BASIC TECHNIQUES OF HARMONIC ANALYSIS

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The purpose of these notes is to introduce a student who is familiar with basics of Lebesgue integration and functional analysis to some of the techniques of harmonic analysis. These notes also contain many exercises, some of which are quite difficult. Special emphasis is made, throughout the notes on interaction of harmonic analysis and other areas of mathematics such as partial differential equations, geometric combinatorics and analytic number theory.

Much of the material in these notes is taken from the lecture notes by Nets Katz, Wilhelm Schlag, Terry Tao, Tom Wolff, and books by Michael Christ, Javier Duoandikoetxea, Chris Sogge and Elias Stein.

Chapter 1: Basic operator bounds

We start out by looking at the inequalities of the form

$$\left(\int_Y |Tf(y)|^p dy\right)^{\frac{1}{p}} \leq C_p \left(\int_X |f(x)|^p dx\right)^{\frac{1}{p}}, \tag{*}$$

where X and Y are measure spaces, and $f : X \to Y$ is a measurable function.

These inequalities have a variety of applications in analysis, partial differential equations and other areas. We shall discuss some of those applications later in the notes.

Let

$$Tf(y) = \int_X K(y,x) f(x) dx, \tag{1.1}$$

where K is a measurable function on $Y \times X$.

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Lemma 1.1 (Schur's lemma). *Let T be as above. Suppose that there exist measurable functions $w_1(y)$ and $w_2(x)$ on Y and X respectively so that*

$$\int_Y w_1(y)|K(y,x)|dy \le C_1 w_2(x), \tag{1.2}$$

and

$$\int_X |K(y,x)|(w_2(x))^{\frac{1}{p-1}}dx \le C_2 (w_1(y))^{\frac{1}{p-1}}. \tag{1.3}$$

Then T is bounded from $L^p(X)$ to $L^q(Y)$, and

$$||T||_{p,p} \le C_1^{\frac{1}{p}} C_2^{\frac{1}{q}}, \tag{1.4}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof of Schur's lemma. It is enough to show that for any $g \in L^q(Y)$,

$$\int_Y |Tf(y)||g(y)|dy \le C_1^{\frac{1}{p}} C_2^{\frac{1}{q}} ||f||_p ||g||_q. \tag{1.5}$$

Indeed,

$$\int_Y |Tf(y)||g(y)|dy = \int_Y \int_X |g(y)||f(x)||K(y,x)|dxdy$$

$$= \int_Y \int_X |g(y)||f(x)||K(x,y)|^{\frac{1}{q}}|K(x,y)|^{\frac{1}{p}}(w_1(y))^{\frac{1}{p}}(w_1(y))^{-\frac{1}{p}}(w_2(x))^{\frac{1}{p}}(w_2(x))^{-\frac{1}{p}}dxdy$$

$$\le \left(\int_Y \int_X |g(y)|^q |K(y,x)|(w_2(x))^{\frac{1}{p-1}}(w_1(y))^{-\frac{1}{p-1}}dxdy\right)^{\frac{1}{q}}$$

$$\times \left(\int_Y \int_X |f(x)|^p |K(y,x)| w_1(y)(w_2(x))^{-1}dxdy\right)^{\frac{1}{p}}$$

$$\le C_1^{\frac{1}{p}} C_2^{\frac{1}{q}} ||f||_p ||g||_q \tag{1.6}$$

and the proof is complete.

Exercise. *State and proof a multi-linear version of Schur's lemma with weights equal to 1.*

Cotlar's lemma. *Let T be a linear operator taking a dense subset of $L^2(X)$ into a dense subset of $L^2(Y)$. Suppose there is a sequence of linear operators $\{T_j\}$ so that*

$$T = \sum_{j=1}^{N} T_j, \tag{1.7}$$

with the limit taken in the strong operator sense. Let $(*)$ denote the adjoint operation. Suppose that there exist constants $c(j)$ and C such that

$$||T_j T_k^*||_{2,2} + ||T_j^* T_k||_{2,2} \leq c(|j-k|), \tag{1.8}$$

and

$$\sum_l \sqrt{c(l)} \leq C. \tag{1.9}$$

Then

$$||T||_{2,2} \leq C. \tag{1.10}$$

Proof of Cotlar's lemma. We have

$$(T^*T)^n = \sum_{j_1,\ldots,j_n=1;k_1,\ldots,k_n=1}^{N} T_{j_1}^* T_{k_1} T_{j_2}^* T_{k_2} \ldots T_{j_n}^* T_{k_n}. \tag{1.11}$$

Let $\sup_{1\leq j\leq N} ||T_j|| = B$. Then

$$||(T^*T)^n|| \leq \sum_{j_1,\ldots,j_n=1;k_1,\ldots,k_n=1}^{N} ||T_{j_1}||^{\frac{1}{2}} ||T_{j_1} T_{k_1}^*||^{\frac{1}{2}} \ldots ||T_{k_{n-1}} T_{j_n}^*||^{\frac{1}{2}} ||T_{j_n}^* T_{k_n}^*||^{\frac{1}{2}} ||T_{k_n}||^{\frac{1}{2}}$$

$$\leq \sum_{j_1,\ldots,j_n=1;k_1,\ldots,k_n=1}^{N} \sqrt{B} c^{\frac{1}{2}}(|j_1-k_1|) c^{\frac{1}{2}}(|k_1-j_2|) \ldots c^{\frac{1}{2}}(|k_{n-1}-j_n|) c^{\frac{1}{2}}(|j_n-k_n|) \sqrt{B}$$

$$\leq NBC^{2n-1}. \tag{1.12}$$

Since T^*T is self-adjoint, the spectral theorem implies that

$$||(T^*T)^n|| = ||T||^{2n}. \tag{1.13}$$

Hence

$$||T|| \leq (NBC^{2n-1})^{\frac{1}{2n}} \to C \tag{1.14}$$

as $n \to \infty$. This completes the proof.

Exercise 1.1 (Calderon-Zygmund operators). *Let* $K : \mathbb{R}^d \setminus (0, \dots, 0)$ *satisfy the following properties:*

$$|K(x)| \leq B|x|^{-d}, \tag{1.15}$$

$$\int_{r<|x|<s} K(x)dx = 0 \text{ for all } 0 < r < s < \infty, \tag{1.16}$$

$$\int_{|x|>2|y|} |K(x) - K(x-y)|dx \leq B \text{ for all } y \neq 0, \tag{1.17}$$

and

$$|\nabla K(x)| \leq B|x|^{-d-1}. \tag{1.18}$$

Let

$$Tf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} K(x-y)f(y)dy, \tag{1.19}$$

initially defined for C_0^1 *functions. Then*

$$||T||_{2,2} \leq CB. \tag{1.20}$$

Exercise 1.2 (First taste of multi-linearity). *Let* T *be the extension of a linear operator taking simple functions on the measure space* X *to functions on measure space* Y*. Suppose* $\{A_j\}_{j \in \mathbb{Z}^+}$ *is a sequence of pairwise disjoint subsets of* X *so that* $X = \cup A_j$*. Let* f_j *denote the restriction of* f *to* A_j*. Let* $T_j f = Tf_j$*, so* $T = \sum T_j$*. Suppose that there exists a constant* $0 < c < 1$ *and a fixed* $A > 0$ *so that for any integers* j, k*, and any function* $f \in L^p(X)$*,*

$$\int_Y |T_j f(y)|^{\frac{p}{2}} |T_k f(y)|^{\frac{p}{2}} dy \leq Ac^{|j-k|} ||f_j||_p^{\frac{p}{2}} ||f_k||_p^{\frac{p}{2}}. \tag{1.21}$$

Then there exists a constant $C = C(A, c, p)$ *such that*

$$||T||_{p,p} \leq C. \tag{1.22}$$

Theorem 1.2 (Marcinkiewicz interpolation theorem). *Let $1 \leq p < \infty$. Let T be as above and suppose that T is weak type (p_0, p_0), which means that*

$$|\{y \in Y : |Tf(y)| > \lambda\}|_Y \leq A_0 \frac{||f||_{p_0}^{p_0}}{\lambda^{p_0}}. \tag{1.23}$$

Suppose that T is also weak-type (p_1, p_1), $p_0 < p_1$, $q < \infty$. Then T is bounded from $L^p(X)$ to $L^p(X)$ for any $p_0 < p < p_1$.

If T is weak type (p_0, p_0) and is bounded from $L^\infty(X)$ to $L^\infty(Y)$, then T is bounded from $L^p(X)$ to $L^p(X)$ for any $p > p_0$.

Proof. It suffices to prove the result with the constants in the (p_0, p_0), (p_1, p_1) and (∞, ∞) inequalities less than or equal to 1.

Observe that

$$\int_Y |Tf(y)|^p dy = p \int_0^\infty \lambda^{p-1} |\{y : |Tf(y)| > \lambda\}|_Y d\lambda. \tag{1.24}$$

Let $f_0 = f\chi_{\{x:|f(x)|>c\lambda\}}$ and $f_1 = f\chi_{\{x:|f(x)|\leq c\lambda\}}$.

It is clear that

$$|\{y : |Tf(y)| > \lambda\}|_Y \leq \left|\left\{y : |Tf_0(y)| > \frac{\lambda}{2}\right\}\right|_Y \leq + \left|\left\{y : |Tf_1(y)| > \frac{\lambda}{2}\right\}\right|_Y. \tag{1.25}$$

Let's handle the case $q = \infty$ first. Let $c = \frac{1}{2A_1}$, where A_1 is a constant such that $||Tf||_\infty \leq A_1 ||f||_\infty$ and $|\left\{y : |Tf_1(y)| > \frac{\lambda}{2}\right\}|_Y = 0$.

Using weak type (p_0, p_0) we have

$$\left|\left\{y : |Tf_0(y)| > \frac{\lambda}{2}\right\}\right|_Y \leq \left(\frac{2A_0}{\lambda} ||f_0||_{p_0}\right)^{p_0}. \tag{1.26}$$

Consequently,

$$||Tf||_p^p \leq p \int_0^\infty \lambda^{p-1-p_0} (2A_0)^{p_0} \int_{\{x:|f(x)|>c\lambda\}} |f(x)|^{p_0} dx d\lambda$$

$$= p(2A_0)^{p_0} \int |f(x)|^{p_0} \int_0^{\frac{|f(x)|}{c}} \lambda^{p-1-p_0} d\lambda dx = \frac{1}{c^{p-p_0}} \frac{p}{p-p_0} (2A_0)^{p_0} (2A_1)^{p-p_0} ||f||_p^p. \tag{1.27}$$

We now deal with the case $p_1 < \infty$. We have

$$||Tf||_p^p \leq p \int_0^\infty \lambda^{p-1-p_0} (2A_0)^{p_0} \int_{\{x:|f(x)|>c\lambda\}} |f(x)|^{p_0} dx d\lambda +$$

$$p\int_0^\infty \lambda^{p-1-p_1}(2A_1)^{p_1}\int_{\{x:|f(x)|\le c\lambda\}}|f(x)|^{p_1}dxd\lambda$$

$$= \left(\frac{p2^{p_0}}{p-p_0}\frac{A_0^{p_0}}{c^{p-p_0}}+\frac{p2^{p_1}}{p_1-p}\frac{A_1^{p_1}}{c^{p-p_1}}\right)||f||_p^p. \tag{1.28}$$

It follows that

$$||Tf||_p \le 2p^{\frac{1}{p}}\left(\frac{1}{p-p_0}+\frac{1}{p_1-p}\right)^{\frac{1}{p}}A_0^{1-\theta}A_1^{\theta}, \tag{1.29}$$

where

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}, \tag{1.30}$$

$0\le\theta\le 1$.

Chapter 2: Stopping time arguments

Hardy-Littlewood Maximal Function: Take 1. Let

$$M_tf(x)=\int_B f(x-ty)dy=\frac{1}{|B_{x,t}|}\int_{B_{x,t}}f(y)dy, \tag{2.1}$$

where B is the ball of unit volume in $\mathbb{R}^d$ centered at the origin, $B(x,t)$ is the ball centered at x of radius t, and f is, for the moment, a compactly supported and infinitely differentiable function. Then the Fundamental Theorem of Calculus says that

$$\lim_{t\to 0}M_tf(x)=f(x). \tag{2.2}$$

Exercise 2.1 (easy). *Carry out the details of* (2.2).

Suppose that $f\in L^p(\mathbb{R}^d$ for some $p\ge 1$. Does (2.2) still hold? Let

$$\mathcal{M}f(x)=\sup_{t>0}|M_tf(x)|. \tag{2.3}$$

Exercise 2.2. *Prove that if* $\mathcal{M}$ *is bounded on* $L^p(\mathbb{R}^d)$, $1<p<\infty$, *then* (2) *holds for almost every* $x\in\mathbb{R}^d$.

Theorem 2.1 (Hardy-Littlewood Maximal Theorem). *Let $\mathcal{M}$ be as above. Then $\mathcal{M}$ is weak type* $(1,1)$.

Corollary 2.2. *$\mathcal{M}$ is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$.*

Proof of Hardy-Littlewood maximal theorem. We shall make use of the following covering lemma.

Lemma 2.3 (Wiener Covering Lemma). *Let $E \subset \mathbb{R}^d$ be measurable and suppose that $E \subset \cup B_j$, where B_j's are balls satisfying $\sup_j diam(B_j) = C_0 < \infty$. Then there is a disjoint sub-collection $\{B_{j_k}\}$ such that*

$$|E| \leq 5^d \sum_k |B_{j_k}|. \tag{2.4}$$

Proof of Wiener Covering Lemma. First chose B_{j_1} such that $diam(B_{j_1}) \geq \frac{1}{2}C_0$. We now proceed inductively. If disjoint balls $B_{j_1} \dots B_{j_k}$ have been selected, we chose, if possible, a ball $B_{j_{k+1}}$ satisfying $diam(B_{j_{k+1}}) \geq \sup_j \{diam(B_j) : B_j \cap B_{j_l} = \emptyset,\ l = 1, \dots, k\}$.

In this way we get a disjoint collection of balls $\{B_{j_k}\}$. If $\sum_k |B_{j_k}| = \infty$, then (2.4) clearly holds, so we assume that the sum is finite.

Let $B^*_{j_k}$ denote the ball with the same center as B_{j_k} but five times the radius. We claim that

$$E \subset \cup_k B^*_{j_k}. \tag{2.5}$$

This would instantly imply (2.4).

To prove the claim it suffices to show that $B_j \subset \cup_k B^*_{j_k}$ if B_j is one of the balls in the covering. This is clear if B_j is one of the B_{j_k}'s, so we assume otherwise.

Observe that $|B_{j_k}| \to 0$ as $k \to \infty$ (why?). Let k be the first integer for which $diam(B_{j_{k+1}}) < \frac{1}{2}diam(B_j)$. It follows that B_j must intersect one of the balls $B_{j_1}, \dots, B_{j_k}$, for if not, it should have been picked instead of $B_{j_{k+1}}$ because its diameter is twice as large. Finally, if $B_j \cap B_{j_l} \neq \emptyset$ for some l, then $B_j \subset B^*_{j_l}$ since $diam(B_{j_l}) \geq \frac{1}{2}diam(B_j)$.

We are now ready to prove the Hardy-Littlewood Maximal Theorem. For a given $\lambda > 0$, let

$$E_\lambda = \{x : \mathcal{M}f(x) > \lambda\}. \tag{2.6}$$

It follows that given $x \in E_\lambda$ there exists B_x centered at x such that

$$\int_{B_x} |f(y)|dy > \lambda |B_x|. \tag{2.7}$$

Wiener Covering lemma implies that there exists points $x_k \in E$ such that the balls B_{k_k} are disjoint and $\sum |B_{x_k}| \geq 5^{-d}|E_\lambda|$. We conclude that

$$|E_\lambda| \leq 5^d \sum_k |B_{x_k}| \leq 5^d \lambda^{-1} \int_{\cup B_{x_k}} |f(y)|dy \tag{2.8}$$

and the theorem follows since the balls are disjoint.

Theorem 2.4 (Calderon Zygmund Decomposition). *Let $f \in L^1(\mathbb{R}^d)$ and $\alpha > 0$. Then we can decompose*

$$f = g + \sum_1^\infty b_k, \tag{2.9}$$

where

$$||g||_1 + \sum_1^\infty ||b_k||_1 \leq 3||f||_1, \tag{2.10}$$

$$|g(x)| < 2^d \alpha \tag{2.11}$$

almost everywhere, and, for certain non-overlapping cubes Q_k

$$b_k = 0 \text{ if } x \notin Q_k \text{ and } \int b_k = 0, \tag{2.12}$$

$$\sum_1^\infty |Q_k| \leq \alpha^{-1}||f||_1. \tag{2.13}$$

We start by dividing $\mathbb{R}^d$ into a lattice of cubes of volume $> \alpha^{-1}||f||_1$. Thus if Q is one of the cubes in the lattice,

$$|Q|^{-1} \int_Q |f(x)|\, dx < \alpha. \tag{2.14}$$

Divide each cube into 2^d non-overlapping cubes and let $Q_{11}, Q_{12}, \ldots$ be the resulting cubes for which (2.14) no longer holds, i.e

$$|Q_{1k}|^{-1} \int_{Q_{1k}} |f(x)|\, dx \geq \alpha. \tag{2.15}$$

Observe that

$$\alpha|Q_{1k}| \leq \int_{Q_{1k}} |f(x)|\, dx < 2^d \alpha |Q_{1k}| \tag{2.16}$$

since $2^d|Q_{1k}| = |Q|$.

Let

$$g(x) = |Q_{1k}|^{-1} \int_{Q_{1k}} f(x)\ dx,\ x \in Q_{1k}, \tag{2.17}$$

and

$$b_{1k}(x) = f(x) - g(x),\ x \in Q_{1k},\ \text{and } b_{1k}(x) = 0,\ x \notin Q_{1k}. \tag{2.18}$$

Now consider all the cubes which are not among $\{Q_{1k}\}$. By construction, each one satisfies (2.14). We divide each such cube as before into 2^d non-overlapping cubes and let $Q_{21}, Q_{22}, \dots$ be the cubes for which the analog of (2.13) holds. We extend the definitions (2.15) and (2.16) for these cubes.

Continuing in this fashion, we obtain a sequence of non-overlapping cubes and and functions b_{jk}. We rearrange the cubes and functions in sequences $\{b_k\}$ and $\{Q_k\}$, respectively.

Extend the definition of g by setting $g(x) = f(x)$ for $x \notin \Omega = \cup Q_k$. Then $f(x) = g(x) + \sum_1^\infty b_k$, so (1) holds. Property (2.10) holds by (2.17), (2.18) and the triangle inequality.

If $x \in \Omega$ then (2.11) holds by (2.16) and (2.17). If $x \notin \Omega$, we can find an arbitrarily small cube Q such that $|Q|^{-1} \int_Q |f(x)| dx < \alpha$. It follows that $|f(x)| < \alpha$ almost everywhere in the complement of Ω, so (2.11) holds.

The property (2.12) holds instantly by construction. To prove property (2.13), observe that the first part of () says that for $Q_k \in \Omega$,

$$|Q_k| \leq \alpha^{-1} \int_{Q_k} |f(x)|\ dx, \tag{2.19}$$

so

$$\sum |Q_k| \leq \sum_k \alpha^{-1} \int_{Q_k} |f(x)|\ dx \leq \alpha^{-1} \int |f(x)|\ dx, \tag{2.20}$$

and we are done.

Chapter 3: More interpolation and some applications of stopping time arguments

Theorem 3.1 (Riesz-Thorin interpolation theorem). *Let T be a linear map from $L^{p_0} \cap L^{p_1} \to L^{q_0} \cap L^{q_1}$ satisfying*

$$||Tf||_{q_j} \leq M_j ||f||_{p_j},\ j = 0, 1, \tag{3.1}$$

with $1 \leq p_j, q_j \leq \infty$. Then, if for $0 < t < 1$, $\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}$, $\frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$,

$$||Tf||_{q_t} \leq M_0^{1-t} M_1^t ||f||_{p_t},\ f \in L^{p_0} \cap L^{p_1}. \tag{3.2}$$

Proof of Riesz-Thorin. If $p_t = \infty$ the result follows from Holder's inequality, so we assume that $p_t < \infty$. It suffices to show that

$$\left|\int Tf(x)g(x)\ dx\right| \leq M_0^{1-t}M_1^t||f||_{p_t}||g||_{q_t'} \tag{3.3}$$

when f and g vanish outside of a set of finite measure and take on a finite number of values. More precisely,

$$f = \sum_1^m a_j\chi_{E_j}\ g = \sum_1^N b_k\chi_{F_k}, \tag{3.4}$$

with $E_j \cap E_{j'} = \emptyset$ and $F_k \cap F_{k'} = \emptyset$ if $j \neq j'$ and $k \neq k'$, respectively. We may also assume that $||f||_{p_t}$ and $||g||_{q_t'}$ are non-zero, so dividing both sides of (3) by norms, it suffices to prove the inequality with $||f||_{p_t} = ||g||_{q_t'} = 1$.

Next, if $a_j = e^{i\theta_j}|a_j|$ and $b_k = e^{i\psi_k}|b_k|$, then, assuming $q_t > 1$, we set

$$f_z = \sum_{j=1}^m |a_j|^{\frac{\alpha(z)}{\alpha(t)}}e^{i\theta_j}\chi_{E_j}, \tag{3.5}$$

and

$$g_z = \sum_{j=1}^N |b_k|^{\frac{1-\beta(z)}{1-\beta(t)}}e^{i\psi_k}\chi_{F_k}, \tag{3.6}$$

where $\alpha(z) = \frac{1-z}{p_0} + \frac{z}{p_1}$, and $\beta(z) = \frac{1-z}{q_0} + \frac{z}{q_1}$. If $q_t = 1$ we modify the definition by taking $g_z \equiv 1$.

It follows that the function

$$F(z) = \int Tf_z g_z dx \tag{3.7}$$

is entire and bounded in the strip $0 \leq Re(z) \leq 1$. Also, $F(t)$ equals the left hand side of (3). By Hadamard's three lines lemma, proved in complex analysis texts, it suffices to prove that

$$|F(z)| \leq M_0,\ Re(z) = 0, \tag{3.8}$$

and

$$|F(z)| \leq M_1,\ Re(z) = 1. \tag{3.9}$$

Exercise 3.1. *State a version of Hadamard's lemma needed here. Prove it using the maximum modulus principle.*

To prove the first inequality, observe that for $y \in \mathbb{R}$, $\alpha(iy) = \frac{1}{p_0} + iy\left(\frac{1}{p_1} - \frac{1}{p_0}\right)$. Consequently,

$$|f_{iy}|^{p_0} = \left| e^{iarg(f)} \cdot |f|^{iy\left(\frac{1}{p_1}-\frac{1}{p_0}\right)} |f|^{\frac{p_t}{p_0}} \right|^{p_0} = |f|^{p_t}, \tag{3.10}$$

and the same type of argument shows that

$$|g_{iy}|^{q_0'} = |g|^{q_t'}. \tag{3.11}$$

Applying (1) in conjunction with the Holder inequality yields

$$|F(iy)| \leq ||Tf_{iy}||_{q_0} ||g_{iy}||_{q_0'} \leq M_0 ||f_{iy}||_{p_0} ||g_{iy}||_{q_0'} = M_0 ||f||_{p_t}^{\frac{p_t}{p_0}} ||g||_{q_t'}^{\frac{q_t'}{q_0'}} = M_0, \tag{3.12}$$

so (3.8) is proved. The same argument gives (3.9), so we are done.

Exercise 3.2 (Young's inequality). *Let*

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y) dy, \tag{3.13}$$

and suppose that

$$\sup_x \left(\int |K(x,y)|^r dy \right)^{\frac{1}{r}}, \sup_y \left(\int |K(x,y)|^r dx \right)^{\frac{1}{r}} \leq C, \tag{3.14}$$

where $r \geq 1$ satisfies

$$\frac{1}{r} = 1 - \left(\frac{1}{p} - \frac{1}{q} \right), \tag{3.15}$$

for some $1 \leq p, q \leq \infty$. It then follows that

$$||Tf||_q \leq C ||f||_p. \tag{3.16}$$

Theorem 3.2 (Hardy-Littlewood-Sobolev inequality). *Let*

$$I_r f(x) = \int_{\mathbb{R}^d} |x-y|^{-\frac{d}{r}} f(y) dy. \tag{3.17}$$

If $r > 1$ and $\frac{1}{r} = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)$ for some $1 < p < q < \infty$, then

$$||I_r f||_{L^q(\mathbb{R}^d)} \leq C_{p,q} ||f||_{L^p(\mathbb{R}^d)}. \tag{3.18}$$

The proof is based on the Calderon-Zygmund decomposition and a series of auxilliary estimates.

Lemma 3.3 (Point-wise estimate). *If $1 \leq p \leq r'$ then*

$$||I_r f||_\infty \leq C_{p,r} ||f||_p^{\frac{p}{r'}} ||f||_\infty^{1-\frac{p}{r'}}, \ f \in L^p \cap L^\infty. \tag{3.19}$$

Lemma 3.5 (CZ prep). *Let $b \in L^1$ be supported in a cube Q and satisfy $\int_Q b(x)dx = 0$. Then*

$$\left(\int_{x \notin Q^*} |I_r b(x)|^r dx \right)^{\frac{1}{r}} \leq C_r ||b||_1. \tag{3.20}$$

Lemma 3.6 (CZ in action). *I_r is weak type $(1, r)$:*

$$|\{x : |I_r f(x)| > \gamma\}| \leq C_r (\gamma^{-1} ||f||_1)^r. \tag{3.21}$$

We now use these lemmas to prove the theorem. At the end, we shall go back and prove the lemmas.

As usual, we may assume that $||f||_p = 1$. We have

$$||I_r f||_q^q = q \int_0^\infty \gamma^{q-1} m(\gamma) d\gamma, \tag{3.22}$$

where

$$m(\gamma) = |\{x : |I_r f(x)| > \gamma\}|. \tag{3.23}$$

Let $f = f_0 + f_1$, where $f_0 = f$ when $|f| > \alpha$ and 0 otherwise. Then by Lemma Point-wise,

$$||I_r f_1||_\infty \leq C_{p,r} ||f_1||_p^{\frac{p}{r'}} ||f_1||_\infty^{1-\frac{p}{r'}} \leq C_{p,r} \alpha^{1-\frac{p}{r'}} = C_{p,r} \alpha^{\frac{p}{q}}. \tag{3.24}$$

By Lemma CZ in action,

$$m(\gamma) \leq |\{x : |I_r f(x)| > \gamma\}| \leq C_r(\gamma^{-1}||f_0||_1)^r \tag{3.25}$$

and we conclude that

$$||I_r f||_q^q \leq C \int_0^\infty \gamma^{q-1}(\gamma^{-1}||f_0||_1)^r d\gamma. \tag{3.26}$$

Let α so that $\frac{\gamma}{2} = C_{p,r}\alpha^{\frac{p}{q}}$. Making a change of variables and applying the Minkowski Integral Inequality, we see that the right hand side of (10) is bounded by

$$C'\left(\int \left(\int_0^{|f(x)|} \alpha^{-1+p-\frac{rp}{q}} d\alpha\right)^{\frac{1}{r}} |f(x)|dx\right)^r. \tag{3.27}$$

Since $q > r$ by assumption, we have

$$\left(\int_0^{|f(x)|} \alpha^{-1+p-\frac{rp}{q}} d\alpha\right)^{\frac{1}{r}} = |f(x)|^{p-1}, \tag{3.28}$$

and we are done since $||f||_p = 1$.

Let's now prove the lemmas.

Proof of Lemma Point-wise. For any $R > 0$,

$$|I_r f(x)| \leq \int_{|y|<R} |y|^{-\frac{d}{r}}|f(x-y)|dy + \int_{|y|>R} |y|^{-\frac{d}{r}}|f(x-y)|dy$$

$$\leq C(R^{d-\frac{d}{r}}||f||_\infty + (R^{d-\frac{dp'}{r}})^{\frac{1}{p'}}||f||_p)$$

$$= C(R^{\frac{d}{r'}}||f||_\infty + R^{-\frac{d}{q}}||f||_p), \tag{3.29}$$

and the lemma follows by choosing R so that the two terms agree.

Proof of Lemma CZ prep. We may assume that Q is the cube of side-length R centered at the origin. Then, for $x \notin Q^*$, the double of Q, we have

$$|I_r b(x)| \leq \int \left||x-y|^{-\frac{d}{r}} - |x|^{-\frac{d}{r}}\right| |b(y)|dy$$

$$\leq CR|x|^{-1-\frac{d}{r}}||b||_1 \tag{3.30}$$

by the mean value theorem. Integrating this inequality yields the result since

$$\left(\int_{x\notin Q^*} |x|^{-r-d}dx\right)^{\frac{1}{r}} = \frac{C}{R}. \tag{3.31}$$

Proof of Lemma CZ in action. Write $f = g + \sum b_k$ by the Calderon-Zygmund lemma. Let $||f||_1 = 1$. By Lemma Point-wise,

$$|I_r g| \leq ||g||_1^{\frac{1}{r'}} ||g||_\infty^{1-\frac{1}{r'}} \leq C\alpha^{1-\frac{1}{r'}} = C\alpha^{\frac{1}{r}}. \tag{3.32}$$

Let $C\alpha^{\frac{1}{r}} = \frac{\gamma}{2}$. Then

$$|\{x : |I_r f(x)| > \gamma\}| \leq |\{x : \sum |I_r b_k(x)| > \frac{\gamma}{2}\}|. \tag{3.33}$$

If $\Omega^* = \cup Q_k^*$, then

$$|\Omega^*| \leq 2^d \alpha^{-1} = C'\gamma^{-r}, \tag{3.34}$$

while Lemma CZ prep gives

$$\left|\left\{x \notin \Omega^* : \sum |I_r b_k(x)| > \frac{\gamma}{2}\right\}\right|^{\frac{1}{r}}$$

$$\leq \left(\frac{\gamma}{2}\right)^{-1} \sum \left(\int_{x\notin\Omega^*} |I_r b_k(x)|^r dx\right)^{\frac{1}{r}} \leq C'\gamma^{-1}. \tag{3.35}$$

Combining the inequalities yields the proof.

We now deduce the d-dimensional version of the Hardy-Littlewood-Sobolev inequality from the one-dimensional version. This technique, which involves freezing variables in an appropriate way, is extremely useful and will come up again.

Let $x = (x', x_d)$. For a fixed x_d,

$$||I_r f(\cdot, x_d)||_{L^q(\mathbb{R}^{d-1})} \leq \int_{-\infty}^{\infty} \left(\int_{\mathbb{R}^{d-1}} \left|\int_{\mathbb{R}^{d-1}} f(y)|x-y|^{-\frac{d}{r}} dy'\right|^q dx'\right)^{\frac{1}{q}} dy_d. \tag{3.36}$$

If we wish to use Young's inequality we must estimate

$$\left(\int_{\mathbb{R}^{d-1}} |x-y|^{-\frac{d}{r}\cdot r} dx'\right)^{\frac{1}{r}} = \left(\int_{\mathbb{R}^{d-1}} |(x', x_d - y_d)|^{-\frac{d}{r}\cdot r} dx'\right)^{\frac{1}{r}} = (C_d|x_d - y_d|^{-1})^{\frac{1}{r}}. \tag{3.37}$$

Young's inequality yields

$$||I_r f(\cdot, x_d)||_{L^q(\mathbb{R}^{d-1})} \leq C \int_{-\infty}^{\infty} |x_d - y_d|^{-\frac{1}{r}} ||f(\cdot, x_d)||_{L^p(\mathbb{R}^{d-1})} dy_d, \tag{3.38}$$

and the Hardy-Littlewood-Sobolev inequality follows from its one-dimensional version.

Let's state this principle in a general form.

Lemma 3.7 (Frozen variable lemma). *Let $K(x,y)$ be a measurable function on $\mathbb{R}^d \times \mathbb{R}^d$. Let*

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy, \tag{3.39}$$

and, for a fixed x_d and y_d define

$$T_{x_d,y_d}g(x') = \int_{\mathbb{R}^{d-1}} K(x',x_d,y',y_d)g(y')dy', \tag{3.40}$$

and suppose that

$$||T_{x_d,y_d}g||_{L^q(\mathbb{R}^{d-1})} \le C_0|x_d - y_d|^{-\frac{1}{r}}||g||_{L^p(\mathbb{R}^{d-1})}. \tag{3.41}$$

Then if $1 < p < q < \infty$ and $\frac{1}{r} = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)$ it follows that

$$||Tf||_{L^q(\mathbb{R}^d)} \le C||f||_{L^p(\mathbb{R}^d)}, \tag{3.42}$$

with $C = C_0 C_{p,q}$, where $C_{p,q}$ is the constant in the one-dimensional Hardy-Littlewood-Sobolev inequality.

Chapter 4: Fourier transform

We now give in to temptation and define the Fourier transform. For $f \in L^1(\mathbb{R}^d)$ define

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)dx. \tag{4.1}$$

Lemma 4.1 (Elementary Fourier facts). *Let $h \in \mathbb{R}^d$, $\tau_h f(x) = f(x+h)$. Then*

$$\widehat{\tau_h f}(\xi) = e^{2\pi i h\cdot\xi}\hat{f}(\xi). \tag{4.2}$$

We have

$$||\hat{f}||_\infty \le ||f||_1. \tag{4.3}$$

$$\text{If } f \in L^1(\mathbb{R}^d), \text{ then } \hat{f} \text{ is uniformly continuous.} \tag{4.4}$$

If $f \in L^1(\mathbb{R}^d)$, then $\hat{f} \to 0$ as $\xi \to \infty$, and, hence, $\hat{f} \in C_0(\mathbb{R}^d)$ (Riemann-Lebesgue lemma).

The equality (4.2), the estimate (4.3), and claim (4.4) follow directly from the definition of the Fourier transform. The Riemann-Lebesgue lemma is established by proving the result for the characteristic function of a cube and passing to the limit.

Our goal is to prove an inversion formula of the form

$$f(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\hat{f}(\xi)d\xi. \tag{4.5}$$

Unfortunately, this integral does not in general converge for $f \in L^1(\mathbb{R}^d)$, so we are led to consider an appropriate sub-space and pass to the limit.

Definition. The space of Schwartz-class functions, $\mathcal{S}(\mathbb{R}^d)$, consists of all $\phi \in C^\infty(\mathbb{R}^d)$ satisfying

$$\sup_x |x^\gamma \partial^\alpha \phi(x)| < \infty \tag{4.6}$$

for all multi-indices α, γ. These semi-norms induce a topology on $\mathcal{S}(\mathbb{R}^d)$ that make it a Frechet space. Observe that $C_0^\infty(\mathbb{R}^d)$ is contained in $\mathcal{S}(\mathbb{R}^d)$.

Let $D_j = \frac{1}{2\pi i}\frac{\partial}{\partial x_j}$.

Lemma. *Let $\phi \in \mathcal{S}$. Then*

$$\widehat{D_j\phi} = -D_j\hat{\phi}. \tag{4.7}$$

Consequently,

$$\xi^\alpha D^\gamma \hat{\phi}(\xi) = \int e^{-2\pi i x\cdot\xi} D^\alpha((-x)^\gamma \phi(x))dx, \tag{4.8}$$

and

$$\sup_\xi |\xi^\gamma \hat{\phi}(\xi)| \le C \sup_x (1+|x|)^{d+1} |D^\gamma(x^\alpha \phi(x))|, \tag{4.9}$$

which implies, in particular, that the Fourier transform maps $\mathcal{S}$ into itself.

The proof is straightforward integration by parts.

Theorem 4.2 (Isomorphism). *The Fourier transform is an isomorphism of $\mathcal{S}$ whose inverse is given by* (5)*.*

The proof is based on a couple of lemmas.

Lemma Symmetry. *If $f, g \in L^1$ then*

$$\int_{\mathbb{R}^d} \hat{f} g dx = \int_{\mathbb{R}^d} f \hat{g} dx. \tag{4.10}$$

Lemma 4.3 (Gaussian). *We have*

$$\int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} e^{-\pi|x|^2} dx = e^{-\pi|\xi|^2}. \tag{4.11}$$

It follows that if $\Gamma_\epsilon(x) = e^{-\pi\epsilon^2|x|^2}$, then $\hat{\Gamma}_\epsilon(\xi) = \epsilon^{-d} e^{-\pi\frac{|\xi|^2}{\epsilon^2}}$. The proofs of the lemmas are routine and are left as exercises.

We now prove Theorem Isomorphism. We have

$$\phi(x) = \int \int e^{2\pi i x\cdot\xi} \hat{\phi}(\xi) d\xi$$

$$= \lim_{\epsilon \to 0} \int e^{2\pi i x\cdot\xi} \hat{\phi}(\xi) e^{-\pi\epsilon^2|\xi|^2} d\xi$$

$$= \lim_{\epsilon \to 0} \epsilon^{-d} \int \phi(x+y) e^{-\pi \frac{|y|^2}{\epsilon^2}} dy = \phi(x) \tag{4.12}$$

since

$$\int e^{-\pi|y|^2} dy = 1 \tag{4.13}$$

and the proof is complete.

Lemma 4.4 (More elementary properties). *If $\phi, \psi \in \mathcal{S}$, then*

$$\int \phi\overline{\psi} = \int \hat{\phi}\overline{\hat{\psi}}, \tag{4.14}$$

$$\widehat{\phi * \psi}(\xi) = \hat{\phi}(\xi)\hat{\psi}(\xi), \tag{4.15}$$

and

$$\widehat{\phi\psi}(\xi) = \hat{\phi} * \hat{\psi}(\xi), \tag{4.16}$$

where

$$\phi * \psi(x) = \int \phi(x-y)\psi(y) dy. \tag{4.17}$$

The proof of (4.14) follows from the inversion formula. The proof of (4.15) results from a straight-forward change of variables followed by Fubini argument, and (4.17) follows from (4.16) using the inversion formula once more.

Chapter 5: Basic facts about distributions

Denote the dual space of $\mathcal{S}$ by $\mathcal{S}'$, the space of tempered distributions.

Fourier transform of distributions. Let $u \in \mathcal{S}'$. Then $\hat{u}$ is defined by the relation

$$\hat{u}(\phi) = u(\hat{\phi}). \tag{5.1}$$

Exercise 5.1. *When $u \in L^1$ (5.1) agrees with the previous definition of the Fourier transform. Moreover, $u \to \hat{u}$ is an isomorphism on $\mathcal{S}'$. Conclude that if $u, \hat{u} \in L^1$, the inversion formula must hold for almost every x.*

Theorem 5.1 (Plancherel and Parseval). *If $u \in L^2$, then $\hat{u} \in L^2$ and*

$$||\hat{u}||_2 = ||u||_2, \tag{5.2}$$

and, whenever $\phi, \psi \in L^2$,

$$\int \phi\overline{\psi} = \int \hat{\phi}\overline{\hat{\psi}}. \tag{5.3}$$

Proof of Plancherel and Parseval. Chose $u_j \in \mathcal{S}$ such that $u_j \to u$ in L^2. Then by (4.14) on this page,

$$||\hat{u}_j - \hat{u}_k||_2^2 = ||u_j - u_k||_2^2 \to 0. \tag{5.4}$$

It follows that $\hat{u}_j \to v \in L^2$ and the continuity of the Fourier transform on $\mathcal{S}'$ forces $v = \hat{u}$. This gives (5.2) and (5.3) just follows from (4.14) and the fact that $\mathcal{S}$ is dense in L^2.

Exercise 5.2. *Verify that $\mathcal{S}$ is dense in L^2.*

Theorem 5.3 (Hausdorff-Young). *Let $1 \le p \le 2$ and let p' be the dual exponent of p. Then if $f \in L^p$ it follows that $\hat{f} \in L^{p'}$ and*

$$||\hat{f}||_{p'} \le ||f||_p. \tag{5.5}$$

Proof of Hausdorff-Young. We know by (4.3) that $||\hat{f}||_\infty \le ||f||_1$, and $||\hat{f}||_2 = ||f||_2$ by Plancherel. The result follows by Riesz-Thorin.

If $u \in \mathcal{S}'(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ and $T : \mathbb{R}^d \to \mathbb{R}^d$ is a linear transformation, we define the pullback of u under T by

$$T^*u = u(T). \tag{5.6}$$

Exercise 5.3. *We have*

$$\widehat{T^* u} = |T|^{-1} \left((T^t)^{-1}\right)^* \hat{u}. \tag{5.7}$$

We say that u is homogeneous of degree σ if $M_t^* u = t^\sigma u$, where $M_t x = tx$. Consequently, if $u \in \mathcal{S}'(\mathbb{R}^d)$ is homogeneous of degree σ, then $\hat{u}$ is homogeneous of degree $-d-\sigma$.

Furthermore, if $Re(\sigma) < -d$, then $\hat{u}$ is continuous. Conclude that if $u \in C^\infty(\mathbb{R}^d \setminus (0,\dots,0))$ is homogeneous, then $\hat{u}$ is also.

Theorem 5.4 (Poisson Summation Formula). *If $\phi \in \mathcal{S}(\mathbb{R}^d)$ then*

$$\sum_{m\in\mathbb{Z}^d} \phi(x+m) = \sum_{m\in\mathbb{Z}^d} \hat{\phi}(m) e^{2\pi i x\cdot m}, \tag{5.8}$$

and, in particular,

$$\sum_{m\in\mathbb{Z}^d} \phi(m) = \sum_{m\in\mathbb{Z}^d} \hat{\phi}(m). \tag{5.9}$$

Proof of Poisson Summation Formula. Let $Q = [0,1]^d$. It is clear that the left hand side of (5.8) converges uniformly in $L^1(Q)$ norm. The Fourier coefficients of this functions are given by

$$g_k = \int_Q e^{-2\pi i x\cdot k} g(x) dx = \int_Q e^{-2\pi i x\cdot k} \sum_{m\in\mathbb{Z}^d} \phi(x+m) dx$$

$$= \sum_{m\in\mathbb{Z}^d} \int_{Q-m} e^{-2\pi i x\cdot k} \phi(x) dx = \hat{\phi}(k). \tag{5.10}$$

On the hand, let h equal to the right hand side of (5.8). It also converges in $L^1(Q)$ norm and the the Fourier coefficients are given by

$$h_k = \int_Q e^{-2\pi i x\cdot k} \sum_{m\in\mathbb{Z}^d} \hat{\phi}(m) e^{2\pi i x\cdot m} = \sum_{m\in\mathbb{Z}^d} \hat{\phi}(m) \int_Q e^{-2\pi i x\cdot(m-k)} dx = \hat{\phi}(k). \tag{5.11}$$

Thus the proof of Poisson Summation Formula would be complete if we prove the following.

Lemma 5.5 (Approximation). *If μ is a Borel measure on $\mathbb{T}^d$ satisfying*

$$\int_{\mathbb{T}^d} e^{-2\pi i x\cdot k} d\mu(x), \tag{5.12}$$

for all $k \in \mathbb{Z}^d$, then $\mu = 0$.

To prove the lemma observe that (5.12) implies that

$$\int_{\mathbb{T}^d} f(x) d\mu(x) = 0 \tag{5.13}$$

for every trigonometric polynomial. Stone-Weierstrass theorem implies that (5.13) holds for every continuous function. Finally, the Riesz Representation Theorem implies that if (5.13) holds for every continuous function then μ is indeed 0.

Exercise 5.4. *If $g \in L^2(\mathbb{T}^d)$ then $\sum_{k\in\mathbb{Z}^d}$ converges to g in L^2 norm and*

$$\int_{\mathbb{T}^d} |g(x)|^2 dx = \sum_{k\in\mathbb{Z}^d} |g_k|^2. \tag{5.14}$$

Chapter 6: Multiplier operators

Let $\triangle$ denote the usual Laplace operator on $\mathbb{R}^d$. Then if f is a function for which the Fourier inversion formula applies, we have

$$-\triangle f(x) = \int e^{2\pi i x\cdot\xi} 4\pi^2 |\xi|^2 \hat{f}(\xi) d\xi. \tag{6.1}$$

Operators of this types are called multiplier operators. More precisely, T_m is a multiplier operator with the multiplier m if

$$T_m f(x) = \int e^{2\pi i x\cdot\xi} m(\xi) \hat{f}(\xi) d\xi. \tag{6.2}$$

Plancherel implies that if $m \in L^\infty$ then $||T_m f||_2 \leq C||f||_2$. The question we ask is, under what conditions if T_m bounded on L^p for $p > 1$.

Theorem 6.1 (Hormander Multiplier Theorem). *Let $m \in L^\infty$. Assume further that, for some integer $s > \frac{d}{2}$,*

$$\sum_{0\leq|\alpha|\leq s} \sup_\lambda \lambda^{-d} ||\lambda^{|\alpha|} D^\alpha \beta(\cdot/\lambda) m(\cdot)||_{L^2(\mathbb{R}^d)} < \infty \tag{6.3}$$

whenever $\beta \in C_0^\infty(\mathbb{R}^d \setminus (0,\dots,0))$. It then follows that for $1 < p < \infty$

$$||T_m f||_{L^p(\mathbb{R}^d)} \leq C_p ||f||_{L^p(\mathbb{R}^d)}. \tag{6.4}$$

Moreover, T_m is weak type $(1,1)$.

By Marcienkiewicz interpolation theorem it suffices to prove the weak $(1,1)$ assertion because the L^2 bounded-ness follows from Plancherel and $2 < p < \infty$ bounded-ness follows from $1 < p < 2$ bounded-ness by duality.

Proof of Hormander Multiplier Theorem. Construct a function $\beta \in C_0^\infty(\mathbb{R}^d \setminus (0,\dots,0))$ such that

$$\sum_{-\infty}^{\infty} \beta(2^{-j}\xi) = 1, \tag{6.5}$$

for $\xi \neq (0, \ldots, 0)$. (Carry out the details of the construction of such a function).

Let

$$m_\lambda = \beta(\xi) m(\lambda \xi). \tag{6.6}$$

Making a change of scale we see that the condition (6.3) is equivalent to

$$\int_{\mathbb{R}^d} \left|(I - \triangle)^{\frac{s}{2}} m_\lambda(\xi)\right|^2 d\xi \leq C. \tag{6.7}$$

Equivalently, if $\hat{K}_\lambda = m$, then

$$\int_{\mathbb{R}^d} |K_\lambda(x)|^2 (1 + |x|^2)^s dx \leq C. \tag{6.8}$$

By Cauchy-Schwartz,

$$\int_{\{x : \max |x_j| > R\}} |K_\lambda(x)| dx \leq C(1 + R)^{\frac{d}{2} - s} \leq C, \tag{6.9}$$

and, by the same reasoning,

$$\int_{\{x : \max |x_j| > R\}} |\nabla K_\lambda(x)| dx \leq C. \tag{6.10}$$

Mean value theorem implies that

$$\int_{\{x : \max |x_j| > R\}} |K_\lambda(x + y) - K_\lambda(x)| dx \leq C|y|. \tag{6.11}$$

Let $f = g + \sum b_k$ denote the usual Calderon-Zygmund decomposition. We first prove that

$$\int_{x \notin Q_k^*} |T_m b_k(x)| dx = \int_{x \notin Q_k^*} |K * b_k(x)| dx \leq C \int |b_k(x)| dx, \tag{6.12}$$

where we assume without loss of generality that

$$Q_k = \{x : \max |x_j| \leq R\}, \tag{6.13}$$

and $\hat{K} = m$.

We have

$$K(x) = \sum_{-\infty}^{\infty} 2^{dj} K_{2^j}(2^j x) \tag{6.14}$$

with convergence in $\mathcal{S}'$.

By (6.9),

$$\int_{x\notin Q_k^*} |\lambda^d K_\lambda(\lambda\cdot) * b_k(x)|dx \le ||b_k||_1 \int_{\{x:\max|x_j|>\lambda R\}} |K_\lambda(x)|dx \le C(R\lambda)^{\frac{d}{2}-s}||b_k||_1, \tag{6.15}$$

which is cool if $R\lambda$ is bounded.

On the other hand,

$$K_\lambda(\lambda\cdot) * b_k(x) = \int (K_\lambda(\lambda(x-y)) - K_\lambda(\lambda x))\, b_k(y)dy, \tag{6.16}$$

so (6.10) implies that

$$\int_{x\notin Q_k^*} |\lambda^d K_\lambda(\lambda\cdot) * b_k(x)|dx \le C(R\lambda)||b_k||_1. \tag{6.17}$$

By the triangle inequality and (6.14),

$$\int_{x\notin Q_k^*} |\lambda^d K(\lambda\cdot) * b_k(x)|dx \le C||b_k||_1 \left(\sum_{2^jR\ge 1} (2^jR)^{\frac{d}{2}-s} + \sum_{2^jR<1} 2^jR \right) \tag{6.18}$$

and (6.12) is proved.

We also have

$$\int |g(x)|^2 dx \le 2^d\alpha \int |g(x)|dx, \tag{6.19}$$

so by Chebyshev

$$|\{x : |T_m g(x)| > \frac{\alpha}{2}\}| \le C\alpha^{-2}||g||_2^2 \le C'\alpha^{-1}||f||_1. \tag{6.20}$$

Finally, $|\Omega^*| \le 2^d\alpha^{-1}||f||_1$ and

$$|\{x \notin \Omega^* : |T_m \sum_k b_k(x)| > \frac{\alpha}{2}\}| \le 2\alpha^{-1} \sum_k \int_{x\notin Q_k^*} |K * b_k(x)|dx \le C'||f||_1 \tag{6.21}$$

and the proof is complete by combining all the inequalities.

Theorem 6.2 (Littlewood-Paley). *Let $S_j f(x)$ be defined by*

$$\widehat{S_j f}(\xi) = \beta(2^{-j}\xi), \tag{6.22}$$

where β is as in the proof of the Hormander multiplier theorem, i.e $\beta \in C_0^\infty(1/2, 2)$ and $\sum \beta(2^j\xi) = 1$. Let

$$Sf(x) = \left(\sum_{-\infty}^{\infty} |S_j f(x)|^2\right)^{\frac{1}{2}}. \tag{6.23}$$

If $1 < p < \infty$ there is a constant C_p such that

$$C_p^{-1}||f||_{L^p(\mathbb{R}^d)} \le ||Sf||_{L^p(\mathbb{R}^d)} \le C_p||f||_{L^p(\mathbb{R}^d)}. \tag{6.24}$$

We first prove that it is enough to show that

$$||Sf||_{L^p(\mathbb{R}^d)} \le C_p||f||_{L^p(\mathbb{R}^d)}. \tag{6.25}$$

Since $\beta(2^{-j}\xi)\beta(2^{-j+l}\xi) = 0$ for $|l| > 1$, Parseval implies that

$$\int f\overline{g} = \sum_{|j-l|\le 1} S_j f\overline{S_l g}. \tag{6.26}$$

Cauchy-Schwartz followed by Holder implies that

$$\left|\int f\overline{g}\right| \le C||Sf||_p||Sg||_{p'}. \tag{6.27}$$

Taking a sup over $||g||_{p'} = 1$ implies that $C_p^{-1}||f||_{L^p(\mathbb{R}^d)} \le ||Sf||_{L^p(\mathbb{R}^d)}$ provided that (6.25) holds.

Rademacher functions. Let $r_0(t) = 1$ on $[0, 1/2]$ and $r_0(t) = -1$ on $(1/2, 1)$. We extend r_0 periodically so that $r_0(1 + t) = r_0(t)$ and set $r_j(t) = r_0(2^j t)$.

Exercise 6.1. *Check that $\{r_j\}$ is an orthonormal system on* $[0, 1]$.

Lemma 6.3 (Khinchin/Rademacher). *Let $F(t) = \sum a_j r_j(t)$. Then*

$$A_p^{-1}||F||_{L^p([0,1])} \le ||F||_{L^2([0,1])} = \left(\sum |a_j|^2\right)^{\frac{1}{2}} \le A_p||F||_{L^p([0,1])}. \tag{6.28}$$

We shall prove the lemma in moment. Let us first prove the theorem. Let $T_t f$ be defined by

$$\widehat{T_t f}(\xi) = \sum_{j=0}^{\infty} r_j(t)\beta(2^{-j}\xi)\hat{f}(\xi) = m_t(\xi)\hat{f}(\xi). \tag{6.29}$$

By the lemma,

$$\left(\sum_0^{\infty} |S_j f(x)|^2\right)^{\frac{p}{2}} \leq A_p \int_0^1 \int_{\mathbb{R}^d} |T_t f(x)|^p dx dt \leq C_p ||f||_p^p. \tag{6.30}$$

The same argument takes care of $\left(\sum_{j<0} |S_j f(x)|^2\right)^{\frac{1}{2}}$ and we are done.

We now prove the lemma. We will actually prove something slightly more general called Khinchin's inequality. Let $\{\omega_n\}_{n=1}^N$ be independent random variables taking values ± 1 with equal probability. We shall prove that

$$\mathbb{E}\left(\left|\sum_1^N a_n \omega_n\right|^p\right) \approx \left(\sum_1^N |a_n|^2\right)^{\frac{p}{2}} \tag{6.31}$$

from which the lemma certainly follows.

We have

$$\mathbb{E}\left(e^{t\sum_n a_n\omega_n}\right) = \Pi_n \mathbb{E}(e^{ta_n\omega_n}) = \Pi_n \frac{1}{2}(e^{ta_n} + e^{-ta_n}), \tag{6.32}$$

where the first equality follows by independence.

Since $\frac{1}{2}(e^x + e^{-x}) \leq e^{\frac{x^2}{2}}$,

$$\mathbb{E}(e^{t\sum_n a_n\omega_n}) \leq e^{\frac{t^2}{2}\sum_n a_n^2}. \tag{6.33}$$

It follows that

$$Prob\left(\sum_n a_n\omega_n \geq \lambda\right) \leq e^{-t\lambda + \frac{t^2}{2}\sum_n a_n^2} \tag{6.34}$$

for any $t > 0$ and $\lambda > 0$.

Taking $t = \frac{\lambda}{\sum_n a_n^2}$ gives

$$Prob\left(\sum_n a_n\omega_n \geq \lambda\right) \leq e^{-\frac{\lambda^2}{2\sum_n a_n^2}}, \tag{6.35}$$

which implies that

$$Prob\left(\left|\sum_n a_n\omega_n\right| \geq \lambda\right) \leq 2e^{-\frac{\lambda^2}{2\sum_n a_n^2}}. \tag{6.36}$$

Since

$$\mathbb{E}(|f|^p) = p\int \lambda^{p-1} Prob(|f| \geq \lambda) d\lambda, \tag{6.37}$$

we have

$$\mathbb{E}\left(\left|\sum_n a_n\omega_n\right|\right) \leq 2p\int \lambda^{p-1} e^{-\frac{\lambda^2}{2\sum_n a_n^2}} d\lambda = 2^{2+\frac{p}{2}} p\gamma(p/2)\left(\sum_n a_n^2\right)^{\frac{p}{2}}, \tag{6.38}$$

and the upper bound is proved.

The lower bound follows from the upper bound and duality. (Carry out the details).

Exercise 6.2. *(Somewhat challenging) Use Khinchin's inequality that Hausdorff-Young inequality is sharp on the scale of L^p spaces. Hint: Take $\phi \in C_0^\infty$ and chose k_js such that $\phi_j = \phi(\cdot - k_j)$ have disjoint supports. Let $\widehat{\phi}_n(\xi) = e^{2\pi i\xi\cdot k_n}\hat{\phi}(\xi)$. Now consider $\sum_{n\le N}\omega_n\phi_n$ and go from there.*

We complete the section on multiplier by proving the Sobolev embedding theorem. For $1 \le p \le \infty$, we define Sobolev spaces $L^p_s(\mathbb{R}^d)$ as a set of all $u \in \mathcal{S}'$ such that $(I-\triangle)^{\frac{s}{2}}u$ is a function and

$$||u||_{L^p_s(\mathbb{R}^d)} = ||(I-\triangle)u)^{\frac{s}{2}}u||_{L^p(\mathbb{R}^d)} < \infty. \tag{6.39}$$

Theorem 6.4 (Sobolev embedding theorem). *If $1 < p \le q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{s}{d}$, then $L^p_s(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ and the inclusion is continuous.*

If $s > \frac{d}{p}$ and $p \ge 1$, then $L^p_s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ and any $u \in L^p_s(\mathbb{R}^d)$ can be modified on a set of measure 0 such that it is continuous.

The proof requires the following basic estimate.

Lemma 6.5 (Kernel estimate). *Let*

$$K_s(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}(1+|\xi|^2)^{-\frac{s}{2}}d\xi. \tag{6.40}$$

If $s > 0$, then K_s is a function. Moreover, for $N > 0$ there exists C_N such that

$$|K_s(x)| \le C_N|x|^{-d+s}(1+|x|)^{-N}. \tag{6.41}$$

We mostly leave the proof as an exercise with the following outline. We have

$$K_s(x) = C_d|x|^{-2m}\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}(-\triangle)^m(1+|\xi|^2)^{-\frac{s}{2}}d\xi. \tag{6.42}$$

If $|x| \ge 1$, (6.42) with $m > \frac{d}{2}$ does the job. If $|x| \le 1$, divide K_s into two pieces using a smooth cut-off function $\rho(\xi/R)$, with support of ρ contained in the unit ball. On the support of $\rho(\xi/R)$, the integral is bounded by CR^{d-s} by brute force. On the support of $1-\rho(\xi/R)$ integrate by parts as in (6.42) and obtain the estimate $C|x|^{-2m}R^{d-s-2m}$. Setting $R = |x|^{-1}$ completes the proof.

We are now ready to complete the proof of Sobolev imbedding theorem. We must to prove that

$$||u||_{L^q(\mathbb{R}^d)} \le C||(I-\triangle)^{\frac{s}{2}}u||_{L^p(\mathbb{R}^d)}. \tag{6.43}$$

We may assume that $s > 0$ since otherwise (6.43) holds instantly. Instead of proving (6.43) we shall prove that

$$||(I-\triangle)^{\frac{s}{2}}u||_{L^q(\mathbb{R}^d)} \le C||v||_{L^p(\mathbb{R}^d)}. \tag{6.44}$$

The kernel of the operator on the left hand side is K_s. By the lemma above,

$$|K_s(x)| \leq C|x|^{-\frac{d}{r}}, \tag{6.45}$$

where $\frac{1}{r} = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)$.

The first part of the Sobolev embedding theorem now follows by the Hardy-Littlewood-Sobolev inequality.

To prove the second part, observe that $K_s \in L^{p'}(\mathbb{R}^d)$ for $s > \frac{d}{p}$. Therefore

$$||u(\cdot + y) - u(\cdot)||_\infty = ||(1 - \triangle)^{\frac{s}{2}}[v(\cdot + y) - v(\cdot)]||_\infty$$

$$\leq C||v(\cdot + y) - v(\cdot)||_p \to 0 \tag{6.46}$$

as $y \to 0$, so u can be modified on the set of measure 0 to make it continuous.

Chapter 7: The method of stationary phase

The purpose of this section is to estimate integrals of the form

$$I(t) = \int_{\mathbb{R}^d} e^{2\pi i t\phi(x)} \psi(x) dx, \tag{7.1}$$

where $\phi : \mathbb{R}^d \to \mathbb{R}$ has some degree of regularity and ψ is a compactly supported function.

The basic idea is that if ϕ oscillates sufficiently fast, the integral defining $I(t)$ has much cancellation, and so $I(t)$ should be small as $t \to \infty$. It is reasonable that derivatives of ϕ should be used to measure the degree of this oscillation.

Theorem 7.1 (Non-stationary phase). *Let $I(t)$ be as above with $\psi \in C_0^\infty(\mathbb{R}^d)$, $\phi \in C^\infty(\mathbb{R}^d)$, and $|\nabla\phi(x)| \geq c > 0$ on the support of ψ. Then for any $N > 0$ there exists C_N such that*

$$|I(t)| \leq C_N(1 + t)^{-N}. \tag{7.2}$$

Proof of Theorem. We have

$$I(t) = \int \frac{1}{2\pi i t\phi_j(x)} \frac{\partial}{\partial x_j} \left(e^{2\pi i t\phi(x)}\right) \psi(x) dx, \tag{7.3}$$

where ϕ_j denotes the partial derivative of ϕ with respect to x_j.

Integration by parts yields,

$$I(t) = \frac{1}{t} \int e^{2\pi i t\phi(x)} \frac{\partial}{\partial x_j} \left(\frac{\psi(x)}{2\pi i \phi_j(x)}\right) dx. \tag{7.4}$$

The result follows by a repeated application of this step.

Exercise. *Show that the above result does not hold, even in a one variable setting if $\psi \in C_0^\infty$ is replaced by the characteristic function of an interval.*

Lemma 7.2 (van der Corput lemma). *Let $I(t)$ be as above with $d = 1$ and $\psi(x) = \chi_{(a,b)}(x)$. Suppose that $\phi \in C^1$, $\phi'(x) \geq 1$, and ϕ' is monotone increasing or decreasing. Then*

$$|I(t)| \leq C(1+t)^{-1}. \tag{7.5}$$

Suppose that $\phi''(t) \geq 1$. Then

$$|I(t)| \leq C(1+t)^{-\frac{1}{2}}. \tag{7.6}$$

In both cases C is independent of a and b.

Proof of lemma. We have

$$\int_a^b e^{2\pi i t\phi(x)} dx = \int_a^b \frac{1}{2\pi i t\phi'(x)} \frac{d}{dx}\left(e^{2\pi i t\phi(x)}\right) dx$$

$$= \frac{e^{2\pi i t\phi(x)}}{2\pi i t\phi'(x)} \,\Big|_a^b - \int_a^b e^{2\pi i t\phi(x)} \frac{d}{dx}\left(\frac{1}{2\pi i t\phi'(x)}\right) dx = A + B. \tag{7.7}$$

It is clear that $|A| \leq Ct^{-1}$. Now the fun begins. We have

$$|B| \leq \int_a^b \left|\frac{1}{2\pi i t\phi'(x)}\right| dx = \left|\int_a^b \frac{1}{2\pi i t\phi'(x)} dx\right| \leq Ct^{-1} \tag{7.8}$$

by the fundamental theorem of calculus.

To prove the second part of the theorem, observe that there is at most one point x_0 such that $\phi'(x_0) = 0$. If no such point exists, we are done by the first part of the theorem. Thus we have

$$\int_a^b e^{2\pi i t\phi(x)} dx = \int_a^{x_a-\delta} e^{2\pi i t\phi(x)} dx + \int_{x_0-\delta}^{x_0+\delta} e^{2\pi i t\phi(x)} dx + \int_{x_0+\delta}^b e^{2\pi i t\phi(x)} dx = A_1 + A_2 + A_3, \tag{7.9}$$

where δ is to be determined later.

We easily see that $|A_2| \leq 2\delta$. To handle A_1 and A_3, observe that on $[a, x_0-\delta]$ or $[x_0+\delta, b]$ we have $\phi'(x) \geq C\delta$ by the mean value theorem. By the first part of the theorem,

$$|A_1|, |A_3| \leq C\delta^{-1}t^{-1}. \tag{7.10}$$

It follows that

$$|I(t)| \leq 2\delta + C\delta^{-1}t^{-1}. \tag{7.11}$$

Choosing $\delta = t^{-\frac{1}{2}}$ completes the proof.

Exercise 7.1. *In the second part of van der Corput theorem, suppose that $\phi^{(m)}(x) \geq 1$ for some $m \geq 2$. Prove that $|I(t)| \leq C(1+t)^{-\frac{1}{m}}$.*

Exercise 7.2 (slightly harder). *Let $\theta \in [0,1]$. Prove that*

$$\left| \sum_a^b e^{2\pi i n^2 \theta} \right| \leq C \left((b-a)\sqrt{\theta} + \frac{1}{\sqrt{\theta}} \right). \tag{7.12}$$

Hint: Use Poisson summation followed by van der Corput theorem. A difficulty will arise along the way, but you will deal with it...

Chapter 8: Basic applications of the method of stationary phase

Schrodinger operator and restriction theory. Consider the Schrodinger equation

$$\triangle u = 2\pi i \frac{\partial u}{\partial t}, \tag{8.1.1}$$

$$u(x,0) = f(x). \tag{8.1.2}$$

Taking the Fourier transform of both sides we get

$$-|\xi|^2 \hat{u} = i\hat{u}_t, \tag{8.1.3}$$

where u_t denotes the time derivative with respect to t.

Solving the differential equation, we see that

$$u(x,t) = \int e^{2\pi i x \cdot \xi + t|\xi|^2} \hat{f}(\xi) d\xi. \tag{8.1.4}$$

A question often asked in the theory of partial differential equations is whether the solution is more regular than the initial data. In the context of the Schrodinger equation one might ask whether there exists $q > 2$ such that

$$\left(\int \int |u(x,t)|^q dxdt \right)^{\frac{1}{q}} \leq C_q ||f||_{L^2(\mathbb{R}^d)}. \tag{8.1.5}$$

This question can be rephrased as follows with a slight change of notation. Let S denote the paraboloid $\{(\xi, |\xi|^2)\}$ in $\mathbb{R}^d$ (instead of $\mathbb{R}^{d+1}$. Let σ denote the Lebesgue measure on S. Let $f \in L^2(\sigma)$. Then

$$\widehat{f\sigma}(x) = \int e^{2\pi i x \cdot \xi} f(\xi) d\sigma(\xi), \tag{8.1.6}$$

so (8.4) may be rephrased as

$$||\widehat{f\sigma}||_{L^q(\mathbb{R}^d)} \leq C_q ||f||_{L^2(\sigma)}. \tag{8.1.7}$$

Let's compute the adjoint of the operator defined by the left hand side of (8.1.7). We have

$$\int \mathcal{R}^* f(x) g(x) dx = \int \widehat{f\sigma}(x) g(x) dx = \int \int e^{-2\pi i x \cdot \xi} f(\xi) d\sigma(\xi) g(x) dx$$

$$= \int \hat{g}(\xi) f(\xi) d\sigma(\xi). \tag{8.1.8}$$

In other words, the adjoint operator is the operator

$$\mathcal{R} g(x) = \hat{g} \mid_S, \tag{8.1.9}$$

the restriction operator.

Continuing this train of thought, we compute

$$\mathcal{R}^* \mathcal{R} g(x) = \int e^{-2\pi i x \cdot \xi} \hat{g}(\xi) d\sigma(\xi) = \int \int e^{-2\pi i (x-y) \cdot \xi} \widehat{\sigma}(y) \hat{g}(\xi) d\xi$$

$$= \int g(x-y) \widehat{\sigma}(y) dy. \tag{8.1.10}$$

In summary, let $Tf(x) = f * \sigma(x)$. Suppose that one can show that

$$||Tf||_{L^{p'}(\mathbb{R}^d)} \leq C_p ||f||_{L^p(\mathbb{R}^d)} \tag{8.1.11}$$

for some $p > 1$. Then

$$||\mathcal{R} f||_{L^2(\sigma)} \leq C_p ||f||_{L^p(\mathbb{R}^d)}, \tag{8.1.12}$$

and, equivalently,

$$||\mathcal{R}^* g||_{L^{p'}(\mathbb{R}^d)} \leq C_p ||g||_{L^2(\sigma)}. \tag{8.1.13}$$

This manipulation is known as the Tomas-Stein trick.

Theorem 8.1.1 (Tomas-Stein restriction theorem). *Suppose that S is a compact smooth hyper-surface with everywhere non-vanishing Gaussian curvature. Let σ denote the Lebesgue measure on S. Then* (8.1.11) *holds with $p \leq \frac{2(d+1)}{d+3}$, and* (8.1.12), (8.1.13) *follow accordingly.*

Notice that this result was motivated by the Schrodinger equation where the corresponding surface S (the paraboloid) was not in fact compact. We shall address this issue below.

Before proceeding to the proof of Tomas-Stein theorem, let us motivate the appearance of the exponent $p = \frac{2(d+1)}{d+3}$. We shall work with the inequality (8.1.12). Since the Gaussian curvature on S does not vanish, it is locally well-approximated by a $\delta \times \delta \times \cdots \times \delta^2$ rectangle. Let $\hat{f}_\delta$ equal to the characteristic function of this rectangle. On one hand,

$$||\mathcal{R} f_\delta||_{L^2(\sigma)} \approx \delta^{\frac{d-1}{2}}. \tag{8.1.14}$$

On the other hand,

$$||f_\delta||_{L^p(\mathbb{R}^d)} \approx \delta^{d+1} \delta^{-\frac{d+1}{p}} \text{ Check this!} \tag{8.1.15}$$

Comparing the exponents, we see that $p \leq \frac{2(d+1)}{d+3}$. Moreover, a careful check of this argument reveals that it has not nothing to do with curvature. In other words, (8.11) cannot hold for any smooth S with $p > \frac{2(d+1)}{d+3}$. This technique is commonly known as Knapp Homogeneity Argument.

Exercise 8.1.1. *Carry out a similar homogeneity argument using inequalities* (8.1.11) *and* (8.1.13) *instead of* (8.1.12)*. Also check the assertion in* (8.1.15)*.*

We are now ready to prove Tomas-Stein theorem. We shall give two proofs and the third proof will pop up later in the notes.

Mockenhaupt's proof of L^2 restriction. We prove (8.1.11). Let β be as in (6.22) (Littlewood-Paley theory). Let

$$\widehat{\sigma}(y) = K_{-\infty}(y) + \sum_{j>0} K_j(y), \tag{8.1.16}$$

where

$$K_j(y) = \beta(2^{-j} y) \widehat{\sigma}(y), \tag{8.1.17}$$

and convolution with $K_{-\infty}$ maps $L^p \to L^q$, for any $1 \leq p < q$ by Young's inequality (check this!).

Let $T_j f(x) = f * K_j(x)$. By the previous chapter,

$$|K_j(x)| \leq C 2^{-j \frac{d-1}{2}}. \tag{8.1.18}$$

It follows that

$$||T_j f||_\infty \le C 2^{-j\frac{d-1}{2}} ||f||_1. \tag{8.1.19}$$

Using (5.7) and the fact that the Fourier transform of a smooth compactly supported function is Schwartz class, we have

$$|\widehat{K}_j(x)| \le C_N 2^{dj} \int (1 + 2^j |\xi - \eta|)^{-N} d\sigma(\eta) \tag{8.1.20}$$

for any $N < \infty$. This quantity is bounded by

$$C_N 2^{dj} \sigma(\{\eta : |\xi - \eta| \le 2^{-j}\}) + \sum_{k>0} C_N 2^{dj} \int_{\{\eta: 2^j|\xi-\eta| \approx 2^k\}} (1 + 2^j |\xi - \eta|)^{-N} d\sigma(\eta)$$

$$\le C_N 2^{dj} 2^{-j(d-1)} + \sum_{k>0} C_N 2^{dj} 2^{-kN} 2^{(k-j)(d-1)} \le C'_N 2^j \tag{8.1.21}$$

since σ is $(d-1)$-dimensional.

By Plancherel, it follows that

$$||T_j f||_2 \le C 2^j ||f||_2. \tag{8.1.22}$$

By Riesz-Thorin interpolation theorem,

$$||T_j f||_{p'} \le C_p 2^{\frac{2j}{p'}} 2^{-j\frac{d-1}{2}(1-\frac{2}{p'})} ||f||_p. \tag{8.1.23}$$

The geometric series converges if $p < \frac{2(d+1)}{d+3}$. This gives us the Tomas-Stein theorem up to the endpoint. Instead of working a bit harder for the endpoint, we give a different proof where the endpoint is more easily attained.

Parameterize S locally as a graph of a smooth function Φ such that $\Phi(0,\dots,0) = 0$ and $\nabla\Phi(0,\dots,0) = (0,\dots,0)$. Let z be a complex parameter

$$m_z(\xi) = \frac{1}{\Gamma(z)} (\xi_d - \Phi(\xi'))_+^{z-1} \psi(\xi'), \tag{8.1.24}$$

where Γ is the standard Gamma function, and ψ is a compactly supported smooth function in a small neighborhood of the origin. The function m_z can be analytically continued to the whole complex plane. Let K_z denote the inverse Fourier transform of m_z. Let $T_z f = f * K_z$.

Exercise 8.1.2. *Check that $K_0(x) = \widehat{\sigma}(x)$.*

When $Re(z) = 1$, m_z is bounded, so Plancherel implies that

$$||T_z f||_2 \leq C||f||_2, \tag{8.1.25}$$

where C is a universal constant.

On the other hand,

$$K_z(x) = \int e^{2\pi i x\cdot\xi} \frac{1}{\Gamma(z)} (\xi_d - \Phi(\xi'))_+^{z-1} \psi(\xi') d\xi. \tag{8.1.26}$$

Making the change of variables $\xi_d \to \xi_d - \Phi(\xi')$, and computing explicitly the Fourier transform of the distribution $\frac{1}{\Gamma(z)}(\xi_d)_+^{z-1}$ we see that $K_z(x) = C\widehat{\sigma}(x)|\xi|_d^{-z}$. It follows by the method of stationary phase and Plancherel that

$$||T_z f||_\infty \leq C||f||_1 \tag{8.1.27}$$

if $Re(z) = -\frac{d-1}{2}$. The theorem now follows from the following analytic interpolation theorem due to E. M. Stein.

Theorem 8.1.2 (Stein's analytic interpolation theorem). *Let T_z be a family of operators indexed by a complex number z with $0 \leq Re(z) \leq 1$. Suppose that this family is analytic in the sense that the function $z \to \int T_z f(x)dx$ is analytic for every measurable function f. Suppose that*

$$||T_z f||_{q_0} \leq C_0||f||_{p_0} \tag{8.1.28}$$

and

$$||T_z f||_{q_1} \leq C_1||f||_{p_1} \tag{8.1.29}$$

with $1 \leq p_j \leq q_j \leq \infty$. Let p_t and q_t be defined as in the statement of Riesz-Thorin interpolation theorem with $t = Re(z)$. Then

$$||T_z f||_{q_t} \leq C_0^{1-t}C_1^t||f||_{p_t}. \tag{8.1.30}$$

The proof of this theorem is similar to the proof of Riesz-Thorin interpolation theorem and is left as an exercise. Later in this course we will need a beautiful endpoint version of this theorem involving spaces H^1 (a close friend of L^1) and BMO (a close friend of L^∞.

Exercise 8.1.3. *Our original motivation for the restriction phenomenon was the Schrodinger equation. We have seen that the solution operator to the Schrodinger equation may be viewed as an extension operator corresponding to the infinite hyperboloid $\{(\xi, |\xi|^2)\}$. While the result we prove above is only stated for convex hyper-surfaces, one can use the analytic interpolation proof of the restriction theorem to show that*

$$\left(\int\int |u(x,t)|^{\frac{2(d+1)}{d-1}} dxdt \right)^{\frac{d-1}{2(d+1)}} \leq C||f||_2, \tag{8.1.31}$$

where $u(x,t)$ is as in (8.1.4). Carry out the details of this argument.

Averages over hyper-surfaces. Consider the initial value for the wave equation

$$\triangle u = \frac{\partial^2 u}{\partial t^2}; \; u(x,0) = 0; \; u_t(x,0) = f(x). \tag{8.2.1}$$

One can check (and you should) that in three dimensions,

$$u(x,t) = ct \int_{S^2} f(x - ty) d\sigma(y), \tag{8.2.2}$$

where S^2 is the unit sphere and $d\sigma$ is the Lebesgue measure on S^2. A more complicated, but similar, formula holds in other dimensions. In the early 70's Strichartz investigated to what extent $u(x,t)$ is more regular than f for a fixed t. More precisely, he asked the following question. Let

$$Uf(x) = \int_{S^{d-1}} f(x-y) d\sigma(y), \tag{8.2.3}$$

where S^{d-1} is the $d-1$-dimensional sphere. What is the range of exponents (p,q) such that

$$||Uf||_{L^q(\mathbb{R}^d)} \leq C_{p,q} ||f||_{L^q(\mathbb{R}^d)}? \tag{8.2.4}$$

Similar question was asked at roughly the same time by W. Littman.

Let us first investigate what the range of bounded-ness should reasonably be. First, it is clear that $p \leq q$ (check this) and that bounded-ness clearly holds for $p = q$ since S^{d-1} is compact. Now take $f_\delta(x) = \chi_{B_\delta}(x)$, where B_δ is the ball of radius δ centered at the origin. We have $||f_\delta||_p \approx \delta^{\frac{d}{p}}$. On the other hand, Uf_δ is bounded below by a constant multiple of δ^{d-1} on an annulus of width approximately δ. It follows that $||Uf||_q \gtrsim \delta^{d-1}\delta^{\frac{1}{q}}$. We conclude that if (8.34) is to hold then

$$\delta^{d-1}\delta^{\frac{1}{q}} \leq C\delta^{\frac{d}{p}}, \tag{8.2.5}$$

which means that

$$\frac{d}{p} \leq d - 1 + \frac{1}{q}. \tag{8.2.6}$$

Dualizing (8.2.6) we obtain

$$\frac{d}{q'} \leq d - 1 + \frac{1}{p'}. \tag{8.2.7}$$

Combining these two sets of restrictions we see that (8.2.4) holds only if $(1/p, 1/q)$ is contained in the closed triangle with the endpoints

$$(0,0), \; (1,1), \; \text{and} \; (d/d+1, 1/d+1). \tag{8.2.8}$$

Observe that the argument we just ran has nothing to do with the sphere. We only needed the fact that the sphere is compact and $d-1$-dimensional. We now turn to positive results.

Theorem 8.2.1 (Littman/Strichartz L^p improving inequality). *Let S be a compact smooth hyper-surface in $\mathbb{R}^d$, $d \geq 2$, with everywhere non-vanishing Gaussian curvature. Let U be defined as above. Then* (8.2.4) *holds with* $(1/p, 1/q)$ *as in* (8.2.8).

The proof is identical to the second proof of the restriction theorem given above. Let $U_z f = K_z * f$, where K_z equals to m_z define above. When $Re(z) = 1$, $U_z : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$, since K_z is bounded when $Re(z) = 1$. By Plancherel and above, $\widehat{K}_z$ is bounded when $Re(z) = -\frac{d-1}{2}$. It follows that $U_z : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ when $Re(z) = -\frac{d-1}{2}$. Analytic interpolation yields the desired result.

In order to clarify the role of curvature, we consider the following two-dimensional situation (though our calculation may be generalized as you shall see). Let

$$Uf(x) = \int_0^2 f(x_1 - s, x_2 - s^m)ds, \tag{8.2.9}$$

where $m \geq 2$. If $m = 2$, the curve $\{(s, s^m) : 0 \leq 2\}$ has non-zero curvature. When $m > 2$, the curvature vanishes at the origin of order $m - 2$. Let

$$U_j f(x) = \int_{2^{-j}}^{2^{-j+1}} f(x_1 - s, x_2 - s^m)ds = 2^{-j} \int_1^2 f(x_1 - 2^{-j}t, x_2 - 2^{-mj}t^m)dt. \tag{8.2.10}$$

It follows that

$$U_j f(x) = 2^{-j} \tau_j U_0 \tau_j^{-1} f(x), \tag{8.2.11}$$

where

$$\tau_j g(x) = g(2^j x_1, 2^{mj} x_2). \tag{8.2.12}$$

Since $||\tau_j g||_p = 2^{j\frac{m+1}{p}}$, we have

$$||U_j f||_q = 2^{-j} 2^{-j\frac{m+1}{q}} ||\tau_j^{-1} f||_p \leq C_{p,q} 2^{-j} 2^{-j\frac{m+1}{q}} 2^{j\frac{m+1}{p}}, \tag{8.2.13}$$

with $(1/p, 1/q)$ as in (8.2.8). The geometric series converges if

$$\frac{1}{p} - \frac{1}{q} < \frac{1}{m+1}. \tag{8.2.14}$$

It follows that U is bounded from $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ if $(1/p, 1/q)$ is as in (8.2.8) and (8.2.14) It also follows that U is not bounded from $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ if $\frac{1}{p} - \frac{1}{q} > \frac{1}{m+1}$. The bounded-ness on the line $\frac{1}{p} - \frac{1}{q} = \frac{1}{m+1}$ does hold (the result is due to Christ, and, independently, to Ricci and Stein) but is not proved here. Can you prove it?

Exercise 8.2.1. *A related way to get a good feel for the role of curvature is as follows. Suppose that* $S = \{x = (x', \Phi(x') : |x'| \leq 1\}$*, where* Φ *is a positive convex function such that* $\Phi(0, \dots, 0) = 0$ *and* $\nabla\Phi(0, \dots, 0) = (0, \dots, 0)$*. Suppose that* $|\{x' : \Phi(x') \leq \delta\}| \leq \delta^r$*,* $r > 0$*,* δ *small. Let* f_δ *be the characteristic function of the set where* $\Phi(x') \leq \delta$ *and* $0 \leq x_d \leq \delta$*. Let* U *be defined as above. Estimate* Uf_δ*. Conclude that* U *is bounded from* $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ *only if* $(1/p, 1/q)$ *is as in* (8.38) *and* $\frac{1}{p} - \frac{1}{q} \leq \frac{r}{r+1}$*. Compare this with* (8.44) *above.*

Distribution of lattice points in convex domains. Our next application of the method of stationary phase has its roots in number theory, and, from a slightly different point of view, in mathematical physics.

Let K be a bounded convex domain containing the origin in its interior. Let

$$N(t) = \#\{tK \cap \mathbb{Z}^d\}, \tag{8.3.1}$$

$d \geq 2$.

Since the boundary of K is $d-1$ dimensional,

$$N(t) = t^d|K| + E(t), \tag{8.3.2}$$

where

$$|E(t)| \leq Ct^{d-1}, \tag{8.3.3}$$

and the case $K = [-1, 1]^d$, t a large integer, shows that we cannot in general do any better. A reasonable question to ask is, can we get a better estimate on $E(t)$ is the boundary of K has some curvature?

Theorem 8.3.1 (Siepinski/van der Corput/Landau lattice point bound). *Suppose that* K *is as above and that boundary of* K *is smooth and has everywhere non-vanishing Gaussian curvature. Then*

$$|E(t)| \leq Ct^{d-2+\frac{2}{d+1}}. \tag{8.3.4}$$

The conclusion of this theorem still holds in two dimensions if the smoothness assumption is eliminated. We are not going to prove this here, but an interested reader is encouraged to pursue the matter. Moreover, if the smoothness assumption is eliminated, the result is sharp in two dimensions. This sharpness is established using a construction due to Jarnik. Details can be found, for example, in a recent book by Martin Huxley. With additional smoothness, better estimate on $E(t)$ are available in two and higher dimensions. These are also described in Huxley's book.

In higher dimensions, it is not known if the theorem holds without the smoothness assumption, nor is it known if the estimate $|E(t)| \leq Ct^{d-2+\frac{2}{d+1}}$ is sharp in any sense. However, the following connection seems highly intriguing. It follows from a result due to George Andrews from the early 60's that if K is a strictly convex body in $\mathbb{R}^d$, then the number of integer lattice points on the boundary of tK is at most $Ct^{d-2+\frac{2}{d+1}}$, and various examples have been constructed over the years to illustrate the sharpness of this bound. Unfortunately, converting these examples into an examples of sharpness of the theorem above (without the smoothness assumption) seems to be quite difficult. Perhaps you would like to give it a try?

We now prove the theorem. We start with the following standard reduction. Let $\rho_0 \in C_0^\infty\left(\frac{1}{4}, 4\right)$ with $\rho_0 \equiv 1$ on $[1,2]$, and let ρ be the radial extension of ρ_0 such that $\int \rho(x)dx = 1$. $\rho_\epsilon(x) = \epsilon^{-d}\rho\left(\frac{x}{\epsilon}\right)$. Let

$$N^\epsilon(t) = \sum_{k\in\mathbb{Z}^d} \chi_{tK} * \rho_\epsilon(k) = t^d|K| + t^d \sum_{k\neq(0,0,\dots,0)} \widehat{\chi}_K(tk)\widehat{\rho}(\epsilon k) = t^d|K| + E^\epsilon(t). \tag{8.3.5}$$

It is not hard to see that there exists $C > 0$ such that

$$N^\epsilon(t - C\epsilon) \leq N(t) \leq N^\epsilon(t + C\epsilon). \tag{8.3.6}$$

It follows that

$$|E(t)| \lesssim |E^\epsilon(t)| + t^{d-1}\epsilon. \tag{8.3.7}$$

We now estimate $E^\epsilon(t)$. We have

$$|E^\epsilon(t)| \leq C_N t^d \sum_{k\neq(0,\dots,0)} t^{-\frac{d+1}{2}}|k|^{-\frac{d+1}{2}}(1+|\epsilon k|)^{-N} \leq C_N t^{\frac{d-1}{2}}\epsilon^{-\frac{d-1}{2}}. \tag{8.3.8}$$

It follows that

$$|E(t)| \leq C_1 t^{d-1}\epsilon + C_2 t^{\frac{d-1}{2}}\epsilon^{-\frac{d-1}{2}}, \tag{8.3.9}$$

and the theorem follows by choosing ϵ so that the two terms are equal.

Exercise 8.3.1. *In the section on the method of stationary phase we proved that if the Gaussian curvature on ∂K does not vanish, then $|\widehat{\sigma}(\xi)| \leq C|\xi|^{-\frac{d-1}{2}}$, where σ is the Lebesgue measure on ∂K. However, in* (8.3.7) *above we actually used the fact that $|\widehat{\chi}_K(\xi)| \leq C|\xi|^{-\frac{d+1}{2}}$. Prove this fact. Hint: Use the divergence theorem to show that*

$$\int_K e^{-2\pi i x\cdot\xi}dx = \frac{C}{|\xi|}\int_{\partial K} e^{-2\pi i x\cdot\xi}\left(\frac{\xi}{|\xi|}\cdot n(x)\right)d\sigma(x), \tag{8.3.10}$$

where $n(x)$ denotes the unit normal on ∂K at x.

Maximal averages over hyper-surfaces. Let S be a smooth compact hyper-surface, and let $A_t f(x) = \int f(x-ty)d\sigma(y)$, where $d\sigma$ is the Lebesgue measure on S. Let $\mathcal{A}f(x) = \sup_{t>0} |A_t f(x)|$. The question we ask is, what is the range of exponents p such that

$$||\mathcal{A}f||_p \leq C_p ||f||_p. \tag{8.4.1}$$

The operator $\mathcal{A}$ is the maximal version of the averaging operator $Tf(x) = A_1 f(x)$ studied above, and one of the main motivations again comes from the study of the wave equation. As we discussed above, in three dimensions the solution operator to the initial value problem (8.31) is given by $ctA_t f(x)$ with $S = S^2$. Establishing (8.4.1) for an exponent p would imply that if the initial data $f \in L^p(\mathbb{R}^d)$, then $\lim_{t\to 0} \frac{u(x,t)}{t} = f(x)$ for almost every x.

The convergence implication underlines another obvious motivation behind (8.4.1), a highly singular generalization of the Fundamental Theorem of Calculus and the Hardy-Littlewood Maximal Theorem. For any $p < \infty$ such that (8.4.1) holds, $A_t f(x) \to cf(x)$ for almost every x, where c is the total mass of σ, if $f \in L^p(\mathbb{R}^d)$.

What range of bounded-ness should we expect? Let us take $S = S^{d-1}$ and let $F(x) = |x|^{-d+1} \log^{-1}\left(\frac{1}{|x|}\right) \chi_{\frac{1}{2}B}(x)$, where B is the unit ball. One one hand, $F \in L^p(\mathbb{R}^d)$ for $p \leq \frac{d}{d-1}$. On the other hand, $A_{|x|}f(x) \equiv \infty$ because S is $d-1$-dimensional. It follows that (8.4.1) holds only if $p > \frac{d}{d-1}$.

Exercise 8.4.1. *Prove that this argument works for any $d-1$-dimensional hyper-surface S.*

The purpose of this sub-section is the prove the following result due to Stein in the case of the sphere, and to Greenleaf in the case of a compact hyper-surface with everywhere non-vanishing Gaussian curvature.

Theorem 8.4.1 (Stein/Greenleaf maximal theorem). *Suppose that S is compact smooth, and has everywhere non-vanishing Gaussian curvature. Let $\mathcal{A}f$ be defined as above. Suppose that $d \geq 3$. Then* (8.4.1) *holds for $p > \frac{d}{d-1}$.*

The much harder two-dimensional result, due to J. Bourgain shall be proved later in these notes after we develop a bit of theory of variable coefficient oscillatory integral operators.

We shall need the following result, interesting in its own right.

Lemma 8.4.2 (Maximal multiplier operators on $L^2(\mathbb{R}^d)$). *Let τ be a compactly supported distribution. Let $\widehat{\tau_t f}(\xi) = \widehat{\tau f}(t\xi)$. Let $\widehat{\tau^k f}(\xi) = \beta(2^{-k}\xi)\widehat{\tau f}(t\xi)$, where β is the Littlewood-Paley cut-off used throughout these notes. Suppose that*

$$\left(\int_1^2 \left|\widehat{\tau^k}(t\xi)\right|^2 dt\right)^{\frac{1}{2}} \leq C_1 2^{-k(\frac{1}{2}+\epsilon)}, \tag{8.4.2}$$

and

$$\left(\int_1^2 \left|\nabla\widehat{\tau^k}(t\xi)\right|^2 dt\right)^{\frac{1}{2}} \leq C_2 2^{-k(\frac{1}{2}+\epsilon)}. \tag{8.4.3}$$

Then

$$\left(\int_{\mathbb{R}^d} |\sup_{t>0} |\tau_t^k f(x)||^2 dx\right)^{\frac{1}{2}} \leq \sqrt{C_1 C_2} 2^{-k\epsilon} ||f||_{L^2(\mathbb{R}^d)}. \tag{8.4.4}$$

In particular,

$$\left(\int_{\mathbb{R}^d} |\sup_{t>0} |\tau_t f(x)||^2 dx\right)^{\frac{1}{2}} \leq \sqrt{C_1 C_2} ||f||_{L^2(\mathbb{R}^d)}. \tag{8.4.5}$$

Lemma 8.4.3 (Maximal multipliers near L^1 lemma). *Let $\tau_t^k f(x)$ be defined as in the previous theorem. Suppose in addition that τ is a $d-1$-dimensional measure in the sense that if $\delta << 1$, $\tau(B_\delta(x_0)) \leq C\delta^{d-1}$, with C independent of δ and x_0, where $B_\delta(x_0)$ denotes the ball of radius δ centered at x_0. Then $\sup_{t>0} |\tau_t^k f(x)|$ is bounded by $C2^k \mathcal{M}_B f(x)$, where $\mathcal{M}_B$ is the Hardy-Littlewood maximal function.*

When σ is the Lebesgue measure on S, the method of stationary phase immediately tells us that the assumptions of the maximal multiplier operators on $L^2(\mathbb{R}^d)$ lemma are satisfied with $\epsilon = \frac{d-2}{2}$. Since the maximal averaging operator is sub-linear, Marcienkiewicz interpolation theorem allows us to interpolate this estimate with that given by the maximal multipliers near L^1 lemma to see that the geometric series converges if $p > \frac{d}{d-1}$. Thus Stein/Greenleaf maximal theorem follows if we can prove the maximal multiplier operators on $L^2(\mathbb{R}^d)$ lemma and also the maximal multipliers near L^1 lemma.

Proof of maximal multiplier operators on $L^2(\mathbb{R}^d)$ lemma.

Orthogonal exponential basis on the unit ball?.

Let K be a bounded domain in $\mathbb{R}^d$. We say that K is spectral if $L^2(K)$ possesses an orthogonal basis of exponentials $\{e^{2\pi i x\cdot a}\}_{a\in A}$. We say that K tiles if there exists a set T such that $\sum_T \chi_K(x-t) \equiv 1$ for almost every $x \in \mathbb{R}^d$. A conjecture due to B. Fuglede formulated in the 70's says that K is spectral if and only if K tiles. Fuglede proved his conjecture in the case when either a tiling set T or a spectrum A is assumed to be a lattice subset of $\mathbb{R}^d$, i.e a set $M\mathbb{Z}^d$, where M is a symmetric invertible matrix with real entries.

Exercise 8.5.1. *Use the Poisson Summation Formula to prove the Fuglede conjecture under the assumption that either a tiling set of a spectrum is a lattice subset of $\mathbb{R}^d$.*

Which domains are spectral? We all know that the unit cube $[0,1]^d$ is spectral. What about the unit ball? The purpose of this section is to use the method of stationary phase and a bit of combinatorics to prove the following:

Theorem 8.5.1 (Ball is not spectral). *The unit ball $B_d = \{x : |x| \leq 1\}$ is not spectral.*

Since B_d certainly does not tile, this theorem is perfectly consistent with the Fuglede conjecture.

Proof. We need the following two facts, one easy, one slightly harder, which we leave as exercises with hints.

Exercise 8.5.2 (easy). *Let A be a spectrum for $L^2(K)$ as defined above. Prove that there exists $c > 0$, depending only on K, such that $|a - a'| \geq c$ for all $a, a' \in A$.*

Exercise 8.5.3 (hard). *Prove that if A is a spectrum for $L^2(K)$, and K has, say, rectifiable boundary, then*

$$\limsup_{R\to\infty} \frac{\#(A \cap [-R,R]^d)}{(2R)^d} = \liminf_{R\to\infty} \frac{\#(A \cap [-R,R]^d)}{(2R)^d} = |K|. \tag{8.5.1}$$

Start out by proving that A is a spectrum for $L^2(K)$ if and only if

$$\sum_A |\widehat{\chi}_K(\xi - a)|^2 \equiv |K|^2 \tag{8.5.2}$$

for almost every $\xi \in \mathbb{R}^d$. Now integrate this quantity over a large ball or a cube. You may find the argument simpler in the case of a ball because of the conveniences afforded by the method of stationary phase. Do this case first and then try to figure out how to use Parseval's type inequality to do the general case.

With these two facts in tow, we are ready to go to work. Consider $A_R = A \cap [-R,R]^d$. By Exercise 8.5.3 we know that $\#A_R \approx R^d$. By the method of stationary phase we know that

Chapter 9: Oscillatory integrals of the second kind: elementary point of view

In Chapter 7 we studied oscillatory integrals of the form (7.1). In this chapter we shall study variable coefficient versions of these integrals. The simplest example is the following.

Theorem 9.1 (Non-degenerate L^2 bound). *Suppose that*

$$\det\left(\frac{\partial^2\phi}{\partial x_j \partial y_k}\right) \neq 0, \tag{9.1}$$

where ϕ is a C^∞ function. Suppose that $a(x,y) \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. Then for $t > 0$,

$$\left|\left|\int e^{2\pi i t\phi(x,y)} a(x,y) f(y) dy\right|\right|_{L^2(\mathbb{R}^d)} \leq C t^{-\frac{d}{2}} ||f||_{L^2(\mathbb{R}^d)}. \tag{9.2}$$

Exercise 9.1 (easy). *Prove that Theorem 9.1 implies Hausdorff-Young.*

Proof of Theorem 9.1.

Chapter 10: Oscillatory integrals of the second kind: a geometric point of view

We start out by defining some basic geometric notions. M is said to be a C^∞ manifold, if M is a Hausdorff space equipped with a countable collection $\{g_j, O_j\}$ satisfying the following properties:
i) $g_j : O_j \to O'_j \subset \mathbb{R}^d$ is a homeomorphism.
ii) Each O_j is an open set and $\cup_j O_j = M$.
iii) $g_{j'} \circ g_j^{-1} : g_j(O_j \cap O'_j) \to g_{j'}(O_j \cap O'_j)$ is a $C^\infty(\mathbb{R}^d)$ function.

We are now ready to define $C^\infty(M)$. We say that $f : M \to \mathbb{R}$ is a C^∞ function if $f(g_j^{-1}(y))$ is a C^∞ function for every j.

We now define a notion of a tangent vector. We say that a continuous linear operator mapping $C^\infty(M)$ to real numbers is a tangenet vector if given $x \in O_j$, there exists $t^j \in \mathbb{R}^d$ such that

$$t(F \circ g_j) = \sum_{i=1}^{d} t_i^j \frac{\partial}{\partial y_i} F(y) \mid_{y=g_j(x)} \tag{10.1}$$

whenever $F \in C^\infty(O'_j)$.