HW 1 Solns

1. First,

$$\sum_{n=0}^{m} |C_n| \le \sum_{n=0}^{m} \sum_{j=0}^{n} |a_j b_{n-j}| \le \sum_{n=0}^{m} |a_n| \sum_{j=0}^{m} |b_j|$$

$$< \left(\sum_{n=0}^{\infty} |a_n|\right)\left(\sum_{n=0}^{\infty} |b_n|\right)$$

Due to abs. convergence of $\sum a_n$ and $\sum b_n$, $\sum c_n$ has to converge absly as well.

For part two of the proof, an almost same argument as that of Theorem 3.50 in Rudin applies here. $\boxtimes$

2. WLOG, assume $a = 0$

① If $0 < s < r$, then for $|z| \le s$, we get

$$\sum |a_n + b_n| |z|^n \le \sum |a_n| s^n + \sum |b_n| s^n < \infty$$

which means the radius of Conv. $\ge \sup_{(0,r)} s = r$

And when $|z| < r$,

$$\sum_{n=0}^{m} (a_n + b_n) z^n = \sum_{n=0}^{m} a_n z^n + \sum_{n=0}^{m} b_n z^n$$

taking $m \to \infty$ and due to abs. Convergence of all 3 series, we get

$$\sum (a_n + b_n) z^n = \sum a_n z^n + \sum b_n z^n$$

② the other series is proved similarly under assistance of the Previous question.

3. a) $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{1}{a}\right|$

b) $$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{1}{a^{2n+1}}\right| = \begin{cases} 0 & , |a|>1 \\ 1 & , |a|=1 \\ \infty & , |a|<1 \end{cases}$$

c) $R = \lim \frac{|a_n|}{|a_{n+1}|} = \left|\frac{1}{k}\right|$

d) $R = (\limsup \sqrt[n]{|a_n|}\,)^{-1} = 1$ as $a_n = 0$ or 1

when n is large.

4. $R = \left(\limsup \sqrt[n]{\frac{n^2}{2^n}}\,\right)^{-1} = 2\left(\limsup \sqrt[n]{n^2}\,\right)^{-1} = 2$

When $z = 1$, Set $S = \sum_{n=0}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Then $$2S = \sum_{n=1}^{\infty} \frac{n^2}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{(n-1)^2}{2^{n-1}} + \sum_{n=1}^{\infty} \frac{2n-1}{2^{n-1}}$$

$$= S + 2\sum_{n=1}^{\infty} \frac{n-1}{2^{n-1}} + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$$

Hence $$S = 2\sum_{n=1}^{\infty} \frac{n-1}{2^{n-1}} + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2\sum_{n=1}^{\infty} \frac{n}{2^n} + 2$$

Now let $B = \sum_{n=1}^{\infty} \frac{n}{2^n}$

Then $2B = \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{n-1}{2^{n-1}} + 2$

$= B + 2$

$\Rightarrow \quad B = 2$

Thus $S = 2 \cdot 2 + 2 = 6$. $\boxtimes$

5. Since $|a_N - a_0| \leq \sum_{n=0}^{N-1} |a_{n+1} - a_n| < \sum |a_{n+1} - a_n| < \infty$,

we have $|a_N| \leq |a_0| + M \quad \left(\sum |a_{n+1} - a_n| \triangleq M \right)$

$\forall N \in \mathbb{N}$

Then $R = \left(\limsup \sqrt[n]{a_n}\right)^{-1}$

$\geq \left(\limsup \sqrt[n]{|a_0| + M}\right)^{-1} = 1$

$\boxtimes$.