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Math 473, Fall 2022, November 28

The Double Dual:

$\alpha \in V$: define L_α on V^* given by

$$L_\alpha(f) = f(\alpha)$$

linear functional on V^*

linearity is immediate

Theorem: V/F finite dimensional.

For each $\alpha \in V$, define $L_\alpha(f) = f(\alpha)$, $f \in V^*$

Then the mapping $\alpha \to L_\alpha$ is an isomorphism from V to V^{**}.

Example (before proof)

$V/\mathbb{R} = \mathbb{R}^2$ $\quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$

For each $\alpha \in \mathbb{R}^2$, $L_\alpha(f) = f(\alpha)$.

②

As we know, $f(\alpha) = a_1\alpha_1 + a_2\alpha_2$ for some a_1, a_2 real numbers.

The map $\alpha \longrightarrow L_\alpha$ is effectively a map from $\alpha = (\alpha_1, \alpha_2)$ to (a_1, a_2) since (a_1, a_2) determines f.

Now, the proof: For each α, L_α is clearly linear. Take $\alpha, \beta \in V$, $c \in F$, and let $\gamma = c\alpha + \beta$.

Then $L_\gamma(f) = f(\gamma) = f(c\alpha + \beta)$

$= cf(\alpha) + f(\beta) = cL_\alpha(f) + L_\beta(f)$, so

$L_\gamma = cL_\alpha + L_\beta$ ✓

Therefore, $\alpha \longrightarrow L_\alpha$ is a linear transformation.

It is non-singular (why?)

Since $\dim V = \dim V^* = \dim V^{**}$,

we are done by theorem 9.

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Corollary: V/F finite dimensional. If L is a linear functional on V^*, then $\exists!\, \alpha \in V$ $\ni L(f) = f(\alpha)$, for every $f \in V^*$.

Corollary: V/F finite dimensional. Then each basis of V^* is the dual of some basis for V.

Proof: Let $\mathcal{B}^* = \{f_1, f_2, \ldots, f_n\}$ be a basis for V^*. We need the following auxilliary result:

Theorem 15: V/F finite dimensional, and $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ basis of V. Then $\exists!$ dual basis $\mathcal{B}^* = \{f_1, f_2, \ldots, f_n\}$ for V^* $\ni f_i(\alpha_j) = \delta_{ij}$.

④

For each linear functional f on V we have

$$f = \sum_{i=1}^{n} f(\alpha_i) f_i,$$ and for each $\alpha \in V$,

$$\alpha = \sum_{i=1}^{n} f_i(\alpha)\alpha_i.$$

Proof of Theorem 15:

We have shown that $\exists!$ basis dual to $\mathcal{B}$. Given $f \in V^*$, $f = \sum_{i=1}^{n} c_i f_i$, and we have already observed that $c_j = f(\alpha_j)$. Similarly, if $\alpha = \sum_{i=1}^{n} x_i \alpha_i \in V$,

$$f_j(\alpha) = \sum_{i=1}^{n} x_i f_j(\alpha_i) = \sum_{i=1}^{n} x_i \delta_{ij} = x_j,$$

so there is a unique expression for α as a linear combo of the α_i's:

$$\alpha = \sum_{i=1}^{n} f_i(\alpha)\alpha_i \quad \checkmark$$

Back to the proof of the Corollary:

$\mathcal{B}^* = \{f_1, f_2, \ldots, f_n\}$ basis of V^*.

By Theorem 15, there is a basis $\{L_1, L_2, \ldots, L_n\}$ for V^{**} $\ni$ $L_i(f_j) = \delta_{ij}$.

By the previous corollary, for each i there is a vector α_i in V $\ni$

$$L_i(f) = f(\alpha_i) \text{ for every } f \in V^*,$$

i.e $\ni$ $L_i = L_{\alpha_i}$

It follows immediately that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis of V, and $\mathcal{B}^*$ is the dual basis.

Theorem 18: If $S \subset V/F$ finite dimensional, then $(S^0)^0$ is the subspace spanned by S.

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Proof: Let $W = \text{span}(S)$. Then $W^0 = S^0$.
We are left to prove that $W = W^{00}$.

By Theorem 16, $\dim W + \dim W^0 = \dim V$

$$\dim W^0 + \dim W^{00} = \dim V^*$$

Since $\dim V = \dim(V^*)$,

$\dim W = \dim W^{00}$. Since $\underset{\text{subspace}}{W \subseteq W^{00}}$,

we must have $W = W^{00}$.

Example: $V = \mathbb{R}^2$ $\quad S = \{(1,0)\}$

$\text{span}(S) = \{(t,0) : t \in \mathbb{R}\}$

$\quad\;\; \shortparallel$

$\quad\;\; W$

$W^0 = \{f \in V^* : f(t,0) = 0\}$

Since $f(x_1, x_2) = a_1 x_1 + a_2 x_2$

W^0 corresponds to $f \ni a_1 = 0$

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What is W^{00}? It is an element L of V^{**} $\ni L(f) = 0$ for $f \in W^0$.

We know that every such L is of the form L_α: $L_\alpha(f) = f(\alpha)$.

We know that $f \in W^0$ are of the form $f(x_1, x_2) = a_1 x_1 + a_2 x_2$ w/ $a_1 = 0$, i.e $f(x_1, x_2) = a_2 x_2$, so $L_\alpha(f) = f(\alpha) = a_2 \alpha_2$. For this to be 0, we must have $\alpha_2 = 0$.

In other words, W^{00} consists of maps $\alpha \longrightarrow L_\alpha$ w/ $\alpha = (\alpha_1, 0)$, which is exactly $\underline{W} = \text{span}(S)$!

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Definition: If V is a vector space, a hypersurface in V is a maximal proper subspace of V.

Theorem 19: If f is a non-zero linear functional on the vector space V, then its null space is a hypersurface. Conversely, every hypersurface in V is a null space of some functional.

Example: $V = \left\{ \{a_i\}_{i=1}^{\infty} ; \sum_{i=1}^{\infty} a_i^2 < \infty \right\}$

$f \in V^*$ $\quad f\left(\{a_i\}_{i=1}^{\infty}\right) = a_1$

Null space $(f) = \underbrace{\left\{ \{a_i\}_{i=1}^{\infty} ; a_1 = 0 \right\}}_{\text{hyperspace}}$

Proof of [illegible]

[illegible]

Proof of Theorem 19:

If f is a non-zero element of V^*, let N_f = its null space.

Let $\alpha \in V \setminus N_f$, i.e. $f(\alpha) \neq 0$.

Claim: $\text{span}(N_f, \alpha) = V$.

Let $\gamma + c\alpha$ ∈ as before ∈ span

N_f $\in F$

Let $\beta \in V$. Define $c = \dfrac{f(\beta)}{f(\alpha)}$

Then $\gamma = \beta - c\alpha \in N_f$ since

$$f(\beta - c\alpha) = f(\beta) - c\, f(\alpha) = 0 \;\checkmark$$

It follows that $\beta \in \text{span}(N_f, \alpha)$ ✓

Now let N be a hyperspace in V,

Fix $\alpha \notin N$. By assumption, $\operatorname{span}(N, \alpha) = V$,

so $\beta = \underbrace{\gamma}_{\in N} + \underbrace{c}_{\in F}\alpha \quad \forall \beta \in V$

as above

If $\beta = \underbrace{\gamma'}_{\in N} + \underbrace{c'}_{\in F}\alpha$, then $(c'-c)\alpha = \gamma - \gamma'$

$\hookrightarrow$ $c = c'$ since otherwise $\alpha \in N$.

It follows that $c = c'$; $\gamma = \gamma'$ ✓

We just proved that if $\beta \in V$, $\exists!\ c \in F \ni$ $\beta - \underbrace{c}_{= g(\beta)}\alpha \in N$. Then g is a linear functional and N is its null space.