Combinatorial methods or integer tiling

Itay Londner
Weizmann institute of science

joint with Izabella Łaba (UBC)

Combinatorics Seminar - University of Rochester

September 2021

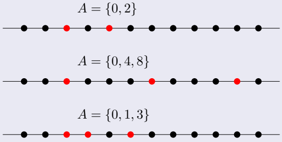
Tiling the integers: an introduction

Tiling the integers with a finite set

Let $A \subset \mathbb{Z}$ be a finite set. We say that A *tiles* $\mathbb{Z}$ *by translations* if $\mathbb{Z}$ can be covered by a union of disjoint translates of A. (There is an infinite set $T \subset \mathbb{Z}$ such that every $x \in \mathbb{Z}$ can be uniquely represented as $x = a + t$, with $a \in A$, $t \in T$.)

Tiling the integers with a finite set

Let $A \subset \mathbb{Z}$ be a finite set. We say that A *tiles* $\mathbb{Z}$ *by translations* if $\mathbb{Z}$ can be covered by a union of disjoint translates of A. (There is an infinite set $T \subset \mathbb{Z}$ such that every $x \in \mathbb{Z}$ can be uniquely represented as $x = a + t$, with $a \in A$, $t \in T$.)



$A = \{0, 2\}$ and $A = \{0, 4, 8\}$ tile $\mathbb{Z}$; $A = \{0, 1, 3\}$ does not.

How to determine whether a given A tiles the integers?

Periodicity

Newman (1977): all tilings of $\mathbb{Z}$ by a finite set A are periodic, with period M.

Tijdeman (1993) + Coven-Meyerowitz (1998): if A tiles the integers, then it also tiles a finite cyclic group $\mathbb{Z}_M$, where M has the same prime factors as $|A|$.

This reduces the problem to the study of tilings of finite cyclic groups $\mathbb{Z}_M = \{0, 1, \ldots, M-1\}$, with addition mod M.
Notation:

$$A \oplus B = \mathbb{Z}_M.$$

Further reductions

We measure distances between elements in $\mathbb{Z}_M$ in terms of the GCD with M. In particular

$$\mathrm{Div}(A) := \{(a - a', M) : \ a, a' \in A\}$$

Further reductions

We measure distances between elements in $\mathbb{Z}_M$ in terms of the GCD with M. In particular

$$\mathrm{Div}(A) := \{(a - a', M) : \; a, a' \in A\}$$

Sands (1979): let $A, B \subset \mathbb{Z}_M$. Then $A \oplus B = \mathbb{Z}_M$ if and only if $|A|\,|B| = M$ and

$$\mathrm{Div}(A) \cap \mathrm{Div}(B) = \{M\}.$$

Geometric representation of tilings

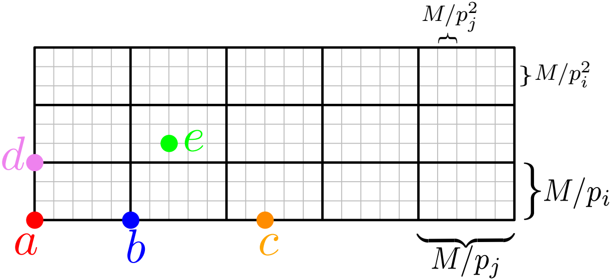
Suppose that $A \oplus B = \mathbb{Z}_M$, with $M = \prod_{i=1}^{K} p_i^{n_i}$, p_i distinct primes, $n_i \geq 1$. By the Chinese Remainder Theorem, we have

$$\mathbb{Z}_M = \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_K{}^{n_K}},$$

which we can represent geometrically as a K-dimensional lattice. Then $A \oplus B$ can be interpreted as a (modular) tiling of that lattice.

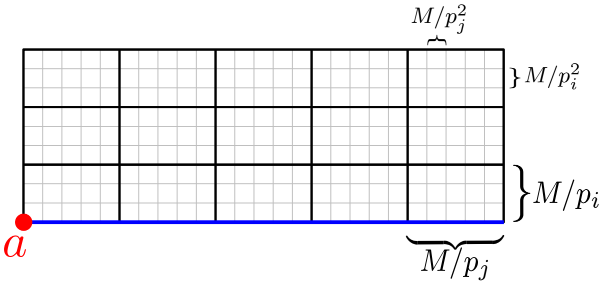
Distances

$$\mathbb{Z}_M = \mathbb{Z}_{p_i^2} \oplus \mathbb{Z}_{p_j^2}, M = p_i^2 p_j^2$$



Geometric representation of sets

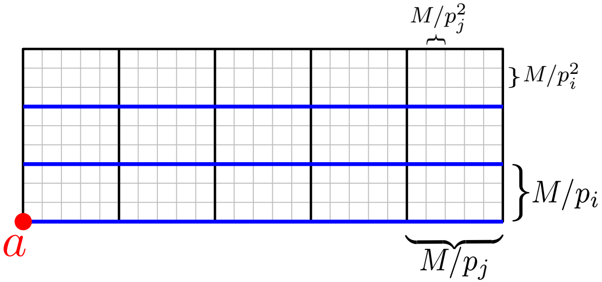
$$\mathbb{Z}_M = \mathbb{Z}_{p_i^2} \oplus \mathbb{Z}_{p_j^2}, M = p_i^2 p_j^2$$



$$\{x \in \mathbb{Z}_M \ : \ p_i^2 | x - a\} = a + p_i^2 \mathbb{Z}_M$$

Geometric representation of sets

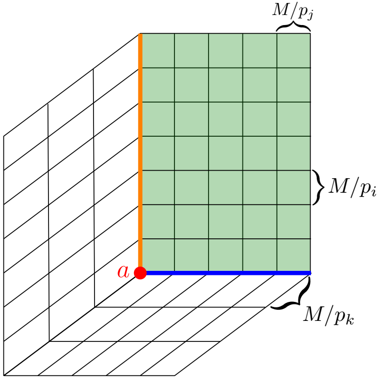
$$\mathbb{Z}_M = \mathbb{Z}_{p_i^2} \oplus \mathbb{Z}_{p_j^2}, M = p_i^2 p_j^2$$



$$\{x \in \mathbb{Z}_M \ : \ p_i | x - a\} = a + p_i\mathbb{Z}_M$$

Geometric representation of sets

$$\mathbb{Z}_M = \mathbb{Z}_{p_i^{n_i}} \oplus \mathbb{Z}_{p_j^{n_j}} \oplus \mathbb{Z}_{p_k^{n_k}}, M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$$



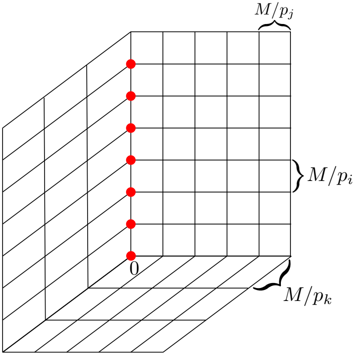
$\{x \in \mathbb{Z}_M \; : \; p_j^{n_j} p_k^{n_k} | x - a\} = a + M/p_i^{n_i} \mathbb{Z}_M$

$\{x \in \mathbb{Z}_M \; : \; p_i^{n_i} p_k^{n_k} | x - a\} = a + M/p_j^{n_j} \mathbb{Z}_M$

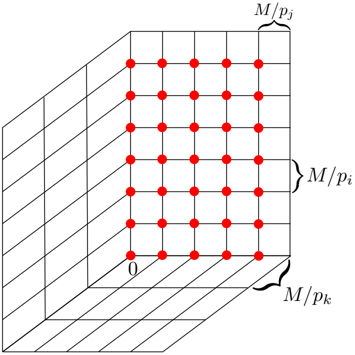
$\{x \in \mathbb{Z}_M \; : \; p_k^{n_k} | x - a\} = a + M/p_i^{n_i} p_j^{n_j} \mathbb{Z}_M$

Geometric representation of sets

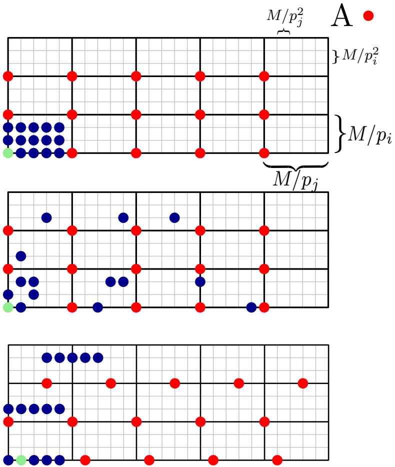
$\{0, M/p_i, 2M/p_i, \ldots, (p_i - 1)M/p_i\}$



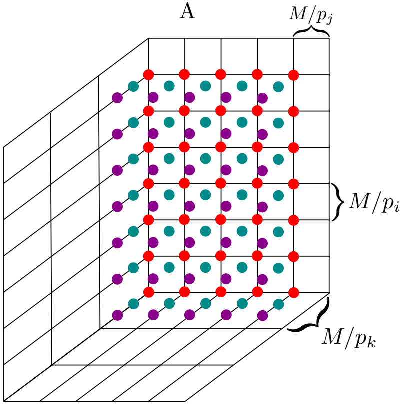
$\{x \in \mathbb{Z}_M : M/p_ip_j | x\}$

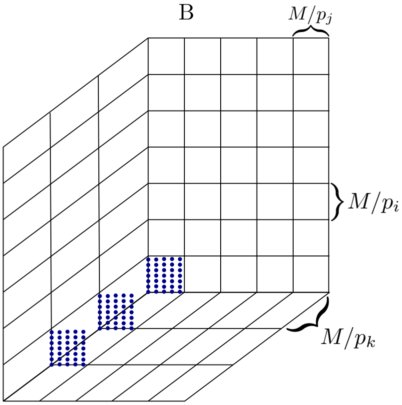


Examples of tilings



Examples of tilings





The Coven-Meyerowitz tiling conditions

Polynomial formulation

By translational invariance, we may assume that $A, B \subset \{0, 1, \dots\}$ and that $0 \in A \cap B$. The *characteristic polynomials* (aka *mask polynomials*) of A and B are

$$A(X) = \sum_{a \in A} X^a, \; B(X) = \sum_{b \in B} X^b.$$

Then $A \oplus B = \mathbb{Z}_M$ is equivalent to

$$A(X)B(X) = 1 + X + \cdots + X^{M-1} \mod (X^M - 1).$$

Cyclotomic polynomials

Recall the s-th *cyclotomic polynomial* is the unique monic, irreducible polynomial $\Phi_s(X)$ whose roots are the primitive s-th roots of unity.

Then the tiling condition $A(X)B(X) = 1 + X + \cdots + X^{M-1}$ mod $(X^M - 1)$ is equivalent to

$$|A||B| = M \text{ and } \Phi_s(X) \mid A(X)B(X) \text{ for all } s|M,\ s \neq 1.$$

Since Φ_s are irreducible, each $\Phi_s(X)$ with $s|M$, $s \neq 1$, must divide at least one of $A(X)$ and $B(X)$.

The Coven-Meyerowitz Theorem (1998)

Let $S_A = \{p^\alpha : \Phi_{p^\alpha}(X) | A(X)\}$. Consider the following conditions.

(T1) $A(1) = \prod_{s \in S_A} \Phi_s(1)$,

(T2) if $s_1, \ldots, s_k \in S_A$ *are powers of different primes, then* $\Phi_{s_1 \ldots s_k}(X)$ *divides* $A(X)$.

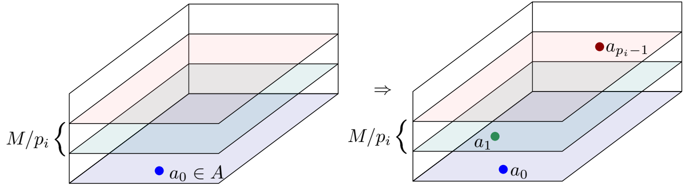
Then:

- if A satisfies (T1), (T2), then A tiles $\mathbb{Z}$;
- if A tiles $\mathbb{Z}$ then (T1) holds;
- if A tiles $\mathbb{Z}$ and $|A|$ has *at most two prime factors*, then (T2) holds.

Cyclotomic polynomials and distribution

Divisibility by prime power cyclotomic polynomials $\Phi_{p_i^\alpha}$ can be interpreted in terms of distribution of the elements of A:

- $\Phi_{p_i}|A \Leftrightarrow A$ is equidistributed mod p_i,
- $\Phi_{p_i^{n_i}}|A \Leftrightarrow A$ is equidistributed mod $p_i^{n_i}$ within residue classes mod $p_i^{n_i-1}$.



$a_0, a_1, a_2, \ldots, a_{p_i-1} \in A$

$p_i^{n_i-1} \parallel a_\nu - a_{\nu'}$

Reformulation of T2

Assume $M = \prod_i p_i^{n_i}$ and let $M_i = M/p_i^{n_i}$. Given $1 \leq \alpha \leq n_i$, we define

$$F_{i,\alpha} = \{0, M_i p_i^{\alpha-1}, 2M_i p_i^{\alpha-1}, \ldots, (p_i - 1)M_i p_i^{\alpha-1}\}$$

Reformulation of T2

Assume $M = \prod_i p_i^{n_i}$ and let $M_i = M/p_i^{n_i}$. Given $1 \leq \alpha \leq n_i$, we define

$$F_{i,\alpha} = \{0, M_i p_i^{\alpha-1}, 2M_i p_i^{\alpha-1}, \ldots, (p_i - 1)M_i p_i^{\alpha-1}\}$$

$$F_{i,\alpha}(X) = \prod_{d|M_i} \Phi_{dp_i^\alpha}(X)$$

Reformulation of T2

Assume $A \oplus B = \mathbb{Z}_M$ and define

$$B^\flat(X) := \prod_i \prod_{\alpha : p_i^\alpha \in S_B} F_{i,\alpha}(X)$$

Reformulation of T2

Assume $A \oplus B = \mathbb{Z}_M$ and define

$$B^\flat(X) := \prod_i \prod_{\alpha : p_i^\alpha \in S_B} F_{i,\alpha}(X)$$

Note that

- for $m|M$ $\Phi_m \nmid B^\flat$ if and only if

$$m = \prod_{i \in I \subset \{1,\ldots,K\}} p_i^{\beta_i} \text{ and } \Phi_{p_i^{\beta_i}} | A \text{ for all } i \, (p_i^{\beta_i} \in S_A)$$

- since $S_A \cup S_B = \{p^\alpha : p^\alpha | M\}$ and disjoint, $B^\flat$ is uniquely determined by A.

Reformulation of T2

Assume $A \oplus B = \mathbb{Z}_M$ and define

$$B^\flat(X) := \prod_i \prod_{\alpha\,:\,p_i^\alpha \in S_B} F_{i,\alpha}(X)$$

Note that

- for $m|M$ $\Phi_m \nmid B^\flat$ if and only if

$$m = \prod_{i \in I \subset \{1,\dots,K\}} p_i^{\beta_i} \text{ and } \Phi_{p_i^{\beta_i}} | A \text{ for all } i \, (p_i^{\beta_i} \in S_A)$$

- since $S_A \cup S_B = \{p^\alpha : p^\alpha | M\}$ and disjoint, $B^\flat$ is uniquely determined by A.

Then

> A satisfies T2 if and only if A is a maximal set satisfying
>
> $$\#(A' \cap B^\flat) = 1 \text{ for all } A'\text{- translation of } A.$$

Reformulation of T2

Example:

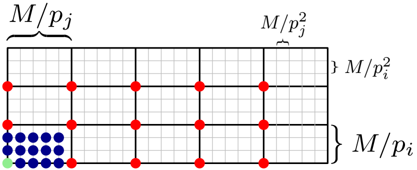
Suppose that $M = p_i^2 p_j^2 p_k^2$ and $A \subset \mathbb{Z}_M, |A| = p_i p_j p_k$ is uniformly distributed modulo p_i, p_j and p_k. Then

The following are equivalent

- A satisfies T2
- any translation of A intersects $p_i p_j p_k \mathbb{Z}_M$ at exactly one point
- A is uniformly distributed modulo $p_i p_j p_k$.

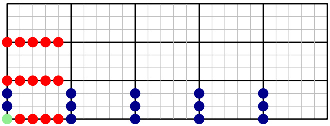
Standard T2 sets

$$A = A^\flat, B = B^\flat$$



$\Phi_{p_i^2}\Phi_{p_j^2}|A$

$\Phi_{p_i}\Phi_{p_j}|B$



$\Phi_{p_i^2}\Phi_{p_j}|A$

$\Phi_{p_i}\Phi_{p_j^2}|B$

A ● B ●

Main result

Main result (Łaba-L 2021)

Theorem. Suppose that $A \oplus B = \mathbb{Z}_M$, with $M = p_i^2 p_j^2 p_k^2$. Then A and B satisfy T2.

Main result (Łaba-L 2021)

Theorem. Suppose that $A \oplus B = \mathbb{Z}_M$, with $M = p_i^2 p_j^2 p_k^2$. Then A and B satisfy T2.

Additionally:

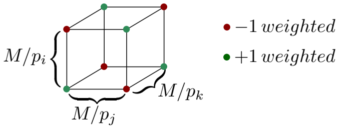
- The proof essentially provides a classification of all tilings of period $M = p_i^2 p_j^2 p_k^2$. (It does not get much more complicated than Szabó-type examples.)
- Methods and some intermediate results extend to more general M.

Cuboids and cyclotomic divisibility

Particular case: Let $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$. An M-cuboid is a weighed set with the mask polynomial

$$\Delta(X) = X^a \prod_i (1 - X^{d_i M/p_i}) \mod (X^M - 1)$$

where $a \in \mathbb{Z}_M$ and $(d_i, p_i) = 1$.

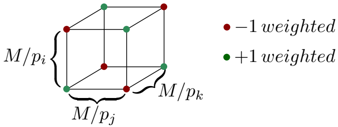


Cuboids and cyclotomic divisibility

Particular case: Let $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$. An M-cuboid is a weighed set with the mask polynomial

$$\Delta(X) = X^a \prod_i (1 - X^{d_i M/p_i}) \mod (X^M - 1)$$

where $a \in \mathbb{Z}_M$ and $(d_i, p_i) = 1$.



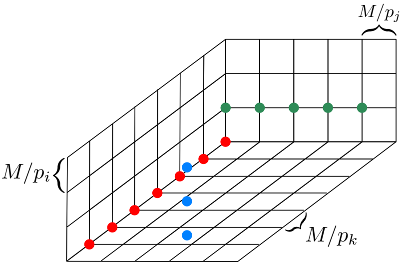
- For $A \subset \mathbb{Z}_M$ we have

$$\Phi_M | A \iff \text{for every } \Delta,\ A|_\Delta \text{ sums up to } 0$$

Fibers

Let $M = \prod_i p_i^{n_i}$. An *M-fiber in the p_i direction* is a set

$$\{a, a + M/p_i, a + 2M/p_i, \ldots, a + (p_i - 1)M/p_i\} \subset \mathbb{Z}_M.$$



- A set $A \subset \mathbb{Z}_M$ is M fibered in the p_i direction if it is a union of disjoint M fibers in that direction.

Application: structure on grids

The cyclotomic Φ_M determines the structures of sets on $\Lambda(x, D(M))$ grids, where $D(M) = M/\prod_i p_i$ and

$$\Lambda(x, D(M)) := \{y \in \mathbb{Z}_M \,:\, D(M) | x - y\}.$$

Application: structure on grids

The cyclotomic Φ_M determines the structures of sets on $\Lambda(x, D(M))$ grids, where $D(M) = M/\prod_i p_i$ and

$$\Lambda(x, D(M)) := \{y \in \mathbb{Z}_M \, : \, D(M)|x - y\}.$$

We prove the following basic structure lemma:
Let $A \subset \mathbb{Z}_M, M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$

Application: structure on grids

The cyclotomic Φ_M determines the structures of sets on $\Lambda(x, D(M))$ grids, where $D(M) = M/\prod_i p_i$ and

$$\Lambda(x, D(M)) := \{y \in \mathbb{Z}_M \ : \ D(M)|x - y\}.$$

We prove the following basic structure lemma:
Let $A \subset \mathbb{Z}_M, M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$

Lemma

If $\Phi_M|A$ and $M/p_i \notin Div(A)$ for some $p_i \neq 2$. Then on every fixed grid $\Lambda(x, D(M))$, the set A is M fibered either in the p_j or the p_k direction.

Application: structure on grids

The cyclotomic Φ_M determines the structures of sets on $\Lambda(x, D(M))$ grids, where $D(M) = M/\prod_i p_i$ and

$$\Lambda(x, D(M)) := \{y \in \mathbb{Z}_M \,:\, D(M)|x - y\}.$$

We prove the following basic structure lemma:
Let $A \subset \mathbb{Z}_M, M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$

Lemma

If $\Phi_M|A$ and $M/p_i \notin Div(A)$ for some $p_i \neq 2$. Then on every fixed grid $\Lambda(x, D(M))$, the set A is M fibered either in the p_j or the p_k direction.

- Note that this lemma does not require the tiling assumption.
- The lemma is not true when $p_i = 2$.

Application: structure on grids

Proof: Fix $a \in A$. We prove first that a has to belong to an M fiber in either the p_j or the p_k directions.

Application: structure on grids

Proof: Fix $a \in A$. We prove first that a has to belong to an M fiber in either the p_j or the p_k directions.
Suppose this is not true and find $x_j, x_k \in \mathbb{Z}_M \setminus A$ with

$$(a - x_\nu, M) = M/p_\nu \text{ for } \nu = j, k.$$

Application: structure on grids

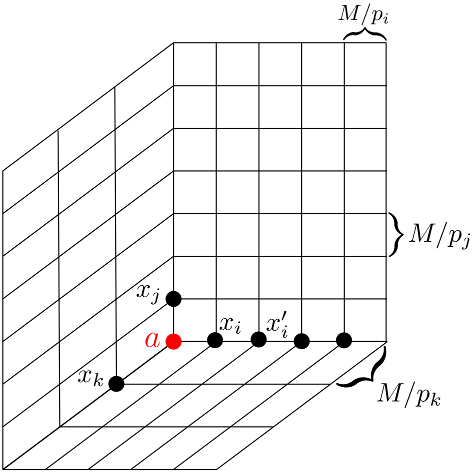
Proof: Fix $a \in A$. We prove first that a has to belong to an M fiber in either the p_j or the p_k directions.
Suppose this is not true and find $x_j, x_k \in \mathbb{Z}_M \setminus A$ with

$$(a - x_\nu, M) = M/p_\nu \text{ for } \nu = j, k.$$

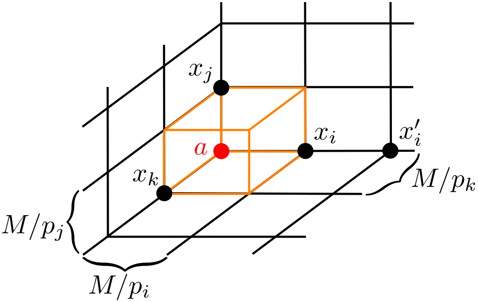
Also, since $p_i \neq 2$ and M/p_i is not a difference in A, we may find $x_i, x_i' \in \mathbb{Z}_M \setminus A$ with

$$(a - x_i, M) = (a - x_i', M) = (x_i - x_i', M) = M/p_i.$$

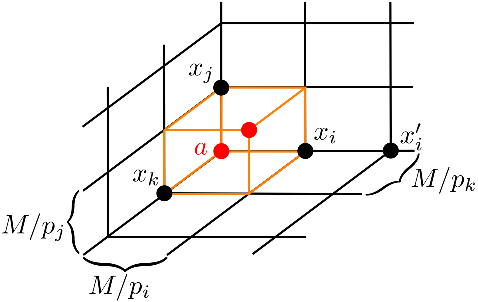
Application: structure on grids



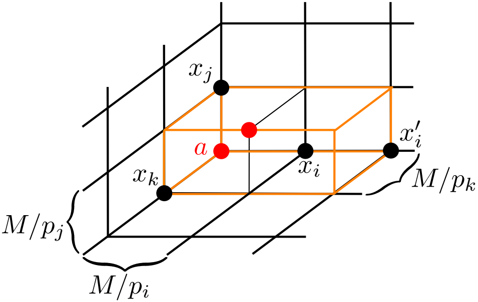
Application: structure on grids



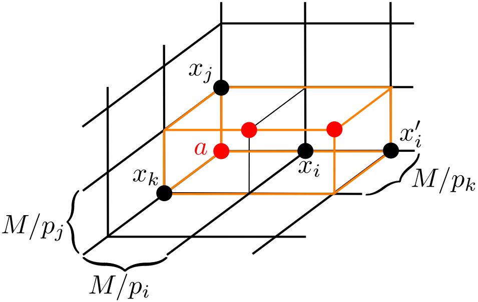
Application: structure on grids



Application: structure on grids



Application: structure on grids

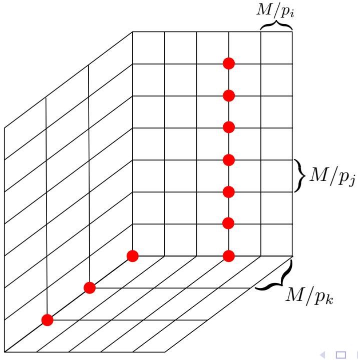


Application: structure on grids

We, therefore, conclude that every element in $A \cap \Lambda(a, D(M))$ is contained in an M fiber in either the p_j or p_k direction.
If the choice of direction is uniform across all elements of $A \cap \Lambda(a, D(M))$ - we're done. Otherwise

Application: structure on grids

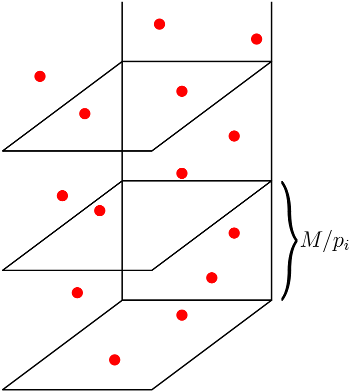
We, therefore, conclude that every element in $A \cap \Lambda(a, D(M))$ is contained in an M fiber in either the p_j or p_k direction.
If the choice of direction is uniform across all elements of $A \cap \Lambda(a, D(M))$ - we're done. Otherwise



Slab reduction

Let $M = \prod_i p_i^{n_i}$ and define the slab

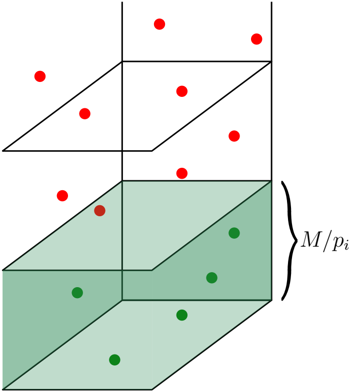
$$A_{p_i} = \{a \in A : 0 \leq a \mod p_i^{n_i} \leq p_i^{n_i-1} - 1\}$$



Slab reduction

Let $M = \prod_i p_i^{n_i}$ and define the slab

$$A_{p_i} = \{a \in A : 0 \le a \mod p_i^{n_i} \le p_i^{n_i - 1} - 1\}$$



Slab reduction

Theorem (Łaba-L, 2021)

Let $A \oplus B = \mathbb{Z}_M$, and $\Phi_{p_i^{n_i}} | A$. The following are equivalent:

(i) For any translate of A' of A we have $A'_{p_i} \oplus B = \mathbb{Z}_{M/p_i}$.

(ii) For every $p_i^{n_i} | m | M$ we have

$$m \in Div(A) \Rightarrow m/p_i \notin Div(B).$$

Remarks:

Slab reduction

Theorem (Łaba-L, 2021)

Let $A \oplus B = \mathbb{Z}_M$, and $\Phi_{p_i^{n_i}}|A$. The following are equivalent:

(i) For any translate of A' of A we have $A'_{p_i} \oplus B = \mathbb{Z}_{M/p_i}$.

(ii) For every $p_i^{n_i}|m|M$ we have

$$m \in Div(A) \Rightarrow m/p_i \notin Div(B).$$

Remarks:

- The case $M = p_i^2 p_j^2 p_k^2$

Slab reduction

Theorem (Łaba-L, 2021)

Let $A \oplus B = \mathbb{Z}_M$, and $\Phi_{p_i^{n_i}} | A$. The following are equivalent:

(i) For any translate of A' of A we have $A'_{p_i} \oplus B = \mathbb{Z}_{M/p_i}$.

(ii) For every $p_i^{n_i} | m | M$ we have

$$m \in Div(A) \Rightarrow m/p_i \notin Div(B).$$

Remarks:

- The case $M = p_i^2 p_j^2 p_k^2$
- Open problem: prove the conclusion of the theorem hold for arbitrary M.
  If true then tiling would imply T2, by induction.

Slab reduction - application

Suppose that every element of A belongs to an M-fiber in the p_i direction, i.e.

$$a + M/p_i, \ldots, a + (p_i - 1)M/p_i \in A \text{ for all } a \in A.$$

We claim that A satisfies the conditions of the slab reduction.

Slab reduction - application

Suppose that every element of A belongs to an M-fiber in the p_i direction, i.e.

$$a + M/p_i, \ldots, a + (p_i - 1)M/p_i \in A \text{ for all } a \in A.$$

We claim that A satisfies the conditions of the slab reduction.

- $(1 + X^{M/p_i} + \ldots + X^{(p_i-1)M/p_i}) | A$, in particular $\Phi_{p_i^{n_i}} | A$.

Slab reduction - application

Suppose that every element of A belongs to an M-fiber in the p_i direction, i.e.

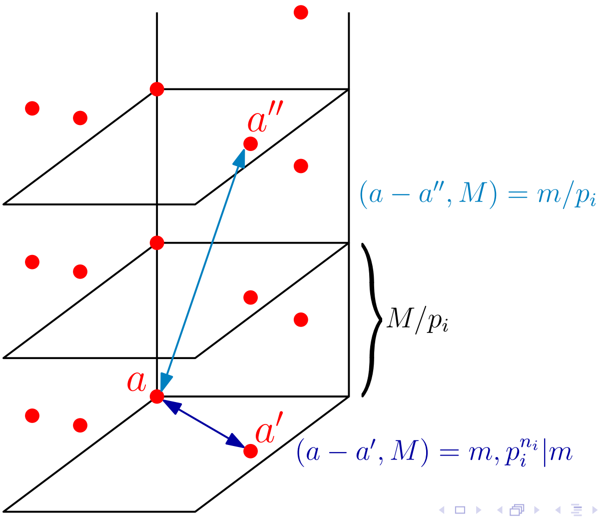
$$a + M/p_i, \ldots, a + (p_i - 1)M/p_i \in A \text{ for all } a \in A.$$

We claim that A satisfies the conditions of the slab reduction.

- $(1 + X^{M/p_i} + \ldots + X^{(p_i-1)M/p_i})|A$, in particular $\Phi_{p_i^{n_i}}|A$.
- A satisfies the condition: for every $p_i^{n_i}|m|M$

$$m \in Div(A) \iff m/p_i \in Div(A).$$

Slab reduction - application



Slab reduction - application

Let $A \oplus B = \mathbb{Z}_M$ with $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$.

Lemma

Assume that Φ_M divides both A and B. Then either A or B has to be M-fibered in some direction. Moreover, if $n_i = n_j = n_k = 2$ then A and B satisfy T2.

Slab reduction - application

Let $A \oplus B = \mathbb{Z}_M$ with $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$.

Lemma

Assume that Φ_M divides both A and B. Then either A or B has to be M-fibered in some direction. Moreover, if $n_i = n_j = n_k = 2$ then A and B satisfy T2.

Proof.

- By PH principle, WLOG, $M/p_i, M/p_j \notin Div(A)$, and $\max\{p_i, p_j\} > 2$

$\square$

Slab reduction - application

Let $A \oplus B = \mathbb{Z}_M$ with $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$.

Lemma

Assume that Φ_M divides both A and B. Then either A or B has to be M-fibered in some direction. Moreover, if $n_i = n_j = n_k = 2$ then A and B satisfy T2.

Proof.

- By PH principle, WLOG, $M/p_i, M/p_j \notin Div(A)$, and $\max\{p_i, p_j\} > 2$
- Since $\Phi_M | A$, by structure lemma A must be M fibered in the p_k direction

$\square$

Structure of proof

Let $M = (p_i p_j p_k)^2$, $|A| = |B| = p_i p_j p_k$.

$$\begin{tikzcd} & \Phi_M | A \arrow[dl] \arrow[dr] & \\ \Phi_M | B & & \Phi_M \nmid B \end{tikzcd}$$

Structure of proof

Let $M = (p_i p_j p_k)^2$, $|A| = |B| = p_i p_j p_k$.

$$
\begin{tikzcd}
 & & \Phi_M \mid A \arrow[dl] \arrow[dr] & \\
 & \Phi_M \mid B \arrow[dl] & & \Phi_M \nmid B \\
\text{A,B satisfy T2} & & &
\end{tikzcd}
$$

Structure of proof

Let $M = (p_i p_j p_k)^2$, $|A| = |B| = p_i p_j p_k$.

$\Phi_M | A$

$\Phi_M | B$

$\Phi_M \nmid B$

A,B satisfy T2

A is fibered on every $M/p_i p_j p_k$-grid
possibly in different directions

Split $A = \cup_i A(i)$
and go down one scale

Structure of proof

Let $M = (p_i p_j p_k)^2$, $|A| = |B| = p_i p_j p_k$.

$\Phi_M | A$

$\Phi_M | B$

A,B satisfy T2

$\Phi_M \nmid B$

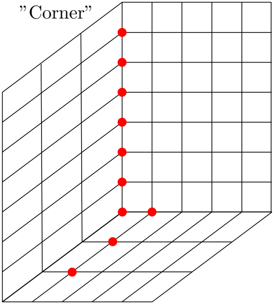
A is fibered on every $M/p_i p_j p_k$-grid
possibly in different directions

Split $A = \cup_i A(i)$
and go down one scale

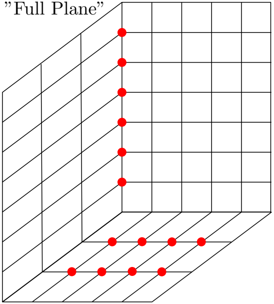
There exists an $M/p_i p_j p_k$-grid Λ
and $A \cap \Lambda$ has one of 4 possible strucutres

Structure of proof

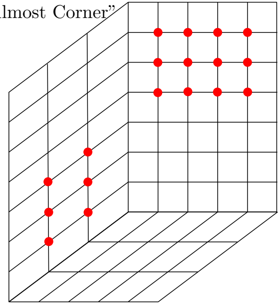
"Corner"



"Full Plane"

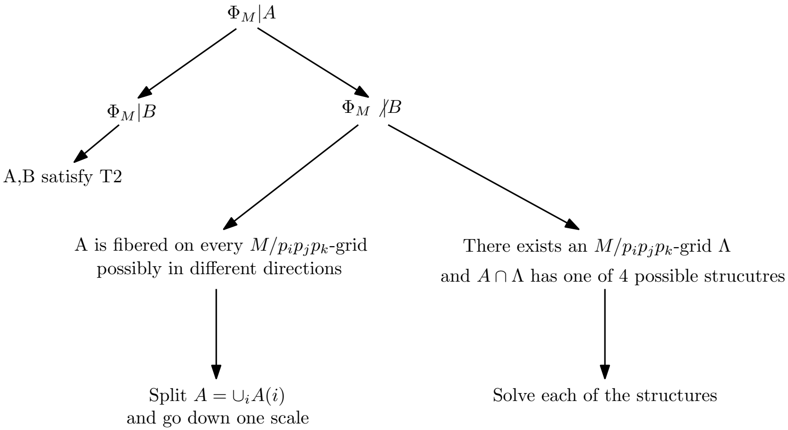


"Almost Corner"



Structure of proof

Let $M = (p_i p_j p_k)^2$, $|A| = |B| = p_i p_j p_k$.



Thank you!