1

Math 173, Fall 2022, October 26

V, W vector spaces over F. A linear transformation from V to W is a function $T: V \longrightarrow W \ni$

$$T(c\alpha + \beta) = \underbrace{c}_{\text{scalar}}\ \underbrace{T\alpha}_{\text{vector}} + \underbrace{T\beta}_{\text{vector}}$$

Example: V = polynomials over $\mathbb{R}$

$$Tf(x) = \underbrace{f'(x)}_{\text{derivative}}$$

Example: $A = m \times n$ matrix over F.

$T(X) = AX$ maps $F^{n \times 1} \longrightarrow F^{m \times 1}$

$U(\alpha) = \alpha A$ maps $F^{m} \longrightarrow F^{n}$

Example: $V = C(\mathbb{R})$

continuous functions from $\mathbb{R} \to \mathbb{R}$

$W = C^1(\mathbb{R})$

once continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}$

$$Tf(x) = \int_0^x f(t)\,dt$$

Then $T : C(\mathbb{R}) \to C^1(\mathbb{R})$

<u>Theorem 1:</u> V finite-dimensional over F

$\{\alpha_1, \ldots, \alpha_n\}$ ordered basis.

W vector space over F, $\beta_1, \ldots, \beta_n$

<u>any</u> vectors in W. Then $\exists !\ T : V \to W$

$\ni T\alpha_j = \beta_j,\ j = 1, 2, \ldots, n$

③

Proof: Given $\alpha \in V$ $\exists!$ n-tuple $(x_1, x_2, \ldots, x_n)$ $\ni$

$$\alpha = x_1\alpha_1 + \ldots + x_n\alpha_n$$

$$T\alpha = x_1\beta_1 + \ldots + x_n\beta_n$$

well-defined rule, but is it linear?

Let $\beta = y_1\alpha_1 + \ldots + y_n\alpha_n \in V$, c = scalar

$c\alpha + \beta = (cx_1 + y_1)\alpha_1 + \ldots + (cx_n + y_n)\alpha_n$, so

$$T(c\alpha + \beta) = (cx_1 + y_1)\beta_1 + \ldots + (cx_n + y_n)\beta_n.$$

$$c(T\alpha) + T\beta = c\sum_{i=1}^{n} x_i\beta_i + \sum_{i=1}^{n} y_i\beta_i$$

$$= \sum_{i=1}^{n} (cx_i + y_i)\beta_i, \text{ so}$$

$$T(c\alpha + \beta) = cT\alpha + T\beta.$$

this gives us existence.

④

Uniqueness: If $U\alpha_j = \beta_j$, $j = 1, 2, \ldots, n$

$$\alpha = \sum_{i=1}^{n} x_i \alpha_i$$

$$U\alpha = U\left(\sum_{i=1}^{n} x_i \alpha_i\right) = \sum_{i=1}^{n} x_i\, U(\alpha_i)$$

$$= \sum_{i=1}^{n} x_i \beta_i,$$ so U is the same rule as T.

Example: $(1, 2)$ $(3, 4)$ basis of $\mathbb{R}^2$

By Theorem 1, $\exists!\ T \ni$

$$T(1,2) = (3,2,1)$$

$$T(3,4) = (6,5,4)$$

What is $T(1,0)$?

$$(1,0) = -2(1,2) + 1 \cdot (3,4)$$

by inspection

5

$T(1,0) = -2(3,2,1) + 1(6,5,4)$

$= (0,1,2)$ ✓

Definition: $T : V \longrightarrow W$

linear transformation; V vector space; W vector space

$\text{Null space}(T) = \{\alpha : T\alpha = 0\}$

$\text{nullity}(T) = \dim(\text{Null space}(T))$

Theorem: $T : V \longrightarrow W$

linear transformation; V vector space; W vector space

Then $\text{rank}(T) + \text{nullity}(T) = \dim(V)$

$\text{rank}(T) = \dim(\text{range}(T))$

(6)

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad (\alpha_1, \alpha_2)$

$T\alpha = \alpha_1 + \alpha_2$ $\quad$ coordinates w/ respect to the standard basis

$T\alpha = 0 \Leftrightarrow \alpha_2 = -\alpha_1$, so

Null space $(T) = \{(t, -t) : t \in \mathbb{R}\}$

nullity $(T) = 1$

Range$(T) = \mathbb{R}$

rank $(T) = \underline{1}$

$\underline{1} + \underline{1} = 2$

$\begin{array}{ccccc} 1 & + & 1 & = & 2 \\ \parallel & & \parallel & & \parallel \\ \text{rank} & & \text{nullity} & & \dim(V) = \dim(\mathbb{R}^2) \end{array}$

7

Proof of Rank-Nullity theorem:

$N =$ Null space of T

$\{\alpha_1, \alpha_2, \dots, \alpha_K\}$ basis of N

Extend to basis of V by

$$\{\underbrace{\alpha_1, \alpha_2, \dots, \alpha_K}, \alpha_{K+1}, \dots, \alpha_n\}$$

Claim: $\{T\alpha_{K+1}, \dots, T\alpha_n\}$ basis of Range(T).

They certainly span since

$T\alpha_j = 0$, $1 \le j \le K$.

Independence: Suppose

$$\sum_{i=K+1}^{n} c_i T\alpha_i = 0$$

(8)

Then $T\left(\sum_{i=k+1}^{n} c_i \alpha_i\right) = 0$

$$\hookrightarrow \alpha = \sum_{i=k+1}^{n} c_i \alpha_i \in N$$

$$\hookrightarrow \alpha = \sum_{i=1}^{k} b_i \alpha_i$$

$$\hookrightarrow \sum_{i=1}^{k} b_i \alpha_i - \sum_{j=k+1}^{n} c_j \alpha_j = 0$$

$$\hookrightarrow b_1 = \dots b_k = c_{k+1} = \dots c_n \equiv 0$$

Let $r = \text{rank}(T)$.

It follows that $r = n - k$

and we are done!