SPHERICAL AVERAGES, DISTANCE SETS, AND LATTICE POINTS ON CONVEX SURFACES

Alex Iosevich and Misha Rudnev

May 24, 2005

Abstract. We prove some lower and upper bounds for the average of the Fourier transforms of measures over dilated boundaries of smooth convex sets. We describe a connection between these estimates and the Erdös/Falconer distance problems, as well as the distribution of lattice points on convex surfaces and related arithmetic problems.

Introduction

The celebrated Falconer distance conjecture says that if the Hausdorff dimension of a compact set E in $\mathbb{R}^d$ is greater than $\frac{d}{2}$, then the Lebesgue measure of $\Delta_K(E)$ is positive, with

$$\Delta_K(E) = \{||x-y||_K : x, y \in E\}, \tag{0.1}$$

where $||\cdot||_K$ is the norm induced by a centrally symmetric convex body with a smooth boundary and everywhere non-vanishing Gaussian curvature. The best known result in this direction is due to Wolff ([W99]), in two dimensions, and Erdogan ([Erd05]), in higher dimensions, who prove that the Lebesgue measure of $\Delta_K(E)$ is indeed positive if the Hausdorff dimension of E is greater than $\frac{d}{2}+\frac{1}{3}$. These results are partly based on the machinery developed by Mattila ([Mat87]), which can be summarized as follows. Mattila proves that if there exists a Borel measure μ supported on E such that

$$M_K(\mu) = \int_1^\infty \left(\int_{\partial K^*} |\widehat{\mu}(t\omega)|^2 d\omega_{K^*} \right)^2 t^{d-1} dt < \infty, \tag{0.2}$$

then the Lebesgue measure of $\Delta_K(E)$ is positive. Here $K^* = \{x \in \mathbb{R}^d : \sup_{y \in K} x \cdot y \leq 1\}$ denotes the convex body dual to K, and $d\omega_{K^*}$ denotes the Lebesgue measure on ∂K^*.

The work was partly supported by a grant from the National Science Foundation and the EPSRC grant GR/S13682/01.

More precisely, Mattila proves this in the case of the Euclidean metric, $K = B_d$, and the relatively straightforward extension to general Ks with a smooth boundary and everywhere non-vanishing curvature is worked out in [HI05].

Mattila proves his result by constructing a measure ν on $\Delta_K(E)$ such that $1 \lesssim \int d\nu(s)$, where here, and throughout the paper, $X \lesssim Y$ means that there exists $C > 0$ such that $X \leq CY$. Similarly, $X \lessapprox Y$, with the controlling parameter $t \gg 1$ means that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $X \leq C_\epsilon t^\epsilon Y$. The notations $X \gtrsim Y$ and $X \gtrapprox Y$ are defined similarly in an obvious way; besides $X \approx Y$ means that $X \lesssim Y$ and $Y \lesssim X$.

By Cauchy-Schwartz and Plancherel,

$$1 \lesssim \left(\int d\nu(s) \right)^2 \leq |\Delta_K(E)| \cdot \int \nu^2(s) ds = |\Delta_K(E)| \cdot \int |\widehat{\nu}(t)|^2 dt. \tag{0.3}$$

Mattila then shows that

$$\widehat{\nu}(t) \approx t^{\frac{d-1}{2}} \int_{\partial K^*} |\widehat{\mu}(t\omega)|^2 d\omega_{K^*}. \tag{0.4}$$

Both Wolff, Erdogan and their predecessors (see the reference list in [Erd05]) obtained an upper bound for $M_K(\mu)$, by first proving bound of the form

$$\int_{\partial K^*} |\widehat{\mu}(t\omega)|^2 d\omega_{K^*} \lesssim t^{-\beta}, \tag{0.5}$$

for some $\beta > 0$ and then arguing by using this, polar coordinates, and Plancherel, that

$$M_K(\mu) \lesssim \int |\widehat{\mu}(\xi)|^2 |\xi|^{-d+(d-\beta)} d\xi = c_{d,\beta} \int \int |x-y|^{-(d-\beta)} d\mu(x) d\mu(y) = I_{d-\beta}(\mu), \tag{0.6}$$

the energy integral of μ of order $-(d-\beta)$. It is a standard result that the latter integral is bounded whenever $d-\beta$ is smaller than the Hausdorff dimension of E.

It has been known for some time that estimates like (0.5) alone cannot yield sufficiently strong bounds on $M_K(\mu)$ to solve the Falconer conjecture. For example, it follows from a result of Sjolin ([Sj93]) that when $d = 2$ and the Hausdorff dimension of E is $s > 1$, the best estimate one can expect is

$$\int_{\partial K^*} |\widehat{\mu}(t\omega)|^2 d\omega_{K^*} \lesssim t^{-\frac{s}{2}}, \tag{0.7}$$

where μ is, as before, a Borel measure on E. Wolff proved (0.7) up to the endpoint. In higher dimensions, Sjolin's construction implies that (see [Sj93]) that the best estimate one can expect is

$$\int_{\partial K^*} |\widehat{\mu}(t\omega)|^2 d\omega_{K^*} \lesssim t^{-\frac{s}{2}-\frac{d-2}{2}}, \tag{0.8}$$

namely there are measures where the inequality (0.8) holds in the opposite direction.

Observe that since one always has the bound $\int_{\partial K^*} |\widehat{\mu}(t\omega)|^2 d\omega_{K^*} \lesssim t^{-s}$, the estimate (0.8) provides non-trivial information only in the range $s > d-2$.

In the forthcoming Theorem 1.5 we demonstrate, among other things, that the best estimate we can expect for the Euclidean metric is,

$$\int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 d\omega \lesssim t^{-s} t^{-1+\frac{2s}{d}}, \tag{0.9}$$

which is always more restrictive than (0.8) if $d \geq 4$ and provides new information in the range $s > \frac{d}{2}$. In dimensions two and three, Sjolin's estimate remains the best restriction known, though we shall below that a variant of (0.9) can be used to recover Sjolin's restriction in two and three dimensions.

Let us also remark on the application of our constructions to upper and lower bounds for L^1-averages

$$\int_{S^{d-1}} |\widehat{\mu}(t\omega)| d\omega, \tag{0.10}$$

though our results in this direction, especially the lower bounds, are considerably less optimal. It follows from (0.9) that the best L^1 estimate one can expect is

$$\int_{S^{d-1}} |\widehat{\mu}(t\omega)| d\omega \lesssim t^{-s} t^{-1+\frac{2s}{d}}. \tag{0.11}$$

This result is new in higher dimensions as no lower bounds for L^1 averages are available there to the best of our knowledge. However, in two dimensions, (0.12) is much weaker than the restriction obtained by Bennett and Vargas ([BV02]) who proved that the best L^1 estimate one can expect is

$$\int_{S^1} |\widehat{\mu}(t\omega)| d\omega \lesssim t^{-\frac{s}{s+2}}. \tag{0.12}$$

Observe that (0.12) does not imply an interesting lower bound for the L^2-average since

$$\left(\int_{S^1} |\widehat{\mu}(t\omega)|^2 d\omega \right)^{\frac{1}{2}} \geq \int_{S^1} |\widehat{\mu}(t\omega)| d\omega \gtrsim t^{-\frac{s}{s+2}}, \tag{0.13}$$

is less restrictive than Sjolin's lower bound given by (0.8).

It is important to note that while (0.7) shows that estimates of the form (0.5) cannot be used to prove the Falconer conjecture in $d = 2$, we are not able to make a similar statement about higher dimensions since the exponent in (0.9) is very well behaved near $s = \frac{d}{2}$. This issue is partially addressed in Theorem 0.5 below in the context of general metrics.

The main purpose of this paper is to prove (0.9), both in the case of averages over the sphere and in more general context, and to explain connections between this problem, the Falconer distance conjecture, and distribution of lattice points on and near convex surfaces.

Section 1: Lower bounds

Our first result is based on the following classical construction. See also [S85] and [BP89] for related materials and background.

Theorem 1.1. *There exists a convex, smooth, centrally symmetric curve $\mathcal{C}$, with everywhere non-vanishing curvature, such that*

$$\#\{t\mathcal{C} \cap \mathbb{Z}^2\} \gtrsim t^{\frac{1}{2}} \tag{1.1}$$

for a sequence of ts tending to infinity.

To take advantage of this result in our context, let A be a Delone set, which means that there exist $C, c > 0$ such that $|a - a'| \geq c$ for all $a \neq a' \in A$, and every cube of side-length C contains at least one point of A. By scaling and throwing away some points, we may assume that A contains exactly one point in every integer translate of the unit cube $[0,1]^d$, which is what we shall do in the sequel.

Let q be a large integer and for $s \in (0, d)$ define

$$f(x) = q^{-d} q^{\frac{d^2}{s}} \sum_{a \in A_q} \phi(q^{\frac{d}{s}}(x - a/q))\phi(a/q), \tag{1.2}$$

where $A_q = A \cap [0, q]^d$, and ϕ is a smooth radial cut-off function supported in the unit ball B_d and identically equal to 1 in a ball of smaller radius. Let $d\mu(x) = f(x)dx$.

Lemma 1.2. *Let*

$$I_\alpha(\mu) = \int\int |x - y|^{-\alpha} d\mu(x) d\mu(y), \tag{1.3}$$

the energy integral of μ. Then $I_s(\mu) \lesssim 1$.

Theorem 1.3. *Let K be the convex body determined by $\mathcal{C}$ from Theorem 0.1. Let $A = \mathbb{Z}^d$. Then for $t \approx q^{\frac{d}{s}}$,*

$$\int_K |\widehat{\mu}(t\omega)|^2 d\omega_K \approx t^{-\frac{1}{2} - \frac{s}{4}}, \tag{1.4}$$

where $d\mu$ is defined as in (0.11) above.

Our construction is motivated by the following result proved in [IL05] and [HI05] in a variety of contexts.

Theorem 1.4. *Suppose that the Lebesgue measure of $\Delta_K(E)$ is positive whenever the Hausdorff dimension of E is greater than s_0. Let A be a Delone set. Then*

$$\#\Delta_K(A_q) \gtrapprox q^{\frac{d}{s_0}}. \tag{1.5}$$

This result provides a partial link between the Falconer distance conjecture and its discrete predecessor, the Erdös distance conjecture which says that if S is a subset of $\mathbb{R}^d$, $d \geq 2$, of cardinality N, then

$$\#\Delta_K(S) \gtrapprox N^{\frac{2}{d}}. \tag{1.6}$$

In the context of Delone sets, this takes the form

$$\#\Delta_K(A_q) \gtrapprox q^2. \tag{1.7}$$

The rough outline of the proof of Theorem 1.4 is the following. Let q_i be a sequence of positive integers such that $q_1 = 2$ and $q_{i+1} > q_i^i$. Let E_i denote the support of $f(x)$ with $q = q_i$. Let $E = \cap_i E_i$. One can show that the Hausdorff dimension of E is s. Observe that

$$|\Delta_K(E_i)| \lesssim q_i^{-\frac{d}{s}} \#\Delta_K(A_{q_i}), \tag{1.8}$$

and the result follows since $|\Delta_K(E_i)|$ is not allowed to go to 0, as $i \to \infty$, by assumption, if $s > s_0$.

In higher dimensions, an analog of Theorem 1.1 is not available, to the best of our knowledge. However, we prove the following conditional statement.

Theorem 1.5. *Let $A = \mathbb{Z}^d$ and let $d\mu(x)$ be as above. Suppose that there exists a convex body K, with ∂K being smooth and having everywhere non-vanishing Gaussian curvature, such that*

$$\#\{t\partial K \cap \mathbb{Z}^d\} \approx t^{d-2+\gamma}, \tag{1.9}$$

for a sequences of ts going to infinity and some $\gamma \in [0,1)$. Then there exists an arbitrarily large t, depending on q, such that

$$\int_{\partial K} |\widehat{\mu}(t\omega)|^2 d\omega_K \gtrsim t^{-s} t^{-1+\frac{2s}{d}} t^{\gamma(1-\frac{s}{d})}. \tag{1.10}$$

The same conclusion holds if the left hand side of (1.9) *is replaced by $\int_{\partial K} |\widehat{\mu}(t\omega)| d\omega_K$.*

Observe that Theorems 1.5 and 1.5 also hold for the Euclidean metric and a Delone set A obtained by the corresponding dilation of the integer lattice, see the remark in the Introduction in [IR05].

In particular, Theorem 1.5 says that if there exists K such that (1.9) holds with $\gamma > 0$, then the Falconer conjecture cannot be proved for Δ_{K^*} simply by considering point-wise estimates for $\int_{\partial K} |\widehat{\mu}(t\omega)|^2 d\omega_K$. Even if $\gamma = 0$, as it is in the case of the Euclidean metric, Theorem 0.5 says that in the range $s > \frac{d}{2}$, the estimate $\int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 d\omega \lesssim t^{-s}$ is not in general valid, and the best estimate we can expect is the one given by (1.10) above.

What is the largest possible γ that can possibly arise this way? It is not difficult to deduce from the following result due to Andrews ([And63]) that $\gamma \leq \frac{2}{d+1}$.

Theorem 1.6. *Let P be a convex polyhedron whose vertices are in $\mathbb{Z}^d$, $d \geq 2$. Then*

$$\# \text{ vertices of } P \lesssim |P|^{\frac{d-1}{d+1}}. \tag{1.11}$$

Observe that $\gamma \leq \frac{2}{d+1}$ is not optimal for smooth K, for which Schmidt ([S85]) conjectured $\gamma = \frac{1}{2} + \epsilon$ in $d = 2$.

In the Euclidean case we have $\gamma = 0$ because of the classical number theoretic result (see e.g. [Lan69]) which says that

$$\#\{t\partial B_d \cap \mathbb{Z}^d\} \lessapprox t^{d-2}, \tag{1.12}$$

and $\lessapprox$ may be replaced by $\lesssim$ for $d \geq 4$ and $\approx$ if $d \geq 5$. The question is, whether one can come up with a more restrictive counter-example by introducing appropriate weighing into the definition of $d\mu$. We hope to address this issue in a subsequent paper.

As we noted above, Theorem 1.5 can be used to recover the sharpness of Wolff's ([W99]) estimate

$$\int_{S^1} |\widehat{\mu}(t\omega)|^2 d\omega \lesssim t^{-\frac{s}{2}}, \tag{1.13}$$

under the assumption that μ is a Borel measure on a compact set of Hausdorff dimension s. Indeed, suppose that one could prove that

$$\int_{S^1} |\widehat{\mu}(t\omega)|^2 d\omega \lesssim t^{-\frac{s}{2}-\epsilon}, \tag{1.14}$$

for some $\epsilon > 0$. Combining this with Theorem 0.5 we obtain the relation

$$\frac{s}{2} + \epsilon \leq s + 1 - s - \gamma \left(\frac{2-s}{2}\right), \tag{1.15}$$

which implies that

$$\gamma \leq 1 - \frac{2\epsilon}{2-s}. \tag{1.16}$$

Letting s tend to 2, we obtain a contradiction as there is no absolute constant that would bound the number of integer lattice points on the circle of radius t, for all t.

Using a similar argument we can check that the estimate

$$\int_{S^2} |\widehat{\mu}(t\omega)|^2 d\omega \lesssim t^{-1-\frac{s}{2}}, \tag{1.17}$$

if true, would be best possible. To see this, suppose that

$$\int_{S^2} |\widehat{\mu}(t\omega)|^2 d\omega \lesssim t^{-1-\frac{s}{2}-\epsilon}, \tag{1.18}$$

for some $\epsilon > 0$. We are then led to the inequality

$$\frac{1}{2} + \frac{s}{2} \leq 1 + s - \frac{2s}{3} - \gamma(1 - s/3), \tag{1.19}$$

which implies that

$$\gamma \leq \frac{1}{2} - \frac{3\epsilon}{3-s}. \tag{1.20}$$

Letting $s \to 3$ we obtain a contradiction as in the two-dimensional case.

Observe that in the case $\epsilon = 0$ we would obtain the restriction $\gamma \leq \frac{1}{2} = \frac{2}{d+1}$, since $d = 3$. This is reasonable in view of Andrews' result (Theorem 1.6) stated above. This leads us to conjecture that in three dimensions, the estimate

$$\int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 d\omega \lesssim t^{-\frac{1}{2}-\frac{s}{2}} \tag{1.21}$$

should hold for any Borel measure μ supported on a compact set E of Hausdorff dimension $s > \frac{3}{2}$.

Section 2: Upper bounds

We now establish upper bounds for $\int_K \left|\widehat{f}(t\omega)\right|^2 d\omega_K$ under the assumption that ∂K is smooth and has everywhere non-vanishing Gaussian curvature.

Theorem 2.1. *Let $A = \mathbb{Z}^d$, $d \geq 2$. Then*

$$\int_{S^{d-1}} \left|\widehat{f}(t\omega)\right|^2 d\omega \lessapprox t^{-s}, \tag{2.1}$$

if $s \leq \frac{d}{2}$, and

$$\int_{S^{d-1}} \left|\widehat{f}(t\omega)\right|^2 d\omega \lessapprox t^{-s} t^{-1+\frac{2s}{d}}, \tag{2.2}$$

if $s > \frac{d}{2}$.

In $d \geq 4$ in the estimates above $\lessapprox$ can be replaced by $\lesssim$. The same conclusion holds if the left hand side of (2.1) and (2.2) is replaced by $\int_{S^{d-1}} |\widehat{\mu}(t\omega)| d\omega$.

Theorem 2.2. *Let $A = \mathbb{Z}^d$, $d \geq 2$. Let K be as above. Then*

$$\int_{\partial K} \left|\widehat{f}(t\omega)\right|^2 d\omega_K \lesssim t^{-s} t^{-1+\frac{2s}{d}} t^{\gamma_d(1-\frac{s}{d})}, \tag{2.3}$$

if $s \geq \frac{d}{2}$, and

$$\int_{\partial K} \left|\widehat{f}(t\omega)\right|^2 d\omega_K \lesssim t^{-s} t^{\gamma_d(1-\frac{s}{d})}, \tag{2.4}$$

where γ_d is a positive number given in Theorem 2.3.

The same conclusion holds if the left hand side of (2.3) *and* (2.4) *is replaced by $\int_{\partial K} |\widehat{\mu}(t\omega)| d\omega_K$.*

Theorem 2.3. *Let $N(R) = \#\{RK \cap \mathbb{Z}^d\}$, $d \geq 2$, where ∂K is smooth and has everywhere non-vanishing curvature. Then*

$$N(R) = |K|R^d + E(R), \tag{2.5}$$

where

$$|E(R)| \lesssim R^{d-2+\gamma_d}, \tag{2.6}$$

where

$$\gamma_2 = \frac{131}{208}, \tag{2.7}$$

$$\gamma_3 = \frac{20}{43}, \tag{2.8}$$

and

$$\gamma_d = \frac{d+4}{d^2+d+2}, \; d \geq 4. \tag{2.9}$$

The estimate (2.7) is due to Huxley ([Hux03]), the estimates (2.8) and (2.9) are due to Müller ([Mul99]).

The point here is that the number of lattice points in the R^{-1} neighborhood of the boundary of RK cannot possible exceed $CR^{d-2+\gamma_d}$ since otherwise one could construct an immediate counterexample to (2.6).

The best known upper bound for a general Delone set A can be obtained using the following result of Erdogan ([Erd05]).

Theorem 2.4. *Let E be a compact subset of $\mathbb{R}^d$ equipped with a Borel measure μ. Then*

$$\int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 d\omega \lesssim t^{-\beta} I_\alpha(\mu), \tag{2.10}$$

where $\beta < \alpha$, if $0 < \alpha \le \frac{d-1}{2}$, $\beta < \frac{d-1}{2}$, if $\frac{d-1}{2} \le \alpha \le \frac{d}{2}$, $\beta < \frac{d+2\alpha-2}{4}$, if $\frac{d}{2} \le \alpha \le \frac{d+2}{2}$, and $\beta < \frac{\alpha+1}{2}$ if $\frac{d+2}{2} \le \alpha \le d$. The proof carries over the case where S^{d-1} is replaced by a boundary of a smooth convex hyper-surface with non-vanishing Gaussian curvature.

It is interesting to note that comparing the upper bound given by Theorem 2.4 and the lower bound given by Theorem 1.5 leads to the trivial restriction

$$\gamma \le 1, \tag{2.11}$$

which always holds since ∂K is $d-1$-dimensional. Hence the Wolff-Erdogan bound (1.13) alone, while tight, cannot be used in order to estimate the number of lattice points on convex curves. (We already know from Theorem 1.6 that $\gamma \le \frac{2}{d+1}$.) In higher dimensions, the bounds of Theorem 2.4 are unlikely to be tight. This raises the question of whether their improvement would make the resulting restriction on γ more meaningful.

Section 3: Synthesis and further connections

In this paper we have produced some arithmetic examples that show that the bound

$$\int_K |\widehat{\mu}(t\omega)|^2 d\omega_K \lesssim t^{-s}, \tag{3.1}$$

or as a matter of fact the stronger bound (0.8)) does not in general hold, even in the case of the Euclidean metric (this would be achieved by scaling the integer lattice into a Delone set $A = \{|a|_K \|a\|^{-1}, a \in \mathbb{Z}^d\}$, where $\|\cdot\|$ is the Euclidean metric. This does not, however, mean that the quantity $M_K(\mu)$ that arises in the study of the Falconer conjecture cannot be bounded. The following result is proved in [IR05].

Theorem 3.1. *Let $A = \mathbb{Z}^d$, $d \ge 3$, (or any other lattice), and suppose that ∂K is smooth and has everywhere non-vanishing Gaussian curvature. Let μ be as above. Then $M_K(\mu) < \infty$ for $s > \frac{d}{2}$ for $d = 3$, and $M_K(\mu) < \infty$ for $s \ge \frac{d}{2}$ for $d \ge 4$.*

In two dimensions the same conclusion can be drawn only in the case of the Euclidean metric or its isometries, i.e when $K = B_2$, the unit disk or an ellipse. The question is wide open in the plane for a general K.

We have seen above that the question of bounding $\int_K |\widehat{\mu}(t\omega)|^2 d\omega_K$ is linked to the problem of studying the discrepancy $E(R) = \#\{RK \cap \mathbb{Z}^d\} - |K|R^d$. It is not difficult to see that the question of bounding $M_K(\mu)$ is linked to the problem of bounding the square average

$$\left(\frac{1}{R} \int_R^{2R} |E(t)|^2 dt \right)^{\frac{1}{2}}. \tag{3.2}$$

This connection is explored thoroughly in [IR05] where the reader can also find a variety of the relevant references and the description of previous work.

We have already pointed out one connection between the estimates we discuss in this paper and discrete problems in Theorem 1.4 above. We now point out another connection by showing that the two dimensional case of Theorem 1.6 follows from the following incidence theorem due to Szemeredi and Trotter ([ST83]), combined with a geometric lemma due to Iosevich and Laba ([IL04]).

Theorem 3.2. *Let X be a set of n points in the plane, and let Y be a set of m curves satisfying the property that any two curves intersect at at most 2 points of X, and that a pair of points in X have at most one curve in Y running through it. Then the number of incidences between elements of X and elements of Y (number of pairs $(p, l) \in X \times Y : p \in l$) is*

$$\lesssim (n + m + (nm)^{\frac{2}{3}}). \tag{3.3}$$

Lemma 3.3. *Let $\mathcal{C}$ is a strictly convex curve in the plane. Then for any $\alpha > 0$ and $x \in \mathbb{R}^2$, then*

$$\#\{\mathcal{C} \cap \alpha\mathcal{C} + x\} \leq 2. \tag{3.4}$$

Combining Theorem 3.2 and Lemma 3.3, we can deduce Theorem 1.6 as follows. Let P be a convex polygon with N lattice vertices. Run a strictly convex curve through the vertices of P such that there are no other lattice points on this curve. Create a family of curves Y obtained by translating this curve by every lattice point inside P. It follows that

$$m = \#Y \approx |P|. \tag{3.5}$$

Let X be the set of lattice points contained in the union of all the translates of P considered above. It follows that

$$n = \#X \approx |P|. \tag{3.6}$$

By Theorem 3.2 and Lemma 3.3, the number of incidences

$$I \lesssim |P|^{\frac{4}{3}}. \tag{3.7}$$

It follows that the number of lattice points on P is $\lesssim \frac{|P|^{\frac{4}{3}}}{|P|} = |P|^{\frac{1}{3}}$, which proves Theorem 1.6 in the case $d = 2$.

The advantage of dealing with the integer lattice lies in the fact that one can use the Poisson Summation Formula, which we quote for the reader's convenience.

Theorem 4.1. *(Poisson Summation Formula). Let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Let L be a lattice in $\mathbb{R}^d$. Then*

$$\sum_{a \in L} f(a) = \sum_{a^* \in L^*} \widehat{f}(a^*), \tag{4.1}$$

whenever both sides converge, where

$$L^* = \{a^* \in \mathbb{R}^d : a \cdot a^* \in \mathbb{Z} \ \forall \ a \in L\}, \tag{4.2}$$

and

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx. \tag{4.3}$$

Using Theorem 4.1 we see that

$$\widehat{f}(\xi) = \widehat{\phi}(q^{-\frac{2}{s}}\xi) q^{-2} \sum_{a \in A} e^{2\pi i a \cdot \frac{\xi}{q}} \phi(a/q) = \widehat{\phi}(q^{-\frac{2}{s}}\xi) \sum_{a \in A} \widehat{\phi}(qa - \xi). \tag{4.4}$$

Let $B_{\frac{1}{q}}(a)$ denote the ball of radius $\frac{1}{q}$ centered at a. Since $|\widehat{\phi}(\xi)| \leq C_N(1+|\xi|)^{-N}$ for any $N > 0$, or alternatively one has the bound $|\widehat{\phi}(\xi)| \lesssim \sum_{j=1}^{\infty} 2^{-Mj} \chi_{B_d}(2^{-j}\xi)$ for some $M > 0$ (where χ_{B_d} denotes the characteristic function of the unit ball), it follows that

$$\int_{\partial K} |\widehat{f}(t\omega)|^2 d\omega_K \approx |\widehat{\phi}_0(q^{-\frac{2}{s}}t)|^2 \int_{\partial K} \sum_{a, a' \in A} \chi_{B_{\frac{1}{q}}(a)}(t\omega) \chi_{B_{\frac{1}{q}}(a')}(t\omega) d\omega_K$$

$$= |\widehat{\phi}_0(q^{-\frac{2}{s}}t)|^2 \int_{\partial K} \sum_{a \in A} \chi_{B_{\frac{1}{q}}(a)}(t\omega) d\omega_K$$

$$\approx |\widehat{\phi}_0(q^{-\frac{2}{s}}t)|^2 \sum_{\frac{t}{q} \leq ||a||_K \leq \frac{t}{q} + \frac{1}{q}} \omega_K \left\{ \omega \in \partial K : \frac{t}{q}\omega_K \in B_{\frac{1}{q}}(a) \right\}$$

$$\approx |_0(q^{-\frac{2}{s}}t)|^2 \sum_{\frac{t}{q} \leq ||a||_K \leq \frac{t}{q} + \frac{1}{q}} \frac{1}{t} \approx |\widehat{\phi}_0(q^{-\frac{2}{s}}t)|^2 \frac{1}{t} \sqrt{\frac{t}{q}}. \tag{4.5}$$

Here, $\phi(\xi) = \phi_0(|\xi|)$, since ϕ is radial, by assumption. We have omitted the standard technical element in the derivation of (4.5), which enables one (using dyadic localization and the Schwartz decay of $\widehat{\phi}$) to essentially assume that $\widehat{\phi}(qa-\xi)$ is supported in a ball of radius $\frac{1}{q}$ around a. In addition, one can always assume that $\widehat{\phi} \geq 0$ by choosing the test function ϕ as a convolution. See [IR05] for more detail on similar estimates involving the Schwartz decay and the issue of endpoints, where exact formulae involving the Bessel functions were used.

Also by the Schwartz decay of $\widehat{\phi}$, we are essentially operating under the restriction $t \lesssim q^{\frac{2}{s}}$. Thus we may choose q as small as $t^{\frac{s}{2}}$. It follows that

$$\int_{\partial K} \left|\widehat{f}(t\omega)\right|^2 d\omega_K \gtrsim t^{-\frac{1}{2}-\frac{s}{4}}, \tag{4.6}$$

as claimed. This completes the proof of Theorem 1.3.

To prove Theorem 1.5, we apply the same argument and arrive at the statement that

$$\int_{\partial K} \left|\widehat{f}(t\omega)\right|^2 d\omega_K \approx \left|\widehat{\phi}_0(q^{-\frac{d}{s}}t)\right|^2 \sum_{\frac{t}{q} \leq ||a||_K \leq \frac{t}{q}+\frac{1}{q}} \frac{1}{t^{d-1}}$$

$$\approx \left|\widehat{\phi}_0(q^{-\frac{2}{s}}t)\right|^2 \frac{1}{t^{d-1}} \left(\frac{t}{q}\right)^{d-2+\gamma}. \tag{4.7}$$

Choosing $q \approx t^{\frac{s}{d}}$ we obtain the conclusion of Theorem 1.5 and complete the proof.

We now prove Lemma 1.2. We have

$$I_s(\mu) = q^{-2d} q^{\frac{2d^2}{s}} \sum_{a,a'} \phi(a/q)\phi(a'/q) \int\int \phi(q^{\frac{d}{s}}(x-a/q))\phi(q^{\frac{d}{s}}(y-a'/q))|x-y|^{-s} dxdy$$

$$= q^{-2d} q^{\frac{2d^2}{s}} \sum_{a} \phi^2(a) \int\int \phi(q^{\frac{d}{s}}(x-a/q))\phi(q^{\frac{d}{s}}(y-a/q))|x-y|^{-s} dxdy \tag{4.8}$$

$$+q^{-2d} q^{\frac{2d^2}{s}} \sum_{a \neq a'} \phi(a/q)\phi(a'/q) \int\int \phi(q^{\frac{d}{s}}(x-a/q))\phi(q^{\frac{d}{s}}(y-a'/q))|x-y|^{-s} dxdy = I + II.$$

Now,

$$I \approx q^{-2d} q^{\frac{2d^2}{s}} \sum_{a \in A_q} \int_{B_{q^{-\frac{d}{s}}}(a/q)} \int_{B_{q^{-\frac{d}{s}}}(a'/q)} |x-y|^{-s} dxdy$$

$$\approx q^{-2d} q^{\frac{2d^2}{s}} \sum_{a \in A_q} q^{-\frac{d^2}{s}} \int_{B_{q^{-\frac{d}{s}}}((0,\dots,0))} |x|^{-s} dx$$

$$\lesssim q^{-2d} q^{\frac{d^2}{s}} q^d \left(q^{-\frac{d}{s}}\right)^{d-s} \lesssim 1. \tag{4.9}$$

On the other hand,

$$II \approx q^{-2d} q^{\frac{2d^2}{s}} \sum_{a \neq a'} \int_{B_{q^{-\frac{d}{s}}}(a/q)} \int_{B_{q^{-\frac{d}{s}}}(a'/q)} |x-y|^{-s} dx dy$$

$$\approx q^{-2d} q^{\frac{2d^2}{s}} q^s \sum_{a \neq a'} |a-a'|^{-s} q^{-\frac{2d^2}{s}} \approx q^{-2d+s} \sum_{a \neq a'} |a-a'|^{-s}$$

$$\approx q^{-2d+s} \sum_{j=1}^{\log(q)} 2^{-js} \sum_{|a-a'| \approx 2^j} 1 \approx q^{-2d+s} \sum_{j=1}^{\log(q)} 2^{-j(2d-s)} \lesssim 1. \tag{4.10}$$

This completes the proof of Lemma 1.2.

Section 5: Proof of Theorem 2.1 and Theorem 2.2

Using (4.7) we see that

$$\int_{S^{d-1}} |\widehat{f}(t\omega)|^2 d\omega \approx |\widehat{\phi}_0(t)|^2 \sum_{\frac{t}{q} \leq |a| \leq \frac{t}{q} + \frac{1}{q}} \frac{1}{t^{d-1}}. \tag{5.1}$$

Using (1.12) and the Pythagorean theorem,

$$\sum_{\frac{t}{q} \leq |a| \leq \frac{t}{q} + \frac{1}{q}} 1 \lessapprox \left(\frac{t}{q}\right)^{d-2}, \tag{5.2}$$

if $s \geq \frac{d}{2}$, and

$$\sum_{\frac{t}{q} \leq |a| \leq \frac{t}{q} + \frac{1}{q}} 1 \lessapprox \left(\frac{t}{q}\right)^{d-2} \frac{t}{q^2}, \tag{5.3}$$

if $s < \frac{d}{2}$. As before, $\lessapprox$ may be replaced by $\lesssim$ for $d \geq 4$.

It follows that if $s \leq \frac{d}{2}$, the right hand side of (5.1) is $\lessapprox t^{-s}$ for $d \leq 3$, and $\lesssim t^{-s}$ if $d \geq 4$. If $s > \frac{d}{2}$, we obtain the bound in terms of $t^{-s} t^{-1+\frac{2s}{d}}$ as claimed. This completes the proof of Theorem 2.1.

To prove Theorem 2.2 we use (4.7) again to see that

$$\int_{\partial K} |\widehat{f}(t\omega)|^2 d\omega_K \approx |\widehat{\phi}_0(t)|^2 \sum_{\frac{t}{q} \le ||a||_K \le \frac{t}{q} + \frac{1}{q}} \frac{1}{t^{d-1}}. \tag{5.4}$$

Arguing as before, if $s \ge \frac{d}{2}$,

$$\int_{\partial K} |\widehat{f}(t\omega)|^2 d\omega_K \lesssim |\widehat{\phi}_0(t)|^2 \frac{1}{t^{d-1}} \left(\frac{t}{q}\right)^{d-2+\gamma_d}, \tag{5.5}$$

and if $s < \frac{d}{2}$,

$$\int_{\partial K} |\widehat{f}(t\omega)|^2 d\omega_K \lesssim |\widehat{\phi}_0(t)|^2 \frac{1}{t^{d-1}} \left(\frac{t}{q}\right)^{d-2+\gamma_d} \frac{t}{q^2}. \tag{5.6}$$

The expression in (5.5) is

$$\lesssim t^{-s} t^{-1+\frac{2s}{d}} t^{\gamma_d (1-\frac{s}{d})}, \tag{5.7}$$

and the expression in (5.6) is

$$\lesssim t^{-s} t^{\gamma_d (1-\frac{s}{d})}, \tag{5.8}$$

and the proof is completed by plugging in the value of γ_d from Theorem 2.3.

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Department of Mathematics, University of Missouri-Columbia, Columbia Missouri 65211 USA

E-mail address: `iosevich @ math.missouri.edu`

Department of Mathematics, University of Bristol, Bristol, England
E-mail address: `m.rudnev @ bris.ac.uk`