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Math 265H September 7, 2022

F = field if it is a set w/ operations $+$ & $\cdot$ satisfying

i) $x, y \in F$, then $x+y \in F$ ~ Addition

ii) $x+y = y+x \quad \forall\, x,y \in F$

iii) $(x+y)+z = x+(y+z)$, $x,y,z \in F$

iv) $\exists\, 0 \in F \ni x+0 = x \quad \forall\, x \in F$

v) To every $x \in F$ corresponds $-x \in F$ $\ni x+(-x) = 0$.

i) $x, y \in F$, then $xy \in F$ ~ Multiplication

ii) $xy = yx$, $\forall\, x,y \in F$

iii) $(xy)z = x(yz) \quad \forall\, x,y,z \in F$

iv) F contains $1 \neq 0 \ni 1 \cdot x = x$ for every $x \in F$

v) If $x \in F$ and $x \neq 0$ then there exists an element $\frac{1}{x} \in F \ni$

$$x \cdot \frac{1}{x} = 1.$$

Distributive Law: $x(y+z) = xy + xz$

Proposition: F field

a) $x+y = x+z \implies y = z$

b) $x+y = x \implies y = 0$.

c) $x+y = 0 \implies y = -x$

d) $-(-x) = x$

Proof: $x+y = x+z$, so

$$y = y+0 = (-x+x)+y = -x+(x+y)$$
$$= -x+(x+z) = (-x+x)+z = 0+z = z$$

proves a)

To prove b), take $z=0$ in a)

To prove c), take $z=-x$ in a)

c) w/ x replaced by $-x$ implies d)

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The same types of arguments yield the following:

a) If $x \neq 0$ and $xy = xz$, then $y = z$.

b) If $x \neq 0$ and $xy = x$, then $y = 1$.

c) If $x \neq 0$ and $xy = 1$, then $y = \frac{1}{x}$.

d) If $x \neq 0$, then $1/(\frac{1}{x}) = x$.

write out the proofs!

Proposition: The field axioms imply:

a) $0x = 0$

b) If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$

c) $(-x)y = -(xy) = x(-y)$

d) $(-x)(-y) = xy$

work through the proofs!

Ordered fields:

An ordered field is a field F which is also an ordered set, such that

i) $x+y < x+z$ if $x, y, z \in F$ and $y < z$

ii) $xy > 0$ if $x \in F$, $y \in F$, $x > 0$, and $y > 0$.

If $x > 0$, x is positive
If $x < 0$, x is negative

ordering relation

Proposition:

a) If $x > 0$, then $-x < 0$

b) If $x > 0$, $y < z$, then $xy < xz$

c) If $x < 0$ and $y < z$, then $xy > xz$

d) If $x \neq 0$, then $x^2 > 0$. For example, $1 > 0$

e) If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.

Go through the proof

It is easy to check that $\mathbb{Q}$ is
rationals
an ordered field.

Theorem: $\exists$ ordered field $\mathbb{R}$ which has the least upper bound property. Moreover, $\mathbb{Q} \subseteq \mathbb{R}$.
i.e. operations of $\oplus$ & $\odot$ coincide

will not be proved in class (Appendix)

Theorem: a) If $x, y \in \mathbb{R}$, and $x > 0$, then $\exists$ positive integer $n \ni nx > y$.

b) If $x, y \in \mathbb{R}$, $x < y$, then $\exists\ p \in \mathbb{Q} \ni$ $x < p < y$.

Proof:

$$A = \{ nx : n = 1, 2, 3, \dots \}$$

If a) is false, y is an upper bound for A.

But then A has a least upper bound

$\alpha = \sup A$ in $\mathbb{R}$.

Since $x>0$, $\alpha - x < \alpha$, so

$\alpha - x$ is not an upper bound for A.

Hence $\alpha - x < mx$ for some positive integer m, so

$\alpha < x + mx = (m+1)x$, which is a contradiction since α is an upper bound of A.

Proof of b) The whole point is that a) essentially implies b)

Since $x<y$, we have $y-x>0$, so $\exists\, n \ni$

$n(y-x) > 1$.

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Also, $\exists$ $\underset{\text{integers}}{m_1, m_2}$ $\ni$ $m_1 > nx$, $m_2 > -nx$

Then $-m_2 < nx \le m_1$ why?

$\exists$ $\underset{\text{integer}}{m}$ $\ni$ $(-m_2 \le m \le m_1)$

w/ $m-1 \le nx \le m$

It follows that

$$nx < m \le 1 + nx < ny$$

$$\Downarrow$$

$$x < \frac{m}{n} < y$$ as desired

Let's now establish the existence of n'th roots of positive reals.

<u>Theorem:</u> For every real $x>0$ and every $n>0$ there is one and only one positive real y such that $y^n = x$.

Proof: Uniqueness is clear since

$$0<y_1<y_2 \Longleftrightarrow y_1^n < y_2^n$$

why?

$$E = \{ t \in \mathbb{R} : t^n < x \}$$

Let $t = \dfrac{x}{1+x}$, i.e $0 \le t \le 1$.

Hence $t^n \le t \le x$. Thus $t \in E$, and $E \neq \emptyset$

why? why?

If $t > 1+x$, then $t^n \ge t > x$, so $t \notin E$.

Thus $1+x$ is an upper bound of E.

The least upper bound property implies that

$y = \sup E$ exists.

We must show that $y^n < x$ and $y^n > x$ both lead to contradictions.

$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \dots + a^{n-1})$,

why?

$a < b$

so $b^n - a^n < (b-a)\,n\,b^{n-1}$.

Assume $y^n < x$. Choose $h \ni 0 < h < 1$

and $h < \dfrac{x - y^n}{n(y+1)^{n-1}}$. (why can such h be chosen?)

Let $a = y$, $b = y + h$. Then

$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1}$

$< x - y^n$.

Thus $(y+h)^n < x$, and $y + h \in E$.

Since $y+h>y$, we have a contradiction.

Now assume $y^n > x$. Let

$$k = \frac{y^n - x}{n y^{n-1}}.$$

Then $0 < k < y$ (why?)

If $t \geq y - k$,

$$y^n - t^n \leq y^n - (y-k)^n < k n y^{n-1} = y^n - x$$

$$\Downarrow$$

$t^n > x$ and $t \notin E$, so

$y-k$ is an upper bound of E.

But $y - k < y$, which is a contradiction

since $y = \sup(E)$.

Hence $y^n = x$, as claimed.