1 Introduction

1.1 Motivation

Inferring glacier thickness has been a long-standing problem for glaciologist. Acquiring such data today is often done with an airborne or sled-mounted ground penetrating radar (GPR). Such equipment can be costly and difficult to acquire, hence the development of inferring methods.

Determining mean ice thickness lets us estimate the total volume of ice in a given glacier, necessary to quantify the water stored. In the context of climate change, a proper estimate of ice thickness is necessary for correctly estimating sea-level change (Farinotti et al., 2016). Many techniques have been developed for this problem, such as "power law" or "scaling" methods, deriving total ice volume from glacier surface area (Bahr, Pfeffer, & Kaser, 2015). Other more modern methods have been developed for inferring distributed ice thickness. Those methods, using theoretical and mathematical models, often need intricate and difficult assumptions such as uniform basal shear stress an estimation of basal velocity (Farinotti et al., 2016). A rapidly increasing number of models are being developed, hence the need of an accurate representation of their performance along various conditions. The Ice Thickness Models Intercomparison eXperiment (Farinotti et al., 2016) or *ITMIX* did exactly this. 17 models were compare in this extensive paper along 21 study sites. This study tremendously helped the development of a global estimate of distributed glacier thickness (Farinotti et al., 2019). The five best performing models out of the experiment needing easily available data were used to make this worldwide estimation. The modelled data is freely available online for glaciers across the world.

The goal of this article is to assess the accuracy of the global glacier thickness estimation over two study sites extensively studied by Andrew Nolan and Tryggvy Unnsteinsson. Ultimately, a model of the distributed ice thickness will be produced using the data learned from comparing the global ice thickness data with field measurements.

2 Background

2.1 *ITMIX*

The Ice Thickness Models Intercomparison eXperiment (Farinotti et al., 2016) is a project launched by the working group on glacier ice thickness estimation, part of the International Association of cryospheric sciences.
The experiment consists of 17 different models tested over 21 cases, providing at least, depending on data availability:

- a glacier outline (**GO**)
- a digital elevation model (**DEM**)

And a combination of:

- the surface mass balance (**SMB**)
- the velocity field ($\vec{\boldsymbol{V}}$)
- the rate of ice thickness change ($\boldsymbol{\frac{\partial h}{\partial t}}$)

Four main types of models were outlined throughout the experiment, each needing specific data.

- Minimization approaches
  - Those models defines ice thickness inversion as a minimization problem. They use a cost function consisting of minimizing the difference between observed and modelled data.
- Mass conserving approaches
  - These methods are based on the principle of mass conservation (Farinotti et al., 2016) The ice flux divergence has to be compensated by the rate of ice thickness change and the climatic mass balance:
  $$\nabla \cdot q = \frac{\partial h}{\partial t} - \dot{b}$$
- Shear-stress based approaches

  - Those approaches rely on some estimation of the basal shear stress τ. They then solve for ice thickness using the shallow ice approximation (Fowler and Larson, 1978)

$$h = \frac{\tau}{f \rho g \sin \alpha}$$

- Velocity-based approaches
  - The models described in this category are based on a form of Glen's flow law (Glen, 1955) and an approximation of either the basal velocity u_b or the depth-averaged profile velocity $\bar{u}$ from the surface velocity u_s.
- Other approaches
  - GCneuralnet(Clarke et al., 2009) is a model based no artificial neural networks. It is based on the assumption that the bedrock topography resembles nearby unglaciarized valleys.
  - Brinkerhoff (Brinkerhoff et al., 2016) is based on Bayesian inference. The idea is that the bed elevation and the ice flux divergence can be described as Gaussian random fields with known covariance but unknown mean.

The various needs of every type of the main categories of models was captured in table 1. To note is that the table indicates the needs of the majority in the models presented, some use more data than the others in the same category.

ITMIX ranked the models from best to worst according to their performance over two rankings:

- Their performance over the glaciers they tested.
- Their performance over all the glaciers, considering the amount of test-cases.

One of the main conclusions out of *ITMIX* is that more data do not necessarily mean a better model. This is probably caused by the high variation in the methodology used by various scientists to capture such data.
As there is limited data available for our study sites, it is important to choose a model corresponding to potentially accessible data. The findings of *ITMIX* confirms that the lack of surface velocity field or rate of ice thickness change over time data isn't too dramatic.

Table 1: Models compared

Model	**GO**	**DEM**	**SMB**	$\vec{V}$	$\frac{\partial h}{\partial t}$
Minimization approach	✓	✓	✓	✓*	
Mass conservation approaches	✓	✓			
Shear-stress based approaches	✓	✓			
Velocity based approaches	✓	✓		✓	
GCneuralnet	✓	✓			
Brinkerhoff	✓	✓	✓	✓	✓

A ✓ with an asterix means that the model can make use of the data but it is not needed.

3 Study sites

3.1 Job glacier

3.1.1 Site description

Job glacier is located upon Mt. Meager, a volcano in southwestern British Columbia. This glacier, like most glaciers across the world, has seen a net negative mass balance in the last ten years (Reyes & Clague, 2004). This decreasing amount of ice means increasing amounts of water flowing along Mt. Meager. In 2010, this caused the largest reported landslide in the history of Canada happened upon Mt. Meager's south flank (Roberti et al., 2018).
Fumaroles emerged from the surface of the glacier in 2016. Roberti et al. (2018) states that the fumarolic activity has probably been active for a long period, but only recorded from the thinning of the glacier.
The research covered by Tryggvi Unnsteinsson aims to understand the dynamical interaction between the glaciological and volcanic systems upon Mt. Meager and the glaciological conditions required for fumarole emergence.

3.1.2 Data available

The data available for this study site consists of a 1 m^2 resolution LIDAR from (?), a glacier outline from Randolph Glacier Inventory and ice thickness measurements from a ground penetrating radar campaign held by Dr. Flowers and her team in September 2018. The ice thickness measurements only cover a small portion of the glacier and the point data can be seen on figure 1.

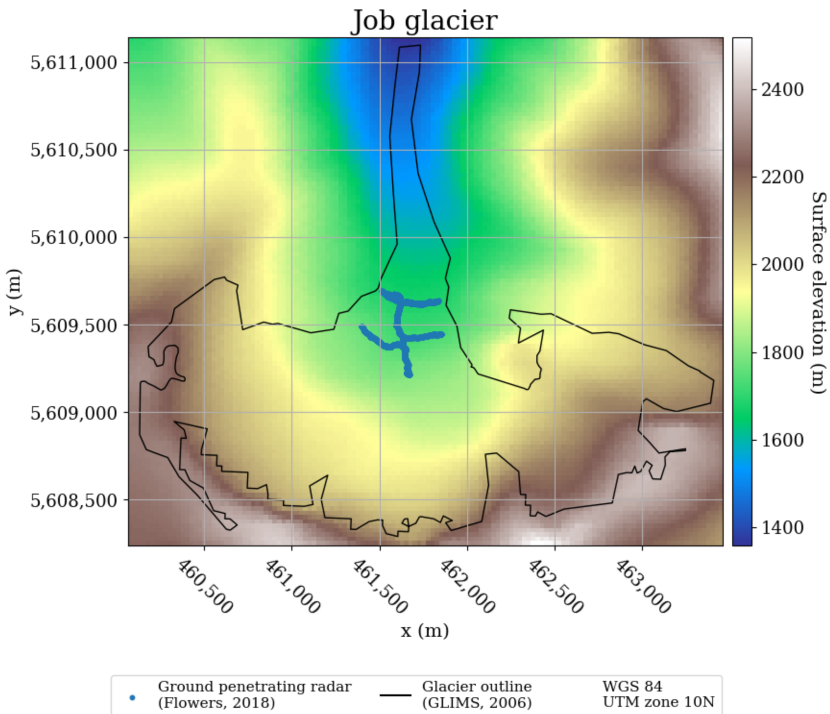


Figure 1: Job glacier's elevation

3.2 Little Kluane glacier

The little Kluane glacier is the nickname given to a small surging glacier part of the bigger Kluane glacier, located in the Kluane national park and reserve, in the St. Elias moutains of Yukon. More stuff need to be said.

3.2.1 Data available

The data available for this study site consists of a DEM we need to find (potentially DEM Arctic v2?)other than the one used by the global thickness model, a glacier outline from Nolan (2020) and ice thickness measurements from a ground penetrating radar campaign held by Dr. Flowers and her team in (??). The ice thickness measurements only cover a small portion of the glacier and the point data can be seen on figure 2.

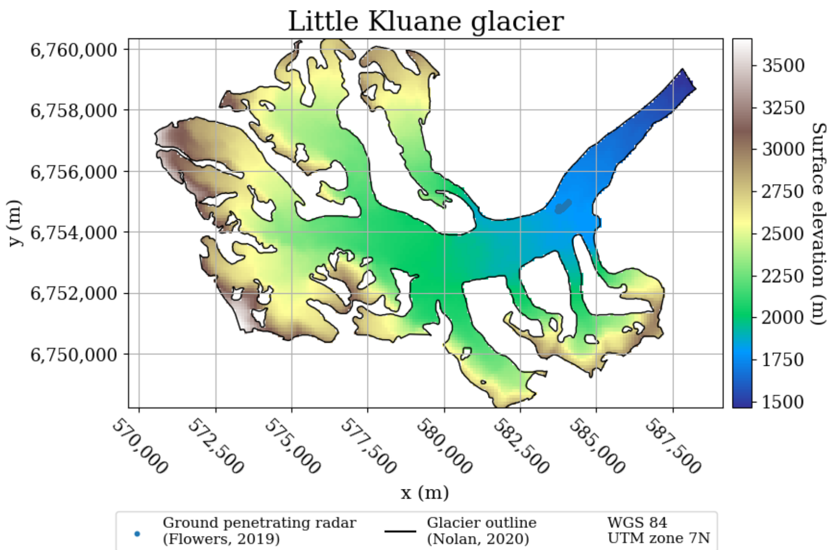


Figure 2: Little Kluane glacier's elevation

4 The global consensus estimate: assessing the error

Following *ITMIX*, Farinotti et al. bettered the earlier global estimate from Huss and Farinotti (2012). In this paper, they use findings from *ITMIX* to infer ice thickness of all the glaciers around the globe. using five different models and weighting each of their results to minimize the error. The computed models are freely available online and this raises a question: if we already have an estimate of ice thickness for our study sites, why should we compute one ourselves?
Having real world ice thickness measurement data can help us better the approximation made by Farinotti et al. (2019). First, we need to assess the error in those models by comparing it to our point data.

4.1 Job glacier

The ice thickness model from Farinotti et al. (2019) can be seen in figure 3.

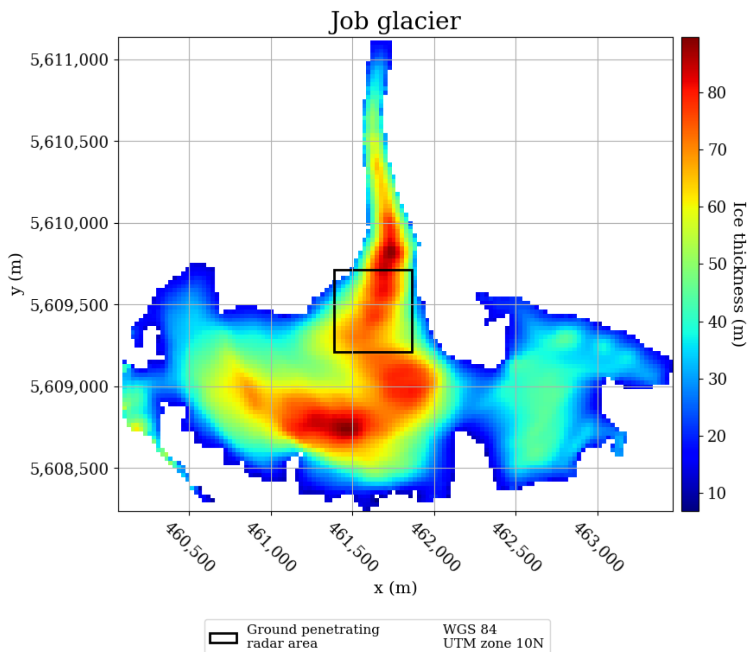


Figure 3: The modelled thickness of Job glacier

The modelled distributed ice thickness puts the thinner ice upon the higher elevations and the thicker ice in the lower areas, all the way down the flow path of the glacier. Also shown on figure 3 is the area covered by the ground penetrating radar point data. We can see the comparison between the field and modelled data on figure 4.

A higher discrepancy seem to be located upon the margin of the glaciers, modelling thicker ice to the west margin and shallower ice to the east margin. To compute the error

$$\Delta h = h' - h$$

we need to transform our point data to pixel data. As we have more points than pixels, the mean value of the point thickness data is taken for each cell. The computed error is shown in figure 5. The positive and negative error in the western and eastern margin respectively seems to be because the model

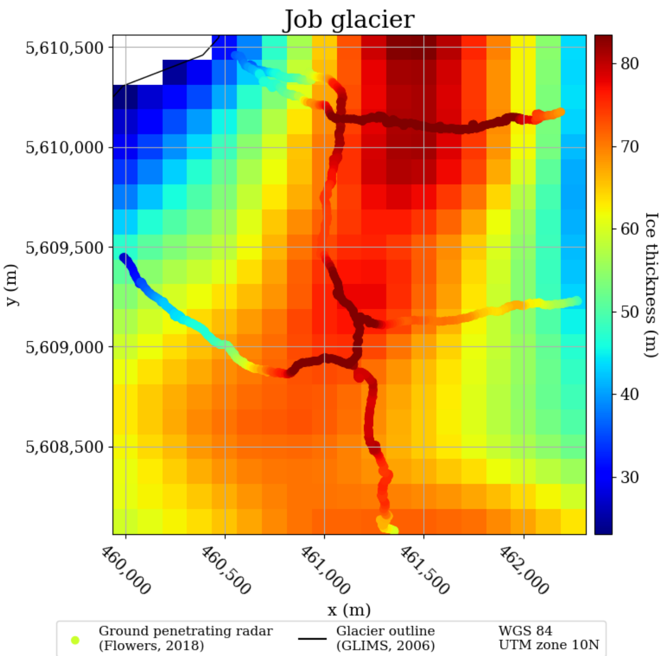


Figure 4: The field and modelled thickness data of Job glacier

doesn't take into account the glacier's flow direction. There might be thicker ice in the eastern region because of the curved nature of the flow line (??). It is also possible to compute the relative distributed error:

$$\delta h = \frac{h' - h}{h} = \frac{\Delta h}{h}$$

This relative error is shown in a similar manner on figure 6. The error is shown in percentage for convenience.
Plotting the mesured thickness on the x-axis and their corresponding modelled thickness on the y-axis gives us a scatter plot showing the overall error distribution. Such a plot is shown on figure 7. If the modelled and measured thickness were identical, they'd be lying upon the $x = y$ line. In Job glacier's case, they seem to be distributed over the line for the shallower measurements and farther under the line for the thicker measurements.

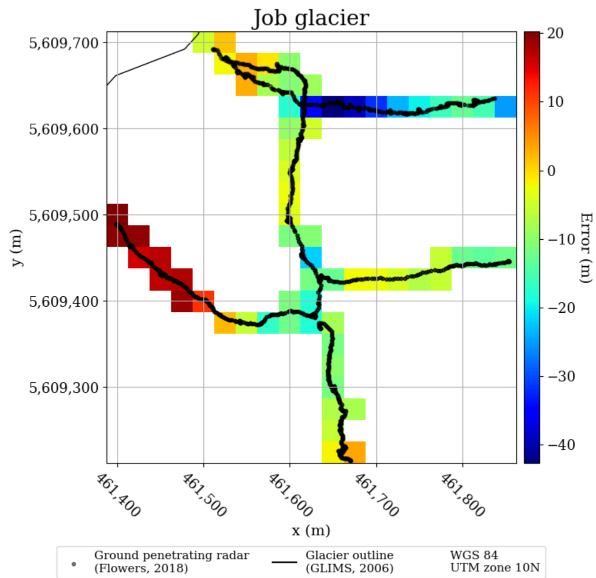


Figure 5: The error in Job glacier's model

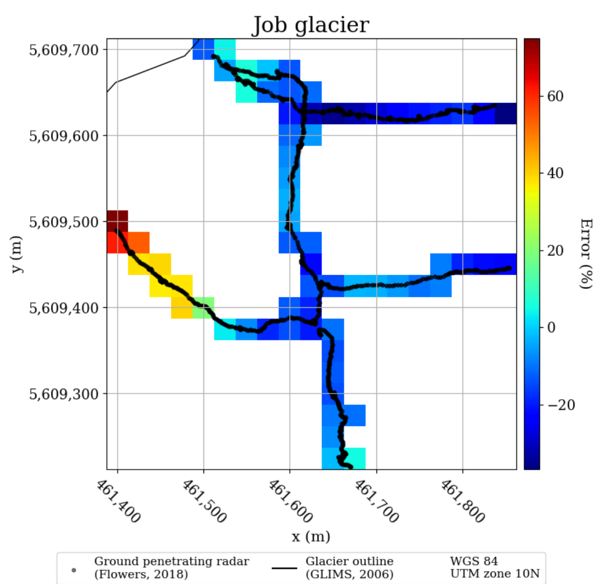


Figure 6: The relative error in Job glacier's model

Figure 7: The measured and modelled thickness of Job glacier

4.2 Little Kluane

Similar plots were made for the Little Kluane glacier. The ice thickness model from Farinotti et al. (2019) can be seen in figure 8.

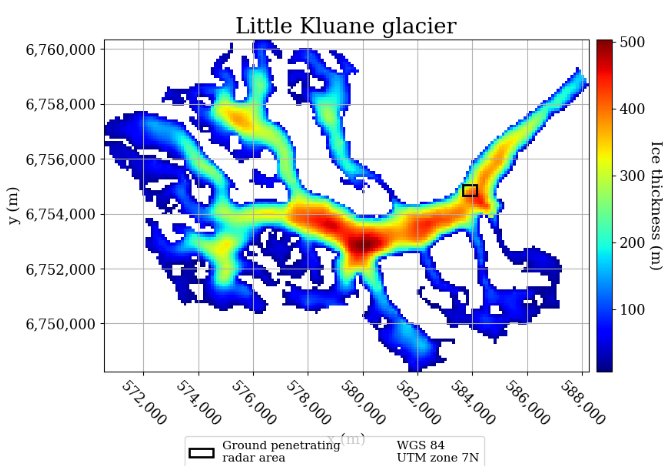


Figure 8: The modelled thickness of Little Kluane glacier

Little Kluane being a bigger glacier than Job, the absolute thickness is much greater. The modelled distributed thickness puts the thicker ice in the lower areas, along the flow lines. Figure 8 also shows the area covered by the ground penetrating radar data. Because the data was taken during its surge, it was difficult for the team to cover a great section without having to cross crevasses, hence the smaller area shown for Little Kluane. In a similar manner to the precedent section, the ice thickness is shown on figure 9. It is hard

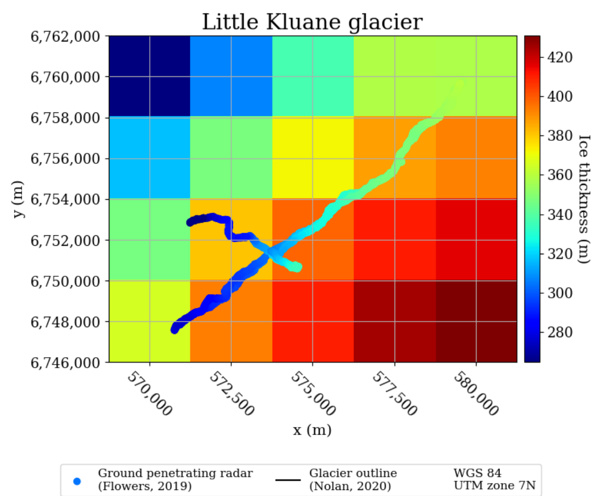


Figure 9: The modelled thickness of Little Kluane glacier

to discern any pattern in this case, but the error is positive overall. From the single cross section line, is is possible to deduce that the ice is probably much thinner at the extremities. The computed error and relative error are shown on figures 10 and 11.

One thing very important to note is the surging nature of the glacier at the times of measurements and the unknown time period covered by the modelled data. The paper coming out in 2019 and the glacier surging in 2018, it is unlikely but possible that they used data from after the surge. More info is needed on this and it is very important.
Similarly to Job glacier, the corresponding measured and modelled ice thick-

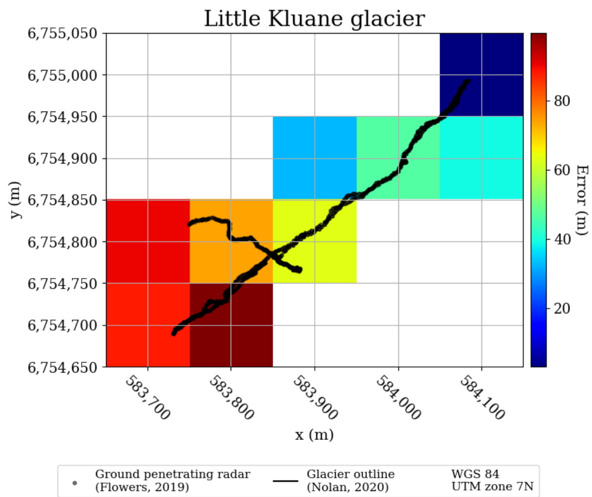


Figure 10: The error in Little Kluane glacier's model

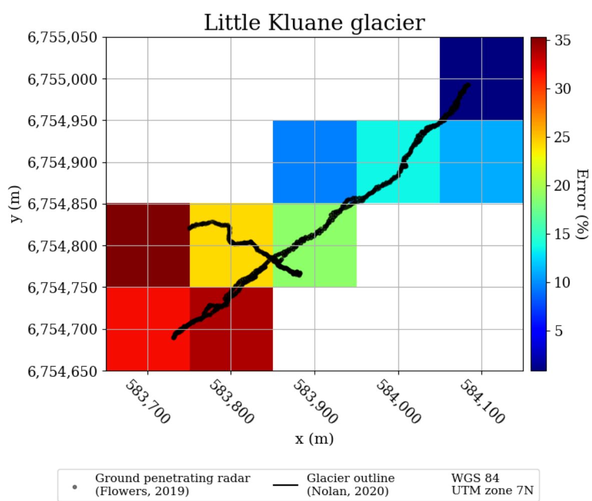


Figure 11: The relative error in Little Kluane glacier's model

ness data is shown on a scatter plot on figure 12. The figure shows us that the error is strictly positive.

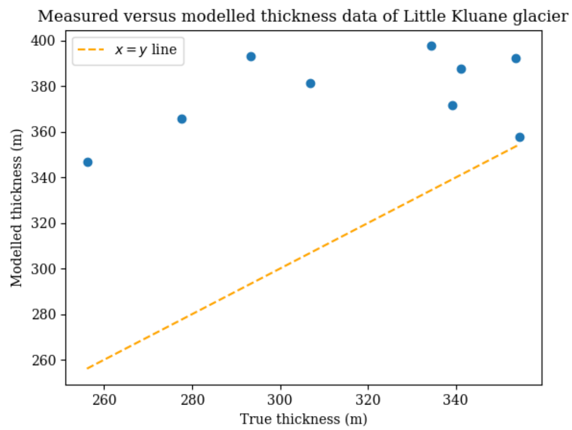


Figure 12: The measured and modelled thickness of Little Kluane glacier

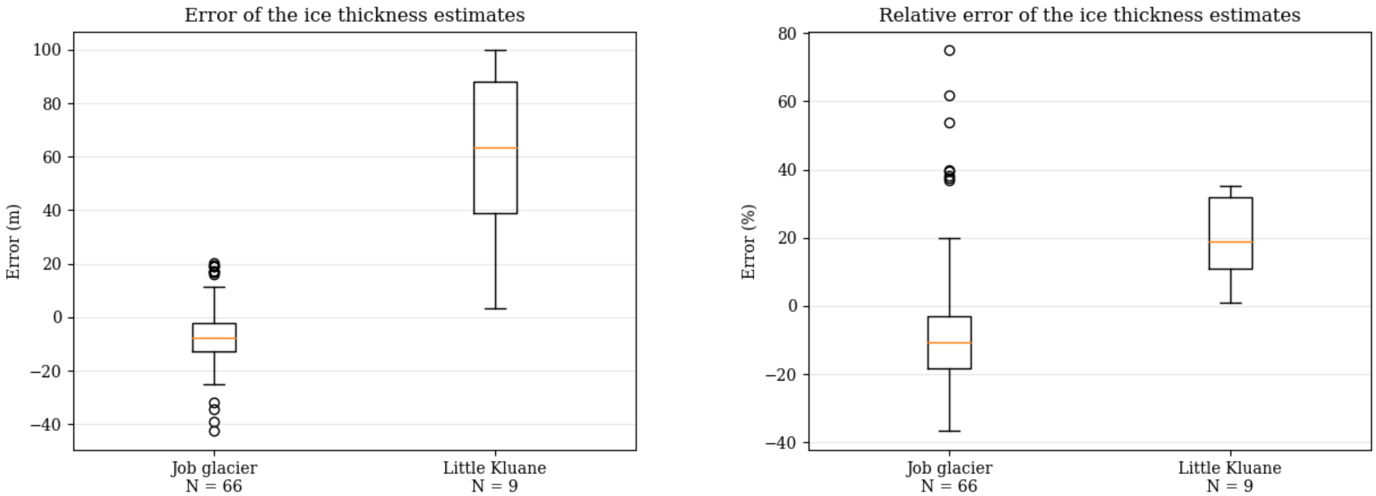
4.3 Error data

The errors between the modelled and observed data from Job and Little Kluane was computed and their main statistics is shown in table 2. Box plots of both the error and the relative error also have been produced and are visible in figure 13.

Table 2: Error data from Job and Little Kluane

Glacier	**Mean**	**Std**	**Median**	**Max**	**Absolute mean**	$\boldsymbol{MSE}$
Job glacier	-7.022	13.3	-7.861	20.27	12.15	226.2
Job glacier (%)	-4.81	22.41	-10.54	74.99	17.77	525.4
Little Kluane glacier	59.73	30.05	63.28	99.75	59.73	4.471e+03
Little Kluane glacier (%)	19.95	11.47	18.92	35.37	19.95	529.3

MSE is the mean square error, computed as $\frac{1}{n}\sum(\Delta h)^2$, where n is the number of cells.



(a) Error

(b) Relative error

Figure 13: Error box plots for Job and Little Kluane glaciers

This confirms us that, for Job glacier's case, the mean error is negative meaning that the ice thickness is mostly underestimated. The percentage standard deviation is however significant (In what way? How is that an absolute metric?) We do see however a few outliers on the box plot, showing that in some areas the thickness is greatly overestimated, up to 80%. This error is located in the western margin of the glacier which is the concave section of the flow line.

For Little Kluane glacier, the error is strictly positive, meaning that the ice thickness is over estimated for every point measured. The model also shows less variation than in Job's case with a smaller percentage standard deviation.

These findings validate the need to better the model with the ice thickness measurements.

5 GlaTe model

The consensus estimate (Farinotti et al., 2019) makes a good estimate of our study cases. However, it would be ideal to be able to use collected

ice thickness data to better the distributed ice thickness model. The freely available online GlaTe model (Langhammer, Grab, Bauder, & Maurer, 2019) aims to do this.
Using a physical model inspired by Clarke et al. (2013), they initially estimate the distributed ice thickness $\hat{h}^{glac}$. With the available ice thickness data h^{GPR}, they reduce the observable error on this approximation by fitting a coefficient α_{GPR} from minimizing a cost function q defined as

$$q = ||h^{GPR} - \alpha_{GPR}\hat{h}^{glac}||^2$$

equivalent of a linear regression with a fixed (and null) intercept. They use this coefficient such that

$$h^{glac} = \alpha_{GPR}\hat{h}^{glac}$$

is a supposedly better overall approximation, giving a bigger weight to the observed data and minimizing the impact of generalized trends in the physical model. Using this better approximation h^{glac}, they then solve this system of equations:

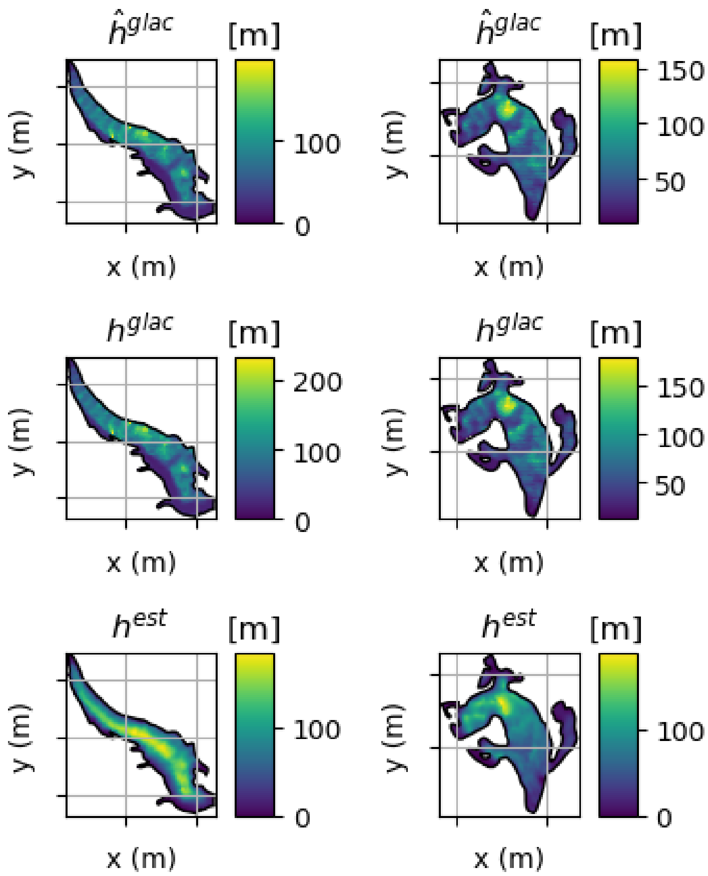
$$\begin{bmatrix} \lambda_1 G \\ \lambda_2 L \\ \lambda_3 B \\ \lambda_4 S \end{bmatrix} h^{est} = \begin{bmatrix} \lambda_1 h^{GPR} \\ \lambda_2 \nabla h^{glac} \\ 0 \\ 0 \end{bmatrix}$$

where h^{est} is the final modelled ice thickness distribution. (I either keep this part and more deeply explain every term or drop it and summarise the process.) The article from Langhammer et al. goes into great details over the process.

The algorithm then outputs three distributed ice thickness maps:

- The physical model $\hat{h}^{glac}$
- The corrected physical model $h^{glac} = \alpha_{GPR}\hat{h}^{glac}$
- The final estimation h^{est}

The GlaTe model could be described as a physical method of ice thickness interpolation. When used with a fairly large and regularly distributed dataset, the modelled ice thickness h^{est} has a very small observable error. This can be observed for North and South glaciers where the GlaTe algorithm was applied on figures 14 and 15.
It seems that h_{est} is a good candidate for am ice thickness model. Scatter-



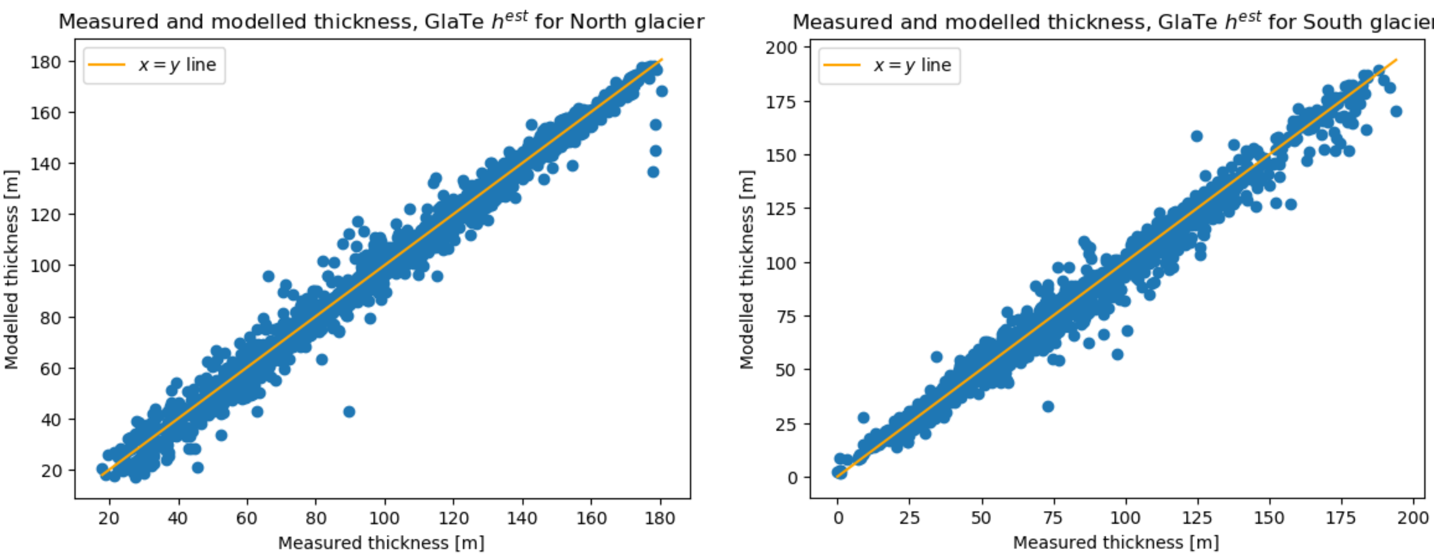
(a) North glacier (b) South glacier

Figure 14: Thickness maps produced with the GlaTe algorithm for North and South glacier.

plots similar to the precedent section also have been produced are visible at figure 15.

While it can be easy to think, referring to figure 15, that the h^{est} model is close to a perfect estimate of the bed ice thickness. However, this is due to the nature of the GlaTe algorithm, reducing the error for known data points only.

If we reduce the number of known points to a smaller sample, as is the case for our other study sites, results can get disappointing. This was tested by arbitrarily selecting a small sample of points on North Glacier and applying



(a) North glacier

(b) South glacier

Figure 15: Measured on modelled thickness for North and South glaciers.

this data trough the GlaTe algorithm. The resulting model was compared against the complete set of data in the same way than the previous figures and can be seen on figures 16 and 17. The plots show that h^{est} ends up severely underestimating the ice thickness. It is then probably not a good idea to use the entire GlaTe algorithm when only a handful of data points is available.

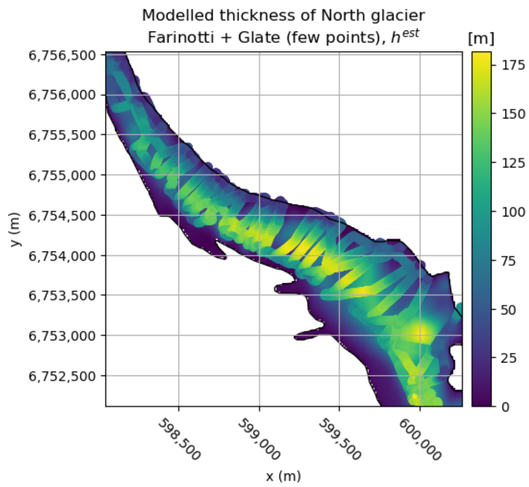


Figure 16: h^{est} and h^{GPR} for North Glacier when only a few data points are set as input

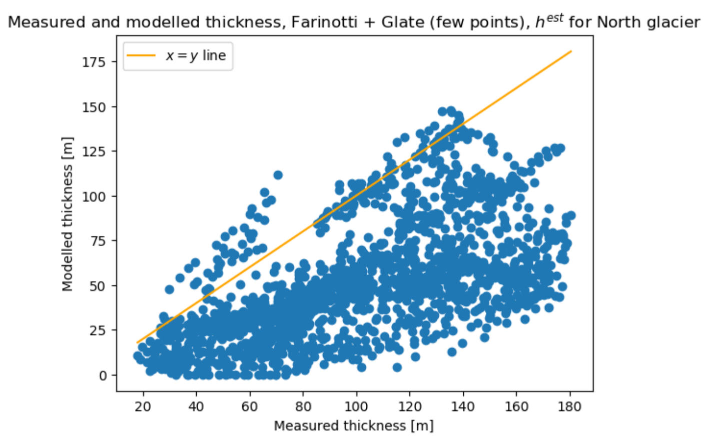


Figure 17: h^{est} and h^{GPR} for North Glacier when only a few data points are set as input

5.1 Verifying α_{GPR}

An interesting approach from the GlaTe algorithm is to scale the modelled ice thickness $\hat{h}^{glac}$ by a scalar α_{GPR}. This scalar, obtained by minimizing the quantity

$$q = ||h^{GPR} - \alpha_{GPR} * \hat{h}^{glac}||^2$$

is then used to obtain a *better* approximation

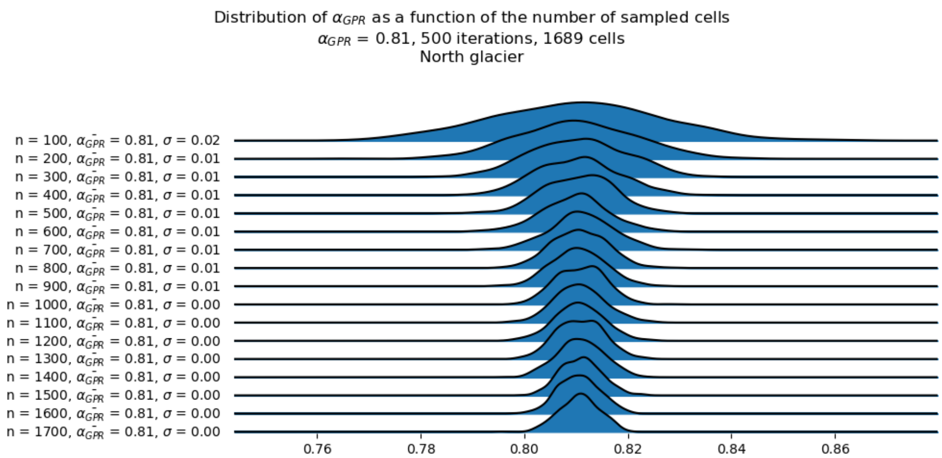
$$h^{glac} = \alpha_{GPR} * \hat{h}^{glac}$$

However, this approach should better our estimation when there is a good coverage of the radar data, which is not the case for our study sites. We can then use our test cases with a better coverage to compute the distribution of the scalar α_{GPR} when randomly sampling GPR points.
By randomly selecting an arbitrary number N of GPR data, their corresponding modelled cells and computing the equivalent α_{GPR} multiple times, we can estimate the distribution of α_{GPR} for N data points[1]. The result of such sampling can be seen on figure 18. The distribution is strongly centered on the *true* value of α_{GPR} and showing a rather small standard deviation for both cases.

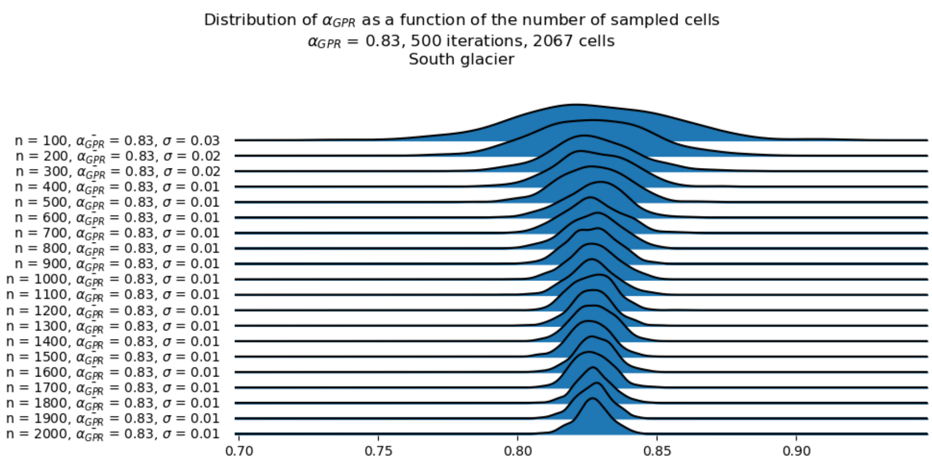
However, to truly represent the locality of our datasets, we need to sample such random data. By randomly selecting a GPR data point and its nearest 50 cells, we can approximate such a distribution. The result of such sampling can be seen on figure 19. The distribution is less strongly centered on the *true* value of α_{GPR} for small values of N and shows a much greater standard deviation. The distribution is also strongly bimodal for increasing values of N.

How can we conclude this section? "We decide to go with it as our best estimate of ice thickness"?

[1]The code used for this sampling is available on GitHub

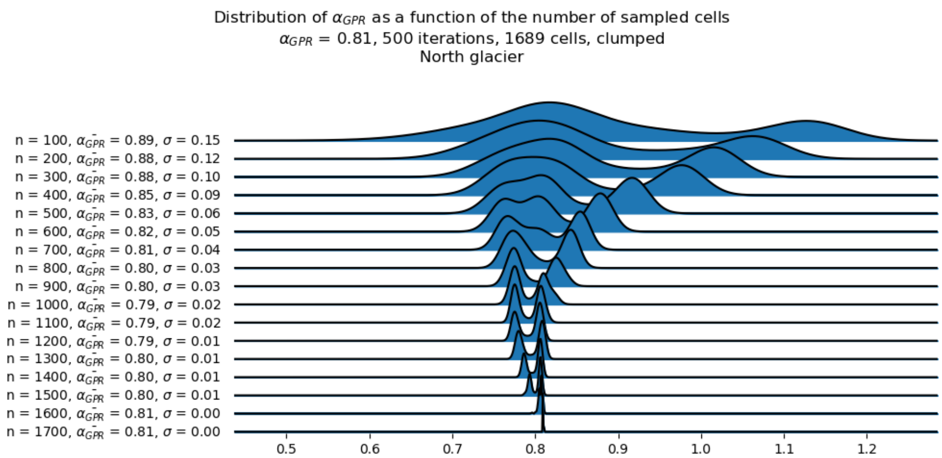


(a) North glacier

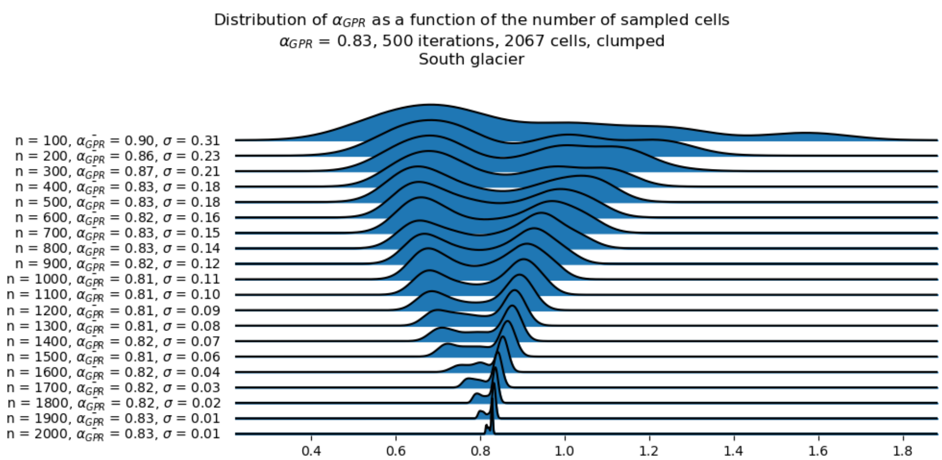


(b) South glacier

Figure 18: Distribution of α_{GPR} for randomly sampled N cells



(a) North glacier



(b) South glacier

Figure 19: Distribution of α_{GPR} for randomly sampled N clumped cells

6 Appendix

6.1 The Kaskawulsh glacier

As part of Young et al. (references needed) article about the Kaskawulsh glacier's mass balance, flux gates area were computed. To compute those flux gates, one needs bed topography data, coming from real word measurements or modelled inference. Ground penetrating radar data for this particular glacier was collected in the summers of 2018 and 2019. Flux gate area was computed by past undergraduate student Rebecca Latto in the summer of 2019 by summing the ice thickness values multiplied 10, assuming a constant distance between the point data of 10 meters.

As part of this ice thickness inference project, the differences between modelled and measured data for the Kaskawulsh glacier was computed. The main goal was to evaluate the potential error that would be made for the flux gate's area if one was to use modelled thickness data. Latto's data was unfortunately not geo-referenced. Her points for a given transect line were separated assuming a constant distance of 10 meters. However, splitting the lines in such a way using QGIS did not work as the algorithm ended up outputting more or less points than that of Latto's data. Therefore a 1 to 1 connection could not be established. The lines were then split up in as many points as there was thickness data for a given line. The distances between the points were then computed, resembling more or less 10 meters[2]. The (x, y) referenced data could then be used to compare the measured thickness with models. Figures representing the measured against the modelled thickness data were produced and can be seen on figures 20 and 21.

To note is that some problems were had noting the orientation of the data. It was assumed that the transects data was orientated down glacier. This can be a problem when computing the error as the differences between the modelled and the observed data. However, it should not be a major problem when computing the differences between the modelled and measured area of the transects. To compute the modelled flux gate areas, modelled ice thickness values $\hat{h}$ were extracted at every (x, y) point were there was measured

[2]The code used for this task and the data produced is available on GitHub

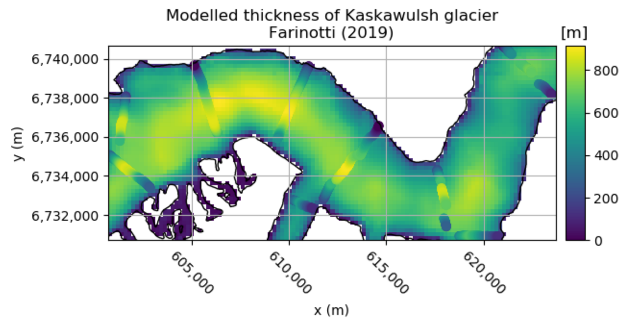


Figure 20: The measured and modelled thickness of Kaskawulsh glacier

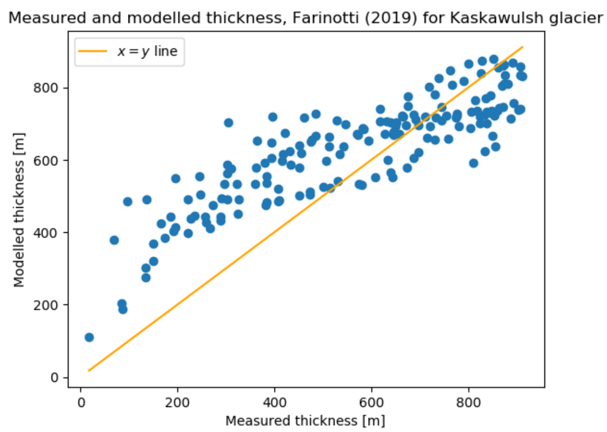


Figure 21: The measured and modelled thickness of Kaskawulsh glacier

data. The distances between the points were computed as

$$d = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

and the area was computed using numpy's trapezoid integration (need reference?) algorithm. The data generated is available on GitHub.

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