A COMBINATORIAL APPROACH TO ORTHOGONAL EXPONENTIALS

A. Iosevich and M. Rudnev

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Abstract. We prove that a symmetric strictly convex set with a smooth boundary in $\mathbb{R}^d$ can possess no more than finitely many orthogonal exponentials, unless $d = 1 \mod (4)$. In the latter case the non-existence theorem is true for a large class of bodies, including the d-dimensional ball. Otherwise, any infinite set of the corresponding exponents necessarily turns out to be a subset of some one-dimensional lattice. We provide examples of convex bodies of revolution in the above dimensions, for which infinite sets of orthogonal exponentials exist.

The analysis is reduced to one dimension by studying the distance set of the putative set of exponents with respect to an appropriate metric. A combinatorial principle due to Erdös lies at the heart of the investigation. According to this principle, if the distance set of an infinite set in $\mathbb{R}^d$ is a subset of the integers, then the set itself is a subset of some one-dimensional lattice.

Introduction

In the study of geometric properties of bodies in the Euclidean space, plane waves naturally play an important role. This role is hard to unitize and in may instances is far from being entirely clear. For instance the set of real plane waves is apparently insufficient as a basis for representation of general solutions of the Helmholtz equation inside a bounded region. Indeed, it is well known to a physicist investigating the quantum billiard problem in a confined planar region, that a mode therein is likely to contain "evanescent" waves, representing a class of super-oscillatory functions, which vary faster than any of their Fourier components. On the other hand, as is pointed out by Berry in [Berry94] one can come up with a representation for such an evanescent wave as a "singular limit of an angular superposition" of plane waves, approximating the former with arbitrary accuracy. The fact is also supported numerically.

For an analyst, the "insufficiency" of plane waves in the above context means that $L^2(K)$, where K is the "billiard domain", does not possess an orthogonal basis of exponentials, or K is not *spectral*. Whether or not a given body is spectral appears to be an important characteristics

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thereof, and has recently been studied rather extensively. E.g. in [Kol00] Kolountzakis proves that a non-symmetric convex domain is not spectral. It was proved in [IKT01] that symmetric convex domains with a point of curvature on the boundary are not spectral, either. Both results are motivated by a paper of Fuglede [Fuglede74] who conjectured that $L^2(K)$ has an orthogonal basis of exponentials if and only if a Lebesgue measurable K tiles $\mathbb{R}^d$ by translation. In the context of convex planar domains, this conjecture has been recently proved in [IKT02].

While at the first glance a problem in analysis, the question of existence of orthogonal exponential bases or sub-bases has a distinctive combinatorial flavor. The definition of orthogonality, combined with the asymptotics of the Fourier transform of the characteristic function of K strongly suggest that the existence of a putative set $A \subset \mathbb{R}^d_*$, where $\mathbb{R}^d_*$ is the dual space to $\mathbb{R}^d$, such that $\{e^{2\pi i x\cdot a}\}_{a\in A}$ are pairwise orthogonal in $L^2(K)$ is closely tied to the properties of the distance set

$$\Delta(A) = \{\rho_*(a-a') : a, a' \in A,\, a \neq a'\},$$

where ρ_* is the dual of the Minkowski functional of K. This brings the study of orthogonal exponentials into the realm of combinatorial distance problems, inexorably connected with the name of P. Erdös. For the modern state of the art, see the recent book [AP95] and the references contained therein. See also [Falc87], [Wolff99] and references therein for closely related continuous analogs of the distance problems, such as the Falconer conjecture. See also [HI2003], [IL2003], and [IL2003II] for some recent work on the connection between the discrete and continuous versions of the distance problem.

In this paper we study the case when K is symmetric, has a smooth boundary and is *strictly convex*. By strict convexity we mean that the boundary ∂K is smooth and has everywhere non-vanishing Gaussian curvature. Then for $d \neq 1 \mod (4)$ we prove non-existence in $L^2(K)$ of infinite-dimensional subspaces allowing orthogonal exponential bases for any of the above K. The case $d = 1$ is obviously exceptional in the above sense; however in dimensions $5, 9, 13, \ldots$ infinite-dimensional spectral sub-spaces may also exist for certain K, as the example that we provide below shows. Interestingly enough, if this is the case, all the corresponding exponents must still necessarily be contained in some one-dimensional lattice. Moreover, this may only occur for K "far enough" (in the sense of Lemma 1.6 in the sequel) from a d-dimensional ball, representing in a sense a "generic" set K for which $card\, A(K) < \infty$. The proof of the main result is based on a principle that goes back to Erdös, which says that if a set of Euclidean distances of a planar set is a subset of the integers, then the set itself is contained in a straight line.

From now on we fix K and without loss of generality assume that it has a unit volume with respect to the Lebesgue measure.

Definition. Let A be a subset of $\mathbb{R}^d_*$ such that the orthogonality relation

$$\widehat{\chi}_K(a-a') = \int_K e^{2\pi i x\cdot(a-a')}dx = \delta_{aa'} \tag{0.1}$$

holds whenever $a, a' \in A$, the right hand side being the Kronecker delta. Then $A = A(K)$ is called a set of *orthogonal exponents* for K.

Definition. A is maximal if for any $a' \in \mathbb{R}^d_* \setminus A$ there exists $a \in A$ such that $\widehat{\chi}_K(a - a') \neq 0$.

By continuity of $\widehat{\chi}_K$, the set $A(K)$ is separated. Namely, there exists a uniform constant $c = c(K) > 0$ such that $|a - a'| \geq c$, for all non-equal $a, a' \in A$. The notation $|\cdot|$ stands henceforth for the Euclidean distance. Indeed, since K is symmetric (0.1) is a cosine transform. Then the constant c is bounded from below in inverse proportionality to the length of the longest diagonal of ∂K, passing through the center. So, if A is a set of orthogonal exponents, its maximal cardinality is $\aleph_0$, and A can be sequentially completed to a maximal set; let us further assume that this is the case. Besides, any translation of A is also a set of orthogonal exponents, so in order to fix A in a certain sense, let's assume that $0 \in A$.

Our main result is the following.

Theorem 1. *Let $K \subset \mathbb{R}^d$ be a closed symmetric convex domain of unit volume with a smooth boundary ∂K, such that the Gaussian curvature does not vanish anywhere on ∂K. Then for $d \neq 1 \mod (4)$, any maximal set $A(K)$ of orthogonal exponents for K is finite. Otherwise, either $A(K)$ is finite, or it is a subset of some one-dimensional lattice in $\mathbb{R}^d_*$.*

Fuglede [Fug74] proved non-existence of an infinite set $A(K)$ when K is a disk in $\mathbb{R}^2$. In the process of revising this manuscript, we became aware of the recent paper [Fug01] extending the result to a disk in $\mathbb{R}^d$ for any $d \geq 2$.

Theorem 1 is driven by strict convexity of K which is used several times throughout the proof. Otherwise, a unit cube in $\mathbb{R}^d$ has a basis of orthogonal exponentials, a cylinder has infinitely many of them located on the symmetry axis. A somewhat less obvious example of a convex set with an orthogonal exponential basis is a hexagon in $\mathbb{R}^2$.

As we mention above, the main tool in the forthcoming proof of Theorem 1 is the following Erdös combinatorial principle on integer distances.

Integer distance principle [Erdös]. *Suppose, T is an infinite point set in $\mathbb{R}^d$ such that the distance set of T is a subset of the set of positive integers $\mathbb{N}$. Then T is contained in a straight line.*

A two-dimensional version of this statement appeared in [Erdös45] and is well known. The higher dimensional version follows from the same proof. Note that the principle is not true for a finite point set T. This follows from the fact that the upper limit for the number of integer solutions of the Diophantine equation $n_1^2 + n_2^2 = n_3^2$ goes to infinity as, say $n_2 \to \infty$, and thus one can score as many points as one wants by choosing members of T as points with coordinates $(0, n_2)$ in the plane for some large n_2, plus all the points $(n_1, 0)$ such that the above equation is satisfied for some n_3.

It is known that the integer distance principle is robust in the sense that it allows for asymptotic versions. For example, a theorem of Kuz'minyh [Kuz77] valid for any $d \geq 2$ requires the distances to be only asymptotically integer with $\limsup_{n\to\infty} n\epsilon(n) = 0$, where $\epsilon(n)$ is the difference between the Euclidean distance between a pair of points and an integer n. This theorem is further generalized in Lemma 1.4 below. In particular, the Euclidean structure of $\mathbb{R}^d$ is irrelevant for the principle in question. Hence, it can be extended to the case when the distance is defined in terms of the dual of the Minkowski functional of K, which appears further in the asymptotic formula (1.1) for $\widehat{\chi}_K$. In fact, in our case $\epsilon(n) = n^{-1}$ and thus the lim sup condition of Kuz'minyh is not satisfied. However, the following condition (0.4) which is in effect a C^1 asymptotic estimate for the distance, turns out to be sufficient for the proof to go through.

Let us repeat that the presence of non-zero curvature is essential for the principle in question. It would certainly not be true for instance, if K were a unit square or rhombus, ρ_* being the l^1 and l^∞ distances, respectively.

Outline of the proof. Let ρ denote the Minkowski functional associated with K. I.e. $\rho(x)$ is a degree one homogeneous function, such that $K = \{x : \rho(x) \leq 1\}$. From convexity of K it follows that $\|x\|_\rho = \rho(x)$ is a norm equivalent to the Euclidean one $|\cdot|$. The latter corresponds to the case when K is a ball.

Let

$$\rho_*(\xi) = \sup_{x\in\partial K} x\cdot\xi \tag{0.2}$$

be the norm dual to ρ. The dual set $K^* = \{\xi : \rho_*(\xi) \leq 1\}$ retains all the essential geometric properties of K. A standard calculation [Herz64] shows that at every point on the boundary of K^* the Gaussian curvature is inversely proportional to the Gaussian curvature at the corresponding point on the boundary of K. See also [Muller99].

The proof proceeds as follows. In Lemma 1.1 we argue that if A is a set of orthogonal exponents for K, then $a, a' \in A$, $a \neq a'$,

$$\rho_*(a - a') = \frac{k}{2} + \frac{d-1}{8} + O\left(|a - a'|^{-1}\right),\ k \in \mathbb{N}, \tag{0.3}$$

as $|a - a'| \to \infty$. The error term can be expanded further to higher orders of asymptotics. It follows that given a fixed pair of $a_0, a_1 \in A$ (in the sequel we assume by default that $a_0 \neq a_1$), for $a \in A$ one has

$$|\rho_*(a_0 - a) - \rho_*(a_1 - a)| = \frac{k}{2} + O\left(|a|^{-2}\right),\ k \in \mathbb{Z}. \tag{0.4}$$

Having established the above two formulas, we proceed to argue by contradiction. Suppose, A is maximal and infinite. Using (0.4) we prove an asymptotic ρ_*-generalization of the integer distances principle (see Lemma 1.4 below) in order to see that all the members of A live precisely on some straight line $L \subset \mathbb{R}^d_*$. This is in contradiction with the formula (0.3), unless the phase shift $\frac{d-1}{8}$ in the later formula is a half-integer itself when $d = 4k+1$, $k \in \mathbb{N}$. In the latter case A shall be (precisely!) a subset of a lattice supported on L.

Then we look at the asymptotics for the zeroes of the Fourier transform of the characteristic function of K, restricted to the line L in order to argue that for many K's, including the d-dimensional ball, a similar contradiction resulting in finiteness of $A(K)$ can still be established. The relevant calculations constitute the scope of Lemma 1.6. However, the non-existence result for all K's is not true if the dimension d is congruent to 1 modulo 4, which we illustrate by an example.

Proof of Theorem 1

The following lemma gives an asymptotic expression for the Fourier transform $\widehat{\chi}_K$ of the characteristic function χ_K of K.

Lemma 1.1. *Let K be as in Theorem 1. Then for any $N \in \mathbb{N}$,*

$$\begin{aligned} \widehat{\chi}_K(\xi) &= \sum_{\alpha=0}^{N} C_\alpha\left(\frac{\xi}{|\xi|}\right) J_{\frac{d}{2}+\alpha}(2\pi\rho_*(\xi))|\xi|^{-\frac{d}{2}-\alpha} + O\left(|\xi|^{-\frac{d+3}{2}+N}\right) \\ &= \tilde{C}_0\left(\frac{\xi}{|\xi|}\right)\sin\left(2\pi\rho_*(\xi) - \pi\frac{d-1}{4}\right)|\xi|^{-\frac{d+1}{2}} + O\left(|\xi|^{-\frac{d+3}{2}}\right). \end{aligned} \tag{1.1}$$

For $\alpha = 0, \dots, N$ the functions $C_\alpha, \tilde{C}_0$ are smooth functions of K alone, with $C_0, \tilde{C}_0$ being strictly positive.

By the $O(\cdot)$ symbol, we mean that the constants buried in it are functions of K alone. Below we shall give the sketch of the proof which will be further revisited in Lemma 1.6. Details can be found in [Herz64] and [Sogge93].

Without loss of generality assume that K is centered at the origin and for a chosen ξ the basis has been chosen such that ξ is directed along the axis, dual to the x_1-axis, namely $\xi = (z, 0, \dots, 0)$. Let $F(x)$ be the cross-sectional area of K along the x_1-axis, i.e.

$$F(x) = \mu_{d-1}(K \cap \{x_1 = x\}),$$

where μ_d is the Lebesgue measure in $\mathbb{R}^d$. Further the notations x and x_1 are identified. One can assume that $F(x)$ is supported on the interval $[-1, 1]$: to make up for this assumption the ensuing formulae should be amended by writing $\rho_*(\xi)$ instead of z. From symmetry of K, the function $F(x)$ is even. Let us compute its Fourier transform $\widehat{F}(z)$. Locally near $x = 1$, by strict convexity of K, there exists a smooth function $x = x(y - y_0)$, where $y = (y_1, \dots, y_d)$ embraces the rest of the coordinates and $(x, y) = (1, y_0)$ is the intersection point of K and the tangent plane $x = 1$. With the notation $y' = y - y_0$, the equation of ∂K can locally be rewritten as

$$1 - x^2 = Qy' \cdot y' + O(|y'|^3), \tag{1.2}$$

where Q is a positive definite quadratic form with constant coefficients and $(\cdot)$ - the Euclidean scalar product. By the Morse lemma, the function in the right hand side can be conjugated (omitting the primes) to $\sum_{i=1}^{d} y_i^2$ via a smooth change of the y-variables. Thus locally near $x = 1$, one can come up with the following expansion for F:

$$F(x) = \chi_{[-1,1]}\int_{|y|\leq\sqrt{1-x^2}} |\mathfrak{J}(y)|dy = \chi_{[-1,1]}\left[\sum_{\alpha=0}^{N} C_\alpha(1-x^2)^{\frac{d-1}{2}+\alpha} + O\left((1-x^2)^{\frac{d+1}{2}+N}\right)\right]. \tag{1.3}$$

Here $\chi_{[-1,1]}$ is the characteristic function of the interval $[-1,1]$ and $\mathfrak{J}(y) = C_0 + O(|y|)$ is the Jacobian of the coordinate change. The constant $C_0 = (\det Q)^{-1/2}$ is positive by the strict convexity assumption. The Jacobian can be Taylor expanded up to order $2N+1$ in powers of y, and by symmetry, odd powers of y contribute zero in the above integral. Away from $x = \pm 1$, the function $F(x)$ is smooth.

Taking the Fourier transform of the above expansion for F, one obtains the following expansion in the Bessel functions:

$$\widehat{F}(z) = \sqrt{\pi} \sum_{\alpha=0}^{N} C_\alpha \Gamma\left(\frac{d+1}{2} + \alpha\right) \left(\frac{1}{\pi z}\right)^{\frac{d}{2}+\alpha} J_{\frac{d}{2}+\alpha}(2\pi z) + O\left(|z|^{\frac{d+3}{2}+N}\right). \tag{1.4}$$

Despite formula (1.3) is local near $x = 1$, formula (1.4) gives correct asymptotics for the Fourier transform $\widehat{F}(z)$, as expression (1.3) may differ from the global expression for $F(x)$ at worst by a C^∞ function supported inside the interval $(-1, 1)$.

Changing the support of the cross-section area function F depending on the direction according to the Minkowski functional $\rho(x)$ results in the appearance of its dual $\rho_*(\xi)$ in the formula (1.1) instead of z above. Change of the direction of the x-axis makes the quantities C_α smooth functions on RP^{d-1}, with C_0 bounded away from zero. These functions also absorb the rest of the constants. The second line of the formula (1.1) follows from the well known asymptotic expansion for the Bessel function $J_{\frac{d}{2}}$ in the principal term of the sum. ■

Then (1.1) yields (0.3) and (0.4). The phase shift in the formula (0.3) will come into play later. The purpose of the following argument is to give an asymptotic ρ_*-version of the integer distances principle.

Let the set of the orthogonal exponents A for K be maximal and countably infinite. From (0.3) we notice that the pairwise ρ_*-distances between the members of A as they get farther from each other, cling to the lattice $\frac{1}{2}\mathbb{Z}$, shifted by $\frac{d-1}{8}$, the shift vanishing for $d = 1 \bmod (4)$. For now, let us ignore the phase shift $\frac{d-1}{8}$. By changing the scale, there is no harm assuming that

$$\forall a, a' \in A,\ a \neq a',\ \rho_*(a - a') = k + O(|a - a'|^{-1}),\ k \in \mathbb{N}. \tag{1.5}$$

Choose a pair of points $a_0, a_1 \in A$. For $a \in A$ consider the possible values for the limit $\lim_{|a|\to\infty} |\rho_*(a_0 - a) - \rho_*(a_1 - a)| \equiv l$. The asymptotic expansion (1.1) imposes stringent constraints on possible location of the "infinitely distant" point a, i.e.

$$|\rho_*(a_0 - a) - \rho_*(a_1 - a)| = l + O(|a|^{-2}), \qquad l \in \{0, 1, \dots, [\rho_*(a_0 - a_1)]\} \subset \mathbb{Z}. \tag{1.6}$$

Above, the bounding constant in the symbol $O(\cdot)$ depends on the distance between a_0 and a_1; $[\cdot]$ stands for the integer part. Formula (1.5) is a somewhat coarser restatement of (0.3), with a change of scale for convenience in order to prove Lemma 1.4. The proof is based on exploiting relation (1.6).

Lemma 1.2. *As $|a| \to \infty$, a point $a \in A$ is either located within a tubular neighborhood of the line connecting the points a_0, a_1 (which implies that $\rho_*(a_0 - a_1) \in \mathbb{N}$) or approaches asymptotically one of ρ_*-hyperboloids $\Gamma(a_0, a_1, \Delta)$ with the foci a_0, a_1 and an integer parameter Δ, $0 \le \Delta \le [\rho_*(a_0 - a_1)]$. A ρ_*-hyperboloid is defined as follows:*

$$\Gamma(a_0, a_1, \Delta) \equiv \{x \in \mathbb{R}^d_* : |\rho_*(x - a_0) - \rho_*(x - a_1)| = \Delta\}. \tag{1.7}$$

Before proving Lemma 1.2 we summarize the relevant properties of ρ_*-hyperboloids. In the sequel, the midpoint O_1 of the section $[a_0, a_1]$ will be referred to as a *vertex* of the ρ_*-hyperboloid $\Gamma(a_0, a_1, \Delta)$ and the line $a_0 a_1$ connecting the foci as its *axis*.

Proposition 1.3.

(1) $\Gamma(a_0, a_1, \Delta)$ *is a straight line coinciding with its axis iff* $\Delta = \rho_*(a_0 - a_1)$.

(2) *Otherwise it is a smooth unbounded hyper-surface, symmetric with respect to the vertex and asymptotic to a bi-infinite cone centered at the vertex. More precisely,*

(i) If $0 < \Delta < \rho_*(a_0 - a_1)$, this hyper-surface has two connected components. The intersection of a connected component with any two-plane containing the foci is a smooth curve intersecting the axis transversely at a point distinct from the vertex and asymptotic to a pair of transverse rays emanating from the vertex on both sides of the axis.

(ii) If $\Delta = 0$, this hyper-surface has one connected component. Its intersection with any two-plane containing the foci is a smooth curve passing through the vertex and asymptotic to a straight line passing through the vertex and intersecting the axis transversely.

The first statement follows from the fact that ρ_* is a strictly convex norm, which implies that the triangle inequality becomes an equality if and only if the three points involved lie on the same line. This is not true in the above mentioned case of the l^1 or l^∞ norm, for example. The transversality and smoothness statements follow from strict convexity and smoothness of the function ρ_* defined in terms of the body K only. The statement about asymptotics is proved as follows.

Fix Δ such that $0 \le \Delta < \rho_*(a_0 - a_1)$ and consider the intersection $\mathcal{H} = \mathcal{H}(\Pi)$ of $\Gamma(a_0, a_1, \Delta)$ with a two-plane Π containing the axis $a_0 a_1$. Then a branch of $\mathcal{H}$ can be smoothly parameterized by $x(t)$, where $x \in \Pi$, $t \in \mathbb{R}$. Clearly, $|x| \to \infty$ as $t \to \infty$. Let $f(x)$ be the restriction of the norm ρ_* to the plane Π. Then one branch of $\mathcal{H}$ is given by $f(x - a_0) - f(x - a_1) = \Delta$. Without loss of generality, suppose that the vertex O_1 is the origin.

Suppose, $f(x) = \epsilon^{-1}$ for some small $\epsilon > 0$. Since f is a homogeneous function of degree one, one has $f(\epsilon(x - a_0)) - f(\epsilon(x - a_1)) = \epsilon\Delta$. Then, $\nabla f(\epsilon x) \cdot (a_1 - a_0) = \Delta + O(\epsilon)$, where $(\cdot)$ is the Euclidean scalar product on Π.

The gradient $\nabla f(\epsilon x)$ is evaluated on the (smooth) boundary of a strictly convex body $K_2 = K^* \cap \Pi$. It is a homogeneous function of degree zero, thus $\nabla f(\epsilon x) = \nabla f(x)$. Then as $\epsilon \to 0$, there exists a limit $x_\Delta \in K_2$ such that $\nabla f(x_\Delta) \cdot (a_1 - a_0) = \Delta$. In other words, as $t \to \infty$, the intersection of the line connecting x with the midpoint O_1 of the section $[a_0 a_1]$ with K_2 limits at x_Δ.

Furthermore, differentiating the equation $f(x(t)-a_0)-f(x(t)-a_1)=\Delta$ with respect to t, and considering the limit as $t\to\infty$, we see that $\dot{x}(t)$ must be perpendicular to the vector $\nabla f(x(t)-a_0)-\nabla f(x(t)-a_1)$. The latter vector approaches a vector tangent to K_2 at the point x_Δ. Hence, there exists a limit for the direction of $\dot{x}(t)$ as $t\to\infty$, so the straight line O_1x_Δ is the asymptote for $x(t)$ as $t\to\infty$. In the same fashion, there exists an asymptote as $t\to-\infty$. From central symmetry, if $0<\Delta<\rho_*(a_0-a_1)$ the same pair of lines are asymptotes for the second connected component of $\mathcal{H}$. ■

Proof of Lemma 1.2. We now prove Lemma 1.2. Consider some $a\in A$ such that $\rho_*(a-a_0)=\epsilon^{-1}$ for a small $\epsilon>0$. Restrict the analysis to some plane Π containing the points aa_0a_1 (they can be on the same line). Let $f(x)$, $x\in\Pi$ be the restriction of the distance ρ_* to the plane Π. The point a lies on the f-circle $f(a-a_0)=\epsilon^{-1}$ in this plane. By (1.6), $f(a-a_1)=\epsilon^{-1}-\Delta+O(\epsilon^2)$, where $\Delta\in\{0,1,\dots,[\rho_*(a_0-a_1)]\}\subset\mathbb{Z}$. We shall consider two cases, $\Delta=[\rho_*(a_0-a_1)]=\rho_*(a_0-a_1)$, and $0\le\Delta<\rho_*(a_0-a_1)$. In the former case, the point a lies at the intersection of the f-circle $f(a-a_0)=\epsilon^{-1}$ and an f-annulus centered at a_1, of radius $\epsilon^{-1}-\rho_*(a_0-a_1)$ and width $O(\epsilon^2)$. The f-circles $f(a-a_0)=\epsilon^{-1}$ and $f(a-a_1)=\epsilon^{-1}-\rho_*(a_0-a_1)$ are tangent to one another at a point b, which is located on the line containing a_0 and a_1. Since an f-circle is locally a parabola (by convexity, non-vanishing curvature assumption and Morse lemma) the point a is contained in a const.$\times\epsilon^2$ rectangle centered at b, where the constant is inversely proportional to $\rho_*(a_0-a_1)$ and depends apart from that only on the upper and lower bounds for the principal curvatures on ∂K.

In the case $0\le\Delta<\rho_*(a_0-a_1)$ the intersection of the f-circle $f(a-a_0)=\epsilon^{-1}$ and the f-annulus centered at a_1, of radius $\epsilon^{-1}-\Delta$ and width $O(\epsilon^2)$ is transverse with the angle of $O(\epsilon)$, which implies the second claim. ■

The following statement is the key aspect of the proof, see the integer distance principle.

Lemma 1.4. *If A is infinite, there exists a straight line L containing it.*

The lemma follows from the following claim.

Claim: For any pair of points $a_0,a_1\in A$ it is impossible to have infinitely many members of A lying outside some tubular neighborhood of the straight line connecting a_0 and a_1.

It's easy to see that the lemma follows from the claim immediately. Indeed, assume the claim and suppose that some point a_2 lies outside the line connecting a_0 and a_1. Then all but finitely many points of A lie in a tubular neighborhood of the line connecting a_0 and a_2. In the same fashion, all but finitely many points of A must lie in a tubular neighborhood of the line connecting a_0 and a_1. The intersection of these two tubular neighborhoods is a bounded set which cannot contain infinitely many members of A due to the fact that A is separated. This argument will be repeated throughout the rest of the proof.

To prove the claim, suppose is not true. Then there exists a pair of points $a_0,a_1\in A$, such that (by Lemma 1.2) there is an infinite set $A_1\subset A$, such that the members $a\in A_1$ approach asymptotically some ρ_*-hyperboloid $\Gamma(a_0,a_1,\Delta)$, for some integer Δ, $0\le\Delta<\rho_*(a_0-a_1)$.

Let C_1 be the asymptotic cone for the above hyperboloid with the vertex O_1. A_1 is an infinite set of points located asymptotically close to C_1. By Lemma 1.2 and Proposition 1.3, one can always find a point $a_2 \in A_1$, close enough to C_1, such that the midpoint O_2 of the segment $[a_0 a_2]$ lies outside the cone C_1.

Then, since the lines $a_0 a_1$ and $a_0 a_2$ are transverse, there must exist an infinite subset $A_2 \subseteq A_1$ of points lying asymptotically close to some ρ_*-hyperboloid $\Gamma(a_0, a_2, \Delta)$ with an integer Δ, $0 \leq \Delta < \rho_*(a_0 - a_2)$. The right inequality is strict, because the line $a_0 a_2$ is transverse to the cone C_1, which in case of equality would lead to a contradiction. Indeed, in the case of equality, one would end up having infinitely many points of a well separated set A, located around a transverse intersection of C_1 with some tubular neighborhood of a straight line, see Lemma 1.2 and Proposition 1.3. This would be a contradiction.

Hence the points of the infinite set A_2 lie asymptotically close to the corresponding cone C_2 with the vertex O_2. Thus, the members of A_2 lie asymptotically close to the intersection $\mathcal{S}_2 \equiv C_1 \cap C_2$ (let also $\mathcal{S}_1 = C_1$). If $\mathcal{S}_2$ is bounded, there is a contradiction with the assumption that A_2 is infinite. Moreover, the intersection of the two cones along $\mathcal{S}_2$ is transverse, for any point thereon is formed by the intersection of a pair of straight lines connecting it to the vertices O_1 and O_2, the latter vertex lying by construction outside the cone C_1. Thus, $\mathcal{S}_2$ is a piecewise smooth unbounded surface of dimension $d-2$.

One can iterate this argument, reducing the dimension. Proceeding by induction, before the ith step, $i \geq 1$, there will be cones C_j, $j = 1, \ldots, i$, with the foci a_0, a_j, all intersecting transversely at a piecewise smooth unbounded surface $\mathcal{S}_i \equiv \bigcap_{j=1}^{i} C_j$ of dimension $d-i$. There will be an infinite set $A_i \subseteq A_1$ of points located asymptotically close to $\mathcal{S}_i$. At this point again one can again pick a point a_{i+1} such that the midpoint O_{i+1} of the segment $[a_0 a_{i+1}]$ lies outside $\bigcup_{j=1}^{i} C_j$, for a_0 is a focus for all the hyperboloids asymptotic to the cones $C_1, \ldots C_i$. Furthermore there is an infinite set $A_{i+1} \subseteq A_i$ of points, asymptotic to some cone C_{i+1}, whose vertex O_{i+1} does not belong to any of the cones $C_1, \ldots, C_i$. Hence, for any point of $\mathcal{S}_i$, the line connecting it with O_{i+1} is transverse to each of the cones $C_1, \ldots, C_i$, thus to their intersection $\mathcal{S}_i$. Therefore, the intersection $\mathcal{S}_{i+1} \equiv C_{i+1} \cap \mathcal{S}_i$ is transverse an should be an unbounded piecewise smooth surface of dimension $d-i-1$.

It becomes impossible to proceed with this construction for $i \geq d$, without a contradiction with the fact that A_1 should be infinite. This proves the claim and the lemma. ■

Proposition 1.5. *If $d \neq 1 \mod (4)$, the set A is finite. Otherwise, if $d = 4k+1$, $k \in \mathbb{N}$, the set A is either finite or it is an infinite subset of a lattice supported on some line L.*

Indeed, if A is infinite, by Lemma 1.4 it is contained in some line L. By fixing $0 \in A$, we ensure that L passes through the origin.

If $d \neq 1 \mod (4)$, take three different points $a_0, a_1, a_2 \in A$ such that a_1 lies between a_0 and a_2 (without loss of generality one can assume $a_0 = 0$). Then $\rho_*(a_2 - a_0) = \rho_*(a_2 - a_1) + \rho_*(a_1 - a_0)$. Since A is separated, the points a_0, a_1, a_2 can be chosen far enough from each other, so that the formula (0.3) cannot be satisfied simultaneously for the quantities $\rho_*(a_2 - a_0), \rho_*(a_2 - a_1)$ and $\rho_*(a_1 - a_0)$, due to the constant phase shift $\frac{d-1}{8}$ in it.

If $d = 1 \mod (4)$, then the phase shift $\frac{d-1}{8}$ is a half-integer itself. In this case, having infinitely many points on the line L would imply that for any pair of points a_0, a_1 there exists a distant point a such that the distances $\rho_*(a-a_0)$ as well as $\rho_*(a-a_1)$ are arbitrarily close to a half-integer. Thus $\rho_*(a_1-a_0)$ is a half-integer itself. ■

This completes the proof of Theorem 1. Finally, let us investigate further the case $d = 4k+1$, $k \in \mathbb{N}$. As in the proof of Lemma 1.1, suppose L is dual to the x-axis. Let $F(x)$ be the cross-sectional area of K along the x-axis. Suppose F is supported on the interval $[-1,1]$. At the points $x = \pm 1$ the function $F(x)$ has a smoothness defect of the same type and order as the function $(1-x^2)^{2k}$, corresponding to the case when K is a d-disk. Consider the Fourier series expansion

$$F(x) = \chi_{[-1,1]} \sum_{n=0}^{\infty} F_n \cos(\pi n x), \tag{1.8}$$

where

$$F_0 = 1, \qquad F_n = \int_{-1}^{1} F(x) \cos(\pi n x) dx, \quad n \in \mathbb{N}. \tag{1.9}$$

The existence of an infinite set of orthogonal exponents $A(K)$ with $0 \in A$, supported on the line L, is equivalent to the existence of an infinite set $E \subset \mathbb{Z}$ such that for all $n \in \mathbb{N} \cap [E \cup (E-E)]$ the corresponding Fourier coefficients $F_n = 0$.

This is of course possible. For example, take E to be a cyclic subgroup of $\mathbb{Z}$. With this in mind, let

$$F(x) = \chi_{[-1,1]} c_k [1 + \cos(\pi x)]^k, \ k \in \mathbb{N}, \tag{1.10}$$

where the constant $c_k > 0$ is chosen to ensure $F_0 = 1$. Then for $n > k$ one has $F_n = 0$ for the Fourier coefficients F_n above. The set E in question will be a cyclic group generated by $k+1$, and take $E = (k+1)\mathbb{N}$, for example. In dimension $d = 4k+1$ take K as a body of revolution

$$r(x) = [\tilde{c}_k (1 + \cos(\pi x))]^{\frac{1}{4}}, \tag{1.11}$$

where r is the radius-vector in $\mathbb{R}^{4k}$. The constant $\tilde{c}_k > 0$ is to yield the above cross-section area $F(x)$. The function $r(x)$ has a negative second derivative for $x \in (-1,1)$, whereas locally near $x = \pm 1$ it has the same type of singularity as the function $\sqrt{1-x^2}$. Thus K obtained this way is strictly convex.

However, for a variety of K's in $\mathbb{R}^{4k+1}$, $k \in \mathbb{N}$, including the case of a ball, the non-existence of an infinite set $A(K)$ is still the case. We thank one of the referees for pointing out that this property is generic in the sense that can be made precise. However, in view of Fuglede's result ([Fug01])we chose to illustrate the point by considering convex bodies close to the Euclidean ball.

Lemma 1.6. *Suppose K is such that the cross-section area $F(x)$ in any direction after a suitable dilation of K can be represented as*

(1.12)

$$F(x) = \chi_{[-1,1]}\left(\sum_{m\geq 2k}^{S} C_m(1-x^2)^m + \epsilon R(x)\right): \quad C_{2k} > 0,\ C_S \neq 0,\ S = 2k+N,\ N \geq 0,$$

where the error term $R(x)$ is a smooth even function of x, such that $D^{(\alpha)}R(1) = 0$, for $\alpha = 0, 1, \ldots, S$. Then for small ϵ the maximal set $A(K)$ is finite.

The proof consists in a direct computation. Using the Bessel function expansion of Lemma 1.1 along with the formula

$$\begin{aligned} J_{m+1/2}(z) &= \sqrt{\tfrac{2}{\pi z}} \Big\{\sin\left(z-\tfrac{\pi}{2}m\right) \textstyle\sum_{l=0}^{[m/2]} \frac{(-1)^l (m+2l)!}{(2l)!(m-2l)!(2z)^{2l}} \\ &+ \cos\left(z-\tfrac{\pi}{2}m\right) \textstyle\sum_{l=0}^{[(m-1)/2]} \frac{(-1)^l (m+2l+1)!}{(2l+1)!(m-2l-1)!(2z)^{2l+1}}\Big\}. \end{aligned} \tag{1.13}$$

we obtain (with the notation $[\cdot]$ for the integer part):

$$\begin{aligned} F_n &= \left(\tfrac{2}{\pi n}\right)^{4k} \textstyle\sum_{s=0}^{N}(-1)^{s+n}(2s+2k+1)!\left(\tfrac{2}{\pi n}\right)^{2s} \\ &\times \textstyle\sum_{l=0}^{s+1}(-1)^l 4^{-l} C_{2s+2k+1-l}\binom{2s+2k+1-l}{l} \\ &+ \epsilon\, O\left(n^{-2\left[\frac{S+1}{2}\right]-2}\right). \end{aligned} \tag{1.14}$$

The quantities, expressed by the sums in the second line of (1.14) bear responsibility for F_n being nonzero. These quantities are listed in the following table. Given $k \geq 1$, let $C_m = 0$ for $m < 2k$, whereas $C_{2k} > 0$. Also $C_S \neq 0$. Take the scalar product of the first row of the table with each subsequent ith row of the table, $i = 2, \ldots$ in order to get a coefficient, multiplying n^{-2i-2} up to a nonzero factor, coming from the first line of the formula (1.14).

$$
\begin{array}{ccccccccc}
-C_2 & C_3 & -C_4 & C_5 & -C_6 & C_7 & \dots & (-1)^S C_{S-1} & (-1)^{S+1} C_S \\
\frac{1}{4}\binom{2}{1} & \binom{3}{0} & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & \frac{1}{16}\binom{3}{2} & \frac{1}{4}\binom{4}{1} & \binom{5}{0} & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & \frac{1}{64}\binom{4}{3} & \frac{1}{16}\binom{5}{2} & \frac{1}{4}\binom{6}{1} & \binom{7}{0} & \dots & 0 & 0 \\
 & & & & \dots & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{4^{S-1}}\binom{S}{S-1}
\end{array}
$$

As $C_{2k} > 0$, $C_S \neq 0$ and ϵ is small enough, it follows that as $n \to \infty$, the absolute value of F_n is asymptotically bounded away from zero by a positive constant times $n^{-2[\frac{S+1}{2}]-2}$. ■

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Bibliography

[AP95] P. Agarwal and J. Pach, *Combinatorial geometry*, Wiley-Interscience Series in Discrete Mathematics and Optimization, a Wiley-Interscience Publication. John Wiley & Sons, Inc., New York. (1995).

[Berry94] M.V. Berry, *Evanescent and real waves in quantum billiards and Gaussian beams*, J. Phys. A: Math. Gen. **27** (1994), L391-L398.

[Erdös45] P. Erdös, *Integral distances*, Bull. Am. Math. Soc. **51** (1945), 996.

[Falc87] K.J. Falconer, *On the Hausdorff dimensions of distance sets*, Mathematika **32** (1985), 206–212.

[Fug01] B. Fuglede, *Orthogonal Exponentials on the Ball*, Expo. Math. **19** (2001), 267-272.

[Fug74] B. Fuglede, *Commuting self-adjoint partial differential operators and a group theoretic problem*, J. Funct. Anal. **16** (1974), 101-121.

[HI2003] S. Hofmann and A. Iosevich, *Circular averages and Falconer/Erdos distance conjecture in the plane for random metrics*, (submitted for publication) (2003).

[Herz64] C.S. Herz, *Fourier transforms related to convex sets*, Ann. of Math. **75** (1962), 81-92.

[IKT01] A. Iosevich, N. Katz, and T. Tao, *Convex bodies with a point of curvature do not have Fourier bases*, Amer. J. Math. **123** (2001), 115-120.

[IKT02] A. Iosevich, N. Katz, and T. Tao, *Fuglede conjecture holds for convex planar domains*, (submitted for publication) (2001).

[IL2003] A. Iosevich and I. Laba, *K-distance, Falconer conjecture, and discrete analogs*, LMS (submitted) (2003).

[IL2003II] A. Iosevich and I. Laba, *Cones over varieties, generalized distance sets and representations of integers*, (in preparation) (2003).

[Kol00] M. Kolountzakis, *Non-symmetric convex domains have no basis of exponentials*, Illinois J. Math. **44** (2000), 542–550.

[Kuz77] A.V. Kuz'minyh, *Integer-valued distances (Russian)*, Dokl. Akad. Nauk SSSR **232** (1977), 1008–1010.

[Muller99] W. Muller, *Lattice points in large convex bodies*, Monatshefte Math. **128** (1999), 315-330.

[Sogge] C.D. Sogge, *Fourier integrals in classical analysis*, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge **105** (1993).

[Wolff99] T. Wolff, *Decay of circular means of Fourier transforms of measures*, Internat. Math. Res. Notices **10** (1999), 547–567.

Alex Iosevich. Department of Mathematics. University of Missouri, Columbia, MO 65211 U.S.A.
E-mail address: iosevich@math.missouri.edu

Mischa Rudnev. School of Mathematics. University of Bristol, University Walk, Bristol BS8 1TW, UK
E-mail address: m.rudnev@bris.ac.uk