Math 173, Fall 2022, October 31

Theorem 3: If A is an $m \times n$ matrix w/ entries in F, then

$$\text{row rank}(A) = \text{column rank}(A)$$

Proof: Let $T : F^{n \times 1} \to F^{m \times 1}$,

$$T(X) = \underbrace{A}_{m \times n} \underbrace{X}_{n \times 1}$$

$$N(T) = \{X : AX = 0\}$$

null space

$$\text{Range}(T) = \{Y : AX = Y \text{ for some } X\}$$

range

Let $A_1, A_2, \ldots, A_n$ columns of A, so

$$AX = x_1 A_1 + \ldots + x_n A_n$$

$\text{Range}(T) =$ column space of A!

2

i.e ~~[illegible]~~

$\text{rank}(A) = \text{column rank}(A)$

By Theorem 2,

$\dim(N(T)) + \text{column rank}(A) = n$

Now consider the solution space of $AX = 0$.

Let R = row-reduced echelon matrix $\sim A$.

Then $RX = 0$ has the same solution space.

Suppose that R has r non-zero rows, then

$RX = 0$ expresses r of the unknowns $x_1, \ldots, x_n$

in terms of the remaining $(n-r)$ unknowns x_j.

Let $k_1 < k_2 < \ldots < k_r$ denote the columns

where the first non-zero entry in the i'th

row takes place. Let J denote the $n-r$

indices that exclude $k_1, k_2, \ldots, k_r$.

3

Then $RX = 0$ has the form

$$x_{k_1} + \sum_{J} c_{1j} x_j = 0$$

c_{ij} scalars

$$\vdots$$

$$x_{k_r} + \sum_{J} c_{rj} x_j = 0$$

All solutions are obtained by assigning arbitrary values to x_j, $j \in J$ and computing x_{k_j}'s as a result.

Define E_j be the solution obtained by taking $x_j = 1$ and 0 otherwise $\forall$ other $i \in J$.

The set of these vectors is linearly independant (easy), so it is enough to check that it spans.

please check!

4

If $T = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}$ is in the solution space,

then so is $M = \sum_{J} t_j E_j$ and

is a solution $\ni$ $x_j = t_j$ for each $j \in J$.

Such a solution is unique, so $M = T$, and

$T \in \operatorname{span}(\{E_j\})$.

Going back to the theorem we are proving, the solution space of $AX = 0$ has dimension $n - r$

By the discussion above, i.e.

$\dim(N(T)) = n - \text{row rank}(A)$

It follows that row rank (A) = column rank (A) ✓

5

Theorem: Let $\bar{V}, W$ - vector spaces over F, field

Let $T, U : V \longrightarrow W$
linear transformations

Then $(T+U)(\alpha) = T\alpha + U\alpha$ is a linear transformation from V to W, and so is $(cT)(\alpha) = c(T\alpha)$.

Definition: $L(V, W)$ = linear transformations from V to $\bar{W}$.

Theorem 5: Let $\bar{V}$ be an n-dimensional vector space over F, and $\bar{W}$ m-dimensional vector space over F. Then $L(V, W)$ is finite dimensional and has dimension $\underline{m \cdot n}$.

6

Proof: Let $\mathcal{B} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ $\mathcal{B}' = \{\beta_1, \dots, \beta_m\}$

order basis of V ordered basis of W

For each (p, q), $1 \le p \le m$, $1 \le q \le n$, define $E^{p,q}: V \to W$ by

$$E^{pq}(\alpha_i) = \begin{cases} 0, & \text{if } i \ne q \\ \beta_p, & \text{if } i = q \end{cases}$$

$$= \delta_{iq} \beta_p$$

By Theorem 1, E^{pq} is a unique linear transformation w/ those properties. We claim that $\{E^{pq}\}$ form a basis of $\mathcal{L}(V, W)$.

Let $T: V \to W$ linear; let

$$T\alpha_j = \sum_{p=1}^{m} \underbrace{A_{pj}}_{\text{coordinates}} \beta_p$$

⑦

We claim that $T = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q}$

Let $U\alpha_j = \sum_p \sum_q A_{pq} E^{p,q}(\alpha_j)$

$= \sum_p \sum_q A_{pq} \delta_{jq} \beta_p = T\alpha_j$, so $U = T$

It follows that $E^{p,q}$ span $L(V, W)$, and we are left to show that they are independent.

Suppose that $U = \sum_p \sum_q A_{pq} E^{p,q}$ is a 0 transformation.

Then $U\alpha_j = 0 \ \forall j$, so

$\sum_{p=1}^{m} A_{pj} \beta_p = 0$, which implies that $A_{pj} = 0 \ \forall p, j$.

Example: $V = \mathbb{R}^2$ $W = \mathbb{R}^2$

Then $L(V, W)$ has dimension $2 \cdot 2 = 4$.

Basis of $V = \{ \underset{e_1}{\underset{\shortparallel}{(1,0)}}, \underset{e_2}{\underset{\shortparallel}{(0,1)}} \}$.

8

Basis of $W = \{ \underset{\overset{\parallel}{e_1}}{(1,0)}, \underset{\overset{\parallel}{e_2}}{(0,1)} \}$

$$E^{p,q}(e_i) = \delta_{iq}\, e_p$$

$E^{1,1}(e_1) = \delta_{11} e_1 = e_1 \qquad E^{1,1}(e_2) = \delta_{21} e_1 = 0$

$E^{1,2}(e_1) = \delta_{12} e_1 = 0 \qquad E^{1,2}(e_2) = \delta_{22} e_1 = e_1$

$E^{2,1}(e_1) = \delta_{11} e_2 = e_2 \qquad E^{2,1}(e_2) = \delta_{21} e_2 = 0$

$E^{2,2}(e_1) = \delta_{12} e_2 = 0 \qquad E^{2,2}(e_2) = \delta_{22} e_2 = e_2$