A COMBINATORIAL APPROACH TO ORTHOGONAL EXPONENTIALS

A. Iosevich and M. Rudnev

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Abstract. We prove that a symmetric strictly convex set with a smooth boundary in $\mathbb{R}^d$ can possess no more than finitely many orthogonal exponentials, unless $d = 1 \mod (4)$. In the latter case the non-existence theorem is true for a large class of bodies, including a d-disk and its perturbations. Otherwise, any infinite set of the corresponding exponents necessarily turns out to be a subset of some one-dimensional lattice. We provide examples of convex bodies of revolution in the above dimensions, for which infinite sets of orthogonal exponentials exist.

The analysis is reduced to one dimension by studying the distance set of the putative set of exponents with respect to an appropriate metric. A combinatorial principle due to Erdös lies at the heart of the investigation. According to this principle, if the distance set of an infinite set in $\mathbb{R}^d$ is a subset of the integers, then the set itself is a subset of some one-dimensional lattice.

We also provide a physical interpretation for the above phenomena.

Introduction

Physicists have long believed that plane waves are not sufficient to completely describe the quantum billiard problem. However, Berry in [Berry94] argues that despite this insufficiency to represent a general solution for an eigenfunction of the Laplacian in the billiard domain K due to the appearance of evanescent waves, the latter can be still arbitrarily well approximated by superpositions of exponentials.

In the language of analysis, the "insufficiency" of plane waves in this context means that $L^2(K)$ does not possess an orthogonal basis of exponentials. Indeed, in [Kol00] Kolountzakis proves that a non-symmetric convex domain does not possess an orthogonal basis of exponentials. It was proved in [IKT01] that symmetric convex domains with a point of curvature on the boundary do not possess orthogonal bases of exponentials either. Both results are motivated by a paper of Fuglede [Fuglede74] who conjectured that $L^2(K)$ has an orthogonal basis of exponentials if and only if K tiles $\mathbb{R}^d$ by translation. In the context of convex planar domains, this conjecture is proved in [IKT02].

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While at the first glance a problem in analysis, the question of existence of orthogonal exponential bases or sub-bases has a distinctive combinatorial flavor. The definition of orthogonality, combined with the asymptotics of the Fourier transform of the characteristic function of K strongly suggest that the existence of a putative set A such that $\{e^{2\pi i x\cdot a}\}_{a\in A}$ are pairwise orthogonal in $L^2(K)$ is closely tied to the properties of the distance set

$$\Delta(A) = \{\rho_*(a-a') : a, a' \in A,\, a \neq a'\},$$

where ρ_* is the Legendre transform of the Minkowski functional of K. This brings the study of orthogonal exponentials into the realm of combinatorial distance problems pioneered by Erdös. See the treatise [AP] and the references contained therein. See also [Falc87], [Wolff99] and references contained therein where a closely related continuous analog known as the Falconer conjecture is treated.

In this paper we study the case when K is symmetric, has a smooth boundary and is *strictly convex*. By strict convexity we mean that the boundary ∂K is smooth and has everywhere non-vanishing Gaussian curvature. Then for $d \neq 1 \mod (4)$ we prove non-existence in $L^2(K)$ of infinite-dimensional subspaces allowing orthogonal exponential bases for any of the above K. However, in dimensions $5, 9, 13, \ldots$ such sub-spaces may exist for certain K, as the examples that we provide below show. If this is the case, all the corresponding exponents must necessarily be contained in some one-dimensional lattice. In the context of quantum billiards, this means that either one can have only finitely many mutually orthogonal plane wave solutions of the Dirichlet problem for the Helmholtz equation in the interior of K, or (for certain K in the exceptional dimensions above) any infinite set of the corresponding wave vectors must be one-dimensional. Our proof is based on a principle that goes back to Erdös, which says that if a set of Euclidean distances of a planar set is a subset of the integers, then the set itself is contained in a straight line.

From now on we fix K and without loss of generality assume that it has a unit volume with respect to the Lebesgue measure.

Definition. Let A be a subset of $\mathbb{R}^d$ such that the orthogonality relation

$$\widehat{\chi}_K(a-a') = \int_K e^{2\pi i x\cdot(a-a')}dx = \delta_{aa'} \tag{0.1}$$

holds whenever $a, a' \in A$, the right hand side being the Kronecker delta. Then $A = A(K)$ is called a set of *orthogonal exponents* for K.

Definition. A is maximal if for any $a' \in \mathbb{R}^d \setminus A$ there exists $a \in A$ such that $\widehat{\chi}_K(a-a') \neq 0$.

By continuity of $\widehat{\chi}_K$, the set $A(K)$ is separated. Namely, there exists a uniform constant $c = c(K) > 0$ such that $|a-a'| \geq c$, for all non-equal $a, a' \in A$. The notation $|\cdot|$ stands for the Euclidean distance. Indeed, since K is symmetric (0.1) is a cosine transform. Then the constant c is bounded from below in inverse proportionality to the length of the longest diagonal of ∂K, passing through the center. So, if A is a set of orthogonal exponents, it can be sequentially completed to a maximal set, and we will further assume that this is the case. Besides, any translation of A is also a set of orthogonal exponents, so in order to fix A in a certain sense, let's assume that $0 \in A$.

Our main result is the following.

Theorem 0.1. *Let $K \subset \mathbb{R}^d$ be a closed symmetric convex domain of unit volume with a smooth boundary ∂K, such that the Gaussian curvature does not vanish anywhere on ∂K. Then for $d \neq 1 \mod (4)$, any maximal set $A(K)$ of orthogonal exponents for K is finite. Otherwise, either $A(K)$ is finite, or it is a subset of some one-dimensional lattice.*

Fuglede [Fug74] proved non-existence of an infinite set $A(K)$ for a disk in $\mathbb{R}^2$. In the process of revising this manuscript, we became aware of the recent paper [Fug01] extending the result to a disk in $\mathbb{R}^d$ for any $d \geq 2$.

Theorem 0.1 is driven by strict convexity of K which is heavily used throughout the proof. Otherwise, a unit cube in R^d has a basis of orthogonal exponentials, a cylinder has infinitely many of them located on the symmetry axis. A somewhat less obvious example of a convex set with an orthogonal exponential basis is a hexagon in R^2.

As we mention above, the main tool in the forthcoming proof of Theorem 0.1 is the following Erdös combinatorial principle on integer distances.

Lemma 0.2. *Suppose, T is an infinite point set in $\mathbb{R}^d$ such that the the distance set of T is a subset of the set of positive integers $\mathbb{N}$. Then T is contained in a straight line.*

A two-dimensional version of this statement appeared in [Erdös45] and is well known. The higher dimensional version follows from the same proof. The higher-dimensional generalization provided by Lemma 0.2 follows from a theorem of Kuz'minyh [Kuz77], which allows any $d \geq 2$ and the distances being only asymptotically integer with $\limsup_{n\to\infty} n\epsilon(n) = 0$, where $\epsilon(n)$ is the difference between the Euclidean distance between a pair of points and an integer n. This theorem is further generalized in Lemma 1.4 below. In particular, the Euclidean structure of $\mathbb{R}^d$ is irrelevant for the principle in question. Hence, it can be extended to the case when the distance is defined in terms of the Legendre transform of the Minkowski functional of K, which appears further in the asymptotic formula (1.1) for $\widehat{\chi}_K$.

We remark that Lemma 0.2 is certainly driven by curvature. It would not be true, for instance, if K were a unit square, i.e if the distance were the "taxi-cab" distance. Indeed, the latter distance between any two points of the integer lattice is an integer.

We start out by giving a brief outline of the proof. Let ρ denote the Minkowski functional associated with K. Namely, $\rho(x)$ is a degree one homogeneous function[1], such that $K = \{x : \rho(x) \leq 1\}$. Let

$$\rho_*(\xi) = \sup_{x \in \partial K} x \cdot \xi \tag{0.2}$$

[1]By Euler's homogeneity relation, $\rho(x)$ is a solution of the first order PDE boundary value problem $x \cdot \nabla\rho = \rho$, $\rho_{\partial K} = 1$, where $x = (x_1, \dots, x_d)$. Then $\|x\|_\rho = \rho(x)$ is a norm equivalent to the Euclidean one $|\cdot|$. The latter corresponds to the case when K is a disc.

be the norm, dual to ρ. It is also equivalent to the Euclidean norm, for the set $K^* = \{\xi : \rho_*(\xi) = 1\}$ dual to K retains all the essential geometric properties of K. It is a standard calculation [Herz64] to show that at every point on the boundary of K^* the Gaussian curvature is inversely proportional to the Gaussian curvature at the corresponding point on the boundary of K, with the proportionality coefficient bounded in terms of K.

The exposition proceeds as follows. In Lemma 1.1 we argue that if A is a set of orthogonal exponents for K, then for non-equal $a, a' \in A$

$$\rho_*(a - a') = \frac{k}{2} + \frac{d-1}{8} + O\left(|a - a'|^{-1}\right), \; k \in \mathbb{N}. \tag{0.3}$$

The error term can be expanded further to higher orders of asymptotics. It follows that given a fixed pair of $a_0, a_1 \in A$ (in the sequel we assume by default that $a_0 \neq a_1$), for $a \in A$ one has

$$|\rho_*(a_0 - a) - \rho_*(a_1 - a)| = \frac{k}{2} + O\left(|a|^{-2}\right), \; k \in \mathbb{Z}. \tag{0.4}$$

Having established the above two formulas, we proceed by *reductio ad absurdum.* Suppose, A is maximal and infinite. Lemma 1.4 provides an asymptotic in the sense of the formula (0.4) ρ_*-generalization of Lemma 0.2 in order to see that all the members of A live precisely on some straight line L. This is in contradiction with the formula (0.3), unless the phase shift $\frac{d-1}{8}$ in the later formula is a half-integer itself when $d = 4k + 1$, $k \in \mathbb{N}$. In the latter case A shall be a subset of a lattice supported on L.

Then we look at the asymptotics for the zeroes of the Fourier transform of the characteristic function of K, restricted to the line L in order to argue that for a variety of K's, including the d-disk, the contradiction still can be obtained. The relevant calculations constitute the scope of Lemma 1.6. However, the non-existence result for all K's is not true in these dimensions, as is illustrated by examples.

Proof of Theorem 0.1

The following lemma gives an asymptotic expression for the Fourier transform $\widehat{\chi}_K$ of the characteristic function χ_K of K.

Lemma 1.1. *Let K be as in Theorem 0.1. Then for any $N \in \mathbb{N}$,*

$$\begin{aligned} \widehat{\chi}_K(\xi) &= \sum_{\alpha=0}^N C_\alpha\left(\tfrac{\xi}{|\xi|}\right) J_{\frac{d}{2}+\alpha}(2\pi\rho_*(\xi))|\xi|^{-\frac{d}{2}-\alpha} + O\left(|\xi|^{-\frac{d+3}{2}+N}\right) \\ &= \tilde{C}_0\left(\tfrac{\xi}{|\xi|}\right) \sin\left(2\pi\rho_*(\xi) - \pi\tfrac{d-1}{4}\right)|\xi|^{-\frac{d+1}{2}} + O\left(|\xi|^{-\frac{d+3}{2}}\right). \end{aligned} \tag{1.1}$$

For $\alpha = 0, \ldots, N$ the functions $C_\alpha, \tilde{C}_0$ are smooth functions of K alone, with $C_0, \tilde{C}_0$ being strictly positive.

By the $O(\cdot)$ symbol, we mean that the constants buried in it are functions of K alone. Below we shall give the sketch of the proof which will be further revisited in Lemma 1.6. Details can be found in [Herz64] and [Sogge93].

Without loss of generality assume that K is centered at the origin and a chosen ξ is directed along the axis, dual to the x-axis, namely $\xi = (z, 0, \dots, 0)$. Let $F(x)$ be the cross-sectional area of K along the x-axis. One can assume that F is supported on the interval $[-1, 1]$: to make up for this assumption the ensuing formulae should be amended by writing $\rho_*(\xi)$ instead of z. From symmetry of K, the function $F(x)$ is even. Let us compute its Fourier transform $\widehat{F}(z)$. Locally near a point where the x-axis intersects ∂K, there exists a smooth function $x = x(y)$, where $y = (y_1, \dots, y_d)$ embraces the rest of the coordinates. It can be rewritten as

$$1 - x^2 = Qy \cdot y + O(|y|^3), \tag{1.2}$$

where Q is a positive definite quadratic form with constant coefficients and $(\cdot)$ - the Euclidean scalar product. By the Morse lemma, the function in the right hand side can be conjugated to $\sum_{i=1}^d y_i^2$ via a smooth change of the y-variables. Thus locally near $x = \pm 1$,

(1.3)
$$F(x) = \chi_{[-1,1]} \int_{|y| \le \sqrt{1-x^2}} |\mathfrak{J}(y)| dy = \chi_{[-1,1]} \sum_{\alpha=0}^{N} \left[C_\alpha (1 - x^2)^{\frac{d-1}{2} + \alpha} + O\left((1 - x^2)^{\frac{d+1}{2} + N} \right) \right].$$

Here $\chi_{[-1,1]}$ is the characteristic function of the interval $[-1, 1]$ and $\mathfrak{J}(y) = C_0 + O(|y|)$ is the Jacobian of the coordinate change. The constant $C_0 = (\det Q)^{-1/2}$ is positive by the strict convexity assumption. The Jacobian can be Taylor expanded up to order $2N + 1$ in powers of y, and by symmetry, odd powers of y contribute zero in the above integral. Away from $x = \pm 1$, the function $F(x)$ is smooth.

Taking the Fourier transform of the above expansion for F, one obtains the following expansion in the Bessel functions:

$$\widehat{F}(z) = \sqrt{\pi} \sum_{\alpha=0}^{N} C_\alpha \Gamma\left(\frac{d+1}{2} + \alpha \right) \left(\frac{1}{\pi z} \right)^{\frac{d}{2} + \alpha} J_{\frac{d}{2} + \alpha}(2\pi z) + O\left(|z|^{\frac{d+3}{2} + N} \right). \tag{1.4}$$

Changing the support of the cross-section area function F depending on the direction according to the Minkowski functional $\rho(x)$ results in the appearance of its dual $\rho_*(\xi)$ in the formula (1.1) instead of z above. Change of the the direction of the x-axis makes the quantities C_α smooth functions on RP^{d-1}, with C_0 bounded away from zero. These functions also absorb the rest of the constants. The second line of the formula (1.1) follows from the well known asymptotic expansion for the Bessel function $J_{\frac{d}{2}}$ in the principal term of the sum. ■

Then (1.1) yields (0.3) and (0.4). The phase shift in the formula (0.3) will come into play later. The purpose of the following argument is to give an asymptotic ρ_*-version of the combinatorial principle, formulated as Lemma 0.2.

Let the set of the orthogonal exponents A for K be maximal and countably infinite. From (0.3) we notice that the pairwise ρ_*-distances between the members of A cling to the lattice $\frac{1}{8}\mathbb{Z}$

as these points get farther from each other. By changing the scale, there is no harm assuming that

$$(1.5) \qquad \forall a, a' \in A,\ a \neq a',\ \rho_*(a - a') = k + O(|a - a'|^{-1}),\ k \in \mathbb{N}.$$

Choose a pair of points $a_0, a_1 \in A$. For $a \in A$ consider the possible values for the limit $\lim_{|a|\to\infty} |\rho_*(a_0 - a) - \rho_*(a_1 - a)| \equiv l$. The asymptotic expansion (1.1) imposes stringent constraints on possible location of the "infinitely distant" point a, i.e.

$$(1.6) \qquad |\rho_*(a_0 - a) - \rho_*(a_1 - a)| = l + O(|a|^{-2}), \qquad l \in \{0, 1, \dots, [\rho_*(a_0 - a_1)]\} \subset \mathbb{Z}.$$

Above $[\cdot]$ stands for the integer part. The formula (1.5) is a somewhat coarser restatement of (0.3), with a change of scale for convenience in order to prove Lemma 1.4. The proof is based on exploiting the relation (1.6).

Lemma 1.2. *As $|a| \to \infty$, a point $a \in A$ is either located within a tubular neighborhood of the line connecting the points a_0, a_1 (which implies that $\rho_*(a_0 - a_1) \in \mathbb{N}$) or approaches asymptotically one of ρ_*-hyperboloids $\Gamma(a_0, a_1, \Delta)$ with foci a_0, a_1 and an integer parameter Δ, $0 \leq \Delta \leq [\rho_*(a_0 - a_1)]$. A ρ_*-hyperboloid is defined as follows:*

$$(1.7) \qquad \Gamma(a_0, a_1, \Delta) \equiv \{x \in \mathbb{R}^d : |\rho_*(x - a_0) - \rho_*(x - a_1)| = \Delta\}.$$

Before proving Lemma 1.2 we summarize the relevant properties of ρ_*-hyperboloids. In the sequel, the midpoint O_1 of the section $[a_0, a_1]$ will be referred to as a *vertex* of the ρ_*-hyperboloid $\Gamma(a_0, a_1, \Delta)$ and the line a_0a_1 connecting the foci as its *axis*.

Proposition 1.3.

(1) $\Gamma(a_0, a_1, \Delta)$ *is a straight line coinciding with its axis iff $\Delta = \rho_*(a_0 - a_1)$.*

(2) *Otherwise it is a smooth unbounded hypersurface, symmetric with respect to the vertex.*

(i) If $0 < \Delta < \rho_*(a_0 - a_1)$, this hypersurface has two connected components. The intersection of a connected component with any two-plane containing the foci is a smooth curve intersecting the axis transversely at a point distinct from the vertex and asymptotic to a pair of transverse rays emanating from the vertex on both sides of the axis.

(ii) If $\Delta = 0$, this hypersurface has one connected component. Its intersection with any two-plane containing the foci is a smooth curve passing through the vertex and asymptotic to a straight line passing through the vertex and intersecting the axis transversely.

The first statement follows from the fact that ρ_* is a norm. The transversality and smoothness statements follow from strict convexity and smoothness of the function ρ_* defined in terms of the body K only. The statement about asymptotics is proved as follows.

Fix Δ such that $0 \leq \Delta < \rho_*(a_0 - a_1)$ and consider the intersection $\mathcal{H} = \mathcal{H}(\Pi)$ of $\Gamma(a_0, a_1, \Delta)$ with a two-plane Π containing the axis a_0a_1. Then a branch of $\mathcal{H}$ can be smoothly parameterized by $x(t)$, where $x \in \Pi$, $t \in \mathbb{R}$. Clearly, $|x| \to \infty$ as $t \to \infty$. Let $f(x)$ be the restriction of the norm ρ_* to the plane Π. Then one branch of $\mathcal{H}$ is given by $f(x-a_0)-f(x-a_1)=\Delta$. Without loss of generality, suppose that the vertex O_1 is the origin.

Suppose, $f(x)=\epsilon^{-1}$ for some small $\epsilon>0$. Since f is a homogeneous function of degree one, one has $f(\epsilon(x-a_0))-f(\epsilon(x-a_1))=\epsilon\Delta$. Then, $\nabla f(\epsilon x)\cdot(a_1-a_0)=\Delta+O(\epsilon)$, where $(\cdot)$ is the Euclidean scalar product on Π.

The gradient $\nabla f(\epsilon x)$ is evaluated on the (smooth) boundary of a strictly convex body $K_2 = K^*\cap\Pi$. It is a homogeneous function of degree zero, thus $\nabla f(\epsilon x)=\nabla f(x)$. Then as $\epsilon\to 0$, there exists a limit $x_\Delta\in K_2$ such that $\nabla f(x_\Delta)\cdot(a_1-a_0)=\Delta$. In other words, as $t\to\infty$, the intersection of the line connecting x with the midpoint O_1 of the section $[a_0a_1]$ with K_2 limits at x_Δ.

Furthermore, differentiating the equation $f(x(t)-a_0)-f(x(t)-a_1)=\Delta$ with respect to t, and considering the limit as $t\to\infty$, we see that $\dot{x}(t)$ must be perpendicular to the vector $\nabla f(x(t)-a_0)-\nabla f(x(t)-a_1)$. The latter vector approaches a vector tangent to K_2 at the point x_Δ. Hence, there exists a limit for the direction of $\dot{x}(t)$ as $t\to\infty$, so the straight line O_1x_Δ is the asymptote for $x(t)$ as $t\to\infty$. In the same fashion, there exists an asymptote as $t\to-\infty$. From central symmetry, if $0<\Delta<\rho_*(a_0-a_1)$ the same pair of lines are asymptotes for the second connected component of $\mathcal{H}$. ■

We now prove Lemma 1.2. Consider some $a\in A$ such that $\rho_*(a-a_0)=\epsilon^{-1}$ for a small $\epsilon>0$. Restrict the analysis to some plane Π containing the points aa_0a_1 (they can be on the same line). Let $f(x)$, $x\in\Pi$ be the restriction of the distance ρ_* to the plane Π. The point a lies on the f-circle $f(a-a_0)=\epsilon^{-1}$ in this plane. By (1.6), $f(a-a_1)=\epsilon^{-1}-\Delta+O(\epsilon^2)$, where $\Delta\in\{0,1,\ldots,[\rho_*(a_0-a_1)]\}\subset\mathbb{Z}$. We shall consider two cases, $\Delta=[\rho_*(a_0-a_1)]=\rho_*(a_0-a_1)$, and $0\le\Delta<\rho_*(a_0-a_1)$. In the former case, the point a lies at the intersection of the f-circle $f(a-a_0)=\epsilon^{-1}$ and an f-annulus centered at a_1, of radius $\epsilon^{-1}-\rho_*(a_0-a_1)$ and width $O(\epsilon^2)$. The f-circles $f(a-a_0)=\epsilon^{-1}$ and $f(a-a_1)=\epsilon^{-1}-\rho_*(a_0-a_1)$ are tangent to one another at a point b, which is located on the line containing a_0 and a_1. Since an f-circle is locally a parabola (by the strict convexity assumption) the point a is contained in a const.$\times\epsilon^2$ rectangle centered at b, where the constant depends only on $\rho_*(a_0-a_1)$ and the bounds for the curvature on ∂K.

In the case $0\le\Delta<\rho_*(a_0-a_1)$ the intersection of the f-circle $f(a-a_0)=\epsilon^{-1}$ and the f-annulus centered at a_1, of radius $\epsilon^{-1}-\Delta$ and width $O(\epsilon^2)$ is transverse with the angle of $O(\epsilon)$, which implies the second claim. ■

The following statement is the key aspect of the proof.

Lemma 1.4. *If A is infinite, there exists a straight line L containing it.*

The lemma follows from the following claim.

Claim: For any pair of points $a_0,a_1\in A$ it is impossible to have infinitely many members of A lying outside some tubular neighborhood of the straight line connecting a_0 and a_1.

The lemma follows from the claim immediately. To see this, assume the claim and suppose that some point a_2 lies outside the line connecting a_0 and a_1. Then all but finitely many points of A lie in a tubular neighborhood of the line connecting a_0 and a_2. In same fashion, all but finitely many points of A must lie in a tubular neighborhood of the line connecting a_0 and a_1. The intersection of these two tubular neighborhoods is a bounded set which cannot contain infinitely many members of A due to the fact that A is separated. This argument will be repeated throughout the rest of the proof.

To prove the claim, suppose is not true. Then there exists a pair of points $a_0, a_1 \in A$, such that (by Lemma 1.2) there is an infinite set $A_1 \subset A$, such that the members $a \in A_1$ approach asymptotically some ρ_*-hyperboloid $\Gamma(a_0, a_1, \Delta)$, for some integer Δ, $0 \le \Delta < \rho_*(a_0 - a_1)$.

Consider $A_1 \cup \{a_0, a_1\}$. Let C_1 be the asymptotic cone for the above hyperboloid with the vertex O_1. A_1 is an infinite set of points located asymptotically close to C_1. By Lemma 1.2 and Proposition 1.3, one can always find a point $a_2 \in A_1$ such that the midpoint O_2 of the segment $[a_0a_2]$ lies outside the cone C_1.

Then, since the lines a_0a_1 and a_0a_2 are transverse, there must exist an infinite subset $A_2 \subseteq A_1$ of points lying asymptotically close to some ρ_*-hyperboloid $\Gamma(a_0, a_2, \Delta)$ with an integer Δ, $0 \le \Delta < \rho_*(a_0 - a_2)$ (the inequality is strict, because the line a_0a_2 is transverse to the cone C_1), hence to the cone C_2 with the vertex O_2. Thus, the members of A_2 lie asymptotically close to the intersection $\mathcal{S}_2 \equiv C_1 \cap C_2$ (let also $\mathcal{S}_1 = C_1$). If the $\mathcal{S}_2$ is bounded, there is a contradiction with the assumption that A_2 is infinite. Moreover, the intersection of the two cones along $\mathcal{S}_2$ is transverse, for any point thereon is formed by the intersection of a pair of straight lines connecting it to the vertices O_1 and O_2, the latter vertex lying by construction outside the cone C_1. Thus, $\mathcal{S}_2$ is a piecewise smooth unbounded surface of dimension $d-2$.

One can continue reducing the dimension by induction. Before the ith step, $i \ge 1$, there will be cones C_j, $j = 1, \dots, i$, with foci a_0, a_j intersecting transversely at a piecewise smooth unbounded surface $\mathcal{S}_i \equiv \bigcap_{j=1}^{i} C_j$ of dimension $d - i$. There will be an infinite set $A_i \subseteq A_1$ of points located asymptotically close to $\mathcal{S}_i$. At this point again one can again pick a point a_{i+1} such that the midpoint O_{i+1} of the segment $[a_0a_{i+1}]$ lies outside $\bigcup_{j=1}^{i} C_j$, for a_0 is a focus for all the hyperboloids asymptotic to the cones $C_1, \dots C_i$. Furthermore there is an infinite set $A_{i+1} \subseteq A_i$ of points, asymptotic to some cone C_{i+1}, whose vertex O_{i+1} does not belong to any of the cones $C_1, \dots, C_i$. Hence, for any point of $\mathcal{S}_i$, the line connecting it with O_{i+1} is transverse to each of the cones $C_1, \dots, C_i$, thus to their intersection. Therefore, the intersection $\mathcal{S}_{i+1} \equiv C_{i+1} \cap \mathcal{S}_i$ is transverse an should be an unbounded piecewise smooth surface of dimension $d - i - 1$.

It becomes impossible to proceed with this construction for $i \ge d$, without a contradiction with the fact that A_1 should be infinite. This proves the claim and the lemma. ■

Proposition 1.5. *If $d \neq 1 \mod (4)$, the set A is finite. Otherwise, if $d = 4k+1$, $k \in \mathbb{N}$, the set A is either finite or it is an infinite subset of a lattice supported on some line L.*

Indeed, if A is infinite, by Lemma 1.4 it is contained in some line L. By fixing $0 \in A$, we ensure that L passes through the origin.

If $d \neq 1 \mod (4)$, take three different points $a_0, a_1, a_2 \in A$ such that a_1 lies between a_0 and a_2. Then $\rho_*(a_2 - a_0) = \rho_*(a_2 - a_1) + \rho_*(a_1 - a_0)$. Since A is separated, the points a_0, a_1, a_2 can be chosen far enough from each other, so that the formula (0.3) cannot be satisfied simultaneously for the quantities $\rho_*(a_2 - a_0), \rho_*(a_2 - a_1)$ and $\rho_*(a_1 - a_0)$, due to the constant phase shift $\frac{d-1}{8}$ in it.

If $d = 1 \mod (4)$, then the phase shift $\frac{d-1}{8}$ is a half-integer itself. In this case, having infinitely many points on the line L would imply that for any pair of points a_0, a_1 there exists a distant point a such that the distances $\rho_*(a - a_0)$ as well as $\rho_*(a - a_1)$ are arbitrarily close to a half-integer. Thus $\rho_*(a_1 - a_0)$ is a half-integer itself. ■

This completes the proof of Theorem 0.1. Finally, let us investigate further the case $d = 4k+1$, $k \in \mathbb{N}$. As in the proof of Lemma 1.1, suppose L is dual to the x-axis. Let $F(x)$ be the cross-sectional area of K along the x-axis. Suppose F is supported on the interval $[-1, 1]$. At the points $x = \pm 1$ the function $F(x)$ has a smoothness defect of the same type and order as the function $(1-x^2)^{2k}$, corresponding to the case when K is a d-disk. Consider the Fourier series expansion

$$F(x) = \chi_{[-1,1]} \sum_{n=0}^{\infty} F_n \cos(\pi n x), \tag{1.8}$$

where

$$F_0 = 1, \qquad F_n = \int_{-1}^{1} F(x) \cos(\pi n x) dx, \quad n \in \mathbb{N}. \tag{1.9}$$

The existence of an infinite set of orthogonal exponents $A(K)$ with $0 \in A$, supported on the line L, is equivalent to the existence of an infinite set $E \subset \mathbb{N}$ such that for all $n \in \mathbb{N} \cap [E \cup (E-E)]$ the corresponding Fourier coefficients $F_n = 0$.

In general, this is possible. E.g. let

$$F(x) = \chi_{[-1,1]} c_k [1 + \cos(\pi x)]^k, \ k \in \mathbb{N}, \tag{1.10}$$

where the constant $c_k > 0$ is chosen to ensure $F_0 = 1$. Then for $n > k$ one has $F_n = 0$ for the the Fourier coefficients F_n above. In dimension $d = 4k+1$ take K as a body of revolution

$$r(x) = [\tilde{c}_k (1 + \cos(\pi x))]^{\frac{1}{4}}, \tag{1.11}$$

where r is the radius-vector in $\mathbb{R}^{4k}$. The constant $\tilde{c}_k > 0$ is to yield the above cross-section area $F(x)$. The function $r(x)$ has a negative second derivative for $x \in (-1, 1)$, whereas locally near $x = \pm 1$ it has the same type of singularity as the function $\sqrt{1-x^2}$. Thus K obtained this way is strictly convex.

However, for a variety of K's in $\mathbb{R}^{4k+1}$, $k \in \mathbb{N}$, the non-existence of an infinite set $A(K)$ is still the case.

Lemma 1.6. *Suppose K is such that the cross-section area $F(x)$ in any direction after a suitable dilation of K can be represented as*

(1.12)

$$F(x) = \chi_{[-1,1]} \left(\sum_{m \geq 2k}^{S} C_m (1-x^2)^m + \epsilon R(x) \right), \qquad C_{2k} > 0,\ C_S \neq 0, \qquad S = 2k+N,\ N \geq 0,$$

where the error term $R(x)$ is a smooth even function of x, with $R^{(\alpha)}(1) = 0$ for $\alpha = 0, 1, \dots, S$. Then for small ϵ the maximal set $A(K)$ is finite.

The proof is a direct computation. Using the Bessel function expansion of Lemma 1.1 along with the formula

$$(1.13) \quad \begin{aligned} J_{m+1/2}(z) &= \sqrt{\frac{2}{\pi z}} \left\{ \sin\left(z - \frac{\pi}{2}m\right) \sum_{l=0}^{[m/2]} \frac{(-1)^l (m+2l)!}{(2l)!(m-2l)!(2z)^{2l}} \right. \\ &\left. + \cos\left(z - \frac{\pi}{2}m\right) \sum_{l=0}^{[(m-1)/2]} \frac{(-1)^l (m+2l+1)!}{(2l+1)!(m-2l-1)!(2z)^{2l+1}} \right\}. \end{aligned}$$

we obtain (with the notation $[\cdot]$ for the integer part):

$$(1.14) \quad \begin{aligned} F_n &= \left(\frac{2}{\pi n}\right)^{4k} \sum_{s=0}^{N} (-1)^{s+n} (2s+2k+1)! \left(\frac{2}{\pi n}\right)^{2s} \\ &\times \sum_{l=0}^{s+1} (-1)^l 4^{-l} C_{2s+2k+1-l} \binom{2s+2k+1-l}{l} \\ &+ \epsilon\, O\left(n^{-2\left[\frac{S+1}{2}\right]-2}\right). \end{aligned}$$

The quantities, expressed by the sums in the second line of (1.14) bear responsibility for F_n being nonzero. These quantities are listed in the following table. Given $k \geq 1$, let $C_m = 0$ for $m < 2k$, whereas $C_{2k} > 0$. Also $C_S \neq 0$. Take the scalar product of the first row of the table with each subsequent ith row of the table, $i = 2, \dots$ in order to get a coefficient, multiplying n^{-2i-2} up to a nonzero factor, coming from the first line of the formula (1.14).

$-C_2$	C_3	$-C_4$	C_5	$-C_6$	C_7	$\dots$	$(-1)^S C_{S-1}$	$(-1)^{S+1} C_S$
$\frac{1}{4}\binom{2}{1}$	$\binom{3}{0}$	0	0	0	0	$\dots$	0	0
0	$\frac{1}{16}\binom{3}{2}$	$\frac{1}{4}\binom{4}{1}$	$\binom{5}{0}$	0	0	$\dots$	0	0
0	0	$\frac{1}{64}\binom{4}{3}$	$\frac{1}{16}\binom{5}{2}$	$\frac{1}{4}\binom{6}{1}$	$\binom{7}{0}$	$\dots$	0	0
				$\dots$				
0	0	0	0	0	0	$\dots$	0	$\frac{1}{4^{S-1}}\binom{S}{S-1}$

As $C_{2k} > 0$, $C_S \neq 0$ and ϵ is small enough, it follows that as $n \to \infty$, the absolute value of F_n is asymptotically bounded away from zero by a positive constant times $n^{-2[\frac{S+1}{2}]-2}$. ■

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Alex Iosevich. Department of Mathematics. University of Missouri, Columbia, MO 65211 U.S.A.

http://www.math.missouri.edu/~iosevich/
E-mail address: `iosevich@math.missouri.edu`

Michael Rudnev. School of Mathematics. University of Bristol, University Walk, Bristol BS8 1TW, UK
E-mail address: `m.rudnev@bris.ac.uk`