PROBLEM 1.1 IN SPIVAK

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We are trying to show that

$$|x| \leq \sum_{i=1}^{n} |x_i|.$$

We start with a couple of simple observations:

- When $n = 1$, the two sides are equal.
- When $n = 2$, the inequality says that the length of the hypotenuse of a right triangle is smaller than the sum of the lengths of the other two sides.

I will write down a proof of the general case, but I will ask you to think about another approach. Let's rewrite the inequality in the form

$$\left(\sum_{i=1}^{n} |x^i|^2\right)^{\frac{1}{2}} \leq \sum_{i=1}^{n} |x^i|.$$

Why did we do that? Because when one side of the inequality is written in one type of a notation and the other side in another, it is often convenient to write them in the same way.

We now square both sides of the inequality. Why? Because there is a square root on the left hand side and those are frequently difficult to deal with. We end up having to prove that

$$\sum_{i=1}^{n} |x^i|^2 \leq \left(\sum_{i=1}^{n} |x^i|\right)^2. \tag{1}$$

We need to relate the two sides of the inequality (1) to one another in some explicit way. Here is what runs through my head when I see expressions of this type. The right hand side is equal to

$$(|x^1| + |x^2| + \cdots + |x^n|)^2.$$

Expanding this expression out is going to bring out $|x^1|^2 + |x^2|^2 + \cdots + |x^n|^2$ plus a bunch of other positive stuff. Why positive? Because when you multiply and add positive expressions, you get positive expressions! This means that the right hand side contains the left hand side plus some other positive stuff, so the right hand side can only be bigger, or the same if the additional stuff is equal to zero. This essentially does the trick, but it is too chatty to be called a proof. But now that we have the right idea, we can work on writing it in a mathematical language.

Write the right hand side of (1) in the form

$$\sum_{i=1}^{n} |x^i| \cdot \sum_{j=1}^{n} |x^j|$$

$$= \sum_{\{(i,j):1 \leq i,j \leq n\}} |x^i||x^j|$$

$$= \sum_{i=1}^{n} |x^i|^2 + \sum_{\{(i,j):i \neq j; 1 \leq i,j \leq n\}} |x^i||x^j|$$

$$= \sum_{i=1}^{n} |x^i|^2 + 2 \sum_{\{(i,j):1 \leq i < j \leq n\}} |x^i||x^j|.$$

The very last step is not difficult, but it may require a moment of reflection. Please make sure that you understand exactly how each step above came out of the previous one.

Plugging the calculation above into (1), we see that (1) comes down to showing that

$$0 \leq 2 \sum_{\{(i,j):1 \leq i < j \leq n\}} |x^i||x^j|,$$

which is, of course, true.

Another approach: One way you can make sure that you truly understand an argument is to give another proof. Please try to write down an argument along the following lines. Prove that case $n = 2$ and then proceed by induction. In dimension 3, for example, you can draw a right triangle as follows. Draw a straight line to the (x^1, x^2) plane from $x = (x^1, x^2, x^3)$. Then draw a straight line from that point $((x^1, x^2, 0)$ to the origin $((0, 0, 0))$. Finally, draw a straight line from the origin back to x. This gives us a right triangle, and by the two-dimensional case, the length of the hypotenuse, otherwise known as $|x|$, is less than or equal to $|x^3|$ plus the length of the segment from the origin to $(x^1, x^2, 0)$. The length of this segment is $\sqrt{|x^1|^2 + |x^2|^2}$. Invoking the two-dimensional case once again, we see that

$$\sqrt{|x^1|^2 + |x^2|^2} \leq |x^1| + |x^2|$$

and we are done with the case $n = 3$. Please write this out for the case of arbitrary n.

When is an inequality an equality? A question that inevitably arises when you prove an inequality is, when is it actually an equality. Note that we proved above that

$$|x|^2 = \sum_{i=1}^{n} |x^i|^2 + 2 \sum_{\{(i,j):1\leq i<j\leq n\}} |x^i||x^j|,$$

i.e

$$|x|^2 = |x|^2 + 2 \sum_{\{(i,j):1\leq i<j\leq n\}} |x^i||x^j|,$$

so the only way we have an equality is if

$$2 \sum_{\{(i,j):1\leq i<j\leq n\}} |x^i||x^j| = 0.$$

Since it is a sum of positive expressions, the only way it is equal to 0 is if each term is 0. This means that for every $i \neq j$, every expression $|x^i||x^j|$ is equal to 0. The only way this can happen is if at most one coordinate x^i is non-zero.