Math 173 September 14, 2022

Matrix multiplication:

Let $A = m \times n$ matrix over F ~ field
$B = n \times p$ matrix over F

Then AB is defined to be the $m \times p$ matrix C w/ entries

$$C_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj}$$

Example: $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ $m=2$ $n=2$

$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $n=2$ $p=1$ $\underline{C \text{ is } 2 \text{ by } 1}$

$$C_{11} = \sum_{r=1}^{2} A_{1r} B_{r1} = A_{11} B_{11} + A_{12} B_{21}$$
$$= 1 \cdot 0 + (-1) \cdot 1 = -1$$

$$C_{21} = \sum_{r=1}^{2} A_{2r} B_{r1} = A_{21} B_{11} + A_{22} B_{21}$$
$$= 0 \cdot 0 + 1 \cdot 1 = 1$$

Therefore, $C = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ ✓

Long view: why multiply matrices?

Let A = $m \times n$ matrix, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

Then AX "transforms" X. It can be viewed as a function from

$F^n \longrightarrow F^m$

k-tuples of elements of F

If B is a k by m matrix over F, then $B(AX)$ can be viewed as a k-tuple in F^k. In this way $B \cdot A$ can be viewed as a function from F^n to F^k.

we will explore this point of view in detail later.

Some notation:

If B is an $\underset{\text{rows}}{n}\times\underset{\text{columns}}{p}$ matrix, the columns are $n\times 1$ matrices $B_1, B_2, \dots, B_p$, defined by

$$B_j = \begin{bmatrix} B_{1j} \\ \vdots \\ \vdots \\ B_{nj} \end{bmatrix} \sim n \text{ rows } 1 \text{ column}$$

$1 \leq j \leq p$.

Then $B = [B_1, B_2, \dots, B_p]$.

Applying the definition of multiplication, if A is an $m\times n$ matrix,

$$AB = [AB_1, AB_2, \dots, AB_p].$$

Example: Multiplication of matrices is not commutative!

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\neq \quad \text{not necessarily}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}$$

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Theorem 8: If A, B, C are matrices over $F \ni$ BC and $A(BC)$ are well-defined, then so are the products AB, $(AB)C$ and $A(BC) = (AB)C$

associativity

Proof: Suppose that B is an $n \times p$ matrix. Since BC is defined, C is a matrix w/ p rows, and BC has n rows. Because $A(BC)$ is defined, so we may assume that A is an $m \times n$ matrix. Thus $A \cdot B$ exists and is an $m \times p$ matrix, so $(AB)C$ exists. Now we prove that $A(BC) = (AB)C$

$$[A(BC)]_{ij} = \sum_r A_{ir} (BC)_{rj} = \sum_r A_{ir} \sum_s B_{rs} C_{sj}$$

$$= \sum_r \sum_s A_{ir} B_{rs} C_{sj} = \sum_s \sum_r A_{ir} B_{rs} C_{sj}$$

$$= \sum_s \left(\sum_r A_{ir} B_{rs} \right) C_{sj} = \sum_s (AB)_{is} C_{sj} = [(AB)C]_{ij}$$

An $m \times m$ matrix is said to be an <u>elementary matrix</u> if it can be obtained from the $m \times m$ identity matrix by means of a single elementary row operation.

2×2 examples:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$$

$$\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \quad c \neq 0 \quad \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \quad c \neq 0$$

Theorem 9: Let e be an elementary row operation and let E be the $m \times m$ matrix $e(I)$. Then for every $m \times n$ matrix A,

$$e(A) = EA.$$

Proof: R1 & R3 arguments are straightforward.

If e is R2, then

$$E_{ik} = \begin{cases} \delta_{ik}, & i \neq r \\ \delta_{rk} + c\,\delta_{sk} \end{cases}$$

why?

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It follows that

$$(EA)_{ij} = \sum_{K=1}^{m} E_{iK} A_{Kj} = \begin{cases} \sum_{K=1}^{m} \delta_{iK} A_{Kj} = A_{ij}, & i \neq r \\ A_{rj} + c A_{sj}, & i = r \end{cases}$$

$$\parallel$$

$$\sum_{K=1}^{m} (\delta_{rK} + c\delta_{sK}) A_{Kj}$$

$$= A_{rj} + c A_{sj}$$

It follows that $EA = e(A)$.

<u>Corollary:</u> A, B $m \times n$ matrices over F. Then B is row-equivalent to A iff $B = PA$, where P = product of $m \times m$ elementary matrices.

Proof: Suppose that $B = PA$, $P = E_s E_{s-1} \dots E_1$ elementary matrices

Then B is row-equivalent to A by a finite inductive chain argument using theorem 9.

Conversely, if B is row-equivalent to A, let $E_1, E_2, \dots E_s$ correspond to row operations $e_1, e_2, \dots, e_s$