MTH 173 HW 4 Solutions

Linus Ge

November 6, 2019

1 Some Common Issues.

Many of the questions on this homework dealt with rather simple concepts. Things like verifying and constructing linear transformations, basic results of rank nullity, singularity, ect. While most people succeeded in understanding the idea behind each question, one must remember this is a proof-based class. Every statement made either needs to either have been proven previously (and stated as such) or justified in one manner or the other. This might even be as simple as attaching the clause "by definition" at the end of a statement.

One example of this might be if I were to state and use the inequality $\frac{x^p}{p} + \frac{y^q}{q} \geq xy$ for $\frac{1}{p} + \frac{1}{q} = 1$ and $x, y \geq 0$ without any justification. While this inequality is true, and I might even consider it to be common knowledge, others may not find obvious and desire justification. So I would need to justify this inequality, perhaps by adding the statement "this follows by taking logarithms and using the concavity of the logarithm function," which may or may not be sufficient depending on the context. Justification does not always need to be a full proof, but a full proof will never hurt.

2 pg.73 question 1

Before actually solving anything, let us state the definition of a linear transformation which we will use in all the problems (there are 2 equivalent ways of defining a linear transformations which the book shows are equivalent, but a specific definition should be specified). $T : V \to W$ is a linear transformation iff $\forall \alpha, \beta \in V$ and $\forall c \in \mathbb{F}$, $T(c\alpha + \beta) = cT(\alpha) + T(\beta)$. This means to prove something is not a linear transformation, it is enough to show $\exists c \in \mathbb{F}$, $\alpha, \beta \in V$ s.t. $T(c\alpha + \beta) \neq cT(\alpha) + T(\beta)$.

2.1 Part a

$T(x_1, x_2) = (x_1 + 1, x_2)$. **This function is not a linear transformation. Let $c = 1$ and $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$ arbitrary. Then $T(c\alpha + \beta) = T(\alpha + \beta) = T(a_1 + b + 1, a_2 + b_2) = (a_1 + b_1 + 1, a_2 + b_2)$, but $cT(\alpha) + T(\beta) = T(\alpha) + T(\beta) = (a_1 + b_1 + 2, a_2 + b_2) \neq (a_1 + b_1 + 1, a_2 + b_2) = T(c\alpha + \beta)$. So T is not a linear transformation by definition.**

2.2 Part b

$T(x_1, x_2) = (x_2, x_1)$. **This function is a linear transformation. Let $c \in \mathbb{R}$, $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$ be arbitrary. Then $T(c\alpha + \beta) = T(ca_1 + b + 1, ca_2 + b_2) = (ca_2 + b_2, ca_1 + b_1)$, and $cT(\alpha) + T(\beta) = (ca_2, ca_1) + (b_2, b_1) = (ca_2 + b_2, ca_1 + b_1) = T(c\alpha + \beta)$. So T is a linear transformation by definition.**

2.3 Part c

$T(x_1, x_2) = (x_1^2, x_2)$. **This function is not a linear transformation. Let $c = 1$ and $\alpha = (1, 1)$, $\beta = (1, 0)$. Then $T(c\alpha + \beta) = T((1, 1) + (1, 0)) = T(2, 1) = (4, 1)$, but $cT(\alpha) + T(\beta) = T(1, 1) + T(1, 0) = (1, 1) + (1, 0) = (2, 1) \neq (4, 1) = T(c\alpha + \beta)$ (also note we are not in $\mathbb{Z}_2$ since it is characteristic 2 while $\mathbb{R}$ is characteristic 0). So T is not a linear transformation by definition.**

2.4 Part d

$T(x_1, x_2) = (\sin(x_1), x_2)$. This function is not a linear transformation. Let $c = 1$ and $\alpha = \beta = (\frac{\pi}{2}, 0)$. Then $T(c\alpha + \beta) = T(\pi, 0) = (0, 0)$, but $cT(\alpha) + T(\beta) = T(\frac{\pi}{2}, 0) + T(\frac{\pi}{2}, 0) = (1, 0) + (1, 0) = (2, 0) \neq (0, 0) = T(c\alpha + \beta)$ (also note we are not in $\mathbb{Z}_2$ since it is characteristic 2 while $\mathbb{R}$ is characteristic 0). So T is not a linear transformation by definition.

2.5 Part e

$T(x_1, x_2) = (x_1 - x_2, 0)$. This function is a linear transformation. Let $c \in \mathbb{R}$, $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$ be arbitrary. Then $T(c\alpha + \beta) = T(ca_1 + b + 1, ca_2 + b_2) = (ca_1 + b_1 - ca_2 - b_2, 0)$, and $cT(\alpha) + T(\beta) = (ca_1 - ca_2, 0) + (b_1 - b_2, 0) = (ca_1 + b_1 - ca_2 - b_2, 0) = T(c\alpha + \beta)$. So T is a linear transformation by definition.

3 pg. 73 question 4

In general, there are two ways of doing this problem. The first is to first note that $(1, -1, 1)$ and $(1, 1, 1)$ are linearly independent since $a(1, -1, 1) + b(1, 1, 1) = (a + b, a - b, a + b) = (0, 0, 0)$ iff $a = b = 0$, then extend it to a basis, order the basis before applying theorem 1. Note this requires explicit details for proving linear independence, that the set you have constructed is a basis, and that the basis can be ordered. Note that theorem 1 uses an ordered basis instead of just any basis, so you must state that the basis is ordered. Not all bases are ordered, with one example being $\{\alpha_i, \alpha_{1+i}, \alpha_{1-i}\}$ for $i \in \mathbb{C}$. This basis has no implicit order since $\mathbb{C}$ is not ordered, so it cannot be used for theorem 1 as it is. Details like these may not seem important in this homework, but this is to help you build good habits going forward, especially when you know very few proven properties about the objects you are dealing with.

The second way of proving existence, which we prove in detail, was to explicitly construct such a linear transformation. This method, while seeming more complicated as it doesn't just cite a theorem, actually requires proving less since you only need to give T, which is $T(x, y, z) = (\frac{x-y}{2}, \frac{x+y}{2})$ and show it does what you want it to. Note $T(1, -1, 1) = (1, 0)$ and $T(1, 1, 1) = (0, 1)$, so T behaves as we want it to. Also note T is linear because $\forall (x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$, $c \in \mathbb{R}$, we have

$$T(cx_1 + y_1, cx_2 + y_2, cx_3 + y_3) = (\frac{cx_1 + y + 1 - cx_2 + y_2}{2}, \frac{cx_1 + y + 1 + cx_2 + y_2}{2})$$

$$= c(\frac{x_1 - x_2}{2}, \frac{x_1 + x_2}{2}) + (\frac{y_1 - y_2}{2}, \frac{y_1 + y_2}{2}) = cT(x_1, x_2, x_3) + T(y_1, y_2, y_3).$$

So T is linear by definition.

4 pg. 73 question 8

To "explicitly describe" is to give the explicit formula for T, one of which is $T(x, y, z) = (x + y, 2y, 2y - x)$. While this is technically the "answer" to the question, we still need to prove this is a linear transformation. Using the definition, $\forall (x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$, $c \in \mathbb{R}$, we have

$$T(cx_1 + y_1, cx_2 + y_2, cx_3 + y_3) = (cx_1 + y + 1 + cx_2 + y_2, 2cx_2 + 2y_2, 2cx_2 + 2y_2 - cx_1 - y_1)$$

$$= c(x_1 + x_2, 2x_2, 2x_2 - x_1) + (y_1 + y_2, 2y_2, 2y_2 - y_1) = cT(x_1, x_2, x_3) + T(y_1, y_2, y_3).$$

So T is linear by definition.

The next very important thing to prove in order to answer the question is that $T(\mathbb{R}^3) = span\{(1, 0, -1), (1, 2, 2)\}$. This is true because $\forall x = a(1, 0, -1) + b(1, 2, 2) \in span\{(1, 0, -1), (1, 2, 2)\}$, $T(a, b, 0) = x$, so $span\{(1, 0, -1), (1, 2, 2)\} \subset T(\mathbb{R}^3)$. For the other containment, $\forall (x, y, z) \in \mathbb{R}^3$, $\exists w = x(1, 0, -1) + y(1, 2, 2) \in span\{(1, 0, -1), (1, 2, 2)\}$ s.t. $T(x, y, z) = w$. So $T(\mathbb{R}^3) \subset span\{(1, 0, -1), (1, 2, 2)\}$ and we have equality by double containment. This completes the proof.

One thing to note is that many of you put the linear transformation into matrix form. While this allows you to use matrices to show things about linear transformations, it does not work well when the linear transformations going forward since while all matrices are linear transformations, not all linear transformations are matrices.

5 pg. 74 Problem 11

For one direction of the statement, let A be the zero matrix. Then we have $Ax = 0\ \forall x \in V$. As T is defined by $Tx = Ax$, $Tx = Ax = 0\ \forall x \in V$, which is by definition the zero transformation.

For the other direction, we prove the contrapositive, which is if A is nonzero, then T is not the zero matrix. If A is nonzero, then A contains some nonzero entry $a_{i,j}$. Now consider the vector $e_j = (x_1, x_2, ..., x_m)$ where $x_j = 1$ and $x_k = 0\ \forall k \neq j$ as a column matrix in V. Then $Ae_j \neq 0$ because the ith entry of Ae_j is $a_{i,j} \neq 0$ by how we defined matrix multiplication with each entry being a dot product so to speak. This means $Te_j = Ae_j \neq 0$, implying T is not the zero transformation by definition as it maps a vector to a nonzero vector.

6 pg. 74 Problem 12

We know by the rank nullity theorem that $\dim(T(V)) + \dim(N(T)) = \dim(V)$. However, because we are given $T(V) = N(T)$, the two spaces must have the same dimension as dimension is well defined. So $\dim(V) = \dim(T(V)) + \dim(N(T)) = 2\dim(T(V)) = 2\dim(N(T))$. Thus $2|\dim(V)$, meaning it is even.

For an example of a linear transformation with $T(V) = N(T)$, we could always use the zero transformation from the zero vector space. However, it is not hard to come up with a different example. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x, y) = (0, x)$. This is a linear transformation (because you always need to prove linearity) because $\forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and $c \in \mathbb{R}$, we have

$$T(cx_1 + y_1, cx_2 + y_2) = (0, cx_1 + y_1) = (0, cx_1) + (0, y_1) = cT(x_1, x_2) + T(y_1, y_2).$$

So T is linear by definition. $T(T(x, y)) = T(0, x) = (0, 0)\ \forall (x, y) \in \mathbb{R}$, so T maps everything in its range to 0, which shows $T(\mathbb{R}^2) \subset N(T)$. To show containment in the other direction, let $T(x, y) = (0, x) = (0, 0)$. Then $x = 0$ and $N(T) = \{(0, y) | y \in \mathbb{R}\}$ because this is exactly the set which T maps to $(0, 0)$. So $\forall (0, x) \in N(T)$, $(0, x) = T(x, 0) \in T(\mathbb{R}^2)$, showing containment in the other direction. This shows the transformation is satisfies the desired conditions.

Note that proving this much was necessary since whenever you are asked to give something which satisfies certain properties, you give what you are asked and prove why it satisfies the desired properties.

7 pg. 74 Problem 13

First, assume $N(T) \cap T(V) = \{0\}$. Let $x \in T(V)$, so $x = T(\alpha)$ for some $\alpha \in V$. If $T(x) = T(T(\alpha)) = 0$, then $x \in N(T)$. But since $N(T) \cap T(V) = \{0\}$, this implies $T(\alpha) = x = 0$. This proves one direction.

Now assume $T(T(\alpha)) = 0$ implies $T(\alpha) = 0$. Let $x \in N(T) \cap T(V)$. We know $x = T(\alpha)$ for some $\alpha \in V$ since $x \in T(V)$. We also know $x \in N(T)$ so $T(x) = T(T(\alpha)) = 0$, so $T(\alpha) = x = 0$ by hypothesis. Since the x chosen was arbitrary in $N(T) \cap T(V)$, we know $N(T) \cap T(V) = \{0\}$ as it is nonempty since subspaces must always contain 0.

8 pg. 83 Problem 3

In order to do this, you could write T as a matrix and find the inverse of that. But doing so would require proving the matrix you have given and T are actually identical, then also proving that the

inverse matrix and T^{-1} are identical, which involves writing more work than you need to do. Instead, let us define T^{-1} from the start as a linear transformation and show it is the inverse function for T.

Let $T(x,y,z) = (3x, x-y, 2x+y+z)$. Then $T^{-1}(x,y,z) = (\frac{x}{3}, \frac{x}{3}-y, z+y-x)$. By definition of an inverse, it would need

$$T^{-1}T(x,y,z) = T^{-1}(3x, x-y, 2x+y+z) = (\frac{3x}{3}, \frac{3x}{x} - x + y, 2x+y+z+(x-y)-3x) = (x,y,z),$$

$$T(T^{-1}(x,y,z)) = T(\frac{x}{3}, \frac{x}{3} - y, z+y-x) = (\frac{3x}{3}, \frac{x}{3} - \frac{x}{3} + y, \frac{2x}{3} + \frac{x}{3} - y + z + y - x) = (x,y,z).$$

Note the above holds $\forall (x,y,z) \in \mathbb{R}^2$, which shows T^{-1} is the inverse function of T by definition.

9 pg. 83 Problem 4

First note $(T^2 - I)(T-3I)$ is a function. Multiplication is composition of functions, and distributivity still holds as T and I are linear by theorem 4, so $(T^2-I)(T-3I) = T^3 - 3T^2 - T + 3I = 0$. This is because

$$T^3 = T(T(T(x,y,z))) = T(T(3x, x-y, 2x+y+z)) = T(9x, 2x+y, 9x+z) = (27x, 7x-y, 29x+y+z),$$

$$T^2(x,y,z) = T(3x, x-y, 2x+y+z) = (9x, 2x+y, 9x+z),$$

$$T(x,y,z) = (3x, x-y, 2x+y+z).$$

This means $(T^2-I)(T-3I)(x,y,z) = (27x, 7x-y, 29x+y+z) - 3(9x, 2x+y, 9x+z) - (3x, x-y, 2x+y+z) + (3x, 3y, 3z) = (0,0,0)$. Note $(x,y,z) \in \mathbb{R}^3$ was arbitrary, so $(T^2-I)(T-3I) = 0$ by definition.

10 pg. 83 Problem 6

It is enough to prove the general theorem without bothering with the special case as the generalization proves the problem in a special case.

Note the general theorem is that if $T : V \to W$ and $U : W \to V$ with $\dim(V) > \dim(U)$, then $UT : V \to V$ is a function which is not invertible. The theorem is true because $N(T) \subset N(UT)$ as $T(\alpha) = 0$ implies $UT(\alpha) = U(T(\alpha)) = U(0) = 0$ by the comment on page 68 of the book. However, using rank nullity, we have $\dim(T(V)) + \dim(N(T)) = \dim(V)$ or $\dim(N(T)) = \dim(V) - \dim(T(V))$ and $T(V) \subset W$ so we have $\dim(N(T)) \geq \dim(V) - \dim(T(V)) \geq 1$ as $\dim(V) > \dim(W)$. So $\dim(N(T)) \geq 1$, meaning $\dim(N(UT)) \geq \dim(N(T)) \geq 1$ as $N(T)$ is a subspace of $N(UT)$, and thus has smaller dimension by corrolary 1 of chapter 2. So $N(UT)$ is nontrivial, implying UT is not invertible. In the special case where $V = \mathbb{R}^3$ and $W = \mathbb{R}^2$, this solves the problem asked.

11 pg. 84 Problem 7

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x,y) = (x-y, 0)$ and $U : \mathbb{R}^2 \to \mathbb{R}^2$ with $U(x,y) = (x,x)$. Note $T(U(x,y)) = T(x,x) = (0,0)$ but $U(T(x,y)) = U(x-y, 0) = (x-y, x-y) \neq (0,0)$ for $x \neq y$. So $TU = 0$ but $UT \neq 0$. Thus, to show we are done, it remains to show T, U are linear transformations. We proved T to be a linear transformation in an earlier question, so it is enough to show U is a linear transformation.

Let $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Then $U(cx_1 + y_1, cx_2, y_2) = (cx_1 + y_1, cx_1, y_1) = c(x_1, x_1) + (y_1, y_1) = cU(x_1, x_2) + U(y_1, y_2)$, so U is linear by definition which completes the proof.

12 pg. 84 Problem 10

For $m < n$, Let $T(x_1, ..., x_n) = (x_1, ..., x_m) \in \mathbb{F}^{m\times 1}$. T is linear as $\forall (x_1, ..., x_n), (y_1, ..., y_n) \in \mathbb{F}^{n\times 1}$, $c \in \mathbb{F}$, $T(cx_1 + y_1, ..., cx_n + y_n) = (cx_1 + y_1, ..., cx_n + y_m) = c(x_1, ..., x_m) + (y_1, ..., y_m) = cT(x_1, ..., x_n) + T(y_1, ..., y_n)$ which means T is linear by definition. Note $\forall (x_1, ..., x_m) \in \mathbb{F}^{m\times 1}$, $T(x_1, ..., x_m, 0, ..., 0) = (x_1, ..., x_m)$, and as $(x_1, ..., x_m, 0, ..., 0) \in \mathbb{F}^{n\times 1}$, we know T is onto. However, $(y_1, ..., y_n) \in \mathbb{F}^{n\times 1}$ with $y_i = 1$ if $i = m + 1$ and $y_i = 0$ otherwise has $T(y_1, ..., y_n) = 0$, but $(y_1, ..., y_n) \neq (0, ..., 0)$ because $y_{m+1} \neq 0$. Thus, $N(T)$ is nontrivial, so T is singular by definition. So T can be onto and singular in this case.

For $m > n$, Let $T(x_1, ..., x_n) = (x_1, ..., x_n, 0, ..., 0) \in \mathbb{F}^{m\times 1}$. T is linear as $\forall (x_1, ..., x_n), (y_1, ..., y_n) \in \mathbb{F}^{n\times 1}$, $c \in \mathbb{F}$, $T(cx_1 + y_1, ..., cx_n + y_n) = (cx_1 + y_1, ..., cx_n + y_n, 0, ..., 0) = c(x_1, ..., x_n, 0, ..., 0) + (y_1, ..., y_n, 0, ..., 0) = cT(x_1, ..., x_n) + T(y_1, ..., y_n)$ which means T is linear by definition. Note $T(x_1, ..., x_n) = (0, ..., 0)$ iff $x_i = 0$ $\forall i$, which means $N(T) = 0$ and T is nonsingular by definition. However, $(x_1, ..., x_m) \in \mathbb{F}^{m\times 1}$ with $x_i = 1$ if $i = n + 1$ and $x_i = 0$ otherwise is not reached by T since $\forall y \in \mathbb{F}^{n\times 1}$, $T(y)$ has the $m + 1$th coordinate as 0. This proves T can be nonsingular, but onto.

13 pg. 84 Problem 11

It is enough to prove that $N(T^2) \subset N(T)$ since that would show $T(T(\alpha)) = T^2(\alpha) = 0$ implies $T(\alpha) = 0$, which we showed in a previous question is equivalent to the $N(T) \cap T(V) = \{0\}$. To prove this fact, let us first use the rank nullity theorem. $\dim(N(T)) + \dim(T(V)) = \dim(V)$, but also $\dim(N(T^2)) + \dim(T^2(V)) = \dim(V)$. We also know $\dim(T^2(V)) = \dim(T(V))$ by hypothesis, so $\dim(N(T)) = \dim(N(T^2))$. Not only this, but we also know $N(T) \subset N(T^2)$ since if $T(\alpha) = 0$, then $T^2(\alpha) = T(T(\alpha)) = T(0) = 0$ by the comment on page 68 of the book. However, by corrollary 1 of chapter 2, we know if $N(T)$ is a proper subspace, it must has strictly smaller dimension. So as the dimensions are equal, the spaces are equal, meaning $N(T^2) \subset N(T)$ which completes the proof.