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Math 173, Fall 2022, November 6

<u>Definition:</u> A group is a set w/ rule that assigns to each pair $x, y \in G$ an element xy $\ni$

i) $x(yz) = (xy)z$

ii) $\exists\, e \in G \ni xe = ex = x \;\forall\, x \in G$

iii) For each $x \in G$, $\exists\, x^{-1} \in G \ni x \cdot x^{-1} = x^{-1} \cdot x = e$.

<u>Example:</u> Given $n \geq 2$, let $G = \{0, 1, \ldots, n-1\}$ w/ rule $xy = x+y \mod n$.

<u>Isomorphism:</u> V, W vector spaces over F; $T: V \longrightarrow W$ 1-1 & onto is called an isomorphism.

Theorem 10: Every n-dim vector space over F is isomorphic to F^n.

Proof: V = n-dim vector space over F, and let $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, an ordered basis.

Define $T: V \longrightarrow F^n$:

$$\alpha = x_1\alpha_1 + \ldots + x_n\alpha_n;$$

$$T\alpha = \underbrace{(x_1, \ldots, x_n)}_{\text{unique!}}$$

We have checked previously that T is linear 1-1 & onto.

From transformations to matrices:

$V = \underbrace{\text{vector space}}_{n\text{-dim}}$ over F & W m-dim over F

$\mathcal{B} = \underbrace{\{\alpha_1, \alpha_2, \ldots, \alpha_n\}}_{\text{ordered basis of } V}$ $\qquad$ $\mathcal{B}' = \underbrace{\{\beta_1, \ldots, \beta_m\}}_{\text{ordered basis of } W}$

$$T\alpha_j = \sum_{i=1}^{m} \underbrace{A_{ij}}_{\text{coordinates of } T\alpha_j \text{ in } \mathcal{B}'} \beta_i$$

$$A = \{A_{ij}\} \equiv \text{matrix of } T \text{ relative to } (\mathcal{B}, \mathcal{B}')$$

How does A determine T?

Let $\alpha = x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n \in V$,

$$T\alpha = T\left(\sum_{j=1}^{n} x_j\alpha_j\right) = \sum_{j=1}^{n} x_j\, T(\alpha_j)$$

$$= \sum_{j=1}^{n} x_j \sum_{i=1}^{m} A_{ij}\beta_i = \sum_{i=1}^{m} \underbrace{\left(\sum_{j=1}^{n} A_{ij}x_j\right)}_{\text{coordinates of } T\alpha} \beta_i$$

Theorem 11: Let V be an n-dimensional vector space over F & W is m-dim over F, w/ ordered bases $\mathcal{B}$, $\mathcal{B}'$ respectively.

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For each linear transformation from $V \xrightarrow{T} W$, there is an $m \times n$ matrix A w/ entries in $F \ni [T\alpha]_{\beta'} = A[\alpha]_{\beta} \quad \forall \alpha \in V$.

Moreover, $T \to A$ is a 1-1 correspondence.

The matrix A above is called the matrix of T relative to bases β, β', and the columns of A are given by

$$A_j = [T\alpha_j]_{\beta'} \qquad j = 1, \dots, n$$

Suppose that $U : V \to W$ linear, and B is a matrix of U relative to the bases β, β', then $cA + B$ is a the matrix of $cT + U$ relative to β, β' since

$$cA_j + B_j = c[T\alpha_j]_{\beta'} + [U\alpha_j]_{\beta'}$$

$$= [cT\alpha_j + U\alpha_j]_{\beta'} = [(cT+U)\alpha_j]_{\beta'}$$

Our observation can be encoded as follows:

Theorem 12: V - n-dim vector space over F, W m-dimensional over F. For each pair of ordered bases $\mathcal{B}, \mathcal{B}'$ for V & W, the function that assigns linear $T: V \longrightarrow W$ to an $m \times n$ matrix A is an isomorphism from $\mathcal{L}(V, W)$ to $F^{m \times n}$.

The case $\mathcal{B} = \mathcal{B}'$: If $V = W$ and the bases coincide, a very interesting situation arises from a practical point of view.

In this case, $T\alpha_j = \sum_{i=1}^{n} A_{ij} \alpha_i$. We have

$$[T\alpha]_{\mathcal{B}} = [T]_{\mathcal{B}} [\alpha]_{\mathcal{B}}$$

matrix A relative to $\mathcal{B}$.

Example: $T: F^2 \to F^2$

$$T(x_1, x_2) = (x_1, 0)$$

$$\beta = \{ \underset{\epsilon_1}{\underset{\shortparallel}{(1,0)}}, \underset{\epsilon_2}{\underset{\shortparallel}{(0,1)}} \}$$

$$T\epsilon_1 = T(1,0) = (1,0) = 1\epsilon_1 + 0\epsilon_2$$

$$T\epsilon_2 = T(0,1) = (0,0) = 0\epsilon_1 + 0\epsilon_2$$

So, $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Example: $V = \left\{ c_0 + c_1 x + c_2 x^2 + c_3 x^3 \;\; c_0, c_1, c_2, c_3 \in \mathbb{R} \right\}$

$f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$, $f_4(x) = x^3$

$$Df_1(x) = 0 = 0f_1 + 0f_2 + 0f_3 + 0f_4$$

$$Df_2(x) = 1 = 1f_1 + 0f_2 + 0f_3 + 0f_4$$

$$Df_3(x) = 2x = 0f_1 + 2f_2 + 0f_3 + 0f_4$$

$$Df_4(x) = 3x^2 = 0f_1 + 0f_2 + 3f_3 + 0f_4$$

It follows that

$$[D]_\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

What about compositions?

$$V \xrightarrow{T} W \xrightarrow{U} Z$$

$$\beta \qquad \beta' \qquad \beta''$$

$$\{\alpha_1, \dots, \alpha_n\} \qquad \{\beta_1, \dots, \beta_m\} \qquad \{\gamma_1, \dots, \gamma_p\}$$

$$[T\alpha]_{\beta'} = A[\alpha]_\beta$$

$$\begin{aligned}[U(T\alpha)]_{\beta''} &= B[T\alpha]_{\beta'} \\ &= BA[\alpha]_\beta\end{aligned}$$

So BA represents $U \circ T$

Another way to see this is

$$(UT)(\alpha_j) = U(T\alpha_j)$$

$$= U\left(\sum_{k=1}^{m} A_{kj}\beta_k\right)$$

$$= \sum_{k=1}^{m} A_{kj}(U\beta_k)$$

$$= \sum_{k=1}^{m} A_{kj}\sum_{i=1}^{p} B_{ik}\gamma_i$$

$$= \sum_{i=1}^{p}\left(\sum_{k=1}^{m} B_{ik}A_{kj}\right)\gamma_i,$$

so $C_{ij} = \sum_{k=1}^{m} B_{ik}A_{kj}$

matrix multiplication