February 4, 2019

G open connected, $b \in \mathbb{C}$ fixed, define $g(z) = \exp(b f(z))$

$$\overset{\parallel}{z^b}$$

yields a branch of z^b if a branch of $\log(z)$ is available.

If we wish to view z^b as a function, this means

$$z^b = \exp(b f(z)), \quad \text{where } f(z) = \log z, \text{ the } \underline{\text{principal}} \text{ branch of the logarithm.}$$

It follows that z^b is analytic, since $\log(z)$ is.

Definition: A <u>region</u> is an open connected subset of the plane.

<u>Cauchy - Riemann equations</u>: Let $f: G \to \mathbb{C}$ be analytic and let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$. We know that

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \quad \text{exists}$$

Take h real. Then

$$\frac{f(x+h+iy) - f(x+iy)}{h} = \frac{u(x+h,y) - u(x,y)}{h} + i\left(\frac{v(x+h,y) - v(x,y)}{h}\right)$$

$$= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y).$$

②

Now let h be imaginary, ie take ih, h real:

$$\frac{f(z+ih)-f(z)}{ih} = -i\left(\frac{u(x,y+h)-u(x,y)}{h}\right) + \frac{v(x,y+h)-v(x,y)}{h}$$

$$\longrightarrow -i\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y)$$

It follows that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ ; \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

If u, v have continuous second derivatives, differentiate and obtain $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

& the same w/ $\underline{\underline{v}}$

harmonicity condition (to be studied in detail later, but why wait?)

Let's try for some kind of a converse. Suppose that

$f(z) = u(z) + iv(z)$, $z = x+iy$, $B(z,r) \subset G$. Let $h = s+it$,

continuous partials assumed; strict

$$u(x+s, y+t) - u(x,y) = \big(u(x+s, y+t) - u(x, y+t)\big) + \big(u(x, y+t) - u(x,y)\big)$$

one variable at a time

mean value theorem

$$= u_x(x+s_1, y+t)\cdot s + u_y(x, y+t_1)\cdot t$$

$|s_1| < s$, $|t_1| < t$

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Let $\varphi(s,t) = [u(x+s, y+t) - u(x,y)] - [u_x(x,y)s + u_y(x,y)t]$

Then $\dfrac{\varphi(s,t)}{s+it} = \dfrac{s}{s+it}\left(u_x(x+s_1, y+t) - u_x(x,y)\right)$

$$+ \frac{t}{s+it}\left(u_y(x, y+t_1) - u_y(x,y)\right)$$

Since the partials are continuous and $|s| \leq |s+it|$, $|t| \leq |s+it|$, $|s_1| < s$, $|t_1| < t$,

$$\lim_{s+it \to 0} \frac{\varphi(s,t)}{s+it} = 0.$$

It follows that

$$u(x+s, y+t) - u(x,y) = u_x(x,y)s + u_y(x,y)t + \varphi(s,t),$$

w/ φ as above.

Similarly,

$$v(x+s, y+t) - v(x,y) = v_x(x,y)s + v_y(x,y)t + \psi(s,t)$$

satisfies the same limit as φ above

$$\frac{f(z+s+it) - f(z)}{s+it} = u_x(z) + iv_x(z) + \frac{\varphi(s,t) + \psi(s,t)}{s+it}$$

$\longrightarrow$ f is differentiable and $f'(z) = u_x(z) + iv_x(z)$.

Moreover, since u_x, v_x continuous, f' is continuous, hence f is <u>analytic</u>.

The precise statement of what we proved is the following.

Theorem: Let u, v be real-valued functions defined on a region G and suppose that u, v have continuous partial derivatives. Then if $f: G \to \mathbb{C}$ is defined by $f(z) = u(z) + iv(z)$, it is analytic iff u, v satisfy Cauchy-Riemann equations.

Harmonic conjugate: G region & $u: G \to \mathbb{R}$ harmonic.

Does there exist $\underbrace{v: G \to \mathbb{R}}_{\text{called harmonic conjugate}}$ $\ni f = u + iv$ is analytic in G?

Theorem: Suppose that $G = \mathbb{C}$ or some open disk. If $u: G \to \mathbb{R}$ is a harmonic function, then u has a harmonic conjugate.

Proof: Let $G = B(\vec{0}, R)$, $0 < R \leq \infty$, $u: G \to \mathbb{R}$ harmonic.

Define $v(x,y) = \int_0^y u_x(x,t)\,dt + \varphi(x)$

to be determined so that C-R are satisfied.

We have — this is the "obvious" choice. Why?

$$V_x(x,y) = \int_0^y u_{xx}(x,t)\,dt + \varphi'(x)$$

not automatic! See Prop IV in 2.1

harmonicity

$$= -\int_0^y u_{yy}(x,t)\,dt + \varphi'(x) = -u_y(x,y) + u_y(x,0) + \varphi'(x)$$

fundamental theorem of calculus

This forces us to mandate $\varphi'(x) = -u_y(x,0)$.

Therefore, $v(x,y) = \int_0^y u_x(x,t)\,dt - \int_0^x u_y(s,0)\,ds$

results in a pair (u,v) satisfying C-R equations.

Analytic functions as mappings:

Suppose that $f: \mathbb{C} \longrightarrow \mathbb{C}$ analytic. If we write $z = x+iy$, we can view f as a transformation of the plane.

Let $f(z) = z^2 = (x+iy)^2 = (x^2-y^2) + i(2xy)$

Consider $L_1 = \{(t+1, t) : t \in \mathbb{R}\}$ & view these

$L_2 = \{(-t+1, t) : t \in \mathbb{R}\}$ as subsets of $\mathbb{C}$

$$L_1 \longrightarrow \left\{\left((t+1)^2 - t^2,\ 2t(t+1)\right)\right\} = \{((2t+1), 2t^2+2t)\}$$

$$L_2 \longrightarrow \{(1-t)^2 - t^2,\ +2t(1-t)\} = \{(-2t+1, 2t-2t^2)\}$$

tanglent line $f(L_1)$: $\{(2, 4t+2)\}$

" " $f(L_2)$: $\{(-2, 2-4t)\}$

Dot product $= -4 + 4 - 16t^2 = 0$ iff $t=0$

This is where L_1 & L_2 intersect!

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We just saw that lines $L_1 \perp L_2$ are mapped to parabolas that are also $\perp$ to each other. We shall return to this theme over and over.

Definition: A path in a region $G \subset \mathbb{C}$ is a continuous function $\gamma : [a,b] \longrightarrow \mathbb{C}$. If γ' exists and is continuous for each $t \in [a,b]$, we call γ a smooth path. γ is piece-wise smooth if it is smooth on a finite collection of sub-intervals (open) covering $[a,b]$ up to finitely many endpoints.

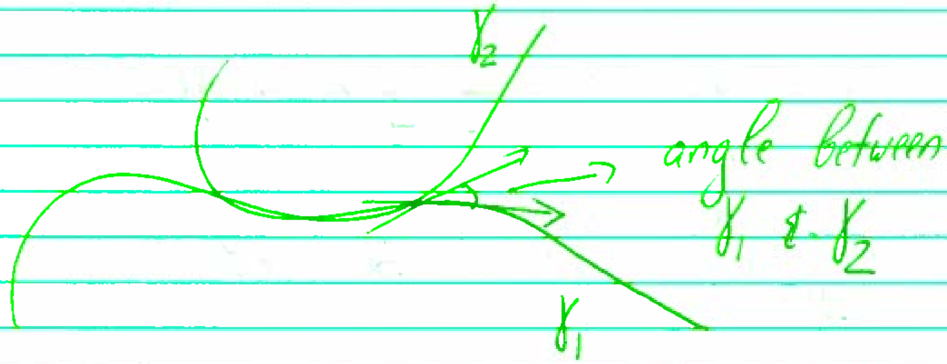
$$\gamma'(t) \equiv \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}$$

ie $Re(\gamma)$, $Im(\gamma)$ have derivatives w/ right & left limits at a, b, respectively.

If $\gamma'(t_0)$ exists, the tangent line goes through $z_0 = \gamma(t_0)$ in the direction $\gamma'(t_0)$. Call $\arg \gamma'(t_0)$, the angle where it is pointing.

If γ_1, γ_2 smooth paths, $\gamma_1(t_1) = \gamma_2(t_2) = z_0$, $\gamma_1'(t_1) \neq 0$, $\gamma_2'(t_2) \neq 0$,

define $\text{angle}(\gamma_1, \gamma_2)_{z_0} = \arg \gamma_2'(t_2) - \arg \gamma_1'(t_1)$



If γ is smooth in G and $f: G \to \mathbb{C}$ is analytic. Then $\sigma = f \circ \sigma$ is a smooth path

and $\sigma'(t) = f'(\gamma(t)) \cdot \gamma'(t)$

Let $z_0 = \gamma(t_0)$ and suppose that $\gamma'(t_0) \neq 0$ & $f'(z_0) \neq 0$. Then

$$\sigma'(t_0) \neq 0 \text{ and } \arg \sigma'(t_0) = \arg f'(z_0) + \arg \gamma'(t_0),$$

which is more conveniently written in the form

$$\arg \sigma'(t_0) - \arg \gamma_1'(t_0) = \arg f'(z_0)$$

This implies the following result:

Theorem: If $f: G \to \mathbb{C}$ is analytic, then f preserves angles at each z_0 where $f'(z_0) \neq 0$.