HOW TO COMPLETE THE SQUARE

This is a quick reminder on how to complete the square. More precisely, suppose that we want to rewrite

$$ax^2 + bx + c \tag{1}$$

in the form

$$M(x-z)^2 + T. \tag{2}$$

We will end up with a formula in a moment, but let's play with (1) and (2) for a moment to see how such a formula comes about. We have

$$M(x-z)^2 + T = Mx^2 - 2Mzx + Mz^2 + T, \tag{3}$$

and we want this expression to (1) for every possible x. This means that all the coefficients must be equal, so

$$M = a, \ -2Mz = b, \text{ and } Mz^2 + T = c. \tag{4}$$

Let's unravel this puzzle piece by piece. The first equality in (4) gives us

$$M = a. \tag{5}$$

We now plug (5) into the the second equality in (4) and obtain

$$z = -\frac{b}{2M} = -\frac{b}{2a}. \tag{6}$$

We must now deal with the third equality in (4). We get

$$c = Mz^2 + T = a \cdot \left(-\frac{b}{2a}\right)^2 + T. \tag{7}$$

We conclude that

$$T = c - a \cdot \left(-\frac{b}{2a}\right)^2 = c - \frac{b^2}{4a}. \tag{8}$$

Putting evverything together we see that

$$M = a, \ z = -\frac{b}{2a}, \ T = c - \frac{b^2}{4a}, \tag{9}$$

and we have ourselves a formula.

Example. *Let* $f(x) = 4x^2 + 24x + 32$*. Complete the square. By the formula,* $a = 4, b = 24$*, and* $c = 32$*. It follows that* $M = 4$*,* $z = -3$*, and* $T = -4$*. It follows that*

$$4x^2 + 24x + 32 = 4(x+3)^2 - 4. \tag{10}$$

Ugly Example. *Sometimes numbers do not divide nicely. Consider* $f(x) = 3x^2 - 5x + 7$*. Applying the formula above we see that*

$$3x^2 - 5x + 7 = 3\left(x - \frac{5}{6}\right)^2 + \frac{59}{12}. \tag{11}$$