Exam # 4: Math 1500

Alex Iosevich

April 26, 2007

Write your solutions on the clear white paper provided by the examiner. Clearly indicate which problem you are solving on a given piece of paper and do not write a solution to more than one problem on a single piece of paper. Show all your work. Good luck!

Problem 1. Graph the following function. Label all the local maxima and minima, inflection points, asymptotes (if any) and regions of concavity (up or down).

$$f(x) = \frac{x^2}{x+1}.$$

$$f(x) = x^4 + 4x^3.$$

Solution: I will write this one out on paper and scan it in... Stay tuned...

Problem 2. Find the dimensions of the rectangle of the largest area that can be inscribed in the circle of radius 4.

Solution: Impose the usual coordinate grid onto the circle and label the vertices of the rectangle with coordinates $(x,y),(x,-y),(-x,y)$, and $(-x,-y)$. Since the rectangle is inscribed in the circle of radius 4, $x^2+y^2=16$, so $y=\sqrt{16-x^2}$. It follows that

$$A = 4xy = 4x\sqrt{16-x^2}.$$

We may as well maximize

$$f(x) = A^2 = 16x^2(16-x^2) = 256x^2 - 16x^4.$$

It follows that

$$f'(x) = 512x - 64x^3 = 64x(8-x^2) = 0$$

if $x=0$ or $x=\pm 2\sqrt{2}$. It is not hard to check that $x=\pm 2\sqrt{2}$ corresponds to the maximum. It follows that the dimensions of the inscribed rectangle of the greatest area in the circle of radius 4 are $4\sqrt{2}$ by $4\sqrt{2}$.

Problem 3. Find the point on the line $y = 4x + 7$ that is closest to the origin.

Solution: The square of the distance from an arbitrary point on the line in question and the origin is $f(x) = x^2 + (4x+7)^2$. It follows that $f'(x) = 2x + 8(4x+7) = 34x + 56 = 0$ if $x = -\frac{28}{17}$. It is not hard to check that this is in fact a local minimum. Plugging this value of x into the equation $y = 4x + 7$ gives us the other coordinate.

Problem 4. Let

$$h(x) = \int_{\cos^3(x)}^{\tan(x)+x+1} \sin^2(\cos^3(t^4+1))dt.$$

Compute $h'(x)$.

Solution: Let $F(x) = \int_{-10}^{x} \sin^2(\cos^3(t^4+1))dt$. Then by FTC, $F'(x) = \sin^2(\cos^3(x^4 + 1))$. Now, $h(x) = F(\tan(x) + x + 1) - F(\cos^3(x))$, so $h'(x) = F'(\tan(x) + x + 1)(\sec^2(x) + 1) - F'(\cos^3(x))(-3\cos^2(x)\sin(x))$, where

$$F'(\tan(x) + x + 1) = \sin^2(\cos^3((\tan(x) + x + 1)^4 + 1)),$$

and

$$F'(\cos^3(x)) = \sin^2(\cos^3((\cos^3(x))^4 + 1)).$$

Problem 5. Compute the following integrals.

$$\int_0^2 x(2 + x^5)dx.$$

$$\int_0^{13} \frac{dx}{(1+2x)^{\frac{2}{3}}}.$$

Solution: We have

$$\begin{aligned} \int_0^2 x(2+x^5)dx &= \int_0^2 (2x + x^6)dx \\ &= x^2 + \frac{x^7}{7}\Big|_0^2 \\ &= 2^2 + \frac{2^7}{7}. \end{aligned}$$

We also have

$$\begin{aligned} \int_0^{13} \frac{dx}{(1+2x)^{\frac{2}{3}}} &= \frac{1}{2}\int_1^{27} \frac{du}{u^{\frac{2}{3}}} \\ &= \frac{3}{2} u^{\frac{1}{3}}\Big|_1^{27} \\ &= \frac{3}{2}(3-1) = 3. \end{aligned}$$