Math 265H

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1. Basic inequalities

A large chunk of mathematical analysis is about estimating complicated quantities quickly and efficiently. Recall that if A, B are real numbers,

$$(1.1) \qquad AB \leq \frac{A^2 + B^2}{2},$$

which follows from the fact that

$$(A - B)^2 \geq 0,$$

expanding the square and regrouping terms.

Now suppose that we have two sequences of real numbers, $\{a_j\}_{j=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$. What can we say about

$$\sum_{j=1}^{\infty} a_j b_j \ ?$$

If we apply the inequality above to each product $a_j b_j$, we see that

$$\sum_{j=1}^{\infty} a_j b_j \leq \frac{1}{2} \sum_{j=1}^{\infty} a_j^2 + \frac{1}{2} \sum_{j=1}^{\infty} b_j^2.$$

Is this inequality any good? Suppose that $b_j = t a_j$ for some non-zero $t \in \mathbb{R}$. Then on the left hand side we have

$$t \sum_{j=1}^{\infty} a_j^2$$

while on the right hand side we have

$$\frac{t^2 + 1}{2} \sum_{j=1}^{\infty} a_j^2,$$

so dividing both sides by $\sum_{j=1}^{\infty} a_j^2$ we end up with a simple inequality

$$t \leq \frac{t^2 + 1}{2}.$$

While true, this inequality is incredibly wasteful when t is very large. It is also quite wasteful when $t = 0$! Furthermore, suppose that we multiply a_j by t and b_j by s. The left hand side becomes

$$ts \sum_{j=1}^{\infty} a_j b_j.$$

The right hand side turns into

$$\frac{t^2}{2} \sum_{j=1}^{\infty} a_j^2 + \frac{s^2}{2} \sum_{j=1}^{\infty} b_j^2$$

and this is just weird. To see why, consider the case when the ratio of t to s is very large. It is not difficult to see that multiplying by scaling factors makes the inequality worse. In such cases, we say, somewhat informally, that the inequality does not scale correctly.

Let's see if we can do a bit better by using our tools a bit more creatively. Let

$$a = \left(\sum_{j=1}^{\infty} a_j^2 \right)^{\frac{1}{2}} \text{ and } b = \left(\sum_{j=1}^{\infty} b_j^2 \right)^{\frac{1}{2}}.$$

Let's estimate

$$\sum_{j=1}^{\infty} \frac{a_j}{a} \cdot \frac{b_j}{b}.$$

Setting

$$A = \frac{a_j}{a} \text{ and } B = \frac{b_j}{b},$$

we see that

$$\sum_{j=1}^{\infty} \frac{a_j}{a} \cdot \frac{b_j}{b} \leq \frac{1}{2} \cdot \frac{\sum_{j=1}^{\infty} a_j^2}{a^2} + \frac{1}{2} \cdot \frac{\sum_{j=1}^{\infty} a_j^2}{b^2}$$

$$= \frac{1}{2} + \frac{1}{2} = 1.$$

Thus we arrive at the Cauchy-Schwartz inequality

(1.2)
$$\sum_{j=1}^{\infty} a_j b_j \leq \left(\sum_{j=1}^{\infty} a_j^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^{\infty} b_j^2 \right)^{\frac{1}{2}}.$$

Let's test this inequality to see if it is any better than our precious attempt. Multiply a_j by t and b_j by s. Then the left hand side is

$$ts \sum_{j=1}^{\infty} a_j b_j$$

while the right hand side is

$$ts \left(\sum_{j=1}^{\infty} a_j^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^{\infty} b_j^2 \right)^{\frac{1}{2}}.$$

We conclude that this inequality **scales correctly**!

PROBLEM 1.1. Prove that the equality in (1.2) holds if and only if there exists a real number t such that $b_j = t a_j$.

Another basic inequality is the one relating the arithmetic means of a finite sequence of positive real numbers, given by

$$\frac{1}{n} \sum_{j=1}^{n} a_j$$

and the geometric mean, given by

$$\left(\prod_{j=1}^{n} a_j \right)^{\frac{1}{n}}.$$

There are many proofs of the fact that

(1.3) $$\frac{1}{n} \sum_{j=1}^{n} a_j \geq \left(\prod_{j=1}^{n} a_j \right)^{\frac{1}{n}},$$

but we shall go through one I find particularly enlightening.

PROBLEM 1.2. Prove (1.3) in the case when $n = 2^k$, $k = 1, 2, \ldots$.

Let's do the case $k = 1$ together. We must show that

$$(a_1 a_2)^{\frac{1}{2}} \leq \frac{a_1 + a_2}{2}.$$

Let $A = \sqrt{a_1}$ and $B = \sqrt{a_2}$. Then this inequality takes the form

$$AB \leq \frac{A^2 + B^2}{2}$$

and this is precisely the inequality (1.1) above. Prove the general case in Problem 1.2 by induction on k.

In view of Problem 1.2, it is enough to prove that if (1.3) holds for $n + 1$, then it holds for n.

PROBLEM 1.3. Assume that

(1.4) $$\left(\prod_{j=1}^{n+1} a_j \right)^{\frac{1}{n+1}} \leq \frac{1}{n+1} \sum_{j=1}^{n+1} a_j.$$

Set $a_{n+1} = \frac{1}{n} \sum_{j=1}^{n} a_j$ and show that (1.4) reduces to (1.3).

2. Why do we need real numbers?

Consider the equation $x^2 = 2$ and let's try to make do with rational numbers. Suppose that

$$x = \frac{m}{n},$$

where m and n are relatively prime integers (which means that they have no prime factors in common).

Then
$$2 = \frac{m^2}{n^2}, \text{ or, equivalently } m^2 = 2n^2.$$
This implies that 2 divides m^2, which can only happen if 2 divides m and so 4 divides m^2. This implies that n^2 is even, which means that n is even. Thus both n and m are even, which contradicts the assumption that they are relatively prime. We conclude that we may not write x in the form $\frac{m}{n}$, where m and n are integers. This means that the solution of the equation $x^2 = 2$, if one exists, is not a rational number. If we want to bother with such equations in the future, we may want to extend the rationals a bit.

PROBLEM 2.1. Do problem 2 on page 22.

3. Suprema and infima

This is Math 171-172 stuff, but you should review it carefully and do the following problems. You should similarly review the field axioms and the basic theorems that follow on pages 5-8.

PROBLEM 3.1. Do problem 6 on page 22.

PROBLEM 3.2. Do problem 7 on page 22.

4. Ordered sets

Let S be a set. An order on S is a relation, denoted by $<$, with the following properties:

i) If $x, y \in S$ then one and only one of the statements
$$x < y, x = y, y < x \text{ is true.}$$
ii) If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

An ordered set is a set where an order is defined. For example, $\mathbb{Q}$, the rational numbers, is an ordered set where $r < s$ if $s - r$ is a positive rational number.

We say that an ordered set S has the **least upper bound property** if the following is true: If $E \subset S$, E is non-empty, is bounded above, then $sup(E)$ exists in S.

PROBLEM 4.1. Prove that rational numbers do not possess the least upper bound property. The proof is in the book, but you should write it out convincingly.

5. The Real Field

THEOREM 5.1. *There exists an ordered field $\mathbb{R}$ which has the least upper bound property and contains $\mathbb{Q}$ as a subfield.*

We shall not prove this theorem in class, but you are encouraged to read the proof in the book.

THEOREM 5.2. *a) If $x, y \in \mathbb{R}$ and $x > 0$, then there is a positive integer n such that $nx > y$. b) If $x, y \in \mathbb{R}$ and $x < y$, then there exists $p \in \mathbb{Q}$ such that $x < p < y$.*

To prove part a), let A be the set of all nx where n runs through the positive integers. If a) were false, then y would be an upper bound for A. But then A has the least upper bound in $\mathbb{R}$, denoted by α. Since $x > 0$, $\alpha - x < \alpha$, and $\alpha - x$ is not an upper bound for A. Hence $\alpha - x < mx$ for some positive integer m. But then $\alpha < x(m+1)$, which is impossible since α is the least upper bound for A.

b) Since $x < y$, we have $y - x > 0$, so by part a) there exists a positive integer n such that

$$n(y - x) > 1.$$

Part a) also implies that there exist positive integers m_1, m_2 such that

$$m_1 > nx \text{ and } m_2 > -nx.$$

It follows that

$$-m_2 < nx < m_1.$$

It follows that there exists an integer m, with $-m_2 \leq m \leq m_1$ such that

$$m - 1 \leq nx < m,$$

so

$$nx < m \leq 1 + nx < ny$$

and we conclude that

$$x < \frac{m}{n} < y$$

as desired.

We now prove the existence of nth roots.

THEOREM 5.3. *For every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$.*

We first prove uniqueness. Observe that $0 < y_1 < y_2$ implies that $y_1^n < y_2^n$ and we are done. We must now prove existence.
Let

$$E = \{t > 0 : t^n < x\}.$$

If $t = \frac{x}{1+x}$, then $0 \leq t \leq 1$, so $t^n \leq t < x$. This shows that E is not empty. If $t > 1 + x$, then $t^n \geq t > x$, so $1 + x$ is an upper bound for E. We conclude that E has the least upper bound y. To prove that $y^n = x$ we will show that both $y^n < x$ and $y^n > x$ are absurd.
Observe that if $0 < a < b$, then

$$b^n - a^n < (b - a)nb^{n-1}.$$

Assume that $y^n < x$. Choose h such that $0 < h < 1$ and

$$h < \frac{x - y^n}{n(y+1)^{n-1}}.$$

Put $a = y, b = y + h$. Then

$$(y + h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n.$$

Thus $(y + h)^n < x$ and $y + h \in E$. Since $y + h > y$, we have a contradiction.

Let us now assume that $y^n > x$. Put

$$k = \frac{y^n - x}{ny^{n-1}}.$$

Then $0 < k < y$. If $t \geq y - k$, we conclude that

$$y^n - t^n \leq y^n - (y - k)^n < kny^{n-1} = y^n - x.$$

Thus $t^n > x$ and $t \notin E$. It follows that $y - k$ is an upper bound of E. But $y - k < y$, which contradicts the fact that y is an upper bound of E. Hence $y^n = x$ and the proof is complete.

6. The Complex Field

A complex number is an ordered pair (a, b) of real numbers. Addition is defined via

$$x + y = (a, b) + (c, d) = (a + c, b + d)$$

and multiplication is given by

$$x \cdot y = (ac - bd, ad + bc).$$

THEOREM 6.1. *Under these operations make complex numbers, denoted by $\mathbb{C}$ into a field with $O = (0, 0)$ and the multiplicative identity given by $(1, 0)$.*

The only non-trivial step is proving the existence of multiplicative inverses. This can be done directly, but we proceed as follows. Define $i = (0, 1)$. Then $(a, b) = a + bi$ by a direct calculation. Formally,

$$\frac{1}{a + bi} = \frac{a - bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 + b^2}.$$

This suggests that the inverse of (a, b) is

$$\left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right),$$

which can be verified by a direct calculation. The only thing that can prevent us from writing this expression down is if $a^2 + b^2 = 0$. But this can happen only if $a = 0$ and $b = 0$. Thus every non-zero zero complex number has a multiplicative inverse.

We have seen above that every complex number can be written in the form $z = a + bi$. We shall write $a = Re(z)$ and $b = Im(z)$. Also define $\bar{z} = a - bi$, the conjugate of z.

THEOREM 6.2. *If z and w are complex numbers, then*
a) $\overline{z + w} = \bar{z} + \bar{w}$.
b) $\overline{zw} = \bar{z} \cdot \bar{w}$.
c) $z + \bar{z} = 2Re(z)$, $z - \bar{z} = 2iIm(z)$,
d) $z \cdot \bar{z}$ is real an positive, except when $z = O$.

DEFINITION 6.3. If z is a complex number, its absolute value $|z|$ is the non-negative square root of $z \cdot \bar{z}$.

How do we know that this object exists and is unique? Look at Theorem 6.2 part d) and Theorem 5.3.

THEOREM 6.4. *Let z and w be complex numbers. Then*
a) $|z| > 0$ *unless* $z = 0$, $|0| = 0$,
b) $|\bar{z}| = |z|$,
c) $|zw| = |z||w|$,
d) $|Re(z)| \le |z|$,
e) $|z + w| \le |z| + |w|$.

The first two parts are trivial. To prove c), observe that

$$zw = (a + ib)(c + id) = (ac - bd, ad + bc).$$

It follows that

$$|zw|^2 = (ac - bd)^2 + (ad + bc)^2 = a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2$$

$$= (a^2 + b^2)(c^2 + d^2) = |z|^2|w|^2$$

and we are done.

To prove part d) it is enough to show that if $z = a + bi$, then

$$|a| \le \sqrt{a^2 + b^2},$$

or, equivalently

$$a^2 \le a^2 + b^2,$$

which is obvious.

To prove e), observe that

$$|z + w|^2 = (z + w)(\bar{z} + \bar{w}) = z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w}$$

$$= |z|^2 + |w|^2 + z\bar{w} + \bar{z}w.$$

Observe that

$$z\bar{w} + \bar{z}w = (a + ib)(c - di) + (a - bi)(c + di)$$

$$= ac + bd + i(bc - ad) + ac + bd + i(ad - bc) = 2(ac + bd) = 2Re(z\bar{w}).$$

It follows that

$$|z + w|^2 = |z|^2 + |w|^2 + 2Re(z\bar{w})$$

$$\le |z|^2 + |w|^2 + |z\bar{w}| = (|z| + |w|)^2.$$

This completes the proof of part e).

We now prove the complex version of the Cauchy-Schwarz inequality.

THEOREM 6.5. *If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are complex numbers, then*

$$\left| \sum_{j=1}^{n} a_j \bar{b_j} \right|^2 \le \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$

To prove this, let $A = \sum_{j=1}^{n} |a_j|^2$, $B = \sum_{j=1}^{n} |b_j|^2$ and $C = \sum_{j=1}^{n} a_j \overline{b_j}$. Then

$$\sum |Ba_j - Cb_j|^2 = \sum (Ba_j - Cb_j)(B\overline{a_j} - \overline{Cb_j})$$
$$= B^2 A - B|C|^2 = B(AB - |C|^2)$$

and it follows that

$$|C|^2 \leq AB,$$

which is what we wanted to prove.

PROBLEM 6.6. Do Problems 8, 11, 12, 13 and 17 on page 23.

7. Higher dimensional Euclidean space

For each positive integer k, let

$$\mathbb{R}^k = \{x = (x_1, \ldots, x_k) : x_j \in \mathbb{R}\}.$$

We shall refer to the elements of $\mathbb{R}^k$ as vectors. Given two vectors x, y in $\mathbb{R}^k$ and $\alpha \in \mathbb{R}$, we define

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_k + y_k) \text{ and } \alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_k).$$

We also define the inner product of two vectors x and y by the relation

$$x \cdot y = x_1 y_2 + x_2 y_2 + \cdots + x_k y_k = \sum_{i=1}^{k} x_i y_i.$$

The norm is defined by the relation

$$|x| = \sqrt{x \cdot x} = \left(\sum_{i=1}^{k} x_i^2 \right)^{\frac{1}{2}}.$$

We now collect all the basic properties in the following result.

THEOREM 7.1. *Suppose that $x, y, z \in \mathbb{R}^k$, and $\alpha \in \mathbb{R}$. Then*
a) $|x| \geq 0$.
b) $|x| = 0$ if and only if $x = (0, \ldots, 0) \equiv \vec{0}$.
c) $|\alpha x| = |\alpha||x|$.
d) $|x \cdot y| \leq |x||y|$.
e) $|x + y| \leq |x| + |y|$.
f) $|x - z| \leq |x - y| + |y - z|$.

The first three items are obvious and the fourth is the consequence of Cauchy-Schwartz. To prove e), observe that

$$|x + y|^2 = (x + y) \cdot (x + y)$$
$$x \cdot x + 2x \cdot y + y \cdot y$$
$$\leq |x|^2 + |y|^2 + 2|x||y|$$
$$= (|x| + |y|)^2$$

and we are done. All that is left now is (f) and it follows from e) by replacing x, y by $x - y, y - z$.

8. Basic topology

DEFINITION 8.1. We say that f is a function from A to B if f associated with each $x \in A$ a $y \in B$. The set A is called a domain of f. The set

$$f(A) \equiv \{f(x) : x \in A\} \subset B$$

is called the range of f.

If $f(A) = B$, we say that f maps A onto B. In this case B is the range of f.

DEFINITION 8.2. If $E \subset B$, we define

$$f^{-1}(E) = \{x \in A : f(x) \in E\}.$$

Note that this notation in no way implies that the inverse of f exists. For example, suppose that A consists of real numbers 1 and -1, B consists of the real number 1 and $f(x) = x^2$. Then $f(x)$ certainly does not have an inverse since it is impossible to decide whether $f^{-1}(1) = 1$ or $f^{-1}(1) = -1$. Nevertheless,

$$f^{-1}(B) = \{x \in A : x^2 = 1\} = \{-1, 1\}.$$

DEFINITION 8.3. In the same way, if $y \in B$,

$$f^{-1}(y) = \{x \in A : f(x) = y\}.$$

If $f^{-1}(y)$ consists of at most one element, we say that f is 1-1. Another way of saying this is $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$, $x_1, x_2, \in A$.

If there is a 1-1 mapping from A ONTO B, we say that A and B can be put into 1-1 correspondence, often denoted by $A \sim B$.

DEFINITION 8.4. For any positive integer n, let

$$J_n = \{1, 2, \ldots, n\}$$

and let J be the set of all positive integers.

For any set A, we say:
a) A is finite if $A \sim J_n$ for some n.
b) A is infinite if A is not finite.
c) A is countable if $A \sim J$.
d) A is uncountable if A is neither finite nor countable.
e) A is at most countable if A is either finite or countable.

PROBLEM 8.5. Let A be the set of integers (not necessarily positive) with the property that given any n in this set $n - 1$ is divisible by 4. Prove that this set is countable without appealing to Theorem 2.8.

DEFINITION 8.6. By a sequence we mean a function $f : J \to \mathbb{R}$. If $f(n) = x_n$, we typically write this sequence in the form $\{x_i\}$ instead of talking about the underlying function f. The values of f are called the elements of the sequence and it is important to note that they need not be distinct.

EXAMPLE 8.7. Simple examples of sequences include $x_n = n$ or $x_n = \frac{1}{n}$. More complicated example is $x_n = p_n$, where p_n denotes the *pth* prime number. An even more amusing example is $x_n = $ the number of triples of integers (x, y, z) which solve the equation

$$x^n + y^n = z^n \text{ where } n = 3, 4, \ldots$$

THEOREM 8.8. *Every infinite subset of a countable set is countable.*

We prove this theorem by a greedy algorithm. Suppose that A is a countable set and E is a subset. Arrange the elements of A in a sequence x_n of distinct elements. Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Let n_2 be the smallest integer greater than n_1 such that $x_{n_2} \in E$. In this way we construct a sequence x_{n_k} which consists of all the elements of E. Setting $f(k) = x_{n_k}$ we obtain a 1-1 correspondence between E and J.

COROLLARY 8.9. *No uncountable set can be a subset of a countable set.*

DEFINITION 8.10. Let A and Ω be sets and suppose that with each element α of A we associate a subset of Ω denoted by E_α.

The set whose elements are the sets E_α shall be denoted by $\{E_\alpha\}$, or a family of sets E_α. We say that

$$x \in \cup_{\alpha \in A} E_\alpha \text{ if } x \in E_\alpha \text{ for at least one } \alpha \in A.$$

If $A = J_n$, the set of positive integers up to n, one usually writes

$$S = \cup_{m=1}^{n} E_m,$$

or

$$S = E_1 \cup E_2 \cup \cdots \cup E_n.$$

If $A = J$, one simply writes

$$S = \cup_{m=1}^{\infty} E_m.$$

The symbol ∞ merely indicates that we are summing over all the elements of J.

The intersection of the sets E_α is defined as the collection of xs that belong to E_α for every $\alpha \in A$. The intersection is denoted by

$$\cup_{j=1}^{\infty} E_\alpha,$$

with analogous adjustments in the case when $A = J_n$ or $A = J$.

THEOREM 8.11. *Let E_n, $n = 1, 2, \ldots$ be a sequence of countable sets, and put*

$$S = \cup_{n=1}^{\infty} E_n.$$

Then S is countable.

To prove this result, write the elements of E_j in the form $\{x_{jk}\}_{k=1}^{\infty}$. Now rearrange and create a set F_1 consisting of x_{11}, the set F_2, consisting of x_{21} and x_{12}, the set F_3 consisting of x_{31}, x_{22} and x_{13}, and so on. Each set F_m is finite and has precisely m elements. Writing elements of F_1, followed by F_2, followed by F_3, and so on, we see that these elements can be written as a sequence. Some of the elements of the sequence may repeat, so there is a subset T of the sequence such that $S \sim T$, which shows that S is at most countable by Theorem 2.8. On the other hand, since $E_1 \subset S$ and E_1 is infinite, we know that S is infinite and thus countable. This completes the proof.

COROLLARY 8.12. *Suppose that A is at most countable, and, for every $\alpha \in A$, B_α is at most countable. Let*

$$T = \cup_{a \in \alpha} B_\alpha.$$

Then T is at most countable.

THEOREM 8.13. *Let A be a countable set and let B_n be the set of all n-tuples $(a_1, \ldots, a_n)$, where $a_k \in A$, and the elements $a_1, \ldots, a_n$ need not be distinct. Then B_n is countable.*

The proof is by induction. The set B_1 is countable by assumption. Assume that B_{n-1} is countable. The elements of B_n are of the form (b, a) where $b \in B_{n-1}$ and $a \in A$. For every fixed b, the set of pairs (b, a) is equivalent ($\sim$) to A and hence countable. It follows that B_n is countable union of countable sets, which makes it countable by Theorem 2.12. By induction, we are done.

PROBLEM 8.14. Do exercises 2,3,4 on page 43.

COROLLARY 8.15. *The set of all rational numbers is countable.*

To prove this we apply Theorem 8.13 with $n = 2$ since rationals are of the form $\frac{a}{b}$, where a, b are integers.

THEOREM 8.16. *The set of sequences whose elements are 0 or 1 is uncountable.*

The proof is by Cantor diagonalization. Assume that the set is countable and write it as a sequence (of sequences) $x_1, x_2, \ldots, x_n, \ldots$. Now construct a sequence x with the property that its nth entry is different from the nth entry of x_n. The resulting sequence is not on the original list and we are done.

PROBLEM 8.17. Start with the interval $[0, 1]$ and remove the interval $(1/3, 2/3)$. Remove the "middle third" of the remaining two intervals, thus obtaining four intervals and keep going. At stage n you will have 2^n intervals. Call the resulting set C_n.

a) Prove that the total length of the intervals in C_n goes to 0 as $n \to \infty$.
b) Let $C = \cap_{n=1}^{\infty} C_n$. Prove that $C \sim [0, 1]$ in the sense that there is a 1-1 and onto function that maps C to $[0, 1]$.

9. Metric spaces

DEFINITION 9.1. We say that X is a metric space if for any two points $p, q \in X$ there exists a real number $d(p, q)$ (distance function) such that
a) $d(p, q) > 0$ if $p \neq q$, $d(p, p) = 0$.
b) $d(p, q) = d(q, p)$.
c) $d(p, q) \leq d(p, r) + d(r, q)$, for any $r \in X$.

EXAMPLE 9.2. The usual distance on $\mathbb{R}^k$, given by

$$|x - y|^2 = \sum_{j=1}^{k} (x_j - y_j)^2$$

is a metric on $\mathbb{R}^k$ viewed as a metric space.

PROBLEM 9.3. Let $C([a, b])$ denote the space of all continuous functions on $[a, b]$. Let

$$d(f, g) = \int_a^b |f(x) - g(x)| dx.$$

Prove that this d is a legitimate distance function in accordance with the definition above. For extra credit, prove the same for

$$\left(\int_a^b |f(x) - g(x)|^2 dx \right)^{\frac{1}{2}}.$$

DEFINITION 9.4. We call (a, b) a segment and $[a, b]$ an interval. The set of points

$$\{x \in \mathbb{R}^k : a_i \leq x_i \leq b_i\}$$

is called a k-cell.

DEFINITION 9.5. We say that $E \subset \mathbb{R}^k$ is convex if $tx + (i - t)y \in E$ whenever $x, y \in E$ and $t \in [0, 1]$.

DEFINITION 9.6. Suppose that $E \subset \mathbb{R}^k$ is convex and bounded. We define the polar body of E by the relation

$$E^* = \{x \in \mathbb{R}^k : |x \cdot y| \leq 1 \ \forall \ y \in E\}.$$

PROBLEM 9.7. a) Prove that the ellipsoid

$$\{x \in \mathbb{R}^k : x_1^2 + 2x_2^2 + \cdots + kx_k^2 \leq 1\} \text{ is convex.}$$

Any proof will do, but if you are able to appropriately adjust the proof for the Euclidean ball, that would be great.

b) For extra credit, let A be an invertible k by k matrix and let $Q_A(x) = Ax \cdot x$. What additional assumptions on A are needed to conclude that

$$\{x \in \mathbb{R}^k : Q_A(x) \leq 1\} \text{ is convex?}$$

PROBLEM 9.8. Assume part b) of the previous exercise if you have not solved it and find a matrix B such that

$$\{x \in \mathbb{R}^k : Q_B(x) \leq 1\} \text{ is a polar body of } \{x \in \mathbb{R}^k : Q_A(x) \leq 1\}.$$

DEFINITION 9.9. Let X be a metric space.
a) A neighborhood of p is a set $N_r(p) = \{x \in X : d(x, p) < r\}$.
b) A point p is a limit point of E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.
c) If $p \in E$ and p is not a limit point, then p is called an isolated point.
d) E is closed if every limit point of E is a point in E.
e) A point p is an interior point if there is a neighborhood N of p such that $N \subset E$.
f) E is open if every point of E is an interior point of E.
g) A complement of E, denoted by E^c is the set of all points in X that are not in E.
h) E is perfect if E is closed and every point of E is a limit point of E.
i) E is bounded if there exists M and $q \in E$ real such that $d(p, q) < M$ for all $p \in E$.
j) E is dense if every point of X is a point of E or a limit point of E (or both).

PROBLEM 9.10. Do problems 7,8,9 on page 43.

THEOREM 9.11. *Every neighborhood is an open set.*

To prove this, let $E = N_r(p)$ and let q be any point in E. Then there exists $h > 0$ such that $d(p,q) = r - h$. For all points s such that $d(q,s) < h$,

$$d(p,s) \leq d(p,q) + d(q,s) < r - h + h = r,$$

so $s \in E$.

It follows that q is an interior point of E.

THEOREM 9.12. *If p is a limit point of E, then every neighborhood of E contains infinitely many points of E.*

To prove this, suppose that there is a neighborhood N of E that contains only finitely many points of E: $q_1, q_2, \ldots, q_n$. Let

$$r = \min_{1 \leq m \leq n} d(p, q_m) > 0.$$

The neighborhood $N_r(p)$ contains no points $q \in E$ such that $q \neq p$, so p cannot be a limit point of E, which is a contradiction.

THEOREM 9.13. *Let $\{E_\alpha\}$ be a collection of sets E_α. Then*

$$(\cup_\alpha E_\alpha)^c = \cap_\alpha E_\alpha^c.$$

To prove this, suppose that x belongs to the left hand side. Then x does not belong to $\cup_\alpha E_\alpha$, which means that $x \notin E_\alpha$ for any α. But this means that $x \in E_\alpha^c$ for every α, which implies that x is an element of the right hand side above. This means that the left hand side is a subset of the right hand side.

Now suppose that $x \in \cap_\alpha E_\alpha^c$. This means that $x \in E_\alpha^c$ for every α, which implies that $x \notin E_\alpha$ for any α. But this means $x \notin \cup_\alpha E_\alpha$, or, in other words, $x \in (\cup_\alpha E_\alpha)^c$. It follows that the right hand side is a subset of the left hand side and we are done.

This result will be a useful tool for us for the remainder of the semester. We now turn our attention to establishing a relation between open and closed sets. But first...

PROBLEM 9.14. Do problems 10 and 11 on page 44.

THEOREM 9.15. *A set is open if and only if its complement is closed.*

Those of you who studied topology before are probably perplexed by this theorem since the way things are defined in topology, this result is automatic. But Rudin defines things a bit differently, with the structure of real numbers in mind.

To prove this theorem, suppose that E^c is closed. Choose $x \in E$. Then x is not a limit point of E^c and $x \notin E^c$. Therefore there exists a neighborhood N of x such that $N \cap E^c$ is empty, which means that $N \subset E$. It follows that E is open.

Now suppose that E is open. Let x be a limit point of E^c. Then every neighborhood of x contains a point of E^c, so x is not an interior point of E. Since E is open, it follows that $x \in E^c$. It follows that E^c is closed and the proof is complete.

COROLLARY 9.16. *A set F is closed if and only if its complement is open.*

THEOREM 9.17. *a) For any collection $\{G_\alpha\}$ of open sets, $\cup_\alpha G_\alpha$ is open.*
b) For any collection $\{F_\alpha\}$ of closed sets, $\cap_\alpha F_\alpha$ is closed.
c) For any finite collection $G_1, \ldots, G_n$ of open sets, $\cap_{i=1}^n G_i$ is open.
d) For any finite collection $F_1, \ldots, F_n$ of closed sets, $\cup_{i=1}^n F_i$ is closed.

Let's prove part a). Suppose that $x \in \cup_\alpha G_\alpha$. Then $x \in G_\alpha$ for some α in the collection of indices. Since G_α is open, there exists a neighborhood N of x contained in G_α. By definition, this neighborhood is contained in $\cup_\alpha G_\alpha$, which proves that $\cup_\alpha G_\alpha$ is open.

Let's now prove b). By Theorem 9.13,

$$(\cap_\alpha F_\alpha)^c = \cup_\alpha F_\alpha^c.$$

The proof now follows from part a).

Let's prove part c). Let $x \in \cap_{i=1}^n G_i$. Since G_i is open, for each i there exists a neighborhood N_i of radius r_i of x contained in G_i. It follows that the neighborhood N of radius $r = \min_i r_i$ is contained in $\cap_{i=1}^n G_i$.

Part d) follows from part c) via Theorem 9.13.

Note that parts c) and d) are only stated for finite collections. Either extend the results to infinite collections or find counter-examples. In the future, please do that automatically.

DEFINITION 9.18. If X is a metric space, if $E \subset X$, and if E' denotes the set of limit points of E in X, then the closer of E is the set

$$\overline{E} = E \cup E'.$$

THEOREM 9.19. *If X is a metric space and $E \subset X$, then*
a) $\overline{E}$ is closed,
b) $E = \overline{E}$ if and only if E is closed.
c) $\overline{E} \subset F$ for every closed set F containing E.

To prove a), note that if $p \in X$ and $p \notin \overline{E}$, then p is neither in E nor is p a limit point of E. Hence p has a neighborhood which does not intersect E. It follows that the complement of $\overline{E}$ is open, so $\overline{E}$ is closed.

To prove b), notice that if $E = \overline{E}$, then a) implies that E is closed. Now suppose that E is closed. Then $E' \subset E$, so $E = \overline{E}$.

To prove c), note that if F is closed and $E \subset F$, then $F' \subset F$, so $E' \subset F$. It follows that $\overline{E} \subset F$ and we are done.

THEOREM 9.20. *Let E be a non-empty set of real numbers which is bounded from above. Let $y = \sup E$. Then $y \in \overline{E}$. Hence $y \in E$ if and only if E is closed.*

To prove this, observe that if $y \in E$, then $y \in \overline{E}$. Assume that $y \notin E$. Then for every $h > 0$ there exists $x \in E$ such that $y - h < x < y$, since otherwise $y - h$ would be an upper bound. Thus y is a limit point of E, so $y \in \overline{E}$.

DEFINITION 9.21. Suppose that $E \subset Y \subset X$. We say that E is open relative to Y if for each $p \in E$ there exists $r > 0$ such that $q \in E$ whenever $d(p, q) < r$ and $q \in Y$.

EXAMPLE 9.22. Let $E = (0, 1)$, $Y = \mathbb{R}$ and $X = \mathbb{R}^2$. Then E is open relative to Y, but not relative to X.

The following result characterizes relatively open sets in a very simple way.

THEOREM 9.23. *Let $Y \subset X$, metric spaces. Then E is open relative to Y if and only if there exists an open set $G \subset X$ such that $E = Y \cap G$.*

To prove this theorem, suppose that E is open relative to Y. Then for every $p \in E$ there exists r_p so that $d(p, q) < r_p$ with $q \in Y$ implies that $q \in E$. Let V_p denote the set of all $q \in X$ such that $d(p, q) < r_p$, and define

$$G = \cup_{p \in E} V_p.$$

Then G is an open subset of X and $E \subset Y \cap G$. By our choice of V_p, we have $V_p \cap Y \subset E$ for each p, so $G \cap Y \subset E$. It follows that $G \cap Y = E$. To prove the other direction, note that if G is open in X and $E = G \cap Y$, every $p \in E$ has a neighborhood in $V_p \subset G$. Then $V_p \cap Y \subset E$, so E is open relative to Y.

10. Compactness

DEFINITION 10.1. An open cover of E in a metric space X is a collection of open sets $\{G_\alpha\}$ such that $E \subset \cup_\alpha G_\alpha$.

EXAMPLE 10.2. Let $X = \mathbb{R}$ and $E = (0, 1]$. Let $G_n = (1/n, 2)$. Then $(0, 1] \subset \cup_{n=1}^\infty G_n$.

DEFINITION 10.3. We say that $K \subset X$, metric space, is compact if every open cover of K contains a finite subcover.

Observe that in the previous example, no finite subcover will work because if N is the largest integer such that G_N is involved in the putative subcover, then any t such that $0 < t < \frac{1}{N}$ is in $(0, 1]$, but is not covered by the subcover.

THEOREM 10.4. *Suppose that $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.*

This result allows us to view compact sets autonomously, in the sense that will become more clear later.

To prove the theorem, suppose that K is compact relative to X, and let $\{V_\alpha\}$ be a collection of sets, open with respect to Y, such that $K \subset \cup_\alpha V_\alpha$. By Theorem 9.23, there are sets G_α, open relative to X such that $V_\alpha = Y \cap G_\alpha$, for all α. Since K is compact relative to X,

$$K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$$

for some finite collection of G_{α_j}s. But since $K \subset Y$,

$$K \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}.$$

This proves that K is compact relative to Y. Conversely, suppose that K is compact relative to Y and let $\{G_\alpha\}$ be an open cover. Define $V_\alpha = G_\alpha \cap Y$. Then

$$K \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$$

for some n by compactness. Since $V_\alpha \subset G_\alpha$, we have

$$K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$$

and the proof is complete.

We now establish a series of tight connection between compact and closed sets.

THEOREM 10.5. *Compact subsets of metric spaces are closed.*

To prove this, suppose that $p \in X$ and $p \notin K$. If $q \in K$, let V_q and V_p denote the neighborhoods of q and p respectively, of radius $< \frac{1}{2}d(p, q)$. Since K is compact, there are finitely many points $q_1, \ldots, q_n$ such that

$$K \subset W_{q_1} \cup W_{q_2} \cup \cdots \cup W_{q_n} = W.$$

Let

$$V = V_{q_1} \cap V_{q_2} \cap \cdots \cap V_{q_n},$$

a neighborhood of p which does not intersect W. Hence $V \subset K^c$ and we are done.

THEOREM 10.6. *Closed subsets of compact sets are compact.*

To prove this, suppose that $F \subset K \subset X$, F closed relative to X and K is compact. Let $\{V_\alpha\}$ be an open cover of F. If F^c is adjoined to $\{V_\alpha\}$, we obtain an open cover of K. Since K is compact, there is an open subcollection of this cover that still covers K, and hence F. If F^c is an element of this subcover, we simply throw it out.

An immediate yet useful consequence of this result is the following.

COROLLARY 10.7. *If F is closed and K is compact, then $F \cap K$ is compact.*

We now continue with the theme that as far as compact sets are concerned, finite covers can handle whatever infinite covers can.

THEOREM 10.8. *If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is non-empty, then $\cap_\alpha K_\alpha$ is non-empty.*

To prove this, fix a member of $\{K_\alpha\}$ and label it K_1. Assume that no point of K_1 belongs to every K_α since otherwise we are done. Let $G_\alpha = K_\alpha^c$. Then these open sets form an open cover of K_1. Since K_1 is compact, there is a finite subcover consisting of, say, $G_{\alpha_1}, \ldots, G_{\alpha_n}$. But this implies that

$$K_1 \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \text{ is empty,}$$

which is a contradiction.

An obvious, but once again important consequence, is the following.

COROLLARY 10.9. *If $\{K_n\}$ is a sequence of non-empty nested compact sets such that $K_{n+1} \subset K_n$, then $\cap_{n=1}^\infty K_n$ is non-empty.*

We now show that compactness precludes the limit points from escaping.

THEOREM 10.10. *If E is an infinite subset of a compact set K, then E has a limit point in K.*

The proof of this fact is almost immediate. If E had no limit point, then every point of E would be contained in a neighborhood containing at most one point of E. This is a cover of E that clearly has no subcover and this is also true for K since $E \subset K$. This contradicts the assumption that K is compact.

If we knew that intervals were compact, the following result would be immediate, but we do not.

THEOREM 10.11. *If $\{I_n\}$ is a sequence of intervals in $\mathbb{R}$, such that $I_{n+1} \subset I_n$, then $\cap_{n=1}^{\infty} I_n$ is not empty.*

To prove this, denote $I_n = [a_n, b_n]$ and let E be the set of a_ns. Then E is non-empty and bounded above by b_1. Let x be the supremum of E. If m and n are positive integers, then

$$a_n \le a_{m+n} \le b_{m+n} \le b_m,$$

so $x \le b_m$ for each m. Since $a_m \le x$, follows that $x \in I_m$ for each m. This completes the proof.

THEOREM 10.12. *Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_{n+1} \subset I_n$, then $\cap_{n=1}^{\infty} I_n$ is non-empty.*

This is an immediately corollary of the previous result and the definition of a k-cell.

THEOREM 10.13. *Every k-cell is compact.*

To prove this, let
$$I = \{x : a_j \le x_j \le b_j; 1 \le j \le k\}.$$
Put
$$\delta = \left\{ \sum_{i=1}^{k} (b_j - a_j)^2 \right\}^{\frac{1}{2}}.$$

Then $|x - y| \le \delta$ if $x, y \in I$. Now suppose that there is an open cover of I that does not have a finite subcover. Let $c_j = \frac{a_j + b_j}{2}$. The intervals $[a_j, c_j]$ and $[c_j, b_j]$ determine 2^k k-cells Q_i whose union is I. At least one of these sets Q_i cannot be covered by any finite subcollection of the cover and we denote this set by I_1. We then subdivide I_1 and continue the process obtaining a sequence I_n with the properties:
   a) $\{I_n\}$ is nested and contained in I.
   b) I_n is not covered by any finite subcollection of the original cover.
   c) If $x, y \in I_n$, $|x - y| \le 2^{-n} \delta$.

By the previous theorem, there is a point x^* in every I_n. For some α, $x^* \in G_\alpha$. Since G_α is open, there exists $r > 0$ such that $|y - x^*| < r$ implies that $y \in G_\alpha$. Choose n large enough so that $2^{-n} \delta < r$ and then c) implies that $I_n \subset G_\alpha$, which contradicts b). We are done.

PROBLEM 10.14. Do problems 12 and 13 on page 44.

We are finitely ready for Heine-Borel!

THEOREM 10.15. *(Heine-Borel) If a set E in $\mathbb{R}^k$ has one of the following three properties, then it has the other two:*
   a) E is closed and bounded.
   b) E is compact.
   c) Every infinite subset of E has a limit point in E.

Let us begin by assuming a). Then $E \subset I$ for some k-cell I and b) follows by above. Using Theorem 10.10 c) follows from b), so it is enough to prove that c) implies a).

Suppose that E is not bounded. Then there exists a sequence $S = \{x_n\}$, $x_n \in E$ with $|x_n| > n$. It follows that S has no limit point in E and we have a contradiction, hence E is bounded. If E is not closed, there exists $x_0 \in \mathbb{R}^k$ which is a limit point of E, but does not live in E. It follows that for each n there exist $x_n \in E$ such that $|x_n - x_0| < \frac{1}{n}$. Let S be the collection of these points. It is clear that S is infinite and x_0 is a limit point. To see that S has no other limit points, let $y \in \mathbb{R}^k$, $y \neq x_0$. Then

$$|x_n - y| \geq |x_0 - y| - |x_n - x_0|$$
$$\geq |x_0 - y| - \frac{1}{n} \geq \frac{1}{2}|x_0 - y|$$

for all but finitely many n. This proves that y is not a limit point, so x_0 is the only limit point of S. In particular, this means that S has no limit points in E, which is a contradiction which establishes that E is closed. This completes the proof of the fact that c) implies a).

THEOREM 10.16. *Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.*

To prove this, observe that since E is bounded, it is a subset of some k-cell I. Since I is compact, E has a limit point in I by Theorem 10.10.

PROBLEM 10.17. Do exercise 16, 17, 22 and 29 on page 44-45. Why doesn't this problem contradict the Heine-Borel theorem?

This is the end of the homework assignment due on Monday, September 28.

11. Perfect sets

THEOREM 11.1. *Let P be a non-empty perfect set in $\mathbb{R}^k$. Then P is uncountable.*

PROBLEM 11.2. Explain clearly why the proof below is essentially the same as the proof of the fact that the set of sequences of 0s and 1s is uncountable.

Let us now turn out attention to the proof. Since P has limit points, P must be infinite. Suppose for contradiction that P is countable and denote its elements by $x_1, x_2, \ldots$.

Let V_1 be any neighborhood of x_1, i.e $V_1 = \{y \in \mathbb{R}^k | y - x_1| < r\}$. Suppose that V_n has been constructed so that $V_n \cap P$ is not empty. Since every point of E is a limit point of E, we have:

i) $\overline{V}_{n+1} \subset V_n$
ii) $x_n \notin \overline{V}_{n+1}$
iii) $V_{n+1} \cap P$ is not empty.

Let $K_n = \overline{V}_n \cap P$. Since $\overline{V}_n$ is closed and bounded, $\overline{V}_n$ is compact. Since $x_n \notin K_{n+1}$, no point of E lies in $\cap_{n=1}^\infty K_n$ is empty. But each K_n is non-empty, by iii), and $K_n \supset K_{n+1}$ by i). This is a contradiction and we are done.

COROLLARY 11.3. *Every interval $[a, b]$ is uncountable. In particular, the set of real numbers is uncountable.*

12. Connected sets

We have a notion of what "connected" means in $\mathbb{R}^k$. Even there things are trickier than they seem, but in general metric spaces we need a whole new approach altogether.

DEFINITION 12.1. We say that $A, B \subset X$, metric space, are separated if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are both empty. A set $E \subset X$ is connected if **it is not** a union of two non-empty separated sets.

THEOREM 12.2. *A subset of the real line $\mathbb{R}$ is connected if and only if it has the following property: If $x \in E, y \in E$ and $x < z < y$, then $z \in E$.*

One direction is almost immediate. If there exist $x < z < y$ such that $x, y \in E$ and $z \notin E$, then E is the union of $E \cap (-\infty, z)$ and $E \cap (z, \infty)$, so E is not connected.

Conversely, suppose that E is not connected. Then there exist A, B non-empty separated sets such that $E = A \cup B$. Pick $x \in A$ and $y \in B$ such that $x < y$ (without loss of generality). Let

$$z = \sup(A \cap [x, y]).$$

Then $z \in \overline{A}$ (why?) It follows that $z \notin B$. In particular, $x \leq z < y$. If $z \notin A$, $x < z < y$ and $z \notin E$. If $z \in A$, then $x \notin \overline{B}$, hence there exists z_1 such that $z < z_1 < y$ and $z_1 \notin B$. Then $x < z_1 < y$ and $z_1 \notin E$.

PROBLEM 12.3. Is it true that a subset of $\mathbb{R}^2$ is connected if and only if given any $x \in E$, $y \in E$ and $z = (1 - t)x + ty$ for some $t \in (0, 1)$, then $z \in E$?

13. Convergent sequences

Definition of a convergent sequence in a general metric space is not particularly different than than classical definition in $\mathbb{R}^d$.

DEFINITION 13.1. We say that a sequence $\{p_n\}$ in X converges if there exists $p \in X$ such that for every $\epsilon > 0$ there exists N such that $n \geq N$ implies that $d(p_n, p) < \epsilon$. In such cases we write

$$\lim_{n \to \infty} p_n = p.$$

The following results gives us a connection between convergent sequences and the notions of the previous sections.

THEOREM 13.2. *Let $\{p_n\}$ be a sequence in a metric space X.*
a) $\{p_n\}$ converges to $p \in X$ if and only if every neighborhood of p contains p_n for all but finitely many values of n.
b) If $p_n \to p$ and $p_n \to p'$, then $p = p'$.
c) If p_n converges, then p_n is bounded.
d) if $E \subset X$ and p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that

$$p = \lim_{n \to \infty} p_n.$$

To prove a), suppose that $p_n \to p$. Then given $\epsilon > 0$, there exists N, such that every neighborhood centered at p of radius ϵ contains every p_n with $n \geq N$. This proves one direction.

Now suppose that every neighborhood of p contains all but finite many values of p_n. This means that given $\epsilon > 0$, there exists N such that all p_n with $n \geq N$ are in the ϵ neighborhood of p. This is because if the set of exceptional p_ns is finite, there exists N such that p_N is the element of this collection with the largest subscript.

To prove b), assume otherwise and take $\epsilon = \frac{1}{2} d(p, p')$.

To prove c) let $\epsilon = 1$.

To prove d) note that since p is a limit point, for each n there exists $p_n \in E$ such that $d(p, p_n) < \frac{1}{n}$.

And now some basic properties of complex sequences.

THEOREM 13.3. *Suppose that $\{s_n\}$, $\{t_n\}$ are complex sequences and*

$$\lim_{n \to \infty} s_n = s, \quad \lim_{n \to \infty} t_n = t.$$

Then
a) $\lim_{n \to \infty}(s_n + t_n) = s + t$.
b) $\lim_{n \to \infty} cs_n = cs$, $\lim_{n \to \infty}(c + s_n) = c + s$, *for any number c.*
c) $\lim_{n \to \infty} s_n t_n = st$.
d) $\lim_{n \to \infty} \frac{1}{s_n} = \frac{1}{s}$, *provided $s_n \neq 0$ and $s \neq 0$.*

The proof of a) and b) is trivial. To prove c), use the fact that

$$s_n t_n - st = (s - s_n)(t_n - t) + s(t_n - t) + t(s_n - s).$$

Part d) follows by similar reasoning.

The next result shows how convergence behaves with respect to components in higher dimensional Euclidean spaces.

THEOREM 13.4. *Suppose that $x_n \in \mathbb{R}^k$ and*

$$x_n = (\alpha_{1,n}, \ldots, \alpha_{k,n}).$$

Then $x_n \to x = (\alpha_1, \ldots, \alpha_k)$ if and only if

$$\lim_{n \to \infty} \alpha_{j,n} = \alpha_j.$$

b) Suppose that x_n, y_n are sequences in $\mathbb{R}^k$, β_n a sequence in $\mathbb{R}$, and $x_n \to x, y_n \to y, \beta_n \to \beta$. Then

$$\lim_{n \to \infty}(x_n + y_n) = x + y; \quad \lim_{n \to \infty} x_n \cdot y_n = x \cdot y, \quad \lim_{n \to \infty} \beta_n x_n = \beta x.$$

The proof follows instantly from the definition and one-dimensional analogs.

PROBLEM 13.5. Do problems 1,2,3,6 on page 78.

14. Subsequences

Given a sequence $\{p_n\}$ consider a sequence n_k of positive integers such that $n_1 < n_2 < n_3 < \ldots$. Then $\{p_{n_k}\}$ is a subsequence of $\{p_n\}$.

Let's collect some basic properties of subsequences.

THEOREM 14.1. *a) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point in X.*

b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

To prove a) note that p_n either takes a finite or an infinite number of values. The finite case is immediate. If p_n takes an infinite number of values $\{p_n\}$ has a limit point p. Choose $n_1 < n_2 < \cdots < n_k$ so that $d(p, p_{n_k}) < \frac{1}{k}$ since every neighborhood of p contains infinitely many points of $\{p_n\}$. It follows that $\{p_{n_i}\}$ converges to p.

Part b) follows from part a) since every bounded sequences lives inside some k-cell and we have shown that k-cells are compact.

THEOREM 14.2. *The sub sequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.*

To prove this, it is enough to show that if q is a limit point of E^*, the set of sub sequential limits of a sequence $\{p_n\}$, then $q \in E^*$. Choose n_1 such that $p_{n_1} \neq q$ and let $\delta = d(q, p_{n_1})$. Suppose that $n_1, n_2, \ldots, n_{i-1}$ have been chosen. Since q is a limit point of E^*, there exists $x \in E^*$ such that $d(x, q) < \delta 2^{-i}$. Since $x \in E^*$, there exists $n_i > n_{i-1}$ such that $d(x, p_{n_1}) < 2^{-i}\delta$. It follows that

$$d(q, p_{n_i}) < 2^{-(i-1)}\delta$$

for $i = 1, 2, \ldots$ which implies that p_{n_i} converges to q, which puts $q \in E^*$.

15. Cauchy sequences

DEFINITION 15.1. A sequence $\{p_n\}$ in a metric space X is said to be a Cauchy sequence if for every $\epsilon > 0$ there is an integer N such that $d(p_n, p_m) < \epsilon$ if $n, m \geq N$.

DEFINITION 15.2. Let $E \subset X$, metric space. We define the diameter of E to be the supremum of the set

$$\{d(p, q), p \in E, q \in E\}.$$

It is often useful to reformulate the Cauchy condition in terms of the diameter of a suitable quantity. More precisely, let E_N be the set consisting of points $p_N, p_{N+1}, \ldots$. Then the sequence is Cauchy if and only if

$$\lim_{N \to \infty} diam(E_N) = 0.$$

THEOREM 15.3. *a) If $\overline{E}$ is a closure of a set E in a metric space X, then*

$$diam(E) = diam(\overline{E}).$$

b) if $\{K_n\}$ is a nested sequence of compact sets such that $\lim_{n \to \infty} diam(K_n) = 0$, then $\cap_{n=1}^{\infty} K_n$ consists precisely of one points.

The proof of b) is not rocket science.

To prove a), observe that $diam(E) \leq diam(\overline{E})$ since $E \subset \overline{E}$. Now consider any two points $x, y \in \overline{E}$. Then there exist $u, v \in E$ such that $d(x, u) < \epsilon$ and $d(y, v) < \epsilon$. It follows that

$$d(x, y) \leq d(x, u) + d(u, v) + d(v, y) < 2\epsilon + d(u, v) \leq 2\epsilon + diam(E).$$

It follows that

$$diam(\overline{E}) \leq 2\epsilon + diam(E),$$

and since this is true for every ϵ,

$$diam(\overline{E}) \leq diam(E).$$

This proves that $diam(E) = diam(\overline{E})$. This completes the proof.

The following result begins the process of establishing a connection between Cauchy sequences and convergence.

THEOREM 15.4. *a) In any metric space, every convergent sequence is a Cauchy sequence.*
b) If X is a compact metric space and $\{p_m\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point in of X.
c) in $\mathbb{R}^k$, every Cauchy sequences converges.

The proof of a) is a standard $\frac{\epsilon}{2}$ argument.

To prove b), let $E_N = \{p_N, p_{N+1}, \dots\}$ and observe that the Cauchy assumption implies that

$$\lim_{N \to infty} diam(\overline{E}) = 0.$$

The sets $\overline{E_n}$ are closed subsets of a compact space and thus compact. They are also nested with diameter going to 0. Thus the intersection consists of a single point p. The proof that p_n converges to p is straightforward.

To prove c) we reduce matters to b) as follows. Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}^k$. Define E_N as above and note that for N large enough, $diam(E_N) < 1$. It follows that the sequence $\{x_n\}$, viewed as a set, is the union of E_N and a finite point set, hence bounded. By Heine-Borel, the closure of this set is compact and c) reduces to b).

DEFINITION 15.5. A metric space in which every Cauchy sequence converges is said to be complete.

PROBLEM 15.6. Let S be the set of real numbers in $[0, 1]$ which arise as roots of polynomials of degree **at most** 2. Prove that S, endowed with the usual metric $d(x, y) = |x - y|$ is not a complete metric space. For **extra credit**, prove the same for polynomials of degree at most $n \geq 3$.

DEFINITION 15.7. A sequence s_n of real numbers is said to be monotonically increasing if $s_{n+1} \geq s_n$ for all n. A sequence s_n of real numbers is said to be monotonically decreasing if $s_{n+1} \leq s_n$.

THEOREM 15.8. *Suppose that $\{s_n\}$ is monotonic. Then s_n converges if and only if s_n is bounded.*

The proof is straightforward and left to the reader.

DEFINITION 15.9. Suppose that for every real M there is an integer N such that $s_n \geq M$ if $n \geq N$. Then we say that

$$s_n \to +\infty$$

with the analogous definition for $s_n \to -\infty$.

DEFINITION 15.10. Let s_n be a sequence of real numbers. Let E be the set of numbers x such that $s_{n_k} \to x$ for some subsequence s_{n_k}. This set contains all sub sequential limits as defined before, plus possible $\pm\infty$. Define

$$s^* = \sup(E)$$

and

$$s_* = \inf(E).$$

We shall refer to s^* and s_* as $\limsup s_n$ and $\liminf s_n$, respectively.

THEOREM 15.11. Let $\{s_n\}$ be a sequence of real numbers. Let E and s^* be as in the definition above. Then

a) $s^* \in E$.

b) If $x > s^*$, then there is an integer N such that $n \geq N$ implies $s_n < x$.

c) s^* is the only number with properties a) and b).