MATH 238: HOMEWORK #2 DUE MONDAY, 10/26/16

ALEX IOSEVICH

Problem #1: Given any positive integer n, construct a set $P \subset \mathbb{R}^2$, not contained in a line, such that

$$\#P = n \text{ and } \Delta(P) \subset \mathbb{Z}. \tag{1}$$

Explain why this does not contradict the Erdős Integer Distance Principle we proved in class.

Hint: Consider the Pythagorean triples $a^2 + b^2 = c^2$. If you need more information about these triples than you currently possess, please use google.

Food for thought: Prove the Erdős Integer Distance Principle in higher dimensions. I recommend starting with three dimensions and going from there. Also, try to come up with a higher dimensional variant of the finite construction in Problem #1.

Problem #2: Let $P \subset \mathbb{R}^2$ such that (1) holds. Prove that there exists $\alpha < 2$ and a universal constant $C > 0$ such that the intersection of P with any square of side-length R contains fewer than CR^α points.

Hint: Proving that $\alpha \leq 2$ is not difficult because our assumptions imply that $|p - q| \geq 1$ for any $p, q \in P$. To obtain an exponent < 2, carefully examine the proof of the Erdős Integer Distance Principle. Keep in mind that since P is finite.

Food for thought: Take α all the way down to 1 and convince yourself that you cannot take it any lower.

Problem #3: Write out a complete proof of Proposition 4.1 in the book.

Problem #4: Exercise 4.4 on page 42 of the book.

Problem #5: Let $S \subset \mathbb{R}^4$ such that $\#S = n$. Let $x = (x_1, x_2, x_3, x_4)$ and define

$$\pi_{12}(S) = \{(x_3, x_4) : x \in S\}.$$

For each $1 \leq i < j \leq 4$, define $\pi_{ij}(S)$ analogously. Prove that

$$(\#S)^3 \leq \prod_{1 \leq i < j \leq 4} \#\pi_{ij}(S).$$

Hint: The assertion you are trying to prove is trivial. The trick is to recognize this from the right point of view.

Food for thought: Let $S \subset \mathbb{R}^d$, $d \geq 2$. Define
$$\pi_1(S) = \{(x_2, x_3, \dots, x_d) : x \in S\}$$
and define π_j analogously. Prove that
$$(\#S)^{d-1} \leq \prod_{j=1}^{d} \#\pi_j(S).$$

Problem #6: Construct a family of graphs with the number of vertices $n \to \infty$ such that $cr(G) \approx \frac{e^3}{n^2}$.