Last time: · The Double Dual, $(V^*)^*$

· For V finite dimensional, $V \to (V^*)^*$ an isomorphism

$\alpha \to L_\alpha$ "natural isomorphism"

$L_\alpha : V^* \to \mathbb{F}$, $L_\alpha(f) = f(\alpha)$

S subset of V, S^0 subspace of V^*, $(S^0)^0$ subspace of $(V^*)^* \cong V$

$S^{00} = \text{span}(S)$ subspace of V

Ex: $V = \mathbb{R}^2$ $\quad S = \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}$ $\quad S^{00} = ?$

Def: For V a vector space, a **hyperspace** W in V is a maximal proper subspace of V

Theorem 19: If $f \in V^*$ (a linear functional) that is non-zero then nullspace(f) is a hyperspace
Conversely, every hyperspace is the nullspace of some functional

Ex: $V = \{ \{a_i\}_{i=1}^{\infty} : \sum_{i=1}^{\infty} a_i^2 < \infty \}$

Let $f(\{a_i\}) = a_1$

Proof:

Lemma: Let $f, g \in V^*$, then g is a scalar multiple of f iff nullspace of g contains the nullspace of f

$$g = cf \iff (f(\alpha) = 0 \Rightarrow g(\alpha) = 0)$$

Theorem 20: Let $g, f_1, \ldots, f_r \in V^*$, null spaces $N, N_1, \ldots, N_r$

$$g \text{ is a linear combination of } f_i \iff N \text{ contains } \bigcap_{i=1} N_i$$

The Transpose

Let V, W vectorspaces over $\mathbb{F}$, $T: V \to W$ linear
$g: W \to \mathbb{F}$, $g \in W^*$

Then $f := g \circ T : V \to W \to \mathbb{F}$ is composition of linear maps, so linear (Theorem 6)

$f \in V^*$

Define function $T^t: W^* \to V^*$
$T^t(g) = g \circ T$

<u>Theorem 21</u>: Let V, W be vectorspaces over $\mathbb{F}$. For each $T \in \mathcal{L}(V, W)$ there exists a unique $T^t \in \mathcal{L}(W^*, V^*)$ called the <u>transpose</u> such that

$\forall g \in W^*, \alpha \in V$ $\quad T^t(g)(\alpha) = g(T(\alpha))$

Ex: $\frac{d}{dx} : \mathcal{P} \longrightarrow \mathcal{P}$ $\qquad \varphi \in \mathcal{P}^*$

$\varphi(p) = p(3)$

$\frac{d}{dx}^t(\varphi)$

Ex: $T : \mathbb{R}^3 \to \mathbb{R}^2$ $\qquad T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+z \\ y+z \end{pmatrix}$ $\qquad \varphi \in (\mathbb{R}^2)^*$ $\quad \varphi\begin{pmatrix} x \\ y \end{pmatrix} = x+y$

$T^t(\varphi)$

$T : V \to W$

nullspace$(T^t) =$

Range? If $f \in \mathcal{R}(T^t)$ $\exists g \in W^*$ s.t. $f = T^t g$

Theorem 22: $T: V \to W$, then

i) nullspace$(T^t) = (\mathcal{R}(T))^0$

If V, W f.d.

ii) $\text{rank}(T^t) = \text{rank}(T)$

iii) $\mathcal{R}(T^t) = (\text{nullspace } T)^0$

Theorem 23: Let V, W be finite dimensional v.s. over $\mathbb{F}$

V: basis $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ dual basis $\mathcal{B}^* = \{f_1, \dots, f_n\}$

W: basis $\mathcal{B}' = \{\beta_1, \dots, \beta_m\}$ dual basis $\mathcal{B}'^* = \{g_1, \dots, g_n\}$

$T: V \to W$, A matrix of T relative to $\mathcal{B}, \mathcal{B}'$
C matrix of T^t relative to $\mathcal{B}^*, \mathcal{B}'^*$

Then $C_{ij} = A_{ji}$

Proof:

Definition: Let A be an $m \times n$ matrix over $\mathbb{F}$. The **transpose** of A is A^t, the $n \times m$ matrix $(A^t)_{ij} = A_{ji}$ (8)

$$A = \begin{pmatrix} 5 & -7 \\ 3 & 8 \\ -4 & 2 \end{pmatrix} \qquad A^t =$$

Properties: $(A+C)^t = A^t + C^t \qquad (\lambda A)^t = \lambda A^t \qquad (AC)^t = C^t A^t$

Theorem 24: Let A be $m \times n$ matrix over $\mathbb{F}$. Then rowrank(A) = columnrank(A)

Proof:

So if T is represented by A, $\operatorname{rank}(T) = \operatorname{rowrank}(A) = \operatorname{columnrank}(A)$ 9

This is $\underline{\operatorname{rank}(A)}$

Change of basis, formula can be confirmed by the transpose and coordinate function

$T: V \to V$ represented by $\underset{\alpha_i, f_i}{B}$ and $\underset{\beta_i, g_i}{B'}$, $A = [T]_B$ $C = [T]_{B'}$