HW6 Soln

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9) u harmonic $\Rightarrow \exists\ f: \mathbb{C} \to \mathbb{C}$ analytic

such that $u(z) = \mathrm{Re}(f(z))$

Define $g(z) = e^{-f(z)}$

Then $|g(z)| = e^{-u(z)} \le 1$ as $u \ge 0$

Hence g is bounded and entire

$\therefore$ g is constant by Liouville's Thm. $\boxed{\checkmark}$

10)

① If $g \equiv 0$, ✓

② $g \not\equiv 0$, then $\exists$ a ball $B \subset G$

s.t $g(a) \neq 0$, $\forall\ a \in B$.

Observe that $\bar{f} = (\bar{f} g) \cdot \frac{1}{g}$ is analytic on

B, if we let $\begin{cases} f(z) = u + iv \\ \bar{f}(z) = u - iv \end{cases}$

Then $\begin{cases} u_x = v_y = -v_y \\ u_y = -v_x = v_x \end{cases}$ on B

$\Rightarrow$ f constant on B

$\Rightarrow$ f constant on G by Cor. 3.8 $\boxed{\checkmark}$

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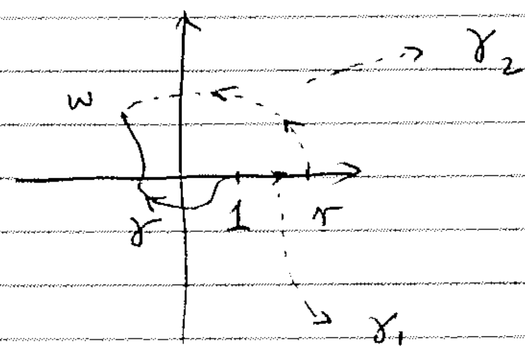
3) $$\int_\gamma \frac{p'(z)}{p(z)}\,dz = \int_\gamma \sum_{j=1}^{n} \frac{dz}{z-z_j}$$

where z_j's are the n roots of p.

$$= 2\pi i \sum_{j=1}^{n} n(\gamma, z_j) = 2\pi i n$$

as $n(\gamma, z_j) = 1$, $j = 1, 2, \cdots n$.

$\square$

4) 

Define $\begin{cases} \gamma_1(t) = 1 + (r-1)t, & t \in [0,1] \\ \gamma_2(t) = re^{i\theta t}, & t \in [0,1] \end{cases}$

We see $\gamma - \gamma_2 - \gamma_1$ forms a closed rect. path in $\mathbb{C} - \{0\}$

By prop 4.1

$$\frac{1}{2\pi i}\int_{\gamma-\gamma_2-\gamma_1} \frac{dz}{z} = k$$

$$\Rightarrow \int_\gamma \frac{dz}{z} = \int_{\gamma_1} \frac{dz}{z} + \int_{\gamma_2} \frac{dz}{z} + 2\pi i k$$

$$= \log r + i\theta + 2\pi i k \qquad \boxtimes$$

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4) ~~With the setting of 5.7 Cauchy's Thm,~~

~~it suffices to show Let a ∈ G − {~~

Let G be an open set in $\mathbb{C}$,

$f: G \to \mathbb{C}$ analytic and γ a closed rect.

curve in G s.t. $n(\gamma; z) = 0 \quad \forall z \in \mathbb{C} - G$

Let $a \in G - \{\gamma\}$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz - n(\gamma; a) f(a)$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(a)}{z-a}\,dz$$

Define $g(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a}, & z \neq a \\ f'(a), & z = a \end{cases}$

Easy to see g is analytic on G.

By Cauchy's Thm

$$\int_\gamma g(z)\,dz = 0$$

Hence the result follows immediately.

$\square$

5) Let $a \in \mathbb{C} - \{\gamma\}$. $\exists$ $R>0$ s.t

$\{\gamma\} \subset B(a;R)$.

Hence $n(\gamma;z) = 0 \quad \forall z \in \mathbb{C} - B(a;R)$

by thm 4.4.

From Cor. 5.9,

$$f^{(k)}(a)\, n(\gamma;a) = \frac{k!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{k+1}}\,dz$$

let $f(z) = 1$, $k \to n-1$

$$\Rightarrow f^{(n-1)}(a)\, n(\gamma;a) = \frac{(n-1)!}{2\pi i}\int_\gamma \frac{1}{(z-a)^n}\,dz = 0$$

for $n \geq 2$.

$$\Rightarrow \int_\gamma \frac{1}{(z-a)^n}\,dz = 0 \quad \text{for } n \geq 2,\ a \notin \{\gamma\}$$

$\square$

7) Let $f(z) = z^n$, $R > 1$.

We have $\{\gamma\} \subset B(1, R)$.

By Cor. 5.9 . Since $n(\gamma ; z) = 0$ $\forall z \in \mathbb{C} - B(1, R)$, we have

$$\int_\gamma \left(\frac{z}{z-1}\right)^n dz = \int_\gamma \frac{f(z)}{(z-1)^n} dz$$

$$= \frac{2\pi i}{(n-1)!} f^{(n-1)}(1) \; n(\gamma ; 1)$$

$$= \frac{2\pi i}{(n-1)!} \cdot n! \cdot 1$$

$$= 2\pi i n$$