September2Math142Fall2020

September 3, 2020

Consider the following practical problem. Suppose that we wish to build a rectangular pen for cattle and we have 100 feet of fencing available. How do we choose the side-lengths of the rectangular pen sich that the enclosed area is as large as possible?

Let us proceed slowly and systematically. The pen is rectangular and we have 1000 feet of fencing. The perimeter of the pen is $2(x+y)$, where x is the length of one side, and y is the length of the other. This gives us the equation $2(x+y)=1000$, hence

$$x+y=50.$$

We wish to maximize the area A, which is given by

$$xy = \text{length times width.}$$

Since $x+y=50$,

$$A = xy = x(50-x),$$

and we wish to find the value of x which makes this area as large as possible.

There are several ways to proceed, and I want to consider two of them because they are both educational and useful.

First, let's graph the function $x(5000-x)$ below.

[2]:
```python
import matplotlib.pyplot as plt
import numpy as np
from sympy import sympify, lambdify
from sympy.abc import x
import warnings; warnings.simplefilter('ignore')

fig = plt.figure(1)
ax = fig.add_subplot(111)



# set up axis
ax.spines['left'].set_position('zero')
ax.spines['right'].set_color('none')
ax.spines['bottom'].set_position('zero')
ax.spines['top'].set_color('none')
ax.xaxis.set_ticks_position('bottom')
ax.yaxis.set_ticks_position('left')
```

```python
# setup x and y ranges and precision
xx = np.arange(-10,100,0.01)

# draw my curve
myfunction=sympify(x*(50-x))
mylambdifiedfunction=lambdify(x,myfunction,'numpy')
ax.plot(xx, mylambdifiedfunction(xx),zorder=100,linewidth=3,color='red')
#plt.axes().set_aspect('equal')

#set bounds
ax.set_xbound(-10,100)
ax.set_ybound(0,700)

plt.show()
```

```
<Figure size 640x480 with 1 Axes>
```

Please note that the scales are not scaled the same in this drawing (otherwise the picture comes out a bit ugly), but the picture makes it clear that the graph achieves a local (and indeed global) maximum somewhere between 20 and 30. There are at least two reasonable ways to find out exactly where this point is.

First, let's just complete the square. We have

$$\begin{aligned} A &= x(50-x) = -x^2 + 50x = -(x^2 - 50x) \\ &= -((x-25)^2 - 625) = -(x-25)^2 + 625. \end{aligned}$$

Since a squared quantity is never negative,

$$-(x-25)^2 + 625 \leq 625$$

and 625 is achievewd when the squared quantity is equal to 0. This happens when $x = 25$ and we see that the largest possible area is 625 and this happens when the length of one side is $x = 25$, and the length of the other is deduced from the formula

$$x + y = 50$$

above, which means that y also equals 25.

It is also possible to attack this problem using calculus. We are trying to maximize $A = x(50-x)$, where $0 \leq x \leq 50$. Why is $x \geq 0$? Because it is not possible to have a pen with a negative side-length. Why is $x \leq 50$? Because the total amount of available fencing is 50.

To maximize $A = x(50-x)$ on $[0, 50]$, we take a derivative and obtain

$$A' = 50 - 2x.$$

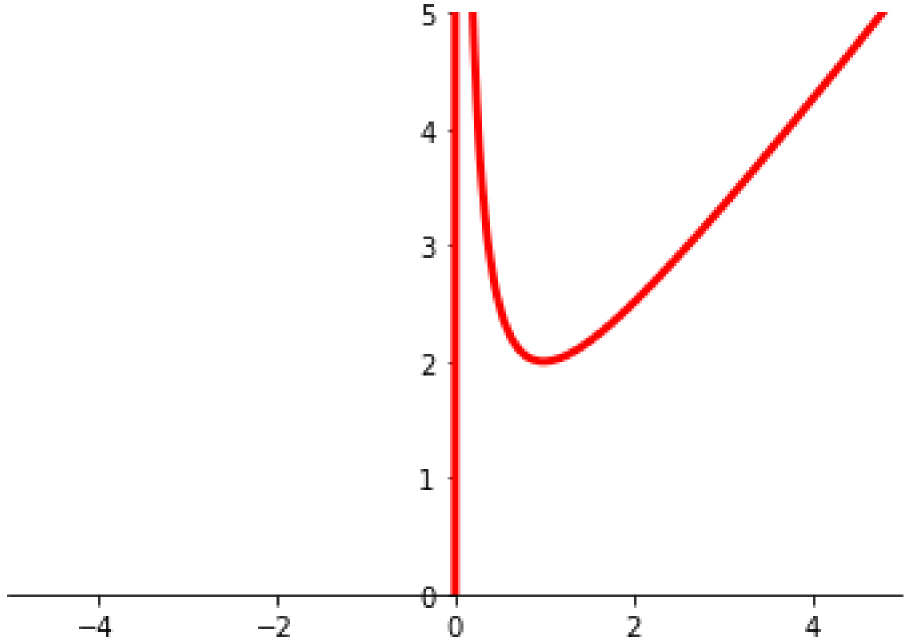
We set it equal to 0 and obtain $x = 25$. This is a local maximum. Why? Because A' is positive to the left of $x = 25$ and negative to the right of $x = 25$ (check!). It turns out that $x - 25$ yields the global maximum as well. We check this by looking at the endpoints $x = 0$ and $x = 50$ which both yield $A = 0$. Hence $x = 25$ yields the global maximum and the largest possible area is $A(25) = 625$.

Let us now turn our attention to a slightly trickier example. Suppose that the area of the pen is mandated to be equal to 100 and we wish to make the perimeter as small as possible.

First of all, why as small as possible and not as large as possible? Consider a rectange with side-lengths 1000 and 1/10. The area is 100 and the perimeter is slightly more than 2000. Now take a rectangle with side-lengths 1000000 and 1/1000. The area is, once again, 100, while the perimeter is slightly larger than 2000000. I think you see the point. We can make perimeter as large as we want while keeping the area equal to 100. This tells us that looking for the smallest possible perimeter probably makes more sense.

Since the area, $A = xy$, the perimeter, $P = 2(x + y) = 2(x + \frac{1}{x})$ and this is the quantity we want to minimize. Before we do anything else, let's graph this function.

[4]:
```
fig = plt.figure(1)
ax = fig.add_subplot(111)


# set up axis
ax.spines['left'].set_position('zero')
ax.spines['right'].set_color('none')
ax.spines['bottom'].set_position('zero')
ax.spines['top'].set_color('none')
ax.xaxis.set_ticks_position('bottom')
ax.yaxis.set_ticks_position('left')

# setup x and y ranges and precision
xx = np.arange(-5,5,0.01)

# draw my curve
myfunction=sympify(x+1/x)
mylambdifiedfunction=lambdify(x,myfunction,'numpy')
ax.plot(xx, mylambdifiedfunction(xx),zorder=100,linewidth=3,color='red')
#plt.axes().set_aspect('equal')

#set bounds
ax.set_xbound(-5,5)
ax.set_ybound(0,5)

plt.show()
```



The graph above shows that the minimum is somewhere between $\frac{1}{2}$ and $\frac{3}{2}$, possibly close to 1. To figure this out exactly, there are two reasonable approaches. Here is one that does not use calculus. Observe that

$$0 \leq \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 = x + \frac{1}{x} - 2.$$

The equality only holds if the left hand side equals 0, which happens if $x = 1$. It follows that

$$x + \frac{1}{x} \geq 2,$$

with the equality holding when $x = 1$. We conclude that

$$P \geq 2\left(x + \frac{1}{x}\right) \geq 4,$$

with the equality holding if $x = 1$. In other words, the minumum value of the perimeter is 1 and it occurs when $x = y = 1$.

Another approach to this problem is using calculus. Since

$$P = 2\left(x + \frac{1}{x}\right),$$

we compute the critical points by taking the derivative. We have

$$P' = 2 - \frac{2}{x^2} = 0 \text{ if } x = \pm 1.$$

We are not interested in negative solutions since side-lengths are positive. At $x = 1$ we have a local minimum since P' is negative to the left of $x = 1$ and positive to the right of $x = 1$ (check!). To check if $x = 1$ is a global minimum, we must check $x = 0$ and $x = 50$, and we conclude that $x = 1$ is indeed a global minimum.

Let is now consider some variants of the problems considered above. Suppose that we wish to build a rectangular pen of a given area and smallest possible perimeter, with one side facing the river where no fencing is needed. In this case, the constraint is

$$A = xy,$$

and we are trying to minimize the perimeter

$$P = x + 2y.$$

Notice that $P = x + 2y$, not $2(x + y)$ becase, once again, we do not need fencing on the side that faces the river. Also note that I did not tell you what the area A is equal to. We are just assuming that the area is fixed and is equal to some positive real number A. Our answer for the optimal values of x and y is going to depend on the value of A.

Since $A = xy$, $y = \frac{A}{x}$, so

$$P = x + \frac{2A}{x}.$$

Using the second method described above, we compute the derivative

$$P' = 1 - \frac{2A}{x^2}$$

and set it equal to 0.

We obtain $x = \sqrt{2A}$ and we can check by looking to the left and the right of this value that we have a local minimum there. If this is also a global minimum, then we see that the amount of fencing is minimized when

$$x = \sqrt{2A}, \text{ and } y = \frac{A}{x} = \frac{\sqrt{2A}}{2}.$$

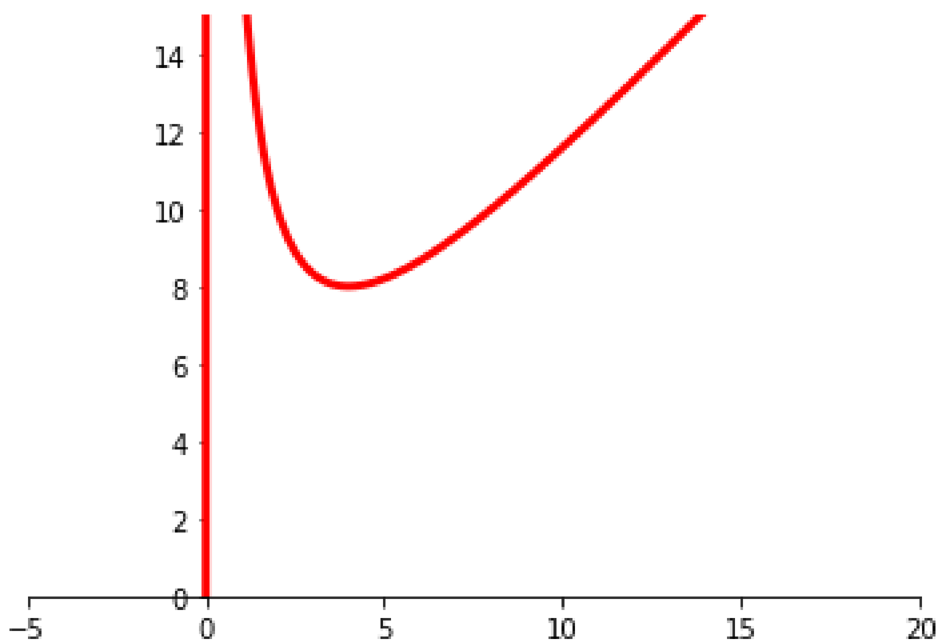
How do we know that we have a global minimum at $x = \sqrt{2A}$? The function

$$P = x + \frac{2}{x}$$

is defined for all positive real numbers. In other words, it is defined on $[0, \infty)$. At both endpoints, this function is infinite, so $x = \sqrt{2A}$ indeed yields the global minimum.

Let's graph this function to get a visual representation of that is going. In the plot below, let's take $A = 8$. According to our calculation, this should yield a global minimum at $x = \sqrt{2A} = \sqrt{16} = 4$.

[6]:
```python
fig = plt.figure(1)
ax = fig.add_subplot(111)


# set up axis
ax.spines['left'].set_position('zero')
ax.spines['right'].set_color('none')
ax.spines['bottom'].set_position('zero')
ax.spines['top'].set_color('none')
ax.xaxis.set_ticks_position('bottom')
ax.yaxis.set_ticks_position('left')

# setup x and y ranges and precision
xx = np.arange(-5,20,0.01)

# draw my curve
myfunction=sympify(x+16/x)
mylambdifiedfunction=lambdify(x,myfunction,'numpy')
ax.plot(xx, mylambdifiedfunction(xx),zorder=100,linewidth=3,color='red')
#plt.axes().set_aspect('equal')

#set bounds
ax.set_xbound(-5,20)
ax.set_ybound(0,15)

plt.show()
```



As you can see, the picture is consistent with our calculations above, as the global minimum does appear to be at least close to $x = 4$.

Let us now consider a related problem with a bit more geometric content. A cylindrical can is to be made to hold 1 liter of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

The can has radius R and height H. The surface area of each of the two caps is πR^2, while the area of the cylindrical part is equal to $2\pi R \cdot H$.

It follows that the total area is equal to

$$2\pi R^2 + 2\pi R \cdot H.$$

Since the can is supposed to hold 1 liter of oil, the volume of the can is equal to 1. The volume of the can is given by the equation

$$\pi R^2 \cdot H.$$

Since this quantity is equal to 1000 (liter is equivalent to 1000 cm^3), we obtain the relation

$$H = \frac{1000}{\pi R^2}.$$

Plugging this back into the formula for the area, we see that the area

$$A = 2\pi R^2 + 2\pi R \cdot \frac{1000}{\pi R^2} = 2\pi R^2 + \frac{2000}{R}.$$

To minimize this quantity, we take a derivative of A with respect to R and set it equal to 0. We have

$$A' = 4\pi R - \frac{2000}{R^2}.$$

Setting this quantity equal to zero we obtain

$$R = \left(\frac{500}{\pi}\right)^{\frac{1}{3}}.$$

Checking to the left and to the right of this value, we check that this is indeed a local minimum. Indeed,

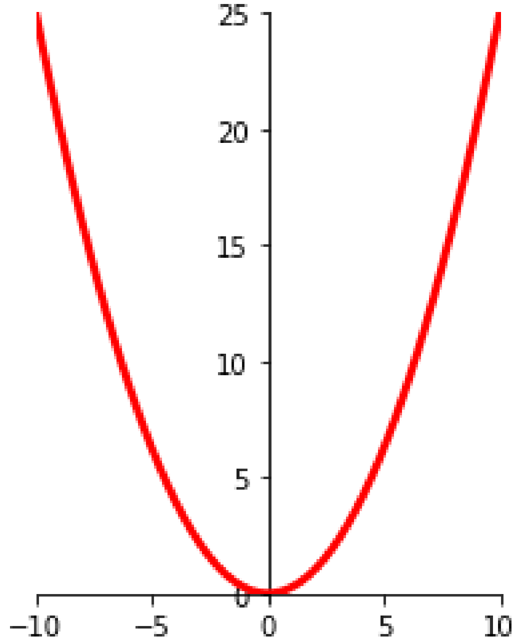
$$A' = \frac{4(\pi R^3 - 500)}{R^2},$$

from which we can read off that %A'$ is negative to the left of critical value, and positive to the right. This means that we first go down the hill and then up the hill, which means that we are at a local minimum.

Why is it a global minimum? Just as in the previous problem, A is defined on the interval $[0, \infty)$, and the values of A at the endpoints are infinite. It follows that we indeed have a local minimum.

Here is another example of yet another type. Find the point on the parabola $y = x^2/4$ that is closest to the point $(5, 0)$. First, let us graph the parabola to get a better idea of what we are after.

[15]:
```
fig = plt.figure(1)
ax = fig.add_subplot(111)



# set up axis
ax.spines['left'].set_position('zero')
ax.spines['right'].set_color('none')
ax.spines['bottom'].set_position('zero')
ax.spines['top'].set_color('none')
ax.xaxis.set_ticks_position('bottom')
ax.yaxis.set_ticks_position('left')

# setup x and y ranges and precision
xx = np.arange(-10,10,0.01)

# draw my curve
myfunction=sympify(x**2/4)
mylambdifiedfunction=lambdify(x,myfunction,'numpy')
ax.plot(xx, mylambdifiedfunction(xx),zorder=100,linewidth=3,color='red')
plt.axes().set_aspect('equal')

#set bounds
ax.set_xbound(-10,10)
ax.set_ybound(0,25)

plt.show()
```



What is the strategy here? Every point on the parabola can be written in the form $(x, x^2/4)$. The distance squared from $(5, 0)$ to $(x, x^2/4)$ is equal to

$$F = (x-5)^2 + \frac{x^4}{16}.$$

Why are we dealing with the distance function squared, instead of the distance function? Because if we minimize the distance squared, we will have minimized the distance as well.

We have

$$F' = 2(x-5) + \frac{x^3}{4},$$

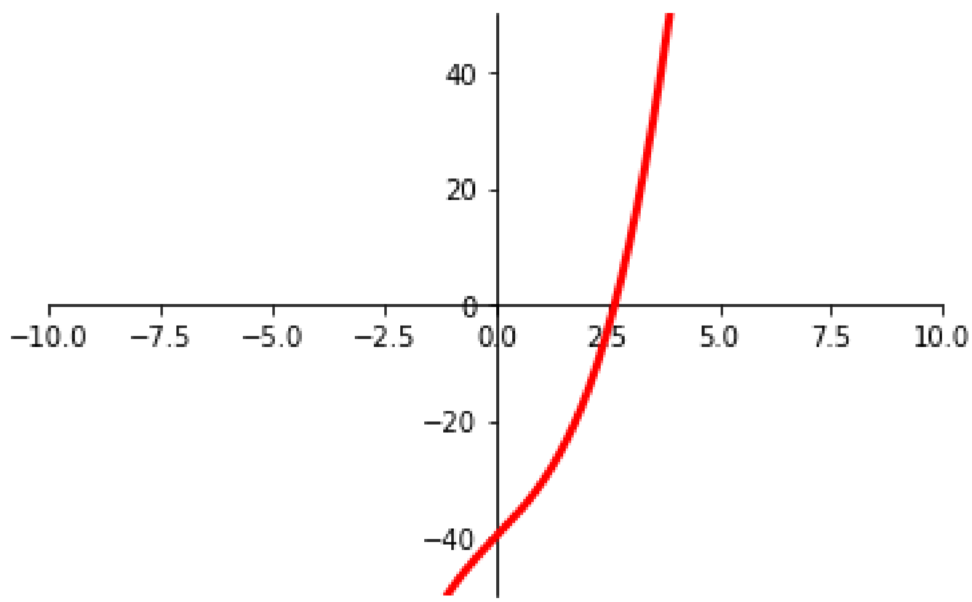
so we must solve the equation

$$f(x) = x^3 + 8x - 40 = 0.$$

So what do we do now? On quizzes and exams in this class, situations like this are not going to arise and all the calculations are going to come out nicely. But I wanted you to see a few examples that are a bit more realistic. It turns out the polynomial $x^3 - 8x - 40$ has exactly one real root and it is, approximately equal to 2.6565. Note that $f(2) = -16$, while $f(3) = 11$, so there must be x_0 somewhere in between such that $f(x_0) = 0$. It turns out that x_4 is, approximately, 2.6565. Also note that $f'(x) = 3x^2 + 8$. This function is always positive, so f is always increasing. In particular, this means that f' does not have any other zeroes (why?)

You can use exactly the same methods as before to check that it is in fact a local and global minimum of the distance function squared. Just for fun, let graph the polynomial $x^3 + 8x - 40$ to see what is going on.

[20]:
```
fig = plt.figure(1)
ax = fig.add_subplot(111)



# set up axis
ax.spines['left'].set_position('zero')
ax.spines['right'].set_color('none')
ax.spines['bottom'].set_position('zero')
ax.spines['top'].set_color('none')
ax.xaxis.set_ticks_position('bottom')
ax.yaxis.set_ticks_position('left')

# setup x and y ranges and precision
xx = np.arange(-10,10,0.01)

# draw my curve
myfunction=sympify(x**3+8*x-40)
mylambdifiedfunction=lambdify(x,myfunction,'numpy')
ax.plot(xx, mylambdifiedfunction(xx),zorder=100,linewidth=3,color='red')
#plt.axes().set_aspect('equal')

#set bounds
ax.set_xbound(-10,10)
ax.set_ybound(-50,50)

plt.show()
```



In class, I am going to work out a similar example where the zeroes of the derivative of the distance function squared will be much easier to compute. But it is worth remembering that in real life, most of these problems are very messy and cannot really be solved without the aid of computational devices. Also, the set up of this problem up to the point where one needs to find the critical points of the distance function squared is always the same.

```
[ ]:
```