MATH 238: HOMEWORK #8 DUE MONDAY, 12/5/2016

ALEX IOSEVICH

Problem #1: Prove that if E is a Besicovitch-Kakeya set in $\mathbb{Z}_p^d$, then

$$E \times E \times \cdots \times E$$

is a Besicovitch-Kakeya set in $\mathbb{Z}_p^{kd}$.

Problem #2: Prove that if $E \subset \mathbb{Z}_p^2$, $p \equiv 3(4)$ contains a circle of every possible non-zero radius, then $\#E \geq cp^2$.

Problem #3: Let $E \subset [0, N]^d \cap \mathbb{Z}^d$, $d \geq 3$, $N > 10^6$. Suppose that for at least $1/100$ of the possible directions determined by $[0, N]^d \cap \mathbb{Z}^d$, E contains a line in each of those direction with at least $N/1000$ points on it. Then

$$\#E \geq cN^{\frac{d+1}{2}}.$$

Problem #4: Let $E \subset S \subset \mathbb{Z}_p^3$, where

$$S = \{x \in \mathbb{Z}_p^3 : ||x|| = 1\},$$

where $||x|| = x_1^2 + x_2^2 + x_3^2$. Prove that if $\#E \geq cp^{\frac{3}{2}}$, then

$$\#\Delta(E) \geq C(c)p.$$

Hint: Recall dot products.

Problem #5: Let $E \subset \mathbb{Z}_p^2$, $p \equiv 3(4)$. Define

$$P(E) = \{(||x^1 - x^2||, ||x^2 - x^3||, ||x^3 - x^4||, ||x^4 - x^1||, ||x^1 - x^3||) : x^j \in E\}.$$

Prove that there exists $\alpha < 2$ such that if $\#E \geq cp^{\alpha}$, then $\#P(E) \geq C(c)p^5$.

Hint: Try to imitate the group action argument for triangles.