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Math 265H, Fall 2022, October 26

Theorem: e is irrational

Proof:

$$e - S_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots$$

$$< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right)$$

geometric series

$$= \frac{1}{(n+1)!} \frac{1}{1 - \frac{1}{n+1}} =$$

$$\frac{1}{(n+1)!} \frac{n+1}{n} = \frac{1}{n!} \frac{1}{n}$$

It follows that $0 < e - s_n < \frac{1}{n! \cdot n}$

awesome approximation

Suppose that e is rational, i.e

$$e = \frac{p}{q}$$

It follows that $0 < (e - s_q)q! < \frac{1}{q}$

By assumption, $eq!$ is an integer.

Since

$$q!\, s_q = q!\left(1 + 1 + \frac{1}{2!} + \dots + \frac{1}{q!}\right)$$

is an integer, we conclude that $q!(e - s_q)$ is an integer that lives in $\left(0, \frac{1}{q}\right)$.

This is difficult to believe ☺.

(3)

Root test: Given $\sum a_n$, let $\alpha = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$

Then

a) If $\alpha < 1$, $\sum a_n$ converges

b) If $\alpha > 1$, $\sum a_n$ diverges.

c) If $\alpha = 1$, could go either way

<u>Proof:</u> If $\alpha < 1$, choose $\beta \ni \alpha < \beta < 1$, and an integer $N \ni \sqrt[n]{|a_n|} < \beta$ for $n \geq N$. Consequently, if $n \geq N$, $|a_n| < \beta^n$

$\Longrightarrow \sum a_n$ converges by the comparison test.

If $\alpha > 1$, Theorem 3.17 yields $\{n_k\} \ni$ $\sqrt[n_k]{|a_{n_k}|} \longrightarrow \alpha$, hence

$|a_n| > 1$ for infinitely many n

$\Longrightarrow \sum a_n$ diverges

If $\alpha = 1$, consider

$$\underbrace{\sum \frac{1}{n}}_{\text{diverges}}, \quad \underbrace{\sum \frac{1}{n^2}}_{\text{converges}}$$

<u>Ratio</u> <u>test:</u> The series $\sum a_n$

a) converges if $\lim \left| \frac{a_{n+1}}{a_n} \right| < 1$,

b) diverges if $\cancel{\lim} \left| \frac{a_{n+1}}{a_n} \right| \geq 1 \ \forall n \geq n_0$ fixed

Proof: a) $\hookrightarrow \exists \beta < 1$ & $N \ni$

$$\left| \frac{a_{n+1}}{a_n} \right| < \beta \text{ for } n \geq N.$$

It follows that

$|a_{N+1}| < \beta |a_N|$

$|a_{N+2}| < \beta |a_{N+1}| < \beta^2 |a_N|$ ...

$|a_{N+p}| < \beta^p |a_N|$

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i.e $|a_n| < |a_N| \beta^{-N} \cdot \beta^n$ for $n \geq N$.

The conclusion follows by comparison test.

If $|a_{n+1}| \geq |a_n|$ for $n \geq n_0$,

$\lim_{n \to \infty} a_n \neq 0$ ✓

If $\lim \left| \frac{a_{n+1}}{a_n} \right| = 1$, we can conclude nothing since $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges.

Key example:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$$

$$\underline{\lim} \frac{a_{n+1}}{a_n} = \lim \left(\frac{2}{3}\right)^n = 0$$

$$\overline{\lim} \frac{a_{n+1}}{a_n} = \lim \frac{1}{2}\left(\frac{3}{2}\right)^n = \infty$$

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$$\overline{\lim_n} \sqrt[n]{a_n} = \lim_n \sqrt[2n]{\frac{1}{3^n}} = \frac{1}{\sqrt{3}}$$

$$\underline{\lim_n} \sqrt[n]{a_n} = \lim_n \sqrt[2n]{\frac{1}{2^n}} = \frac{1}{\sqrt{2}}$$

So root test $\Rightarrow$ convergence and ratio test does not apply.

Another key example:

$$\left(\frac{1}{2}+1\right)+\left(\frac{1}{8}+\frac{1}{4}\right)+\left(\frac{1}{32}+\frac{1}{16}\right)+\cdots$$

$$\underline{\lim} \frac{a_{n+1}}{a_n} = \frac{1}{8} \qquad \overline{\lim} \frac{a_{n+1}}{a_n} = 2$$

and $\lim \sqrt[n]{a_n} = \frac{1}{2}$

Root test wins again!

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Root test vs Ratio test:

$\{c_n\} \in \mathbb{R}$ positive

i) $\underline{\lim} \frac{c_{n+1}}{c_n} \leq \underline{\lim} \sqrt[n]{c_n}$

ii) $\overline{\lim} \sqrt[n]{c_n} \leq \overline{\lim} \frac{c_{n+1}}{c_n}$

Proof of ii) (i) is the same, more or less)

$$\alpha = \overline{\lim} \frac{c_{n+1}}{c_n}$$

If $\alpha < \infty$, choose $\beta > \alpha$.

Then $\frac{c_{n+1}}{c_n} \leq \beta$ for $n \geq N$ (some N)

$\hookrightarrow c_{N+k+1} \leq \beta c_{N+k} \quad k = 0, 1, \ldots, p-1$

$\hookrightarrow c_{N+p} \leq \beta^p c_N$

$\hookrightarrow c_n \leq c_N \beta^{-N} \beta^n, \quad n \geq N$

It follows that

$$\sqrt[n]{c_n} \le \sqrt[n]{c_N \beta^{-N}} \cdot \beta$$

$$\Rightarrow \lim \sqrt[n]{c_n} \le \beta$$

By Theorem 3.20 (b)

$$\Rightarrow \overline{\lim} \sqrt[n]{c_n} \le \alpha$$

Power Series:

$$\sum_{n=0}^{\infty} c_n z^n \quad \text{is called a power series}$$

c_n complex numbers

Summation by parts:

$\{a_n\}$ $\{b_n\}$ sequences

$$A_n = \sum_{k=0}^{n} a_k \qquad n \ge 0 \qquad A_{-1} = 0$$

$$\sum_{p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

Proof:

$$\sum_{p}^{q} a_n b_n = \sum_{p}^{q} (A_n - A_{n-1}) b_n =$$

$$\sum_{p}^{q} A_n b_n - \sum_{p-1}^{q-1} A_n b_{n+1}$$

$$= \sum_{p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$