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Math 173, Fall 2022, December 5

$T: V \longrightarrow W$

$g \in W^*$. Let $f(\alpha) = g(T\alpha)$, i.e

$T^t(g) = f$

transpose

$V = W = \mathbb{R}^2$ $\qquad \alpha = (\alpha_1, \alpha_2)$

$$T\alpha = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a_{11}\alpha_1 + a_{12}\alpha_2 \\ a_{21}\alpha_1 + a_{22}\alpha_2 \end{pmatrix}$$

Take $g \in W^*$, i.e $g(x_1, x_2) = c_1 x_1 + c_2 x_2$

$T^t(g) = f$, where

$$f(\alpha) = g(a_{11}\alpha_1 + a_{12}\alpha_2, a_{21}\alpha_1 + a_{22}\alpha_2)$$

$$= c_1 a_{11}\alpha_1 + c_1 a_{12}\alpha_2 + c_2 a_{21}\alpha_1 + c_2 a_{22}\alpha_2$$

$$= (c_1 a_{11} + c_2 a_{21})\alpha_1 + (c_1 a_{12} + c_2 a_{22})\alpha_2$$

②

g is determined by c_1, c_2

f is determined by $(c_1 a_{11} + c_2 a_{21})$ & $(c_1 a_{12} + c_2 a_{22})$

Note that

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_{11}c_1 + a_{21}c_2 \\ a_{12}c_1 + a_{22}c_2 \end{pmatrix}$$

$\underbrace{\hphantom{xxxxxx}}_{\text{"classical" transpose}}$

Theorem 21: $V, W / F$. For each $T \in \mathcal{L}(V, W)$, $\exists !\, T^t \in \mathcal{L}(W^*, V^*) \ni T^t g(\alpha) = g(T\alpha)$

Theorem 22: $V, W / F$, $T \in \mathcal{L}(V, W)$. Then Null space $(T^t) = (\text{Range}(T))^0$. If V, W finite dimensional, then

i) $\text{rank}(T^t) = \text{rank}(T)$

ii) $\text{Range}(T^t) = (\text{Null space } T)^0$

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Theorem 23: $V, W/F$ $\mathcal{B}$ = basis for V, $\mathcal{B}^*$ dual basis, $\mathcal{B}'$ = basis for W, $\mathcal{B}'^*$ dual basis. $T \in \mathcal{L}(V, W)$; A matrix relative to $\mathcal{B}, \mathcal{B}'$; B matrix relative to $\mathcal{B}'^*, \mathcal{B}^*$. Then

$$B_{ij} = A_{ji}.$$

Theorem 24: A $m \times n$ matrix over F. Then the row rank of A is equal to the column rank of A.

Proof: Let $\mathcal{B}$ be the standard ordered basis for F^n and $\mathcal{B}'$ the ordered standard basis for F^m.

$T: F^n \to F^m$ w/ matrix A, i.e

$$T(x_1, \dots, x_n) = (y_1, y_2, \dots, y_m)$$

$$y_i = \sum_{j=1}^{n} A_{ij} x_j$$

The column rank (A) = rank (T), by construction. By the same reasoning, the column rank (A^t), which is just row rank (A), is equal to rank (T^t). Theorem 22 says that

rank (T) = rank (T^t), and we

are done!

<u>Determinants</u>: D: $n \times n$ matrices over F

$\longrightarrow$ scalar in F

D is n-linear if for each i, $1 \leq i \leq n$, D is a linear function of the i'th row when the other rows are fixed, i.e

$D(\alpha_1, \ldots, c\alpha_i + \alpha_i', \alpha_{i+1}, \ldots, \alpha_n) =$

$cD(\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \alpha_n) + D(\alpha_1, \alpha_2, \ldots, \alpha_i', \ldots, \alpha_n)$

Example: $D(A) = A_{11} A_{22} \dots A_{nn}$

Lemma: A linear combo of n-linear functions is linear.

Proof: Follows from the definition.

Example: $D(A) = A_{11} A_{22} - A_{12} A_{21}$

2×2 matrix

Definition: D n-linear. We say that D is alternating if

i) $D(A) = 0$ whenever 2 rows are equal.

ii) If A' is obtained from A by interchanging 2 rows, $D(A') = -D(A)$.

Definition: D is a determinant function if D is n-linear, alternating, and $D(I) = 1$.

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Lemma: Let D be a 2-linear function such that $D(A)=0$ $\forall$ 2×2 matrices w/ equal rows. Then D is alternating.

Proof: $D(\alpha+\beta, \alpha+\beta) = D(\alpha,\alpha) + D(\alpha,\beta) + D(\beta,\alpha) + D(\beta,\beta)$

It follows that $D(\alpha,\beta) = -D(\beta,\alpha)$.

Lemma: Let D be an n-linear function on $n\times n$ matrices over F. Suppose that $D(A)=0$ whenever two adjacent rows are equal. Then D is alternating.

Proof: We must show that $\underbrace{D(A')}_{\text{2 rows interchanged}} = -D(A)$

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Let B be obtained by interchanging rows i and j of A, where $i < j$.

We begin by interchanging row i with row $i+1$ and continue until we have

$$\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_j, \alpha_i, \alpha_{j+1}, \dots, \alpha_n$$

$K = j - i$ interchanges of adjacent rows.

Now move α_j to the i'th position using $K-1$ interchanges of adjacent rows.

So B is obtained from A by $K + (K-1) = 2K-1$ interchanges of adjacent rows. Thus

$$D(B) = (-1)^{2K-1} D(A) = -D(A).$$

Suppose that A is any $n \times n$ matrix w/ $\alpha_i = \alpha_j$, $i \neq j$. If $j = i+1$, then A has two equal adjacent rows, so $D(A) = 0$.

If $j > i+1$, interchange α_j and α_{i+1}, so $D(B) = 0$ for the resulting matrix B.

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But, $D(B) = -D(A)$, so $D(A) = 0$

Definition: A $n \times n$ matrix over F.

$A(i|j) = (n-1) \times (n-1)$ matrix obtained by deleting i'th row and j'th column.

If D is $(n-1)$-linear function,

$D_{ij}(A) = D[A(i|j)]$.

Theorem: $n > 1$ D $(n-1)$-linear. For each j, $1 \leq j \leq n$,

$$E_j(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} D_{ij}(A)$$

is an alternating n-linear function on $n \times n$ matrices A. If D is a determinant, then so is each $\underline{E_j}$.