Math 173, Fall 2022 Oct 24

Theorem 10: Let R be a non-zero row-reduced echelon matrix. Then the non-zero rows of R form a basis of the row space of R.

Proof: Let $\rho_1, \rho_2, \ldots, \rho_r$ = non-zero rows of R

$\rho_i = (R_{i1}, R_{i2}, \ldots, R_{in})$

$\{\rho_i\}$ span rowspace (R), so it is enough to show that they are linearly independent.

Proof of linear independence:

Since R is row-reduced echelon matrix, $\exists\, k_1, k_2, \ldots, k_r \ni$ for $i \leq r$,

a) $R(i, j) = 0$ if $j < k_r$

b) $R(i, k_j) = \delta_{ij}$ c) $k_1 < k_2 < \ldots < k_r$

Suppose $\beta = (b_1, b_2, \dots, b_n) \in$ row space(R), i.e

$\beta = c_1\rho_1 + \dots + c_r\rho_r$

We have $b_{k_j} = \sum_{i=1}^{r} c_i R(i, k_j)$ ~ why?

$= \sum_{i=1}^{r} c_i \delta_{ij} = c_j$, so $b_{k_j} = c_j$.

In particular, if $\beta = 0$, i.e

$c_1\rho_1 + \dots + c_r\rho_r = 0$, then $c_j = b_{k_j} = 0$,

which implies linear independence.

Theorem 11: Let m, n positive integers and F field. Suppose $W \subset F^n$ (subspace), and $\dim W \leq m$. Then there is precisely one $m \times n$ row-reduced echelon matrix over F which has W as its row space.

Proof: There is at least one $m \times n$ row-reduced echelon matrix w/ row space W.

Since $\dim W \leq m$, we can find

$\alpha_1, \alpha_2, \dots, \alpha_m$ that span W.
in W

Let $A = m \times n$ matrix w/ rows $\alpha_1, \dots, \alpha_m$
and let R be a row-reduced echelon matrix
equivalent to A. Then the row space of R
is W. ~ this establishes existence

Uniqueness: Let R = row reduced echelon
w/ W as row space.
Let $\rho_1, \rho_2, \dots, \rho_r$ = non-zero rows of R
w/ leading non-zero entry of ρ_i in column k_i.
$\{\rho_i\}$ form a basis of W and
(as proved above), if $\beta = (b_1, \dots, b_n) \in W$
$\beta = c_1\rho_1 + \dots + c_n\rho_n$, $c_i = b_{k_i}$, i.e

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$$\beta = \sum_{i=1}^{r} b_{k_i} \rho_i$$

In other words, β is determined

if one knows b_{k_i}, $i=1,2,\dots,r$.

As an important example, ρ_s is the unique vectors in W $\ni$ k_sth coordinate is 1 and the rest are 0.

Suppose $\beta \in W$, $\beta \neq \vec{0}$.

Claim: The first non-zero coordinate of β occurs in one of the columns k_s.

To see this, let

$\beta = \sum_{i=1}^{r} b_{k_i} \rho_i$, $\beta \neq \vec{0}$, we have

$$\beta = \sum_{i=s}^{r} b_{k_i} \rho_i, \quad b_{k_s} \neq 0$$

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It follows that $R_{ij} = 0$ if $i > s$
and $j \leq k_s$.

Thus $\beta = (0, \ldots, 0, \underbrace{b_{k_s}}_{\text{first non-zero coordinate}}, \ldots, b_n)$

Moreover, ρ_s has a non-zero k_s coordinate.
& $\rho_s \in W$ by construction.

We now see that R is determined by W
and the procedure is as follows:

$\beta = (b_1, \ldots, b_n) \in W$

If $\beta \neq \vec{0}$, the first non-zero entry
is in some column t, i.e

$\beta = (0, \ldots, 0, b_t, \ldots, b_n)$, $b_t \neq 0$.

Arrange $k_1 < k_2 < \ldots < k_r$.

6 For each k_s $\exists!\ \rho_s \in W$ w/ 1 in the k_s slot & 0 otherwise.

Then $R = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ & the proof is complete.

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$\mathbb{R}^3$ $(1,0,0)$ $(0,1,0)$

$(2,3,0)$ ~ span W

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 0 \end{pmatrix}$$ quickly reduces to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$k_1 = 1$ $k_2 = 2$

$r = 2$

Let $\beta = \underset{\displaystyle \overset{\shortparallel}{\;}}{(b_1, b_2, b_3)} = (-1, 2, 0) \in W$

$\shortparallel$

$\underset{\displaystyle \overset{\shortparallel}{c_1}}{-1} \cdot (1,0,0) + \underset{\displaystyle \overset{\shortparallel}{c_2}}{2} \cdot (0,1,0)$

$c_1 = b_{k_1} = b_1$ $c_2 = b_{k_2} = b_2$

So $\beta = (b_{k_1}, b_{k_2}, 0)$