Soln #7

Pg 96

4) Consider $r : [0,1] \to G$ a closed rectifiable curve in G.

Define $\Gamma : [0,1]^2 \to G$ by

$$\Gamma(s,t) := (1-t)\, r(s) + t \cdot \frac{r(s)}{|r(s)|}$$

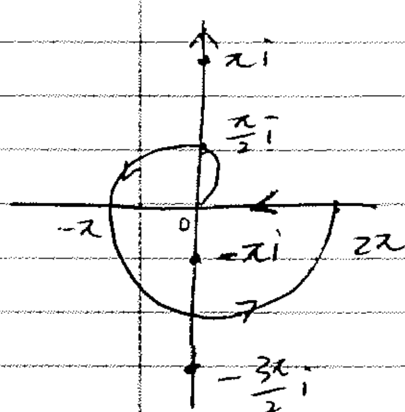
Easy to check $\begin{cases} \Gamma(s,0) = r(s) \\ |\Gamma(s,1)| = 1 \end{cases}$, $\forall s \in [0,1]$

Hence conclusion. $\square$

6) $$\int_\gamma \frac{dz}{z^2+\pi^2} = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-\pi i} - \frac{1}{2\pi i}\int_\gamma \frac{dz}{z+\pi i}$$

$$= n(\gamma; \pi i) - n(\gamma; -\pi i)$$

$$= 0 - 1 = -1$$



11) Let $f(z) = e^{z} - e^{-z}$

~~a)~~

$$f^{(3)}(0)\, n(\gamma;0) = \frac{3!}{2\pi i}\int_\gamma \frac{f(z)}{z^4}dz$$

$$\Rightarrow \int_\gamma \frac{f(z)}{z^4}dz = \frac{2\pi i}{3} n(\gamma;0)$$

a) $\frac{2\pi i}{3}$ b) $\frac{4\pi i}{3}$ c) $\frac{4\pi i}{3}$

Pg 99

3)

$\Leftarrow$) Since f analytic, and $f^{(m-1)}(a) = \cdots = f(a) = 0$

$$f(z) = \sum \frac{f^{(n)}(a)}{n!}(z-a)^n = \sum_{n=m} \frac{f^{(n)}(a)}{n!}(z-a)^n$$

$$= (z-a)^m \cdot P(z) \quad \text{where } P(a) \neq 0$$

as $f^{(m)}(a) \neq 0$

Therefore a is of multi. m.

$\Rightarrow$) If a is a zero of multi. m,

then $f(z) = (z-a)^m g(z)$, $g(a) \neq 0$

and g analytic.

Easy to check $f^{(m-1)}(a) = \cdots = f(a) = 0$.

Also have $f^{(m)}(a) = m!\, g(a) \neq 0$ $\boxed{\checkmark}$

4) By Cor 7.6, $f^{-1} : f(G) \to \mathbb{C}$ is

analytic and $(f^{-1})'(w) = f'(z)^{-1}$ where $w = f($

~~we have~~

Since $f^{-1} f(z) = z$, we have

$f^{-1'}(f(z)) \cdot f'(z) = 1$, $\forall z \in G$

So $f'(z) \neq 0$ $\boxed{\checkmark}$

Pg 110

1)

a) $\lim_{z\to 0}(z-0)f(z) = \lim_{z\to 0} \sin z = 0$

Rem. Sing. , $f(0)=1$

b) Simple pole / $\frac{1}{z}$

c) rem. sing. / $f(0)=0$

Mainly use Cor 1.18

d) ess. sing. / $\mathbb{C}-\{0\}$

e) pole / $\frac{1}{z}$

f) ess. sing. / $\mathbb{C}$

g) pole / $-\frac{1}{z}$

h) pole / $-1/z$

i) ess. / $\mathbb{C}$

j) ess / $\mathbb{C}$

2) Easy to verify that

$$\cancel{f(z)=}\quad r(z) = \frac{1}{3(z^2+z+1)} + \frac{2}{3(z-1)^2}.$$

As $z^2+z+1 = (z-\omega)(z-\omega^2)$, $\omega = e^{\frac{2}{3}\pi i}$

we can further decompose $r(z)$ into

$$r(z) = \frac{1}{3(z-\omega)(z-\omega^2)} + \frac{2}{3(z-1)^2}$$

$$= \frac{1}{3(\omega-\omega^2)} \cdot \left(\frac{1}{z-\omega} - \frac{1}{z-\omega^2} \right) + \frac{2}{3(z-1)^2}$$

4)

a) For $0<|z|<1$,

$$f(z) = \frac{1}{2}\cdot\frac{1}{z} - \frac{1}{z-1} + \frac{1}{z-2}\cdot\frac{1}{2}$$

$$= \sum_{n=-1}^{\infty} \left(1 - \frac{1}{2^{n+2}}\right) z^n$$

b) For $1 < |z| < 2$

$$\frac{1}{z-2} = -\frac{1}{2} \cdot \frac{1}{1 - z/2}$$

$$= -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1 - 1/z} = \frac{1}{z} \cdot \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} z^{-n-1}$$

$$\Rightarrow f(z) = \frac{1}{2} z^{-1} - \sum_{n=0}^{\infty} z^{-n-1} - \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

$$= \sum a_n z^n$$

$$a_n = \begin{cases} -1, & n < -1 \\ -1/2 & n = -1 \\ -2^{-n-2} & n \geq 0 \end{cases}$$

c)

$$\frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1 - 2/z} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$$

$$\Rightarrow f(z) = \sum_{n=2}^{\infty} \left(-1 + 2^{n-2}\right) z^{-n}$$

11) By $e^{x} = \sum_{n=0} \frac{1}{n!} x^{n}$

we have $e^{1/z} = \sum_{n=0} \frac{1}{n!} z^{-n}$

$$= \sum_{-\infty}^{\infty} a_n z^{n} \quad \text{with} \quad a_n = \begin{cases} \frac{1}{n!}, & n \leq 0 \\ 0, & n > 0 \end{cases}$$