GEOMETRY OF THE GAUSS MAP AND LATTICE POINTS IN CONVEX DOMAINS - PRELIMINARY REPORT

L. Brandolini, L. Colzani, A. Iosevich, A. Podkorytov, and G. Travaglini

June 12, 2000

Abstract. Let Ω be a convex planar domain, and let $N_\theta(R) = card\{R\Omega_\theta \cap \mathbb{Z}^2\}$, where Ω_θ denotes the rotation of Ω by θ. We prove, up to a small logarithmic transgression, that $N_\theta(R) = |\Omega|R^2 + O(R^{\frac{2}{3}})$, for almost every rotation. We also obtain a refined result based on the fractal structure of the image of the boundary of Ω under the Gauss map.

Introduction

Let Ω be a convex planar domain, and let $N(R) = card\{R\Omega \cap \mathbb{Z}^2\}$. It was observed by Gauss that $N(R) = |\Omega|R^2 + D(R)$, where $|D(R)| \lesssim R$, since the discrepancy $D(R)$ cannot be larger than the number of lattice points that live a distance at most $1/\sqrt{2}$ from the boundary of Ω. Here, and throughout the paper, $A \lesssim B$ means that there exists a uniform C, such that $A \leq CB$. Similarly, $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. In general, this estimate cannot be improved, as can be seen by taking Ω to be a square with sides parallel to the axis. However, the purpose of this paper is to show that the remainder term is better for almost every rotation of the domain.

If the boundary of Ω has everywhere non-vanishing Gaussian curvature, better estimates for the remainder term are possible. It is a classical result that, in that case, $|D(R)| \lesssim R^{\frac{2}{3}}$. An example due to Jarnik shows that without further assumptions, this result is best possible. See, for example, [Landau15] and [Jarnik25]. If the boundary is assumed to have a certain degree of smoothness, further improvements have been obtained, culminating in a result due to Huxley, see [Huxley96], which says that if the boundary of Ω is five times differentiable and has curvature bounded below by a fixed constant, then $|D(R)| \lesssim R^{\frac{46}{73}}$. It was observed by Hardy that one cannot do better than $R^{\frac{1}{2}}$ times an appropriate power of the logarithm.

We have noted that the trivial estimate, $|D(R)| \lesssim R$, cannot, in general, be improved. Moreover, it was proved by Randol in [Randol66], that if, for example, Ω is given by the

Research supported in part by the NSF grant DMS97-06825 and CNR grants

equation $x_1^m + x_2^m \leq 1$, $m > 2$, then $|D(R)| \lesssim R^{\frac{m-1}{m}}$, and $\frac{m-1}{m}$ cannot be replaced by any smaller number. On the other hand, Colin de Verdiere showed in [ColinDeVerdiere77] that if the boundary of Ω has finite order of contact with its tangent lines, then, for almost every rotation of Ω, the corresponding error term $|D(R,\theta)| \lesssim R^{\frac{2}{3}}$. This result was extended to a certain class of domains, where the order of contact is infinite, by the third author in [Iosevich99]. This raises the obvious question of whether this result holds for an arbitrary convex planar domain. Up to a logarithmic transgression, we answer this question in the affirmative. This is the substance of our first result.

Theorem 0.1. *Let Ω be a convex domain, and let $\delta > \frac{1}{2}$. Define*

$$\mathcal{M}(\theta) = \sup_{R \geq 2} \log^{-\delta}(R) R^{-\frac{2}{3}} |D(R,\theta)|, \tag{0.1}$$

where $D(R,\theta)$ is the discrepancy corresponding to the domain Ω dilated by R and rotated by the angle θ. Then $\mathcal{M} \in Weak - L^2(S^1)$, and, in particular, $\mathcal{M}(\theta) < \infty$ for almost every θ.

Theorem 0.1 is begging to be generalized for the following reason. A result due to Skriganov, see [Skriganov98], says that if Ω is a polygon, $|D(R,\theta)| \lesssim \log^{1+\epsilon}(R)$, for any $\epsilon > 0$, for almost every θ. There is much room between this result and $R^{\frac{2}{3}} \log^{\frac{1}{2}+\epsilon}(R)$ we obtain above, and it makes one ask which geometric properties are in play here. We address this issue in the following way. At every point of a convex set there is the left and the right tangent. Therefore, at every point we have the left(-) and the right(+) normal. Let $\mathcal{N}^{\pm} : \partial\Omega \to S^1$ denote the Gauss maps, which take each point on the boundary of Ω to the right/left unit normal at that point. Our second main result is the following.

Theorem 0.2. *Let Ω be a convex domain. Let $\mathcal{N}^{\pm}$ be the Gauss maps defined above, and let $\mathcal{N}(\partial\Omega) = \mathcal{N}^+(\partial\Omega) \cup \mathcal{N}^-(\partial\Omega)$. Suppose that for any sufficiently small ϵ,*

$$|\{\theta \in S^1 : dist(\theta, \mathcal{N}(\partial\Omega)) \leq \epsilon\}| \lesssim \epsilon^{1-d}. \tag{0.2}$$

Let

$$\mathcal{M}(\theta) = \sup_{R \geq 2} \log^{-\delta}(R) R^{-\frac{2d}{2d+1}} |D(R,\theta)|. \tag{0.3}$$

Then $\mathcal{M} \in L^1(S^1)$ if $d > 0$ and $\delta > 1$, or, if $d = 0$ and $\delta > 3$. In particular, $\mathcal{M}(\theta) < \infty$ for almost every θ.

The estimate (0.2) implies that the upper Minkowski dimension of $\mathcal{N}(\partial\Omega)$ is at least d. Conversely, if the upper Minkowski dimension of $\mathcal{N}(\partial\Omega)$ is d, then the estimate (0.2) holds, up to an arbitrarily small power of ϵ, i.e $|\{\theta \in S^1 : dist(\theta, \mathcal{N}(\partial\Omega)) \leq \epsilon\}| \lesssim \epsilon^{1-d-\eta}$, for any $\eta > 0$.

The conclusion of Theorem 0.2 can be improved under additional assumptions. For example, see [BrandIosTravag00], if Ω is contained in a disc, and the set of the irregular points (the points where the curvature function is not defined) of Ω is contained in the boundary of this disc, one can replace $\frac{d}{d+1}$ in Lemma 2.1, (see below), by $\frac{d}{2}$. We can use this fact to change the exponent $\frac{2d}{2d+1}$ in Theorem 0.2 to $\frac{2d}{d+2}$, under this strengthened assumption.

Theorem 0.2 is stated in terms of the estimate (0.2) for the sake of simplicity. The Theorem could be restated somewhat more precisely in terms of the properties of the distribution function $|\{\theta \in S^1 : dist(\theta, \mathcal{N}(\partial\Omega)) \leq \epsilon\}|$. The condition (0.2) only holds in the case $d = 0$ if Ω is a polygon. As we noted above, a better result is known for polygons. We include this case for the sake of completeness, and also because it can be extended to polygons with infinitely many sides, as we shall see in the examples below.

Example 1. *Our first example illustrates the case $d > 0$ of Theorem 0.2. Consider a polygon with infinitely many sides, where the slopes of the normals to the sides form a sequence $\{j^{-\alpha}\}_{j=1,2,\dots}$. It is not hard to see that the upper Minkowski dimension of $\mathcal{N}(\partial\Omega)$ is $\frac{1}{1+\alpha}$ and also (0.2) holds with the same exponent.*

Example 2. *We now consider the case of a polygon with infinitely many sides, such that the slopes of the normals form a lacunary sequence, for example, $\{2^{-j}\}_{j=0,1,\dots}$. In this case, the upper Minkowski dimension of $\mathcal{N}(\partial\Omega)$ is 0, whereas the estimate (0.2) does not hold with $d = 0$, though it holds for every $d > 0$. Theorem 0.2 says that $\sup_{R\geq 2} R^{-\nu}|D(R,\theta)|$ is finite for almost every θ. However, as we foreshadowed above, we can do better if we work directly with the quantity $|\{\theta \in S^1 : dist(\theta, \mathcal{N}(\partial\Omega)) \leq \epsilon\}|$, instead of the condition (0.2). It is not difficult to see that, in this case,*

$$|\{\theta \in S^1 : dist(\theta, \mathcal{N}(\partial\Omega)) \leq \epsilon\}| \approx \epsilon \log(1/\epsilon). \tag{0.4}$$

The estimate (0.4) along with the proof of Theorem 0.2 yields the conclusion of Theorem 0.2 with $d = 0$ and $\delta > 4$.

We note that Theorem 0.2 does not apply only to polygons, with finitely, or infinitely many sides. In fact, it is not difficult to construct examples of convex domains, where $\mathcal{N}(\partial\Omega)$ has upper Minkowski dimension $0 < d < 1$, which are not polygons. It is just a matter of constructing an appropriate increasing function, for example a Cantor-Lebesgue type function, which defines the tangent vector field.

As we noted above, for a general fixed unrotated convex domain Ω, the estimate $|D(R)| \lesssim R^{\frac{2}{3}}$ cannot be improved. However, this does not imply that a better result may not hold for almost every rotation of a convex domain. We conjecture that for almost every θ, $|D(R,\theta)| \lesssim R^{\frac{1}{2}+\epsilon}$, $\epsilon > 0$, the same as the supposed range for the disc. This belief is supported by the fact

$$\left(\int_{S^1}\int_{\mathbb{T}^2} |D(R,\theta,\tau)|^2 d\tau d\theta\right)^{\frac{1}{2}} \lesssim R^{\frac{1}{2}}, \tag{0.5}$$

where $\mathbb{T}^2$ denotes the two-dimensional torus, and $D(R,\theta,\tau)$ denotes the discrepancy corresponding to the case where a convex domain Ω is rotated by θ, and translated by τ. See [BrandRigTravag98], Theorem 6.2.

The main technical ingredient

The main ingredient in the proof of Theorem 0.1 and Theorem 0.2 is the following maximal stationary phase estimate for the Fourier transform of the characteristic function of Ω, which is interesting in its own right. Under the analyticity assumption, this estimate is implied by a result obtained by Svensson in [Svensson71]. However, lack of any smoothness assumption, besides convexity, involves considerable difficulties.

Theorem 0.3. *Let Ω be a convex domain. Then*

$$|\{\theta \in S^1 : \sup_{R \geq 0} R^{\frac{3}{2}} |\widehat{\chi}_\Omega(R\theta)| > t\}| \lesssim t^{-2}. \tag{0.6}$$

We shall also need the following decay estimate which we shall use to prove Theorem 0.2.

Theorem 0.4. *Let Ω be a convex domain. Then*

$$|\widehat{\chi}_\Omega(R\theta)| \lesssim R^{-2}(dist(\theta, \mathcal{N}(\partial\Omega))^{-1}. \tag{0.7}$$

This paper is organized as follows. In Section I, we prove Theorem 0.1 using Theorem 0.3 and the Poisson Summation Formula. In Section II, we prove Theorem 0.2 using an appropriate modification of Theorem 0.3, and Theorem 0.4. In Section III, we prove Theorem 0.3, and in Section IV, we prove Theorem 0.4.

Section I: Proof of Theorem 0.1

Let ψ be a smooth positive radial function of mass 1 supported in the unit disc centered at the origin. Let $\psi_\epsilon(x) = \epsilon^{-2}\psi(\epsilon^{-1}x)$. Define

$$N(R, \theta, \epsilon) = \sum_{k \neq (0,0)} \chi_{R\theta^{-1}\Omega} * \psi_\epsilon(k), \tag{1.1}$$

and let

$$D(R, \theta, \epsilon) = N(R, \theta, \epsilon) - R^2|\Omega|. \tag{1.2}$$

Lemma 1.1. *We have*

$$D(R, \theta, \epsilon) = R^2 \sum_{k \neq (0,0)} \widehat{\chi}_\Omega(R\theta k)\widehat{\psi}(\epsilon k). \tag{1.3}$$

Proof. This is precisely the Poisson summation formula.

□

Lemma 1.2. *We have*

$$D(R-\epsilon,\theta,\epsilon)-(2R\epsilon-\epsilon^2)|\Omega| \le D(R,\theta) \le D(R+\epsilon,\theta,\epsilon)-(2R\epsilon+\epsilon^2)|\Omega|. \tag{1.4}$$

Proof. We may assume that Ω contains the origin. We have

$$\chi_{(R-\epsilon)\theta^{-1}\Omega} * \psi_\epsilon(k) \le \chi_{R\theta^{-1}\Omega}(k) \le \chi_{(R+\epsilon)\theta^{-1}\Omega} * \psi_\epsilon(k), \tag{1.5}$$

along with

$$N(R-\epsilon,\theta,\epsilon) \le N(R,\theta) \le N(R+\epsilon,\theta,\epsilon), \tag{1.6}$$

and the result follows.

□

Lemma 1.3. *By Lemma 1.1 we have*

$$|\{\theta \in S^1 : \sup_{2^j \le R \le 2^{j+1}} R^{-\frac{2}{3}}|D(R,\theta)| > t\}| \lesssim t^{-2}. \tag{1.7}$$

Proof. We have

$$\sup_{2^j \le R \le 2^{j+1}} |D(R,\theta,\epsilon)| \le$$

$$\epsilon^{3/2}2^{(j+1)/2} \sum_{k\neq(0,0)} |\epsilon k|^{-\frac{3}{2}}|\widehat{\psi}(\epsilon k)| \sup_{2^j \le R \le 2^{j+1}} |R\theta k|^{\frac{3}{2}}|\widehat{\chi}_\Omega(R\theta k)|. \tag{1.8}$$

Since the function $\sup_{2^j \le R \le 2^{j+1}} |R\theta k|^{\frac{3}{2}}|\widehat{\chi}_\Omega(R\theta k)|$ is uniformly in $Weak-L^2$, by Theorem 0.3, the sum is also in this space, with the norm controlled by

$$\epsilon^{3/2}2^{j/2} \sum_{k\neq(0,0)} |\epsilon k|^{-\frac{3}{2}}|\widehat{\psi}(\epsilon k)| \lesssim 2^{j/2}\epsilon^{-1/2}. \tag{1.9}$$

The result now follows from Lemma 1.2 by taking $\epsilon = 2^{-j/3}$.

□

We are now ready to complete the proof of Theorem 0.1. Observe that

$$\sup_{R\ge 2} \log^{-\delta}(R) R^{-\frac{2}{3}}|D(R,\theta)| \lesssim \left\{ \sum_{j=1}^{\infty} j^{-2\delta} \sup_{2^j \le R \le 2^{j+1}} R^{-\frac{4}{3}}|D(R,\theta)|^2 \right\}^{\frac{1}{2}}. \tag{1.10}$$

The function $\sup_{2^j \le R \le 2^{j+1}} R^{-\frac{4}{3}}|D(R,\theta)|^2$ is uniformly in $Weak-L^1$, and can therefore be summed by the sequence $j^{-2\delta}$, $2\delta > 1$. The conclusion of Theorem 0.1 follows.

Section II: Proof of Theorem 0.2

Lemma 2.1. *Under the assumptions of Theorem 0.2, we have*

$$\int_{S^1}\left[\sup_{2^j\le R\le 2^{j+1}} R^{2-\frac{d}{d+1}}|\widehat{\chi}_\Omega(R\theta)|\right]d\theta \lesssim 1, \tag{2.1}$$

if $d>0$, and if $d=0$,

$$\int_{S^1}\left[\sup_{2^j\le R\le 2^{j+1}} R^{2}|\widehat{\chi}_\Omega(R\theta)|\right]d\theta \lesssim j. \tag{2.2}$$

Proof. Applying Theorem 0.3, we have

$$\int_{\{d(\theta,\mathcal{N}(\partial\Omega))\le 2^{-j/d+1}\}}\left[\sup_{2^j\le R\le 2^{j+1}} R^{2-\frac{d}{d+1}}|\widehat{\chi}_\Omega(R\theta)|\right]d\theta \lesssim$$

$$2^{j((1/2)-(d/d+1))}\int_{\{d(\theta,\mathcal{N}(\partial\Omega))\le 2^{-j/d+1}\}}\left[\sup_{2^j\le R\le 2^{j+1}} R^{3/2}|\widehat{\chi}_\Omega(R\theta)|\right]d\theta \lesssim$$

$$2^{j((1/2)-(d/d+1))}|\{\theta\in S^1: d(\theta,\mathcal{N}(\partial\Omega))\le 2^{-j/d+1}\}|^{\frac{1}{2}} \lesssim 1. \tag{2.3}$$

Moreoever, by Theorem 0.4, we have

$$\int_{\{d(\theta,\mathcal{N}(\partial\Omega))>2^{-j/d+1}\}}\left[\sup_{2^j\le R\le 2^{j+1}} R^{2-\frac{d}{d+1}}|\widehat{\chi}_\Omega(R\theta)|\right]d\theta \lesssim$$

$$2^{-jd/d+1}\int_{\{d(\theta,\mathcal{N}(\partial\Omega))>2^{-j/d+1}\}}(d(\theta,\mathcal{N}(\partial\Omega)))^{-1}d\theta \lesssim$$

$$2^{-jd/d+1}\sum_{h=0}^{\infty}\left(2^{h-j/d+1}\right)^{-1}|\{\theta\in S^1: 2^{h-j/d+1}\le d(\theta,\mathcal{N}(\partial\Omega))\le 2^{h+1-j/d+1}\}| \lesssim$$

$$2^{-jd/d+1}\sum_{h=0}^{\infty}\left(2^{h-j/d+1}\right)^{-1}\left(2^{h-j/d+1}\right)^{1-d} \lesssim 1. \tag{2.4}$$

Observe that when $d=0$, it suffices to sum the series in the range $0\le h\lesssim j$. This completes the proof of Lemma 2.1. □

Lemma 2.2. *Under the assumptions of Theorem 0.2, we have*

$$\int_{S^1}\left[\sup_{2^j\le R\le 2^{j+1}} R^{-\frac{2d}{2d+1}}|D(R,\theta)|\right]d\theta \lesssim 1, \tag{2.5}$$

if $d>0$.

If $d=0$, we have

$$\int_{S^1}\left[\sup_{2^j\le R\le 2^{j+1}} |D(R,\theta)|\right]d\theta \lesssim j^2. \tag{2.6}$$

Proof. Assume that $d>0$. We have

$$\int_{S^1}\left[\sup_{2^j\le R\le 2^{j+1}} |D(R,\theta,\epsilon)|\right]d\theta \lesssim \epsilon^{2-(d/d+1)}2^{j(d/d+1)}\times$$

$$\sum_{k\ne(0,0)} |\epsilon k|^{-2+(d/d+1)}|\widehat{\psi}(\epsilon k)| \int_{S^1}\left[\sup_{2^j\le R\le 2^{j+1}} |R\theta k|^{2-(d/d+1)}|\widehat{\chi}_\Omega(R\theta k)|\right]d\theta \lesssim$$

$$\epsilon^{-d/d+1}2^{jd/d+1}. \tag{2.7}$$

We also have

$$\int_{S^1}\left[\sup_{2^j\le R\le 2^{j+1}} R^{-\frac{2d}{2d+1}}|D(R,\theta)|\right]d\theta \lesssim$$

$$2^{-2dj/2d+1}\left(\int_{S^1}\left[\sup_{2^j\le R\le 2^{j+1}} |D(R,\theta,\epsilon)|\right]d\theta + 2^j\epsilon\right) \lesssim$$

$$2^{-2dj/2d+1}(\epsilon^{-d/d+1}2^{jd/d+1}+2^j\epsilon). \tag{2.8}$$

Chosing $\epsilon = 2^{-j/2d+1}$ yields the result. The proof in the case $d=0$ is similar and is left to the reader.

□

We are now ready to complete the proof of Theorem 0.2. We have

$$\sup_{R\ge 2} \log^{-\delta}(R) R^{-2d/2d+1}|D(R,\theta)| \lesssim$$

$$\sum_{j=1}^{\infty} j^{-\delta} \sup_{2^j\le R\le 2^{j+1}} R^{-2d/2d+1}|D(R,\theta)|. \tag{2.9}$$

In view of Lemma 2.2, if $d>0$, the series converges for $\delta>1$, and if $d=0$, one must take $\delta>3$. This completes the proof of Theorem 0.2.

Section III: Proof of Theorem 0.3

We start out by arguing that we may take Ω with a smooth boundary and everywhere non-vanishing curvature, so long as the constants in our argument do not depend on curvature and the degree of smoothness. Indeed, suppose that $\sup_{R>0} R^{\frac{3}{2}}|\widehat{\chi}_\Omega(R\theta)|$ is not in $weak-L^2(S^1)$. This means that given $k>0$, there exists $N>0$, such that the $weak-L^2(S^1)$ norm of $\sup_{0<R<N} R^{\frac{3}{2}}|\widehat{\chi}_\Omega(R\theta)|$ is at least k. Approximate Ω by a sequence Ω_n of convex domains such that the boundary of each Ω_n is smooth and has everywhere non-vanishing curvature. We arrange things so that $|\Omega-\Omega_n|\leq\frac{1}{n}$. This can be achieved, for example, by approximating Ω by polygons, and then approximating the polygons by convex bodies with the given properties. At this point, the claim follows by taking n sufficiently large.

We fix a direction θ, and without loss of generality we assume $\theta=(1,0)$. Then

$$\widehat{\chi}_\Omega(R,0)=\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\chi_\Omega(x_1,x_2)dx_2\right)e^{-2\pi i x_1 R}dx_1=\widehat{h}(R), \tag{3.1}$$

where $h(s)$ denotes the length of the segment obtained by intersecting Ω with the line $x_1=s$. This function is concave on a suitable interval $[a,b]$. Applying Lemma 3.7 of [BrandRigTravag98], we obtain

$$|\widehat{\chi}_\Omega(R\theta)|\leqslant\frac{c}{R}\left(h\left(a+\frac{1}{2R}\right)+h\left(b-\frac{1}{2R}\right)\right)=\frac{c}{R}\left(\mu\left(\theta,\frac{1}{2R}\right)+\mu\left(-\theta,\frac{1}{2R}\right)\right), \tag{3.2}$$

where $\mu(\theta,\varepsilon)$ denotes the length of the chord $C(\theta,\epsilon)=\{x\in\Omega: x\cdot\theta=S_\theta-\epsilon\}$, and $S_\theta=sup_{x\in\Omega}x\cdot\theta$. We are therefore reduced to studying the maximal function

$$\mu^*(\theta)=\sup_{\varepsilon>0}\frac{1}{\sqrt{\varepsilon}}\mu(\theta,\varepsilon). \tag{3.3}$$

Observe $C(\theta,0)$ is a single point, which we denote by $z(\theta)$. Let new θ_o be a fixed direction. With a mild abuse of notation let $\theta=e^{i\theta}$. We denote by $\lambda(\theta)$ the arc-length on $\partial\Omega$ between $z(\theta_o)$ and $z(\theta)$. Let

$$\lambda^*(\theta)=\sup_{\theta\neq\phi}\frac{\lambda(\theta)-\lambda(\phi)}{\theta-\phi}. \tag{3.4}$$

We shall need the following estimate.

Lemma 3.1. *We have*

$$[\mu^*(\theta)]^2\lesssim\lambda^*(\theta). \tag{3.5}$$

Proof. We chose coordinates such that the boundary of Ω is locally a graph of a function $f(x)$, with $f(0)=f'(0)=0$. Without loss of generality, take $x>0$, the other case being similar. Observe that

$$\mu^*(\theta)\leq 2\sup_x\frac{x^2}{f(x)}. \tag{3.6}$$

By the mean value theorem,

$$\sup_x \frac{x^2}{f(x)} \leq \sup_z \frac{2z}{f'(z)} \leq \sup_z \frac{2}{f'(z)} \int_0^z \sqrt{1+(f'(t))^2}dt \tag{3.7}$$

$$= \sup_\psi \frac{2}{\psi} \int_0^{(f')^{-1}(\psi)} \sqrt{1+(f'(t))^2}dt = 2\sup_\psi \frac{\lambda(\psi+\theta)-\lambda(\theta)}{\psi} = 2\lambda^*(\theta). \tag{3.8}$$

This completes the proof of Lemma 3.1. $\square$

Theorem 0.3 now follows from the classical Hardy-Littlewood maximal theorem, which we state in the following form.

Lemma 3.2. *Let λ be and increasing bounded function on the interval $[a,b]$. Then for every $t>0$,*

$$|\{\theta : \lambda^*(\theta) > t\}| \leqslant \frac{\lambda(b)-\lambda(a)}{t}. \tag{3.5}$$

Section IV: Proof of Theorem 0.4

By Lemma 3.8 in [BrandRigTravag98], we have

$$|\widehat{\chi}_\Omega(R\theta)| \lesssim \left(|A_\Omega(R^{-1},\theta)| + |A_\Omega(R^{-1},\theta+\pi)|\right), \tag{4.1}$$

where $A_\Omega(R^{-1},\theta) = \{x\in\Omega : S_\theta - \epsilon < x\cdot\theta < S_\theta\}$, where $S_\theta = \sup_{x\in\Omega} x\cdot\theta$, as in the proof of Theorem 0.3.

Without loss of generality we may assume that $\theta = -\frac{\pi}{2}$. We can also assume that the boundary of Ω passes through the origin, and the Ω lies in the upper half plane. In a neighborhood of the origin, the boundary of Ω is described by a convex function, say $y=\phi(x)$, satisfying $\phi(x)\geq 0$, and $\phi(0)=0$. Let $\phi'(0-)$ and $\phi'(0+)$ denote the left and the right derivatives at the origin. Since we assume that $\theta \notin \mathcal{N}(\partial\Omega)$, we must have $\phi'(0-) < 0 < \phi'(0+)$. Let

$$C = \{(x,y)\in\mathbb{R}^2 : y > \phi'(0-)x \text{ and } y > \phi'(0+)x\}. \tag{4.2}$$

By convexity, $\Omega\subset C$. It follows that

$$|A_\Omega(R^{-1},\theta)| \leq \frac{1}{R^2\phi'(0+)} + \frac{1}{R^2|\phi'(0-)|} \leq \frac{2}{R^2\min\{\phi'(0+),|\phi'(0-)|\}}. \tag{4.3}$$

The proof follows since

$$\min\{\phi'(0+),|\phi'(0-)|\} \approx dist(\theta,\mathcal{N}(\partial\Omega)). \tag{4.4}$$

$\square$

References

[BrandIosTravag00] L. Brandolini, A. Iosevich, and G. Travaglini, *Average decay of the Fourier transform and the Minkowski dimension*, (in preparation) (2000).

[BrandRigTravag98] L. Brandolini, M. Rigoli, and G. Travaglini, *Average decay of Fourier transforms and geometry of convex sets*, Revista Mat. Iber. **14** (1998), 519-560.

[ColinDeVerdiere77] Y. Colin de Verdiere, *Nombre de points entiers dans une famille homothtique de domains de R*, Ann. Sci. cole Norm. Sup. **10** (1977), 559–575.

[Huxley96] M. N. Huxley, *Area, lattice points, and exponential sums*, Clarendon Press, Oxford (1996).

[Iosevich99] A. Iosevich, *Generalized diophanitine conditions and the distribution of lattice points in convex planar domains*, submitted for publication (1999).

[Jarnik25] V. Jarnik, *Uber die Gitterpunkte auf Konvexen kurven*, Math. Zeitschrift **24** (1925), 500-518.

[Landau15] E. Landau, *Uber die Gitterpunkte in einem Kreise (II)*, Gottinger Nachrichten (1915), 161-171.

[Randol66] B. Randol, *A lattice-point problem II*, Trans. Amer. Math. Soc. **125** (1966), 101–113.

[Svensson71] I. Svensson, *Estimates for the Fourier transform of the characteristic function of a convex set*, Ark. f. Mat. **9** (1971), 11-22.

[Skriganov98] M. M. Skriganov, *Ergodic theory on $SL(n)$, Diophantine approximations and anomalies in the lattice point problem*, Invent. Math. **132** (1998), 1–72.