MTH 174 Homework #2 Solutions

Pages 17-18, #s 2,3,5,7,9 ages 22-25, #s 10,11,14,16 Page 28, #17,22 Pages 33-34, 29,32,34

Problem 2: Show that f is independent of the second variable if and only if there is a function $g:\mathbb{R}\to\mathbb{R}$ such that $f(x,y)=g(x)$. What is $f'(a,b)$ in terms of g'?

- First suppose that f is independent of the second variable. Then $f(x,y_1)=f(x,y_2)\ \forall y\in\mathbb{R}$ so in particular, $f(x,y)=f(x,0) \Rightarrow g(x)=f(x,0)=f(x,y)\ \forall y$.
- On the other hand suppose that $f(x,y)=g(x)$ for some $g(x)$. then clearly f depends only on x so $f(x,y_1)=f(x,y_2)\ \forall y_2\in\mathbb{R}$.
- We aren't allowed to use tools like the chain rule, so we have to use the definitions.

$$\lim_{(h,k)\to(0,0)}\left|\frac{f(a-h,b-k)-f(a,b)-\lambda(h,k)}{(h,k)}\right| = \lim_{(h,k)\to(0,0)}\left|\frac{g(a-h)-g(a)-g'(a)\cdot h}{(h,k)}\right| \le \lim_{h\to 0}\left|\frac{g(a-h)-g(a)-g'(a)\cdot h}{h}\right| = 0$$

If $g(x)$ is differentiable $\Rightarrow \lambda(h,k)$, the derivative of f, is $=(g'(a),0)$ in matrix form.
Notice that $f(x,y)=(g(x),c)$ for some constant c. So the "Jacobian" is $(g'(x),0)$, which matches our computation.

Problem 7: Let $f:\mathbb{R}^n\to\mathbb{R}$ be a function such that $|f(x)|<|x|^2$. Show that f is differentiable at 0.

Let zero be the vector $(0,\dots,0)\in\mathbb{R}^n$ + $h\in\mathbb{R}^n$. then

$$\lim_{h\to 0}\left|\frac{f(0+h)-f(0)+\lambda(h)}{h}\right| \le \lim_{h\to 0}\frac{|f(h)|+|f(0)|+|\lambda(h)|}{|h|} \le \lim_{h\to 0}\left(\frac{|h|^2}{|h|}+\frac{|\lambda(h)|}{|h|}\right)$$

$$= \lim_{h\to 0}\frac{|\lambda(h)|}{|h|} \Rightarrow \text{we may let } |\lambda(h)|=0 \text{ + conclude } f \text{ is differentiable}$$

with $f'=0$.

Problem 9: Two functions $f,g:\mathbb{R}\to\mathbb{R}$ are equal up to n^{th} order at a if $\lim_{h\to a}\frac{f(a+h)-g(a+h)}{h^n}=0$

a) Show that f is differentiable at a iff $\exists$ a function g of the form $g(x)=a_0+a_1(x-a)$ such that f and g are equal up to first order at a.

- Suppose $\exists$ a function $g(x)=a_0+a_1(x-a)$ equal to f up to first order. Then

$$\lim_{h\to 0}\frac{f(a+h)-g(a+h)}{h} = \lim_{h\to 0}\frac{f(a+h)-a_0+a_1(a+h-a)}{h} = \lim_{h\to 0}\frac{f(a+h)-a_0+a_1h}{h}=0$$

$$\Rightarrow \lim_{h\to 0}[f(a+h)-a_0+a_1h]=0 \Rightarrow f(a)=a_0 \quad \text{So we have:}$$

$$\lim_{h\to 0}\frac{f(a+h)-f(a)+a_1h}{h} = \lim_{h\to 0}\frac{f(a+h)-a_0+a_1h}{h} = \lim_{h\to 0}\frac{f(a+h)-g(a+h)}{h}=0$$

$\Rightarrow f'(a)$ exists and is equal to a_1.

- Next suppose that f is differentiable at a. then $\lim_{h\to 0}\frac{f(a+h)-f(a)-\lambda h}{h}=0$

Set $g(x)=f(a)+\lambda(x-a)$. then $g(a+h)=f(a)+\lambda h \Rightarrow \lim_{h\to 0}\frac{f(a+h)-f(a)-\lambda h}{h}=\lim_{h\to 0}\frac{f(a+h)-g(a+h)}{h}=0$

so $g=f$ up to first order. (this is a good way to think about what differentiable <u>means</u>

b) If $f'(a), \dots, f^{(n)}(a)$ exist, show that f + the function g defined by $g(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$ are equal up to n^{th} order.

$$g^k(x) = \begin{cases} \sum_{i=k}^{n} \frac{f^{(i)}(a)}{(i-k)!}(x-a)^{i-k} & , k \le n \\ 0 & , k > n \end{cases} \Rightarrow g^k(a) = \begin{cases} f^{(k)}(a) & , k \le n \\ 0 & , k > n \end{cases}$$

Then apply L'Hospital's Rule to $\lim_{h\to 0} \frac{f(a+h) - g(a+h)}{h^n}$ $\Rightarrow$ f + g are equal up to n^{th} order.

Problem 11: Find f' for the following (where $g: \mathbb{R} \to \mathbb{R}$) is continuous:

a) $f(x,y) = \int_a^{x+y} g$ Let $h(t) = \int_a^t g$. Then $f(x,y) = h(u(x,y))$ where $u(x,y) = x+y$.

Then $f'(x,y) = [D_1 h\circ u(x,y), D_2 h\circ u(x,y)] = [g(x+y), g(x+y)]$

b) $f(x,y) = \int_a^{x\cdot y} g \Rightarrow f'(x,y) = [y g(xy), x g(xy)]$

c) $f(x,y,z) = \int_{x^y}^{\sin(x\sin(y\sin z))} g = \int_a^{\sin(x\sin(y\sin(z)))} g - \int_a^{x^y} g$

$= g(\sin(x\sin(y\sin z))) \cdot D\sin(x\sin(y\sin(z))) - g(x^y)\cdot Dx^y$

$= g(\sin(x\sin(y\sin z)))\cos(x\sin(y\sin(z)))[\sin(y\sin z), x\cos(y\sin z)\sin z, x\sin(y\sin z)y\cos z]$

$- g(x^y)[yx^{y-1}, x^y \ln(x), 0]$.

Problem 16: Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$ is differentiable and has a differentiable inverse. Show that $(f^{-1})'(a) = [f'(f^{-1}(a))]^{-1}$. Hint: $f\circ f^{-1}(x) = x$

$(f\circ f^{-1})'(x) = f'(f^{-1}(x))\cdot(f^{-1})'(x) = 1 \Rightarrow (f^{-1})'(x) = [f'(f^{-1}(x))]^{-1}$ ✓

Problem 22: If $f: \mathbb{R}^2 \to \mathbb{R}$ and $D_2 f = 0$ show that f is independent of the 2^{nd} variable. If $D_1 f = D_2 f = 0$, show that f is constant

- $f(x,y) = r \in \mathbb{R}$ $\forall$ $(x,y) \in \mathbb{R}^2$. Suppose $D_2 f = 0$.

Assuming f is differentiable, the mean value theorem $\Rightarrow$ for any (a,b) (open interval) $\exists$ $c \in (a,b)$ such that $\frac{f(x,a) - f(x,b)}{a-b} = D_2 f(x,c) = 0 \Rightarrow f(x,a) = f(x,b)$ $\forall$ $a,b \in \mathbb{R}$

$\Rightarrow f$ is independent of the 2^{nd} variable

- If $D_1 f = 0$ <u>and</u> $D_2 f$ then f must be independent of the first and second variable and thus $f(x_1,y_1) = f(x_2,y_2) = $ constant.

Problem 29: Let $f: \mathbb{R}^n \to \mathbb{R}$. For $x \in \mathbb{R}^n$, the limit $\lim_{t \to 0} \frac{f(a+tx)-f(a)}{t}$ if it exists, is denoted by $D_x f(x)$ and is called the directional derivative of f at a in the direction x

a) Show that $D_i f(a) = D_{e_i} f(a)$. ($i \in \mathbb{Z}$ + e_i is a unit vector)

$$\lim_{t \to 0} \frac{f(a+te_i)-f(a)}{t} = \lim_{t \to 0} \frac{f(a_1, \dots, a_i+t, \dots, a_n)-f(a)}{t} = D_i f(a) \text{ by definition.}$$

b) Show that $D_{tx} f(a) = t D_x f(a)$

$$\lim_{t \to 0} \frac{f(a+t^2x)-f(a)}{t} = \lim_{t \to 0} \frac{f(a+t^2x)-f(a)}{t^2} \cdot t = t D_x f(a)$$

c) If f is differentiable at a show that $D_x f(a) = Df(a)(x)$ and therefore $D_{x+y} f(a) = D_x f(a) + D_y f(a)$

- let $x \neq 0$. $0 = \lim_{t \to 0} \left| \frac{f(a+tx)-f(a)-Df(a)(tx)}{tx} \right| = \lim_{t \to 0} \left| \frac{f(a+tx)-f(a)-tDf(a)(x)}{tx} \right|$

$$= \lim_{t \to 0} \left| \frac{f(a+tx)-f(a)}{t} - Df(a)(x) \right| \cdot \frac{1}{|x|} \Rightarrow Df(a)(x) = \frac{f(a+tx)-f(a)}{t} = D_x f(a).$$

- If $x=0$, $D_x f(a) = 0$. + $Df(a)(x) = \lim_{t \to 0} \left| \frac{f(a+tx)-f(a) - Df(a)(x)}{t} \right| = \lim_{t \to 0} \left| \frac{Df(a)(x)}{t} \right| = 0$

$\Rightarrow Df(a)(x)$ is also zero. ✓

Problem 34: A function $f: \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree m if $f(tx) = t^m f(x)$ $\forall x$. If f is also differentiable, show that

$$\sum_{i=1}^n x^i D_i f(x) = m f(x) \qquad \text{Hint: If } g(t) = f(tx), \text{ find } g'(1)$$

- Let $g(t) = f(tx) = t^m f(x)$
- $g'(t) = x f'(tx) = m t^{m-1} f(x) \Rightarrow g'(1) = f'(x) = m f(x)$

$$\Rightarrow x \cdot f'(x) = \sum_{i=1}^n x^i D_i f(x) = m f(x)$$