Basic skills: summation by parts, finite sums I

Alex Iosevich

March 2020

Who is this video for?

- This video is the first from my "Basic Skills" series.

Who is this video for?

- This video is the first from my "Basic Skills" series.
- The idea is to go over a series concepts and techniques that undergraduate mathematics majors repeatedly encounter.

Who is this video for?

- This video is the first from my "Basic Skills" series.
- The idea is to go over a series concepts and techniques that undergraduate mathematics majors repeatedly encounter.
- Statistics, physics, computer science, chemistry and engineering majors may find these videos help as well.

Who is this video for?

- This video is the first from my "Basic Skills" series.
- The idea is to go over a series concepts and techniques that undergraduate mathematics majors repeatedly encounter.
- Statistics, physics, computer science, chemistry and engineering majors may find these videos help as well.
- Most of these videos will be accessible to advanced high school students.

A bit more motivation

- Calculus is not a prerequisite for watching this video. However, the ideas we will go over will be quite helpful when you take calculus.

A bit more motivation

- Calculus is not a prerequisite for watching this video. However, the ideas we will go over will be quite helpful when you take calculus.

- If you have already taken calculus, you know to calculate integrals like

$$\int_a^b x \cdot 2^x dx.$$

A bit more motivation

- Calculus is not a prerequisite for watching this video. However, the ideas we will go over will be quite helpful when you take calculus.

- If you have already taken calculus, you know to calculate integrals like

$$\int_a^b x \cdot 2^x dx.$$

- Since calculus is often taught as a collection of mechanical tricks, you have never been exposed to the analogous sum

$$\sum_{k=a}^{b} k \cdot 2^k,$$

and this is the type of an issue we are going to address in the video series.

Geometric series

- One of the most important objects in mathematics is a geometric series. This is a series of the form

Geometric series

- One of the most important objects in mathematics is a geometric series. This is a series of the form

- $$1 + A + A^2 + \cdots + A^n,$$

Geometric series

- One of the most important objects in mathematics is a geometric series. This is a series of the form
- $$1 + A + A^2 + \cdots + A^n,$$
- where A is a real number $\neq 0, 1$, and n is a positive integer.

Geometric series

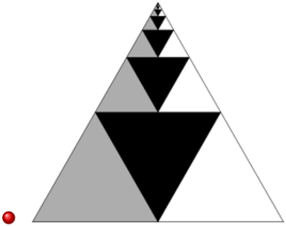
- One of the most important objects in mathematics is a geometric series. This is a series of the form
- $$1 + A + A^2 + \cdots + A^n,$$
- where A is a real number $\neq 0, 1$, and n is a positive integer.
- The geometric series need not start at 1, so

Geometric series

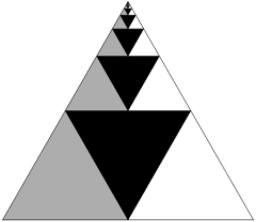
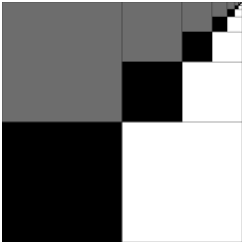
- One of the most important objects in mathematics is a geometric series. This is a series of the form
- $$1 + A + A^2 + \cdots + A^n,$$
- where A is a real number $\neq 0, 1$, and n is a positive integer.
- The geometric series need not start at 1, so
- $$A^k + A^{k+1} + \cdots + A^n$$

  is also a geometric series, where k is a positive integer $< n$.

Geometric series-simple diagrams from wikipedia



Geometric series-simple diagrams from wikipedia

- 
- 

Summing the geometric series

- Using the summation notation, the geometric series can be written as

Summing the geometric series

- Using the summation notation, the geometric series can be written as
- $$\sum_{j=k}^{n} A^j.$$

Summing the geometric series

- Using the summation notation, the geometric series can be written as
-
$$\sum_{j=k}^{n} A^j.$$
- How do we evaluate this series? First, let
$$\square = A^k + A^{k+1} + \cdots + A^n.$$

Summing the geometric series

- Using the summation notation, the geometric series can be written as
- $$\sum_{j=k}^{n} A^j.$$
- How do we evaluate this series? First, let
$$\square = A^k + A^{k+1} + \cdots + A^n.$$
- Then
$$A \cdot \square = A^{k+1} + A^{k+2} + \cdots + A^n + A^{n+1}.$$

Summing the geometric series (continued)

- Subtracting $\Box$ from $A \cdot \Box$, we see that

$$A \cdot \Box - \Box = A^{n+1} - A^k,$$

Summing the geometric series (continued)

- Subtracting $\square$ from $A \cdot \square$, we see that

$$A \cdot \square - \square = A^{n+1} - A^k,$$

- This implies that

$$\square = \frac{A^{n+1} - A^k}{A - 1}.$$

Summing the geometric series (continued)

- Subtracting $\square$ from $A \cdot \square$, we see that

$$A \cdot \square - \square = A^{n+1} - A^k,$$

- This implies that

$$\square = \frac{A^{n+1} - A^k}{A - 1}.$$

- Here is a simple example to give ourselves a sanity check. According to our formula,

Summing the geometric series (continued)

- Subtracting $\square$ from $A \cdot \square$, we see that

$$A \cdot \square - \square = A^{n+1} - A^k,$$

- This implies that

$$\square = \frac{A^{n+1} - A^k}{A - 1}.$$

- Here is a simple example to give ourselves a sanity check. According to our formula,

-

$$1 + 2 + \cdots + 2^4 = 2^5 - 1 = 31,$$

which is, indeed, true!

Why did the $\square$ idea work?

- When something works in mathematics, we are tempted not to question our good fortune and move on.

Why did the □ idea work?

- When something works in mathematics, we are tempted not to question our good fortune and move on.

- However, themes tend to recur, so it is useful to understand what happened.

Why did the □ idea work?

- When something works in mathematics, we are tempted not to question our good fortune and move on.
- However, themes tend to recur, so it is useful to understand what happened.
- The key observation behind what we did is that multiplying a geometric series $A^k + A^{k+1} + \cdots + A^n$ by A

Why did the □ idea work?

- When something works in mathematics, we are tempted not to question our good fortune and move on.
- However, themes tend to recur, so it is useful to understand what happened.
- The key observation behind what we did is that multiplying a geometric series $A^k + A^{k+1} + \cdots + A^n$ by A
- yields another geometric series

$$A^{k+1} + A^{k+2} + \cdots + A^n + A^{n+1}$$

which differs from the original geometric series in only two entries.

Spicing up the geometric series

- Suppose that instead of geometric series above, we consider the following fancier sum

Spicing up the geometric series

- Suppose that instead of geometric series above, we consider the following fancier sum
- $$1 \cdot A + 2 \cdot A^2 + 3 \cdot A^3 + \cdots + n \cdot A^n,$$

Spicing up the geometric series

- Suppose that instead of geometric series above, we consider the following fancier sum
- $$1 \cdot A + 2 \cdot A^2 + 3 \cdot A^3 + \cdots + n \cdot A^n,$$
- where, as before, A is a non-zero real number.

Spicing up the geometric series

- Suppose that instead of geometric series above, we consider the following fancier sum

- $$1 \cdot A + 2 \cdot A^2 + 3 \cdot A^3 + \cdots + n \cdot A^n,$$

- where, as before, A is a non-zero real number.

- In summation notation, this sum takes the form

Spicing up the geometric series

- Suppose that instead of geometric series above, we consider the following fancier sum
- $$1 \cdot A + 2 \cdot A^2 + 3 \cdot A^3 + \cdots + n \cdot A^n,$$
- where, as before, A is a non-zero real number.
- In summation notation, this sum takes the form
- $$\sum_{k=1}^{n} k \cdot A^k.$$

Just how spicy is it?

- Suppose that we just keep shaking our heads and refuse to accept the fact that the series above is not a geometric series?

Just how spicy is it?

- Suppose that we just keep shaking our heads and refuse to accept the fact that the series above is not a geometric series?

- To perpetuate our delusion, we write

$$A + A^2 + \cdots + A^n,$$

Just how spicy is it?

- Suppose that we just keep shaking our heads and refuse to accept the fact that the series above is not a geometric series?
- To perpetuate our delusion, we write

$$A + A^2 + \cdots + A^n,$$

- but then we notice that this does not add up to what we need since A^2 needs to be multiplied by two, not one, and so on.

Just how spicy is it?

- Suppose that we just keep shaking our heads and refuse to accept the fact that the series above is not a geometric series?
- To perpetuate our delusion, we write

$$A + A^2 + \cdots + A^n,$$

- but then we notice that this does not add up to what we need since A^2 needs to be multiplied by two, not one, and so on.
- But we persist and try to correct by adding

$$A^2 + A^3 + \cdots + A^n.$$

Just how spicy is it? (continued)

- The correction term we added helped a bit. We now have one factor of A, which is correct, and two factors of A^2, which is again correct, but we only have two factors of A^3 and we need three, and so on.

Just how spicy is it? (continued)

- The correction term we added helped a bit. We now have one factor of A, which is correct, and two factors of A^2, which is again correct, but we only have two factors of A^3 and we need three, and so on.

- But we are persistent, so we add

$$A^3 + A^4 + \cdots + A^n.$$

Just how spicy is it? (continued)

- The correction term we added helped a bit. We now have one factor of A, which is correct, and two factors of A^2, which is again correct, but we only have two factors of A^3 and we need three, and so on.

- But we are persistent, so we add

$$A^3 + A^4 + \cdots + A^n.$$

- We are starting to see what is going on. While our series is not geometric, we can express it as a sum of a bunch of geometric series.

Just how spicy is it? (continued)

- The correction term we added helped a bit. We now have one factor of A, which is correct, and two factors of A^2, which is again correct, but we only have two factors of A^3 and we need three, and so on.

- But we are persistent, so we add

$$A^3 + A^4 + \cdots + A^n.$$

- We are starting to see what is going on. While our series is not geometric, we can express it as a sum of a bunch of geometric series.

- Let us fully write out the case $n = 3$.

Just how spicy is it? (continued some more)

- In the case $n = 3$ we have

$$A + 2 \cdot A^2 + 3 \cdot A^3.$$

Just how spicy is it? (continued some more)

- In the case $n = 3$ we have

$$A + 2 \cdot A^2 + 3 \cdot A^3.$$

- This expression equals $\Box_1 + \Box_2 + \Box_3$, where

Just how spicy is it? (continued some more)

- In the case $n = 3$ we have
$$A + 2 \cdot A^2 + 3 \cdot A^3.$$

- This expression equals $\square_1 + \square_2 + \square_3$, where

-
$$\square_1 = A + A^2 + A^3,$$

Just how spicy is it? (continued some more)

- In the case $n = 3$ we have

$$A + 2 \cdot A^2 + 3 \cdot A^3.$$

- This expression equals $\square_1 + \square_2 + \square_3$, where

- $$\square_1 = A + A^2 + A^3,$$

- $$\square_2 = A^2 + A^3,$$

Just how spicy is it? (continued some more)

- In the case $n = 3$ we have

$$A + 2 \cdot A^2 + 3 \cdot A^3.$$

- This expression equals $\square_1 + \square_2 + \square_3$, where
-

$$\square_1 = A + A^2 + A^3,$$

-

$$\square_2 = A^2 + A^3,$$

- and

$$\square_3 = A^3.$$

Cutting through the spice

- In general, let
$$\triangle = A + 2 \cdot A^2 + \cdots + n \cdot A^n.$$

Cutting through the spice

- In general, let

$$\triangle = A + 2 \cdot A^2 + \cdots + n \cdot A^n.$$

- Then

$$\triangle = \square_1 + \square_2 + \cdots + \square_n,$$

Cutting through the spice

- In general, let

$$\triangle = A + 2 \cdot A^2 + \cdots + n \cdot A^n.$$

- Then

$$\triangle = \square_1 + \square_2 + \cdots + \square_n,$$

- where

$$\square_k = A^k + \cdots + A^n.$$

Cutting through the spice

- In general, let

$$\triangle = A + 2 \cdot A^2 + \cdots + n \cdot A^n.$$

- Then

$$\triangle = \square_1 + \square_2 + \cdots + \square_n,$$

- where

$$\square_k = A^k + \cdots + A^n.$$

- It is a very good time to recall that we have shown above that

$$\square_k = \frac{A^{n+1} - A^k}{A - 1}.$$

Cutting through the spice (continued)

- We must now sum up all the $\Box_k$s. How do we do that?

Cutting through the spice (continued)

- We must now sum up all the $\square_k$s. How do we do that?

- Looking at the expression for $\square_k$ we see that we must sum up

Cutting through the spice (continued)

- We must now sum up all the $\Box_k$s. How do we do that?

- Looking at the expression for $\Box_k$ we see that we must sum up

- $$\frac{A^{n+1}-A^1}{A-1}+\frac{A^{n+1}-A^2}{A-1}+\cdots+\frac{A^{n+1}-A^n}{A-1}=$$

Cutting through the spice (continued)

- We must now sum up all the $\square_k$s. How do we do that?
- Looking at the expression for $\square_k$ we see that we must sum up
- $$\frac{A^{n+1}-A^1}{A-1}+\frac{A^{n+1}-A^2}{A-1}+\cdots+\frac{A^{n+1}-A^n}{A-1}=$$
- $$=\frac{nA^{n+1}}{A-1}-\frac{1}{A-1}(A+A^2+\cdots+A^n)$$

Cutting through the spice (continued)

- We must now sum up all the $\square_k$s. How do we do that?
- Looking at the expression for $\square_k$ we see that we must sum up
- $$\frac{A^{n+1} - A^1}{A-1} + \frac{A^{n+1} - A^2}{A-1} + \cdots + \frac{A^{n+1} - A^n}{A-1} =$$
- $$= \frac{nA^{n+1}}{A-1} - \frac{1}{A-1}(A + A^2 + \cdots + A^n)$$
- $$= \frac{nA^{n+1}}{A-1} - \frac{(A^{n+1} - A)}{(A-1)^2}.$$

Cutting through the spice (finale)

- Note that we used the formula for $\square_k$ repeatedly above.

Cutting through the spice (finale)

- Note that we used the formula for $\Box_k$ repeatedly above.
- In order to keep good habits, let's compute through an example. According to our formula, taking $A = 2$ and $n = 3$,

Cutting through the spice (finale)

- Note that we used the formula for $\square_k$ repeatedly above.

- In order to keep good habits, let's compute through an example. According to our formula, taking $A = 2$ and $n = 3$,

- $$2 + 2 \cdot 2^2 + 3 \cdot 2^3 = 3 \cdot 16 - (16 - 2) = 48 - 14 = 34,$$

  which is true.

Cutting through the spice (finale)

- Note that we used the formula for $\square_k$ repeatedly above.
- In order to keep good habits, let's compute through an example. According to our formula, taking $A = 2$ and $n = 3$,
- $$2 + 2 \cdot 2^2 + 3 \cdot 2^3 = 3 \cdot 16 - (16 - 2) = 48 - 14 = 34,$$

  which is true.
- In order to built up these skills further, we need to go back and redo all these calculations using the summation notation.

Diving into the summation notation

- Let us compute

$$\sum_{j=k}^{n} A^j.$$

Diving into the summation notation

- Let us compute

$$\sum_{j=k}^{n} A^j.$$

- Following the prescription from above, we consider

$$A \cdot \sum_{j=k}^{n} A^j = \sum_{j=k}^{n} A^{j+1}.$$

Diving into the summation notation

- Let us compute

$$\sum_{j=k}^{n} A^j.$$

- Following the prescription from above, we consider

$$A \cdot \sum_{j=k}^{n} A^j = \sum_{j=k}^{n} A^{j+1}.$$

- We want to subtract $\sum_{j=k}^{n} A^j$ from

$$A \cdot \sum_{j=k}^{n} A^j = \sum_{j=k}^{n} A^{j+1}.$$

Changing the index of summation

- The technical problem we are facing is that in considering the expression

$$\sum_{j=k}^{n} A^{j+1} - \sum_{j=k}^{n} A^{j},$$

Changing the index of summation

- The technical problem we are facing is that in considering the expression

$$\sum_{j=k}^{n} A^{j+1} - \sum_{j=k}^{n} A^{j},$$

- we see that the summands are of a slightly different form!

Changing the index of summation

- The technical problem we are facing is that in considering the expression

$$\sum_{j=k}^{n} A^{j+1} - \sum_{j=k}^{n} A^{j},$$

- we see that the summands are of a slightly different form!

- We can fix the problem as follows. Let $m = j + 1$. Then since j ranges from k to n, m ranges from $k + 1$ to $n + 1$.

Changing the index of summation

- The technical problem we are facing is that in considering the expression
$$\sum_{j=k}^{n} A^{j+1} - \sum_{j=k}^{n} A^{j},$$

- we see that the summands are of a slightly different form!

- We can fix the problem as follows. Let $m = j + 1$. Then since j ranges from k to n, m ranges from $k + 1$ to $n + 1$.

- It follows that
$$\sum_{j=k}^{n} A^{j+1} = \sum_{m=k+1}^{n+1} A^{m}.$$

"Dummy" variable

- It is very important to internalize the fact that the letter m is a "dummy variable". Once you execute the sum, nobody is going to know whether you used the letter m or any other letter in the English alphabet or the Tibetan alphabet for that matter!

"Dummy" variable

- It is very important to internalize the fact that the letter m is a "dummy variable". Once you execute the sum, nobody is going to know whether you used the letter m or any other letter in the English alphabet or the Tibetan alphabet for that matter!

- In particular,

$$\sum_{m=k+1}^{n+1} A^m = \sum_{j=k+1}^{n+1} A^j.$$

"Dummy" variable

- It is very important to internalize the fact that the letter m is a "dummy variable". Once you execute the sum, nobody is going to know whether you used the letter m or any other letter in the English alphabet or the Tibetan alphabet for that matter!

- In particular,

$$\sum_{m=k+1}^{n+1} A^m = \sum_{j=k+1}^{n+1} A^j.$$

- It follows that

$$A \cdot \sum_{j=k}^{n} A^j - \sum_{j=k}^{n} A^j = \sum_{j=k+1}^{n+1} A^j - \sum_{j=k}^{n} A^j.$$

Double summation

- We can now see that most of the terms are going to cancel, leaving us with
$$A^{n+1} - A^k,$$
as before.

- We can now see that most of the terms are going to cancel, leaving us with

$$A^{n+1} - A^k,$$

as before.

- Putting everything together, we see that

$$(A-1)\sum_{j=k}^{n} A^j = A^{n+1} - A^k,$$

Double summation

- We can now see that most of the terms are going to cancel, leaving us with

$$A^{n+1} - A^k,$$

as before.

- Putting everything together, we see that

$$(A-1)\sum_{j=k}^{n} A^j = A^{n+1} - A^k,$$

- and we conclude that

$$\sum_{j=k}^{n} A^j = \frac{A^{n+1} - A^k}{A-1},$$

as before.

Double summation (continued)

- We now go ahead and redo the calculation for

$$\triangle = A + 2 \cdot A^2 + \cdots + n \cdot A^n = \sum_{k=1}^{n} k \cdot A^k.$$

Double summation (continued)

- We now go ahead and redo the calculation for

$$\triangle = A + 2 \cdot A^2 + \cdots + n \cdot A^n = \sum_{k=1}^{n} k \cdot A^k .$$

- As we saw before,

$$\triangle = \square_1 + \cdots + \square_n ,$$

Double summation (continued)

- We now go ahead and redo the calculation for

$$\triangle = A + 2 \cdot A^2 + \cdots + n \cdot A^n = \sum_{k=1}^{n} k \cdot A^k.$$

- As we saw before,

$$\triangle = \square_1 + \cdots + \square_n,$$

- where

$$\square_k = \sum_{j=k}^{n} A^j.$$

Double summation (continued)

- To put it another way,

$$\triangle = \sum_{k=1}^{n} \sum_{j=k}^{n} A^j,$$

a double sum.

Double summation (continued)

- To put it another way,

$$\triangle = \sum_{k=1}^{n} \sum_{j=k}^{n} A^j,$$

a double sum.

- But we have a formula for the inner sum, so

$$\triangle = \sum_{k=1}^{n} \frac{A^{n+1} - A^k}{A - 1}$$

Double summation (continued)

- To put it another way,

$$\triangle = \sum_{k=1}^{n} \sum_{j=k}^{n} A^j,$$

a double sum.

- But we have a formula for the inner sum, so

$$\triangle = \sum_{k=1}^{n} \frac{A^{n+1} - A^k}{A - 1}$$

-

$$= \frac{nA^{n+1}}{A-1} - \frac{1}{A-1} \sum_{k=1}^{n} A^k$$

Conclusion

- $$\frac{nA^{n+1}}{A-1} - \frac{(A^{n+1}-A)}{(A-1)^2},$$

  same as before.

Conclusion

- $$\frac{nA^{n+1}}{A-1} - \frac{(A^{n+1} - A)}{(A-1)^2},$$
  same as before.

- Before ending this video, let consider an interesting special case of this formula where $A = \frac{1}{2}$.

Conclusion

- $$\frac{nA^{n+1}}{A-1} - \frac{(A^{n+1}-A)}{(A-1)^2},$$
  same as before.

- Before ending this video, let consider an interesting special case of this formula where $A = \frac{1}{2}$.

- We obtain
  $$\sum_{k=1}^{n} \frac{k}{2^k} = -\frac{n}{2^n} - \frac{1}{2^{n-1}} + 2.$$

Conclusion

- As $n \to \infty$, $-\frac{n}{2^n} - \frac{1}{2^{n-1}} + 2 \to 2$, and we conclude that

$$\sum_{n=1}^{\infty} \frac{k}{2^k} = 2.$$

Conclusion

- As $n \to \infty$, $-\frac{n}{2^n} - \frac{1}{2^{n-1}} + 2 \to 2$, and we conclude that

$$\sum_{n=1}^{\infty} \frac{k}{2^k} = 2.$$

- Perhaps we did this a bit too quickly. Why is $\frac{n}{2^n}$ tend to 0 as n tends to ∞? What is the precise definition of $\sum_{k=1}^{\infty} \frac{k}{2^k}$?

Conclusion

- As $n \to \infty$, $-\frac{n}{2^n} - \frac{1}{2^{n-1}} + 2 \to 2$, and we conclude that
$$\sum_{n=1}^{\infty} \frac{k}{2^k} = 2.$$

- Perhaps we did this a bit too quickly. Why is $\frac{n}{2^n}$ tend to 0 as n tends to ∞? What is the precise definition of $\sum_{k=1}^{\infty} \frac{k}{2^k}$?

- We shall address all these issues and more in the subsequent videos.

Conclusion

- As $n \to \infty$, $-\frac{n}{2^n} - \frac{1}{2^{n-1}} + 2 \to 2$, and we conclude that
$$\sum_{n=1}^{\infty} \frac{k}{2^k} = 2.$$
- Perhaps we did this a bit too quickly. Why is $\frac{n}{2^n}$ tend to 0 as n tends to ∞? What is the precise definition of $\sum_{k=1}^{\infty} \frac{k}{2^k}$?
- We shall address all these issues and more in the subsequent videos.
- See you soon!!