LOOSE COMMENTS ON CHAPTER 8

January 29, 2005

Cauchy-Schwartz inequality. *We have*

$$x \cdot y \leq ||x|| \cdot ||y||. \tag{1}$$

An alternate proof of this inequality is the following. Let $X_j = \frac{x_j}{||x||}, Y_j = \frac{y_j}{||y||}$. Let $X = (X_1, \ldots, X_n, Y = (Y_1, \ldots, Y_n)$. Observe that

$$\sum_{j=1}^{n} X_j^2 = \sum_{j=1}^{n} Y_j^2 = 1. \tag{2}$$

It is enough to prove that

$$X \cdot Y \leq 1. \tag{3}$$

Observe that

$$0 \leq (X_j - Y_j)^2 = X_j^2 + Y_j^2 - 2X_jY_j. \tag{4}$$

It follows that

$$X_jY_j \leq \frac{X_j^2 + Y_j^2}{2}. \tag{5}$$

We conclude that

$$X \cdot Y = \sum_{j=1}^{n} X_jY_j \leq \sum_{j=1}^{n} \frac{X_j^2 + Y_j^2}{2} = \frac{1}{2}\sum_{j=1}^{n} X_j^2 + \frac{1}{2}\sum_{j=1}^{n} Y_j^2 = \frac{1}{2} + \frac{1}{2} = 1. \tag{6}$$

This completes the proof.

Simple applications of Cauchy-Schwartz. Perhaps the most immediate application is the triangle inequality:

$$||x+y|| \leq ||x|| + ||y||. \tag{7}$$

The proof writes itself. We have

$$||x+y||^2 = (x+y)\cdot(x+y) = x\cdot x + 2x\cdot y + y\cdot y = ||x||^2 + ||y||^2 + 2x\cdot y$$

$$\leq ||x||^2 + ||y||^2 + 2||x||\cdot||y|| = (||x|| + ||y||)^2, \tag{8}$$

and the proof is complete.

Comparisons.... So we defined $||x||$, $||x||_1$, and $||x||_\infty$. How are they related to each other? First observe that

$$||x||_1 = \sum_{j=1}^{n} |x_j|\cdot 1 = x\cdot(1,1,\ldots,1) \leq \sqrt{n}\cdot||x||, \tag{9}$$

by Cauchy-Schwartz, and we conclude that

$$||x||_1 \leq \sqrt{n}||x||. \tag{10}$$

Our next brilliant insight is that

$$\sum_{j=1}^{n} |x_j|^2 = |x_1|^2 + \cdots + |x_n|^2 \leq (|x_1| + \cdots + |x_n|)^2, \tag{11}$$

which implies that

$$||x|| \leq ||x||_1. \tag{12}$$

We summarize what we discovered so far as follows:

$$||x||_1 \leq \sqrt{n}\cdot||x|| \leq \sqrt{n}\cdot||x||_1. \tag{13}$$

More comparisons... We have

$$\sum_{j=1}^{n} |x_j|^2 \leq n\cdot\max_{1\leq j\leq n} |x_j|^2 = n\cdot||x||_\infty^2, \tag{14}$$

and we conclude that

$$||x|| \leq \sqrt{n}\cdot||x||_\infty. \tag{15}$$

On the other hand,

$$||x||_\infty \leq \sum_{j=1}^{n} |x_j| = ||x||_1 \leq \sqrt{n}\cdot||x||, \tag{16}$$

so we obtain

$$||x|| \leq \sqrt{n}\cdot||x||_\infty \leq n\cdot||x||. \tag{17}$$

So everything in the world is comparable!

Some loose thoughts on dot product and cross product. By the law of cosines (state it and prove it!) we have

$$||a-b||^2 = ||a||^2 + ||b||^2 - 2||a|| \cdot ||b|| \cdot \cos(\theta), \tag{18}$$

where θ is the angle between a and b. Since

$$||a-b||^2 = ||a||^2 + ||b||^2 - 2a \cdot b, \tag{19}$$

it follows that

$$\cos(\theta) = \frac{a \cdot b}{||a|| \cdot ||b||}. \tag{20}$$

It follows, among other things, that a and b are perpendicular if $a \cdot b = 0$. A related thought is how to write down an equation of a plane in $\mathbb{R}^3$. Let $V = (A, B, C)$ be a vector in $\mathbb{R}^3$. Then

$$Ax + By + Cz = 0 \tag{21}$$

is clearly the equation of the plane through the origin perpendicular to V. Suppose that we want an equation of the plane parallel to this one but not passing through the origin. We just make it

$$Ax + By + Cz = D > 0. \tag{22}$$

Why is this? Give a complete and rigorous explanation.

O.K., on to cross products... Given vectors a and b in $\mathbb{R}^3$, we want to come up with a vector $a \times b$ which is perpendicular to a and b! This is accomplished as follows. Define

$$a \times b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1), \tag{23}$$

and check by brute force that $a \cdot (a \times b) = b \cdot (a \times b) = 0$, which, by our previous observations implies that $a \times b$ is perpendicular to both a and b. Another point of view is the following. Consider a matrix where the first row is (i, j, k), the second row is (a_1, a_2, a_3), and the third row is (b_1, b_2, b_3). Then the determinant of this matrix is the vector $a \times b$ if i, j, k are interpreted in the usual way. Can you use elementary linear algebra to deduce that $a \times b$ given from this point of view is perpendicular to both a and b without performing any direct calculations? Recall row and column manipulations...

Basic cross-product identity. We have

$$a \times b = ||a|| \cdot ||b|| \cdot \sin(\theta), \tag{24}$$

where θ is the angle between a and b. How do we see this? Well, we first prove that

$$||a \times b||^2 = ||a||^2 \cdot ||b||^2 - (a \cdot b)^2. \tag{25}$$

This follows by a direct calculation that I very much want to carry out now... An immediate consequence of (25) is that

$$||a \times b||^2 = ||a||^2 \cdot ||b||^2 - ||a||^2 \cdot ||b||^2 \cdot \cos^2(\theta) = ||a||^2 \cdot ||b||^2 \cdot \sin^2(\theta), \tag{26}$$

and we are done.