Exam # 1: Math 1500

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Write your solutions on the clear white paper provided by the examiner. Clearly indicate which problem you are solving on a given piece of paper and do not write a solution to more than one problem on a single piece of paper. Show all your work. Good luck!

Problem 1. Let $f(x) = -7x^2 - 28x - 18$. Complete the square and graph the resulting parabola labeling the vertex. Indicate if the parabola opens up or down and briefly explain why.

Solution: We have

$$\begin{aligned} f(x) &= -7\left(x^2 + 4x + \frac{18}{7}\right) \\ &= -7\left((x+2)^2 - 4 + \frac{18}{7}\right) \\ &= -7(x+2)^2 + 10. \end{aligned}$$

The parabola opens down because of the negative sign and the vertex is at $(-2, 10)$.

Problem 2. Determine if the following limits exist. If a given limit exists, determine its value. If it does not exist, explain why.

$$a)\ \lim_{x \to -1} \frac{x+1}{|x+1|}.$$

$$b)\ \lim_{x \to 3} \frac{\sqrt{2x+3} - 3}{x - 3}.$$

Solution: The first limit does not exist because the limit from the left is -1 and the limit from the right is $+1$. The second limit is

$$\lim_{x \to 3} \frac{\sqrt{2x+3} - 3}{x - 3} = \lim_{x \to 3} \frac{\sqrt{2x+3} - 3}{x - 3} \cdot \frac{\sqrt{2x+3} + 3}{\sqrt{2x+3} + 3}$$

$$= \lim_{x\to 3} \frac{2x+3-9}{x-3} \cdot \frac{1}{\sqrt{2x+3}+3}$$

$$= \lim_{x\to 3} \frac{2(x-3)}{x-3} \cdot \frac{1}{\sqrt{2x+3}+3}$$

$$= 2\lim_{x\to 3} \frac{1}{\sqrt{2x+3}+3} = \frac{1}{3}.$$

Problem 3. Let

$$g(x) = \begin{cases} \frac{x^2-9}{x+3}, & x \neq -3 \\ -5, & x = -3 \end{cases}$$

Is this function continuous at $x = -3$? If it is continuous, explain why. If it is not continuous, explain why not.

Solution: Continuity of $g(x)$ at -3 would mean that

$$\lim_{x\to -3} g(x) = g(-3).$$

By definition of $g(x)$ above, $g(-3) = -5$. On the other hand,

$$\lim_{x\to -3} g(x) = \lim_{x\to -3} \frac{x^2-9}{x+3}$$

$$= \lim_{x\to -3} \frac{(x-3)(x+3)}{x+3} = \lim_{x\to -3} x-3 = -6.$$

Since $-5 \neq -6$, $g(x)$ is not continuous at $x = -3$.

Problem 4. Let $f(x) = 2x^2 + 1$. Compute

$$\lim_{t\to 0} \frac{f(1+t) - f(1)}{t}.$$

Solution We have

$$\lim_{t\to 0} \frac{2(1+t)^2 + 1 - 2\cdot 1^2 - 1}{t}$$

$$= \lim_{t\to 0} \frac{4t+2t^2}{t} = \lim_{t\to 0} 4 + 2t = 4.$$

Problem 5. Let $f(x) = \sin^2(x)$, $g(x) = \frac{1}{\sqrt{|x|+1}}$, and $h(x) = \frac{x}{x^2+1}$. Compute $f(g(h(x)))$. DO NOT SIMPLIFY!

Solution: We have

$$g(h(x)) = \frac{1}{\sqrt{|\frac{x}{x^2+1}| + 1}},$$

so

$$f(g(h(x)) = \sin^2\left(\frac{1}{\sqrt{|\frac{x}{x^2+1}| + 1}}\right).$$