MAXIMAL AVERAGES OVER FRACTALS

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August 19, 2000

Abstract. Let $E_d = \{x = r\omega \in \mathbb{R}^d : r \in E\}$, where E is a compact one-dimensional set of Hasudorff dimension α with some appropriate metric uniformity assumptions. Let $A_t f(x) = \int f(x - ty) d\mu_d(y)$, and let $\mathcal{A}f(x) = \sup_{t>0} A_t f(x)$, where μ is a measure on E and μ_d is its "rotated" analog. We prove that $\mathcal{A} : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ for $p > \frac{d-\alpha}{d-1}$, the range that "interpolates" between the Hardy-Littlewood maximal function and the Stein's spherical maximal operator. The result holds even though the Fourier transform $\widehat{\mu}$ may have no decay at infinity, and, consequently, the decay of $\widehat{\mu_d}$ is no better than that of the Fourier transform of the Lebesgue measure on the $d-1$-dimensional sphere.

Section 0: Introduction

Let S be a smooth hypersurface in $\mathbb{R}^d$. Let

$$A_t f(x) = \int_S f(x - ty) d\sigma(y), \tag{0.1}$$

where $d\sigma$ denotes the Lebesgue measure on S, and let

$$\mathcal{A}f(x) = \sup_{t>0} A_t f(x). \tag{0.2}$$

The basic unsolved probelm is, given a hypersurface S, what is the sharp range of exponents so that

$$||\mathcal{A}f||_{L^p(\mathbb{R}^d)} \leq C_p ||f||_{L^p(\mathbb{R}^d)}. \tag{0.3}$$

If S has everywhere non-vanishing Gaussian curvature then the estimate (0.3) holds if and only if $p > \frac{d}{d-1}$. See [St], [Gr]. If the Gaussian curvature is allowed to vanish, numerous

Research supported in part by NSF grant DMS00-87339 and an ENSERC grant

results are available, though the general picture is far from complete. See, for example, [CoMa], [SoSt], [NSW], [I1], [I2], [IS1], [IS2], [ISS].

More generally we can define

$$A_t f(x) = \int_E f(x - ty) d\nu(y), \tag{0.4}$$

where E is a subset of $\mathbb{R}^d$ of Hausdorff dimension α, not necessarily an integer, and ν is a measure carried by E. Suppose that $0 < \mu(E) < \infty$ and re-scale μ so that $\mu(E) = 1$. Let $\mathcal{A}f(x)$ be defined as in (0.2) above. As before, we ask for the sharp range of exponents such that the estimate (0.3) holds.

Definition. Let (E, μ) be as in the first paragraph. We say that μ is *locally uniformly α-dimensional* if $\mu(B_R(x)) \leq CR^\alpha$, where $B_R(x)$ is a ball of radius $R \leq 1$ centered at $x \in E$ and C is independent of x and R.

Definition. Let μ denote a Hausdorff measure on $E \subset [1, 2]$. Let σ_r denote the rotationally invariant probability measure on the sphere of radius r. Let $\mu_d = \int_0^1 \sigma_r d\mu(r)$ denote the corresponding rotionally invariant measure on the set $E_d = \{x = r\omega \in \mathbb{R}^d : r \in E\}$.

Theorem 0.1. *Suppose that μ is locally uniformly α-dimensional. Let*

$$A_t f(x) = \int f(x - ty) d\mu_d(y), \tag{0.5}$$

and let

$$\mathcal{A}f(x) = \sup_{t>0} A_t f(x). \tag{0.6}$$

Then

$$||\mathcal{A}f||_{L^p(\mathbb{R}^2)} \leq C_p ||f||_{L^p(\mathbb{R}^2)} \quad \textit{for} \quad p > \frac{d-\alpha}{d-1}. \tag{0.7}$$

When $\alpha = 0$, this is precisely the range of boundedness of the Hardy-Littlewood maximal operator. When $\alpha = 1$, this is the range of boundedness of Stein's spherical maximal operator.

Taking $F(x) = |x|^{-d+1+\alpha} \log^{-1}\left(\frac{1}{|x|}\right)$ times the characteristic function of the ball of radius $\frac{1}{2}$ shows that $\mathcal{A}$ is unbounded on $L^p(\mathbb{R}^d)$ if $p \leq \frac{d}{d-1+\alpha}$. The two estimates agree if $\alpha = 0$ or 1, but the $p > \frac{d-\alpha}{d-1}$ result is worse for other values of α. A gap remains. The possible nature of this gap can be illustrated as follows. Consider the maximal operator associated to the analytic family of measures given by

$$\mu_\alpha = \frac{1}{\Gamma(\alpha)} (1 - |x|^2)^{\alpha - 1}. \tag{0.9}$$

If $\alpha = 0$, we get the spherical maximal operator. It is bounded on $L^p(\mathbb{R}^d)$ for $p > \frac{d}{d-1}$ as we mentioned above. If $\alpha = 1$, we get the Hardy-Littlewood maximal operator. It is bounded on $L^p(\mathbb{R}^d)$ for $p > 1$. Interpolating we see that for $0 < Re(\alpha) < 1$, the maximal averaging operator is bounded on $L^p(\mathbb{R}^d)$ for $p > \frac{d}{d-1+\alpha}$.

It is not hard to check that μ_α is α-dimensional, and

$$|\widehat{\mu}_\alpha(\xi)| \leq C(1+|\xi|)^{-\alpha}. \tag{0.10}$$

On the other hand, let E_m denote the Cantor-like subset of $[0,1]$ consisting of real numbers whose base m expansions have only 0's and 1's. Let μ_m denote the probability measure on E_m. It is not difficult to check that

$$\widehat{\mu_m}(\xi) = \Pi_{j=0}^{\infty} \cos\left(\frac{\pi\xi}{2 \cdot m^j}\right). \tag{0.11}$$

Taking a sequence of powers of m, we see that $\mu_m(\xi)$ does not tend to 0 as $\xi \to \infty$. However, we are saved by the fact that the proof of L^2 boundedness of maximal averaging operators, see Lemma 1.3 below, does not use the decay rate of $\widehat{\mu}$, but rather the "average"

$$\left(\int_1^2 |\widehat{\mu}(t\xi)|^2 dt\right)^{\frac{1}{2}}. \tag{0.12}$$

In the case $\mu = \mu_m$, see Lemma 1.1 below, one can show that (0.12) is bounded by

$$C|\xi|^{-\frac{\alpha_m}{2}}, \tag{0.13}$$

where

$$\alpha_m = \frac{\log(2)}{\log(m)}, \tag{0.14}$$

the Hausdorff dimension of E_m.

For more general Hausdorff measures, see Lemma 1.1 below, the same result holds with α_m replaced by the Hausdorff dimension of the corresponding set.

It is well known (see, for example, [Wolff], that the power of ξ in (0.13) cannot be imporoved beyond the index in (0.14) since the dimension of E_m is precisely α_m. This means that in order to improve the $\frac{d-\alpha}{d-1}$ index in Theorem 0.1, we must use something more than just the decay of (0.12).

Note that the phenomenon we are exploiting to obtain Theorem 0.1 is purely fractal in nature. We are using the fact that even though $\widehat{\mu_d}$ decays only of order $-\frac{d-1}{2}$ at infinity, the square function

$$\left(\int_1^2 |\widehat{\mu_d}(t\xi)|^2 dt\right)^{\frac{1}{2}} \tag{0.15}$$

decays of order $-\frac{d-1+\alpha}{2}$, where α is the Hausdorff dimension of μ. This phenomenon cannot take place if we consider the standard maximal averaging operator

$$\sup_{t>0} \left| \int_S f(x-ty)d\sigma(y) \right|, \tag{0.16}$$

where S is a smooth compact hypersurface in $\mathbb{R}^d$ and $d\sigma$ is the restriction of Lebesgue measure to S. By the method of stationary phase, $\widehat{d\sigma}$ decays slowly in the directions normal to the points in S where the Gaussian curvature vanishes. This means, in particular, that multiplying by t and averaging as in (0.15) cannot possibly improve the decay. One can improve the decay by using a non-istropic dilation, for example, $\rho_t f(x) = f(tx_1, \dots, tx_{d-1}, t^2 x_d)$. See [IS2]. The idea here is that non-isotropic dilations "rotate" the surface in such a way that "bad" normals are encountered very infrequently. In contrast, the improved estimate for the square function in (0.15) is purely metric in nature.

Section I: Proof of Theorem 0.1

The proof of Theorem 0.1 is based on the following sequence of lemmas.

Lemma 1.1. *Let (E,μ) be as above. Then*

$$\left(\int_1^2 |\widehat{\mu}(t\xi)|^2 dt \right)^{\frac{1}{2}} \leq C(1+|\xi|)^{-\frac{\alpha}{2}}, \tag{1.1}$$

and, consequently,

$$\left(\int_1^2 |\widehat{\mu_d}(t\xi)|^2 dt \right)^{\frac{1}{2}} \leq C(1+|\xi|)^{-\frac{d+1+\alpha}{2}}. \tag{1.2}$$

Moreover, the same estimates hold if $\widehat{\mu}(t\xi)$ is replaced by $\nabla\widehat{\mu}(t\xi)$.

Lemma 1.2. *Let μ_d^k be defined by the formula $\widehat{\mu_d^k}(\xi) = \widehat{\mu_d}(\xi)\phi_k(\xi)$, where ϕ_k is the usual Paley-Littlewood cutoff, and μ_d is as in the statement of the Theorem 0.1. Let*

$$A_t^k f(x) = \int f(x-ty)d\mu_d^k(y), \tag{1.3}$$

and

$$\mathcal{A}^k f(x) = \sup_{t>0} A_t^k f(x). \tag{1.4}$$

Then

$$||\mathcal{A}^k f||_{L^p(\mathbb{R}^2)} \leq C 2^{k(1-\alpha)} ||f||_{L^p(\mathbb{R}^2)} \quad \textit{for} \quad p>1. \tag{1.5}$$

The following result was proved in [IS], Theorem 8. See also, [SoSt] and [CoMa].

Lemma 1.3. *Suppose that τ is a distribution supported in a ball B of radius C_1 with $|\hat{\tau}(\xi)| \leq C_1$ and $\max\{|x| : x \in supp\ \tau\} \leq C_2$. Suppose, moreover, that*

$$\left\{ \int_1^2 |\hat{\tau}(t\xi)|^2 dt \right\}^{\frac{1}{2}} \leq C_1(1+|\xi|)^{-\frac{1}{2}}\gamma(\xi), \tag{1.6}$$

and

$$\left\{ \int_1^2 |\nabla\hat{\tau}(t\xi)|^2 dt \right\}^{\frac{1}{2}} \leq C_1(1+|\xi|)^{-\frac{1}{2}}\gamma(\xi), \tag{1.7}$$

where γ is bounded and non-increasing on $[0,\infty)$, and $\sum_{n=0}^{\infty}\gamma(2^n) < \infty$. Let $\hat{\tau}_t(\xi) = \hat{\tau}(t\xi)$. Let

$$\mathcal{M}_\tau f(x) = \sup_{t>0} |f * \tau_t(x)|. \tag{1.8}$$

Then

$$||\mathcal{M}_\tau f||_{L^2(\mathbb{R}^d)} \leq C\sqrt{C_1C_2}||f||_{L^2(\mathbb{R}^d)} \tag{1.9}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$, the usual Schwartz class.

Lemma 1.4. *With the notation of Lemma 0.2, we have*

$$||\mathcal{A}^k f||_{L^2(\mathbb{R}^2)} \leq C2^{-k\left(\frac{d-2+\alpha}{2}\right)}||f||_{L^2(\mathbb{R}^2)}. \tag{1.10}$$

Synthesis. Lemma 1.1 and Lemma 1.3 combine to imply Lemma 1.4. Interpolating Lemma 1.2 and Lemma 1.4 and summing up the geometric series we obtain the conclusion of Theorem 0.1 unless $d=2$ and $\alpha=0$. This is the only case when the exponent in (1.10) is not negative and our interpolation scheme does not work. This makes sense because a special case of the situation where $d=2$, $\alpha=0$, is Bourgain's circular maximal operator. The proof that this operator is bounded for $p>2$ certainly cannot be carried out using the methods of this paper.

In order to handle the case $d=2$, $\alpha=0$, we dominate our operator by Bourgain's Circular Maximal Operator in the following way. Recall that

$$\mu_2 = \int_0^1 \sigma_r d\mu(r), \tag{1.11}$$

where σ_r is the rotation invariant measure probability measure on the circle of radius r.

Let $\tau_t g(x) = g(tx)$. Consider the "linearized" maximal operator

$$\mathcal{M}f(x) = A_{t(x)}f(x), \tag{1.12}$$

where $t(x)$ is an arbitrary measurable function of x. It is enough to prove that $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^2)$ for $p>2$. We have

$$|\mathcal{M}f(x)| = |\tau_{t(x)}\mu_d * f(x)| \leq \int_0^1 |d\sigma_{rt(x)} * f(x)|d\mu(r) \leq ||\mu_d|||\mathcal{T}f(x)|, \tag{1.13}$$

where $\mathcal{T}f(x)$ is Bourgain's Circular Maximal Operator, and $||\mu_d||$ denotes the total mass of μ_d. This completes the proof of Theorem 0.1.

Section II: Proof of Lemma 1.1 and Lemma 1.2

Proof of Lemma 1.1. By Fubini and the definition of the Fourier transform, we see that if $|\xi|$ is large,

$$\int_1^2 |\widehat{\mu}(t\xi)|^2 dt = \int\int\int_1^2 e^{2\pi i(x-y)t\xi} dt d\mu(x) d\mu(y) \tag{2.1}$$

$$= \int\int e^{3\pi i(x-y)\xi} \frac{\sin(\pi(x-y))}{\pi(x-y)\xi} d\mu(x) d\mu(y) \tag{2.2}$$

$$\leq \int\int \frac{d\mu(x)d\mu(y)}{1+|x-y||\xi|} \tag{2.3}$$

(2.4)
$$\leq \sum_{j=0}^{\log_2(|\xi|)} \int\int_{\{2^j \leq |x-y||\xi| \leq 2^{j+1}\}} \frac{d\mu(x)d\mu(y)}{1+|x-y||\xi|} + \sum_{j=\log_2(|\xi|)}^{\infty} \int\int_{\{2^j \leq |x-y||\xi| \leq 2^{j+1}\}} \frac{d\mu(x)d\mu(y)}{1+|x-y||\xi|}$$

$$+ \int\int_{\{|x-y||\xi| \leq 1\}} d\mu(x)d\mu(y) = I + II + III. \tag{2.5}$$

It is not hard to see that I is bounded by

$$C\sum_{j=1}^{\infty} 2^{-j} \left(\frac{2^j}{|\xi|}\right)^{\alpha} \leq C|\xi|^{-\alpha}, \tag{2.6}$$

since μ is locally uniformly α dimensional. On the other hand, II is bounded by

$$C \sum_{\log_2(|\xi|)}^{\infty} 2^{-j} \leq C|\xi|^{-1}. \tag{2.7}$$

The expression III is bounded by

$$\int_{\{|x-y| \leq \frac{1}{|\xi|}\}} d\mu(x)d\mu(y) \leq C|\xi|^{-\alpha} \tag{2.8}$$

since μ is, by assumption, locally uniformly α-dimensional.

This completes the proof of the first assertion of Lemma 1.1.

To prove the second assertion of Lemma 1.1 we write, with $|\xi| = r$,

$$\widehat{\mu_d}(t\xi) = c\int_E \left(\int_{S^{d-1}} e^{-isw\cdot t\xi} d\omega \right) d\mu(s) \tag{2.9}$$

$$= C\int_E J_{\frac{d-2}{2}}(trs)(trs)^{-\frac{d-2}{2}} d\mu(s) \tag{2.10}$$

$$= C\left(\int_E e^{itrs}(trs)^{-\frac{d-1}{2}} d\mu(s) + \int_E e^{-itrs}(trs)^{-\frac{d-1}{2}} d\mu(s) \right) \tag{2.11}$$

$$+O((tr)^{-\frac{d+1}{2}}), \tag{2.12}$$

using the well known asymptotics of the Bessel function $J_{\frac{d-2}{2}}$ together with our assumption that μ is a measure on the set $E \subset [1,2]$.

At this point, the second assertion of Lemma 1.1 follows from the first assertion applied to the measure $s^{-\frac{d-1}{2}} d\mu(s)$.

Proof of Lemma 1.2. We have

$$\widehat{\phi_k} * \mu_d(x) \leq 2^{dk} \int \frac{d\mu_d(y)}{(1+2^k|x-y|)^N} \tag{2.13}$$

$$\leq \left(\sum_{-\infty}^{-1} + \sum_{0}^{k} + \sum_{k}^{\infty} \right) \int_{\{2^j \leq 2^k|x-y| \leq 2^{j+1}\}} 2^{dk} \frac{d\mu_d(y)}{(1+2^k|x-y|)^N} = I + II + III. \tag{2.14}$$

Using the fact that μ is locally unformly α dimensional, we see that

$$I = \int_{\{2^k|x-y|\leq 1\}} 2^{dk} \frac{d\mu_d(y)}{(1+2^k|x-y|)^N} \leq C2^{dk}2^{-k(d-1+\alpha)}\chi_B(x) \leq C2^{k(1-\alpha)}\chi_B(x), \tag{2.15}$$

where B is the ball of radius 5 centered at the origin.

Similarly,

$$II \leq C\sum_{j=0}^{k} 2^{dk}2^{(j-k)(d-1+\alpha)}2^{-jN} \leq C2^{k(1-\alpha)}\chi_B(x). \tag{2.16}$$

On the other hand,

$$III \leq C\sum_{j=k}^{\infty} 2^{dk}2^{-jN}|B|2^{(j-k)d}\frac{\chi_{2^{j-k}B}(x)}{|2^{j-k}B|} = \sum_{j=k}^{\infty} 2^{j(d-N)}|B|\frac{\chi_{2^{j-k}B}(x)}{|2^{j-k}B|}. \tag{2.17}$$

It follows that $\mathcal{A}^k$ is dominated by $C2^{(1-\alpha)k}$ times the Hardy-Littlewood maximal function. Lemma 1.2 follows.

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