DISCRETE SUBSETS OF $\mathbb{R}^2$ AND THE ASSOCIATED DISTANCE SETS

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Abstract. We prove that a well-distributed subset of $\mathbb{R}^2$ can have a distance set Δ with $\#(\Delta \cap [0,N]) \leq CN^{3/2-\epsilon}$ only if the distance is induced by a polygon K. Furthermore, if the above estimate holds with $\epsilon = 1/2$, then K can have only finitely many sides.

Basic definitions

Separated sets. We say that $S \subset \mathbb{R}^d$ is separated if there exists $c_{separation} > 0$ such that $||x-y|| \geq c_{separation}$ for every $x, y \in S$, $x \neq y$. Here and throughout the paper, $||x|| = \sqrt{x_1^2 + \cdots + x_d^2}$ is the standard Euclidean distance.

Well-distributed sets. We say that $S \subset \mathbb{R}^d$ is well-distributed if there exists a $C_{density} > 0$ such that every cube of side-length $C_{density}$ contains at least one element of S.

K-distance. Let K be a bounded convex set, symmetric with respect to the origin. Given $x, y \in \mathbb{R}^d$, define the K-distance, $||x-y||_K = \inf\{t : x-y \in tK\}$.

K-distance sets. Let $A \subset \mathbb{R}^d$. Define $\Delta_K(A) = \{||x-y||_K : x, y \in A\}$, the K-distance set of A.

K-well-distributed sets. We say that $S \subset \mathbb{R}^d$ is K-well-distributed if there is a constant $C_{K,density}$ such that every translate of $C_{K,density}K$ contains at least one element of S.

Notation: $A \lessapprox B$, with respect to a large parameter N, means that there exists a positive constant C_ϵ such that $A \leq C_\epsilon N^\epsilon B$ for any $\epsilon > 0$. Similarly, $A \lesssim B$ means that there exists a $C > 0$ such that $A \leq CB$, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

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Distance sets play an important role in combinatorics and its applications to analysis and other areas. See, for example, [PA95] and the references contained therein. Perhaps the most celebrated classical example is the Erdős Distance Problem, which asks for the smallest possible cardinality of $\Delta_{B_2}(A)$ if $A \subset \mathbb{R}^2$ has cardinality $N < \infty$ and B_2 is the Euclidean unit disc. Erdős conjectured that $\#\Delta_{B_2}(A) \gtrapprox N$. The best known result to date in two dimensions is due to Solymosi and Tóth, who prove in [ST01] that $\#\Delta_{B_2}(A) \gtrsim N^{\frac{6}{7}}$. For a survey of higher dimensional results see [PA95] and the references contained therein. For applications of distance sets in analysis see e.g. [IKP99], where distance sets are used to study the question of existence of orthogonal exponential bases.

The situation changes drastically if the Euclidean disc B_2 is replaced by a convex planar set with a "flat" boundary. For example, suppose that $K = [-1,1]^2$, which corresponds to the "taxi-cab" or $l^1(\mathbb{R}^2)$ distance. Let $A = \{m \in \mathbb{Z}^2 : 0 \leq m_i \leq N^{\frac{1}{2}}\}$. Then $\#A \approx N$, and it is easy to see that $\#\Delta_K(A) \approx N^{\frac{1}{2}}$, which is much less than what is known to be true for the Euclidean distance. In fact, it follows from an argument due to Erdős ([Erd46]; see also [I01]) that the estimate $\#\Delta_K(A) \gtrsim N^{\frac{1}{2}}$ holds for any K.

The example in the previous paragraph shows that the properties of the distance set very much depend on the underlying distance. One way of bringing this idea into sharper focus is the following. Let S be a separated subset of $\mathbb{R}^2$, α-dimensional in the sense that

$$\#(S \cap [-N,N]^2) \approx N^{\alpha}. \tag{0.1}$$

If Erdős' conjecture holds, then $\#\Delta_{B_2}(S \cap [-N,N]^2) \gtrapprox N^{\alpha}$; in particular, if $\alpha > 1$ then $\Delta_{B_2}(S)$ cannot be separated. This formulation expresses the Erdős Distance Conjecture in the language of the Falconer Distance Conjecture (see e.g. [Wolff02]) which says that if a compact set $E \subset \mathbb{R}^2$ has Hausdorff dimension $\alpha > 1$, then $\Delta_{B_2}(E)$ has positive Lebesgue measure. On the other hand, we have seen above that if $K = [0,1]^2$, the distance set $\Delta_K(S)$ can be separated for a 2-dimensional set S (e.g. $S = \mathbb{Z}^2$).

The purpose of this paper is to address the following question: if S is a well-distributed subset of $\mathbb{R}^d$, for what K can $\Delta_K(S)$ be separated? We conjecture that this is only possible if K is a polygon with finitely many sides. This is indeed what we prove in two dimensions; furthermore, a weaker conclusion that ∂K is the closure of a union of (possibly infinitely many) line segments remains true if the separation assumption is replaced by a weaker condition on the density of $\#\Delta_K(S)$.

Theorem 0.1. *Let S be well-distributed subset of $\mathbb{R}^2$, and let $\Delta_{K,N}(S) = \Delta_K(S) \cap [0,N]$.*

(i) Assume that $\underline{\lim}_{N\to\infty} \#\Delta_{K,N}(S) \cdot N^{-3/2} = 0$. Then K is a polygon (possibly with infinitely many sides).

(ii) If moreover $\#\Delta_{K,N}(S) \lesssim N^{1+\alpha}$ for some $0 < \alpha < 1/2$, then the number of sides of K whose length is greater than δ is bounded[1] by $C\delta^{-2\alpha}$.

[1] The trivial estimate would be $C\delta^{-1}$.

(iii) If $\#\Delta_{K,N}(S) \lesssim N$ (in particular, this holds if $\Delta_K(S)$ is separated), then K is a polygon with finitely many sides.

The assumptions of Theorem 0.1 can be weakened slightly in a technical way, see Lemmas 1.1–1.2 below; this, however, does not improve the values of the exponents in the theorem.

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Proof of Theorem 0.1

Let $S \subset \mathbb{R}^2$ be a well-distributed set. Rescaling if necessary, we may assume that S is K-well-distributed with $C_{K,density} < 1/2$. We identify $\mathbb{R}^2$ with $\mathbb{C}$ via $z = re^{i\theta}$, and denote by C_{θ_1,θ_2} the cone $\{re^{i\theta} : \theta_1 < \theta < \theta_2\}$. We also write $\Gamma = \partial K$. A *line segment* will always be assumed to have non-zero length.

Theorem 0.1 is an immediate consequence of Lemmas 1.1 and 1.2: it suffices to observe that the assumptions of Theorem 0.1 (i), (ii), (iii) imply those of Lemma 1.1, Lemma 1.2(ii) and (i), respectively. Let $\lambda(N) = \#\Delta_K(S) \cap (N-2, N+2))$, and let $L(N) = \min\{\lambda(n) : N \leq n \leq kN\}$ for some $k > 1$.

Lemma 1.1. *Let S be as above. Assume that $\underline{\lim}_{N\to\infty} L(N)N^{-1/2} = 0$. Then for any $\theta_1 < \theta_2$ the curve $\Gamma \cap C_{\theta_1,\theta_2}$ contains a line segment.*

Lemma 1.2. *Let S be as above.*

(i) If $L(N) \lesssim 1$, then Γ may contain only a finite number of line segments such that no two of them are collinear.

(ii) If $L(N) \lesssim N^\alpha$ for some $0 < \alpha < 1/2$, then the number of sides of K whose length is greater than δ is bounded by $C\delta^{-2\alpha}$.

We now prove Lemmas 1.1 and 1.2. The main geometrical observation is contained in the next lemma.

Lemma 1.3. *Let $\Gamma = \partial K$, where $K \subset \mathbb{R}^2$ is convex. Let $\alpha > 0$, $x \in \mathbb{R}^2$, $x \neq 0$.*

(i) If $\Gamma \cap (\alpha\Gamma + x)$ contains three distinct points, at least one of these points must lie on a line segment contained in Γ.

(ii) $\Gamma \cap (\alpha\Gamma + x)$ cannot contain more than 2 line segments such that no two of them are collinear.

We will first prove Lemmas 1.1 and 1.2, assuming Lemma 1.3; the proof uses a variation on an argument of Moser [Mo]. The proof of Lemma 1.3 will be given later in this section.

Fix 2 points $P, Q \in S$; translating S if necessary, we may assume that $P = -Q$ and that $d_K(P, 0) < 1$. Let

$$A_N = \{x \in \mathbb{R}^2 : d_K(x, 0) \in (N-1, N+1)\}. \tag{1.1}$$

For all N large enough (depending on θ_1, θ_2) the set $A_N \cap C_{\theta_1,\theta_2}$ contains at least $cN(\theta_2 - \theta_1)$ points of S.

Observe that all of the distances

$$d_K(s, P),\ d_K(s, Q) :\ s \in S \cap A_N \tag{1.2}$$

lie in $(N-2, N+2)$, hence the number of distinct distances in (1.2) is bounded by $\lambda(N)$.

Proof of Lemma 1.1. We may assume that $0 < \theta_2 - \theta_1 < \pi/2$. Fix θ_1', θ_2' so that $\theta_1 < \theta_1' < \theta_2' < \theta_2$, and let $C = C_{\theta_1,\theta_2}$, $C' = C_{\theta_1',\theta_2'}$. Then for all N large enough we have

$$C' \cap A_N \subset (C + P) \cap (C + Q) \tag{1.3}$$

and

$$\#(S \cap C' \cap A_N) \geq cN(\theta_2' - \theta_1'). \tag{1.4}$$

Let

$$\{d_1, \dots, d_l\} = \{d_K(s, P) :\ s \in S \cap A_N\},\ \{d_1', \dots, d_{l'}'\} = \{d_K(s, Q) :\ s \in S \cap A_N\}, \tag{1.5}$$

where $d_j \neq d_k$, $d_j' \neq d_k'$ if $j \neq k$. Then $l, l' \leq \lambda(N)$ (see (1.2)). We have

$$S \cap C' \cap A_N \subset S \cap C' \cap \bigcup_{i,j} \Gamma_i \cap \Gamma_j', \tag{1.6}$$

where $\Gamma_i = d_i\Gamma + P$, $\Gamma_j' = d_j'\Gamma + Q$. By our assumption, we may choose N so that $N \geq 10\lambda^2(N)c^{-1}(\theta_2' - \theta_1')^{-1}$. Then there are i, j such that $\#(S \cap C' \cap \Gamma_i \cap \Gamma_j') \geq 10$. It follows from Lemma 1.3(i) that at least one of the points in $S \cap C' \cap \Gamma_i$ lies on a line segment I contained in $C' \cap \Gamma_i$. By (1.3), I is contained in $d_i\Gamma \cap C$, hence $\Gamma \cap C$ contains the line segment $d_i^{-1}I$.

Proof of Lemma 1.2. Suppose that $L(N) \lesssim N^\alpha$ for some $0 \leq \alpha < 1/2$, and that Γ contains line segments $I_1, \dots, I_M$ of length at least $\delta > 0$, all pointing in different directions.

We will essentially continue to use the notation of the proof of Lemma 1.1. Choose P, Q as above, and let C_m denote cones $C_m = C_{\theta_m,\theta_m'}$ such that $\theta_m < \theta_m'$, $\Gamma \cap C_m \subset I_m$, and $\theta_m' - \theta_m \geq c\delta$. Let also $C_m' \subset C_m$ be slightly smaller cones. Our assumption on $L(N)$ implies that we may choose $N \approx \delta^{-1}$ so that $\lambda(N) \leq cN^\alpha$, each sector $C_m' \cap A_N$ contains at least 10 points of S, and

$$A_N \cap C_m' \subset (C_m + P) \cap (C_m + Q). \tag{1.7}$$

Let also $d_i, d_j', \Gamma_i, \Gamma_j'$ be as above. Then for each m

$$S \cap A_N \cap C_m' \subset \bigcup_{i,j} \Gamma_i \cap \Gamma_j' \cap C_m'. \tag{1.8}$$

If N is large enough, $\Gamma_i \cap C'_m \subset C_m$ and $\Gamma'_j \cap C'_m \subset C_m$, hence the set on the right is a union of line segments parallel to I_m. It must contain at least one such segment, since the set on the left is assumed to be non-empty. Therefore the set

$$\bigcup_{i,j} \Gamma_i \cap \Gamma'_j \tag{1.9}$$

contains at least M line segments pointing in different directions, one for each m. But on the other hand, by Lemma 1.3(ii) any $\Gamma_i \cap \Gamma'_j$ can contain at most two line segments that do not lie on one line. It follows that the set in (1.9) contains at most $2\lambda^2(N) \leq c^2 N^{2\alpha}$ line segments in different directions, hence $M \leq 2c^2 N^{2\alpha}$. Since Γ can contain at most two parallel line segments that do not lie on one line, the number of line segments in Lemma 1.2 is bounded by $4c^2 N^{2\alpha}$ as claimed.

Proof of Lemma 1.3. We first prove part (i) of the lemma. Suppose that P_1, P_2, P_3 are three distinct points in $\Gamma \cap (\alpha\Gamma + x)$. We may assume that they are not collinear, since otherwise the conclusion of the lemma is obvious. We have $P_1, P_2, P_3 \in \Gamma$ and $P'_1, P'_2, P'_3 \in \Gamma$, where $P'_j = \alpha^{-1}(P_j - x)$. Let T and T' denote the triangles $P_1P_2P_3$ and $P'_1P'_2P'_3$, and let K' be the convex hull of $T \cup T'$. Since $K' \subset K$ and all of the points P_j, P'_j lie on $\Gamma = \partial K$, they must also lie on $\partial K'$.

Observe that $\partial K'$ consists of some number of the edges of the triangles T, T' and at most 2 additional line segments. If $\partial K'$ contains at most one of P_iP_j and $P'_iP'_j$ for each i, j, K' is a polygon with at most 5 edges, hence at most 5 vertices. Thus if the 6 points P_j, P'_j lie on $\partial K'$, at least three of them must be collinear, and one of them must be P_j for some j (otherwise the P'_j would be collinear). If these three points are distinct, then Γ contains the line segment joining all of them, and we are done. Suppose therefore that they are not distinct. It suffices to consider the cases when $P_1 = P'_1$ or $P_1 = P'_2$. If $P_1 = P'_1$, then we must have $\alpha \neq 1$ and P_1, P_2, P'_2 are distinct and collinear; if $P_1 = P'_2$, then P'_1, P_1, P_2 are distinct and collinear. Thus at least three of the points P_j, P'_j are distinct and collinear, and we argue as above.

It remains to consider the case when $\partial K'$ contains both P_iP_j and $P'_iP'_j$ for some i, j. The outward unit normal vector to P_iP_j and $P'_iP'_j$ is the same, hence all four points P_i, P_j, P'_i, P'_j are collinear, at least three of them are distinct, and Γ contains a line segment joining all of them.

Part (ii) of the lemma is an immediate consequence of the following. Let (x_1, x_2) denote the rectangular coordinates in the plane.

Lemma 1.4. *Let I be a line segment contained in $\Gamma \cap (\alpha\Gamma + u)$, where $u = (c, 0)$.*

(i) If $\alpha = 1$, then I is parallel to the x_1-axis.

(ii) If $\alpha \neq 1$, then the point $(\frac{c}{1-\alpha}, 0)$ lies on the straight line containing I.

Proof of Lemma 1.4. Part (i) is obvious; we prove (ii). If I lies on the line $x_2 = ax_1 + b$, then so does $\alpha I + u$. But on the other hand $\alpha I + u$ lies on the line

$$x_2 = \alpha\left(a\frac{x_1 - c}{\alpha} + b\right) = ax_1 - ac + \alpha b. \tag{1.10}$$

It follows that $b = \alpha b - ca$, hence $-\frac{b}{a} = \frac{c}{1-\alpha}$. But $-b/a$ is the x_1-intercept of the line in question.

Similarly, if I lies on the line $x_1 = b$, then $\alpha I + u$ lies on the lines $x_1 = b$ and $x_1 = \alpha b + c$, hence $b = \frac{c}{1-\alpha}$.

To finish the proof of Lemma 1.3 (ii), it suffices to observe that in both of the cases (i), (ii) of Lemma 1.4 the boundary of a convex body cannot contain three such line segments if no two of them lie on one line.

References

[Erdos46] P. Erdős, *On sets of distances of n points*, Amer. Math. Monthly **53** (1946), 248-250.

[I01] A. Iosevich, *Curvature, combinatorics and the Fourier transform*, Notices of the AMS **46 no. 6** (2001), 577-583.

[IKP99] A. Iosevich, N. Katz, and S. Pedersen, *Fourier basis and the Erdős distance problem*, Math. Research Letters **6** (1999).

[Mo] L. Moser, *On the different distances determined by n points*, Amer. Math. Monthly **59** (1952), 85-91.

[PA95] J. Pach and and P. Agarwal, *Combinatorial Geometry*, Wiley-Interscience Series (1995).

[ST01] J. Solymosi and Cs. D. Tóth, *Distinct distances in the plane*, Discrete Comput. Geom. **25** (2001), 629-634.

[Wolff02] T. Wolff, *Lecture notes in harmonic analysis (revised)* (2002).

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