Tyler Perlman HW 2 Answers MTH 173

Page 26. Question 1:

Because all matrices are row equivalent to a matrix in row-reduced echelon form (by a finite sequence of elementary row operations), we can find some R such that $R = PA$ where R is in row reduced echelon form, matrix A is a given matrix, and P is the product of elementary ~~row opera~~ matrices (and thus invertible by Theorem 11).

As such, we can write $R = E_n E_{n-1} E_{n-2} \cdots E_2 E_1 A$ where $E_1, \ldots, E_n$ are the elementary row operations used to row-reduce A in the form of elementary matrices. By associativity of matrix multiplication, we can write:

$$R = (E_n E_{n-1} \cdots E_2 E_1) A$$

We then denote $(E_n E_{n-1} \cdots E_2 E_1) = P$

Because $P = E_n \cdots E_1 = (E_n \cdots E_1) I$
and $R = (E_n \cdots E_1) A$,

we can find both P & R at the same time by simply row reducing the augmented matrix

$$[A \mid I] \xrightarrow{\text{Row Ops.}} [R \mid P]$$

and after applying our elementary row operations reach

• Note: I must be an appropriately sized square matrix for $I \cdot A$ to be well defined

• Note: If A is invertible, $R = I$ and $P = A^{-1}$

Using the method justified above:

$$\left[\begin{array}{cccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 3 & 5 & 0 & 1 & 0 \\ 1 & -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{cccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 5 & 1 & 1 & 0 \\ 0 & -4 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 + 2R_2]{R_1 \to R_1 - R_2} \left[\begin{array}{cccc|ccc} 1 & 0 & -3 & -5 & 0 & -1 & 0 \\ 0 & 2 & 4 & 5 & 1 & 1 & 0 \\ 0 & 0 & 8 & 11 & 1 & 2 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 \cdot \frac{1}{8}]{R_2 \to R_2 \cdot \frac{1}{2}} \left[\begin{array}{cccc|ccc} 1 & 0 & -3 & -5 & 0 & -1 & 0 \\ 0 & 1 & 2 & 5/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 11/8 & 1/8 & 1/4 & 1/8 \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 + 3R_3} \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & -7/8 & 3/8 & -1/4 & 3/8 \\ 0 & 1 & 0 & -1/4 & 1/4 & 0 & -1/4 \\ 0 & 0 & 1 & 11/8 & 1/8 & 1/4 & 1/8 \end{array}\right] \Rightarrow R = \begin{bmatrix} 1 & 0 & 0 & -7/8 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 11/8 \end{bmatrix}$$

$$P = \begin{bmatrix} 3/8 & -1/4 & 3/8 \\ 1/4 & 0 & -1/4 \\ 1/8 & 1/4 & 1/8 \end{bmatrix}$$

Page 26. Question 3:

By the method verified in the ~~proof~~ solution of Question 1:

look at augmented matrices $\left[\begin{array}{c|c} A & I \end{array}\right]$ and if we can put this in the form $\left[\begin{array}{c|c} I & P \end{array}\right]$ by row reduction, then A invertible and $P = A^{-1}$.

- Matrix 1:

$$\left[\begin{array}{ccc|ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 4 & -1 & 2 & 0 & 1 & 0 \\ 6 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 0 & -11 & 4 & -2 & 1 & 0 \\ 0 & -11 & 4 & -3 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to -\frac{1}{11}R_3]{R_1 \to \frac{1}{2}R_1,\ R_2 \to -\frac{1}{11}R_2} \left[\begin{array}{ccc|ccc} 1 & 5/2 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -4/11 & 2/11 & -1/11 & 0 \\ 0 & 1 & -4/11 & 3/11 & 0 & -1/11 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 - \frac{5}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 9/22 & 1/22 & 8/11 & 0 \\ 0 & 1 & -4/11 & 2/11 & -1/11 & 0 \\ 0 & 0 & 0 & 1/11 & 1/11 & -1/11 \end{array}\right]$$

$\left[\begin{array}{c|c} A & I \end{array}\right] \to \left[\begin{array}{c|c} R & P \end{array}\right]$ where $R \neq I$, This implies that A is not row equivalent to the 3×3 identity, and thus not invertible.

- Matrix 2:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 2 & 4 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 5 & -2 & -3 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - 5R_3]{R_1 \to R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 8 & -3 & 1 & -5 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{R_2}{8}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3/8 & 1/8 & -5/8 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3/8 & 1/8 & -5/8 \\ 0 & 1 & 0 & -3/4 & 1/4 & -1/4 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -3/4 & 1/4 & -1/4 \\ 0 & 0 & 1 & -3/8 & 1/8 & -5/8 \end{array}\right]$$

$$\Longrightarrow \quad A \text{ invertible and } A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -3/4 & 1/4 & -1/4 \\ -3/8 & 1/8 & -5/8 \end{bmatrix}$$

Page 26 Question 4:

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$$

For what X do we have a scalar c s.t.

$$AX = cX \, ?$$

- **Method 1:**

$$\begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_1 \\ x_1 + 5x_2 \\ x_2 + 5x_3 \end{bmatrix} \overset{\text{want}}{=} c \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}$$

$$\Longrightarrow \begin{aligned} 5x_1 &= cx_1 & &\Rightarrow c = 5 \text{ or } x_1 = 0 \\ x_1 + 5x_2 &= cx_2 & &\Rightarrow x_1 = 0 \text{ and } (c = 5 \text{ or } x_2 = 0) \\ x_2 + 5x_3 &= cx_3 & &\Rightarrow x_2 = 0 \text{ and } (c = 5 \text{ or } x_3 = 0) \end{aligned}$$

$\Longrightarrow$

- $X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ with any $c \in \mathbb{R}$ (Trivial Solution)

- $X = \begin{bmatrix} 0 \\ 0 \\ \lambda \end{bmatrix}$ with $\lambda \in \mathbb{R}$ and $c = 5$ $\left(\text{All scalar multiples of } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$

- **Method 2:**

$$A_{3\times 3} \cdot X_{3\times 1} = (AX)_{3\times 1} \quad \text{and} \quad c_{\text{scalar}} \cdot X_{3\times 1} = (cX)_{3\times 1}$$

with both AX and cX being well-defined 3×1 matrices.

We now treat cX as $(c \cdot I)X$ and will do simple algebraic manipulations:

$$AX = cIX \Longrightarrow AX - cIX = 0$$

$$\Longrightarrow (A - cI)X = 0$$

$$\Longrightarrow \begin{bmatrix} 5-c & 0 & 0 \\ 1 & 5-c & 0 \\ 0 & 1 & 5-c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (5-c)x_1 \\ x_1 + (5-c)x_2 \\ x_2 + 5-c\, x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

will yield the same equations and conclusions as above

Page 27 Question 10:

Prove that if A is an $m \times n$ matrix and B is an $n \times m$ matrix and $n < m$, then AB is not invertible

Because A is a matrix with m rows and n columns with $m > n$, if we were to put it into row reduced echelon form, we know that there can be at most n rows with non-zero entries, or equivalently, we will always have at least $(m-n)$ ~~zero~~ rows of all zeroes. Thus, we can write that: (using Theorem 12 to say P invertible)

$$PA = R \implies A = P^{-1}R \quad \text{where } R = \begin{bmatrix} R' \\ \hline 0 \end{bmatrix}$$

where ~~and~~ R' is an ~~max~~ $m \times n$ matrix in ~~and~~ row reduced echelon form and the 0 represents $(m-n)$ rows of n zeroes, and ~~[illegible]~~ P is the product of elementary $m \times m$ matrices. (See Solution to problem 1 for more details).
Moving forward, we will multiply the above expression on the right by B:

$$AB = (P^{-1}R)B$$

this is well defined because $P^{-1}R$ & A are $m \times n$ and B is $n \times m$

By the associativity of matrix multiplication:

$$AB = P^{-1}(RB)$$

R is $m \times n$ and B is $n \times m$

Now, we will look at the product RB:

$$RB_{ij} = \sum_{r=1}^{n} R_{ir} B_{rj} \quad \forall \ 1 \le i, j \le m$$

For RB_{ij} when $i > n$, $R_{ir} = 0$. As such, $RB_{ij} = 0$ when $i > n$. Thus, the bottom $m-n$ rows of RB will contain m zeroes each.

Page 27. Question 10 continued.

Going back to the equation

$$AB = P^{-1}(RB)$$, by left multiplication on both sides by P, we get

$$P(AB) = RB.$$

We now know, however, that RB has at least $(m-n)$ zero rows, and is thus not invertible. By the corollary to Theorem 10, we know that the product of invertible matrices is invertible. The contrapositive of this would be that if the product of matrices is not invertible, then (at least) one of the matrices in the product must not be invertible.
Using this with the knowledge that P is invertible and RB is not, we can conclude that AB is not invertible. $\square$

Page 34. Question 6

~~$\forall z \in \{f(z)\} \, z \in \mathbb{C}$~~

V is the set of complex-valued functions over the Real line s.t. $f(-t) = \overline{f(t)} \ \forall t \in \mathbb{R}$

Show that $(V, +, \cdot)$ where $(f+g)(t) \equiv f(t) + g(t)$
$(cf)t \equiv c f(t)$

is a vector space over $\mathbb{R}$.

Give an example of a function in V which is not real-valued.

- First, I will give the example function: $f(t) = e^{it}$

$$f(-t) = e^{-it} = \cos(-t) + i\sin(-t) = \cos(t) - i\sin(t)$$

$$\overline{f(t)} = \overline{e^{it}} = \overline{(\cos(t) + i\sin(t))} = \cos(t) - i\sin(t) \ \checkmark$$

- Next, I shall proceed with the proof.

- $\mathbb{R}$ is our field of scalars
- $\{f(t) : \mathbb{R} \to \mathbb{C} \mid f(-t) = \overline{f(t)}\}$ is our set of vectors
- $(f+g)(t) = f(t) + g(t) \underset{\substack{\text{by commutativity} \\ \text{of addition of} \\ \text{Complex numbers}}}{=} g(t) + f(t) = (g+f)(t) \ \checkmark$

- $(f+(g+h))(t) = f(t)+(g+h)(t) = f(t)+g(t)+h(t) = (f+g)(t)+h(t)$
$= ((f+g)+h)(t)$ ✓

- define $O(t) = 0 \quad \forall t \in \mathbb{R}$

$(f+O)(t) = f(t) + O(t) = f(t)$ ✓

is $O(t) \in V$?

$O(-t) = 0$
$\overline{O(t)} = \overline{0} = 0$ ✓

- for $f(t)$, define $-f(t) = -(f(t))$

$(-f)(-t) = -f(-t)$, $\overline{(-f)(t)} = \overline{-f(t)} = \overline{-1 \cdot f(t)} = \overline{-1} \cdot \overline{f(t)} = -1 \cdot \overline{f(t)}$
$= -\overline{f(t)}$

$f(-t) = \overline{f(t)} \implies -(f(-t)) = -(\overline{f(t)})$ ✓

$\implies -f \in V$ ✓

$(f + -f)(t) = f(t) - f(t) = 0 = O(t)$ ✓

- $(1f)(t) = 1f(t) = f(t)$ ✓

- $((c_1c_2)f)(t) = (c_1c_2)f(t) = c_1(c_2f(t)) = c_1(c_2f)(t)$ ✓

- $(c(f+g))(t) = c[f(t)+g(t)] = cf(t)+cg(t) = (cf+cg)(t)$ ✓

- $((c_1+c_2)f)(t) = (c_1+c_2)f(t) = c_1f(t) + c_2f(t) = (c_1f+c_2f)(t)$ ✓ $\square$

Page 34 Question 7

$V = \{(x,y) \in \mathbb{R}\times\mathbb{R}\}$ $\quad F = \mathbb{R}$
$(x,y)+(x_1,y_1) = (x+x_1, 0)$
$c(x,y) = (cx, 0)$

- Field $= \mathbb{R}$
- $V = \{(x,y) \in \mathbb{R}\times\mathbb{R}\}$
- $(x_1,y_1)+(x_2,y_2) = (x_1+x_2, 0) = (x_2+x_1, 0) = (x_2,y_2)+(x_1,y_1)$ ✓
- $(x_1,y_1)+((x_2,y_2)+(x_3,y_3)) = (x_1,y_1)+(x_2+x_3, 0) = (x_1+x_2+x_3, 0)$
$((x_1,y_1)+(x_2,y_2))+(x_3,y_3) = (x_1+x_2, 0)+(x_3,y_3) = (x_1+x_2+x_3, 0)$ ✓

Page 34 Question 7 continued

~~$(x_1, y_1) = (x_1, y_1)$~~

· Propose $(x_1, \alpha) + (x_1, y_1) = (0, 0) \ \forall \alpha \in \mathbb{R}$

$$(x_1 - x_1, 0) = (0, 0)$$

as such, $-v$ not unique $\Rightarrow$ not a vector space

· Also no unique zero vector: $(0, \alpha) \ \forall \alpha \in R$ will be an additive identity.

Not vector space

Page 39 ~~Question~~ Question 3

If $(-3,1,0,-1)$ is in the span of the other vectors, then it is a linear combination of them, and thus, they will all be linearly dependant, so we will look at the augmented matrix:

$$\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ \\ \end{array} \left[\begin{array}{cccc} 2 & -1 & 3 & 2 \\ -1 & 1 & 1 & -3 \\ 1 & 1 & 9 & -5 \\ \hline -3 & 1 & 0 & -1 \end{array}\right]$$

and will row reduce while never swaping a row with R_4.

If we wind up with R_4 having all zeroes, it is a linear combination of the over rows, and thus $(-3,1,0,-1)$ is in the span of the vectors.

$$\left[\begin{array}{cccc} 2 & -1 & 3 & 2 \\ -1 & 1 & 1 & -3 \\ 1 & 1 & 9 & -5 \\ \hline -3 & 1 & 0 & -1 \end{array}\right] \xrightarrow[R_4 \to R_4 + 3R_3]{R_1 \to R_1 - 2R_3,\ R_2 \to R_2 + R_3} \left[\begin{array}{cccc} 0 & -3 & -15 & 12 \\ 0 & 2 & 10 & -8 \\ 1 & 1 & 9 & -5 \\ \hline 0 & 4 & 27 & -16 \end{array}\right] \xrightarrow[R_4 \to \frac{R_4}{4}]{R_1 \to R_1 \cdot -\frac{1}{3},\ R_2 \to R_2 \cdot \frac{1}{2}} \left[\begin{array}{cccc} 0 & 1 & 5 & -4 \\ 0 & 1 & 5 & -4 \\ 1 & 1 & 9 & -5 \\ \hline 0 & 1 & \frac{27}{4} & -4 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 - R_2]{R_1 \to R_1 - R_2,\ R_3 \to R_3 - R_2} \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 4 \\ 1 & 0 & 4 & -1 \\ \hline 0 & 0 & \frac{7}{4} & 0 \end{array}\right] \xrightarrow[R_4 \to R_4 \cdot \frac{4}{7}]{R_3 \leftrightarrow R_1} \left[\begin{array}{cccc} 1 & 0 & 4 & -1 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - 4R_4,\ R_2 \to R_2 - 5R_4} \left[\begin{array}{cccc} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right]$$

$R_4 \neq (0,0,0,0)$, so $(-3,1,0,-1)$ not in the span of the other vectors.

- Note: if you allow for the swapping of R_4 with other rows, you may (falsly) believe that it is in the span, while, in actuality, the 3 vectors given were already linearly dependant.

Page 39 Question 4

To solve this system, we will first row reduce the matrix A in $AX=0$ (don't need an augmented matrix because "Y" = 0)

$$\begin{bmatrix} 2 & -1 & 4/3 & -1 & 0 \\ 1 & 0 & 2/3 & 0 & -1 \\ 9 & -3 & 6 & -3 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 4/3 & -1 & 0 \\ 1 & 0 & 2/3 & 0 & -1 \\ 9 & -3 & 6 & -3 & -3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 9R_2]{R_1 \to R_1 - 2R_2} \begin{bmatrix} 0 & -1 & 0 & -1 & 2 \\ 1 & 0 & 2/3 & 0 & -1 \\ 0 & -3 & 0 & -3 & 6 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 0 & -1 & 0 & -1 & 2 \\ 1 & 0 & 2/3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_1 \to -R_1]{R_3 \to \frac{R_3}{6}} \begin{bmatrix} 0 & 1 & 0 & 1 & -2 \\ 1 & 0 & 2/3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 & 1 & -2 \\ 1 & 0 & 2/3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_3 \to R_2} \begin{bmatrix} 1 & 0 & 2/3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -2 \end{bmatrix}$$

$\Rightarrow$ We now only need to solve the system

$$\begin{cases} x_1 + \frac{2}{3}x_3 - x_5 = 0 \\ x_2 + x_4 + 2x_5 = 0 \end{cases}$$

So, to find solutions, lets choose arbitrary values for x_1, x_2, and x_5: $x_1 = a$, $x_2 = b$, $x_3 = c$

$\Rightarrow$ $a + \frac{2}{3}x_3 - c = 0$, $b + x_4 + 2c = 0$ $\Rightarrow$ $x_3 = \frac{3}{2}(c-a)$, $x_4 = -(2c+b)$

$$\boxed{W = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -3/2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3/2 \\ -2 \\ 1 \end{bmatrix} \right)}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 3/2c - 3/2a \\ -2c - b \\ c \end{bmatrix} \; \forall a,b,c \in \mathbb{R} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ -3/2 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 3/2 \\ -2 \\ 1 \end{bmatrix}$$

Page 40 Question 5:

Which are subspaces of the space of all $n\times n$ matrices over F? $(n \geq 2)$

- a) All invertible A? **No!** The zero matrix is not the product of elementary matrices. (Does not row reduce to I)
  $\Rightarrow$ No zero in the set. $(I - I = 0$; not closed under addition$)$.

b) All non-invertible A? **No!** Not closed under addition.

Example:
$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = I_{n\times n}$$

not invertible — Invertible

Also no multiplicative identity because $I_{n\times n}$ is invertible.

c) All matrices A s.t. $AB = BA$ for a fixed matrix B?

Check $(cA + A')B \stackrel{?}{=} B(cA + A')$

$$(cA+A')B = cAB + A'B = cBA + BA' = B(cA + A') \checkmark$$

Yes! This is a subspace.

d.) All matrices s.t. $A^2 = A$?

$$(cA + A')^2 = (cA + A')(cA + A') = cA(cA + A') + A'(cA + A')$$
$$= c^2A^2 + cAA' + cA'A + A'^2 = c^2A + A' + c(AA' + A'A)$$

$I^2 = I$, so lets look at $(cI + I)^2 = ((c+1)I)^2 = (c+1)^2 I^2 = (c+1)^2 I$

is $(c+1)I = (c+1)^2 I$? **No!**

for example: $I + I = 2I \quad 2I \neq 4I$
$(I+I)^2 = 4I$

counter example, so no! not a subspace.

Page 40 Question 8:

$V_e = \{ f(x) \mid f(-x) = f(x) \}$

$V_o = \{ f(x) \mid f(-x) = -f(x) \}$

a.) • is V_e a subspace of V?

$f, g \in V_e$

$(cf+g)(-x) = (cf)(-x) + g(-x) = cf(-x) + g(-x) = cf(x) + g(x)$

$= (cf)(x) + g(x) = (cf+g)(x)$ ✓

Yes!

• is V_o a subspace of V?

$(cf+g)(-x) = (cf)(-x) + g(-x) = cf(-x) + g(-x) = -cf(x) - g(x)$

$= (-cf)(x) + (-g)(x) = (-cf-g)(x) = (-(cf+g))(x)$ ✓

b.) Want to show that any function in V can be expressed as the linear combination of elements of $V_e \cup V_o$.

I will be doing this by construction:

~~Propose that~~

• Define $f_e(x) \equiv \dfrac{f(x) + f(-x)}{2}$

$f_o(x) \equiv \dfrac{f(x) - f(-x)}{2}$

• I propose that $f_e(x)$ is even and $f_o(x)$ is odd

→ $f_e(-x) = \dfrac{f(-x) + f(-(-x))}{2} \overset{\text{commutative}}{=} \dfrac{f(x) + f(-x)}{2} = f_e(x)$ ✓

→ $f_o(-x) = \dfrac{f(-x) - f(-(-x))}{2} \overset{\text{commute}}{=} \dfrac{-f(x) + f(-x)}{2} = -\left(\dfrac{f(x) - f(-x)}{2}\right)$

$= -f_o(x)$ ✓

Now, simply add: $(f_e + f_o)(x) = f_e(x) + f_o(x) = \frac{1}{2}\Big((f(x) + f(-x)) + (f(x) - f(-$

$$= \frac{1}{2}\left(2f(x)+0\right) = \left(\frac{1}{2}\cdot 2\right)f(x) = (1)f(x) = f(x)$$

Thus, $\forall f(x) \in V$, $f(x) = (f_o + f_e)(x)$

where $f_o \in V_o$; $f_e \in V_e$

$\Rightarrow V_e + V_o = V$ $\square$

c.) $V_e \cap V_o \stackrel{?}{=} \{0\}$

~~Let $f(x) \in V_e \cup V_o$~~

Let $f(x) \in V_e \cap V_o$

Then $f(-x) = f(x)$
and
$f(-x) = -f(x)$

~~This implies (by transitive property of equivalence relations)~~

transitivity
$\Rightarrow$ $f(x) = -f(x) \quad \forall x \in \mathbb{R}$

$\Rightarrow f(x) + f(x) = 0$

$\Rightarrow (2f)(x) = 0$

$\Rightarrow f(x) = 0$

$\Rightarrow$ ~~$\$$~~ $V_e \cap V_o = \{0\}$ $\square$

Page 48 Question 1

Let α_1 & α_2 be linearly dependant vectors.
This means that there exist $c_1, c_2 \in F$ s.t.
c_1 & c_2 not both zero

$c_1\alpha_1 + c_2\alpha_2 = 0$

$\Rightarrow c_1\alpha_1 = -c_2\alpha_2$

• if $c_1 \neq 0$:

$\Rightarrow \alpha_1 = -\frac{c_2}{c_1}\alpha_2$ (by $\frac{c_2}{c_1}$ I mean $c_1^{-1}c_2$)

Thus, α_1 is a scalar multiple of α_2.

• if $c_1 = 0$: $0\alpha_1 + c_2\alpha_2 = 0$

$\Rightarrow c_2\alpha_2 = 0$

$\Rightarrow \alpha_2 = 0$ or $c_2 = 0$

$\uparrow$ trivial, and not allowed by definition

$\Rightarrow \alpha_2 = 0\alpha_1$

$\Rightarrow \alpha_2$ is a scalar multiple of α_1 $\square$

Page 48 Question 2

If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ linearly independant, ~~from def pg. 48 #4~~, zero cannot be expressed as a linear combination of them: $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = 0$ (other than the trivial case)

Thus, if we look at this as being in the form

$$(0,0,0,0) = c_1(1,1,2,4) + c_2(2,-1,-5,2) + c_3(1,-1,-4,0) + c_4(2,1,1,6)$$

$$\Longleftrightarrow A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So, if the matrix $A = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$

is row equivalent to the identity, then $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are linearly independant because $c_1 = c_2 = c_3 = c_4 = 0$ will be the only solution.

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 4R_1]{R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & -2 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 2R_2]{R_3 \to R_3 - 3R_2} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{R_2}{-3}} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 2/3 & 1/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 2/3 & 1/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

Non-trivial solutions exist (c_3 & c_4 can be anything and we get 2 vectors spanning our possible constants)

$\Rightarrow$ $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are linearly dependant.

0

Page 48 Question 5

3 vectors in $\mathbb{R}^3$ that are linearly dependant, but any pairs of them are linearly independant.

I propose: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

- $a(1,0,0) + b(0,1,0) = (0,0,0)$

  $\Rightarrow$ $a = 0$, $b = 0$ ✓

- $a(1,0,0) + b(1,1,0) = (0,0,0)$

  $\Rightarrow$ $a + b = 0$, $b = 0$ $\Rightarrow$ $a = 0$, $b = 0$ ✓

- $a(1,1,0) + b(0,1,0) = (0,0,0)$

  $\Rightarrow$ $a = 0$, $a + b = 0$ $\Rightarrow$ $b = 0$, $a = 0$ ✓

$a(1,0,0) + b(0,1,0) + c(1,1,0) = 0$

$a + c = 0$
$b + c = 0$ $\Rightarrow$ any ~~triple~~ scalars $(\varepsilon, \varepsilon, -\varepsilon)$ will produce a non-trivial solution.