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Math 173 Fall 2022 September 19

Corollary of Theorem 9:

A, B $m \times n$ matrices over F. ~ field

Then A is row-equivalent to B iff

$B = PA$, where P = product of $m \times m$ elementary matrices.

Proof: Suppose $B = PA$ where $P = E_s E_{s-1} \dots E_1$,

E_i = elementary matrix

Then $E_1 A$ is row-equivalent to A, so

$E_s \dots E_1 A$ is row-equivalent to A.

Conversely, suppose that B is row-equivalent to A. Let $E_1, E_2, \dots, E_s$ be the elementary matrices corresponding to some sequence of elementary row operations which carry A to B. Then

$$B = (E_s \dots E_1) A$$ ✓

Invertible matrices:

Definition: A $n \times n$ matrix over F. ~ field

An $n \times n$ matrix $B \ni BA = I$ is called a left inverse of A, an $n \times n$ matrix $B \ni AB = I$ is called a right inverse of A. If $AB = BA = I$, B is called a two-sided inverse of A, and A is said to be invertible.

Lemma: If A has left-inverse B and right inverse C, then $B = C$.

Proof: Suppose $BA = I$ & $AC = I$. Then $B = BI = B(AC) = (BA)C = I \cdot C = C$ ✓

So if A has a left & right inverses, they are the same and called A^{-1}.

Theorem 10: Let A and B be $n \times n$ matrices over F

i) If A is invertible, so is A^{-1}, and
$(A^{-1})^{-1} = A$

ii) If both A & B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: i) If A is invertible, $\exists A^{-1} \ni$
$A \cdot A^{-1} = A^{-1} \cdot A = \underline{I}$, so there is nothing to prove.

ii) $AB(B^{-1}A^{-1}) = A(BB^{-1}) \cdot A^{-1}$
$= A \cdot \underline{I} \cdot A^{-1} = \underline{I}$

Similarly, $B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B$
$= B^{-1}\underline{I}B = \underline{I}$ ✓

Theorem 11: An elementary matrix is invertible.

Proof: E = elementary matrix corresponding to elementary operation e. If e_1 is the inverse operation of e (Theorem 2), and $E_1 = e_1(I)$, then

$$EE_1 = e(E_1) = e(e_1(I)) = I$$

$$\& \; E_1 E = e_1(E) = e_1(e(I)) = I$$

It follows that $E_1 = E^{-1}$

Examples:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}$$

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Theorem 12: A $n \times n$ matrix

i) A is invertible.

ii) A is row-equivalent to the $n \times n$ identity matrix.

iii) A is a product of elementary matrices.

Proof: Let R = row-reduced eschelon matrix $\sim A$. By Theorem 9,

$$R = \underbrace{E_k \dots E_1}_{\text{elementary matrices}} A$$

It follows that $A = E_1^{-1} \dots E_k^{-1} R$.

It follows that A is invertible iff R is invertible.

Since R is a square matrix, R is invertible iff each row has a non-zero entry, i.e. $R = I$.

It follows that A is invertible iff $R = I$, and if $R = I$, $A = E_k^{-1} \dots E_1^{-1}$.

Corollary: If A is an invertible $n \times n$ matrix, and if a sequence of elementary row operations leads A to I, the same row operations lead I to A^{-1}.

Corollary: A, B $m \times n$ matrices. Then B row-equivalent to A iff $B = PA$, where P = invertible $m \times m$ matrix. why?

Example:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

row2 $\to$ row2 $+ (-1)$ row 1

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$