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The distribution of stress in the neighbourhood of a crack in an elastic solid

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The distribution of stress produced in the interior of an elastic solid by the opening of an internal crack under the action of pressure applied to its surface is considered. The analysis is given for 'Griffith' cracks (§ 2) and for circular cracks (§ 3), it being assumed in the latter case that the applied pressure varies over the surface of the crack. For both types of crack the case in which the pressure is constant over the entire crack surface is considered in some detail, the stress components being tabulated and the distribution of stress shown graphically. The effect of a crack (of either type) on the stress produced in an elastic body by a uniform tensile stress is considered and the conditions for rupture deduced.

1. Introduction

1·1. A theory of rupture of non-ductile materials such as glass has been put forward by A. A. Griffith (1920, 1924) on the basis of the existence of a large number of cracks in the interior of the solid body. The fundamental concept of the Griffith theory is that the bounding surfaces of a solid possess a surface tension, just as those of a liquid do, and that when a crack spreads the decrease in the strain energy is balanced by an increase in the potential energy due to this surface tension. The calculation of the effect of the presence of a crack on the energy of an elastic body is based on Inglis's solution (Inglis 1913*a*, *b*) of the two-dimensional equations of elastic equilibrium in the space bounded by two concentric ellipses, the crack being then taken to be an ellipse of zero eccentricity. Denoting the surface tension of the material of the solid body by T, the width of the crack by $2c$, and the Young's modulus of the material of the body by E, Griffith showed that, in the case of plane stress, the crack will spread when the stress P, applied normally to the direction of the crack, exceeds the critical value

$$P_c = \sqrt{\frac{2ET}{\pi c}}. \tag{1·1·1}$$

The use of the equations of elastic equilibrium restricts the analysis to materials which obey Hooke's law fairly closely. This will not be true in general, as the high concentration of stress at the edges of the crack will induce a certain amount of plastic flow. Because of the smallness of the region to which the plastic flow is confined the distribution of stress at points remote from the edges of the crack will not be affected and the strain energy will not differ appreciably from that derived from the solution of the elastic equations. A formula differing from (1·1·1) by a factor 0·8 has been derived by Orowan (1934) from rather similar assumptions. In the case of plane strain the formula (1·1·1) is replaced by*

$$P_c = \sqrt{\frac{2ET}{\pi c(1-\sigma^2)}}. \tag{1·1·2}$$

1·2. Griffith's theory of rupture has recently been extended to three dimensions by Sack (1946). Observing that the length of internal cracks does not greatly exceed their width, Sack calculates the conditions of rupture for a solid containing a plane crack bounded by a circle—a 'penny-shaped' crack—when one of the principal stresses is acting normally to the plane of the crack. By treating the crack as an oblate spheroid whose elliptic section has zero eccentricity Sack establishes that rupture will occur when the tensile stress P normal to the crack exceeds the critical value

$$P_c = \sqrt{\frac{\pi ET}{2c(1-\sigma^2)}}, \tag{1·2·1}$$

where c is the radius of the crack, and, as before, T denotes the surface tension of the material of the body, and E its Young's modulus.

The three-dimensional model introduced by Sack thus gives a critical tensile stress differing from the Griffith value (1·1·1) by a factor $\pi/\{2(1-\sigma^2)^{\frac{1}{2}}\}$, and from (1·1·2) by a factor $\frac{1}{2}\pi$.

1·3. In the present paper main consideration will be given to the distribution of stress in the neighbourhood of a circular crack of the type considered by Sack. As is usual it is assumed that the crack is of such dimensions that its depth exceeds the radius of molecular action at all points not very near its edges. The mathematical theory of elasticity then gives the components of stress accurately at all points of the elastic body other than those in the immediate vicinity of the edges of the crack. If the crack is sufficiently large, the error incurred in the calculation of the strain energy is then negligible, for the stress near the edge of the crack is proportional to $r^{-\frac{1}{2}}$, so that the strain energy in any small sphere whose centre coincides with the edge is finite and negligible in comparison with the strain energy in the rest of the solid. The elastic body is assumed to consist of homogeneous isotropic material and to be so large that its dimensions are very much greater than the radius of the crack; the crack may then be considered to be situated in the interior of an infinite elastic medium. In the first instance the elastic body is considered to be deformed by an internal pressure acting across the surfaces of the crack, the effect of tensile stresses applied to a body with an internal crack free from stress being deduced later.

* This is given incorrectly in Griffith (1924) as

$$P_c = \sqrt{\left[\frac{2ET}{\pi c}(1-\sigma^2)\right]}.$$

To afford a comparison with the three-dimensional case the distribution of stress in the neighbourhood of a crack in two dimensions (plane strain) is first considered (§ 2); this corresponds to the two-dimensional Griffith model. The elastic body is supposed to be deformed by the opening of the crack under the action of a uniform hydrostatic pressure acting over its surface. The calculations are based on a solution of the elastic equations given by Westergaard (1939); analytical expressions for the components of stress and for the principal shearing stress are derived from Westergaard's stress function. The advantage of the analysis given here is not only that it is simpler than Inglis's but that Cartesian co-ordinates are employed throughout, thus facilitating the interpretation of the results. To give some idea of the distribution of stress in the body the intensity of the principal shearing stress is computed at a network of points in the two-dimensional co-ordinate plane and the system of 'isochromatic' lines (contours of equal principal shearing stress) plotted in the neighbourhood of the crack. The solution corresponding to the opening of an internal crack under the influence of a tensile stress applied to the surface of the body is then indicated and the Griffith criterion (1·1·2) derived.

In the case of a circular crack (§ 3) it is first of all assumed that the internal pressure is a function of the distance from the centre of the crack. Sack's calculations are based on Neuber's solution (Neuber 1934) of the equations of elastic equilibrium in oblate spheroidal co-ordinates. This method could no doubt be adapted to the case of a variable internal pressure but would probably be rather laborious. Even in the case of constant applied pressure considered by Sack the expressions for the components of stress do not lend themselves readily to computation; and, as in the case of Inglis's analysis, the choice of co-ordinate system makes the interpretation of the results somewhat difficult. In the analysis given below cylindrical polar co-ordinates are used. The solution of the equations of elastic equilibrium in these co-ordinates by the method of Hankel transforms, developed recently by Harding & Sneddon (1945), is employed to reduce the problem to that of the solution of a pair of dual integral equations. A relation giving the shape of the crack in terms of the pressure applied to its surface is derived from the known solution of this pair of equations. The converse problem, that of determining the distribution of internal pressure necessary to preserve a given shape of crack, is also considered.

The distribution of stress in the case in which the applied internal pressure remains constant over the entire surface of the crack is considered in some detail. It is first shown that in this instance the crack, assumed to be originally an infinitely thin crevice, has, after the application of the pressure, the shape of a very flat ellipsoid of revolution. The components of stress are calculated at various points in the interior of the elastic body, and the results given in a set of tables which enable the stress components at any point to be obtained by interpolation; the variation of the various stress components in certain planes parallel to the crack is also illustrated graphically. The principal shearing stress is tabulated in a similar fashion and the contours of equal principal shearing stress plotted to show the distribution of stress in the neighbourhood of the crack.

Finally, the effect of a circular crack, free from stress, on the distribution of stress in an elastic body subjected to a uniform tensile stress is considered, and Sack's criterion (1·2·1) derived on energy considerations.

1·4. Since an exact analysis of the stresses near the base of a crack using a model of the kind introduced by Griffith and Sack would be a necessary preliminary to the consideration of such theoretical problems as the determination of the velocity of propagation of cracks, it is perhaps of some interest to quote here expressions for the components of stress in the vicinity of the edge of the crack. In the case of a two-dimensional crack, if the origin of co-ordinates is chosen to be at the edge of the crack and take axes of co-ordinates (ξ, η) along the axis of the crack and perpendicular to that direction, then the shape of the crack is given approximately by the formula

$$\eta^2 = -b\xi. \tag{1·4·1}$$

Taking polar co-ordinates defined by the relations

$$\xi = \delta \cos \psi, \quad \eta = \delta \sin \psi,$$

then in the immediate vicinity of the crack the components of stress are given by the expressions

$$\left.\begin{aligned}\sigma_x &= p_0\left(\frac{c}{2\delta}\right)^{\frac{1}{2}} (\tfrac{3}{4}\cos\tfrac{1}{2}\psi + \tfrac{1}{4}\cos\tfrac{5}{2}\psi), && (1·4·1)\\ \sigma_y &= p_0\left(\frac{c}{2\delta}\right)^{\frac{1}{2}} (\tfrac{5}{4}\cos\tfrac{1}{2}\psi - \tfrac{1}{4}\cos\tfrac{5}{2}\psi), && (1·4·2)\\ \tau_{xy} &= p_0\left(\frac{c}{8\delta}\right)^{\frac{1}{2}} \sin\psi\cos\tfrac{3}{2}\psi. && (1·4·3)\end{aligned}\right\}$$

It follows immediately from these formulae that the principal shearing stress is given by the equation

$$\tau = p_0\left(\frac{c}{8\delta}\right)^{\frac{1}{2}} \sin\psi. \tag{1·4·4}$$

The most striking feature of the analysis in the three-dimensional case is that the expressions for the components of stress in the neighbourhood of the crack differ from those of the two-dimensional case by a numerical factor only. The stresses are derived from (1·4·1–3) by replacing $\sigma_x, \sigma_y, \tau_{xy}$ on the left-hand sides by $\frac{1}{2}(\pi\sigma_r)$, $\frac{1}{2}(\pi\sigma_2)$, $\frac{1}{2}(\pi\tau_{2r})$, where δ and ψ have a similar interpretation in three dimensions. The hoop stress σ_θ has no analogue in two dimensions; it is given by the relation

$$\sigma_\theta = \frac{4\sigma p_0}{\pi}\left(\frac{c}{2\delta}\right)^{\frac{1}{2}} \cos\tfrac{1}{2}\psi. \tag{1·4·5}$$

2. The two-dimensional model

2·1. It has been shown by Westergaard (1939)* that the stress function

$$Z = p_0\left(\frac{z}{\sqrt{(z^2-c^2)}}-1\right), \quad (z = x+iy), \tag{2·1·1}$$

describes the distribution of stress in the interior of an infinite 'two-dimensional' elastic medium produced by the opening of an internal crack from $z=-c$ to $z=c$ (c being real) under the action of a uniform liquid pressure p_0 in the crack as the only load. If the value of the normal component of the displacement, u_y, be denoted by w when $y=0$, then it can readily be shown that for $|x|<c$,

$$\left(\frac{x}{c}\right)^2+\left(\frac{w}{\epsilon}\right)^2 = 1, \tag{2·1·2}$$

where the value of the semi-minor axis ϵ is given by the relation

$$\epsilon = 2(1-\sigma^2)\,p_0c/E. \tag{2·1·3}$$

Poisson ratio

Equation (2·1·2) shows that as a result of the elastic strain the shape of the internal crack is elliptic. Furthermore, equation (2·1·1) shows that w is zero for $|x|>c$.

If one adopts the notation

$$z = re^{i\theta}, \quad z-c = r_1e^{i\theta_1}, \quad z+c = r_2e^{i\theta_2}, \tag{2·1·4}$$

then the components of stress are given by the equations

$$\left.\begin{aligned}\tfrac{1}{2}(\sigma_x+\sigma_y) &= \quad ReZ \quad = p_0\left\{\frac{r}{r_1^{\frac{1}{2}}r_2^{\frac{1}{2}}}\cos\left(\theta-\tfrac{1}{2}\theta_1-\tfrac{1}{2}\theta_2\right)-1\right\},\\ \tfrac{1}{2}(\sigma_y-\sigma_x) &= \quad yImZ' = p_0\frac{r\sin\theta}{c}\left(\frac{c^2}{r_1r_2}\right)^{\frac{3}{2}}\sin\tfrac{3}{2}(\theta_1+\theta_2),\\ \tau_{xy} &= -yReZ' = p_0\frac{r\sin\theta}{c}\left(\frac{c^2}{r_1r_2}\right)^{\frac{3}{2}}\cos\tfrac{3}{2}(\theta_1+\theta_2).\end{aligned}\right\} \tag{2·1·5}$$

By means of these expressions the components of stress at any point in the medium can be calculated; the maximum shearing stress across any plane through the point (x, y) can then be obtained from the relation

$$\tau^2 = (\tfrac{1}{2}\sigma_y-\tfrac{1}{2}\sigma_x)^2+\tau_{xy}^2. \tag{2·1·6}$$

Substituting from (2·1·5) into (2·1·6) then, for the maximum shearing stress,

$$\tau = p_0\frac{r\sin\theta}{c}\left(\frac{c^2}{r_1r_2}\right)^{\frac{3}{2}}. \tag{2·1·7}$$

* *Note added in proof.* The solution of the elastic equations due to Westergaard, and used in §2 of this paper, corresponds to a uniform distribution of pressure along the surfaces of the crack. By using a method similar to that of §3, but involving Fourier cosine transforms instead of Hankel transforms, it is possible to extend the analysis to cases in which the pressure along the crack is variable. This has been done and will be published shortly in *The Quarterly of Applied Mathematics* by Sneddon, I. N. & Elliott, H. A. under the title: 'The opening of a Griffith crack under internal pressure.'

It is easily verified from the equations (2·1·5) that when $y = 0$ and $-c < x < c$

$$\sigma_x = \sigma_y = -p_0, \quad \tau_{xy} = 0,$$

and that when $|x| > c$

$$\sigma_x = \sigma_y = p_0\left(\frac{x}{\sqrt{(c^2-x^2)}} - 1\right), \quad \tau_{xy} = 0.$$

Thus if $x-c$ is small and positive the normal component of stress $[\sigma_y]_{y=0}$ is given approximately by the expression

$$[\sigma_y]_{y=0} \doteqdot p_0\sqrt{\frac{c}{2(x-c)}}. \tag{2·1·8}$$

To determine the distribution of stress in the immediate vicinity of the crack r_1 is taken to be a small quantity, δ say; to conform with the notation of § 1·4 take θ_1 to be ψ, but it should be noted that there are no restrictions on the magnitude of ψ. Since δ/c is assumed small then, from (2·1·4),

$$\left.\begin{aligned} r &= c+\delta\cos\psi, & r_2 &= 2c+\delta\cos\psi, \\ \theta &= \delta\sin\psi/c, & \theta_2 &= \delta\sin\psi/2c, \end{aligned}\right\} \tag{2·1·9}$$

to the first order of small quantities. Substituting from (2·1·9) into equations (2·1·5) then the expressions

$$\tfrac{1}{2}(\sigma_x+\sigma_y) = p_0\left[\left(\frac{c}{2\delta}\right)^{\frac{1}{2}}\cos\tfrac{1}{2}\psi\left\{1-\frac{3\delta}{4c}+O(\delta^2/c^2)\right\}-1\right], \tag{2·1·10}$$

$$\tfrac{1}{2}(\sigma_y-\sigma_x) = p_0\left(\frac{c}{8\delta}\right)^{\frac{1}{2}}\sin\psi\left\{\sin\tfrac{3}{2}\psi-\frac{3\delta}{4c}\sin\tfrac{1}{2}\psi+O(\delta^2/c^2)\right\}, \tag{2·1·11}$$

$$\tau_{xy} = p_0\left(\frac{c}{8\delta}\right)^{\frac{1}{2}}\sin\psi\left\{\cos\tfrac{3}{2}\psi-\frac{3\delta}{4c}\cos\tfrac{1}{2}\psi+O(\delta^2/c^2)\right\}, \tag{2·1·12}$$

are obtained for the determination of the components of stress near the edge of the crack. Similarly, one obtains from equation (2·1·7)—or directly from equations (2·1·11) and (2·1·12)—the expression

$$\tau = p_0\left(\frac{c}{8\delta}\right)^{\frac{1}{2}}\sin\psi\left\{1-\frac{3\delta}{4c}\cos\psi+O(\delta^2/c^2)\right\} \tag{2·1·13}$$

for the principal shearing stress. If terms in δ/c of order greater than $-\frac{1}{2}$ are neglected in these expressions then one obtains the approximate formulae (1·4·1–3). Again, putting $x = c+\xi$, $w = \eta$, $b = 2\epsilon^2/c$ in equation (2·1·2) then the approximate relation

$$\eta^2 = -b\xi \tag{2·1·14}$$

is obtained on neglecting terms of the second order in ξ.

2·2. To give some idea of the distribution of stress in the neighbourhood of the crack, the maximum shearing stress τ was calculated for several values of x and y by means of equation (2·1·7). The results are embodied in table 1 and the variation of τ with x and y shown graphically in figure 1. In the case of plane strain (which is the case considered here) the condition of constant energy of distortion states (Nadai 1931, p. 184) that plastic flow is initiated when τ reaches the value k, where the constant k is related to s_0, the yield point in tension of the material, by the equation

$$3k^2 = s_0^2. \tag{2·2·1}$$

The maximum shear theory gives precisely the same condition except that now

$$2k = s_0. \tag{2·2·2}$$

Thus in both cases the inception of plastic flow can be predicted from the behaviour of the function τ defined by equation (2·1·7).

TABLE 1. VARIATION OF τ/p_0 WITH x AND y IN THE TWO-DIMENSIONAL PROBLEM

y \ x	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0	3·0
0·2	0·189	0·199	0·236	0·327	0·546	0·785	0·405	0·179	0·094	0·058	0·037	0·009
0·4	0·320	0·332	0·372	0·444	0·534	0·543	0·400	0·248	0·153	0·099	0·068	0·017
0·6	0·378	0·386	0·408	0·439	0·456	0·428	0·346	0·252	0·176	0·124	0·089	0·024
0·8	0·381	0·384	0·390	0·395	0·386	0·354	0·298	0·235	0·178	0·134	0·101	0·031
1·0	0·354	0·354	0·352	0·345	0·329	0·299	0·262	0·210	0·171	0·135	0·106	0·036
1·2	0·315	0·313	0·308	0·298	0·281	0·256	0·225	0·192	0·159	0·130	0·106	0·039
1·4	0·275	0·273	0·267	0·257	0·242	0·222	0·198	0·172	0·147	0·124	0·103	0·042
1·6	0·238	0·236	0·231	0·222	0·209	0·193	0·174	0·154	0·134	0·115	0·099	0·044
1·8	0·206	0·205	0·200	0·192	0·182	0·169	0·154	0·138	0·123	0·107	0·093	0·044
2·0	0·179	0·178	0·174	0·168	0·159	0·149	0·137	0·124	0·112	0·099	0·087	0·044
3·0	0·095	0·094	0·093	0·091	0·088	0·084	0·080	0·076	0·071	0·066	0·061	0·039

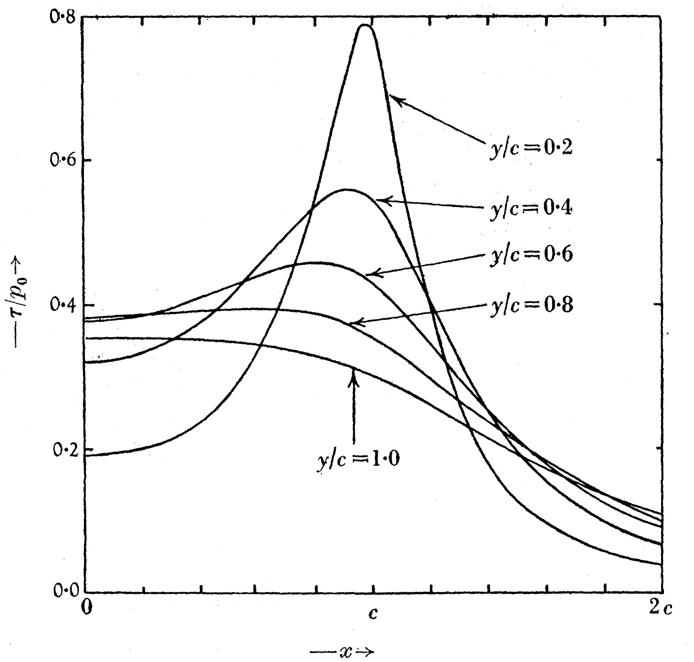

FIGURE 1. The variation of the principal shearing stress, τ, with x and y.

A convenient method of showing the variation of this function and of visualizing the distribution of stress in the interior of the medium consists of plotting the contours of equal principal shearing stress, i.e. constructing the family of curves $\tau = \alpha p_0$, or

$$c^2 y^4 = \alpha\{(x-c)^2+y^2\}^{\frac{3}{2}}\{(x+c)^2+y^2\}^{\frac{3}{2}}, \tag{2·2·3}$$

where α is a parameter. These curves are the 'isochromatic lines' of photoelasticity. The family of curves was constructed by determining from table 1 and figure 1 the points in the vicinity of the crack at which the parameter α has the values 0·7, 0·5, 0·4, 0·3 and 0·2. The result is shown in figure 2. The shape of the isochromatic lines in the neighbourhood of the edge of the crack can readily be deduced from equation (2·1·13); in polar co-ordinates δ and ψ with origin at $x = c$ the equation of an isochromatic curve reduces to the simple form

$$\delta = \beta \sin^2 \psi, \tag{2·2·4}$$

where $\beta = c/8\alpha^2$. Thus near the edge of the crack the isochromatics are curves of two loops situated symmetrically with regard to the axis of the crack.

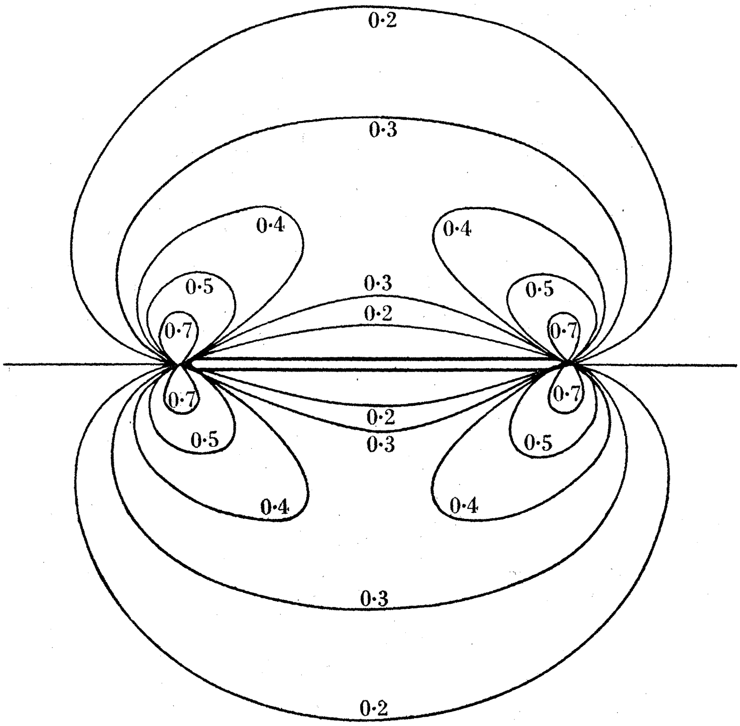

FIGURE 2. The isochromatic lines in the vicinity of a Griffith crack.

The fact that all the isochromatic lines pass through the points $(\pm c, 0)$ shows that at both of these points the principal shearing stress is infinite; thus even for small internal pressures p_0 plastic flow occurs at the corners of the crack to remove this infinite stress. There is, in fact, no purely *elastic* solution of the problem; if, however, the internal pressure is not too large the region of plastic flow will be small and will not appreciably affect the distribution of stress at points in the solid at a distance from the corners of the crack.

Although there seems to be no photoelectric determinations of the stress distribution in the vicinity of a crack there is a certain degree of similarity between the theoretical isochromatic lines of figure 2 and those in the vicinity of a tunnel as determined experimentally by Farquharson (Frocht 1941, p. 347). The shape of the tunnel differs appreciably from that of a crack of small transverse thickness, but the resemblance of the shape of the isochromatic lines in the neighbourhood of the lower straight edge of the tunnel to that of the lines of figure 2 is striking.

2·3. Now consider the energy of the crack. When the semi-minor axis of the crack is ϵ the internal pressure has the value

$$p = \frac{E}{2(1-\sigma^2)}\left(\frac{\epsilon}{c}\right).$$

Considering an element of length of the crack it is found that the work done in increasing the semi-minor axis to $\epsilon + d\epsilon$ is

$$2p\,dx\,dw = \frac{E}{1-\sigma^2}\left(\frac{\epsilon+\frac{1}{2}d\epsilon}{c}\right)dx\sqrt{\left(1-\frac{x^2}{c^2}\right)}\,d\epsilon,$$

so that the energy of the crack is

$$W = \frac{2E}{(1-\sigma^2)\,c}\int_0^c\sqrt{\left(1-\frac{x^2}{c^2}\right)}\,dx\int_0^\epsilon \epsilon\,d\epsilon$$

$$= \frac{\pi E\epsilon^2}{4(1-\sigma^2)} = \frac{\pi(1-\sigma^2)\,p_0^2c^2}{E}, \qquad (2\cdot3\cdot1)$$

where p_0 refers to the final pressure.

The theory of the Griffith crack, considers the distribution of stress in the vicinity of a crack when the crack is opened under the influence of an average tension p_0 applied to the surface of the body; the surface of the crack is assumed to be free from stress. The solution of this problem is obtained by superposing on the solution (2·1·1) a second solution

$$Z = +p_0. \qquad (2\cdot3\cdot2)$$

This ensures that $\sigma_x = \sigma_y = p_0$ for large values of x and y, and that $\sigma_x = \tau_{xy} = 0$ across the surface of the crack. Equation (2·3·1) then shows that the presence of a crack of length $2c$ lowers the potential energy of the solid by an amount W. On the other hand, the crack has a surface energy

$$U = 4cT, \qquad (2\cdot3\cdot3)$$

where T is the surface tension of the material. Thus the total diminution of the potential energy due to the presence of the crack is

$$W - U = \pi(1-\sigma^2)\,p_0^2c^2/E - 4cT.$$

The condition
$$\frac{\partial}{\partial c}(W-U) = 0$$

that the crack may extend then leads to the equation

$$c = \frac{2ET}{\pi p_0^2(1-\sigma^2)}.$$

If c is less than this value or if p_0 exceeds the critical value p_c given by the relation

$$p_c = \sqrt{\left\{\frac{2ET}{\pi c(1-\sigma^2)}\right\}}, \tag{2·3·4}$$

the crack will become unstable and spread. Equation (2·3·4) is the (corrected) Griffith criterion for rupture in the case of plane strain, equation (1·1·2) above.

3. The three-dimensional model

3·1. In the three-dimensional case assume that a crack is created in the interior of an infinite elastic medium and that it is 'penny-shaped', occupying the circle $r^2 = x^2+y^2 = c^2$ in the plane $z = 0$. An examination of the equations of § 2·1 shows immediately that in the case of the two-dimensional model the distribution of stress in the neighbourhood of a crack of length $2c$ lying along the axis $y = 0$, is identical to that in the semi-infinite elastic medium $y \geqslant 0$ when the conditions on the boundary $y = 0$ are prescribed by the relations

$$\left.\begin{array}{ll}\text{(i)} & \tau_{xy} = 0 \quad \text{for all values of } x, \\ \text{(ii)} & \sigma_y = -p_0 \;\; (|x| < c), \quad u_y = 0 \;\; (|x| > c).\end{array}\right\} \tag{3·1·1}$$

In the three-dimensional case assume that the distribution of stress in the neighbourhood of the crack is the same as that produced in a semi-infinite elastic medium when certain conditions are prescribed on its bounding plane.

For a crack of the shape described above there is symmetry about the z-axis, so cylindrical co-ordinates r, θ, z may be employed; in this co-ordinate system the displacement vector assumes the form $(u_r, 0, u_z)$, and the stress in the interior of the medium is specified completely by the stress components $\sigma_r, \sigma_z, \sigma_\theta, \tau_{rz}$, the remaining components being identically zero.

By analogy with equations (3·1·1) it may now be assumed that the distribution of stress in the neighbourhood of the crack is the same as that produced in the interior of the semi-infinite elastic solid $z \geqslant 0$ by the boundary conditions

$$\left.\begin{array}{ll}\text{(i)} & \tau_{rz} = 0 \quad \text{for all values of } r, \\ \text{(ii)} & \sigma_z = -p(r) \;\; (r < c), \quad u_z = 0 \;\; (r > c),\end{array}\right\} \tag{3·1·2}$$

on the plane $z = 0$. In the first instance assume that the applied internal pressure p is a function of r; the case of a constant internal pressure will be examined later.

The only restriction on the function $p(r)$ imposed by the subsequent analysis is that it is such that the integral

$$\int_0^c rp(r)\, J_0(\xi r)\, dr$$

exists for all real values of the parameter ξ.

The conditions (3·1·2) do not correspond to a hydrostatic pressure in the crack in all cases. In the analyses of Griffiths and Sack the crack is assumed to be of very small depth, so that the surface of the crack may be taken to be coincident with the z-axis; in this instance the pressure across the surface of the crack is $-\sigma_z$ and the conditions (3·1·2) are exact. Even in cases where the depth of the crack is taken into account the distribution of stress given by the conditions (3·1·2) will not differ much from that given by a hydrostatic pressure inside the crack—except possibly near the boundary of the crack. It will be seen later (§ 3·5) that in the case where the internal pressure is constant throughout the crack and the medium is incompressible the conditions (3·1·2) correspond exactly to a hydrostatic pressure.

With the axial symmetry the equations of elastic equilibrium reduce in the absence of body forces to (Love 1934, p. 90)

$$\frac{\partial \sigma_r}{\partial r}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r} = 0, \tag{3·1·3}$$

$$\frac{\partial \tau_{rz}}{\partial r}+\frac{\partial \sigma_z}{\partial z}+\frac{\tau_{rz}}{r} = 0. \tag{3·1·4}$$

It has been shown (Harding & Sneddon 1945) that these equations and the conditions of compatibility are satisfied by the forms

$$\sigma_r = \int_0^\infty \xi\{\lambda G''' + (\lambda+2\mu)\,\xi^2 G'\}\, J_0(\xi r)\, d\xi - \frac{2(\lambda+\mu)}{r}\int_0^\infty \xi^2 G' J_1(\xi r)\, d\xi, \tag{3·1·5}$$

$$\sigma_z = \int_0^\infty \xi\{(\lambda+2\mu)\, G''' - (3\lambda+4\mu)\,\xi^2 G'\}\, J_0(\xi r)\, d\xi, \tag{3·1·6}$$

$$\sigma_\theta = \lambda\int_0^\infty \xi\{G''' - \xi^2 G'\}\, J_0(\xi r)\, d\xi + \frac{2(\lambda+\mu)}{r}\int_0^\infty \xi^2 G' J_1(\xi r)\, d\xi, \tag{3·1·7}$$

$$\tau_{rz} = \int_0^\infty \xi^2\{\lambda G'' + (\lambda+2\mu)\,\xi^2 G\}\, J_1(\xi r)\, d\xi, \tag{3·1·8}$$

and as noted above

$$\tau_{z\theta} = \tau_{r\theta} = 0. \tag{3·1·9}$$

In these expressions λ and μ are Lamé's elastic constants and the function $G(\xi, z)$ is a solution of the ordinary differential equation

$$\left(\frac{d^2}{dz^2}-\xi^2\right)^2 G = 0. \tag{3·1·10}$$

The constants of integration introduced into the solution of this fourth-order equation are determined from the imposed boundary conditions. The solutions given above are such that all the components of stress tend to zero as $r \to \infty$; to ensure further that all these components tend to zero as $z \to \infty$ assume a solution of equation (3·1·10) of the form

$$G = (A + Bz)\, e^{-\xi z}, \tag{3·1·11}$$

where A and B are functions of the parameter ξ.

The corresponding expressions for the non-vanishing components of the displacement vector are

$$u_r = \frac{\lambda + \mu}{\mu} \int_0^\infty \xi^2 G' J_1(\xi r)\, d\xi, \tag{3·1·12}$$

$$u_z = \int_0^\infty \xi \left\{ G'' - \frac{\lambda + 2\mu}{\mu} \xi^2 G \right\} J_0(\xi r)\, d\xi. \tag{3·1·13}$$

To determine the values of the arbitrary functions A and B of equation (3·1·11) it is necessary to evaluate τ_{rz}, u_z, σ_z on the plane $z = 0$. By repeated differentiation of (3·1·11) and by use of the results

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \quad \mu = \frac{E}{2(1+\sigma)} \tag{3·1·14}$$

expressing the Lamé constants in terms of the Young's modulus E and the Poisson ratio σ, then, on substituting $z = 0$ into equation (3·1·8),

$$[\tau_{rz}]_{z=0} = \frac{E}{(1+\sigma)(1-2\sigma)} \int_0^\infty \xi^3 (\xi A - 2\sigma B)\, J_1(\xi r)\, d\xi.$$

Inverting this result by means of the Hankel inversion theorem, then

$$\xi^2 (\xi A - 2\sigma B) = \frac{(1+\sigma)(1-2\sigma)}{E} \int_0^\infty r [\tau_{rz}]_{z=0} J_1(\xi r)\, d\xi.$$

If the boundary conditions $[\tau_{rz}]_{z=0} = 0$ holds for all values of r it follows that

$$\xi A = 2\sigma B \tag{3·1·15}$$

between the two constants of integration A and B. Substituting from equation (3·1·15) into equation (3·1·11), differentiating with respect to z and then putting $z = 0$ it is found from equations (3·1·6) and (3·1·13) that

$$[\sigma_z]_{z=0} = \frac{E}{(1+\sigma)(1-2\sigma)} \int_0^\infty \xi^3 B J_0(\xi r)\, d\xi, \tag{3·1·16}$$

$$[u_z]_{z=0} = -\frac{2(1-\sigma)}{1-2\sigma} \int_0^\infty \xi^2 B J_0(\xi r)\, d\xi. \tag{3·1·17}$$

Writing (3·1·11) in the form

$$G = \frac{B(c\xi)}{\xi} (2\sigma + \xi z)\, e^{-\xi z} \tag{3·1·18}$$

and then putting $\eta = \xi c$ in equations (3·1·16) and (3·1·17), one obtains finally

$$[\sigma_z]_{z=0} = \frac{E}{c^4(1+\sigma)(1-2\sigma)} \int_0^\infty \eta f(\eta) J_0(\rho\eta)\, d\eta \tag{3·1·19}$$

and

$$[u_z]_{z=0} = -\frac{2(1-\sigma)}{c^3(1-2\sigma)} \int_0^\infty f(\eta) J_0(\rho\eta)\, d\eta, \tag{3·1·20}$$

where

$$f(\eta) = \eta^2 B(\eta) \tag{3·1·21}$$

and $\rho = r/c$.

3·2. Inserting the boundary conditions

$$z = 0,\ u_z = 0\ \ (r > c), \quad \sigma_z = -p(r)\ \ (r < c)$$

into equations (3·1·19) and (3·1·20), the pair of integral equations

$$\left.\begin{aligned} &\int_0^\infty \eta f(\eta) J_0(\rho\eta)\, d\eta = g(\rho) \quad (0 < \rho < 1), \\ &\int_0^\infty f(\eta) J_0(\rho\eta)\, d\eta = 0 \qquad (\rho > 1), \end{aligned}\right\} \tag{3·2·1}$$

is obtained for the determination of the unknown function $f(\eta)$ and hence of the constant $B(c\xi)$ of equation (3·1·18); in the former of these two equations is written

$$g(\rho) = -\frac{(1+\sigma)(1-2\sigma)}{E} c^4 p(\rho c). \tag{3·2·2}$$

Dual integral equations of the type (3·2·1) have been considered by Titchmarsh (1937, p. 334) and Busbridge (1938); the solution of the pair (3·2·1) can be derived easily from Busbridge's analysis in the form

$$f(\eta) = \frac{2}{\pi} \int_0^1 \mu \sin \mu\eta\, d\mu \int_0^1 \frac{\rho g(\rho\mu)}{\sqrt{(1-\rho^2)}}\, d\rho. \tag{3·2·3}$$

Substituting from (3·2·3) into (3·1·20) and making use of the result (Watson 1944, p. 405)

$$\int_0^\infty \sin \mu\eta J_0(\rho\eta)\, d\eta = \begin{cases} 0 & (\rho > \mu), \\ (\mu^2 - \rho^2)^{-\frac{1}{2}} & (\rho < \mu), \end{cases}$$

it is found that when the applied pressure is $p(r)$ the value of the normal component of the surface displacement is given by the equation

$$[u_z]_{z=0} = \frac{4(1-\sigma^2)c}{\pi E} \int_\rho^1 \frac{\mu\, d\mu}{\sqrt{(\mu^2-\rho^2)}} \int_0^1 \frac{x p(x\mu c)}{\sqrt{(1-x^2)}}\, dx \quad (\rho < 1). \tag{3·2·4}$$

If, for example, the applied pressure is given by the power series

$$p(r) = p_0 \sum_{n=0}^\infty \alpha_n \left(\frac{r}{c}\right)^n, \quad (0 < r < c), \tag{3·2·5}$$

then substituting into equation (3·2·3) one obtains

$$f(\eta) = -\frac{(1+\sigma)(1-2\sigma)}{\sqrt{\pi} E} c^4 p_0 \sum_{n=0}^\infty \frac{\Gamma(1+\frac{1}{2}n)}{\Gamma(\frac{3}{2}+\frac{1}{2}n)} \alpha_n \int_0^1 \mu^{n+1} \sin \mu\eta\, d\mu, \tag{3·2·6}$$

so that the normal component of the surface displacement now has the value

$$[u_z]_{z=0} = \frac{2}{\sqrt{\pi}}(1-\sigma^2)\frac{p_0}{E}\sum_{n=0}^{\infty}\frac{\Gamma(1+\frac{1}{2}n)}{\Gamma(\frac{3}{2}+\frac{1}{2}n)}\alpha_n I_n\left(\frac{r}{c}\right)^{n+1} \quad (0<r<c), \tag{3·2·7}$$

where I_n denotes the integral

$$I_n = c\int_1^{c/r}\frac{\eta^{n+1}}{\sqrt{(\eta^2-1)}}\,d\eta. \tag{3·2·8}$$

Integrating by parts it can readily be shown that I_n satisfies the recurrence relation

$$I_n = \frac{n}{n+1}I_{n-2}+\frac{1}{n+1}\left(\frac{c}{r}\right)^{n+1}(c^2-r^2)^{\frac{1}{2}}. \tag{3·2·9}$$

By means of these equations the value of the normal component of the surface displacement can be calculated when the law of variation of the applied pressure is known.

The value of the radial component of the surface displacement is found from equation (3·1·12) to be

$$[u_r]_{z=0} = \frac{1}{c^3}\int_0^{\infty}f(\eta)\,J_1(\rho\eta)\,d\eta, \tag{3·2·10}$$

whence by equations (3·2·2) and (3·2·3) and making use of the result (Watson 1944, p. 405) we obtain

$$\int_0^{\infty}J_1(\rho\eta)\sin\mu\eta\,d\eta = \begin{cases}\dfrac{\mu}{\rho}(\rho^2-\mu^2)^{-\frac{1}{2}} & (\mu<\rho),\\ 0 & (\mu>\rho),\end{cases} \tag{3·2·11}$$

then, in the general case,

$$[u_r]_{z=0} = -\frac{2(1+\sigma)(1-2\sigma)c}{\pi E}\int_0^1\mu^2\,d\mu\int_{\mu}^1\frac{p(\rho\mu c)\,d\rho}{\sqrt{\{(1-\rho^2)(\rho^2-\mu^2)\}}}. \tag{3·2·12}$$

It will be observed from this equation that the radial component of the surface displacement is zero when the medium is incompressible ($\sigma = \frac{1}{2}$).

3·3. If it be supposed that the applied pressure $p(r)$ is constant over a circular area of radius $a \leqslant c$, then

$$p(r) = \begin{cases}p_0 & (0<r<a),\\ 0 & (a<r<c).\end{cases} \tag{3·3·1}$$

Substituting into equation (3·2·4) one obtains the expression

$$w = \frac{4p_0(1-\sigma^2)}{\pi E}(c^2-r^2)^{\frac{1}{2}}\{1-(1-a^2/c^2)^{\frac{1}{2}}\} \tag{3·3·2}$$

for the normal component of the surface displacement, $w = [u_z]_{z=0}$. If $w = \epsilon$ when $r = 0$ then

$$\epsilon = \frac{4(1-\sigma^2)p_0c}{\pi E}\{1-(1-a^2/c^2)^{\frac{1}{2}}\}, \tag{3·3·3}$$

and equation (3·3·2) may be written in the form

$$\frac{r^2}{c^2}+\frac{w^2}{\epsilon^2} = 1, \tag{3·3·4}$$

showing that for all values of $a \leqslant c$, the crack resulting from the application of an internal pressure is ellipsoidal in shape provided the applied pressure is constant. For a point force applied at the centre of symmetry allow the radius a to tend to zero and the applied pressure p_0 to tend to infinity in such a way that the total force, $\pi a^2 p_0$, remains constant, P say; in this instance one obtains from equation (3·3·3)

$$\epsilon = \frac{2(1-\sigma^2)}{\pi^2}\frac{P}{cE}.$$

If the pressure p_0 acts over a circle of radius c, i.e. over the entire surface of the crack, and produces a displacement ϵ of the centre of the circle then

$$p_0 = \frac{\pi E\epsilon}{4(1-\sigma^2)\,c}, \tag{3·3·5}$$

from which it will be observed that p_0 is constant for constant values of the ratio ϵ/c. Now consider the work done in forming an ellipsoidal depression of circular section of radius c and of depth ϵ. When the depth of the depression is ϵ the pressure is given by equation (3·3·5); when the depth of the depression has increased to $\epsilon + d\epsilon$ the pressure will have increased to $\pi E(\epsilon + d\epsilon)/[4(1-\sigma^2)\,c]$, so that in bringing about a change $d\epsilon$ of the depth of the ellipsoidal depression the average pressure is

$$p_0 = \frac{\pi E(\epsilon + \frac{1}{2}d\epsilon)}{4(1-\sigma^2)\,c}.$$

Considering a ring element of surface intersecting the plane $\theta = 0$ at the points P, P' (see figure 3) it is found that the work done in making a small normal displacement dw is

$$p_0\, 2\pi r\, dr\, dw = \frac{\pi E(\epsilon + \frac{1}{2}d\epsilon)}{4(1-\sigma^2)\,c}\, 2\pi r\, dr\, d\epsilon (1 - r^2/c^2)^{\frac{1}{2}}$$

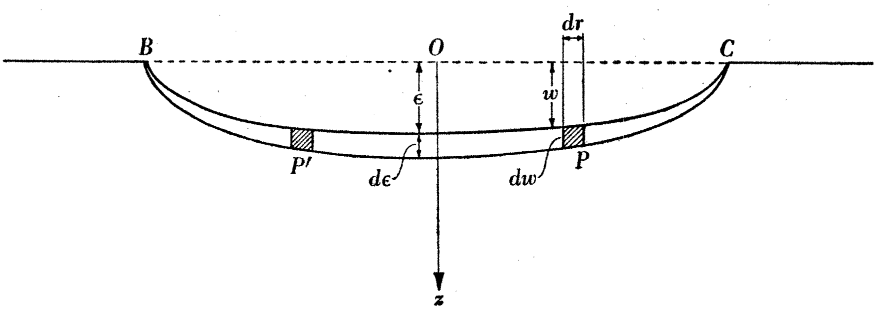

FIGURE 3

by equation (3·3·2). Thus the total work done in forming the ellipsoidal depression on the surface of a semi-infinite elastic solid is

$$W_1 = \frac{\pi^2 E}{2(1-\sigma^2)} \int_0^c r(1-r^2/c^2)^{\frac{1}{2}}\,dr \int_0^\epsilon \epsilon\,d\epsilon,$$

which gives

$$W_1 = \frac{\pi^2 E c \epsilon^2}{12(1-\sigma^2)}.$$

The energy of a crack formed by internal pressure is twice this energy; multiplying the last result by 2 and substituting for the value of ϵ in terms of the final pressure p_0 one obtains for the energy of a crack of radius c the formula

$$W = 2W_1 = \frac{8(1-\sigma^2)}{3E} p_0^2 c^3. \qquad (3\cdot3\cdot6)$$

3·4. It was observed above (equations (3·3·3) and (3·3·4)) that as long as the internal applied pressure is constant the crack will be ellipsoidal in shape whatever the radius of the circle over which it is applied. It is natural then to try to determine whether cracks of a shape other than ellipsoidal are possible, and if they are to determine the distribution of internal pressure necessary to preserve their shape. This is equivalent to assuming that the value of $[u_z]_{z=0}$ is known when $r \leqslant c$, but that the function $p(r)$ is unknown. One might then regard equation (3·2·4) as an integral equation determining $p(r)$ when $[u_z]_{z=0}$ is known. It is, however, simpler to consider the problem of determining the distribution of stress in a semi-infinite elastic medium bounded by the plane $z = 0$ when the surface value of the normal component of the displacement vector is prescribed for all values of r. The analysis proceeds along similar lines except that now the boundary conditions

$$\text{(i)} \quad \tau_{zr} = 0, \quad \text{for all values of } r,$$

$$\text{(ii)} \quad u_z = \begin{cases} w(r) & (r < c), \\ 0 & (r > c), \end{cases}$$

replace the set (3·1·2). Substituting the condition (ii) into equation (3·1·17) then

$$\int_0^\infty \xi^2 B J_0(\xi r)\,d\xi = -\begin{cases} -\dfrac{(1-2\sigma)}{2(1-\sigma)} w(r) & (r < c), \\ 0 & (r > c). \end{cases}$$

Inverting this result by means of the Hankel inversion theorem

$$\xi B = -\frac{(1-2\sigma)}{2(1-\sigma)} \int_0^c r w(r)\, J_0(\xi r)\,dr. \qquad (3\cdot4\cdot1)$$

Denoting the integral on the right-hand side by $\overline{w}(\xi)$ then, from equation (3·1·16), it is found that

$$[\sigma_z]_{z=0} = -\frac{E}{2(1-\sigma^2)} \int_0^\infty \xi^2 \overline{w}(\xi)\, J_0(\xi r)\,d\xi. \qquad (3\cdot4\cdot2)$$

Equation (3·4·2) gives the value of the applied pressure on the circle $r \leqslant c$ which must be applied to maintain the prescribed surface displacement.

For example, if it is assumed that

$$w(r) = \epsilon(1-r^2/c^2), \tag{3·4·3}$$

then from equation (3·4·1) it follows that

$$\overline{w}(\xi) = 2\epsilon J_2(c\xi)/\xi^2,$$

and so by equation (3·4·2) that the normal component of the surface stress is given by the expression

$$[\sigma_z]_{z=0} = -\frac{E\epsilon}{1-\sigma^2}\int_0^\infty J_2(c\xi)\, J_0(r\xi)\, d\xi.$$

Using the recurrence relation

$$J_2(c\xi) = J_0(c\xi) - \frac{2}{c}\frac{\partial}{\partial\xi} J_1(c\xi),$$

after an integration by parts it is found that

$$\int_0^\infty J_0(r\xi)\, J_2(c\xi)\, d\xi = \int_0^\infty J_0(r\xi)\, J_0(c\xi)\, d\xi - \frac{2r}{c}\int_0^\infty J_1(r\xi)\, J_1(c\xi)\, d\xi.$$

Now by Gübler's formula (Watson 1944, p. 410) the value of the first integral is found to be

$$\left(\frac{1}{c}\right) {}_2F_1\left[\tfrac{1}{2}, \tfrac{1}{2};\ 1;\ \frac{r^2}{c^2}\right] = \frac{2}{\pi c} K(r/c) \quad (r<c),$$

in the usual notation for elliptic integrals (Jahnke-Emde 1938, p. 85). The second integral could also be evaluated by the Gübler formula, but the hypergeometric series involved in this instance is not readily summed. The integration can be carried out more easily by making use of Neumann's result

$$J_1(\xi r)\, J_1(\xi c) = \frac{1}{\pi}\int_0^\pi J_0(\xi R)\cos\theta\, d\theta, \quad R^2 = c^2 + r^2 - 2cr\cos\theta.$$

Multiplying both sides of this equation by $\exp\{-\xi z\}$ and integrating with respect to ξ from 0 to ∞, then

$$\int_0^\infty e^{-\xi z} J_1(c\xi)\, J_1(r\xi)\, d\xi = \frac{1}{\pi}\int_0^\pi \cos\theta\, d\theta \int_0^\infty J_0(\xi R)\, e^{-\xi z}\, d\xi = \frac{1}{\pi}\int_0^\pi \frac{\cos\theta\, d\theta}{\sqrt{(R^2+z^2)}}.$$

Letting z tend to zero on both sides of the last equation one obtains the required result

$$\int_0^\infty J_1(\xi r)\, J_1(\xi c)\, d\xi = \frac{1}{\pi}\int_0^\pi \frac{\cos\theta\, d\theta}{R}.$$

The integral on the right-hand side can easily be evaluated in terms of elliptic integrals; with the usual notation for complete elliptic integrals, then

$$\frac{2r}{c}\int_0^\infty J_1(\xi r)\, J_1(\xi c)\, d\xi = \frac{2\{1+(r/c)^2\}}{\pi(c+r)} K(k) - \frac{2(c+r)}{\pi c^2} E(k),$$

where $$k^2 = 4rc/(r+c)^2.$$

Thus the displacement (3·4·3) is maintained by an applied internal pressure

$$p(r) = -[\sigma_z]_{z=0} = \frac{2E\epsilon}{\pi c(1-\sigma^2)} f(\rho), \tag{3·4·4}$$

where the function $f(\rho)$ is defined by the equation

$$f(\rho) = \frac{1+\rho^2}{1+\rho} K(k) - K(\rho) - (1+\rho)\, E(k) \tag{3·4·5}$$

and $\rho = r/c$, $k^2 = 4\rho/(1+\rho)^2$. The variation of the function $f(\rho)$ with ρ is shown graphically in figure 4. From this curve it follows that the applied internal force is *compressive* in the body of the crack, but that near the circular edge of the crack the applied force becomes *tensile* until round the edge it is both infinite and tensile. This is precisely the kind of variation which would be predicted on physical grounds for a crack of this shape.

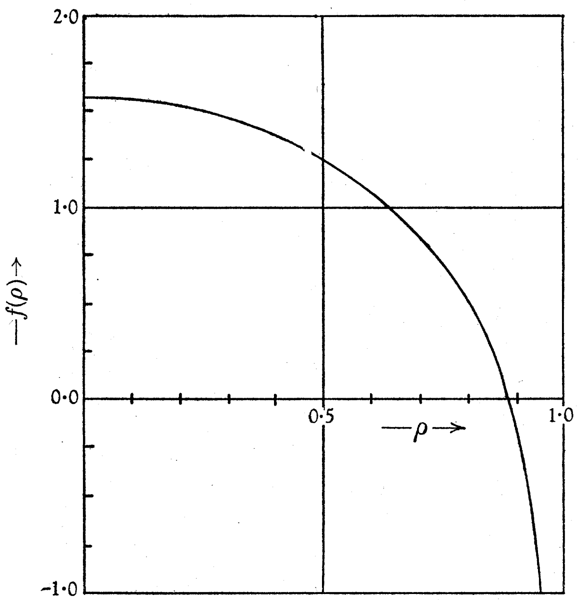

FIGURE 4. Variation of the function $f(\rho)$ with ρ.

3·5. Now return to the problem of determining the distribution of stress in the interior of the medium, in the case where the applied pressure $p(r)$ is a constant p_0 over the entire surface area $r \leqslant c$ of the crack. Putting $p(\rho c) = p_0$, then from equations (3·2·2) and (3·2·3),

$$f(\eta) = \frac{2p_0 c^4(1+\sigma)(1-2\sigma)}{\pi E} \frac{d}{d\eta}\left(\frac{\sin\eta}{\eta}\right). \tag{3·5·1}$$

Substituting this value for $f(\eta)$ into equation (3·1·19) it follows that, for the normal component of stress across the plane $z = 0$,

$$[\sigma_z]_{z=0} = \frac{2p_0}{\pi}\left[\rho\int_0^\infty \sin\eta J_1(\rho\eta)\, d\eta - \int_0^\infty \frac{\sin\eta}{\eta} J_0(\rho\eta)\, d\eta\right]. \tag{3·5·2}$$

By means of the results (Watson 1944, p. 405)

$$\int_0^\infty \frac{\sin\eta}{\eta} J_0(\rho\eta)\, d\eta = \begin{cases} \sin^{-1}(1/\rho) & (\rho \geqslant 1), \\ \tfrac{1}{2}\pi & (\rho \leqslant 1), \end{cases}$$

$$\int_0^\infty \rho \sin\eta J_1(\rho\eta)\, d\eta = \begin{cases} (\rho^2-1)^{-\frac{1}{2}} & (\rho > 1), \\ 0 & (\rho < 1), \end{cases}$$

it is verified that $\sigma_z = -p_0$ when $\rho < 1$, and for the value of the normal component of the surface stress when $\rho > 1$ the expression

$$[\sigma_z]_{z=0} = -\frac{2p_0}{\pi}\left[\sin^{-1}\frac{1}{\rho} - \frac{1}{\sqrt{(\rho^2-1)}}\right] \tag{3·5·3}$$

is obtained. It will be observed that when $\rho = 1$ the stress $[\sigma_z]_{z=0}$ is infinite and that when $\rho - 1$ is small and positive

$$[\sigma_z]_{z=0} \doteqdot \frac{p_0}{\pi}(\rho-1)^{-\frac{1}{2}}, \tag{3·5·4}$$

showing that there is a large *tensile* stress in the neighbourhood of the circle $\rho = 1$, $z = 0$. Furthermore, as $\rho \to \infty$,

$$[\sigma_z]_{z=0} \sim \frac{4p_0}{3\pi}\rho^{-3}, \tag{3·5·5}$$

the stress being still tensile. Since

$$\frac{d}{d\rho}[\sigma_z]_{z=0} = -\frac{2p_0}{\pi}\{\rho(\rho^2-1)^{\frac{3}{2}}\}^{-1}$$

does not change sign in the region $\rho > 1$, it follows that the normal component of the surface stress $[\sigma_z]_{z=0}$ never becomes compressive in that region. The similarity of the expression (3·5·4) to that for the corresponding stress in the two-dimensional analysis (equation (2·1·8)) is also of interest.

For the values of the other stress components on the plane $z = 0$ one obtains from equations (3·1·5)–(3·1·8) the expressions

$$[\sigma_r + \sigma_\theta]_{z=0} = (1+2\sigma)[\sigma_z]_{z=0},$$

$$[\sigma_r - \sigma_\theta]_{z=0} = \frac{2(1-\sigma)p_0}{\pi}\int_0^\infty \eta J_2(\rho\eta)\frac{d}{d\eta}\left(\frac{\sin\eta}{\eta}\right) d\eta$$

$$= \frac{2(1-2\sigma)p_0}{\pi}\left[\int_0^\infty J_2(\rho\eta)\cos\eta\, d\eta - \int_0^\infty J_2(\rho\eta)\frac{\sin\eta}{\eta}\, d\eta\right].$$

Evaluating the integrals in the square bracket it is found that the bracket vanishes when $\rho < 1$ and has the value

$$-(\rho^2-1)^{-\frac{1}{2}}$$

when $\rho > 1$. Thus if $\rho < 1$ and $z = 0$, then

$$\sigma_\theta = \sigma_r = -(\sigma + \tfrac{1}{2})p_0,$$

and, if $\rho > 1$, $z = 0$, $\sigma_r = \frac{2p_0}{\pi}\left[(\rho^2-1)^{-\frac{1}{2}} - (\sigma+\frac{1}{2})\sin^{-1}\frac{1}{\rho}\right]$,

$$\sigma_\theta = \frac{2p_0}{\pi}\left[2\sigma(\rho^2-1)^{-\frac{1}{2}} - (\sigma+\tfrac{1}{2})\sin^{-1}\frac{1}{\rho}\right].$$

It will be noted that in the case $\sigma = \frac{1}{2}$ (incompressible medium) $\sigma_r = \sigma_z = \sigma_\theta = -p_0$ $(\rho < 1)$, so that in this instance the internal pressure is truly hydrostatic.

3·6. The evaluation of the stress components in the interior of the medium is similar to that involved in the solution of the Boussinesq problem for a cylinder (Sneddon 1946). Obtaining the value of $B(c\xi)$ from equations (2·1·21) and (3·5·1) and substituting from equation (3·1·18) into equations (3·1·12) and (3·1·13), then, for the non-vanishing components of the displacement vector,

$$u_r = \frac{2p_0c}{\pi E}(1+\sigma)\int_0^\infty (1-2\sigma-\zeta\eta)\frac{d}{d\eta}\left(\frac{\sin\eta}{\eta}\right)e^{-\zeta\eta}J_1(\rho\eta)\,d\eta, \tag{3·6·1}$$

$$u_z = -\frac{4p_0c(1-\sigma^2)}{\pi E}\int_0^\infty\left(1+\frac{\zeta\eta}{2(1-\sigma)}\right)\frac{d}{d\eta}\left(\frac{\sin\eta}{\eta}\right)e^{-\zeta\eta}J_0(\rho\eta)\,d\eta, \tag{3·6·2}$$

where $\rho = r/c$ and $\zeta = z/c$. Similarly, from equations (3·1·5)–(3·1·8) one obtains for the components of stress at a general point in the interior of the elastic solid

$$\sigma_z = \frac{2p_0}{\pi}\{C_1^0(\rho,\zeta) - S_0^0(\rho,\zeta) + \zeta C_2^0(\rho,\zeta) - \zeta S_1^0(\rho,\zeta)\}, \tag{3·6·3}$$

$$\tau_{zr} = \frac{2p_0\zeta}{\pi}\{C_2^1(\rho,\zeta) - S_1^1(\rho,\zeta)\}, \tag{3·6·4}$$

$$\sigma_r + \sigma_\theta + \sigma_z = \frac{4(1+\sigma)}{\pi}p_0\{C_1^0(\rho,\zeta) - S_0^0(\rho,\zeta)\}, \tag{3·6·5}$$

where C_n^m and S_n^m denote the integrals

$$C_n^m(\rho,\zeta) = \int_0^\infty \eta^{n-1}e^{-\zeta\eta}J_m(\rho\eta)\cos\eta\,d\eta, \quad S_n^m(\rho,\zeta) = \int_0^\infty \eta^{n-1}e^{-\zeta\eta}J_m(\rho\eta)\sin\eta\,d\eta. \tag{3·6·6}$$

The fourth relation for the complete determination of all the components of stress is found from equations (3·1·5) and (3·1·7) to be

$$\sigma_\theta - \sigma_r = 2(\lambda+\mu)\int_0^\infty \xi^3 G'\left\{\frac{2J_1(\xi r)}{\xi r} - J_0(\xi r)\right\}d\xi.$$

Putting $n = 1$, $z = \xi r$ in the recurrence formula

$$J_{n-1}(z) + J_{n+1} = \frac{2n}{z}J_n(z), \tag{3·6·7}$$

then

$$\sigma_\theta - \sigma_r = 2(\lambda+\mu)\int_0^\infty \xi^3 G' J_2(\xi r)\,d\xi, \tag{3·6·8}$$

whence follows the relation

$$\sigma_\theta - \sigma_r = \frac{2p_0}{\pi}[(1-2\sigma)\{C_1^2(\rho,\zeta) - S_0^2(\rho,\zeta)\} - \zeta\{C_2^2(\rho,\zeta) - S_1^2(\rho,\zeta)\}]. \tag{3·6·9}$$

Along the axis of symmetry, $r = 0$, it is found that $\tau_{rz} = 0$, $\sigma_r - \sigma_\theta = 0$, so that the maximum shearing stress across any plane through a point on the axis of symmetry is given by the expression $|\tau|$, where

$$\tau = \tfrac{1}{2}(\sigma_z - \sigma_r) = -\frac{p_0}{2\pi}\left\{(1-2\sigma)\left(\int_0^\infty \cos\eta\, e^{-\zeta\eta}\,d\eta - \int_0^\infty \frac{\sin\eta}{\eta} e^{-\zeta\eta}\,d\eta\right)\right.$$
$$\left. + 3\zeta\left(\int_0^\infty \eta\cos\eta\, e^{-\zeta\eta}\,d\eta - \int_0^\infty \sin\eta\, e^{-\zeta\eta}\,d\eta\right)\right\}.$$

Evaluating the integrals

$$\tau = -\tfrac{1}{4}(1-2\sigma)\,p_0 + \frac{p_0}{2\pi}\left[(1-2\sigma)\left(\frac{\zeta}{1+\zeta^2} + \tan^{-1}\zeta\right) - \frac{6\zeta}{(1+\zeta^2)^2}\right]. \qquad (3\cdot6\cdot10)$$

Differentiating this expression with respect to ζ then

$$\frac{d\tau}{d\zeta} = \frac{4p_0}{\pi}\frac{(5-\sigma)\,\zeta^2 - (1+\sigma)}{(1+\zeta^2)^3},$$

showing that $|\tau|$ increases from the value $\frac{1}{4}(1-2\sigma)\,p_0$ at $\zeta = 0$ to a maximum at $\zeta = \{(1+\sigma)/(5-\sigma)\}^{\frac{1}{2}}$ and then decreases steadily to zero at $\zeta = \infty$.

Writing, in the general case, $w = \zeta + i$, then

$$Z_n^m = C_n^m - iS_n^m = \int_0^\infty \eta^{n-1} e^{-w\eta} J_m(\rho\eta)\,d\eta,$$

and the integrals Z_n^m can be evaluated by means of the formulae

$$\int_0^\infty \eta^{n-1} e^{-w\eta} J_m(\rho\eta)\,d\eta = \begin{cases} \dfrac{(n-m+1)!}{(w^2+\rho^2)^{\frac{1}{2}n}}\, P_{n-1}^m\left(\dfrac{w}{\sqrt{(w^2+\rho^2)}}\right) & (m \leqslant n-1), \\[2ex] \dfrac{(n+m-1)!}{(w^2+\rho^2)^{\frac{1}{2}n}}\, P_{n-1}^{-m}\left(\dfrac{w}{\sqrt{(w^2+\rho^2)}}\right) & (m > n-1), \end{cases}$$

where P_n^m denotes the associated Legendre function.

In this way one obtains the results

$$Z_1^0 = (\rho^2+w^2)^{-\frac{1}{2}}, \qquad Z_2^0 = w(\rho^2+w^2)^{-\frac{3}{2}},$$
$$Z_0^1 = \frac{1}{\rho}\{(\rho^2+w^2)^{\frac{1}{2}} - w\}, \quad Z_1^1 = \frac{1}{\rho} - \frac{W}{\rho\sqrt{(w^2+\rho^2)}},$$
$$Z_2^1 = \rho(\rho^2+w^2)^{-\frac{3}{2}},$$

and by virtue of equation (3·6·7) the further relation

$$Z_n^2 = \frac{2}{\rho} Z_{n-1}^1 - Z_n^0. \qquad (3\cdot6\cdot11)$$

Adopting the notation

$$\left.\begin{aligned} r^2 &= 1+\zeta^2, & \zeta\tan\theta &= 1, \\ R^2 &= (\rho^2+\zeta^2-1)^2 + 4\zeta^2, & 2\zeta\cot\phi &= \rho^2+\zeta^2-1, \end{aligned}\right\} \qquad (3\cdot6\cdot12)$$

and equating real and imaginary parts one obtains the formulae

$$\left.\begin{aligned}
C_0^1 &= R^{-\frac{1}{2}}\cos\tfrac{1}{2}\phi, & S_1^0 &= R^{-\frac{1}{2}}\sin\tfrac{1}{2}\phi,\\
C_2^0 &= rR^{-\frac{3}{2}}\cos(\tfrac{3}{2}\phi-\theta), & S_2^0 &= rR^{-\frac{3}{2}}\sin(\tfrac{3}{2}\phi-\theta),\\
C_0^1 &= \frac{1}{\rho}(R^{\frac{1}{2}}\cos\tfrac{1}{2}\phi-\zeta), & S_0^1 &= \frac{1}{\rho}(1-R^{\frac{1}{2}}\sin\tfrac{1}{2}\phi),\\
C_1^1 &= \frac{1}{\rho}-\frac{r}{\rho}R^{-\frac{1}{2}}\cos(\theta-\tfrac{1}{2}\phi), & S_1^1 &= \frac{r}{\rho}R^{-\frac{1}{2}}\sin(\theta-\tfrac{1}{2}\phi),\\
C_2^1 &= \rho R^{-\frac{3}{2}}\cos\tfrac{3}{2}\phi, & S_2^1 &= \rho R^{-\frac{3}{2}}\sin\tfrac{3}{2}\phi,
\end{aligned}\right\} \tag{3·6·13}$$

and from (3·6·12)

$$\left.\begin{aligned}
C_1^2 &= \frac{2}{\rho}C_0^1-C_1^0, \quad S_1^2 = \frac{2}{\rho}S_0^1-S_1^0,\\
C_2^2 &= \frac{2}{\rho}C_1^1-C_2^0.
\end{aligned}\right\} \tag{3·6·14}$$

It only remains to evaluate S_0^0 and S_0^2.

Integrating the expressions for Z_1^0, Z_0^1 with respect to w, then

$$\int_0^\infty \frac{1-e^{-wx}}{x}J_0(\rho x)\,dx = \log\frac{\sqrt{(\rho^2+w^2)}+w}{\rho}, \tag{3·6·15}$$

$$\int_0^\infty \frac{1-e^{-wx}}{x^2}J_1(\rho x)\,dx = \frac{1}{2\rho}\left[w\sqrt{(w^2+\rho^2)}-w^2+\rho^2\log\frac{\sqrt{(\rho^2+w^2)}+w}{\rho}\right], \tag{3·6·16}$$

it being assumed in both cases that $\rho \neq 0$. With the notation of (3·6·12) then from (3·6·16) by equating imaginary parts

$$S_0^0 = \tan^{-1}\frac{R^{\frac{1}{2}}\sin\tfrac{1}{2}\phi+r\sin\theta}{R^{\frac{1}{2}}\cos\tfrac{1}{2}\phi+r\cos\theta} \quad (\rho\neq 0). \tag{3·6·17}$$

Using the recurrence relation (3·6·7) it is deduced, from equations (3·6·15) and (3·6·16), that if $\rho \neq 0$,

$$\int_0^\infty \frac{1-e^{-wx}}{x}J_2(\rho x)\,dx = \frac{1}{\rho^2}\{w\sqrt{(w^2+\rho^2)}-w^2\} = wZ_0^1/\rho.$$

Putting $w=\zeta+i$, $Z_0^1 = C_0^1-iS_0^1$ and equating imaginary parts, then

$$S_0^2 = \frac{1}{\rho}(C_0^1-\zeta S_0^1). \tag{3·6·18}$$

3·7. Before proceeding to the determination of the distribution of stress in the medium at points remote from the crack consider the expressions for the stress components at points in the immediate vicinity of the periphery of the crack. In this way results are obtained analogous to those of § 1·4; with a notation similar to that of that section the integrals of § 3·6 are computed when $\zeta = \delta \sin \psi / c$, and $\rho = 1 + \delta \cos \psi / c$. Then

$$\left.\begin{aligned} r &= 1, & \phi &= \psi - \tfrac{1}{2}\delta \sin \psi / c, \\ \theta &= \tfrac{1}{2}\pi - \delta \sin \psi / c, & R &= 2\delta(c + \delta \cos \psi)/c^2. \end{aligned}\right\} \tag{3·7·1}$$

Substituting from equations (3·7·1) into the formulae (3·6·13–18) and retaining only those terms which are of first or higher order in $(c/\delta)^{\frac{1}{2}}$, it follows that

$$S_1^1 = C_1^0 = -C_1^2 = \left(\frac{c}{2\delta}\right)^{\frac{1}{2}} \cos \tfrac{1}{2}\psi, \quad C_2^1 = \left(\frac{c}{2\delta}\right)^{\frac{3}{2}} \cos \tfrac{3}{2}\psi,$$

$$C_2^0 = -C_2^2 = \left(\frac{c}{2\delta}\right)^{\frac{3}{2}} \sin \tfrac{3}{2}\psi, \qquad S_1^0 = -S_1^2 = \left(\frac{c}{2\delta}\right)^{\frac{1}{2}} \sin \tfrac{1}{2}\psi,$$

the other integrals being negligible to this approximation. With these values for the integrals involved the following expressions are obtained from equations (3·6·3–5) and (3·6·9) for the stress components σ_r, σ_z and τ_{rz}:

$$\sigma_r = \frac{2p_0}{\pi}\left(\frac{c}{2\delta}\right)^{\frac{1}{2}} \{\tfrac{3}{4} \cos \tfrac{1}{2}\psi + \tfrac{1}{4} \cos \tfrac{5}{2}\psi\}, \tag{3·7·2}$$

$$\sigma_z = \frac{2p_0}{\pi}\left(\frac{c}{2\delta}\right)^{\frac{1}{2}} \{\tfrac{5}{4} \cos \tfrac{1}{2}\psi - \tfrac{1}{4} \cos \tfrac{5}{2}\psi\}, \tag{3·7·3}$$

$$\tau_{zr} = \frac{p_0}{\pi}\left(\frac{c}{2\delta}\right)^{\frac{1}{2}} \sin \psi \cos \tfrac{3}{2}\psi. \tag{3·7·4}$$

It will be observed that these formulae differ from the expressions for the components of stress in the two-dimensional case (equations (1·4·1–3)) only by the presence of a factor $2/\pi$. There is no parallel in the two-dimensional analysis to the hoop stress σ_θ; in the three-dimensional case its value near the edge of the crack is given by the equation

$$\sigma_\theta = \frac{4\sigma p_0}{\pi}\left(\frac{c}{2\delta}\right)^{\frac{1}{2}} \cos \tfrac{1}{2}\psi. \tag{3·7·5}$$

3·8. By means of the formulae of § 3·6 the components of stress at any point $r = \rho a$, $z = \zeta a$, in the interior of the elastic body can be calculated for any prescribed value of the Poisson ratio. The results of these calculations for the case where the Poisson ratio has the value 0·25 and where both ρ and ζ assume a sequence of values are given in tables 2–5. The variation of the various components of stress in certain planes is shown graphically in figures 5–8. The rapid decay of the stress components σ_r and σ_θ as z increases is illustrated by figures 5 and 6; the variation with z of the normal component of stress, σ_z, is much more gradual as is seen from figure 7. Perhaps the most striking feature of these results is the marked similarity between the variation of the shearing stress τ_{zr} and that of the principal shearing stress τ in the two-dimensional case (compare figures 1 and 8).

TABLE 2. THE VARIATION OF σ_r/p_0 WITH ρ AND ζ

ζ \ ρ	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0
0·0	−0·750	−0·750	−0·750	−0·750	−0·750	∞	0·010	−0·055	−0·068	−0·069	−0·066
0·2	−0·446	−0·433	−0·386	−0·297	−0·180	−0·070	0·007	0·060	0·072	0·054	0·039
0·4	−0·214	−0·207	−0·160	−0·121	−0·087	−0·057	−0·041	−0·019	0·000	0·018	0·014
0·6	−0·075	−0·068	−0·060	−0·050	−0·039	−0·029	−0·019	−0·013	−0·007	0·000	0·007
0·8	−0·006	−0·001	0·007	0·013	0·022	0·014	0·006	0·001	0·000	−0·016	−0·026
1·0	0·017	0·015	0·012	0·010	0·005	−0·011	−0·022	−0·026	−0·028	−0·031	−0·027
1·2	0·031	0·029	0·026	0·023	0·022	0·019	0·015	0·009	−0·001	−0·007	−0·009
1·4	0·030	0·028	0·025	0·020	0·015	0·009	0·005	0·002	0·000	−0·001	−0·001
1·6	0·028	0·027	0·025	0·020	0·015	0·011	0·007	0·003	0·001	0·000	−0·001
1·8	0·024	0·023	0·022	0·019	0·015	0·010	0·007	0·004	0·001	0·001	0·000
2·0	0·021	0·020	0·018	0·016	0·013	0·010	0·006	0·005	0·003	0·002	0·000
3·0	0·009	0·009	0·008	0·008	0·007	0·006	0·005	0·004	0·003	0·002	0·002
4·0	0·004	0·004	0·004	0·004	0·003	0·003	0·003	0·003	0·002	0·002	0·002

TABLE 3. THE VARIATION OF σ_θ/p_0 WITH ρ AND ζ

ζ \ ρ	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0
0·0	−0·750	−0·750	−0·750	−0·750	−0·750	∞	0·010	−0·055	−0·068	−0·069	−0·066
0·2	−0·446	−0·438	−0·412	−0·344	−0·242	−0·100	0·045	0·022	0·010	0·005	0·004
0·4	−0·214	−0·219	−0·178	−0·131	−0·067	−0·014	0·025	0·018	0·013	0·009	0·006
0·6	−0·075	−0·072	−0·051	−0·009	0·016	0·020	0·023	0·013	0·007	0·006	0·005
0·8	−0·006	−0·009	−0·017	−0·023	−0·019	−0·008	0·009	0·011	0·014	0·017	0·012
1·0	0·017	0·019	0·026	0·033	0·030	0·026	0·022	0·018	0·017	0·015	0·003
1·2	0·031	0·039	0·045	0·053	0·061	0·058	0·039	0·021	0·011	0·019	0·006
1·4	0·030	0·035	0·039	0·044	0·048	0·054	0·048	0·041	0·033	0·025	0·016
1·6	0·028	0·032	0·037	0·040	0·043	0·046	0·042	0·038	0·033	0·029	0·021
1·8	0·024	0·027	0·029	0·032	0·034	0·036	0·038	0·036	0·032	0·028	0·020
2·0	0·021	0·023	0·025	0·027	0·029	0·031	0·032	0·030	0·027	0·025	0·019
3·0	0·009	0·009	0·010	0·010	0·011	0·011	0·013	0·012	0·011	0·010	0·010
4·0	0·004	0·005	0·005	0·006	0·006	0·006	0·006	0·006	0·006	0·005	0·005

TABLE 4. THE VARIATION OF σ_z/p_0 WITH ζ AND ρ

ζ \ ρ	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0
0·0	−1·000	−1·000	−1·000	−1·000	−1·000	∞	0·333	0·143	0·080	0·050	0·034
0·2	−0·987	−0·985	−0·975	−0·937	−0·729	0·232	0·358	0·166	0·089	0·055	0·035
0·4	−0·917	−0·912	−0·862	−0·742	−0·451	−0·006	0·189	0·150	0·095	0·060	0·040
0·6	−0·788	−0·768	−0·700	−0·560	−0·334	−0·081	0·068	0·097	0·080	0·053	0·041
0·8	−0·639	−0·617	−0·548	−0·429	−0·263	−0·108	0·002	0·048	0·055	0·048	0·038
1·0	−0·508	−0·482	−0·424	−0·333	−0·237	−0·113	−0·032	0·014	0·018	0·022	0·030
1·2	−0·386	−0·373	−0·329	−0·262	−0·185	−0·109	−0·049	−0·009	0·012	0·021	0·023
1·4	−0·302	−0·286	−0·256	−0·210	−0·154	−0·094	−0·055	−0·022	−0·001	0·010	0·015
1·6	−0·230	−0·224	−0·201	−0·168	−0·129	−0·090	−0·055	−0·027	−0·010	0·001	0·008
1·8	−0·180	−0·175	−0·159	−0·138	−0·108	−0·079	−0·054	−0·032	−0·010	−0·005	0·002
2·0	−0·142	−0·139	−0·127	−0·111	−0·091	−0·070	−0·050	−0·033	−0·019	−0·012	−0·003
3·0	−0·052	−0·051	−0·049	−0·045	−0·040	−0·035	−0·029	−0·024	−0·019	−0·014	−0·010
4·0	−0·025	−0·024	−0·023	−0·022	−0·021	−0·019	−0·017	−0·014	−0·013	−0·011	−0·009

TABLE 5. THE VARIATION OF τ_{zr}/p_0 WITH ρ AND ζ

$\zeta \backslash \rho$	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0
0·0	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000
0·2	0·000	−0·020	−0·051	−0·121	−0·327	−0·420	0·066	0·052	0·030	0·017	0·011
0·4	0·000	−0·055	−0·126	−0·233	−0·354	−0·319	−0·097	0·003	0·021	0·019	0·014
0·6	0·000	−0·074	−0·155	−0·239	−0·292	−0·252	−0·133	−0·041	−0·006	0·018	0·007
0·8	0·000	−0·074	−0·145	−0·202	−0·225	−0·200	−0·131	−0·069	−0·029	−0·009	0·000
1·0	0·000	−0·061	−0·119	−0·159	−0·170	−0·156	−0·117	−0·075	−0·046	−0·021	−0·009
1·2	0·000	−0·050	−0·092	−0·123	−0·133	−0·124	−0·100	−0·076	−0·047	−0·028	−0·016
1·4	0·000	−0·040	−0·070	−0·092	−0·102	−0·097	−0·084	−0·065	−0·048	−0·032	−0·020
1·6	0·000	−0·029	−0·053	−0·070	−0·078	−0·077	−0·069	−0·057	−0·044	−0·032	−0·022
1·8	0·000	−0·021	−0·040	−0·053	−0·060	−0·061	−0·057	−0·049	−0·040	−0·031	−0·023
2·0	0·000	−0·016	−0·031	−0·041	−0·047	−0·049	−0·046	−0·042	−0·035	−0·029	−0·022
3·0	0·000	−0·005	−0·009	−0·012	−0·015	−0·017	−0·018	−0·018	−0·017	−0·016	−0·014
4·0	0·000	−0·002	−0·004	−0·005	−0·006	−0·007	−0·007	−0·008	−0·008	−0·008	−0·008

The principal stresses at any point are determined by the roots of the discriminating cubic

$$\begin{vmatrix} \sigma-\sigma_r & 0 & \tau_{zr} \\ 0 & \sigma-\sigma_\theta & 0 \\ \tau_{zr} & 0 & \sigma-\sigma_z \end{vmatrix} = 0, \tag{3·8·1}$$

so that they are equal to

$$\sigma_\theta, \quad \tfrac{1}{2}(\sigma_r+\sigma_z) \pm \{(\tfrac{1}{2}\sigma_r-\tfrac{1}{2}\sigma_z)^2+\tau_{rz}^2\}^{\frac{1}{2}}. \tag{3·8·2}$$

With the values of the stress components given in tables 2–5 the components of principal stress may readily be deduced from the formulae (3·8·2). The numerical value of the principal shearing stress can then be determined from the fact that it is equal to one-half of the algebraic difference between the maximum and the minimum components of the principal stress. The results of such a calculation are embodied in table 6, and the variation of the principal shearing stress in planes parallel to the plane of the crack exhibited graphically in figure 9. In general outline this variation is similar to that of the principal shearing stress in the two-dimensional case.

TABLE 6. THE VARIATION OF THE PRINCIPAL SHEARING STRESS (DIVIDED BY p_0) WITH ρ AND ζ

$\zeta \backslash \rho$	0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0
0·0	0·125	0·125	0·125	0·125	0·125	∞	0·172	0·099	0·074	0·059	0·050
0·2	0·270	0·277	0·299	0·342	0·427	0·446	0·188	0·085	0·051	0·033	0·021
0·4	0·351	0·356	0·373	0·388	0·398	0·320	0·151	0·085	0·047	0·033	0·023
0·6	0·357	0·358	0·356	0·349	0·327	0·254	0·140	0·069	0·044	0·032	0·025
0·8	0·317	0·316	0·313	0·299	0·266	0·201	0·131	0·073	0·040	0·032	0·032
1·0	0·263	0·256	0·248	0·234	0·209	0·164	0·117	0·075	0·044	0·031	0·009
1·2	0·209	0·207	0·200	0·188	0·168	0·139	0·105	0·077	0·048	0·030	0·017
1·4	0·169	0·162	0·157	0·147	0·123	0·110	0·089	0·066	0·048	0·032	0·022
1·6	0·129	0·129	0·125	0·117	0·107	0·092	0·076	0·059	0·045	0·032	0·023
1·8	0·102	0·100	0·099	0·095	0·086	0·076	0·064	0·052	0·040	0·031	0·023
2·0	0·082	0·081	0·079	0·075	0·070	0·063	0·054	0·046	0·037	0·030	0·022
3·0	0·030	0·030	0·030	0·029	0·028	0·026	0·025	0·023	0·020	0·018	0·016

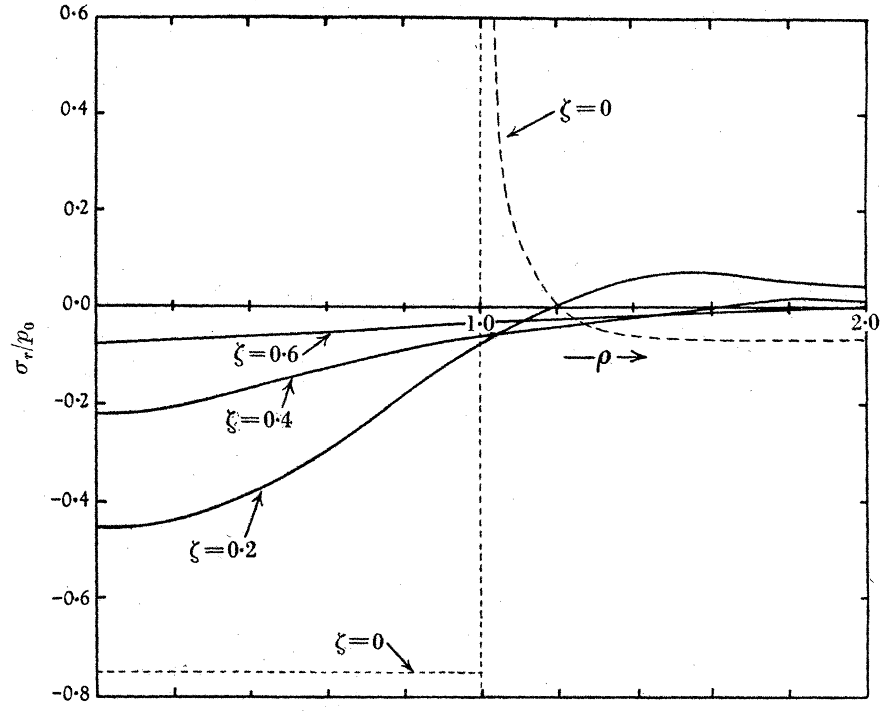

FIGURE 5. The variation of the radial component of stress, σ_r, with ρ and ζ. The broken curve shows the variation of σ_r in the plane of the crack ($z=0$).

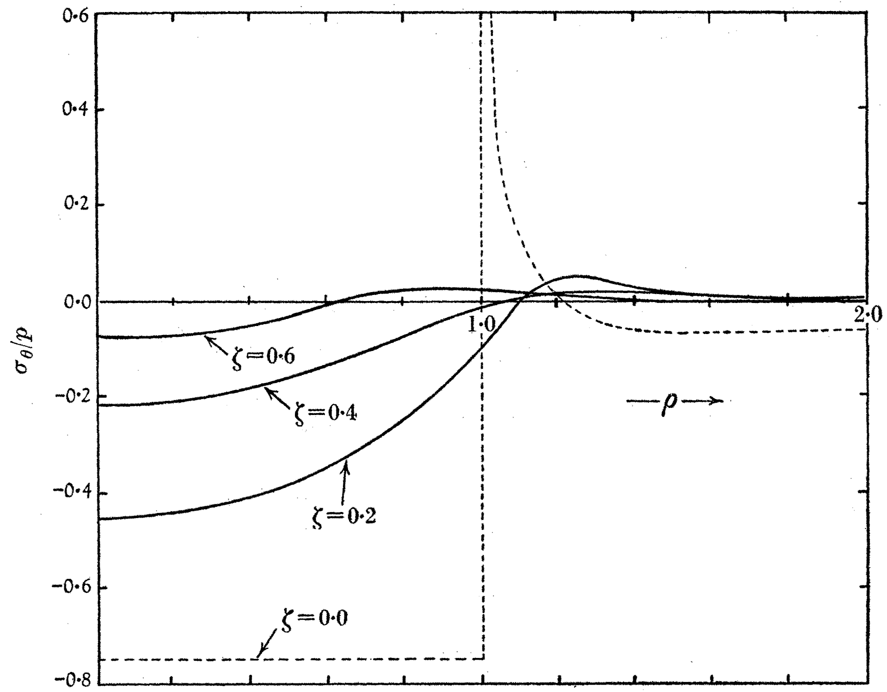

FIGURE 6. The variation of the hoop stress, σ_θ, with ρ and ζ. The broken curve shows the variation of σ_θ in the plane of the crack ($z=0$).

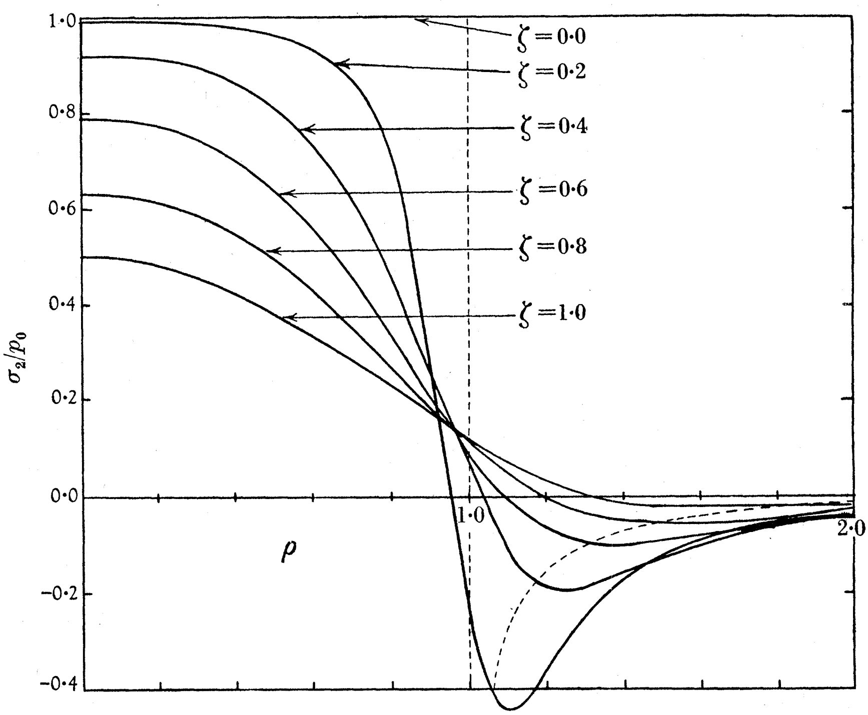

FIGURE 7. The variation of the normal component of stress, σ_z, with ρ and ζ. The broken curve shows the variation of σ_z in the plane of the crack ($z = 0$).

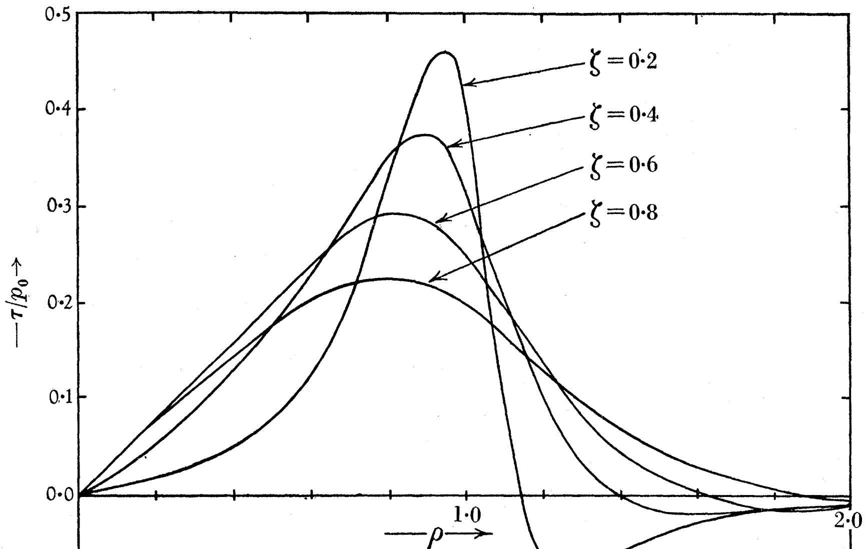

FIGURE 8. The variation of the shearing stress, τ_{zr}, with ρ and ζ.

The most convenient way of visualizing the distribution of stress in an elastic body is probably to draw the contours of equal principal shearing stress—the three-dimensional analogue of the isochromatic lines of photoelasticity. By means of table 6 and figure 9, points in the z-r plane at which the principal shearing stress

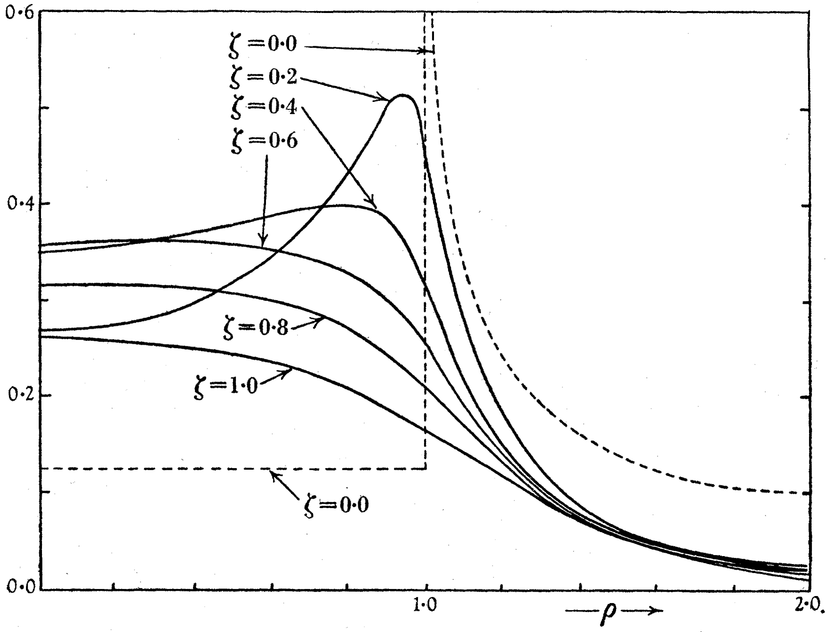

FIGURE 9. The variation of the principal shearing stress, τ, with ρ and ζ. The broken curve shows the variation of τ in the plane of the crack.

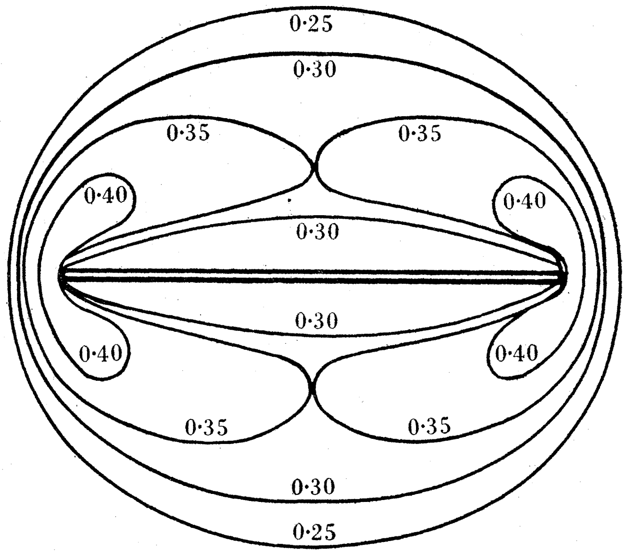

FIGURE 10. The contours of equal principal shearing stress in the vicinity of a circular crack.

reaches the values 0·25, 0·30, 0·35, 0·40 can readily be determined (in terms of p_0 as unit), and by drawing the curves through corresponding values the contours of (equal) principal shearing stress can be plotted. The result is shown in figure 10. It will be observed that except in the vicinity of the edges of the crack—and apart from numerical values—these contours of equal principal shearing stress are very like the isochromatic lines of the two-dimensional case (figure 2). The main difference between the two systems of curves lies in their behaviour near the edges of the crack. In the two-dimensional case the points $x = \pm c$ are nodes separating two loops of the curves (cf. the curves corresponding to the values $\tau/p_0 = 0{\cdot}4, 0{\cdot}5, 0{\cdot}7$ in figure 2), but in the three-dimensional case the point $r = c$ is a simple point of the curve, the curve cutting the r-axis again at a point where $r > c$ (cf. curves with τ/p_0 in figure 10). It should, of course, be remembered that these 'contours' are in reality *surfaces* of equal principal shearing stress obtained by rotating figure 10 about the z-axis.

As in the two-dimensional case the principal shearing stress becomes infinite at the edges of the crack, indicating that a certain amount of plastic flow will occur and hence that there is in reality no purely elastic problem. Except in the neighbourhood of the crack, however, the elastic stresses are dominant and the energy relation (3·3·6) approximates very closely to the true result.

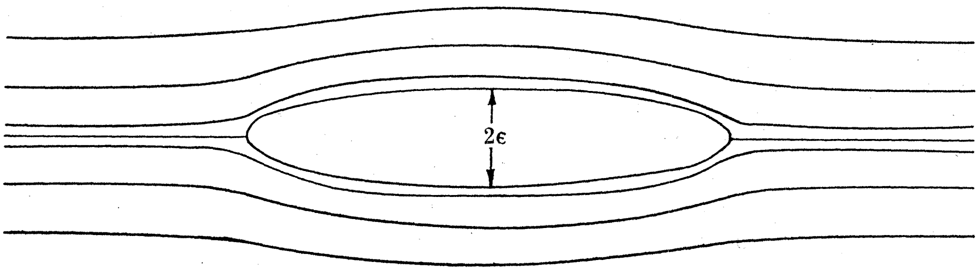

FIGURE 11. The normal displacement of planes parallel to the plane of an ellipsoidal crack of small depth. The depth of the crack, 2ϵ, has been greatly exaggerated in this diagram to show the variation of the normal displacement more clearly.

3·9. The distribution of stress in the vicinity of a crack can also be illustrated by calculating the normal component of the displacement vector for points in the elastic body and then drawing the normal displacement of planes which were originally parallel to the plane of the crack. Since the normal component of stress σ_z is infinitely large at the edge of the crack it might at first sight appear that the displacement u_z should also be infinite in that region; that this is not so can readily be seen from an examination of equation (3·6·2). It follows immediately from this equation that the normal component of the displacement vector is given by the formula

$$u_z = \epsilon\left[1-\rho S_0^1-\zeta S_0^0-\frac{\zeta}{2(1-\sigma)}(\zeta S_1^0+\rho S_1^1-S_0^0)\right] \qquad (3{\cdot}9{\cdot}1)$$

with the notation of (3·6·6). By means of this formula the variation of the normal displacement along the planes $z/c = 0{\cdot}05, 0{\cdot}2, 0{\cdot}4$ was calculated. The results are shown graphically in figure 11, which shows the position of the planes $z/c = 0{\cdot}05, 0{\cdot}2,$

0·4 after the crack has been opened out to a depth 2ϵ, by a uniform internal pressure; in this diagram the ratio ϵ/c has been very much exaggerated to show more clearly the variation of the normal displacement in planes near to the crack.

Approximate formulae for the determination of the normal component of the displacement vector in planes close to the crack can be derived from equation (3·9·1). If ρ is small, then

$$J_0(\rho\eta) = 1-\tfrac{1}{4}\rho^2\eta^2, \quad J_1(\rho\eta) = \tfrac{1}{2}\rho\eta$$

may be written in the integrals for C_n^m and S_n^m; making this substitution, evaluating the integrals now involved and retaining terms of (3·9·1) of order two or less one obtains the approximate result

$$u_z = \epsilon\{f_1(\zeta)-\rho^2 f_2(\zeta)\}+O(\epsilon\rho^4), \tag{3·9·2}$$

where the functions $f_1(\zeta)$ and $f_2(\zeta)$ are defined by the relations

$$\left.\begin{aligned} f_1(\zeta) &= 1-\frac{(1-2\sigma)}{2(1-\sigma)}\zeta\left(\frac{\pi}{2}-\tan^{-1}\zeta\right)-\frac{1}{2(1-\sigma)}\frac{\zeta^2}{1+\zeta^2}, \\ f_2(\zeta) &= \left(\frac{1}{2}-\frac{3-\sigma}{2(1-\sigma)}\zeta^2\right)(1+\zeta^2)^{-3}. \end{aligned}\right\} \tag{3·9·3}$$

By means of equation (3·9·2) may be calculated the normal displacement at points just off the axis of symmetry $r = 0$.

Similarly, by taking ρ to be large the integrals involved by means of equations (3·6·13) can be evaluated; since as a first approximation one may take $R = \rho$, $\phi = 2\zeta/\rho^2$ in these equations. Thus for ρ large and $\zeta \ll \rho$ one has the approximate formula

$$u_z = \frac{\epsilon\zeta}{\rho}\left\{1-\frac{\zeta}{2(1-\sigma)}\right\}+O\left(\frac{\epsilon}{\rho^2}\right). \tag{3·9·4}$$

The value of the displacement vector in the neighbourhood of the periphery of the crack $r = c$ can be determined by the methods of § 3·7. With the notation of that section

$$u_z = \epsilon\sqrt{\left(\frac{2\delta}{c}\right)}\sin\tfrac{1}{2}\psi\left[1-\frac{1+\cos\psi}{4(1-\sigma)}\right], \tag{3·9·5}$$

where $z = \delta\sin\psi$ and $r = c+\delta\cos\psi$.

The relation of the approximate formulae (3·9·2–5) to the exact result (3·9·1) is shown in figure 12. In this diagram the value of the displacement u_z for points along the plane $z/c = 0·05$ are shown; the full line gives the values of the displacement as calculated by equation (3·9·1) for $\sigma = 0·25$; the broken lines show the values given by the approximate formulae, for the same value of the Poisson ratio. It will be observed that the agreement is good within the range specified. It is to be expected that for smaller values of δ the expression (3·9·5) will give more accurate results.

3·10. The effect of a circular crack on the strength of a brittle solid subjected to a uniform tensile stress can be deduced immediately from the analysis of §§ 3·5 and 3·6. Assume that the radius c of the crack (which may be taken to be situated in the plane $z = 0$ with its centre at the origin of co-ordinates) is considerably smaller than the dimensions of the elastic body. Then replace the finite elastic body by an infinite elastic body with a crack in its interior which is subjected to the stress system

$$\sigma_z = p_0, \quad \sigma_r = \sigma_\theta = p_1 \tag{3·10·1}$$

when r is very large. Since the surface of the crack may be assumed to be free from external forces the further boundary conditions

$$\sigma_z = 0, \quad \tau_{zr} = 0 \tag{3·10·2}$$

follow when $z = 0$ and $r < c$.

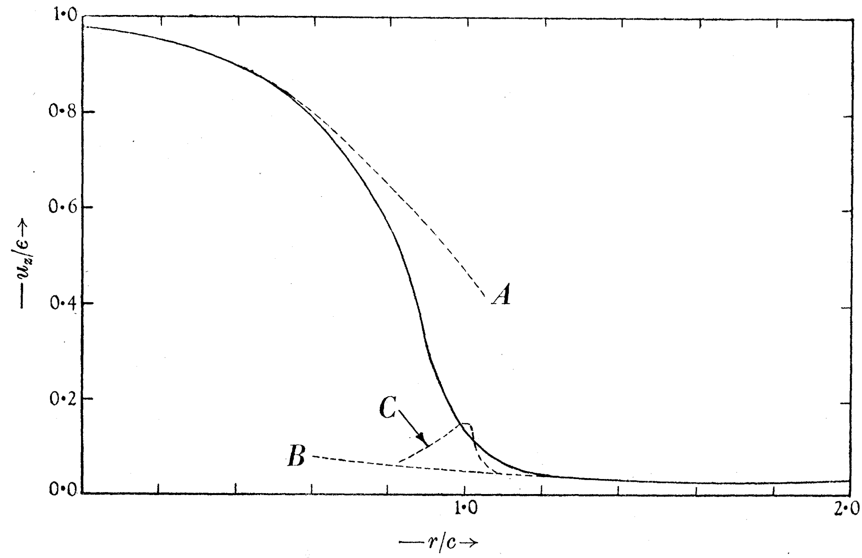

FIGURE 12. The variation of the normal displacement u_z in the plane $z = 0·05$ c.c. The full curve corresponds to the exact values calculated from equation (3·9·1). The broken curves correspond to the approximate formulae—A to (3·9·2), B to (3·9·4) and C to (3·9·5).

If the solution $\qquad \tau_{zr} = 0, \quad \sigma_z = p_0, \quad \sigma_r = \sigma_\theta = p_1 \qquad$ (3·10·3)

be added to equations (3·6·3), (3·6·4), (3·6·5) and (3·6·9), the expressions so obtained for the components of stress satisfy the equations of equilibrium (3·1·3) and (3·1·4) and the boundary conditions (3·10·1) and (3·10·2). The distribution of stress in the interior of the solid can then be determined from the results of § 3·7 by a very simple calculation, or from figures 5–8 by a simple change of scale along the axis corresponding to the stress component.

It follows immediately from equation (3·3·6) that if a brittle body is acted upon by uniform stress $\sigma_r = \sigma_\theta = p_1$, $\sigma_z = p_0$, the presence of a circular crack of radius c lowers the potential energy of the medium by an amount

$$W = \frac{8(1-\sigma^2)\,p_0^2 c^3}{3E} \tag{3·10·4}$$

which is independent of the value of p_1. The crack has also a surface energy

$$U = 2\pi c^2 T, \tag{3·10·5}$$

where T denotes the surface tension of the material of the elastic body. The condition that the crack may spread is

$$\frac{\partial}{\partial c}(W-U) = 0$$

or

$$c = \frac{\pi E T}{2p_0^2(1-\sigma^2)},$$

so that the crack will become unstable and spread if p_0 exceeds the critical value

$$p_{cr} = \sqrt{\frac{\pi E T}{2c(1-\sigma^2)}}.$$

This is the criterion derived by Sack (1946) (equation (1·2·1) above).

I should like to take this opportunity of expressing my thanks to Professor N. F. Mott for acquainting me with this problem and for various suggestions received during many helpful discussions on the theory of cracks. I am also indebted to Mr R. Sack, of the H. H. Wills Physical Laboratory, the University of Bristol, for putting his unpublished work at my disposal, and to Miss June Tenwick for her assistance in checking the calculations of § 3·8.

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