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Math 265H, September 21, 2022

Last time: A set E is open iff its complement is closed.

Corollary: A set F is closed iff its complement is open.

Theorem:

a) Union of open sets is open

b) Intersection of closed sets is closed

c) Intersection of finitely many open sets is open.

d) Union of finitely many closed sets is closed.

Proof:

a) $G = \bigcup G_\alpha$ If $x \in G$, $x \in G_\alpha$ for some α, so a nhood of x is contained in G_α, hence in $\bigcup_\alpha G_\alpha$

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b) $\{F_\alpha\}$ closed

$$\left(\bigcap_\alpha F_\alpha\right)^c = \bigcup_\alpha (F_\alpha^c) \quad \text{open by a)}$$

open

$\bigcap_\alpha F_\alpha$ is closed!

c) Let $H = \bigcap_{i=1}^{n} G_i$ finiteness is essential!

Let $x \in H \Leftrightarrow x \in G_i \ \forall i$

$\exists$ nhood $N_{r_i}(x)$ contained in G_i

Let $r = \min_{1 \le i \le n} r_i$. Then $N_r(x)$ is contained in every G_i, hence contained in $\bigcap_{i=1}^{n} G_i$

d) Follows from c) once again by taking

$$\left(\bigcup_{i=1}^{n} F_i\right)^c = \bigcap_{i=1}^{n} F_i^c$$

Definition: X metric space, $E \subset X$, E' = set of limit points of E in X

$\overline{E} = E \cup E'$

closure of E in X

Theorem: X metric space, $E \subseteq X$, then

a) $\overline{E}$ is closed

b) $E = \overline{E}$ iff E is closed.

c) $\overline{E} \subset F$ for every closed set $F \subset X \ni E \subset F$.

Proof: a) If $p \in X$ and $p \notin \overline{E}$, then p is neither a point of E nor a limit point of E. Hence p has a nhood that does not intersect E $\Rightarrow$ $\overline{E}^c$ is open $\leftarrow$ So $\overline{E}$ is closed.

④

b) If $E = \overline{E}$, E is closed by a)
If E is closed, $E' \subset E$, so $E = \overline{E}$ ✓

c) If F is closed and $F \supset E$, then
$F \supset F'$, hence $F \supset E' \Rightarrow F \supset \overline{E}$.

<u>Theorem</u> Let $E \neq \emptyset$, $E \subseteq \mathbb{R}$ bounded above.
Let $y = \sup(E)$. Then $y \in \overline{E}$. Hence $y \in E$ if E is closed.

Proof: If $y \in E$, $y \in \overline{E}$. Assume that $y \notin E$. For every $\epsilon > 0$ there is a point $x \in E \ni y - h < x < y$, else $y - h$ would be an upper bound of E.

Thus y is a limit point of E and we are done!

<u>Compact sets</u>: We say that $\{G_\alpha\}$ is an open cover of E if each G_α is open and

$$E \subset \bigcup_\alpha G_\alpha.$$

We say that E is compact if every cover of E contains a finite subcover.

$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ example

Finite subcover is any collection

$$G_{\alpha_1} \cup G_{\alpha_2} \cup \dots G_{\alpha_n}, \quad G_{\alpha_j} \subset \{G_\alpha\}$$

<u>Theorem</u>: $K \subset Y \subset X$. Then K is compact in X iff K is compact in Y.

<u>Proof</u>: If K is compact in X, and cover $\{G_\alpha\}$ of K has a finite subcover.

Now consider a collection $\{V_\alpha\}$ of open sets in Y that cover K.

Suppose for a moment that we can find G_α open in X $\ni$ $V_\alpha = Y \cap G_\alpha$.

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Then, using compactness of K in X,

$K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}$ for some finite collection of G_{α_j}'s

Since $K \subset Y$, it is still true that

$$K \subset \underbrace{G_{\alpha_1} \cap Y}_{= V_{\alpha_1}} \cup \dots \underbrace{G_{\alpha_n} \cap Y}_{V_{\alpha_n}}$$

and we have compactness in Y.

Conversely, suppose that K is compact in Y & let $\underbrace{\{G_\alpha\}}_{\text{open in } X}$ cover $\underline{\underline{K}}$.

Let $V_\alpha = Y \cap G_\alpha$. Then

$K \subset V_{\alpha_1} \cup V_{\alpha_2} \cup \dots \cup V_{\alpha_n}$ ~ finite subcollection

& since $V_\alpha \subset G_\alpha$, we see that

$K \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$ & we are done!

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We still need to prove that if $Y \subset X$, metric space

then $E \subset Y$ is open in Y iff
$$E = Y \cap G \text{ for some open } G \text{ in } X$$

Proof:

Suppose that E is open relative to Y. Then for each $p \in E$ $\exists\, r_p > 0$ $\ni$
$$d(p,q) < r_p,\ q \in Y \Longrightarrow q \in E.$$

Let $V_p = \{q \in X : d(p,q) < r_p\}$ &

$$\text{let } G = \bigcup_{p \in E} V_p \sim \text{open by above!!}$$

By construction, $E \subset G \cap Y$.

By our choice of V_p, $V_p \cap Y \subset E$ $\forall p \in E$,

so $G \cap Y \subset E$, i.e $E = G \cap Y$ & one direction is done.

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Conversely, if G is open in X and $E = G \cap Y$, every $p \in E$ has a neighborhood $V_p \subset G$. Then $V_p \cap Y \subset E$, so E is open in Y.

<u>Theorem:</u> Compact subsets of metric spaces are closed.

Proof: Let K be compact in X. We shall prove that K^c is open.

Let $p \in X$, $p \notin K$. If $q \in K$, let V_q & W_q be nhoods of p & q of radius $< \frac{1}{2} d(p, q)$

Since K is compact,

$$K \subset W_{q_1} \cup W_{q_2} \cup \dots \cup W_{q_n} = W.$$

If $V = V_{q_1} \cap \dots \cap V_{q_n}$, then V is a neighborhood of p which does not intersect $\underline{W}$.

open!

Hence $V \subset K^c$, so p is an interior point of K^c

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Theorem: Closed subsets of compact sets are compact.

Proof: Suppose $F \subset K \subset X$, F closed in X and K is compact.

Let $\{V_\alpha\}$ be an open cover of F

we need a finite subcover

Note that $\{V_\alpha\} \cup F^c$ covers K, so a finite subcollection covers F and we are done.

Corollary: If F is closed and K is compact, then $F \cap K$ is compact.

Proof: By above, $F \cap K$ is closed, and since $F \cap K \subset \underset{\text{compact}}{K}$, previous theorem $\Rightarrow$ $F \cap K$ is compact.

Theorem: If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X & the intersection of every finite subcollection of $\{K_\alpha\}$ is non-empty, then $\bigcap K_\alpha$ is non-empty.

Proof: Fix K_1 in $\{K_\alpha\}$ & define $G_\alpha = K_\alpha^c$. Assume that no point of K_1 belongs to every K_α. Then $\{G_\alpha\}$ form an open cover of K_1, and since K_1 is compact,

$K_1 \subset G_{\alpha_1} \cup G_{\alpha_2} \cdots \cup G_{\alpha_n}$ n finite

$\longrightarrow K_1 \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_n}$ is empty

$\longrightarrow$ contradiction!