MTH 174 Homework 5 solutions

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Problem 4-1: Let $e_1, \dots, e_n$ be the usual basis of $\mathbb{R}^n$ and let $\varphi_1, \dots, \varphi_n$ be the dual basis.

a) Show that $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(e_{i_1}, \dots, e_{i_k}) = 1$. What would the right side be if we forgot the coefficient $\frac{(k+\ell)!}{k!\,\ell!}$ did not appear in the definition of $\wedge$?

By theorem 4-4 part (3), $\varphi_{i_1} \wedge \varphi_{i_2} \wedge \cdots \wedge \varphi_{i_k}(e_{i_1}, \dots, e_{i_k}) = k!\,\mathrm{Alt}(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})(e_{i_1}, \dots, e_{i_k})$

$$= k! \cdot \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}\sigma \cdot (\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})(e_{\sigma(i_1)}, \dots, e_{\sigma(i_k)}) = \sum_{\sigma \in S_k} \varphi_{i_1}(e_{\sigma(i_1)}) \cdots \varphi_{i_k}(e_{\sigma(i_k)}) = 1$$

If $\frac{(k+\ell)!}{k!\,\ell!}$ did not appear in the definition of the wedge product, then we would get $k!$

b) Show that $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(v_1, \dots, v_k)$ is the determinant of the $k \times k$ minor of $\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}$ obtained by selecting columns $i_1, \dots, i_k$

- Since $\{e_i\}$ is a basis we can write $v_i = \sum_{j=1}^{n} a_{ij} e_j$. We have

$$\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(v_1, \dots, v_k) = \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}\Big(\sum_{j=1}^{n} a_{1j} e_j, \dots, \sum_{j=1}^{n} a_{kj} e_j\Big) = \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}\Big(\sum_{j=1}^{k} a_{1 i_j} e_{i_j}, \dots, \sum_{j=1}^{k} a_{2 i_j} e_{i_j}\Big)$$

$$= \det(a_{\ell i_j})\, \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(e_{i_1}, \dots, e_{i_k}) = \det(a_{\ell i_j}) \text{ where } 1 \le \ell, j \le k$$ ✓

Problem 2: If $f: V \longrightarrow V$ is a linear transformation and $\dim V = n$, then $f^*: \Lambda^n(V) \longrightarrow \Lambda^n(V)$ must be multiplication by some constant c. Show that $c = \det f$.

Recall that $\dim \Lambda^n(V) = 1$ & it's a vector space. This is why f^* is multiplication by a constant c. Let $\mathcal{E} = \{e_i\}$ be a basis for V & $\mathcal{E}^* = \{\varphi_i\}$ the dual basis. We just calculate:

$$f^*(\varphi_1 \wedge \cdots \wedge \varphi_n)(e_1, \dots, e_n) = \varphi_1 \wedge \cdots \wedge \varphi_n(f(e_1), \dots, f(e_n)) = \det(f)(\varphi_1 \wedge \cdots \wedge \varphi_n)(e_1, \dots, e_n)$$

The fact that forms are multilinear $\Rightarrow$ this is true for any $(x_1, \dots, x_n) \in V^n$. ✓

Problem 3: If $\omega \in \Lambda^n(V)$ is the volume element determined by the inner product T & the measure μ, and $w_1, \dots, w_n \in V$, show that $|\omega(w_1, \dots, w_n)| = \sqrt{\det(g_{ij})}$ where $g_{ij} = T(w_i, w_j)$. Hint: If $v_1, \dots, v_n$ is an orthonormal basis and $w_i = \sum_{j=1}^{n} a_{ij} v_j$ show that $g_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj}$

Following the hint let $V = \{v_i\}$ be an orthonormal basis and $w_i = \sum_{j=1}^{n} a_{ij} v_j$

Theorem 4-6 $\Rightarrow \omega(w_1, \dots, w_n) = \det(a_{ij}) \cdot \omega(v_1, \dots, v_n) = \det(a_{ij})$

- Let G be the matrix $g_{ij} = T(w_i, w_j)$ & A the matrix a_{ij}

Then $AA^T = G \Rightarrow \det(G) = \det(A)^2 \Rightarrow$ done.

Problem 5: If $c: [0,1] \longrightarrow (\mathbb{R}^n)^n$ is continuous and each $(c^1(t), \dots, c^n(t))$ is a basis for $\mathbb{R}^n$, show that $[c^1(0), \dots, c^n(0)] = [c^1(1), \dots, c^n(1)]$.

$$\left.\begin{array}{l} \bullet \text{ Let } c^i(0) = \sum_{j=1}^{n} a_{ij}(t)\, c^j(t) \quad (0 \le t \le 1) \\ \bullet \text{ Let } A(t) = a_{ij}(t) \end{array}\right\} (c^1(0), \dots, c^n(0))^T = A(t)\,(c^1(t), \dots, c^n(t))^T$$

$\Rightarrow$ take determinants of both sides. Note that $\det(A(t))$ is continuous + doesn't change signs $\Rightarrow$ done

Problem 10: If $w_1, \ldots, w_{n-1} \in \mathbb{R}^n$ show that $|w_1 \times \cdots \times w_{n-1}| = \sqrt{\det(g_{ij})}$ where $g_{ij} = \langle w_i, w_j \rangle$

- $\langle w, z \rangle = \varphi(w) = \det(v_1, \ldots, v_{n-1}, w)^T$
- Let $z = w_1 \times \cdots \times w_n$ & $\hat{z} = \frac{z}{|z|}$. Then define $\varphi \in \Lambda^{n-1}(V)$ by $\varphi(x_1, \ldots, x_{n-1}) = \det(x_1, \ldots, x_{n-1}, \hat{z})$ so that $\varphi(w_1, \ldots, w_{n-1}) = |z|$
- Let $V = \text{span}(w_1, \ldots, w_{n-1})$. Let $(v_1, \ldots, v_{n-1})$ be an orthonormal basis for V so that $(v_1, \ldots, v_{n-1}, \hat{z})$ is an orthonormal basis for $\mathbb{R}^n \Rightarrow \varphi(v_1, \ldots, v_{n-1}) = \pm 1 \Rightarrow$ if $[v_1, \ldots, v_{n-1}] = \mu$ & $\langle , \rangle = \varphi$ is a volume element so by Theorem 4-6 we are done.

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Problem 13:

a) If $f: \mathbb{R}^n \to \mathbb{R}^m$ and $g: \mathbb{R}^m \to \mathbb{R}^n$, show that $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)^* = f^* \circ g^*$

$$(g \circ f)_*(v|_p) = (D(g \circ f)(p)(v))_{g \circ f(p)} = (Dg(f(p)) \circ Df(p)(v))_{g \circ f(p)} = g_*[(Df(p)(v))_{f(p)}] = g_* \circ f_*(v|_p)$$

(b) If $f, g: \mathbb{R}^n \to \mathbb{R}$ show that $d(f \cdot g) = f \cdot dg + g \cdot df$.

This is a direct consequence of Leibniz rule

Problem 17: If $f: \mathbb{R}^n \to \mathbb{R}^n$ define a vector field $\bar{F}$ by $\bar{F}(p) = f(p)_p \in \mathbb{R}^n_p$

(a) Show that every vector field F on $\mathbb{R}^n$ is of the form $\bar{F}$ for some f.

- Let $f = (f^1, \ldots, f^n)$. Then $\bar{F}(p) = f(p)_p = (f^1(p), \ldots, f^n(p))_p = f^1(p)(e_1)_p + \cdots + f^2(p)(e_n)_p = f(p)_p$

(b) Show that $\text{div} \bar{F} = \text{trace } f'$

- $\text{div} \bar{F} = \sum_{i=1}^n D_i f^i$ where f^i are the components of $\Rightarrow$ done.

Problem 20: Let $f: U \to \mathbb{R}^n$ be a differentiable function with a differentiable inverse $f^{-1}: f(U) \to \mathbb{R}^n$. If every closed form on U is exact, show that the same is true for $f(U)$.

- Let α be a closed form on $f(U)$. Set $\beta = f^* \alpha$. Then $d\beta = df^* \alpha = f^*(d\alpha) = f^*(0) = 0$

$\Rightarrow \beta$ is closed $\Rightarrow$ by assumption β is exact $\Rightarrow \exists\, g \in C^\infty(U)$ such that $\beta = dg$.

$\Rightarrow d((f^{-1})^* g) = (f^{-1})^*(dg) = (f^{-1})^* \beta = (f^{-1})^*(f^* \alpha) = (f \circ f^{-1})^* \alpha = \alpha$

$\Rightarrow \alpha = d((f^{-1})^* g)$ is exact.

Problem 21: Prove that, on the set where θ is defined, we have $d\theta = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$

* θ is defined in problem 3-41 as

$$\theta(x,y) = \begin{cases} \operatorname{atan}(y/x) & x>0 \quad y>0 \\ \pi + \operatorname{atan}(y/x) & x<0 \\ 2\pi + \operatorname{atan}(y/x) & x>0 \quad y<0 \\ \pi/2 & x=0 \quad y>0 \\ 3\pi/2 & x=0 \quad y<0 \end{cases}$$

This is a staightforward application of the definitions & only involves computing $\frac{\partial}{\partial x} \operatorname{atan}(y/x)$ and $\frac{\partial}{\partial y} \operatorname{atan}(y/x)$.

which is easy

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Problem 23: For $R > 0$ and n an integer, define the singular 1-cube $c_{R,n}: [0,1] \to \mathbb{R}^2 - 0$ by $c_{R,n}(t) = (R\cos 2\pi n t, R \sin 2\pi n t)$. Show that there is a singular 2-cube $c: [0,1]^2 \to \mathbb{R}^2 - 0$ such that $c_{R_1,n} - c_{R_2,n} = \partial c$

- Define $c: [0,1]^2 \to \mathbb{R}^2 - \{0\}$ by $c(x,y) = x c_{R_1,n} - (1-y) c_{R_2,n}$

Then $\partial c = (-1) c(0,y) + (-1)^2 c(1,y) + (-1)^2 c(x,0) + (-1)^3 c(x,1) = c_{R_1,n} - c_{R_2,n}$