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$$(*)\quad \begin{cases} A_{11}x_1 + A_{12}x_2 + \dots + A_{1n}x_n = y_1 \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \dots + A_{mn}x_n = y_m \end{cases}$$

$$\{A_{ij}\}_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

m = # of equations

n = # of variables

A_{ij}'s, x_j's & y_i's all lives in some field $\underline{\underline{F}}$.

$\{x_1, x_2, \dots, x_n\}$ is called a solution of the system $(*)$.

If $y_1 = y_2 = \dots = y_m = 0$, the system is called homogeneous.

Example: $F = \mathbb{R}$

$x_1 + x_2 = 2$
$x_1 - x_2 = 0$

Multiply the second equation by -1:

$-x_1 + x_2 = 0$ and add to the first equation

We obtain $(x_1 + x_2) + -1(x_1 - x_2)$

$= 2 - 0 = 2$

$\cancel{x_1} + x_2 - \cancel{x_1} + x_2 = 2$

$2x_2 = 2$ $x_2 = 1$

Plug this into either of the equations and obtain $x_1 = 1$

(3)

In general, select $c_1, c_2, \dots, c_m \in F$, then

$$(c_1 A_{11} + c_2 A_{21} + \dots + c_m A_{m1}) x_1 + \dots + (c_1 A_{1n} + c_2 A_{2n} + \dots + c_m A_{mn}) x_n$$
$$= c_1 y_1 + \dots + c_m y_m.$$

called a linear combination of the equations in (*).

Consider a system

$$\begin{aligned} B_{11} x_1 + B_{12} x_2 + \dots + B_{1n} x_n &= z_1 \\ &\vdots \qquad\qquad (**) \\ B_{k1} x_1 + B_{k2} x_2 + \dots + B_{kn} x_n &= z_k \end{aligned}$$

where each of the equations in (**) is a linear combination of equations in (*)

Then every solution of (*) is a solution of (**). But not necessarily vice-versa!

¿why?

We say that two systems are equivalent if each equation in each system is a linear combination of the equations in the other system.

<u>Theorem:</u> Equivalent systems of linear equations have exactly the same solutions.

the goal now is to create a method for creating equivalent systems that leads to solving these systems quickly and efficiently.

Abbreviate (*) by writing

$$AX = Y$$

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & & & \\ \vdots & & & \\ A_{m1} & A_{m2} & & A_{mn} \end{bmatrix}$$ matrix of coefficients

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

Elementary row operations:

1. Multiplication of one row by a non-zero scalar c.

2. Replacement of the r'th row of A by row r plus c times row s, c any scalar and $r \neq s$.

3. Interchange two rows of A.

Example: $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Multiply second row by -1:

$$\begin{pmatrix} 1 & 1 \\ -1 & +1 \end{pmatrix}$$

Replace second row by second row plus first row:

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

⑥

Multiply the first row by -2:

$$\begin{pmatrix} -2 & -2 \\ 0 & 2 \end{pmatrix}$$

Replace the first row by the first row plus the second row:

$$\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

Divide the first row by -2 and the second row by 2:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we'll generalize these ideas later

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Quantifying elementary operations:

$$A \longrightarrow e(A)$$

A: $m \times n$ matrix (m rows, n columns)

$e(A)$: A after an elementary row operation

1. $e(A)_{ij} = A_{ij}$ if $i \neq r$,

   $e(A)_{rj} = c A_{rj}$

   r'th row; multiplication of the r'th row

2. $e(A)_{ij} = A_{ij}$ if $i \neq r$,

   nothing happens away from the r'th row

   $e(A)_{ij}$ if $i \neq r$, $e(A)_{rj} = A_{rj} + c A_{sj}$

   A_{rj}: r'th row; A_{sj}: s'th row

3. $e(A)_{ij} = A_{ij}$ if i is different from $\underline{\underline{r}}$ and $\underline{\underline{s}}$

$$e(A)_{rj} = A_{sj}, \quad e(A)_{sj} = e(A)_{rj}$$

Are these processes reversible?

Theorem 2: To each elementary operation (row) e there corresponds an elementary operation e_1 <u>of</u> <u>the</u> <u>same</u> <u>type</u> $\ni$ $e(e_1(A)) = A$ for each $\underline{\underline{A}}$.

$\int$

the inverse of each row operation exists and is a row operation of the same type.

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Definition: A, B $m \times n$ over a field F.

rows columns

We say that B is row-equivalent to A if B can be obtained from A by a finite sequence of elementary row operations.

by Theorem 2 above, B row-equivalent to $A \Longleftrightarrow A$ row-equivalent to B.

Proof:

1. e_1 = operation that multiplies rth row by $c^{-1} = \frac{1}{c}$

recall that $c \neq 0$

2. e_1 = operation that replaces r'th row by the r'th row plus $(-c)$ times the s'th row.

3. $e_1 = e$! just reverse the rows back!

<u>Theorem 3:</u> If A & B are row-equivalent $m \times n$ matrices, $AX = 0$ and $BX = 0$ have the same solutions.

$$0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \qquad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Proof:

$$A = A_0 \longrightarrow A_1 \longrightarrow \dots \longrightarrow A_K = B$$

elementary row operations

It is enough to show that

$A_j X = 0$ and $A_{j+1} X = 0$ have the same solutions.

Suppose that B is obtained from A by a single elementary operation. Then each equation in $BX = 0$ is a linear combination of equations in $AX = 0$. The converse is also true by Theorem 3. Hence the conclusion follows by Theorem 1.