1

Math 265H, Fall 2022, November 2

Definition: X, Y metric spaces; $E \subset X$, $f: E \to Y$, p = limit point of E. We say that

$$\lim_{x \to p} f(x) = q \quad \text{if for every } \epsilon > 0$$

$$\exists\, \delta > 0 \ni d_Y(f(x), q) < \epsilon$$

$$\text{for all } x \ni 0 < d_X(x, p) < \delta.$$

Theorem: X, Y, E, f, p as above. Then

$$\lim_{x \to p} f(x) = q \quad \text{iff}$$

$$\lim_{n \to \infty} f(p_n) = q \quad \forall \{p_n\} \ni p_n \neq p \ \ \& \ \lim_{n \to \infty} p_n = p.$$

Proof: Suppose $\lim_{x \to p} f(x) = q$. Choose $p_n \to p$, $p_n \neq p$.

Let $\epsilon > 0$ be given. Then $\exists\, \delta > 0 \ni d_Y(f(x), q) < \epsilon$ if $x \in E$ and $0 < d_X(x, p) < \delta$.

(2)

There exists $N \ni n > N \Longleftrightarrow 0 < d_X(p_n, p) < \delta$.

Thus for $n > N$, $d_Y(f(p_n), q) < \epsilon$ & we are done.

Conversely, suppose that $\lim_{x \to p} f(x) = q$ does not hold.

Then $\exists\ \epsilon > 0 \ni \forall \delta > 0\ \exists\ x \in E$ for which $d_Y(f(x), q) \geq \epsilon$ w/ $d_X(x, p) < \delta$.

Let $\delta_n = \frac{1}{n}$ and this yields a sequence $\{p_n\}$ such that $\lim_{n \to \infty} f(p_n) = q$ does not hold.

Corollary: If f has a limit at p, this limit is unique.

Theorem: $E \subset X$, metric space, p limit point of E, $f, g : E \longrightarrow \mathbb{C}$, and

$$\lim_{x \to p} f(x) = A, \quad \lim_{x \to p} g(x) = B.$$

Then a) $\lim_{x \to p} (f+g)(x) = A+B$

b) $\lim_{x \to p} (fg)(x) = A \cdot B$

c) $\lim_{x \to p} \frac{f}{g}(x) = \frac{A}{B}$ if $B \neq 0$

Definition: X, Y metric spaces, $E \subset X$, $p \in E$ $f: E \to Y$. Then f is continuous at p if for every $\epsilon > 0$ $\exists$ $\delta > 0$ $\ni$ $d_Y(f(x), f(p)) < \epsilon$ $\forall x \in E \ni d_X(x,p) < \delta$.

must be defined at p!!

In other words, f is continuous at p iff $\lim_{x \to p} f(x) = f(p)$ ✓

(4)

Theorem: X, Y, Z metric spaces, $E \subset X$,

$$f: E \longrightarrow Y, \quad g: f(E) \longrightarrow Z$$

$$h(x) = g(f(x)) \quad x \in E. \qquad \underline{h \equiv g \circ f}$$

If f is continuous at p, & g is continuous at $f(p)$, then h is continuous at p.

Proof: Let $\epsilon > 0$ be given. Since g is continuous at $f(p)$, there is $\eta > 0 \ni$

$d_Z(g(y), g(f(p))) < \epsilon$ if $d_Y(y, f(p)) < \eta$, $y \in f(E)$

Since f is continuous at, $\exists\, \delta > 0 \ni$

$d_Y(f(x), f(p)) < \eta$ if $d_X(x, p) < \delta$ and $x \in E$.

We conclude that $d_Z(h(x), h(p)) =$

$$d_Z(g(f(x)), g(f(p))) < \epsilon$$

if $d_X(x, p) < \delta$ and $x \in E$.

5

Theorem: $f: X \to Y$ is continuous
$\underbrace{\hphantom{X \to Y}}_{\text{metric spaces}}$

on X iff $f^{-1}(V)$ is open in X for every open set V in Y.

Proof: Suppose f is continuous on X and V is an open set in Y. The goal is to show that every point of $f^{-1}(V)$ is an interior point of $f^{-1}(V)$. Consider $p \in X$ and $f(p) \in V$. Since V is open, $\exists\ \epsilon > 0 \ni y \in V$ if $d_Y(f(p), y) < \epsilon$. By continuity, this implies that $\exists\ \delta > 0 \ni d_Y(f(x), f(p)) < \epsilon$ if $d_X(x, p) < \delta$. It follows that $x \in f^{-1}(V)$ if $d_X(x, p) < \delta$. This implies that p is an interior point of $f^{-1}(V)$.

6

<u>Corollary:</u> $f: X \longrightarrow Y$ is continuous
$\quad$ metric spaces

iff $f^{-1}(C)$ is closed in X for every closed set C in Y.

<u>Theorem:</u> f, g complex continuous on X, metric space.

Then $f+g$, fg & f/g are continuous on X.

Proof: At limit points, this was already proved. At isolated points, there is nothing to prove.

Theorem: a) $f_1, f_2, \ldots, f_K$ real functions on X, metric space, and $\vec{f}(x) = (f_1(x), \ldots, f_K(x))$
$x \in X$

Then f is continuous iff each f_i is continuous.

b) If $\vec{f}, \vec{g}$ are continuous from X to $\mathbb{R}^K$, then $\vec{f}+\vec{g}$ and $\vec{f} \cdot \vec{g}$ are continuous on X.

dot product

7

Proof: $|f_i(x) - f_i(y)| \leq |\vec{f}(x) - \vec{f}(\vec{y})|$
$= \left(\sum_{i=1}^{K} |f_i(x) - f_i(y)|^2\right)^{\frac{1}{2}}$ and part a) follows

By the previous theorem and part a), part b) follows as well.

With basics out of the way, let's connect continuity and compactness.

Definition: $f: E \to \mathbb{R}^K$ is bounded if there is $M \ni |f(x)| \leq M \quad \forall x \in E$.

Theorem: Suppose $f: X \to Y$ continuous.
(2 metric spaces)

Then if X is compact, $f(X)$ is compact.

Proof: Let $\{V_\alpha\}$ be an open cover of $f(X)$.
Since f is continuous, $f^{-1}(V_\alpha)$ is open for each α. By compactness,
$X \subset f^{-1}(V_{\alpha_1}) \cup \dots \cup f^{-1}(V_{\alpha_n})$.

(8)

Since $f(f^{-1}(E)) \subset E$ for every $E \subset Y$, it follows that $f(X) \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$.

<u>Note:</u> $f(f^{-1}(E)) \subset E$ if $E \subset Y$.

Similarly, if $F \subset X$, $\underset{\{x:\ f(x) \in f(F)\}}{\underset{\parallel}{f^{-1}(f(F))}} \supset F$.

Certainly $F \subset$ this set, but there may be other points there as well.