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Maximum modulus principle (final version:) Let G be a region in $\mathbb{C}$ and f an analytic function on G. Suppose that $\exists M \ni \overline{\lim_{z \to a}} |f(z)| \leq M$ $\forall a \in \partial_\infty G$.

Then $|f(z)| \leq M \quad \forall z \in G$.

$$\overline{\lim_{z \to a}} f(z) = \lim_{r \to 0^+} \sup \{ f(z) : z \in G \cap B(a, r) \}$$

$a \in G$ $\quad f : G \to \mathbb{R}$

or $a = \infty$.

$$\partial_\infty G = \begin{cases} \partial G, & \text{if } G \text{ bounded} \\ \partial G \cup \{\infty\} & \text{if } G \text{ unbounded} \end{cases}$$

Proof: Let $\delta > 0$ & let

$$H = \{z \in G : |f(z)| > M + \delta\}$$

Since $|f|$ is continuous, H is open

must show that H is empty

Since $\overline{\lim_{z \to a}} |f(z)| \leq M \quad \forall a \in \partial_\infty G$,

there is $B(a, r) \ni |f(z)| < M + \delta$

$\forall z \in G \cap B(a, r)$.

It follows that $\overline{H} \subset G$,

The same is true if G is unbounded & $a = \infty$.

It follows that H is bounded and $\overline{H}$ is compact.

This means that the second version of the maximum modulus principle applies.

Observe that for $z \in \partial H$, $|f(z)| = M+\delta$ since $\overline{H} \subset \{z : |f(z)| \geq M+\delta\}$

It follows that $H = \emptyset$, or f is constant. But if f is constant, $H = \emptyset$ by assumption.

This completes the proof.

It is time to start reaping the rewards of maximum principle and there are many!

Schwarz's lemma:

Let $D = \{z : |z| < 1\}$ and suppose that f is analytic in D w/

i) $|f(z)| \leq 1$, $z \in D$

ii) $f(0) = 0$

Then $|f'(0)| \leq 1$ & $|f(z)| \leq |z|$ $\forall z \in D$

Moreover, if $|f'(0)| = 1$ or if $|f(z)| = |z|$ for <u>some</u> $z \in D$,

$f(w) = cw$ $\forall w \in D$.

To prove this, define

$g: D \to \mathbb{C}$ by $g(z) = \dfrac{f(z)}{z}$, $z \neq 0$

& $g(0) = f'(0)$.

Then g is analytic in D. By the maximum modulus principle,

$|g(z)| \leq r^{-1}$ if $|z| \leq r$, $0 < r < 1$.

Letting $r \to 1 \Rightarrow$

$|g(z)| \leq 1 \Rightarrow |f(z)| \leq |z|$

& $|f'(0)| = |g(0)| \leq 1$.

If $|f(z)| = |z|$ for some $z \in D$, $z \neq 0$ or

if $|f'(0)| = 1$, then

g assumes its max inside D

$\Longrightarrow g(z) = c$

$\Longrightarrow f(z) = cz$ & we are done!

We will now use this to classify all conformal maps of the disk to itself.

For $|a| < 1$, define

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}, \quad \text{analytic for } |z| < |a|^{-1}.$$

$$\varphi_a(\varphi_{-a}(z)) = z = \varphi_{-a}(\varphi_a(z))$$

calculation

So $\varphi_a : D \longrightarrow D$ 1-1

$$|\varphi_a(e^{i\theta})| = \left|\frac{e^{i\theta}-a}{1-\bar{a}e^{i\theta}}\right| =$$

$$\left|\frac{e^{i\theta}-a}{e^{-i\theta}-\bar{a}}\right| = 1, \text{ so}$$

$$\varphi_a(\partial D) = \partial D.$$

In summary, if $|a|<1$:

i) φ_a is a 1-1 mapping of D

ii) φ_{-a} is the inverse of φ_a

iii) $\varphi_a : \partial D \longrightarrow \partial D$

iv) $\varphi_a(a) = 0$ v) $\varphi_a'(0) = 1 - |a|^2$

vi) $\varphi_a'(a) = (1-|a|^2)^{-1}$

quotient rule

Suppose that f is analytic on D w/ $|f(z)| \leq 1$. Assume that $|a|<1$ & $f(a) = \alpha$.

Let $g = \varphi_\alpha \circ f \circ \varphi_a^{-1} : D \longrightarrow D$

$$g(0) = \varphi_\alpha(f(a)) = \varphi_\alpha(\alpha) = 0.$$

By Schwarz's lemma

$$|g'(0)| \leq 1.$$

We can go farther.

$$g'(0) = (\varphi_\alpha \circ f)'(\varphi_{-a}(0)) \cdot \varphi_{-a}'(0)$$

$$= (\varphi_\alpha \circ f)'(a)(1-|a|^2)$$

$$= \varphi_\alpha'(\alpha) f'(a)(1-|a|^2)$$

$$= \frac{1-|a|^2}{1-|\alpha|^2} f'(a)$$

$$\Longrightarrow |f'(a)| \leq \frac{1-|\alpha|^2}{1-|a|^2}$$

w/ equality when $|g'(0)|=1$

$\overset{\text{Schwarz}}{\Longrightarrow}$

$$f(z) = \varphi_{-\alpha}(c\,\varphi_a(z)), \quad |c|=1$$

for $|z|<1$.

Theorem: Let $f : \mathbb{D} \to \mathbb{D}$ be a 1-1 analytic map of $\mathbb{D}$ onto $\mathbb{D}$ & $f(a) = 0$. Then $\exists$ c w/ $|c| = 1$ $\ni$ $f = c\varphi_a$.

Proof: Since f is 1-1, $\exists$ g $\ni$ $g(f(z)) = z \implies g'(0) f'(a) = 1$.

By above,

$$|f'(a)| \leq (1-|a|^2)^{-1} \text{ \&}$$

$$|g'(0)| \leq 1-|a|^2 \qquad (g(0) = a)$$

$$\implies |f'(a)| = (1-|a|^2)^{-1}$$

$$\implies f = c\varphi_a, \quad |c| = 1.$$