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two level system driven with periodical force

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1 Schrodinger equation and Hamiltonian

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$$
$$\text{where} \qquad |\psi(t)\rangle = \begin{pmatrix} \psi_\uparrow(t) \\ \psi_\downarrow(t) \end{pmatrix} \tag{1}$$

The Hamiltonian is considered to be peridical here, which reads $\hat{H}(t) = \hat{H}(t+T)$. The general solution can be expressed by the evolution operator $\hat{U}(t, t_0)$ as

$$|\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle \tag{2}$$

,where

$$\hat{U}(t, t_0) = \hat{T}e^{-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'} \tag{3}$$

We have

$$|\psi(t)\rangle = \hat{U}(T+\tau, \tau)\hat{U}(\tau, 0)|\psi(0)\rangle = \hat{U}(T, 0)|\psi(\tau)\rangle \tag{4}$$

, where $0 < \tau < T$. The key point here is that $\hat{U}(T+\tau, \tau)$ has no τ dependence, which is the consequence of the peridicity. Since $\hat{U}(T, 0)$ is an unitary operator, it can be diagonalized as

$$\hat{S}\hat{U}(T, 0)\hat{S}^\dagger = \begin{pmatrix} e^{-i\epsilon_1 T/\hbar} & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & e^{-i\epsilon_n T/\hbar} \end{pmatrix} \tag{5}$$

, where ϵ_i called “quasi-energy” satisfying $-\frac{\pi}{T} < \epsilon_i/\hbar < \frac{\pi}{T}$ or $-\frac{\hbar\omega}{2} < \epsilon_i < \frac{\hbar\omega}{2}$. Next, $|\psi(t)\rangle$ can be expanded using the Eigen states of $\hat{U}(T, 0)$ denoted as $|\psi_i(t)\rangle$, which satisfies $\hat{U}(T, 0)|\psi_i(t)\rangle = e^{-\frac{i\epsilon_i T}{\hbar}}|\psi_i(t)\rangle = |\psi_i(t+T)\rangle$. We can rewrite $|\psi_i(t)\rangle = e^{-\frac{i\epsilon_i t}{\hbar}}|u_i(t)\rangle$, then

$$|\psi_i(t+T)\rangle = e^{-i\epsilon_i T/\hbar}e^{-i\epsilon_1 t/\hbar}|u_i(t+T)\rangle = e^{-i\epsilon_i T/\hbar}|\psi_i(t)\rangle = e^{-i\epsilon_i T/\hbar}e^{-i\epsilon_1 t/\hbar}|u_i(t)\rangle \Rightarrow |u_i(t+T)\rangle = |u_i(t)\rangle \tag{6}$$

The following equation is called Floquest theorem, which says that the general solution of equation (1) can be written as

$$|\psi(t)\rangle = \sum_{i=1}^{n} c_n|\psi_i(t)\rangle \tag{7}$$

, where each lieanerly independent eigen-function $|\psi_i(t)\rangle$ can be factorised into a phase factor $e^{-i\epsilon_i t}$ and a periodical part $|u_i(t)\rangle$ with quasi-nenergy ϵ_i as

$$|\psi_i(t)\rangle = e^{-i\epsilon_i t/\hbar}|u_i(t)\rangle \qquad \text{with} \qquad |u_i(t)\rangle = |u_i(t+T)\rangle \tag{8}$$

Since $|u_i(t)\rangle$ is a periodical function of time, we can express it by Fourier tranformation as

$$|u_i(t)\rangle = \sum_l e^{-i\omega_l t}|a_{li}\rangle$$

Then we can obtain the effective Eigen equation for $|u_i(t)\rangle$, which can be written as

$$\begin{aligned} i\hbar\frac{d}{dt}|\psi(t)\rangle = \epsilon_i e^{-i\epsilon_i t/\hbar}|u_i(t)\rangle + e^{-i\epsilon_i t/\hbar} i\hbar\frac{d}{dt}|u_i(t)\rangle = \hat{H}(t)e^{-i\epsilon_i t/\hbar}|u_i(t)\rangle \\ \Rightarrow \qquad \epsilon_i|u_i(t)\rangle = \left[\hat{H}(t) - i\hbar\frac{d}{dt}\right]|u_i(t)\rangle \\ \Rightarrow \qquad \epsilon_i|a_{mi}\rangle = \sum_l \left[\hat{H}(\omega_m - \omega_l) + \hbar\omega_l\delta_{ml}\right]|a_{li}\rangle = \sum_l \hat{H}^{\text{Floquet}}_{ml}|a_{li}\rangle \\ \text{with}\, \omega_l = \frac{2\pi l}{T} = l\omega \end{aligned} \tag{9}$$

2 apply to a simple two-level system

The simplest quantum system is two-level system, which only contains two quantum states coupled by electromagnetic wave. The corresponding Shrodinger equation to be solved can be written as

$$\begin{aligned} i\hbar\frac{d}{dt}\begin{pmatrix} \psi_\uparrow(t) \\ \psi_\downarrow(t) \end{pmatrix} = \hat{H}(t)\begin{pmatrix} \psi_\uparrow(t) \\ \psi_\downarrow(t) \end{pmatrix} \\ \text{with} \quad \hat{H}(t) = \begin{pmatrix} \frac{1}{2}\hbar\omega_0 & E\cos(\omega t) \\ E\cos(\omega t) & -\frac{1}{2}\hbar\omega_0 \end{pmatrix} \end{aligned} \tag{10}$$

Because $\hat{H}(t)$ is peridical in time, the Floquet theorem tell us

$$\begin{pmatrix} \psi_\uparrow(t) \\ \psi_\downarrow(t) \end{pmatrix} = e^{-i\epsilon_i t/\hbar}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \quad \text{with}\, a(t+T) = a(t) \quad b(t+T) = b(t) \tag{11}$$

According to equation (9), we can obtain the corresponding Floquet Hamiltonian as

$$\hat{H}^{\text{Floquet}} = \begin{pmatrix} \dots & \frac{E}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{E}{2} & \frac{1}{2}\hbar\omega_0 + \hbar\omega & 0 & 0 & \frac{E}{2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\hbar\omega_0 + \hbar\omega & \frac{E}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{E}{2} & \frac{1}{2}\hbar\omega_0 & 0 & 0 & \frac{E}{2} & 0 \\ 0 & \frac{E}{2} & 0 & 0 & -\frac{1}{2}\hbar\omega_0 & \frac{E}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{E}{2} & \frac{1}{2}\hbar\omega_0 - \hbar\omega & 0 & 0 \\ 0 & 0 & 0 & \frac{E}{2} & 0 & 0 & -\frac{1}{2}\hbar\omega_0 - \hbar\omega & \frac{E}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{E}{2} & \dots \end{pmatrix} \tag{12}$$

the corresponding eigen vector of ϵ_i can be written as

$$\begin{pmatrix} a_{N,i} \\ b_{N,i} \\ \ldots \\ a_{1,i} \\ b_{1,i} \\ a_{0,i} \\ b_{0,i} \\ a_{-1,i} \\ b_{-1,i} \\ \ldots \\ a_{-N,i} \\ b_{-N,i} \end{pmatrix} \tag{13}$$

and the general solution can be written as

$$\begin{pmatrix} \psi_\uparrow(t) \\ \psi_\downarrow(t) \end{pmatrix} = C_1 e^{-i\epsilon_1 t/\hbar} \begin{pmatrix} \sum_{l=-N}^{l=N} a_{l,1} e^{-il\omega t} \\ \sum_{l=-N}^{l=N} b_{l,1} e^{-il\omega t} \end{pmatrix} + C_2 e^{-i\epsilon_2 t/\hbar} \begin{pmatrix} \sum_{l=-N}^{l=N} a_{l,2} e^{-il\omega t} \\ \sum_{l=-N}^{l=N} b_{l,2} e^{-il\omega t} \end{pmatrix} \tag{14}$$

3 Question 0: the rotational wave approximation (RWA)

RWA is probably the most famous approximation in quantum optics, which is valid for the case that $\omega_0 \sim \omega$. In that case we can keep only the $a_{0,i}$ and $b_{1,i}$ to be none zero. The corresponding Floquet effective Eigen problem becomes,

$$\begin{pmatrix} -\frac{1}{2}\hbar\omega_0 + \hbar\omega & \frac{E}{2} \\ \frac{E}{2} & \frac{1}{2}\hbar\omega_0 \end{pmatrix} \begin{pmatrix} b_{1,i} \\ a_{0,i} \end{pmatrix} = \epsilon_i \begin{pmatrix} b_{1,i} \\ a_{0,i} \end{pmatrix} \tag{15}$$

Please fix $\hbar\omega$ and solve the above equation under RWA and plot the quasi-enenrgy ϵ_i as a function of $\hbar\omega_0$ for different coupling strength E.

4 Question I:

There are two solutions with quasi-energy $-\frac{1}{2}\hbar\omega < \epsilon_1 < \epsilon_2 < \frac{1}{2}\hbar\omega$ for given energy level difference $\hbar\omega_0$. Please write a computer code to calculate two quasi-energies as a function of $\hbar\omega_0$ and plot them in a single plot.

5 Question II:

Suppose the initial condition at t=0 is

$$|\psi(t)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{16}$$

please plot the final solution ($a(t)$ and $b(t)$) for several different ω_0 and study how the behavior of $a(t)$ and $b(t)$ will change with different ω_0.

6 Question III:

Add another EM wave with different frequency 5ω and the Hamiltnian now becomes

$$\hat{H}(t) = \begin{pmatrix} \frac{1}{2}\hbar\omega_0 & E\cos(\omega t) + E'\cos(5\omega t) \\ E\cos(\omega t) + E'\cos(5\omega t) & -\frac{1}{2}\hbar\omega_0 \end{pmatrix} \tag{17}$$

Please study the above two problems again.