MTH 174 Homework 4 Solutions

Page 56, problems 17, 18, 20

Problem 17: If C is a bounded set of measure zero and $\int_A \chi_C$ exists, show that $\int_A \chi_C = 0$.
Hint: Use problem 3-8.

- Recall that χ_C is the characteristic function of C: $\chi_C(x) = \begin{cases} 0 & x \notin C \\ 1 & x \in C \end{cases}$
- Problem 3-8 says that $[a_1,b_1] \times \cdots \times [a_n,b_n]$ does not have content zero if $a_i < b_i$ for each i.
- Theorem 3-6 says that if C is compact & has measure zero then it also has content zero.

These two facts together $\Rightarrow$ C cannot contain any closed closed rectangle.

$\Rightarrow$ If $\{S_i\}$ is any collection of rectangles on A, χ_C attains zero on S_i for each i.

$\Rightarrow$ $L(f,P) = 0$ for all partitions P $\Rightarrow$ if χ_C is integrable on A, $\int_A \chi_C = 0$.

Problem 18: If $f: A \to \mathbb{R}$ is non-negative and $\int_A f = 0$, show that $\{x : f(x) \neq 0\}$ has measure zero.
Hint: Prove that $\{x : f(x) > 1/n\}$ has content zero.

- A subset A of $\mathbb{R}^n$ has content zero if for every $\varepsilon > 0$ there is a finite cover $\{U_i\}$ of A by closed rectangles such that $\sum_{i=1}^{n} v(U_i) < \varepsilon$
- Let $A_n = \{x : f(x) > 1/n\}$ and $\varepsilon > 0$. We can always find a partition P of A such that

$$\varepsilon > U(f,P) = \sum_{S \in P} M_S(f) \geq \sum_{S \in P,\, S \cap A_n \neq \emptyset} M_S(f) > \sum_{S \in P,\, S \cap A_n \neq \emptyset} \frac{1}{n} \cdot v(S)$$

So taking lim $\varepsilon \to 0$ gives us what we want.

Problem 20: Show that an increasing function $f: [a,b] \to \mathbb{R}$ is integrable on $[a,b]$.

- In a previous homework (problem 3-12) we showed that if $f: [a,b] \to \mathbb{R}$ is increasing then the set of discontinuities of f on $[a,b]$ has measure zero.
- Theorem 3-8: Let A be a closed rectangle and $f: A \to \mathbb{R}$ a bounded function. Let $B = \{x : f \text{ is not continuous at } x\}$. Then f is integrable if and only if B is a set of measure zero.
- These two facts together directly imply the result.

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Problem 25: Use induction on n to show that $[a_1,b_1] \times \cdots \times [a_n,b_n]$ is not a set of measure zero (or content 0) if $a_i < b_i$ for each i. * Isn't this equivalent to problem 3-8? *

- When $n = 1$, we just have $[a_1,b_1]$, a closed, compact interval in $\mathbb{R}$. Then theorem 3-5 tells us $[a_1,b_1]$ does not have content zero since for any finite cover $\{U_i\}$ of $[a_1,b_1]$, $\sum v(U_i) \geq b-a$.
- Next suppose for $n \leq N$, $[a_1,b_1] \times \cdots \times [a_n,b_n]$ is not a set of measure zero. Then if $\{U_{ij}\}$ is a cover of $[a_j,b_j]$, $\prod_{j=1}^{N} \left(\sum_i v(U_{ij})\right) \geq \prod_{j=1}^{N} (b_j - a_j) > 0$ and Obviously $\prod_{j=1}^{N+1} \left(\sum_i v(U_{ij})\right) = \sum_i v(U_{i,N+1}) \cdot \prod_{j=1}^{N} \left(\sum_i v(U_{ij})\right)$ which is not zero since neither term is zero and the real numbers do not have zero divisors.

Problem 27: If $f:[a,b]\times[a,b]\to\mathbb{R}$ is continuous, show that $\int_a^b\int_a^y f(x,y)\,dx\,dy = \int_a^b\int_x^b f(x,y)\,dy\,dx$

- The "Remark" after Fubini's Theorem says that $\int_{A\times B} f = \int_B \left(\mathcal{L}\int_A f(x,y)dx\right)dy = \int_B \left(\mathcal{U}\int_A f(x,y)dx\right)dy$.

But f is continuous so $\mathcal{L}=\mathcal{U}$. Also $[a\times b]\times[a\times b]$ is compact so f is integrable $\Rightarrow$ we can apply this remark.

* You guys must fill in the details of the proof for this remark if you go this route. This will essentially amount to modifying the proof for Fubini's theorem.

Problem 28: Use Fubini's Theorem to give an easy proof that $D_{1,2}f = D_{2,1}f$ if these are continuous.
Hint: If $D_{1,2}f(a)-D_{2,1}f(a)>0$ there is a rectangle containing a such that $D_{1,2}f - D_{2,1}f>0$ on A.

- Suppose $D_{1,2}f(a)-D_{2,1}f(a)>0$ & let $S=[a,b]\times[c,d]$ be a rectangle containing a, with $D_{1,2}f - D_{2,1}f > 0$ on S. Fubini's theorem gives us:

$$\int_S D_{1,2}f - D_{2,1}f = \int_{[a,b]}\int_{[c,d]} D_{1,2}f(x,y) - D_{2,1}f(x,y)\,dy\,dx$$

$$= \int_{[a,b]} (D_1 f(x,d) - D_1 f(x,c))\,dx - \int_{[a,b]} (D_2 f(b,y) - D_2 f(a,y))\,dy$$

$$= f(b,d) - f(b,c) - f(a,d) + f(a,c) - f(b,d) + f(a,c) + f(a,d) - f(a,c) = 0 \quad \text{contradiction.}$$

- a similar argument applies when $D_{1,2}f - D_{2,1}f < 0$.

Page 66 problem 37

Problem 37:
a) Suppose that $f:(0,1)\to\mathbb{R}$ is a non-negative continuous function. Show that $\int_{(0,1)} f$ exists if and only if $\lim_{\varepsilon\to 0}\int_\varepsilon^{1-\varepsilon} f$ exists.

- Suppose $\int_0^1 f$ exists. Then by definition there is an admissible cover $\mathcal{O}=\{U_i\}$ of open sets U_i contained in $(0,1)$ and a partition Φ subordinate to $\mathcal{O}$ such that $\sum_{\varphi\in\Phi}\int_{(0,1)}\varphi f$ converges. We may choose $U_\varepsilon = \{(\varepsilon, 1-\varepsilon)\}$ & $\mathcal{O}=\{U_\varepsilon\}_{\varepsilon>0}$. $\Phi=\{\varphi_\varepsilon\}$ where $\varphi_\varepsilon = 0$ outside of U_ε. Now refine our definitions even further by letting $\{\varepsilon_i\}_{i=1}^\infty$ be a sequence of ε_i's $\to 0$ as $i\to\infty$, & $\mathcal{O}=\{U_{\varepsilon_i}\}$, $\Phi=\{\varphi_{\varepsilon_i}\}$. Then $\sum_{i=1}^k \int_{(0,1)}\varphi_{\varepsilon_i} f = \int_{\varepsilon_i}^{1-\varepsilon_i} f$. This does not change the actual value of the sum by 3-12. Also as long as f is bounded on each $(\varepsilon_i, 1-\varepsilon_i)$ we have shown that $\int_0^1 f$ exists $\iff \lim_{\varepsilon\to\infty}\int_\varepsilon^{1-\varepsilon} f$ exists.

b) Let $A_n = [1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}}]$. Suppose that $f:(0,1)\to\mathbb{R}$ satisfies $\int_{A_n} f = \frac{(-1)^n}{n}$ and $f(x)=0$ $\forall x\notin$ any A_n. Show that $\int_{(0,1)} f$ does not exist but $\lim_{\varepsilon\to 0}\int_{(\varepsilon,1-\varepsilon)} f = \log 2$.

* Careful! This problem is impossible. Can you see why? We must specify that f is either positive or negative on A_n but not both.

- I guess we should also assume that f is bounded on each A_n in which case the integral is the usual integral:

$$\lim_{\varepsilon\to 0}\int_\varepsilon^{1-\varepsilon} f(x)\,dx = \sum_{n=1}^\infty \int_{A_n} f(x)\,dx = \sum_{n=1}^\infty \frac{(-1)^n}{n} = -\log 2$$

Spivak also forgot this negative sign!

- To show $\int_{(0,1)} f$ does not exist, Let $U_i = (1-\frac{1}{2^i - 1}, 1-\frac{1}{2^{i+1}+1})$ & Φ subordinate to $\mathcal{O}=\{U_i\}_{i\geq 1}$. $\int_{(0,1)} f$ exists $\iff \sum_{i=1}^\infty \int_{(0,1)}\varphi_i\cdot|f|$ converges but $\sum_{i=1}^{n-1}\int_{A_i}|f| \leq \sum_{i=1}^n \int_{(0,1)}\varphi_i|f|$. But $\int_{A_i}|f| \geq 1/n \Rightarrow$ the series diverges & $\int_{(0,1)} f$ does not exist.

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Problem 39: Use Theorem 3-14 to prove Theorem 3-13 without the assumption $\det g'(x) \neq 0$

Theorem 3-13: Let $A \subset \mathbb{R}^n$ be an open set and $g: A \to \mathbb{R}^n$ a 1-1 continuously differentiable function such that $\det g'(x) \neq 0$ $\forall x \in A$. If $f: g(A) \to \mathbb{R}$ is integrable, then
$$\int_{g(A)} f = \int_A (f \circ g)|\det g'|$$

• The idea is we don't need $\det g'(x) \neq 0$ since by Sard's theorem, the set $B = \{x \in A : g'(x) = 0\}$ is "small enough" that it doesn't cause us problems. We have:
$$\int_{g(A)} f = \int_{g(A)-g(B)} f + \int_{g(B)} f = \int_{g(A)-g(B)} f \quad \text{since } \int_E f = 0 \text{ when } E \text{ has measure zero.}$$

• Now we may apply Theorem 3-13 since $\det g'(x) \neq 0$ on $E = g(A) - g(B)$:
$$\int_{g(E)} f = \int_E (f \circ g)|\det g'|$$

Problem 41: Define $f: \{r : r > 0\} \times (0, 2\pi) \to \mathbb{R}^2$ by $f(r,\theta) = (r\cos\theta, r\sin\theta)$.

a) Show that f is 1-1, compute $f'(r,\theta)$, and show that $\det f'(r,\theta) \neq 0$ for all (r,θ). Show that $f(\{r : r>0\} \times (0,2\pi))$ is the set A from problem 2-23.

• $A = \{(x,y) \in \mathbb{R}^2 : x < 0, \text{ or } x \geq 0 \text{ and } y \neq 0\}$
$$f'(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \Rightarrow \det f' = r\cos^2\theta + r\sin^2\theta = r \neq 0 \text{ since } r > 0$$

• $\left.\begin{array}{l} r\cos\theta = r'\cos\theta' \\ r\sin\theta = r'\sin\theta' \end{array}\right\} r = r'$ & sin & cos are both 1-1 on $(0, 2\pi]$ => f is 1-1.

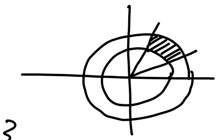
• To show that $f(\{r : r>0\} \times (0,2\pi)) = A$, just think about it: The only thing in $\mathbb{R}^2$ excluded from both sets is the positive x-axis.

b) If $P = f^{-1}$ show that $P(x,y) = (r(x,y), \theta(x,y))$ where $r(x,y) = \sqrt{x^2+y^2}$, $\theta = \begin{cases} \tan^{-1}(y/x) & x,y > 0 \\ \pi + \tan^{-1}(y/x) & x < 0 \\ 2\pi + \tan^{-1}(y/x) & x > 0, y < 0 \\ \pi/2 & x = 0, y > 0 \\ 3\pi/2 & x = 0, y < 0 \end{cases}$

Find $P'(x,y)$.

This is a standard & straight-forward computation.

c) Let $C \subset A$ be the region between the circles of radii r_1 & r_2 & the half-lines through zero making angles θ_1 & θ_2 with the x-axis. If $h(x,y) = g(r(x,y), \theta(x,y))$ show that
$$\int_C h = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} r g(r,\theta)\, d\theta\, dr$$



Note that $|\det P'| = \sqrt{x^2+y^2} = r(x,y)$, and $C = P(A)$ for $A = \{(r,\theta) : r_1 \leq r \leq r_2, \theta_1 \leq \theta \leq \theta_2\}$
$$\Rightarrow \int_C h = \int_{P(A)} h = \int_A h \circ P \cdot |\det P'| = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} r \cdot g(r,\theta)\, d\theta\, dr \text{ by Theorem 3-13.}$$

If $B_r = \{(x,y) : x^2 + y^2 \leq r^2\}$, show that $\int_{B_r} h = \int_0^r \int_0^{2\pi} r g(r,\theta)\, d\theta\, dr$.

This is a direct consequence of the above, just with $r_1 \to 0$ & $r_2 = r$, $\theta_1 \to 0$ & $\theta_2 = 2\pi$. The parts of the new domain "left out" will not effect the integral since they have measure zero.

d) If $C_r = [-r,r] \times [-r,r]$, show that $\int_{B_r} e^{-(x^2+y^2)}\,dx\,dy = \pi(1-e^{-r^2})$

and $\int_{C_r} e^{-(x^2+y^2)}\,dx\,dy = \left(\int_{-r}^{r} e^{-x^2}\,dx\right)^2$

This integral is one of Alex's favorite of all time and he is highly likely to put something related to this on an exam.

- It follows from c) (with $h(x,y) = e^{-(x^2+y^2)}$ + $g(r,\theta) = e^{-r^2}$) that

$$\int_{B_r} e^{-(x^2+y^2)} = \int_0^r \int_0^{2\pi} re^{-r^2}\,d\theta\,dr$$

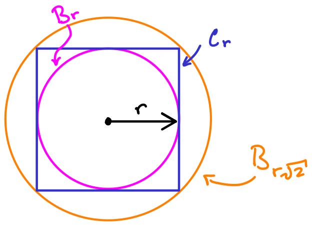
u-sub w/ $u = r^2$

$$= 2\pi \int_0^r re^{-r^2}\,dr = 2\pi\left(-\tfrac{1}{2}e^{-r^2}\right)\Big|_0^r = \pi(1-e^{-r^2})$$

- Similarly, $\int_{C_r} e^{-(x^2+y^2)}\,dx\,dy = \int_{-r}^{r}\int_{-r}^{r} e^{-(x^2+y^2)}\,dx\,dy = \int_{-r}^{r}\int_{-r}^{r} e^{-x^2}e^{-y^2}\,dx\,dy = \left(\int_{-r}^{r} e^{-x^2}\,dx\right)^2$ ∇

You have to admit this is pretty cool.

e) Prove that $\lim_{r\to\infty} \int_{B_r} e^{-(x^2+y^2)}\,dx\,dy = \lim_{r\to\infty} \int_{C_r} e^{-(x^2+y^2)}\,dx\,dy$ and conclude that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$



d) Tells us that $\lim_{r\to\infty} \int_{B_r} e^{-(x^2+y^2)}\,dx\,dy = \pi$ since the term involving r is $e^{-r^2} \to 0$ as $r \to \infty$. $\Rightarrow \lim_{r\to\infty} \int_{C_r} e^{-(x^2+y^2)}\,dx\,dy = \pi$ by the "squeeze theorem"

$\Rightarrow \int_{-r}^{r} e^{-x^2}\,dx = \sqrt{\pi}$ by the second part of part d ∇