MATH 238: HOMEWORK #6 DUE MONDAY, 10/31/16

ALEX IOSEVICH

Problem #1: Let $\nu_{f,g}(t) = \sum_{x \cdot y = t} f(x)g(y)$, where $f, g : \mathbb{Z}_p^d \to \mathbb{R}$, $d \geq 2$. Suppose that $f, g \geq 0$. For $1 \leq p < \infty$, define

$$||f||_p = \left(\sum_{x \in \mathbb{Z}_p^d} |f(x)|^p \right)^{\frac{1}{p}}.$$

Prove that

$$\nu(t) = ||f||_1 ||g||_1 p^{-1} + R_{f,g}(t),$$

where

$$|R_{f,g}(t)| \leq ||f||_2 ||g||_2 p^{\frac{d-1}{2}}.$$

Problem #2: With $\nu_{f,g}(t)$ as above, formulate and prove a bound for $\sum_t \nu_{f,g}^2(t)$ in analogy with the bound we obtained in class in the case when $f(x) = g(x) = E(x)$, where $E \subset \mathbb{Z}_p^2$. Note that I am asking for this bound in $\mathbb{Z}_p^d$, $d \geq 2$.

Use this bound to show that if $A \subset \mathbb{Z}_p$ such that $\#A \geq cp^{\frac{d}{2d-1}}$, then

$$\#dA^2 \equiv \# \{A \cdot A + A \cdot A + \cdots + A \cdot A\} \geq C(c)p.$$

Problem #3: Let $p \equiv 3 \mod 4$. Let $O_2(\mathbb{Z}_p)$ denote the group of two by two matrices M, with entries in $\mathbb{Z}_p$, such that $M^t M = I_2$ and $det(M) = 1$. Suppose that $x, y \in \mathbb{Z}_p^2$, $x \neq (0,0)$, $y \neq (0,0)$, such that $||x|| = ||y||$. Then there exists $M \in O_2(\mathbb{Z}_p)$ such that $y = Mx$.

Problem #4: Prove that if $t \neq 0$ and $f : \mathbb{Z}_p^2 \to \mathbb{R}$, $p \equiv 3 \mod 4$, then

$$\left(\sum_{||m||=t} \left|\widehat{f}(m)\right|^2 \right)^{\frac{1}{2}} \leq C \left(\frac{1}{p^2} \sum_{x \in \mathbb{Z}_p^2} |f(x)|^{\frac{4}{3}} \right)^{\frac{3}{4}}.$$

Problem #5: Let $E \subset \mathbb{Z}_p^2$, $p \equiv 3 \mod 4$. Define an equivalence relation $\sim$ on $\mathbb{Z}_p^2$, $z \sim w$, $z, w \in \mathbb{Z}_p^2$, $z \neq (0,0)$, $w \neq (0,0)$, if $z = tw$ for some $t \in \mathbb{Z}_p$. Let $\mathcal{D}(E)$ (the direction set of E) denote the set of equivalence relations of elements of

$$E - E = \{x - y : x, y \in E\}.$$

Prove that if $\#E > p$, then $\mathcal{D}(E) = \mathcal{D}(\mathbb{Z}_p)$.

Hint: Prove that if $\mathcal{D}(E) \neq \mathcal{D}(\mathbb{Z}_p)$, then there exist $v, w \in \mathbb{Z}_p^2$ such that $v \cdot w = 0$ and

$$E = \{tv + f(t)w : t \in \mathbb{Z}_p\},$$

where $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is some function.

In particular, this will imply that $\#E = p$, instantly giving you what you want. Heuristically, what the hint says is that if a direction is missing, you can express your set as a graph with respect to some coordinate system. This is why a semi-circle is a graph, but the full circle is not.