Falconer conjecture, spherical averages and discrete analogs

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ABSTRACT. The Falconer conjecture says that if a compact set in $\mathbb{R}^d$ has Hausdorff dimension $> \frac{d}{2}$, then the Euclidean distance set $\Delta(E) = \{|x-y| : x, y \in E\}$ has positive Lebesgue measure. Mattila showed that this result would follow from proving that there exists a Borel measure μ on E such that $\int_1^\infty \left(\int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 d\omega \right)^2 t^{d-1} dt < \infty$ provided that the Hausdorff dimension of E exceeds $\frac{d}{2}$. In this paper we prove that this estimate holds on average. We also use average variants of the Falconer conjecture to deduce corresponding discrete analogs for the asymptotic version of the Erdos distance problem.

Introduction

Let $E \subset [0,1]^d$, $d \geq 2$. The well-known conjecture of Falconer says that if the Hausdorff dimension of E exceeds $\frac{d}{2}$, then the distance set $\Delta(E) = \{|x-y| : x, y \in E\}$ has positive Lebesgue measure.

The initial result in this direction was proved by Falconer ([**Falconer86**]) who showed that $\Delta(E)$ has positive Lebesgue measure if the Hausdorff dimension of E exceeds $\frac{d+1}{2}$. This result was later improved in all dimensions by Bourgain ([**Bourgain94**]). The best known result in the plane is due to Wolff ([**Wolff99**]) who proved that $\Delta(E)$ has positive Lebesgue measure provided that the Hausdorff dimension of E is greater than $\frac{4}{3}$.

Mattila developed the following beautiful approach to the Falconer distance problem. See, for example, [**Wolff02**], for a thorough description. Also, see the outline in the section entitled "Basic Idea" below. Mattila proved that it suffices to show that if the Hausdorff dimension of E is greater than $\frac{d}{2}$, then there exists a Borel measure on E such that

(*) $$\int_1^\infty \left(\int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 d\omega \right)^2 t^{d-1} dt < \infty.$$

In this paper we show that $(*)$ holds on average in the context of studying distance sets with respect to suitable perturbations of the Euclidean metric. Our main result is the following.

THEOREM 0.1. *Let $E \subset [0,1]^d$ be a set of Hausdorff dimension greater than $\frac{d}{2}$. Let $K_{a,O}$ denote the ellipsoid with eccentricities $a_1, a_2, \ldots, a_d$, $1 \le a_j \le 2$, rotated by $O \in SO(d)$, and let $\Delta_{a,O}(E)$ denote the corresponding distance set. Then*
i) $(*)$ holds on average in the sense that

$$(0.1) \qquad \int_{SO(d)} \int_{[0;1]^d} \int_1^\infty \left(\int_{S^{d-1}} |\widehat{\mu}(t\omega_{a,O})|^2 d\omega \right)^2 t^{d-1} dt \, da \, dH_d < \infty,$$

where dH_d is the Haar measure on $SO(d)$ and $\omega_{a,O}$ denotes standard coordinates on the ellipsoid $\frac{x_1^2}{a_1^2} + \cdots + \frac{x_d^2}{a_d^2} = 1$ rotated by $O \in SO(d)$.
ii) Let K be any bounded convex set symmetric with respect to the origin such that the boundary ∂K is smooth. Suppose that the Hausdorff dimension of E is greater than 1. Then the Lebesgue measure of $\Delta_{aK}(E)$ is positive for almost every $a \in [1,2]^d$, where $aK = \{(a_1 x_1, \ldots, a_d x_d) : x \in K\}$ and $\Delta_K(E) = \{||x - y||_K : x, y \in K\}$ where $||z||_K = \inf\{t : z \in tK\}$, the distance induced by K.
iii) Let $K = B_d$, the unit Euclidean ball. Suppose that the Hausdorff dimension of E is greater than 2. Then $\Delta_{aK}(E)$ contains an interval for almost every $a \in [1,2]^d$.

As we shall see below, Part ii) is sharp in the sense that for each $\alpha < 1$ there exists $E \subset [0,1]^d$ of Hausdorff dimension α such that $\Delta_{a,O}(E)$ has measure zero for every $(a,O) \in [1,2]^d \times SO(d)$. Both Part ii) and Part iii) rely heavily on ideas of Solomyak ([**Solomyak98**]), and Peres and Schlag (PeresSchlag00).

Part i) is a generalization of the two-dimensional version proved in [**HofIos03**] using similar methods, though the geometry of higher dimensions is considerably more complicated.

Application to the Asymptotic Erdos Distance Conjecture. The classical Erdos Distance Conjecture says that if $S \subset \mathbb{R}^d$ and $\#S = N$, then for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $\#\Delta(S) \ge C_\epsilon N^{\frac{2}{d} - \epsilon}$. See [**PA95**] and references contained therein for the history of the problem and statements of known results in various dimensions. In analysis applications (see e.g. [**IKP99**], [**IKT01**]), the following variant often arises:

ASYMPTOTIC ERDOS DISTANCE CONJECTURE. *Let $A \subset \mathbb{R}^d$ be separated in the sense that $|a - a'| \ge c_0 > 0$, $a \ne a'$, and well-distributed in the sense that every cube of radius $C_0 > 0$ contains at least one point of A. Let*

$$(0.2) \qquad\qquad\qquad A_R = A \cap [0, R]^d.$$

Then for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$(0.3) \qquad\qquad\qquad \#\Delta(A_R) \ge C_\epsilon R^{2-\epsilon}.$$

We first repeat the observation made in [**HofIos03**] and [**IoLa2003**] that the Falconer Distance Conjecture implies the Asymptotic Erdos Distance Conjecture. We shall then see that Theorem 0.1 implies that if A is a well-distributed set in $\mathbb{R}^d$

then for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $\#\Delta_{aK}(A_R) \geq C_\epsilon R^{d-\epsilon}$ for almost every $a \in [1,2]^d$, if ∂K is smooth.

We start out with the following construction due to Falconer. Let q_i be a sequence of positive integers such that $q_1 > 1$ and $q_{i+1} \geq q_i^i$. Let

$$(0.4) \qquad E_i = \{x \in [0,1]^d : \exists p \in [0,q_i]^d \cap \mathbb{Z}^d \text{ so that } |x_k - p_k/q_i| \leq q_i^{-\frac{d}{s}}\}$$

where $x = (x_1, \ldots, x_d)$, $p = (p_1, \ldots, p_d)$, and $0 < s < d$. A standard argument (see e.g. [**Falconer85**], Theorem 8.15) shows that the Hausdorff dimension of $E = \cap E_i$ is s. (Note that the argument in Falconer's book is one-dimensional, but the same argument works in higher dimensions). On the other hand,

$$(0.5) \qquad \lambda(\Delta(E_i)) \lesssim q_i^2 q_i^{-\frac{d}{s}},$$

where λ denotes the Lebesgue measure, since $\#\Delta([0,q_i]^d \cap \mathbb{Z}^d) \lesssim q_i^2$ due to the fact $0 \leq n_1^2 + \cdots + n_d^2 \leq dq_i^2$. Since $q_i^2 q_i^{-\frac{d}{s}} \to 0$ as $q_i \to \infty$ for $s < \frac{d}{2}$, we see that the standard Falconer Distance Conjecture cannot hold for $s < \frac{d}{2}$.

However, if the Euclidean distance is replaced by an arbitrary distance, all we can say is that $\#\Delta([0,q_i]^d \cap \mathbb{Z}^d) \lesssim q_i^d$ since the integer lattice is translation invariant. In particular, it follows that for the above construction, $\Delta_{a,O}(E)$ has zero Lebesgue measure for every $(a,O) \in SO(d)$ if the Hausdorff dimension of E is smaller than 1.

Now let E_i be defined as above with $\mathbb{Z}^d$ replaced by a well-distributed set A. Using the proof of the aforementioned Theorem 8.15 in [**Falconer85**] one can check that the Hausdorff dimension of $E = \cap E_i$ is still s. Now suppose that $\#\Delta(A_{R_i}) \lesssim R_i^\beta$ for a sequence of $\{R_i\}$ going to infinity. Rarify this sequence, if necessary, so that it satisfies the conditions on the sequence $\{q_i\}$ above. Thus we may assume that $\#\Delta(A_{q_i}) \lesssim q_i^\beta$. It follows that $\lambda(\Delta(E_i)) \lesssim q_i^\beta q_i^{-\frac{d}{s}}$. If $\beta < 2$, we can find $s > \frac{d}{2}$ so that $q_i^\beta q_i^{-\frac{d}{s}} \to 0$ as $q_i \to \infty$. This would say that the Lebesgue measure of $\Delta(E)$ is zero even though the Hausdorff dimension of E is greater than $\frac{d}{2}$. This is impossible if the Falconer Distance Conjecture is true. Thus Falconer Distance Conjecture implies the Asymptotic Erdos Distance Conjecture.

We now state the following consequence of Theorem 0.1., Part ii).

COROLLARY 0.2. *Let A be a separated well-distributed subset of $\mathbb{R}^d$. Suppose that K is a bounded symmetric convex set with a smooth boundary. Then for every $\epsilon > 0$ and almost every $a \in [1,2]^d$, there exists $C_\epsilon > 0$ such that $\#\Delta_{aK}(A_R) \geq C_\epsilon R^{d-\epsilon}$.*

The proof of Corollary 0.2 follows directly from the argument in the preceding paragraph. We also note that Corollary 0.2 is sharp, at least up to the factor of $C_\epsilon R^\epsilon$. To see this, take $A = \mathbb{Z}^d$. Since the lattice is translation invariant, there is no harm in just counting the number of distances in A_R to the origin, which, of course, cannot exceed R^d.

Observe that this is a stronger result than what can be obtained point-wise. If $A = \mathbb{Z}^d$, then it is clear that $\#\Delta(A_R) \leq R^2$, and, in fact, the same lower bound also holds, up to logarithms in lower dimensions. This is the same dichotomy we encountered for the average version of the Falconer distance problem where the point-wise result cannot hold if the Hausdorff dimension of E is smaller than $\frac{d}{2}$,

whereas the average version over ellipsoids (Theorem 0.1 Part ii)) holds as long as the Hausdorff dimension of E is greater than 1.

Basic reductions

Let $d\nu_0$ denote the measure on $\Delta_K(E)$ defined by

$$(1.1) \qquad \int f(t) d\nu_0(t) = \int\int f(||x-y||_K) d\mu(x) d\mu(y),$$

where $d\mu$ is the Frostman measure on E. Recall that the Frostman measure is a probability measure such that $\mu(B_\delta(x_0)) \lesssim \delta^\alpha$, for any α less than the Hausdorff dimension of E. See, for example, [**Wolff02**] for a proof that such a measure always exists. Here, and throughout the paper, $a \lesssim b$ means that there exists a positive constant C such that $a \leq Cb$. Similarly, $a \approx b$ means that $a \lesssim b$ and $b \lesssim a$.

In order to prove that the Lebesgue measure of $\Delta_K(E)$ is positive, it suffices to prove that $\widehat{\nu} \in L^2(1,\infty)$ since Plancherel would then imply that ν is an L^2 density, i.e the Lebesgue measure times an L^2 function whose support contains $\Delta_K(E)$. Let β be a smooth cutoff function supported in $[1/2,4]$, identically equal to 1 in $[1,2]$, with $\sum \beta(2^k \cdot) \equiv 2$. Thus it suffices to prove that

$$(1.2) \qquad \sum_{k,k',n} \int \int\int_{S_{k,k'}} e^{2\pi i t(||x-y||_K - ||x'-y'||_K)} d\mu^* \beta(2^{-n}t) dt < \infty,$$

where $d\mu^* = d\mu(x) d\mu(y) d\mu(x') d\mu(y')$ and

$$(1.3) \ \ S_{k,k'} = \{(x,y,x',y') \in E\times E\times E\times E : ||x-y||_K \approx 2^{-k}; \ ||x'-y'||_K \approx 2^{-k'}\}.$$

Integrating in t we obtain,

$$(1.4) \qquad 2^n \int_{S_{k,k'}} \widehat{\beta}(2^n(||x-y||_K - ||x'-y'||_K)) d\mu^*.$$

It is not difficult to see that we may take $k \approx k'$, and $k < n/100$, since $\mu \times \mu(\{(x,y) : ||x-y||_K \lesssim 2^{-n}\}) \lesssim 2^{-n\alpha}$. It is worth noting that the only property of $\widehat{\beta}$ that plays an important role here is that it decays rapidly away from a fixed neighborhood of the origin. This is a general property of Fourier transforms of smooth compactly supported functions. See, for example, [**Sogge93**], for this and similar integration by parts arguments.

Thus we are led to study the following object:

$$(1.5) \qquad \sum_{k \approx k' \leq n/100} 2^n \int_{S_{k,k'}} \widehat{\beta}(2^n(||x-y||_K - ||x'-y'||_K)) d\mu^*.$$

We remark that the expression (1.5) is quite natural. In order to obtain good estimates on the integral above, one has to prove a quantitative version of the heuristic that distances do not repeat very often, even approximately. For if they did, the distance set could indeed be very small.

In order to establish the criterion described in $(*)$ above, Mattila ([**Mattila87**]) modified the measure $d\nu$ by considering $e^{i\pi \frac{d-1}{8}} t^{-\frac{d-1}{2}} d\nu(t) + e^{-i\pi \frac{d-1}{8}} |t|^{-\frac{d-1}{2}} d\nu(-t)$.

He then proved using the method of stationary phase in the context of the Euclidean metric (see [**HofIos03**] for a general description) that the statement that $\widehat{\nu} \in L^2$ is equivalent to the statement that

$$(1.6) \qquad \int_1^\infty \left(\int_{\partial K^*} |\widehat{\mu}(t\omega)|^2 d\omega_{K^*} \right)^2 t^{d-1} dt < \infty,$$

where $K^* = \{\xi : \sup_{x \in K} x \cdot \xi \leq 1\}$, is the dual body of K, and $d\omega_K$ denotes the Lebesgue measure on ∂K^*. It follows that in order to prove (1.6), it suffices to bound the following variant of (1.5):

$$\sum_{k \approx k' \leq n/100} 2^n \int_{S_{k,k'}} \widehat{\beta}(2^n(||u||_K - ||u'||_K))|u|^{-\frac{d-1}{2}}|u'|^{-\frac{d-1}{2}} d\mu^*$$

$$(1.7) \qquad \approx \sum_{k \approx k' \leq n/100} 2^n 2^{k(d-1)} \int_{S_{k,k'}} \widehat{\beta}(2^n(||u||_K - ||u'||_K)) d\mu^*,$$

where here and for the remainder of the paper $u = x - y$ and $u' = x' - y'$.

The aforementioned results of Bourgain and Wolff were obtained using Mattila's approach. More precisely, they used restriction theorem technology to obtain good estimates on

$$(1.8) \qquad \int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 d\omega.$$

In two dimensions, this approach can only prove that the distance set is of positive Lebesgue measure under the assumption that the Hausdorff dimension of the underlying set is greater than $\frac{4}{3}$, which is precisely what Wolff did. In higher dimensions, it is not clear whether optimal bounds on (1.8) alone are sufficient to prove Falconer's conjecture. In all dimensions, it seems quite possible that $(*)$ (or, more generally, (1.6)) may lead to the solution of the Falconer conjecture.

Proof of Theorem 0.1

Proof of Part i). We now specialize (1.7) to our situation. Let $|| \cdot ||_{a,O}$ denote the norm generated by the ellipsoid with eccentricities $a = (a_1, \ldots, a_d) \in [1,2]^d$, rotated by $O \in SO(d)$.

It suffices to prove that
$$(2.1)$$
$$\sum_{k \approx k' \leq n/100} 2^n 2^{k(d-1)} \int \int \int_{S_{k,k'}} \widehat{\beta}(2^n(||u||_{a,O} - ||u'||_{a,O})) d\mu^* \psi(a) da dH_d < \infty,$$

where ψ is a smooth cutoff function supported in $[1/2, 4]^d$, identically equal to 1 in $[1,2]^d$, and dH_d is the Haar measure on $SO(d)$.

Let $\omega = \frac{u}{|u|}$ and $\omega' = \frac{u'}{|u'|}$. Let $A_1, \ldots, A_{d-1}$, and $B_1, \ldots, B_{d-1}$ denote the standard polar coordinate angles of ω and ω' respectively. Decompose $j_l 2^{-(n-k)} \leq A_l \leq (j_l + 1)2^{-(n-k)}$ and $j'_l 2^{-(n-k)} \leq B_l \leq (j'_l + 1)2^{-(n-k)}$. Let $j = (j_1, \ldots, j_{d-1})$, $j' = (j'_1, \ldots, j'_{d-1})$ and define $S_{k,k',j,j'}$ accordingly.

LEMMA 2.1. *For $(x, y, x', y') \in S_{k,k',j,j'}$ and any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that*

$$\int \lambda(\{a \in [1,2]^d : |\|u\|_{a,O} - \|u'\|_{a,O}| \leq 2^{-n}\}) dH_d$$

(2.2)

$$\leq C_\epsilon 2^{n\epsilon} \min \left\{ 1, \Pi_{l=1}^{d-1} \frac{1}{|j_l - j_l'|^{\frac{1}{d-1}}} \left(\Pi_{i1 \neq i2} \frac{j_{i2} j_{i2}'}{|j_{i1} j_{i2}' - j_{i1}' j_{i2}|} \right)^{\frac{1}{d-1}} \left(2^{-(n-k)} \right)^{\frac{d-2}{2}} \right\}.$$

Lemma 2.1 follows by induction using the two-dimensional case and the following version of the box lemma.

LEMMA 2.2. *Let $\pi_{i1,i2}$ denote the projection operator onto $(i1, i2)$-axis. Let S be a measurable compact set. Then*

(2.3) $$\lambda(S) \leq \Pi_{i1 \neq i2} \lambda(\pi_{i1,i2}(S))^{\frac{1}{d-1}}.$$

To prove Lemma 2.2, observe that $\chi_S(x) \leq \Pi_{i1 \neq i2} \chi_{\pi_{i1,i2}(S)}(x_{i1}, x_{i2})$ and successively apply Holder's inequality.

To apply Lemma 2.2 to obtain Lemma 2.1, we need the fact that in two dimensions, for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $\int |\{a \in [1,2]^d : |\|u\|_{a,O} - \|u'\|_{a,O}| \leq 2^{-n}\}| dH_2 \leq C_\epsilon 2^{n\epsilon} \sum_{j_1 \neq j_1'} \frac{1}{|j_1 - j_1'|}$. This is verified by a direct calculation using the fact that $dH_2 = d\phi$, since $SO(2) = S^1$.

It follows that given $\epsilon > 0$ there exists $C_\epsilon > 0$ such that the expression in (2.1) is $\leq C_\epsilon 2^{n\epsilon} 2^n 2^{k(d-1)} \times$

$$\sum_{k \approx k' \leq n/100} \sum_{j \neq j'} \min \left\{ 1, \Pi_{l=1}^{d-1} \frac{1}{|j_l - j_l'|^{\frac{1}{d-1}}} \left(\Pi_{i1 \neq i2} \frac{j_{i2} j_{i2}'}{|j_{i1} j_{i2}' - j_{i1}' j_{i2}|} \right)^{\frac{1}{d-1}} \left(2^{-(n-k)} \right)^{\frac{d-2}{2}} \right\}$$

(2.4)

$$\times \int_{S_{k,k',j,j'}} d\mu^* \leq C_\epsilon 2^{n\epsilon} 2^n 2^{k(d-1)} \sum_{k \approx k' \leq n/100} \left(2^{(n-k)} \right)^{d-2} \left(2^{-(n-k)} \right)^{\frac{d-2}{2}} \sum_{j'} \int_{S_{k,k',j,j'}} d\mu^*,$$

where $S_{k,k',j,j'}$ is defined before the statement of Lemma 2.1.

If we fix x', y' and y, x is contained in a ball of radius 2^{-n} whose μ measure is bounded by $C 2^{-n\alpha}$ since μ is the Frostman measure. If we sum in j' and integrate in x', we see that this integral is bounded by the μ measure of the annulus of width radius 2^{-k} and width 2^{-k}, which yields $C 2^{-k\alpha}$ since the annulus in question is contained in a ball of radius 2^{-k+1}. It follows that the expression (2.4) is

(2.5) $$\leq C_\epsilon 2^{n\epsilon} \sum_{k \approx k' \leq n/100} 2^{\frac{nd}{2}} 2^{-n\alpha} 2^{-k\alpha} 2^{-\frac{(d-2)k}{2}} 2^{k(d-1)} < \infty$$

if $\alpha > \frac{d}{2}$, and the proof is complete.

Proof of the part ii). Without loss of generality let us assume that the origin is in E. We can also assume, after perhaps applying a linear transformation that E is contained in the first quadrant and there is a cone L with the vertex at the origin such that positive portion of E is contained in L and all partial derivatives of the function $f(x) = \|x\|_K$ in that cone are bounded from above and below.

The proof of ii) will be based on the following version of the projection theorem due to Solomyak ([**Solomyak98**]).

THEOREM 2.3. *Let E be a Borel subset of $\mathbb{R}^d$. Let U be a locally compact topological space with a Borel measure μ, finite on compact subsets. Suppose that we have a family of maps $\Phi_a : E \to \mathbb{R}$ for $a \in U$, with the following properties:*

$$(2.6) \qquad |\Phi_a(x) - \Phi_a(y)| \leq c(a)|x - y|, \quad \text{for} \quad x, y \in E \quad \text{and} \quad a \in U,$$

and for any compact set $G \subset U$ and any $r > 0$

$$(2.7) \qquad \mu\{a \in G, |\Phi_a(x) - \Phi_a(y)| \leq r\} \leq K_G \min\left(1, \frac{r}{|x - y|}\right).$$

Then $\lambda(\Phi_a(E)) > 0$ for μ-a.e. $a \in U$, provided that the Hausdorff dimension of E is greater than 1.

Set $\Phi_a(x) = \rho_a(x) = \rho(a_1 x_1, a_2 x_2, \dots, a_d x_d) = ||x||_{\frac{1}{a}K}$. Using the mean value theorem and the boundedness of partial derivatives from above, we get

$$|\Phi_a(x) - \Phi_a(y)| = |\rho_a(x) - \rho_a(y)| \leq$$

$$|\rho(a_1 x_1, a_2 x_2, \dots, a_d x_d) - \rho(a_1 y_1, a_2 x_2, \dots, a_d x_d)| +$$
$$|\rho(a_1 y_1, a_2 x_2, \dots, a_d x_d) - \rho(a_1 y_1, a_2 y_2, a_3 x_3, \dots, a_d x_d)| + \dots$$
$$|\rho(a_1 y_1, a_2 y_2, \dots, a_{d-1} y_{d-1}, a_d x_d) - \rho(a_1 y_1, a_2 y_2, \dots, a_d y_d)|$$

$$(2.8) \qquad \lesssim \sum_{j=1}^{d} a_j |x_j - y_j| \lesssim |x - y|,$$

and so (2.6) is verified. On the other hand

$$|\Phi_a(x) - \Phi_a(y)| = |\rho_a(x) - \rho_a(y)| \gtrsim \left(\sum_{j=1}^{d} a_j^2 (x_j - y_j)^2\right)^{1/2}$$

$$\gtrsim \sum_{j=1}^{d} a_j |x_j - y_j|$$

$$(2.9) \qquad \gtrsim (a_1^2 + a_2^2 + \dots + a_d^2)^{1/2} |x - y|.$$

In the first line of (2.9) we have used the mean value theorem and boundedness all partial derivatives from below. On the second line we have used the fact that l^1 norm and l^2 norm are comparable, and on the last line we have used the fact that angle between $(a_1, a_2, \dots, a_d)$ and $(|x_1 - y_1|, \dots, |x_d - y_d|)$ less than $\frac{\pi}{2}$. So we have

$$\mu\left\{a \in [1, 2]^d, |\Phi_a(x) - \Phi_a(y)| \leq r\right\}$$

$$(2.10) \qquad \leq \mu\left\{a \in [1, 2]^d, \left(\sum_{j=1}^{d} a_j^2\right)^{1/2} \lesssim \frac{r}{|x - y|}\right\} \lesssim \frac{r^d}{|x - y|^d}.$$

It follows that

$$(2.11) \qquad \mu\{a \in [1, 2]^d, |\Phi_a(x) - \Phi_a(y)| \leq r\} \lesssim \min\left(1, \frac{r}{|x - y|}\right).$$

This completes the proof of Part ii).

Proof of Part iii). We shall prove the following slightly stronger version of the claim in Part iii). Let $K = B_d$, the unit ball. Suppose that the Hausdorff dimension of E is greater than two. Then for every $x \in \mathbb{R}^d$, $\Delta_{aK}^x(E) = \{\|x - y\|_{aK}, y \in E\}$ contains an interval for almost every $a \in [1, 2]^d$. Moreover, for each $d > 2$, there exists a Borel set $E \subset \mathbb{R}^d$ such that the Hausdorff dimension of E equals two, whereas $\Delta_{aK}^0(E)$ has an empty interior for almost all $a \in [1, 2]^d$. The proof of this is based on the following result due to Peres and Schlag ([**PeresSchlag00**], Corollary 6.2).

THEOREM 2.4. *Suppose $E \subset \mathbb{R}^d$ is a Borel set of Hausdorff dimension greater than $2k$. Let $G(d, k)$ denote the Grassmanian of k-dimensional subspaces of $\mathbb{R}^d$. Then for a.e. $\pi \in G(d, k)$ the projections of E onto π have non-empty interior.*

As above we assume without loss of generality that x in the definition of $\Delta_{K_a}^x(E)$ is the origin. It follows by a direct calculation (see also, for example, [**Falconer85**]) that the Hausdorff dimension of E equals the Hausdorff dimension of E^2, where $E^2 = \{(x_1^2, \ldots, x_d^2) : (x_1, \ldots, x_d) \in E\}$. Writing out the definition of $\Delta_{aK}^x(E)$, we see that

$$(2.12) \qquad (\Delta_{aK}^0(E))^2 = \left\{ \sum_{j=1}^{d} \frac{x_j^2}{a_j^2} : (x_1, \ldots, x_d) \in E \right\}$$

$$= c(a)(projection\ of\ E^2\ onto\ e_a),$$

where e_a is the vector on S^{d-1} with the jth component equal to $\left(\frac{a_j^{-2}}{(\sum_{l=1}^{d} a_l^{-4})^{1/2}} \right)$ and $c(a) = (\sum_{l=1}^{d} a_l^{-4})^{1/2}$. An application of Theorem 2.4 with $k = 1$ completes the proof.

To prove the second part of iii) we take a Besicovich set A in $\mathbb{R}^2$, i.e. a set of measure zero that contains a full line in every direction. See, for example, [**Falconer85**] for a thorough description of the topic and concrete examples. Define

$$(2.13) \qquad B = \{\mathbb{R}^2 \setminus \cup_{r \in \mathbb{Q}^2}(r + A)\} \cap \{x \in \mathbb{R}^2, x_i > 0, i = 1, 2\}$$

as a subset of the (x_1, x_2) coordinate plane of $\mathbb{R}^d$. Here $\mathbb{Q}$ denotes the set of rational numbers.

We claim that the set $E = \{(x_1^{1/2}, \ldots, x_d^{1/2}), (x_1, \ldots, x_d) \in B\}$ is the set whose existence is claimed in the second part of iii). Indeed, the dimension of E is 2. To see this, observe that B is clearly two-dimensional since it is a subset of positive measure of two-dimensional Euclidean space. The set E is the image of B under a mapping which is Lipschitz except at a single point. It follows (see e.g. [**Falconer85**]) that the Hausdorff dimension of E is at least two. Since E is contained in the two-dimensional Euclidean space, the Hausdorff dimension of E is exactly two. Moreover, we have

$$(\Delta_{aK}^0(E))^2 = \left\{ \sum_{j=1}^{d} \frac{x_j^2}{a_j^2} : (x_1, \ldots, x_d) \in E \right\}$$

$$(2.14) \qquad = c(a)(projection\ of\ B\ on\ e_a),$$

where e_a and $c(a)$ as above.

We conclude that projections of B onto lines which do not lie in the (x_1, x_2)-plane do not contain intervals. To see this observe that the pre-image of a putative interval on such a projection is a strip in the (x_1, x_2)-plane. Such a strip is missing a line by definition of B. This proves that there exists a point in the putative interval which is in fact not there. This contradiction completes the proof.

References

References

[Bourgain94]   J. Bourgain, *Hausdorff dimension and distance sets*, Israel J. Math. **87** (1994), 193-201.

[Falconer85]   K. J. Falconer, *The geometry of fractal sets*, Cambridge University Press (1985).

[Falconer86]   K. J. Falconer, *On the Hausdorff dimensions of distance sets*, Mathematika **32** (1986), 206-212.

[HofIos03]   S. Hofmann and A. Iosevich, *Circular averages and Falconer/Erdos distance conjecture in the plane for random metrics*, Proc. Amer. Math. Soc. (submitted) (2003).

[IKP99]   A. Iosevich, N. Katz, and S. Pedersen, *Fourier basis and the Erdős distance problem*, Math. Research Letters **6** (1999).

[IKT01]   A. Iosevich, N. Katz, and T. Tao, *Convex bodies with a point of curvature do not have Fourier bases*, Amer. J. Math. **123** (2001), 115-120.

[IoLa2003]   A. Iosevich and I. Laba, *K-distance sets and Falconer conjecture*, PLMS (submitted) (2003).

[Mattila87]   P. Mattila, *Spherical averages of Fourier transforms of measures with finite energy: dimensions of intersections and distance sets*, Mathematika **34** (1987), 207-228.

[PA95]   J. Pach and and P. Agarwal, *Combinatorial Geometry*, Wiley-Interscience Series (1995).

[PeresSchlag00]   Y. Peres and W. Schlag, *Smoothness of projections, Bernoulli convolutions, and the dimension of exceptions*, Duke Math. J. **102** (2000), 193-251.

[Rudin87]   W. Rudin, *Real and complex analysis*, Third edition. McGraw-Hill Book Co., New York (1987).

[Sjolin93]   P. Sjolin, *Estimates of spherical averages of Fourier transforms and dimensions of sets*, Mathematika **40** (1993), 322-330.

[Sogge93]   C. D. Sogge, *Fourier integrals in classical analysis*, Cambridge Tracts in Mathematics, Cambridge University Press **105** (1993).

[Solomyak98]   B. Solomyak, *Measure and dimension for some fractal families*, Math. Proc. Cambridge Philos. Soc. **124** (1998), 531-546.

[Wolff99]   T. Wolff, *Decay of circular means of Fourier transforms of measures*, International Mathematics Research Notices **10** (1999), 547-567.

[Wolff02]   T. Wolff, *Lectures in Harmonic Analysis*, California Institute of Technology Class Lectures Notes (revised by I. Laba) (2002).

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