Math 173, Oct 5, 2022

Definition: If $S_1, S_2, \dots, S_K \subseteq V$ ~ vector space, the set of all sums

$$\alpha_1 + \alpha_2 + \dots + \alpha_K, \quad \alpha_j \in S_j$$

is called $S_1 + S_2 + \dots + S_K$

$$\| $$

$$\sum_{i=1}^{K} S_i$$

If W_i, $i = 1, 2, \dots, K$ are subspaces, $\sum_{i=1}^{K} W_i$ is a subspace spanned by $\bigcup_{i=1}^{K} W_i$.

Example: $F = \mathbb{C}$ ~ complex numbers.

W_1 = 2x2 matrices of the form

$$\begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$$

W_2 = 2x2 matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

Then $W_1 + W_2 = V$ = vector space of all 2x2 matrices over $\mathbb{C}$ since

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \checkmark$$

(2)

$$W_1 \cap W_2 = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}, \; x \in \mathbb{C} \right\}$$

Example: V = space of all polynomial functions over F.

Let $S = \{f_0, f_1, f_2, \ldots, f_n, \ldots\}$

$$f_n(x) = x^n, \quad n = 0, 1, 2, \ldots$$

$$V = \text{span}(S).$$

Bases and dimension:

V vector space over F. A subset S of V is linearly dependent if $\exists$ distinct $\alpha_1, \alpha_2, \ldots, \alpha_n \in S$ and scalars $c_1, c_2, \ldots, c_n$ in F, not all 0, $\ni$

$$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = 0.$$

A set which is not linearly dependent is called linearly independent.

Definition: V = vector space. A basis for V is a linearly independent set of vectors in V which spans V. The space V is finite-dimensional if it has a finite basis.

Example: $F = \mathbb{C}$,

$$\left.\begin{aligned} \alpha_1 &= (3, 0, -3) \\ \alpha_2 &= (-1, 1, 2) \\ \alpha_3 &= (4, 2, -2) \\ \alpha_4 &= (2, 1, 1) \end{aligned}\right\} \text{ linearly dependent}$$

$$2\alpha_1 + 2\alpha_2 - \alpha_3 + 0 \cdot \alpha_4 = 0$$

$$\begin{aligned} \epsilon_1 &= (1, 0, 0) \\ \epsilon_2 &= (0, 1, 0) \\ \epsilon_3 &= (0, 0, 1) \end{aligned} \sim \text{ linearly independent}$$

If $c_1\epsilon_1 + c_2\epsilon_2 + c_3\epsilon_3 = 0$,

$(c_1, c_2, c_3) = (0, 0, 0) \hookrightarrow c_1 = c_2 = c_3 = 0$ ✓

If $V = F^n$,

$$\begin{aligned} \epsilon_1 &= (1, 0, \ldots, 0) \\ \epsilon_2 &= (0, 1, \ldots, 0) \\ &\vdots \\ \epsilon_n &= (0, 0, \ldots, 0) \end{aligned} \sim \text{ standard basis}$$

why does it span?

A particularly important example:

$A = m \times n$ matrix; S = solution space for

$$AX = 0$$

R = row-reduced echelon matrix equivalent to A.

Then S = solution space for $RX = 0$.

$RX = 0$ has the form

$$x_{k_1} + \sum_{J} \underbrace{c_{1j}}_{\text{scalars}} x_j$$

you have seen this before!

$$\vdots$$

$$x_{k_r} + \sum_{\underbrace{J}_{\text{remaining variables}}} c_{rj} x_j$$

Solutions are obtained by assigning arbitrary values to x_j, $j \in J$.

Let E_j = solution obtained by setting $x_j = 1$ & the rest of the $x_{j'} = 0$ for $j' \in J$.

(5)

The set $\{E_j\}_{j \in J}$
is linearly independent (why?)
We claim that it is a basis for the solution space of $RX = 0$.

It is enough to check that it spans since it is already linearly independent.
If the column matrix T is as above,
$N = \sum_J t_j E_j$ is in the solution space
if $F \begin{pmatrix} t_{j_1} \\ \vdots \\ t_{j_{|J|}} \end{pmatrix} \in$ solution space

N is the solution such that $x_j = t_j$ for each $j \in J$.
The solution w/ this property is unique,
so $N = T$ and $T \in \text{span}(\{E_j\})$.

Example: Infinite basis

V = polynomial functions over $\mathbb{R}$ ~ real numbers

$f_n(x) = x^n$, so $\{f_0, f_1, f_2, \ldots, f_n, \ldots\}$ span V

Why are they independent?

Suppose that

$$c_0 f_0 + c_1 f_1 + \ldots + c_n f_n = 0, \text{ ie}$$

$$c_0 + c_1 x + \ldots + c_n x^n = 0 \quad \forall x!$$

degree n $\hookrightarrow$ $\leq n$ solutions

$\hookrightarrow$ contradiction.