CONVEX BODIES WITH A POINT OF CURVATURE DO NOT HAVE FOURIER BASES

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Abstract. We prove that no smooth symmetric convex body Ω with at least one point of non-vanishing Gaussian curvature can admit an orthogonal basis of exponentials. (The non-symmetric case was proven in [Kol]). This is further evidence of Fuglede's conjecture, which states that such a basis is possible if and only if Ω can tile $\mathbb{R}^d$ by translations.

Introduction and statement of results

Let Ω be a domain in $\mathbb{R}^d$, i.e., Ω is a Lebesgue measurable subset of $\mathbb{R}^d$ with finite non-zero Lebesgue measure. We say that a set $\Lambda \subset \mathbb{R}^d$ is a *spectrum* of Ω if $\{e^{2\pi i x\cdot\lambda}\}_{\lambda\in\Lambda}$ is an orthogonal basis of $L^2(\Omega)$.

Conjecture A. [Fug] A domain Ω admits a spectrum if and only if it is possible to tile $\mathbb{R}^d$ by a family of translates of Ω.

Fuglede proved this conjecture under the additional assumption that the tiling set or the spectrum are lattice subsets of $\mathbb{R}^d$. In general, this conjecture is nowhere near resolution, even in dimension one. It has been the subject of recent research, see for example [JoPe2] and [LaWa].

In this paper we shall address the following special case of Conjecture A.

Conjecture B. *Suppose that Ω is a convex body with at least one point of non-vanishing Gaussian curvature. Then Ω does not admit a spectrum.*

The set Ω is called *symmetric* with respect to a point $x_0 \in \mathbb{R}^d$ if $y \in \Omega$ implies that $2x_0 - y \in \Omega$. In [Kol], Kolountzakis proved Conjecture B under the assumption that Ω is not symmetric with respect to any point. However, the symmetric case appears resistant to these methods.

In [IKP], (Theorem 1), the authors proved Conjecture B in the case where Ω is the ball in $\mathbb{R}^d$, $d > 1$. By generalizing the arguments of [IKP], we show

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Theorem 0.1. *Suppose that Ω is a symmetric convex body in $\mathbb{R}^d$, $d \geq 2$. If the boundary of Ω is smooth, then Ω does not admit a spectrum. The same conclusion holds in $\mathbb{R}^2$ if the boundary of Ω is piece-wise smooth, and has at least one point of non-vanishing Gaussian curvature.*

By Gauss-Bonnet theorem, a smooth convex hypersurface has at least one point of non-vanishing Gaussian curvature. Thus Theorem 0.1 verifies Conjecture B for smooth Ω, and for piece-wise smooth $\Omega \subset \mathbb{R}^2$.

The proof of Theorem 0.1 is based on the geometry of the set

$$Z_\Omega = \left\{ \xi \in \mathbb{R}^d : \hat{\chi}_\Omega(\xi) = \int_\Omega e^{-2\pi i \xi \cdot x} \, dx = 0 \right\}. \tag{0.1}$$

The relevance of this set lies in the trivial observation that for any spectrum Λ of Ω, we have

$$\lambda - \lambda' \in Z_\Omega \text{ for all } \lambda, \lambda' \in \Lambda, \lambda \neq \lambda'. \tag{0.2}$$

For any $\eta \in \mathbb{R}^d$ and ball B, define the set $X_{\Omega,\eta,B}$ by

$$X_{\Omega,\eta,B} = Z_\Omega \cap B \cap (Z_\Omega - \eta) \cap (B - \eta). \tag{0.3}$$

Definition. We say that a set $S \subset \mathbb{R}^d$ is 1-separated if $\inf\{|x-y| : x, y \in S\} \geq 1$.

Define the entropy $\mathcal{E}(X_{\Omega,\eta,B})$ to be the largest number of 1-separated points one can place inside $X_{\Omega,\eta,B}$.

Theorem 0.1 now follows immediately from the following two propositions.

Proposition 0.2. *If Ω is a spectral set and B is a ball of radius $R \gg 1$, then there exists an $|\eta| \sim 1$ such that*

$$\mathcal{E}(X_{\Omega,\eta,B}) \sim R^d. \tag{0.4}$$

Proposition 0.3. *Let Ω be as in Theorem 0.1. Then for every $R \gg 1$ there exists a ball B of radius R and $\epsilon > 0$ such that*

$$\mathcal{E}(X_{\Omega,\eta,B}) \lesssim R^{d-\epsilon} \tag{0.5}$$

for all $|\eta| \sim 1$.

The proof of Proposition 0.3 will show that $\epsilon = 1$ if the boundary of Ω is smooth. If $d = 2$ and the boundary of Ω is piece-wise smooth, then $\epsilon = \frac{1}{2}$.

To illustrate these propositions we give two examples. When Ω is a cube, then Z_Ω is a union of hyperplanes, and $X_{\Omega,\eta,B}$ can be the union of $O(R)$ hyperplanes in B with total entropy about R^d. However, when Ω is a sphere, Z_Ω is the union of spheres, and $X_{\Omega,\eta,B}$ is the union of $O(R)$ $d-2$-dimensional spheres in B, with total entropy about R^{d-1}.

Techniques similar to those used to prove Proposition 0.2 can be used to show the non-existence of spectra for other types of domains than convex bodies with a point of non-zero curvature. In a subsequent paper the authors will address this issue in the context of convex polygons.

Notation. Throughout the paper, $a \sim b$, $a, b > 0$, means that there exist positive constants c_1 and c_2, such that $c_1 a \le b \le c_2 a$. Similarly, $a \lesssim b$, $a, b > 0$, means that there exist a positive constant c, such that $a \le cb$.

Proof of Proposition 0.2

Let Ω, B, R be as in Proposition 0.2. Let Λ denote the putative spectrum for Ω.

Let B_1, B_2 be any balls of radius $\sim R$ such that

$$B_1 - B_2 \subset B. \tag{1.1}$$

Since $\hat{\chi}_\Omega$ is smooth and non-vanishing at the origin, we see that $dist(0, Z_\Omega) \gtrsim 1$. From (0.2) we thus have

$$|\lambda - \lambda'| \gtrsim 1 \text{ for all } \lambda, \lambda' \in \Lambda, \lambda \neq \lambda'. \tag{1.2}$$

We also have the density property

$$\# B_i \cap \Lambda \gtrsim R^d \tag{1.3}$$

for $i = 1, 2$; see e.g. [Lan], [Beu], [IoPe2].

From these two properties we may find $\lambda_1, \lambda_2 \in B_1$ so that

$$|\lambda_1 - \lambda_2| \sim 1. \tag{1.4}$$

From (0.2) and (1.1) we have

$$\lambda - \lambda_1, \lambda - \lambda_2 \in Z_\Omega \cap B \tag{1.5}$$

for all $\lambda \in B_2 \cap \Lambda$. We can re-arrange this as

$$X_{\Omega, \lambda_2 - \lambda_1, B} \supset (B_2 \cap \Lambda) - \lambda_2. \tag{1.6}$$

The claim then follows from (1.2), (1.3), and (1.4). $\square$

Proof of Proposition 0.3

We first prove Proposition 0.3 in the case where the boundary of Ω is smooth. We shall then explain how the proof can be modified in two dimensions to yield the conclusion of the theorem under the assumption that the boundary of Ω is piece-wise smooth and has at least one point of non-vanishing Gaussian curvature.

Let Ω be as in Theorem 0.1; we may assume that Ω is symmetric around the origin. Our main tool will be the method of stationary phase.

Our starting point is the formula

$$\widehat{\chi}_\Omega(\xi) = (i|\xi|)^{-1} \int_{\partial\Omega} e^{2\pi i x\cdot\xi} \left(\frac{\xi}{|\xi|} \cdot n(x)\right) d\sigma(x) \tag{2.1}$$

of Herz [H], where n denotes the unit outward normal vector to $\partial\Omega$ and $d\sigma$ denotes the Lebesgue measure on $\partial\Omega$.

Let x_0 be a point on $\partial\Omega$ with non-vanishing Gaussian curvature. By the symmetry assumption $-x_0$ is also in Ω. Let ψ be a smooth cutoff function supported in a small neighborhood of x_0. Let $\mathcal{C}$ denote the cone of vectors normal to Ω on the support of ψ. If the boundary of Ω is smooth, integration by parts yields, for $\xi \in \mathcal{C}$,

$$(i|\xi|)\widehat{\chi}_\Omega(\xi) = \int_{\partial\Omega} e^{2\pi i x\cdot\xi} \left(\frac{\xi}{|\xi|} \cdot n(x)\right) (\psi(x) + \psi(-x)) d\sigma(x) + O((1 + |\xi|)^{-N}), \tag{2.2}$$

where N is an arbitrary constant.

Fix $R \gg 1$, and let B be a ball of radius R in $\mathcal{C}$ which is a distance $\sim R$ from the origin. A stationary phase calculation (see e.g. [H]) gives

$$\int_{\partial\Omega} e^{2\pi i x\cdot\xi} \frac{\xi}{|\xi|} \cdot n(x)(\psi(x) + \psi(-x))d\sigma(x) = a(\xi) \cos\left(P(\xi) - \frac{\pi d}{4}\right) + O(R^{-\frac{d+1}{2}}) \tag{2.3}$$

for all $\xi \in B$, where $|a(\xi)| \sim R^{-\frac{d-1}{2}}$ and

$$P(\xi) = \sup_{x\in\partial\Omega} x \cdot \xi. \tag{2.4}$$

Combining this with (2.2) we thus see that

$$\left|\cos\left(P(\xi) - \frac{\pi d}{4}\right)\right| \lesssim R^{-1} \tag{2.5}$$

for all $\xi \in Z_\Omega \cap B$.

Fix $|\eta| \sim 1$. By definition, $\xi \in X_{\Omega,\eta,B}$ implies that ξ and $\xi + \eta$ are in $Z_\Omega \cap B$. From (2.5) we thus see that

$$dist(P(\xi + \eta) - P(\xi), \pi\mathbb{Z}) \lesssim R^{-1} \tag{2.6}$$

for all $\xi \in X_{\Omega,\eta,B}$.

From (2.4) we deduce that P is smooth on B, ∇P is homogeneous of degree 0 and non-degenerate on B. In fact, as ξ ranges over B, ∇P parametrizes a piece of $\partial\Omega$ where the Gaussian curvature does not vanish. A direct computation in the style of [H] shows that the Hessian quadratic form of P is homogeneous of degree -1 and has rank $d-1$. Moreover, the eigenvalues of this form at ξ are bounded by the constant multiple of the reciprocal of the smallest principal curvature at x, where x is the unique point where ξ is normal to $\partial\Omega$. (See also [So], Ch. 1). At this point, Taylor's theorem gives

$$dist(\nabla P(\xi) \cdot \eta, \pi\mathbb{Z}) \lesssim R^{-1}. \tag{2.7}$$

Now, our observation imply that $\nabla P(\xi) \cdot \eta$ is bounded independent of R, from which we deduce there are only a finite number, independent of R, of elements of $\pi\mathbb{Z}$ which are relevant in this distance calculation.

We next observe that our conclusions about $\nabla P \cdot \eta$, combined with (2.7), imply that $\xi/|\xi|$ lies within $O(R^{-1})$ of a finite number, independent of R, of $d-2$-dimensional surfaces in S^{d-1}. Thus ξ is contained in finitely many, independent of R, subsets of B of measure $\approx R^{d-1}$. By definition of entropy (see above), it follows that $\mathcal{E}(X_{\Omega,\eta,B}) \lesssim R^{d-1}$, which is the conclusion of Proposition 0.3 with $\epsilon = 1$.

If the boundary of Ω is piece-wise smooth, in any dimension, and has at least one point of non-vanishing Gaussian curvature, then $O(R^{-\frac{d+1}{2}})$ in (2.3) must be replaced with $O(R^{-1})$. In two dimensions this leads one to replace $O(R^{-1})$ in (2.5), (2.6), and (2.7) by $O(R^{-\frac{1}{2}})$, which yields the conclusion of Proposition 0.3 with $\epsilon = \frac{1}{2}$.

$\square$

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