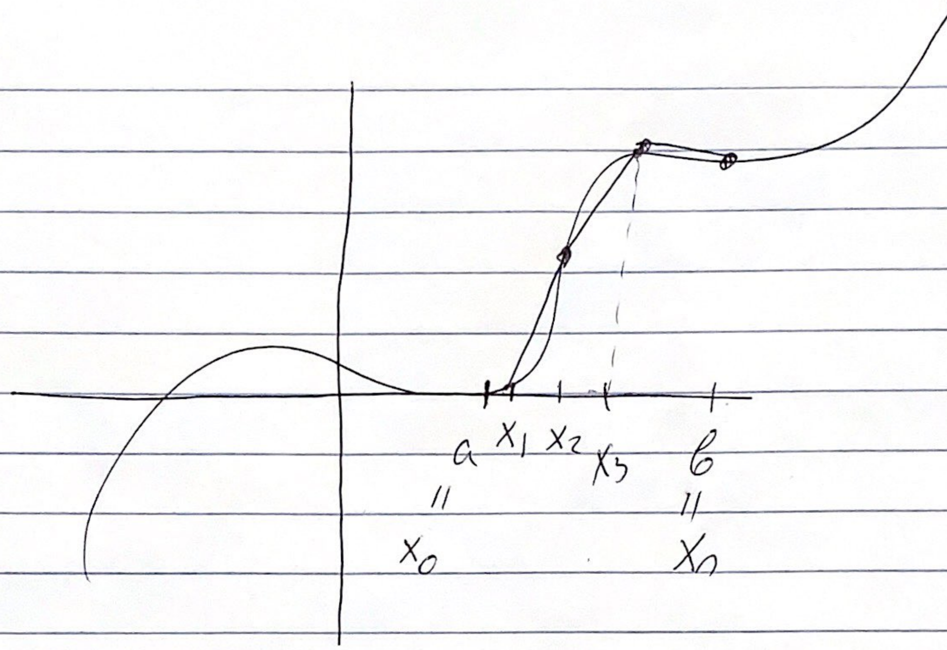
1

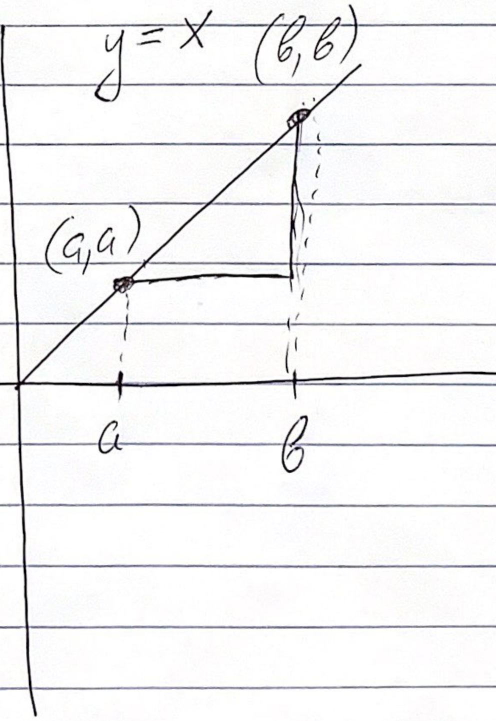



Distance $= \sqrt{(b-a)^2 + (b-a)^2}$

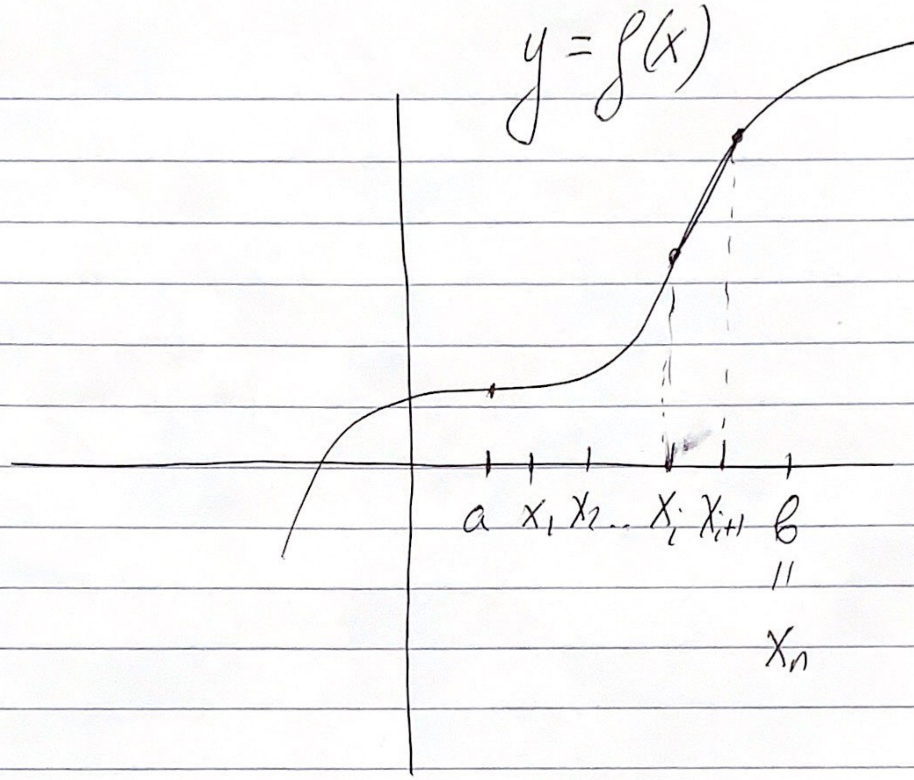
$\parallel$

length $= \sqrt{2} \cdot (b-a)$

Pythagorean theorem



(2)




$$(x_i, f(x_i)) \qquad (x_{i+1}, f(x_{i+1}))$$

$$\text{Distance} = \sqrt{(x_{i+1} - x_i)^2 + (f(x_{i+1}) - f(x_i))^2}$$

$$f(x_{i+1}) - f(x_i) = f'(x_i^*)(x_{i+1} - x_i)$$

$$\text{Distance} = \sqrt{(x_{i+1} - x_i)^2 \left(1 + (f'(x_i^*))^2\right)}$$

$$(x_{i+1} - x_i)\sqrt{1 + (f'(x_i^*))^2}$$

(3)

Total length =

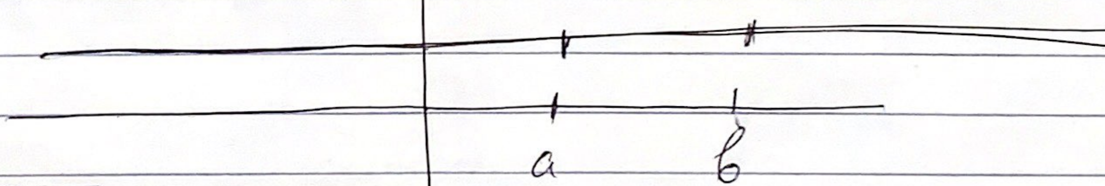
$$\sum_{i=0}^{n} (x_{i+1} - x_i) \sqrt{1 + (f'(x_i^*))^2}$$

In the limit we obtain

$$\int_a^b \sqrt{1+(f'(x))^2}\, dx$$

$f(x) = 1 \quad f'(x) = 0$

$y = 1$



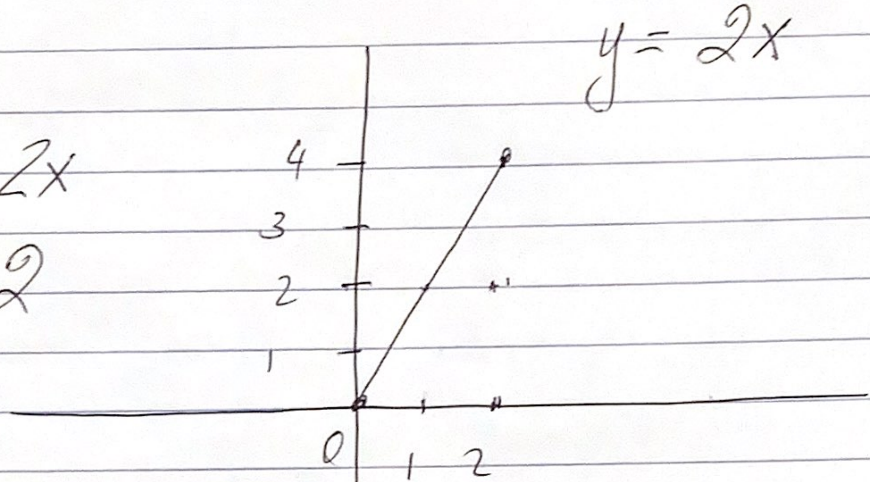

$$\int_a^b \sqrt{1+0^2}\, dx = b-a$$

length = $b-a$

(4)

$f(x) = 2x$

$f'(x) = 2$




length $= \sqrt{2^2 + 4^2} = 2\sqrt{5}$

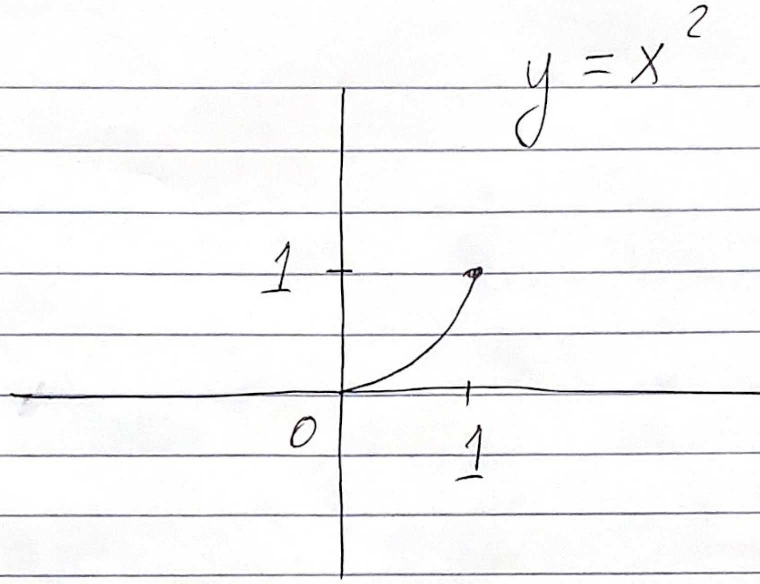
$$L = \int_0^2 \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_0^2 \sqrt{1+4}\, dx = \int_0^2 \sqrt{5}\, dx = 2\sqrt{5}$$

(5)

$f(x) = x^2$

$f'(x) = 2x$




$$L = \int_0^1 \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_0^1 \sqrt{1 + 4x^2}\, dx$$

$$x = \frac{1}{2}\tan\theta \qquad dx = \frac{1}{2}\sec^2\theta\, d\theta$$

$$\int_0^{\tan^{-1}(2)} \sqrt{1 + \tan^2\theta} \cdot \frac{1}{2}\sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\tan^{-1}(2)} \sec\theta \cdot \sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\tan^{-1}(2)} \sec^3\theta\, d\theta$$

⑥

$$\frac{1}{2}\int_0^{\tan^{-1}(2)} \sec^3\theta \, d\theta = \frac{1}{2}\int_0^{\tan^{-1}(2)} \underbrace{\sec^2\theta}_{g'(\theta)} \cdot \underbrace{\sec\theta}_{f(\theta)} \, d\theta$$

$$\int_0^{\tan^{-1}(2)} f(\theta) g'(\theta)\, d\theta = f(\theta) g(\theta) \Big|_0^{\tan^{-1}(2)} - \int_0^{\tan^{-1}(2)} f'(\theta) g(\theta)\, d\theta$$

$g'(\theta) = \sec^2\theta \qquad g(x) = \tan\theta$

$f(\theta) = \sec\theta \qquad f'(\theta) = \sec\theta \tan\theta$

$$\sec\theta \tan\theta \Big|_0^{\tan^{-1}(2)} - \int_0^{\tan^{-1}(2)} \sec\theta \tan\theta \cdot \tan\theta \, d\theta$$

$$= \sec\theta \tan\theta \Big|_0^{\tan^{-1}(2)} - \int_0^{\tan^{-1}(2)} \sec\theta \tan^2\theta \, d\theta$$

$$= \sec\theta \tan\theta \Big|_0^{\tan^{-1}(2)} - \int_0^{\tan^{-1}(2)} \sec\theta (\sec^2\theta - 1) \, d\theta$$

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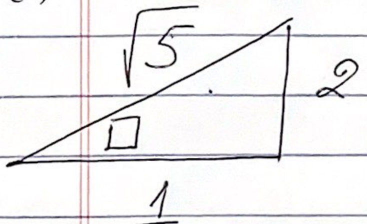
$$\int_0^{\tan^{-1}(2)} \sec^3\theta = \sec\theta\tan\theta \Big|_0^{\tan^{-1}(2)} - \int_0^{\tan^{-1}(2)} \sec^3\theta\, d\theta$$

$$+ \int_0^{\tan^{-1}(2)} \sec\theta\, d\theta$$

$$2\int_0^{\tan^{-1}(2)} \sec^3\theta\, d\theta = \sec\theta\tan\theta \Big|_0^{\tan^{-1}(2)} + \int_0^{\tan^{-1}(2)} \sec\theta\, d\theta$$

$$\int_0^{\tan^{-1}(2)} \sec^3\theta\, d\theta = \underbrace{\frac{1}{2}\sec\theta\tan\theta \Big|_0^{\tan^{-1}(2)}}_{\text{I}} + \underbrace{\frac{1}{2}\int_0^{\tan^{-1}(2)} \sec\theta\, d\theta}_{\text{II}}$$

$\tan^{-1}(2) = \square$




$$\text{I} = \frac{1}{2}\sec\theta\tan\theta \Big|_0^{\tan^{-1}(2)} = \frac{1}{2}\cdot\sqrt{5}\cdot 2 - 0$$

$$= \sqrt{5}$$

(8)

$$\underline{\text{II}} = \int_0^{\tan^{-1}(2)} \sec\theta \, d\theta = \int_0^{\tan^{-1}(2)} \sec\theta \, \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

$$= \int_0^{\tan^{-1}(2)} \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

$u = \sec\theta + \tan\theta \qquad du = (\sec^2\theta + \sec\theta\tan\theta)\,d\theta$

$$\int_1^{2+\sqrt{5}} \frac{du}{u} = \ln(|u|)\Big|_1^{2+\sqrt{5}} = \ln(2+\sqrt{5})$$

$$\text{Length} = \frac{1}{2}(\underline{\text{I}} + \underline{\text{II}}) =$$

$$\frac{\sqrt{5} + \ln(2+\sqrt{5})}{2}$$

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$x = y^2 \qquad (0,0) \qquad (1,1)$

$$L = \int_0^1 \sqrt{1+4y^2}\, dy$$

$y = x^2 - \ln(x) \qquad (1,1) \qquad (e, e^2-1)$

$\quad \parallel$

$f(x)$

$$f'(x) = 2x - \frac{1}{x}$$

$$(f'(x))^2 = 4x^2 + \frac{1}{x^2} - 4$$

$$1 + (f'(x))^2 \qquad \underbrace{\int_1^e \sqrt{4x^2 + \frac{1}{x^2} - 3}\, dx}_{\text{completes the setup!}}$$

Length =

(10)

$$y = \ln(1-x^2) \qquad 0 \le x \le \frac{1}{2}$$

$$\overset{\shortparallel}{f(x)}$$

$$f'(x) = \frac{-2x}{1-x^2}$$

$$1 + (f'(x))^2 = 1 + \frac{4x^2}{(1-x^2)^2}$$

$$\frac{(1-x^2)^2 + 4x^2}{(1-x^2)^2} = \frac{1-2x^2+x^4+4x^2}{(1-x^2)^2} =$$

$$\frac{1+2x^2+x^4}{(1-x^2)^2} = \frac{(1+x^2)^2}{(1-x^2)^2}$$

$$L = \int_0^{\frac{1}{2}} \sqrt{\frac{(1+x^2)^2}{(1-x^2)^2}}\, dx = \int_0^{\frac{1}{2}} \frac{1+x^2}{1-x^2}\, dx$$

(11)

$$\int_0^{\frac{1}{2}} \frac{1+x^2}{1-x^2}\,dx$$

$$\begin{array}{r}
-1 \\
1-x^2 \,\overline{\big)\, x^2+1} \\
x^2-1 \\
\hline
2
\end{array}$$

$$\frac{1+x^2}{1-x^2} = -1 + \frac{2}{1-x^2}$$

$$\underbrace{\int_0^{\frac{1}{2}} -1\,dx}_{I} + 2\underbrace{\int_0^{\frac{1}{2}} \frac{dx}{1-x^2}}_{II}$$

$$I = -\frac{1}{2}$$

(12)

$$\frac{\pi}{2} = 2\int_0^{\frac{1}{2}} \frac{dx}{1-x^2}$$

$$\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$$

$$1 = A(1+x) + B(1-x)$$

$x=1$: $\quad 2A = 1 \quad A = \frac{1}{2}$

$x=-1$: $\quad 2B = 1 \quad B = \frac{1}{2}$

$$2\int_0^{\frac{1}{2}} \left(\frac{1}{2}\frac{1}{1-x} + \frac{1}{2}\frac{1}{1+x}\right) dx$$

$$= \int_0^{\frac{1}{2}} \frac{dx}{1-x} + \int_0^{\frac{1}{2}} \frac{dx}{1+x}$$

$$= \ln(1-x)\Big|_0^{\frac{1}{2}} + \ln(1+x)\Big|_0^{\frac{1}{2}}$$

$$= \ln\left(\frac{1}{2}\right) + \ln\left(\frac{3}{2}\right)$$

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$\epsilon > 0$

$f(x) = \frac{1}{\sqrt{x}}$

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{dx}{\sqrt{x}} =$$

$$\lim_{\epsilon \to 0} 2x^{\frac{1}{2}} \Big|_\epsilon^1 = \lim_{\epsilon \to 0} \left(2 - 2\sqrt{\epsilon}\right)$$

$$= 2$$

Why can't we just ignore the fact that $\frac{1}{\sqrt{x}}$ is infinite at 0?

$$\int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x} \Big|_0^1 = 2$$

14

$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^{1} = -1-1 = -2$$

WRONG