper to an equivalent problem in potential theory.

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from (3·4) and (4·6) by differentiations. For numerical calculations it is convenient to express some of the integrals in terms of Jacobian elliptic functions by writing

$$\lambda = a^2 \operatorname{cn}^2 u/\operatorname{sn}^2 u = a^2(\operatorname{sn}^{-2} u - 1), \tag{4·8}$$

where $\operatorname{sn} u$ is the Jacobian elliptic function which has real and imaginary periods $4K$, $2iK'$ respectively, corresponding to the modulus k and complementary modulus k', where

$$ka = (a^2-b^2)^{\frac{1}{2}}, \quad ak' = b. \tag{4·9}$$

The variable u takes all real values between 0 and K.

The integral in (4·5) diverges when $\lambda = 0$, so it is convenient to express this integral in the alternative form

$$\begin{aligned}\frac{\partial\phi}{\partial Z} &= AZ\left\{\frac{2}{\sqrt{\{Q(\lambda)\}}} - \int_\lambda^\infty \frac{2s+a^2+b^2}{(a^2+s)(b^2+s)\sqrt{\{Q(s)\}}}\,ds\right\}\\ &= \frac{2AZ}{ab^2}\left\{\frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} - E(u)\right\},\end{aligned} \tag{4·10}$$

where

$$E(u) = \int_0^u \operatorname{dn}^2 t\,dt, \tag{4·11}$$

in order to evaluate $\partial\phi/\partial Z$ when $\lambda = 0$. Differentiating $\partial\phi/\partial Z$ with respect to Z with the help of (4·3) gives

$$\begin{aligned}\frac{\partial^2\phi}{\partial Z^2} &= A\left\{\frac{2\lambda^{\frac{1}{2}}[\lambda(a^2b^2-\mu\nu)-a^2b^2(\mu+\nu)-(a^2+b^2)\mu\nu]}{a^2b^2(\lambda-\mu)(\lambda-\nu)(a^2+\lambda)^{\frac{1}{2}}(b^2+\lambda)^{\frac{1}{2}}} - \int_\lambda^\infty \frac{2s+a^2+b^2}{(a^2+s)(b^2+s)\sqrt{\{Q(s)\}}}\,ds\right\}\\ &= A\left\{\frac{2\lambda^{\frac{1}{2}}[\lambda(a^2b^2-\mu\nu)-a^2b^2(\mu+\nu)-(a^2+b^2)\mu\nu]}{a^2b^2(\lambda-\mu)(\lambda-\nu)(a^2+\lambda)^{\frac{1}{2}}(b^2+\lambda)^{\frac{1}{2}}} - \frac{2}{ab^2}\left[E(u) - \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}\right]\right\}.\end{aligned} \tag{4·12}$$

When $\mu = 0$, i.e. $Z = 0$, outside the ellipse, it can be verified from (4·5) or (4·10) that $\partial\phi/\partial Z = 0$. When $\lambda = 0$, i.e. $Z = 0$, inside the ellipse, $u = K$, and, from (4·12),

$$\frac{\partial^2\phi}{\partial Z^2} = -\frac{2A}{ab^2}E(K) = \frac{p}{8\mu'}. \tag{4·13}$$

Hence the constant A is given by

$$A = -\frac{ab^2p}{16\mu' E(K)}. \tag{4·14}$$

Also, when $\lambda = 0$,

$$w = -8(1-\eta)\frac{\partial\phi}{\partial Z} = -\frac{16(1-\eta)A\mu^{\frac{1}{2}}\nu^{\frac{1}{2}}}{a^2b^2} = -\frac{16(1-\eta)A}{ab}\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}^{\frac{1}{2}}, \tag{4·15}$$

which is the value of the normal displacement over all points of the elliptical crack.

The solution of the problem has now been taken to a point at which the displacements and stresses at any point of the medium may be found by straightforward calculations.

5. *The elliptical punch problem.* Consider a semi-infinite medium $Z \geqslant 0$ indented by a flat-ended elliptical cylindrical punch. It is assumed that there is zero shear stress on the plane boundary $Z = 0$ so that the general formulae (3·4) will still be applicable. The remaining conditions to be satisfied on the plane boundary are

$$\left.\begin{aligned}&\text{(i)}\ \ w = -8(1-\eta)\frac{\partial\phi}{\partial Z} = \epsilon \quad (\lambda = 0),\\ &\text{(ii)}\ \ \frac{\partial^2\phi}{\partial Z^2} = 0 \qquad\qquad\quad (\mu = 0),\end{aligned}\right\} \tag{5·1}$$

where ϵ is a small constant. Condition (ii) ensures that there is zero normal stress on the boundary outside the punch.

Since $\partial\phi/\partial Z$ is harmonic it is seen that $\partial\phi/\partial Z$ represents the velocity potential of the perfect fluid flow through an elliptical aperture in a thin rigid boundary. The solution of this problem (2) is

$$\frac{\partial\phi}{\partial Z} = -\frac{\epsilon a}{16(1-\eta)K}\int_\lambda^\infty \frac{ds}{\sqrt{\{Q(s)\}}} = -\frac{\epsilon}{8(1-\eta)}\frac{u}{K}, \tag{5·2}$$

where u is given by (4·8). It can be verified that this satisfies the conditions (5·1). In order to complete the solution of the problem it is necessary to find a harmonic function ϕ such that $\partial\phi/\partial Z$ is given by (5·2). So far this function has not been found. It is, however, possible to obtain all the components of displacement and stress in (3·4) except D and Φ. In particular, the value of $\widehat{zz}$ on the boundary $Z = 0$ under the elliptical punch is

$$\widehat{zz} = -8\mu'\frac{\partial^2\phi}{\partial Z^2} = -\frac{\mu'\epsilon}{(1-\eta)bK}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-\frac{1}{2}} \quad (\lambda = 0). \tag{5·3}$$

On integrating this over the ellipse the total pressure P exerted by the punch is found to be

$$P = -\iint \widehat{zz}\,dx\,dy = \frac{2\pi\mu' a\epsilon}{(1-\eta)K}. \tag{5·4}$$

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