January 28, 2019

Proposition: Let $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence $R>0$.
Then

a) For each $K\geq 1$ the series

$$\sum_{n=K}^{\infty} n(n-1)\dots(n-K+1)\, a_n (z-a)^{n-K}$$

has radius of convergence R.

Differentiation of power series is not a problem

b) f is infinitely differentiable on $B(a,R)$ & $f^{(K)}(z)$ is given by a), at least in the range $K\geq 1$ & $|z-a|<R$.

c) For $n\geq 0$, $a_n = \frac{1}{n!} f^{(n)}(a)$.

if you do not have Taylor's theorem in $\mathbb{R}$ at the tips of your fingers, please review it carefully!

Proof: The case $K=2$ follows from $K=1$, and so on.
$K=1$: We have assumed that $\overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}} = R^{-1}$. We must compute $\overline{\lim_{n\to\infty}} |n a_n|^{\frac{1}{n-1}} = \overline{\lim_{n\to\infty}} n^{\frac{1}{n-1}} \cdot \overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n-1}}$.

The first limit is 1 (homework). We must show that

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n-1}} = R^{-1}.$$

Let $(R')^{-1} = \overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n-1}}$. Then R' is the radius of convergence of $\sum_{n=1}^{\infty} a_n z^{n-1}$

$$= \sum_{n=0}^{\infty} a_{n+1} z^n.$$

We will play R' against R from this point on.

Observe that $z\sum_{n=0}^{\infty} a_{n+1} z^n + a_0 = \sum a_n z^n$. It follows that if $|z|<R'$, then $\sum |a_n z^n| = |a_0| + \left(\sum |a_{n+1} z^n|\right)\cdot|z| < \infty$, which implies that $\underline{R' \le R}$.

If $|z|<R$ & $|z|\neq 0$, then $\sum |a_n z^n| < \infty$ and

$\sum |a_{n+1} z^n| = |z|^{-1}\sum |a_n z^n| - |z|^{-1}|a_0|$, so

$R \le R'$. This implies that $R=R'$, proving part a).

Proof of part b) For $|z|<R$, let $\underbrace{g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}}_{\text{the presumed derivative}}$,

$\underbrace{S_n(z) = \sum_{k=0}^{n} a_k z^k}_{\text{it is easier to differentiate finite sums}}$

and $\underbrace{R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k}_{\text{remainder}}$

Fix a point $w \in B(\vec{0}, R)$ and fix r with $|w| < r < R$. We must show that $f'(w)$ exists and is equal to $g(w)$.

Let $\delta > 0 \ni B(w, \delta) \subset B(\vec{0}, r)$

arbitrary, except for

Let $z \in B(w, \delta)$. Then

$$\frac{f(z) - f(w)}{z - w} - g(w) = \left[\frac{S_n(z) - S_n(w)}{z - w} - S_n'(w)\right] + \left[S_n'(w) - g(w)\right] + \left[\frac{R_n(z) - R_n(w)}{z - w}\right] \qquad (*)$$

Now we note that the expression with R_n must be the easiest part. We have

$$\frac{R_n(z) - R_n(w)}{z - w} = \frac{1}{z - w} \sum_{k=n+1}^{\infty} a_k (z^k - w^k)$$

$$= \sum_{k=n+1}^{\infty} a_k \left(\frac{z^k - w^k}{z - w}\right)$$ long division should help

Observe that

$$\left|\frac{z^k - w^k}{z - w}\right| = |z^{k-1} + z^{k-2} w + \ldots + z w^{k-2} + w^{k-1}| \le k r^{k-1},$$

$$\text{so } \left|\frac{R_n(z) - R_n(w)}{z - w}\right| \le \underbrace{\sum_{k=n+1}^{\infty} |a_k| k r^{k-1}}_{\text{tail of a convergent sum since } r < R.}$$

This implies that for $\epsilon > 0$ $\exists N_1 \ni$ for $n \ge N_1$

$$\left|\frac{R_n(z) - R_n(w)}{z - w}\right| < \frac{\epsilon}{3}.$$

Also, $\lim s_n'(w) = g(w)$, so $\exists\, N_2$ w/ $|s_n'(w) - g(w)| < \frac{\epsilon}{3}$, whenever $n \geq N_2$. Now let $n = \max\{N_1, N_2\}$ and choose $\delta > 0 \ni \left| \frac{s_n(z) - s_n(w)}{z-w} - s_n'(w) \right| < \frac{\epsilon}{3}$, whenever

$$0 < |z-w| < \delta.$$

Putting everything together in $(*)$ using the triangle inequality, we see that $\left| \frac{f(z)-f(w)}{z-w} - g(w) \right| < \epsilon$.

Part c) follows from part a) (check carefully).

<u>What you should remember</u>: The key to understanding $\frac{f(z)-f(w)}{z-w} - g(w)$ is the formula $(*)$. The first term is small for trivial reasons since s_n is a finite sum. The second term is small by definition of convergence. The third term is <u>the key</u>. The key technical idea there is the expression for $\frac{z^k - w^k}{z-w}$ via long division.

<u>Memory lane</u>: What is the first $\frac{\epsilon}{3}$ argument you remember from analysis? How is it related to the one above? If you do not remember one precisely, you need to open your undergraduate analysis book <u>now</u>. Mathematics is a cumulative discipline.

Corollary: If the series $\sum\limits_{n=0}^{\infty} a_n(z-a)^n$ has radius of convergence $R>0$, then $f(z)=\sum\limits_{n=0}^{\infty} a_n(z-a)^n$ is analytic in $B(a;R)$.

Everybody's favorite example is $\exp(z)=\sum\limits_{n=0}^{\infty}\frac{z^n}{n!}$ since the radius of convergence is ∞. On the other hand, $\sum\limits_{n=0}^{\infty} z^n$ is analytic in $B(\vec{0},1)$.

Proposition: If G is open and connected, and $f: G \to \mathbb{C}$ is differentiable with $f'(z)=0 \ \forall z \in G$, then f is constant. We will prove this result on Wednesday.

An important mental exercise: Lean back in your chair, close your eyes and think through every result covered in this class so far, every proof and every homework problem. If you cannot do this without paper, do not despair! Just go through everything several more times.