p48-49

#8 Let $A_1 = \begin{Bmatrix} 1 & 0 \\ 0 & 0 \end{Bmatrix}$, $A_2 = \begin{Bmatrix} 1 & 0 \\ 1 & 0 \end{Bmatrix}$, $A_3 = \begin{Bmatrix} 1 & 1 \\ 0 & 0 \end{Bmatrix}$, $A_4 = \begin{Bmatrix} 0 & 0 \\ 0 & 1 \end{Bmatrix}$

s.t. $A_1^2 = \begin{Bmatrix} (1+0) & (1\cdot 0+0) \\ (0+0) & (0+0) \end{Bmatrix} = A_1$, $A_2^2 = \begin{Bmatrix} (1+0\cdot 1) & (1\cdot 0+0) \\ (1+0\cdot 1) & (1\cdot 0+0) \end{Bmatrix} = A_2$,

$A_3^2 = \begin{Bmatrix} (1+1\cdot 0) & (1+1\cdot 0) \\ (0\cdot 1+0) & (0\cdot 1+0) \end{Bmatrix} = A_3$, $A_4^2 = \begin{Bmatrix} 0 & 0 \\ (0+0\cdot 1) & (0+1) \end{Bmatrix} = A_4$

Thus $A_i = A_i^2$.

Let $\begin{Bmatrix} a & b \\ c & d \end{Bmatrix} = \lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3 + \lambda_4 A_4 \Rightarrow$

$\begin{Bmatrix} a & b \\ c & d \end{Bmatrix} = \begin{Bmatrix} (\lambda_1+\lambda_2+\lambda_3) & \lambda_3 \\ \lambda_2 & \lambda_4 \end{Bmatrix} \Rightarrow \lambda_3 = b,\ \lambda_2 = c,\ \lambda_4 = d \Rightarrow \lambda_1 = a-b-c.$

Then by $\lambda_i \in F$, A_i spans V.

If $a=b=c=d$, $\lambda_3 = b = 0$, $\lambda_4 = d = 0$, $\lambda_2 = c = 0$, $\lambda_1 = a-b-c = 0-0-0=0$

$\Rightarrow 0 = \lambda_i \Rightarrow A_i$ linearly independent $\Rightarrow A_i$ a basis.

#10) Claim if $\alpha_k \in span(\{\alpha_1, \alpha_2 \ldots \alpha_{k-1}, \alpha_{k+1} \ldots \alpha_r\})$,

$span(\{\alpha_1 \ldots \alpha_{k-1}, \alpha_{k+1}, \ldots \alpha_r\}) = span(\{\alpha_1 \ldots \alpha_r\})$.

Let $S = \{a_n\}_{n=1}^{r}$, $S_k = S/\{\alpha_k\}$. By $\alpha_k \in span(S_k)$, $\exists$

$\lambda_1 \ldots \lambda_r$ s.t. $\alpha_k = \sum_{n\neq k} \lambda_n \alpha_n$.

Note $span(S_k) \subseteq span(S)$ by $S_k \subseteq S$.

Let $v \in span(S)$. Then $\exists\ \varphi_n$ s.t. $v = \sum_n \varphi_n \alpha_n =$

$\left(\sum_{n\neq k} \varphi_n \alpha_n\right) + \varphi_k \alpha_k = \left(\sum_{n\neq k} \varphi_n \alpha_n\right) + \varphi_k \sum_{n\neq k} \lambda_n \alpha_n = \left(\sum_{n\neq k} (\varphi_n + \varphi_k \lambda_n)\alpha_n\right) \in$

$span(S_k) \Rightarrow v \in span(S_k) \Rightarrow span(S) \subseteq span(S_k) \Rightarrow span(S) = span(S_k)$.

Use the following alogrithm:

1. Let $k=1$.
2. If $\alpha_k \in span(S_k)$, remove α_k from S and relable. Note $span(S_k) = span(S)$. If α_k was removed, go to 1.
3. If $k<r$, let $k=k+1$, go to 2

Note the algrithm will complete in at most r^2 steps, as it can iterate through r times before removing an element, and can remove at most r elements.

If the algorithm produces some S', note $\text{span}(S') = \text{span}(S)$, as elements were removed in step 2.

If $\exists\ \alpha_k \in S'$ s.t. $\alpha_k \in \text{span}(S')$, it would have been removed. Note then $0 \notin S'$

If $\exists\ \lambda_1 \ldots \lambda_r$ s.t. $\lambda_1 \alpha_1 + \ldots + \lambda_r \alpha_r = 0$ and $\exists\ \lambda_k \neq 0$
$\alpha_k = \sum_{n \neq k} \frac{\lambda_n}{\lambda_k} \alpha_n. \Rightarrow \alpha_k \in \text{span}(S'_k)$. Contradiction.

Then S' is linearly independent, $\text{span}(S') = \text{span}(S) = V \Rightarrow$
S' a basis. by $S' \subseteq S$, S' finite $\Rightarrow V$ finite dimensional.

12) Let A_{kl} be 0 on all entries except the one in the kth row and lth column. Claim $\{A_{kl}\}$ is a basis for $1 \leq k \leq m$, $1 \leq l \leq n$.

Let B a $m \times n$ matrix. Note $B = \sum_{k=1}^{m} \sum_{l=1}^{n} B_{kl} \cdot A_{kl}$, so $\{A_{kl}\}$ spans.

If $\sum_{k=1}^{m} \sum_{l=1}^{n} \lambda_{kl} \cdot A_{kl} = B$, $B = 0$, $\quad \lambda_{kl} = B_{kl} = 0 \ \forall k, l$.

Thus $\{A_{kl}\}$ a basis, $|\{A_{kl}\}| = m \cdot n$. Done.

14) Suppose $\{\alpha_n\}_{n=1}^{r}$ is a finite basis.

Define $f: \mathbb{Q}^r \to \mathbb{R}$ as $f(\vec{x}) = \sum_{n=1}^{r} x_n \alpha_n$.
Claim f a surjection. By $\{\alpha_n\}$ a basis, $\forall\, x \in \mathbb{R}$, $\exists\, \lambda_n$ $1 \le n \le r$ s.t. $\sum_{n=1}^{r} \lambda_n \alpha_n = x$. Let $y = (\lambda_1, \lambda_2, \dots, \lambda_r) \in \mathbb{Q}^r$. Then $f(y) = x \Rightarrow$ f surjective.

Then the cardinality of $|\mathbb{Q}^r| \ge |\mathbb{R}|$. Note $\mathbb{Q}^r$ is the finite cross-product of countable sets $\Rightarrow \mathbb{Q}^r$ countable $\Rightarrow \mathbb{R}$ countable. However $\mathbb{R}$ is uncountable. Contradiction.

Thus $\mathbb{R}$ does not have a finite basis $\Rightarrow \mathbb{R}$ infinite dimensional.

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#2)
$$\left[\begin{array}{ccc|c} 2i & 2 & 0 & 1 \\ 1 & -1 & 1+i & 0 \\ 0 & 1 & 1-i & 1 \end{array}\right] \begin{array}{l} \to R_1 - 2R_3 \\ \to R_2 + R_3 \\ \end{array} \left[\begin{array}{ccc|c} 2i & 0 & 2i-2 & -1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1-i & 1 \end{array}\right] \xrightarrow{R_1 - 2iR_2}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & -2i-2 & -1-2i \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1-i & 1 \end{array}\right] \to R_1/(-2i-2) \left[\begin{array}{ccc|c} 0 & 0 & 1 & \frac{3}{4}+\frac{i}{4} \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1-i & 1 \end{array}\right]$$

$$\begin{array}{l} \to R_1 \\ \to R_2 - 2R_1 \\ \to R_3 - (1-i)R_1 \end{array} \left[\begin{array}{ccc|c} 0 & 0 & 1 & \frac{3}{4}+\frac{i}{4} \\ 1 & 0 & 0 & -\frac{1}{2}-\frac{i}{2} \\ 0 & 1 & 0 & \frac{i}{2} \end{array}\right] \begin{array}{l} \to R_2 \\ \to R_3 \\ \to R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-1}{2}-\frac{i}{2} \\ 0 & 1 & 0 & i/2 \\ 0 & 0 & 1 & \frac{3}{4}+\frac{i}{4} \end{array}\right]$$

Then the coordinate matrix is $\begin{bmatrix} \frac{-1}{2}-\frac{i}{2} \\ i/2 \\ \frac{3}{4}+\frac{i}{4} \end{bmatrix}$

#4) a) Note to be a basis, $\{a_1, a_2\}$ need to span W and be linearly independent.

Suppose $\exists \lambda_1, \lambda_2$ s.t. $\lambda_1 a_1 + \lambda_2 a_2 = 0 \Rightarrow$
$(\lambda_1 + (1+i)\lambda_2, \lambda_2, i\lambda_1 - \lambda_2) = 0 \Rightarrow \lambda_2 = 0 \Rightarrow \lambda_1 + (1+i)\lambda_2 = 0 = \lambda_1$.

Thus $0 = \lambda_1 = \lambda_2$, so $\{a_1, a_2\}$ linearly independent. By def, $W = \text{span}(\{a_1, a_2\}) \Rightarrow \{a_1, a_2\}$ a basis for W.

b) Note $-i \cdot a_1 + a_2 = (-i, 0, 1) + (1+i, 1, -1) = (1, 1, 0) = b_1$
also, $(2-i)a_1 + ia_2 = (2-i, 0, 1+2i) + (i-1, i, -i) = (1, i, 1+i) = b_2$.
Thus $b_1, b_2 \in W$.

Suppose $\exists \lambda_1, \lambda_2$ s.t. $\lambda_1 b_1 + \lambda_2 b_2 = 0$. Then $(\lambda_1 + \lambda_2, \lambda_1 + i\lambda_2, (1+i)\lambda_2) = 0 \Rightarrow$
$(1+i)\lambda_2 = 0 \Rightarrow \lambda_2 = 0. \Rightarrow 0 = \lambda_1 + \lambda_2 = \lambda_1 \Rightarrow 0 = \lambda_1 = \lambda_2 \Rightarrow b_1, b_2$ linearly independent. Let $V = \text{span}(b_1, b_2)$. Note $V \subseteq W \Rightarrow V$ finite-dimensional
Let $n = \dim V$. Note $n \leq \dim(W)$ by T 5, corollary 1. (pg 46).
Then $2 \geq n$. By Theorem 5, pg 45, $\{b_1, b_2\}$ is part of a finite basis for $W \Rightarrow |\{b_1, b_2\}| = 2 \leq n \Rightarrow 2 = \dim V = \dim W$

Then by Corollary 1, pg 46, we must have V not a proper subspace of W. By V a subspace of W have $V = W$. Thus $\{b_1, b_2\}$ span W, linearly independent, and thus are a basis.

c) Note, $\left(\frac{1-i}{2}\right)b_1 + \frac{1+i}{2} b_2 = \left(\frac{1-i}{2}, \frac{1-i}{2}, 0\right) + \left(\frac{1+i}{2}, \frac{i-1}{2}, i\right) = (1, 0, i) = a_1$
Also, $\left(\frac{i+3}{2}\right)b_1 + \left(\frac{i-1}{2}\right)b_2 = \left(\frac{i+3}{2}, \frac{i+3}{2}, 0\right) + \left(\frac{i-1}{2}, \frac{-1-i}{2}, -1\right) = (1+i, 1, -1) = a_2$
Then $a_1 = \left\{\frac{1-i}{2}, \frac{1+i}{2}\right\}$, $a_2 = \left\{\frac{i+3}{2}, \frac{i-1}{2}\right\}$.

pg 54 #6) a) Suppose $\exists\ \lambda_1, \lambda_2, \lambda_3$ s.t. $\lambda_1 + \lambda_2 e^{-ix} + \lambda_3 e^{ix} = 0$

Note for $x=0$, we see $\lambda_1 + \lambda_2 + \lambda_3 = 0 \Rightarrow \lambda_1 = -\lambda_2 - \lambda_3$.

For $x = \pi$, $\lambda_1 - \lambda_2 - \lambda_3 = 0 \Rightarrow \lambda_1 = -\lambda_1 = \lambda_1 = 0$.

For $x = \frac{\pi}{2}$, have $0 + \lambda_2 \cdot -i + \lambda_3 \cdot i \Rightarrow \lambda_2 = \lambda_3$

Then $\forall x$, $0 + \lambda_2 e^{-ix} + \lambda_2 e^{ix} = \lambda_2 \cdot (\cos(-x) + i\sin(-x) + \cos(x) + i\sin(x)) =$
$\lambda_2 \cdot (\cos(x) - i\sin(x) + \cos(x) + i\sin(x)) = \lambda_2 2\cos(x) = 0 \Rightarrow \lambda_2 = 0 = \lambda_3 = \lambda_1$

Thus $1, e^{ix}, e^{-ix}$ linearly independent.

b) Want $1 = \sum_{i=1}^{3} P_{i1} f_i$. Note $1 = 1 \cdot 1 + 0 \cdot e^{-ix} + 0 \cdot e^{ix}$ works.

Note $\cos(x) = 0 \cdot 1 + \frac{1}{2} e^{-ix} + \frac{1}{2} e^{ix} = \frac{1}{2}\cos x - \frac{i}{2}\sin x + \frac{1}{2}\cos(x) + i\sin(x)$.

Note $0 \cdot 1 + \frac{i}{2} e^{-ix} + \frac{-i}{2} e^{ix} = \frac{i}{2}\cos x + \frac{1}{2}\sin x - \frac{i}{2}\cos x + \frac{1}{2}\sin x = \sin x$

Then $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{i}{2} \\ 0 & \frac{1}{2} & \frac{-i}{2} \end{bmatrix}$ works.

#7) Suppose $\exists\ \lambda_1, \lambda_2, \lambda_3$ s.t. $\lambda_1 1 + \lambda_2 (x+t) + \lambda_3 (x+t)^2 = c_0 + c_1 x + c_2 x^2 \Rightarrow$
$c_0 + c_1 x + c_2 x^2 = \lambda_1 + \lambda_2 x + \lambda_2 t + \lambda_3 x^2 + \lambda_3 2tx + \lambda_3 t^2 \Rightarrow (\lambda_1 + \lambda_2 t + \lambda_3 t^2)$
$+ (\lambda_2 + \lambda_3 2t) x + (\lambda_3) x^2 \Rightarrow$

$c_2 = \lambda_3$

$c_1 = \lambda_2 + \lambda_3 2t = \lambda_2 + c_2 2t \Rightarrow \lambda_2 = c_1 - 2tc_2$

$c_0 = \lambda_1 + \lambda_2 t + \lambda_3 t^2 = \lambda_1 + tc_1 - 2t^2 c_2 + c_2 t^2 = \lambda_1 + tc_1 - c_2 t^2 \Rightarrow \lambda_1 = c_0 - tc_1 + c_2 t^2$

Then $\lambda_1, \lambda_2, \lambda_3 \in F$ and exist, so $\{1, x+t, (x+t^2)\}$ span.

Also, if $c_0 = c_1 = c_2$, $\lambda_3 = c_2 = 0$, $\lambda_2 = c_1 - 2c_2 t = 0$, $\lambda_1 = c_0 - tc_1 + c_2 t^2 = 0$.

Thus $\{1, (x+t), (x+t)^2\}$ linear independent. Thus they are a basis.

Furthermore, we have already found the coordinates, $\{c_0 - tc_1 + c_2 t^2,$
$c_1 - 2tc_2, c_2\}$.

pg66 #3) Note that the rowspace of $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ is the same as their RRE form, by cor of T11

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 3 & 4 & -2 & 5 \\ 1 & 4 & 0 & 9 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2+3R_1} \begin{bmatrix} -1 & 0 & 1 & 2 \\ 0 & 4 & 1 & 11 \\ 0 & 4 & 1 & 11 \end{bmatrix} \xrightarrow[R_3-R_2]{-R_1} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 4 & 1 & 11 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2/4}$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & \frac{1}{4} & \frac{11}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Suppose we have some linear equation
$ax_1 + bx_2 + cx_3 + dx_4 = 0$
$Ax_1 + Bx_2 + Cx_3 + Dx_4 = 0$ so that only these two work.

Note $a - c - 2d = 0$, $b + \frac{c}{4} + \frac{11}{4}d = 0$. Let $c, d = 1$. Then $a = -3, b = 3$ work.

$-3x_1 - 3x_2 + x_3 + x_4 = 0$. Let $c = -1, d = 1$

$-x_1 - \frac{5}{2}x_2 - x_3 + x_4 = 0$. Note both $(1,0,-1,-2)$ and $(0,1,\frac{1}{4},\frac{11}{4})$ work and $(-3,3,1,1)$ and $(-1,\frac{5}{2},-1,1)$ clearly linearly independent, so these work.

#5) Note the row space will be the same as in the row-reduced form.

$$\begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ -1 & 2 & -4 & 2 & 0 \\ 2 & -1 & 5 & 2 & 1 \\ 2 & 1 & 3 & 5 & 2 \end{bmatrix} \xrightarrow[R_4-2R_1]{R_2+R_1,\ R_3-2R_1} \begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 2 & -2 & 3 & -1 \\ 0 & -1 & 1 & 0 & 3 \\ 0 & 1 & -1 & 3 & 0 \end{bmatrix} \xrightarrow[R_3+R_4]{R_2-2R_4} \begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 1 & -1 & 3 & 0 \end{bmatrix} \xrightarrow{R_2+R_3}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 1 & -1 & 3 & 0 \end{bmatrix} \xrightarrow[R_3/3]{R_2/2} \begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 3 & 0 \end{bmatrix} \xrightarrow[R_3-R_2]{R_1+R_2} \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{bmatrix} \xrightarrow[R_4-3R_3]{R_1-R_3}$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

Let $R_1 = (1,0,2,0,0)$. Let $R_2 = (0,1,-1,0,0)$, $R_3 = (0,0,0,1,0)$, $R_4 = (0,0,0,0,1)$. Then vectors are of form $\sum_{k=1}^{4} b_k R_k$ for $b_k \in F$.

pg 66 #6 a) Note By Theorem 10, we can find the basis from the row-reduced matrix

$$\begin{bmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix} \xrightarrow[R_4-6R_2]{R_1-3R_2,\ R_3-2R_2} \begin{bmatrix} 0 & 0 & 3 & 15 & 3 \\ 1 & 7 & -1 & -2 & -1 \\ 0 & 0 & 2 & 10 & 3 \\ 0 & 0 & 5 & 25 & 6 \end{bmatrix} \xrightarrow{R_1/3}$$

$$\begin{bmatrix} 0 & 0 & 1 & 5 & 1 \\ 1 & 7 & -1 & -2 & -1 \\ 0 & 0 & 2 & 10 & 3 \\ 0 & 0 & 5 & 25 & 6 \end{bmatrix} \xrightarrow[R_4-5R_1]{R_2+R_1,\ R_3-2R_1} \begin{bmatrix} 0 & 0 & 1 & 5 & 1 \\ 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_4-R_3]{R_1-R_3}$$

$$\begin{bmatrix} 0 & 0 & 1 & 5 & 0 \\ 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1]{R_2} \begin{bmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then a basis is $\{\alpha_1 = (1, 7, 0, 3, 0);\ \alpha_2 = (0, 0, 1, 5, 0);\ \alpha_3 = (0, 0, 0, 0, 1)\}$

b) Note $V = \text{span}(\alpha_1, \alpha_2, \alpha_3) \Rightarrow \vec{v} \in V \iff \exists a, b, c \in F$ s.t.
$\vec{v} = a\alpha_1 + b\alpha_2 + c\alpha_3 = (a, 7a, b, 3a + 5b, c)$.

c) Note $a = x_1,\ b = x_3,\ c = x_5$ by part b) $\Rightarrow$ the coordinates are $\begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix}$.

#7) Let R be the row-reduced Echelon form of A. Claim $\exists Z$ s.t. $R|Z$ is the row-reduced echelon form of $A|Y$. Note that we can generate the row reduced matrices of A, $A|Y$ by using the row-reduction algorithm. The row reduction algorithm begins with the left-most column and then works right, so the $m+1$ column will not be considered until the first m columns have been reduced. Since the first m columns of $A|Y$ are A, we will end with the first m rows reduced $\Rightarrow$ the first M columns are R. Let Z be the $m+1$st collumn.

$\Rightarrow$) Let K be the number of non-zero rows in R (these will be the first K rows as R row-reduced Echelon).

Note then in $R|Z$, the $K+1$ through nth rows are 0 off of Z. Row operations don't effect the solutions, so $A|Y$ has a solution $\Rightarrow$ $R|Z$ has a solution $\Rightarrow$ if a row is 0 off of Z, its entry in Z is 0 (as it is the sum of 0 elements). Thus the $K+1$ through nth rows are 0 rows. Note as the first K rows of R are non-zero, the first K rows of $R|Z$ are non-zero. Thus $R|Z$ has K-non zero rows.

By Theorem 10, the non-zero rows of R, $R|Z$ form basis for A, $A|Y$ respectively $\Rightarrow$ rowrank$(A) = K$, row rank$(A|Y) = K \Rightarrow$ row rank$(A) =$ rowrank$(A|Y)$.

$\Leftarrow$) Suppose rowrank$(A) =$ rowrank$(A|Y) = K$.
As above, R, $R|Z$ must have exactly K non-zero rows.
Note then the first K rows of $R|Z$ have the Kth row of R non-zero, and the other rows are all 0.

The algorithm on pg 14 shows this is sufficient for $R|Z$ to have solutions, and thus $A|Y$ to have them.