ON COMBINATORIAL COMPLEXITY OF CONVEX SEQUENCES

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Abstract. We show that the equation

(*) $$b_{i_1}+b_{i_2}+\dots+b_{i_d}=b_{i_{d+1}}+\dots+b_{i_{2d}}$$

has $O\left(N^{2d-2+2^{-d+1}}\right)$ solutions for any strictly convex sequence $\{b_i\}_{i=1}^N$ without any additional arithmetic assumptions. The proof is based on weighted incidence theory and an inductive procedure which allows us to effectively deal with higher dimensional interactions. We also explain a connection between this problem and the Falconer distance problem in geometric measure theory.

Section 1: Introduction and statement of results

Consider a sequence of real numbers $\{b_i\}_{i=1}^N$. It is a classical problem in number theory to determine the number, $\mathfrak{N}_d=\mathfrak{N}_d(N)$, of solutions of the equation

$$b_{i_1}+b_{i_2}+\dots+b_{i_d}=b_{i_{d+1}}+\dots+b_{i_{2d}}. \tag{1.1}$$

See, for example a book by Nathanson ([Nath96]) or a survey by Heath-Brown ([HB02]), and the references contained therein for a thorough description of various algebraic and combinatorial aspects of this problems. The properties of $\mathfrak{N}_d$ depend on geometric and arithmetic properties of the sequence $\{b_i\}$. For instance, if $b_i=i$, the number of solutions of (1.1) is $\approx N^{2d-1}$.[1]

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[1] Here and throughout the paper the notations $a\lesssim b$, or $a=O(b)$ means that there exists $C>0$ such that $a\le Cb$, and $a\approx b$ means that $a\lesssim b$ and $b\lesssim a$. Similarly, $a\lessapprox b$ with a parameter N means that for every $\epsilon>0$ there exists $C_\epsilon>0$ such that $a\le C_\epsilon N^\epsilon b$. The notation $a\gtrsim b$ ($a\gtrapprox b$) means $b\lesssim a$ ($a\ge C_\varepsilon N^{-\varepsilon}b$).

More interesting bounds are available if the sequence $\{b_i\}$ is a strictly convex, i.e. the differences $b_{i+1} - b_i$ increase with i, and points (i, b_i) lie on a strictly convex curve in $\mathbb{R}^2$. For example, if $b_i = i^2$, and $d \geq 4$, one has $\mathfrak{N}_d \lesssim N^{2d-2}$, and the same estimate with an appropriate power of $\log(N)$ is valid if $d = 2, 3$. Similarly, elementary number-theoretical considerations imply that $\mathfrak{N}_2 \lessapprox N^2$ in the case $b_i = i^k$, $k = 3, 4, \ldots$. The simple examples above show that for a general strictly convex sequence, the best one can hope for is an estimate of the form $\mathfrak{N}_d \lessapprox N^{2d-2}$. There are no examples known to the authors where $\mathfrak{N}_d \gtrsim N^{2d-2+\varepsilon}$, for any $\varepsilon > 0$.

Under additional arithmetic assumptions, the situation may change drastically. For example, it is conjectured that if $d = 2$ and $k \geq 5$, the equation (1.1) only has trivial solutions. Similarly, in higher dimensions it is conjectured that if d is fixed and k is sufficiently large, then (1.1) holds only if $b_{i_l} = b_{i_{l+d}}$, up to permutation. See [HB02] and the references contained therein. These conjectures and the known positive results in this context show that the bound $\mathfrak{N}_d \lessapprox N^{2d-2}$ can at times be significantly improved. Another natural, non-integer example illustrating the power of arithmetic considerations can be constructed as follows. Let $\{k_i\}_{i=1}^N$ be a sequence of square free positive integers, and $b_i = \sqrt{k_i}$. A theorem due to Besicovitch ([Bes40]) says that these numbers are linearly independent over $\mathbb{Q}$. It follows that in this case $\mathfrak{N}_d \approx N^d$.

The main thrust of this paper is to obtain the bound on $\mathfrak{N}_d$ under the assumption of strict convexity *without any additional arithmetic assumptions.* It is reasonable to conjecture that for every strictly convex sequence $\{b_i\}$, $\mathfrak{N}_d \lessapprox N^{2d-2}$. We prove that this estimate is asymptotically true with an exponentially vanishing error as $d \to \infty$. More precisely, we show (see Theorem 1 below) that $\mathfrak{N}_d \lesssim N^{2d-2+2^{-d+1}}$. Konyagin ([Ko00]) proved[2] this estimate in the two-dimensional case. His approach is based on the Szemerédi-Trotter incidence theorem. A part of his motivation is a result due to Karatsuba ([Kar98]) which says that if $\{b_j\}_{j=1}^N$ is a convex sequence and $\Delta(N) = \mathfrak{N}_2(N)/N^3$, then

$$\int_0^1 \left| \sum_{j=1}^N \gamma(x) e^{2\pi i b_j \beta} \right| d\beta \gtrsim \Delta^{-\frac{1}{2}}(N), \tag{1.2}$$

for any $\gamma(x)$ with $|\gamma(x)| = 1$. The issue was further explored by Garaev ([Gar00]) who gave a direct (without using the methods of combinatorial geometry) proof of Konyagin's estimate, which may well lead to further progress on this problem.

When $d > 2$, one is naturally led to consider higher dimensional incidence theory, where significant complications arise. In this paper we address this issue by a introducing an appropriate weighted version of the Szemerédi-Trotter incidence theorem and induction by dimension. The weights come into play for $d > 2$, and dealing with them requires an effective divide-and-conquer approach, which is one of the central points of this paper.

Statement of results. Fix a convex sequence $\{b_i\}_{i=1}^N$, N large, let $\mathcal{N} \equiv \{1, 2, \ldots, N\}$), and let $f : \mathbb{R} \to \mathbb{R}$ be a fixed strictly convex function such that $f(i) = b_i$. Let $\mathcal{B} =$

[2]This estimate is also implicit in the results of the paper of Elekes et al., [ENR99].

$\{b_1, \ldots, b_N\}$. Uniformity of the ensuing estimates is understood in the sense that that none of the constants, hidden in the estimates, and different for different d, depend on the specific sequence $\{b_i\}_{i\in\mathcal{N}}$ or f.

We need the following notation. The bounds for the quantities $\mathfrak{N}_d$ can be obtained by studying the set

$$(1.3)\qquad \mathcal{C}_d \;\equiv\; \underbrace{\mathcal{B}+\ldots+\mathcal{B}}_{d\text{ times}} \;=\; \{c|\, c = b_{i_1}+\ldots+b_{i_d},\, \forall(i_1,\ldots,i_d)\in\mathcal{N}^d\}.$$

More precisely, for some $c\in\mathcal{C}_d$ we shall refer to the quantity

$$(1.4)\qquad \nu(c) \;=\; \frac{|\{(i_1,\ldots,i_d)\in\mathcal{N}^d:\, b_{i_1}+\ldots+b_{i_d}=c\}|}{d!}$$

as the *weight* of c. [3]

We have

$$(1.5)\qquad \sum_{c\in\mathcal{C}_d}\nu(c) = \frac{1}{d!}N^d,$$

the right-hand side being the *net weight*, and

$$(1.6)\qquad \mathfrak{N}_d = \sum_{c\in\mathcal{C}_d}\nu^2(c).$$

Suppose the set $\mathcal{C}_d = \{c_1, c_2, \ldots, c_t, \ldots\}$ has been ordered by non-increasing weight. We shall see that in order to estimate $\mathfrak{N}_d$, it is sufficient to estimate the minimum cardinality $|\mathcal{C}_d|$ along with a majorant for the *weight distribution* function, i.e. a decreasing function $\mathfrak{n}(t)$ such that $\mathfrak{n}(t)\geq\nu(c_t)$. The inverse, also decreasing function $\mathfrak{n}^{-1}$ would provide the bound[4]

$$(1.7)\qquad \mathfrak{n}^{-1}(s) \;\geq\; |\;\mathcal{C}_{d,s}\equiv\{c\in\mathcal{C}_d:\,\nu(c)\geq s\}|.$$

Our main result is the following.

Theorem 1. *For $d\geq 2$, let $\alpha = 2(1-2^{-d})$, $\beta = d-\frac{4}{3}(1-2^{-d})$. Then*

$$(1.8)\qquad |\mathcal{C}_d| \;\gtrsim\; N^\alpha,$$

$$(1.9)\qquad \mathfrak{n}(t) \;\lesssim\; N^\beta t^{-1/3},$$

[3] In (1.4) above and throughout the paper the notation $|\cdot|$ denotes the cardinality of a (finite) set $\cdot$.

[4] Note that $\mathfrak{n}^{-1}$ is simply the distribution function for $\mathfrak{n}$ in the measure-theoretical sense.

$$\mathfrak{N}_d \lesssim N^{2d-\alpha}. \tag{1.10}$$

Remark. The main estimates of the theorem are (1.9) and (1.10), the latter one being the estimate on the number of solutions of the diophantine equation (1.1). The estimate (1.8) on cardinality of the sumset $\mathcal{C}_d$ has been included in the statement for the sake of completeness. This estimate is implicit in [ENR99], Ch. 4 and is based on the repeated application of the classical Szemerédi-Trotter theorem. It is insufficient, however, to obtain estimates (1.9) and (1.10), dealing the weight distribution within the set $\mathcal{C}_d$. This necessitated the development of certain weighted incidence estimates included in the sequel.

In Section 5 of this paper, we improve the lower bound on $\mathcal{C}_d$ under addition curvature assumptions using Fourier analysis.

The proof of Theorem 1 applies to a more general class of problems, such as counting the number of integer solutions of the equation of the form $F(i_1, \dots, i_d) = F(i_{d+1}, \dots, i_{2d})$, where F is a strictly convex function of d variables which satisfies additional, fairly mild, assumptions on its coordinate lower dimensional sections.

It is interesting to contrast Theorem 1 with the following well-known result from additive number theory, due to Freiman ([Frei73]).

Freiman's theorem. *Let $\mathcal{B} \subset \mathbb{Z}$ have cardinality N, and suppose that $|\mathcal{B}+\mathcal{B}| \leq CN$. Then $\mathcal{B}$ is contained in a proper s-dimensional progression*[5] *P of length at most KN, where s and K depend only on C.*

The estimate (1.8) gives us $|\mathcal{B}+\mathcal{B}| \gtrsim N^{\frac{3}{2}}$, and it is conjectured that the right bound is $\gtrapprox N^2$. From the point of view of Freiman's theorem, a power estimate is reasonable: if $\{b_j\}_{j=1}^N$ happens to be a sequence of integers, the strict convexity assumption guarantees that $\mathcal{B}$ is not contained in an s-dimensional arithmetic progression. However, a tighter connection with Freiman's theorem, explaining the above exponent would be quite valuable. See [Gr02] for a description of Freiman's theorem and related results on the structure of sumsets. See also [KT99] and [KT01] for the description of related ideas in the context of the Kakeya problem and the Falconer conjecture. Also see [Bo01] for the description of related issues in the context of Λ_p sets.

In Section 5 below we give a proof of a weaker though more robust version of (1.8) using Fourier analysis and results related to the Falconer distance problem. This approach allows one to obtain estimates on the size of $\mathcal{C}_d$ with an additional restriction that elements of this set be separated on a scale depending on N. Similar connections are explored in [HI2003], [IL2003], and [IL2004]. See also [Mag02] for a connection between diophantine equations and ergodic theory.

In the case when the sequence $\{b_i\}_{i\in\mathcal{N}}$ is integer-valued, the estimate (1.10) implies an

[5] The set $P = \left\{x_0 + \sum_{j=1}^s \lambda_j x_j : 0 \leq \lambda_j < l_j\right\}$ with $x_0, \dots, x_s \in \mathbb{Z}$ and $\lambda_1, \dots, \lambda_s; l_1, \dots, l_s \in \mathbb{Z}^+$, is said to be an s-dimensional arithmetic progression of length $l = \prod_{j=1}^s l_j$. P is proper if $l = |P|$.

estimate for the L_p-norm of trigonometric polynomials with frequencies in $\{b_i\}_{i\in\mathcal{N}}$, i.e the Dirichlet kernel associated with the sequence $\{b_i\}$.

Corollary 2. *If $\{b_i\}_{i\in\mathcal{N}} \subset \mathbb{Z}$, let*

$$f_N(\theta) = \sum_{j=1}^{N} e^{2\pi i b_j \theta}.$$

Then

$$\|f_N\|_{2d} \equiv \left(\int_0^{2\pi} |f_N(\theta)|^{2d} d\theta\right)^{\frac{1}{2d}} = O\left(N^{1-\frac{1-2^{-d}}{d}}\right). \tag{1.11}$$

Remark. By expanding the square we see that (1.10) and (1.11) are essentially identities when $d = 1$. When $d > 1$ observe that (1.11) is much stronger than the estimate that can be obtained by interpolating the case $d = 1$ and $d = \infty$ using Holder's inequality.

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Section 2: Incidence theorems

As we mention in the introduction, the main tool used in [ENR99] and [Ko00] is the theorem of Szemerédi and Trotter ([ST83]) bounding the number of incidences between a collection of points and straight lines in the Euclidean plane. The theorem was extended to the case of points and hyper-planes or spheres (with some natural restrictions on the arrangements) by Clarkson et al. ([CEGSW90]), see also the references therein. It provides a powerful tool for solving problems in geometric combinatorics. See also the books by Pach and Agarwal ([PA95]) and Matoušek ([Ma02]) for an exhaustive description of this subject and related issues. It was observed by Székely ([Sz97]) that the geometric graph theory can deliver a short formal proof of the following statement of the Szemerédi-Trotter incidence theorem in dimension two, with the set of lines generalized to a class of curves satisfying generic intersection hypotheses.[6] From this point on, we shall use the terms "lines" and "curves" interchangeably.

Theorem 3 [Szemerédi-Trotter, Székely]. *Let $(\mathcal{L},\mathcal{P})$ be an arrangement*[7] *of m curves and n points in $\mathbb{R}^2$. Suppose that no more than μ curves pass through any pair of points of $\mathcal{P}$ and that any two curves of $\mathcal{L}$ intersect at no more than ν points of $\mathcal{P}$. Then the total number of incidences*

$$I = |\{(l,p) \in \mathcal{L}\times\mathcal{P} : p \in l\}| \lesssim (\mu\nu)^{\frac{1}{3}}(mn)^{\frac{2}{3}} + m + \mu n. \tag{2.1}$$

[6] There is nothing to prevent one from generalizing the ambient space $\mathbb{R}^2$ to a general two-manifold of finite genus.

[7] By the arrangement we further mean an embedding, or drawing of the curves and points in the plane.

In the case of points and straight lines, $\mu = \nu = 1$. Let us further refer to this as the *simple intersection* case, where

$$I \lesssim (mn)^{\frac{2}{3}} + m + n. \tag{2.1a}$$

The quantities μ and ν, if bounded independently of m and n, may be viewed as constants which get absorbed into the $\lesssim$ signs. Consequently, the assumption that any two curves intersect at a *finite* (i.e. independent of m, n) number of points, and that through any two points there pass no more than a *finite* number of curves mean essentially that one has the estimate given by (2.1a) rather than the one given by (2.1).

In the simple intersection case, the number of incidences I for the arrangement $(\mathcal{L}, \mathcal{P})$ can be expressed in terms of the counting function δ_{lp} for the arrangement. More precisely,

$$I = \sum_{l\in\mathcal{L},p\in\mathcal{P}} \delta_{lp},$$

where $\delta_{lp} = 1$ if $p \in l$, and 0 otherwise.

For our applications it will be useful to give an essentially equivalent formulation of Theorem 3. We shall refer to μ and ν as maximum weights. The numbers m and n shall will referred to as net weights.

For the remainder of the paper, let us always consider the simple intersection case. However, given some μ, ν, we assign to each line $l \in \mathcal{L}$ and each point $p \in \mathcal{P}$ the weights $\mu(l) \in \{1, \dots, \mu\}$ and $\nu(p) \in \{1, \dots, \nu\}$, respectively (although the individual weights certainly don't have to be integer), so that

$$\sum_{l\in\mathcal{L}} \mu(l) = m, \quad \sum_{p\in\mathcal{P}} \nu(p) = n.$$

Let us call such a weight assignment a *weight distribution with the maximum weights* (μ, ν) *and the net weights* (m, n). A single pair $(l, p) \in \mathcal{L}\times\mathcal{P}$ will have the weight $w_{lp} = \mu(l)\nu(p)\delta_{lp}$. Let the number of weighted incidences be defined by

$$I \equiv \sum_{l\in\mathcal{L},p\in\mathcal{P}} w_{lp}. \tag{2.2}$$

A weighted version of Theorem 3 is formulated as follows.

Theorem 3a. *For all simple intersection arrangements* $(\mathcal{L}, \mathcal{P})$ *with net weight* (m, n)*, for all weight distributions with maximum weights* (μ, ν)*, one has*

$$I \lesssim (\mu\nu)^{\frac{1}{3}} (mn)^{\frac{2}{3}} + \nu m + \mu n. \tag{2.3}$$

Theorem 3a is shown to straightforwardly follow from Theorem 3 in the Appendix, after a simple weight rearrangement argument. Note that for the right hand side of (2.3) one has

$$(\mu\nu)^{\frac{1}{3}}(mn)^{\frac{2}{3}} + \nu m + \mu n = \mu\nu\left[\left(\frac{m}{\mu}\frac{n}{\nu}\right)^{\frac{2}{3}} + \frac{m}{\mu} + \frac{n}{\nu}\right], \tag{2.4}$$

which suggests that the maximum number of weighted incidences is achieved when there are $\frac{m}{\mu}$ lines and $\frac{n}{\nu}$ points with uniformly distributed weights.

Note that unless the weights are distributed uniformly, neither $|\mathcal{L}|$, nor $|\mathcal{P}|$ enter the estimate (2.3). Suppose, for example that it is known that $|\mathcal{L}| \gg m\mu^{-1}$, that is the majority of the lines have weights, smaller than μ. Can one use a divide-and-conquer approach to take advantage of the average weight $\bar{\mu} = \frac{m}{|\mathcal{L}|}$ in the formula (2.3)? As far as the equation (1.1) is concerned, the answer is yes. Ii is stated by Lemma 6, which is central for the proof of Theorem 1. Note that the maximum weight for the elements of $\mathcal{C}_d$ can be bounded trivially by N^{d-1}, or, less trivially, by $N^{d\frac{d-1}{d+1}}$ using the following theorem of Andrews ([An63]) (see also [BL98]).

Theorem 4 [Andrews]. *The number of vertices of a convex lattice polytope*[8] *in* $\mathbb{R}^d$ *of volume* V *is* $O\left(V^{\frac{d-1}{d+1}}\right)$.

Remark. The proof of Theorem 1 below, driven by the weighted incidence technology, also yields an upper bound for the number of solutions of the following generalization of the problem considered in this paper:

$$b_{i_1} + b_{i_2} + \ldots + b_{i_d} = a_{j_1} + a_{j_2} + \ldots + a_{j_D}$$

of equation (1.1), where $\{a_i\}_{i\in\mathcal{N}}$ is another convex sequence and $2 \leq d \leq D$.

Section 3: Proof of Theorem 1

The proof is by induction on d, starting from $d = 2$. Let

$$\gamma = \{(x, f(x)) : x \in [1, N]\}, \text{ and } \gamma_{\mathcal{N}} = \{(i, f(i)) : i \in \mathcal{N}\} \tag{3.1}$$

The case $d = 2$:

Lemma 5. *We have*

$$|\mathcal{C}_2| \gtrsim N^{3/2}, \tag{3.2}$$

[8] A lattice polytope is a polytope with vertices in the integer lattice $\mathbb{Z}^d$.

and

$$|\mathcal{C}_s| = |\{c \in \mathcal{C}_2 : \nu(c) \geq s\}| \lesssim N^3 s^{-3}. \tag{3.3}$$

Proof. Define

$$\mathcal{N}_2 \equiv \mathcal{N} + \mathcal{N}.$$

Consider the set of points $\mathcal{P} = \mathcal{N} \times \mathcal{B} + \gamma_{\mathcal{N}} = \mathcal{N}_2 \times \mathcal{C}_2$ and the set of curves $\mathcal{L} = \gamma + \mathcal{N}_2 \times \mathcal{B}$. Convexity implies that the arrangement $(\mathcal{L}, \mathcal{P})$ satisfies the simple intersection condition.

Since $|\mathcal{P}| \lesssim N^2$, the number of incidences I for this arrangement can be estimated by the non-linear term in formula $(2.1a)$, i.e

$$I \lesssim N^{4/3}(|\mathcal{P}|)^{2/3}. \tag{3.4}$$

On the other hand, each curve of $\mathcal{L}$ contains at least N points of $\mathcal{P}$ (that is why $\mathcal{P}$ has been taken as $\mathcal{N}_2 \times \mathcal{B}$ rather than simply $\mathcal{N} \times \mathcal{B}$). It follows that $I \gtrsim N^3$, and

$$N|\mathcal{C}_2| \approx |\mathcal{P}| \gtrsim N^{5/2},$$

which implies (3.2).

Let $\mathcal{P}_s = \{p \in \mathcal{P} : \nu(p) \geq s\}$, where $\nu(p)$ is the number of curves of the arrangement $\mathcal{L}$ intersecting at the point p. Applying the estimate (3.4) for the number of incidences for the arrangement $(\mathcal{L}, \mathcal{P}_s)$, with $|\mathcal{P}_s|$ substituting $\mathcal{P}$ and comparing it with the lower bound $s|\mathcal{P}_s|$, we get

$$|\mathcal{C}_s| \approx N^{-1}|\mathcal{P}_s| \lesssim N^3 s^{-3},$$

which implies (3.3).

In view of (3.2) let $\bar{\nu} = \sqrt{N}$ be the (approximate) upper bound for the average weight per element of $\mathcal{C}_2$. By (3.3) the weight distribution function in the (ordered) set $\mathcal{C}_2$ satisfies

$$\nu(c_t) \lesssim \mathfrak{n}(t) = N t^{-1/3}.$$

It follows that for the set $\mathcal{C}_{2,\bar{\nu}}$, containing those $O(N^{3/2})$ elements of $\mathcal{C}_2$, whose weights may exceed $\bar{\nu}$, one has

$$\sum_{c \in \mathcal{C}_{2,\bar{\nu}}} \nu(c)^2 \lesssim N^2 \int_1^{N^{3/2}} t^{-2/3} dt \approx N^{5/2}. \tag{3.5}$$

On the other hand, for the complement $\mathcal{C}^c_{2,\bar{\nu}}$ of $\mathcal{C}_{2,\bar{\nu}}$ in $\mathcal{C}_2$, where the weight does not exceed $\bar{\nu}$, one has

$$\sum_{c \in \mathcal{C}^c_{2,\bar{\nu}}} \nu(c)^2 \lesssim \bar{\nu} \sum_{c \in \mathcal{C}_2} \nu(c) \approx N^{5/2}, \tag{3.6}$$

as the total weight of $\mathcal{C}_2$ is approximately N^2. This proves the formulae $(1.8-1.10)$ for $d=2$.

Remark. Formulas (3.5) and (3.6) are easily understood in the sense of the defining formulae $(1.4-1.7)$, dealing with the weight distribution function $\nu(c)$ in the set $\mathcal{C}_2$, with the known L_1 norm of ν, the net weight. The quantity $\mathfrak{N}_2$ in question, which is the square of the L_2 norm of $\nu(c)$ can be bounded as follows. One naturally partitions the domain $\mathcal{C}_2$ in two subsets. In the first subset, where $\nu(c)$ is likely to exceed an upper bound for its average $\bar{\nu}$ (obtained as the net weight divided by the lower bound for $|\mathcal{C}_2|$) one uses the (strictly decreasing, concave) majorant $\mathfrak{n}(t)$ for $\nu(c_t)$ and gets (3.5). The integral of $\nu^2(c)$ over the second subset, where $\nu(c)\lesssim\bar{\nu}$ is bounded by the product of the L_1 norm of the function $\nu(c)$ ($\approx N^2$) and the L_∞ norm $\bar{\nu}=\sqrt{N}$ over the subset: this is (3.6). The same tactics is used in the following main part of the proof. The tricky part there is getting the tight enough majorant $\mathfrak{n}(t)$.

The case $d\Rightarrow d+1$:

In order to characterize the weight distribution function $\nu(c)$, for $c\in\mathcal{C}_{d+1}$, consider the equation

$$f(i_1)\;+\;[\,f(i_2)+\ldots+f(i_{d+1})]\;=\;c. \tag{3.7}$$

Let $u\in\mathcal{C}_d$. Extend (3.7) to the system of equations

$$\begin{cases} f(i_1)+u=c, \\ \quad i_1\;+j=k, \end{cases}\qquad \forall\,(i_1,j,k,u,c)\in\mathcal{N}\times\mathcal{N}_2\times\mathcal{N}_2\times\mathcal{C}_d\times\mathcal{C}_{d+1}. \tag{3.8}$$

Note that $\mathcal{C}_{d+1}$ is considered as a set, rather than multi-set. The elements of the set $\mathcal{C}_d=\{u_1,u_2,\ldots,u_t,\ldots\}$ are endowed with non-increasing weights, with some weight distribution function $\mu(u)$. In particular, the L_1 norm of $\mu(u)$, $\|\mu\|_1$, over $\mathcal{C}_d$ is $O(N^d)$, the L_∞ norm is $O(N^{d\frac{d-1}{d+1}})$, by the aforementioned Andrews theorem, and (by the induction assumption) that there is a majorant

$$\mu(u_t)\;\lesssim\;\mathfrak{m}(t)\;=\;N^{\beta_d}t^{-1/3}, \tag{3.9}$$

where $\beta_d=d-\frac{4}{3}(1-2^{-d})$. There is also the estimate (1.8) for the cardinality of $\mathcal{C}_d$, enabling one to introduce the upper bound for the average weight $\bar{\mu}$ in $\mathcal{C}_d$ as follows:

$$\frac{\|\mu\|_1}{|\mathcal{C}_d|}\;\lesssim\;\bar{\mu}\;=\;N^{d-\alpha_d}, \tag{3.10}$$

with $\alpha_d=2-2^{-d+1}$.

The number of solutions of (3.7) is not smaller than the number of solutions of (3.8), divided by N. The number of solutions of (3.8) can be estimated in terms of the number of weighted incidences I between the weighted set $\mathcal{L}$ of the curves, given by the translations γ_{ju} of the curve γ defined by (3.1), by the elements of $\mathcal{N}_2 \times \mathcal{C}_d$ and the set $\mathcal{P} = \mathcal{N}_2 \times \mathcal{C}_{d+1}$. Thus the problem essentially boils down to the same scheme as it was in the case $d = 2$, except that *weighted* incidences should be counted in order to verify estimates (1.9) and (1.10). Verification of (1.8) is easier: it requires only the available (through the induction assumption) lower bound $|\mathcal{C}_d| \gtrsim N^{\alpha_d}$ and the use of (2.1a) and was done in [ENR99] (and in the case $d = 2$, see (3.4) and the formula that follows it). The corresponding estimate (3.12) can be also obtained via the ensuing Lemma 6, which we have chosen to do in order to show that the lemma by itself is tight.

In our consideration, each translated curve γ_{ju} would inherit the weight $\mu(u)$ of the corresponding element $u \in \mathcal{C}_d$. The following lemma is central for the rest of the proof.

Lemma 6. *Under the assumptions* (3.9) *and* (3.10) *on the weight distribution function* $\mu(u)$ *in the set* $\mathcal{C}_d$*, the number of incidences for the above defined arrangement* $(\mathcal{L}, \mathcal{P})$*, describing the solutions of the system* (3.8) *is bounded as follows:*

$$I \lesssim \bar{\mu}^{1/3} N^{2(d+1)/3} (N|\mathcal{C}_{d+1}|)^{2/3}. \tag{3.11}$$

Lemma 6 shows that in order to count the weighted incidences in the arrangement $(\mathcal{L}, \mathcal{P})$, instead of the maximum weight upper bound $O(N^{d\frac{d-1}{d+1}})$ in the set $\mathcal{L}$ (transferred from $\mathcal{C}_d$), given by the Andrews theorem, one can use the formula (2.3) with the (smaller) average weight majorant $\bar{\mu}$ for the quantity μ, as well as (naturally) the net weight $m = N^{d+1}$ and the maximum point weight $\nu = 1$ in $\mathcal{P}$. The proof of Lemma 6 is given in the next section. We shall now use it to complete the proof of Theorem 1.

Assuming Lemma 6, we compare its estimate (3.11) with the fact that on each curve of $\mathcal{L}$ there lies at least N points of $\mathcal{P}$, thus $I \geq N^{d+2}$, because N^{d+1} is approximately the net weight of $\mathcal{L}$. Comparing the powers of N, we get[9]

$$|\mathcal{C}_{d+1}| \gtrsim N^{2-2^{-d}} = N^{\alpha_{d+1}}. \tag{3.12}$$

This leads us to define the upper bound for the average weight in $\mathcal{C}_{d+1}$

$$\bar{\nu} = N^{d+1-\alpha_{d+1}}. \tag{3.13}$$

Let $\mathcal{P}_s = \{p \in \mathcal{P} : \nu(p) \geq s\}$, where $\nu(p)$ is now defined as the total weight of all the curves of the arrangement $\mathcal{L}$ intersecting at the point p. Clearly $\mathcal{P}_s = \mathcal{N}_2 \times \mathcal{C}_{d+1,s}$, where $\mathcal{C}_{d+1,s}$ is the subset of $\mathcal{C}_{d+1}$, consisting of all those elements whose weight is not smaller than s. In order to estimate $|\mathcal{C}_{d+1,s}|$, formula (2.1a) cannot be used, as one has to take

[9] As we have mentioned earlier, one can do without Lemma 6 in order to get (3.12). Namely, if I is the number of (non-weighted in this case) incidences for the arrangement $(\mathcal{L}, \mathcal{P})$, then similarly to the case $d = 2$, one has $N^2|\mathcal{C}_d| \lesssim I \lesssim (N|\mathcal{C}_{d+1}|)^{2/3}(N|\mathcal{C}_d|)^{2/3}$, which implies the bound (3.12) for $|\mathcal{C}_{d+1}|$, under the assumption that $|\mathcal{C}_d| \gtrsim N^{\alpha_d}$. This was done in [ENR].

into account the individual weight of each curve $\gamma_{ju} \in \mathcal{L}$, passing through the given point p. Instead, weighted incidences have to be dealt with, and Lemma 6 enables one use the average weight $\bar{\mu}$ in the estimate, rather than the maximum weight $\mu \gg \bar{\mu}$.

In view of this, we proceed by comparing the lower bound $sN|\mathcal{C}_{d+1,s}|$, for the number of weighted incidences for the arrangement $(\mathcal{L}, \mathcal{P}_s)$ with (3.11), in which $|\mathcal{C}_{d+1,s}|$ substitutes $\mathcal{C}_{d+1}$. This yields

$$|\mathcal{C}_{d+1,s}| \lesssim N^{1-\alpha_d} \left(\frac{N^d}{s}\right)^3. \tag{3.14}$$

If $s = \bar{\nu}$, defined by (3.10), it follows that

$$|\mathcal{C}_{d+1,\bar{\nu}}| \lesssim N^{\alpha_{d+1}}, \tag{3.15}$$

which is the same as the right-hand side in (3.12), and complies with (1.8). Inversion of (3.14) yields the majorant for the weight distribution function $\nu(c)$ for $c \in \mathcal{C}_{d+1}$:

$$\nu(c_t) \lesssim \mathfrak{n}(t) = N^{d-\frac{1-2^{-d+1}}{3}} t^{-1/3} = N^{\beta_{d+1}} t^{-1/3}, \tag{3.16}$$

as is claimed by (1.9).

The final step of the proof follows the remark at the end of the $d = 2$ section. Namely one partitions

$$\mathcal{C}_{d+1} = \mathcal{C}_{d+1,\bar{\nu}} \cup \mathcal{C}^c_{d+1,\bar{\nu}},$$

the first piece containing "heavy" elements, and estimates

$$\sum_{c \in \mathcal{C}^c_{d+1,\bar{\nu}}} \nu^2(c) \lesssim N^{d+1}\bar{\nu} = N^{2(d+1)-\alpha_{d+1}}, \tag{3.17}$$

as well as

$$\sum_{c \in \mathcal{C}_{d+1,\bar{\nu}}} \nu^2(c) \lesssim N^{2\beta_{d+1}} \int_1^{N^{\alpha_{d+1}}} t^{-2/3} dt \approx N^{2(d+1)-\alpha_{d+1}}. \tag{3.18}$$

The estimates (3.17) and (3.18) are consistent with (1.10). Thus the proof of Theorem 1 is complete.

Section 4: Proof of Lemma 6

The objective is to partition the set

$$\mathcal{C}_d = \bigcup_{i=0}^{M} \mathcal{C}_{d,\mu_i} \tag{4.1}$$

into M (a fairly large number of) pieces, trying to make each one of them as large as possible, yet having control over the number of weighted incidences it can possibly be responsible for. For simplicity let

$$\mathcal{C} \equiv \mathcal{C}_d, \; \mathcal{C}_i \equiv \mathcal{C}_{d,\mu_i}.$$

The partition is required to have the following property:

$$\mu(c) \lesssim \mu_i, \; \forall c \in \mathcal{C}_i, \tag{4.2}$$

where the strictly decreasing sequence μ_i will start out from

$$\mu_0 = N^{d\frac{d-1}{d+1}},$$

(the maximum weight granted by the Andrews theorem[10]) and go down geometrically to the average weight $\bar{\mu}$ in $\mathcal{C}$, specified in (3.10). The number M is chosen in such a way that μ_M gets sufficiently close to $\bar{\mu}$, so that the effect of the difference between them can be swallowed by the $\lesssim$ symbol. The sequence $\{\mathcal{C}_i\}$ will be constructed, using the weight distribution majorant (3.9).

By the general estimate (2.3) of Theorem 3a in order to prove the lemma, it suffices to show that

$$\left(\tilde{I} \equiv \sum_{i=0}^{M} \mu_i^{\frac{1}{3}} m_i^{2/3}\right) \lesssim \left(\bar{I} \equiv \bar{\mu}^{1/3} m^{2/3}\right), \tag{4.3}$$

where $m = N^d$ is the net weight of $\mathcal{C}$, and m_i is the net weight of each $\mathcal{C}_i$, for $i = 0, \ldots, M$. Indeed, it easy to see that the linear terms coming from the bound (2.3) are irrelevant: the first linear term is N^{d+1}, being the total weight of the set of lines $\mathcal{L}$, defined by the system of equations (3.8); the second linear term, relative to the set $\mathcal{C}_i$ will be equal to $\mu_i N^{d+1}$, as N^{d+1}, is also the net weight of the set of points $\mathcal{P}$, defined by the system of equations (3.8). Both terms will be dominated by the quantity $\tilde{I}$ defined by (4.3) by construction. This is shown explicitly in the end of the proof.

The weights m_i are to be estimated via μ_i, using the inverse formula for the majorant (3.9), i.e

$$|\{c \in \mathcal{C} : \mu(c) \geq s\}| \lesssim \mathfrak{m}^{-1}(s) = N^{3\beta_d} s^{-3}, \;\; \beta_d = d - \frac{4}{3}(1 - 2^{-d}). \tag{4.4}$$

Note that the majorant (3.9) is good for nothing as far as the elements c of $\mathcal{C}$, such that $\mu(c) \lesssim \bar{\nu}$ are concerned. Indeed, a calculation yields

$$\int_{\bar{\mu}}^{\infty} \mathfrak{m}^{-1}(s)\, ds \approx m,$$

[10]In fact, one can see from the proof that the use of the Andrews theorem is unnecessary: one can simply start out with $\mu_0 = N^d$, which is the net weight of $\mathcal{C}$.

where $m \approx N^d$ is the net weight of $\mathcal{C}$.

Also for the terms in the sum in the right-hand side of (4.3) denote

$$\tilde{I}_i \;\equiv\; \mu_i^{\frac{1}{3}} m_i^{2/3}.$$

The sets $\mathcal{C}_i$ and the number M are to be chosen such that

$$\tilde{I}_i \;\lesssim\; N^{-\varepsilon_i} \bar{I}, \tag{4.5}$$

for some geometrically vanishing sequence of small positive numbers $\{\varepsilon_i\}_{i=0}^{M-1}$. This prompts the choice

$$M \;\approx\; \log\log N, \tag{4.6}$$

as then $\varepsilon_{M-1} \approx \varepsilon_0\, e^{-\log\log N} \;\approx\; \frac{1}{\log N}$, so for a sufficiently small, yet $O(1)$ value of ε_0,

$$N^{\varepsilon_{M-1}} \;\approx\; 1 \;\text{ and }\; \sum_{i=0}^{M-1} N^{-\varepsilon_i} \approx \int_1^{\log\log N} N^{-\varepsilon_0\, exp(-t)} dt \lesssim \int_1^\infty \frac{e^{-z}}{z} dz \;\approx\; 1. \tag{4.7}$$

Let us describe the first step of the construction. Let a number δ_0 be defined via $\mu_0 = N^{\delta_0} \bar{\mu}$. Define the weight m_0 of the set $\mathcal{C}_0$ implicitly via (4.5), i.e.

$$\mu_0^{1/3} m_0^{2/3} \;\approx\; N^{-\varepsilon_0} \bar{\mu}^{1/3} m^{2/3},$$

which yields

$$m_0 = N^{-\frac{1}{2}(3\varepsilon_0 + \delta_0)}\, m. \tag{4.8}$$

Then the weight of any element c in the complement $\mathcal{C}_0^c$ of $\mathcal{C}_0$ in $\mathcal{C}$ should be bounded from above by some quantity μ_1, which can be defined implicitly from

$$\int_{\mu_1}^\infty \mathfrak{m}^{-1}(s)\, ds \;=\; m_0.$$

This yields

$$\mu_1 = \bar{\mu} N^{\delta_1},\; \delta_1 = \frac{1}{4}(3\varepsilon_0 + \delta_0). \tag{4.9}$$

Clearly, for ε_0 small enough, say $\varepsilon_0 = \frac{1}{9}\delta_0$, one has $\delta_1 \le \frac{1}{3}\delta_0$.

The procedure is now repeated for the set $\mathcal{C}_0^c$, where the maximum weight is bounded in terms of μ_1, rather than μ_0, which will result in some set $\mathcal{C}_1$ having been pulled out of it, such that the maximum weight in the complement of $\mathcal{C}_1$ in $\mathcal{C}_0^c$ is bounded in terms of some $\mu_2 \ll \mu_1$, and so on. After having done it $M-1$ times, the set $\mathcal{C}$ will be partitioned, according to (4.1), where the last member of the partition $\mathcal{C}_M$ is the complement of the union $\bigcup_{i=0}^{M-1} \mathcal{C}_i$ in $\mathcal{C}$. For $i = 1, \ldots, M$ the maximum individual element weight in $\mathcal{C}_i$ is bounded similarly to (4.9), namely

$$\mu_i = \bar{\mu}\, N^{\delta_i}, \;\; \delta_i = \frac{1}{4}(3\varepsilon_{i-1} + \delta_{i-1}). \tag{4.10}$$

Then, if the quantities ε_i vanish geometrically, with the ratio exceeding, for instance 9, then $\delta_i \le \delta_0 e^{-i}$, $i = 1, \ldots, M$.

By construction, each set $\mathcal{C}_i$, for $i = 0, \ldots, M-1$ would create the number of weighted incidences I_i for the arrangement $(\mathcal{L}, \mathcal{P})$, bounded (by (3.11) and (4.5)) by

$$I_i \;\lesssim\; N^{2d+2^{-d}-\varepsilon_i+1}.$$

Note that in comparison with (1.10) one has $d \to d+1$, which accounts for an extra N here, as the quantity $\mathfrak{N}_d$ equals N^{-1} times the number of incidences for the arrangement $(\mathcal{L}, \mathcal{P})$, introduced apropos of the system of equations (3.8), rather than equation (3.7).

As each $\varepsilon_i \le 1$, the right hand side of the last expression will exceed the maximum for the linear term in the estimate (2.3), applied to the arrangement $(\mathcal{L}, \mathcal{P})$, as the latter can be bounded simply via

$$\mu_0 N^{d+1} \;\lesssim\; N^{\frac{2d^2}{d+1}}.$$

Finally, by (4.7)

$$\mu_M \;\lesssim\; \bar{\mu},$$

and thus the remaining set $\mathcal{C}_M$, as well as (also by (4.7)) the union $\bigcup_{i-1}^{M-1} \mathcal{C}_i$ will not be responsible for more incidences than specified by the right-hand side of (3.11). This completes the proof of Lemma 6.

Section 5: The Falconer distance problem and convex sequences

In this section we recover some of the estimates implied by Theorem 1 using Fourier analysis and connect the arithmetic problem studied in this paper to the Falconer distance conjecture in geometric measure theory.

The Falconer distance problem (see e.g. [Fa86] asks whether the Lebesgue measure $\mathcal{L}^1$ of the distance set $\Delta(E) = \{|x-y| : x, y \in E\}$, $|x| = \sqrt{x_1^2 + \cdots + x_d^2}$, $d \ge 2$, is positive provided that the Hausdorff dimension dim of $E \subset [0,1]^d$ is sufficiently large. It is conjectured that the conclusion should hold provided that $dim(E) > \frac{d}{2}$, and there exists and an arithmetic example, based on the integer lattice and diophantine approximation shows that such a result would be the best one possible. Namely, Falconer ([Fa86]) proved the first result in this direction, having shown that $\mathcal{L}^1(\Delta(E)) > 0$ in $\mathbb{R}^d$, provided that $dim(E) > \frac{d+1}{2}$.

The best known result in two dimensions is due to Wolff ([Wo99]) who proved that $\mathcal{L}^1(\Delta(E)) > 0$ provided that $dim(E) > \frac{4}{3}$. See also previous improvements due to Bourgain ([Bo94]). In higher dimensions, the best known estimate is due to Erdogan ([Er03]) who proved that $\mathcal{L}^1(\Delta(E)) > 0$ provided that

$$dim(E) > \frac{d(d+2)}{2(d+1)}. \tag{5.1}$$

Moreover, Erdogan's proof makes it clear that the Euclidean distance in the definition of the distance set may be replaced by any distance ρ such that the level set $\{x : \rho(x) = 1\}$ is smooth, convex and has curvature, bounded from below.

Recall the definition of the set $\mathcal{C}_d$ from (1.3): $c \in \mathcal{C}_d$ should satisfy the equation

$$b_{i_1} + b_{i_2} + \cdots + b_{i_d} = c.$$

Let f be the convex function, underlying the sequence $\{b_i\}$, i.e. such that

$$f(i) = b_i. \tag{5.2}$$

Let $q_1 = 2$, and $q_{j+1} = q_j^j$. Let

$$E_j = \left\{ x \in [0,1]^d : \left| x_k - \frac{p_k}{q_j} \right| \leq q_j^{-\frac{d}{s}}, \text{ for every } p = (p_1, \dots, p_d) \in \mathbb{Z}^d \cap [0, q_j]^d \right\}, \tag{5.3}$$

for some $s \in (0, d)$.

Let $E = \cap E_j$. It follows from Theorem 8.17 in [Fa85] that the Hausdorff dimension of E is s.

Define $\rho_f(x) = f(x_1) + \cdots + f(x_d)$ and $\Delta_{\rho_f}(E) = \{\rho(x-y) : x, y \in E\}$. Then f can be chosen such that (5.2) is satisfied and

$$|\Delta_{\rho_f}(E_j)| \lesssim q_j^{-\frac{d}{s}} \times |\{\rho_f(z-w) : z, w \in \mathbb{Z}^d \cap [0, q_j]^d\}|. \tag{5.4}$$

Suppose that there exists a subsequence of the q_js going to infinity such that

$$|\{\rho_f(z-w) : z, w \in \mathbb{Z}^d \cap [0, q_j]^d\}| \lesssim q_j^{\beta}. \tag{5.5}$$

Plugging this estimate into (5.4) we see that

$$|\Delta_{\rho_f}(E_j)| \lesssim q_j^{-\frac{d}{s}} q_j^{\beta}. \tag{5.6}$$

Suppose that the function f can be chosen such that (5.2) holds, the level set

$$\{x : \rho_f(x) = 1\} \tag{5.7}$$

is smooth and has curvature bounded from below. It follows from (5.1) that $\Delta_{\rho_f}(E)$ has positive Lebesgue measure if $s > \frac{d(d+2)}{2(d+1)}$. However, this is in direct contradiction with (5.6) if $\beta < \frac{2(d+1)}{d+2}$. In the language of Theorem 1, we have just proved that

$$|\mathcal{C}_d| \gtrapprox N^{2-\frac{2}{d+2}}, \tag{5.8}$$

which somewhat recovers (1.8) of Theorem 1 in the case $d = 2$. Observe that we actually proved a little more, namely that the number of $N^{-\frac{d-1}{d+1}}$-*separated values* of $\mathcal{C}_d$ is bounded from below by $N^{2-\frac{2}{d+2}}$, not just the total number of values.

If we use the special structure of E, we can do even better. While the Falconer conjecture is not solved in general, it is solved for Salem sets, which are sets of dimension $s < d$ which carry a Borel measure μ such that $|\widehat{\mu}(\xi)| \lessapprox |\xi|^{\frac{s}{2}}$. See [Mat87]. It follows from a theorem Kaufman ([Kauf81]) that the set E constructed in (5.3) above is indeed a Salem set, and we conclude that

$$|\mathcal{C}_d| \gtrapprox N^2, \ \forall d \geq 2. \tag{5.9}$$

Appendix: Proof of Theorem 3a

Without loss of generality, one can assume that all the weights are integers, the net line weight m is a multiple of the maximum line weight μ, and the net point weight n is a multiple of the maximum point weight ν. Then the formula (2.4) is equivalent to the bound (2.1a), for the number of incidences between m/μ lines and n/ν points, provided that each incidence would be counted $\mu\nu$ times. In other words, for the uniform weight distribution there is nothing to prove.

Otherwise, consider some arrangement $(\mathcal{L}, \mathcal{P})$ and suppose, that the weight distribution over, say $\mathcal{P}$ is not uniform. Then there exist $p_1, p_2 \in \mathcal{P}$, such that $\nu(p_1) < \nu(p_2) < \nu$. For $p \in \mathcal{P}$ let

$$w_p = \sum_{l \in \mathcal{L}} \mu(l)\delta_{lp},$$

be the total weight of all the lines incident to p. If $w(p_1) > w(p_2)$, first change the weight distribution by swapping the values $\nu(p_1)$ and $\nu(p_2)$ over the points p_1 and p_2. Then modify the weight distribution by changing $\nu(p_1) \to \nu(p_1) - 1$ and $\nu(p_2) \to \nu(p_2) + 1$. If $\nu(p_1)$ has become zero, remove p_1 from $\mathcal{P}$. As the result, the weight distribution has been modified, so that the number of incidences has increased, and the net weight has stayed constant. Continue this (greedy) procedure, until the weight distribution over $\mathcal{P}$ has become uniform; then do the same thing with the set $\mathcal{L}$. At each single step, the number of incidences will have increased. However, as the result, one still ends up with the upper bound (2.1a), as only m/μ lines and n/ν points remain.

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