Math 173, Fall 2022, Monday, October 17

Theorem 5: If $\underbrace{W \subseteq V}_{\text{subspace}} \sim$ finite dimensional, then every linearly independent subset of W is finite and is contained in a basis of V.

Corollary 1: If $\underbrace{W \subset V}_{\text{proper subspace}} \sim$ finite dimensional, then W is finite dimensional and $\dim(W) < \dim(V)$.

Corollary 2: In a finite dimensional vector space V every non-empty linearly independent set of vectors is a part of a basis.

Corollary 3: Let A be an $n \times n$ matrix over F, and row vectors are linearly independent. Then A is invertible.

Proof: Let $W = \text{span}\{\overbrace{\alpha_1, \dots, \alpha_n}^{\sim \text{ rows of } A}\}$

Then $\dim(W) = n$, so $W = F^n$ by Corollary 1.

It follows that $\exists \{B_{ij}\} \in F \ni$

$$\underbrace{\epsilon_i}_{\text{standard bases}} = \sum_{j=1}^{n} B_{ij} \alpha_j, \quad 1 \leq i \leq n$$

$$\Longrightarrow BA = I \checkmark$$

Theorem 6: $\underbrace{W_1, W_2}_{\text{subspaces}} \subset \underbrace{V}_{\text{finite dimensional}}$

Then $W_1 + W_2$ is finite-dimensional and

$$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$$

Example (before proof)

$W_1, W_2 \subset \mathbb{R}^3$ $\quad W_1 = \{(x, y, z) : x + y + z = 0\}$

$W_2 = \{(x, y, z) : z = 0\}$

$$W_1 \cap W_2 = \underbrace{\{(x, y, z) : z = 0, \ y = -x\}}_{\text{1-dimensional}}$$

$W_1 + W_2$ consists of vectors of the form

$(a, b, -a-b) + (c, d, 0)$

$= (a+c, b+d, -a-b)$

In particular, every vector of the form

$(a+c, d, -a)$ is there

Since c is arbitrary, we get every

vector of the form $(c', d, -a)$, and

since a is arbitrary, we get every

(c', d, a'), so $W_1 + W_2 = \mathbb{R}^3$

$$\underset{\substack{\shortparallel \\ 2}}{\dim W_1} + \underset{\substack{\shortparallel \\ 2}}{\dim W_2} = \underset{\substack{\shortparallel \\ 1}}{\dim (W_1 \cap W_2)} + \underset{\substack{\shortparallel \\ 3}}{\dim (W_1 + W_2)}$$

✓

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Proof: $W_1 \cap W_2$ has a basis $\{\alpha_1, \dots, \alpha_k\}$

Extend it to the basis of W_1:

$$\{\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \dots, \beta_m\}$$

" " " " " W_2:

$$\{\alpha_1, \alpha_2, \dots, \alpha_k, \gamma_1, \dots, \gamma_n\}$$

$$W_1 + W_2 = \text{span}\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m, \gamma_1, \dots, \gamma_n\}$$

but are they independent?

Suppose that

$$\sum x_i \alpha_i + \sum y_j \beta_j + \sum z_r \gamma_r = 0$$

Then $\sum z_r \gamma_r = -\sum x_i \alpha_i - \sum y_j \beta_j$

$\hookrightarrow \sum z_r \gamma_r \in W_1$

But $\sum z_r \gamma_r \in W_2$, so

$\sum z_r \gamma_r = \sum c_i \alpha_i$

some scalars

Since $\{\alpha_1, \ldots, \alpha_k, \gamma_1, \ldots, \gamma_n\}$ is independent,

$z_r \equiv 0$, $c_i \equiv 0$ In particular,

$\sum x_i \alpha_i + \sum y_j \beta_j = 0$, and since $\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_n\}$ is independent,

$x_i \equiv 0$, $y_j \equiv 0$. It follows that

$\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n\}$ is a basis for $W_1 + W_2$.

It follows that

$\dim W_1 + \dim W_2 = (k+m) + (k+n)$

$= k + (m+k+n) = \dim(W_1 \cap W_2) + \dim(W_1 \cup W_2)$

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Coordinates: $V = F^n$

$\alpha = (x_1, x_2, \dots, x_n) = \sum x_i \epsilon_i$

standard vectors

we shall think of x_i's as coordinates and the order matters!

But there are other bases!!

Let $\mathcal{B} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis of V written in a particular order

Claim: $\exists!$ n-tuple $(x_1, x_2, \dots, x_n) = X$ $\ni$

$\alpha = \sum_{i=1}^{n} x_i \alpha_i$ ~ basis from above

X is unique because if $\alpha = \sum_{i=1}^{n} z_i \alpha_i$,

$\sum (z_i - x_i) \alpha_i = 0 \Rightarrow z_i \equiv x_i$ ✓

by independence.

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We shall call $\{x_i\}$s coordinates of α relative to the ordered basis

$$\beta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$$

If $\alpha = \sum_{i=1}^{n} x_i \alpha_i$, $\beta = \sum_{i=1}^{n} y_i \alpha_i$, then

$\alpha + \beta$ has coordinates $x + y$ ✓

$c\alpha$ has coordinates cx ✓

And this is where fun really begins!

More on that on Wednesday!