January 30, 2019

Proposition: If G is open and connected, and $f: G \to \mathbb{C}$ is differentiable with $f'(z) = 0 \ \forall z \in G$, then f is constant.

Proof: Fix $z_0 \in G$ and let $\omega_0 = f(z_0)$. Put

$A = \{ z \in G : f(z) = \omega_0 \} \longrightarrow$ we will show that A is both open and closed. (Do you remember topology? If not, crack open Munkres)

Let $z \in G$ & let $z_n \to z$, $z_n \in A$. Since $f(z_n) = \omega_0$ for each $n \geq 1$, and f is continuous, we get $f(z) = \omega_0$, or $z \in A$

This proves that A is closed in G.

$\underbrace{\hphantom{f(z)=\omega_0}}_{\text{Key property of continuity.}}$

Now fix $a \in A$, and let $\epsilon > 0$ be such that $B(a, \epsilon) \subset G$. If $z \in B(a, \epsilon)$, set

$g(t) = f(tz + (1-t)a)$, $0 \leq t \leq 1$. Then

$$\frac{g(t)-g(s)}{t-s} = \frac{g(t)-g(s)}{(t-s)z+(s-t)a} \cdot \frac{(t-s)z+(s-t)a}{t-s}$$

Letting $t \to s$, the chain rule yields

$$\lim_{t \to s} \frac{g(t)-g(s)}{t-s} = f'(sz + (1-s)a) \cdot (z-a) = 0.$$

We just proved that $g'(s) = 0$ for $0 \leq s \leq 1$, which implies that g is constant. Hence

$$f(z) = g(1) = g(0) = f(a) = \omega_0.$$

In other words, $B(a, \epsilon) \subset A$, so A is open

Since A is open and closed, $A = G$.

③

Going back to $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ ~ real coefficients

This implies that $e^{\bar{z}} = \overline{e^z}$. In particular, if θ is real,

$|e^{i\theta}|^2 = e^{i\theta} e^{-i\theta} = e^0 = 1$. In general, $|e^z|^2 = e^z e^{\bar{z}} = e^{z+\bar{z}} = e^{2Re(z)}$

We just proved that $|e^z| = e^{Re(z)}$.

We are ready to define $\cos z, \sin z$:

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots + (-1)^n \frac{z^{2n}}{(2n)!} + \dots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots + (-1)^n \frac{z^{2n-1}}{(2n-1)!} + \dots$$

radius of convergence $= \infty$, as before

It is not difficult to see that

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \qquad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

$$\xRightarrow{\text{instantly}} \cos^2 z + \sin^2 z = 1 \quad \text{and}$$

$$e^{iz} = \cos z + i \sin z$$

It follows that $z = |z| e^{i\theta}$, $\theta = \arg(z)$.

Also, $|e^z| = e^{Re(z)}$ & $\arg e^z = Im(z)$.

So far, the properties are quite analogous to the real case, but this is about to end. The key difference is the notion of periodicity, which is where we turn our attention.

What is not immediately obvious is whether the same k works for each $z \in G$. To understand what is going on, let

$$h(z) = \frac{f(z) - g(z)}{2\pi i} \longrightarrow \text{continuous \& takes on } \underline{\text{integer values}}$$

Since G is connected, $h(G)$ is connected, hence

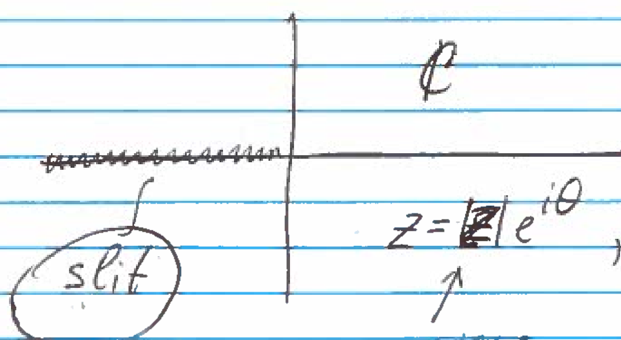
$$\exists\, k \in \mathbb{Z} \ni f(z) + 2\pi k i = g(z).$$

In summary, we get the following result.

Proposition: If $G \subset \mathbb{C}$ is open and connected and f is a branch of $\log(z)$ on G, then the totality of branches of $\log(z)$ are the functions $f(z) + 2\pi k i$, $k \in \mathbb{Z}$.

Characterizations are useful, but what do these branches actually look like?

Let $G = \mathbb{C} - \{z \in \mathbb{R} : z \leq 0\}$



$z = |z| e^{i\theta}$, $-\pi < \theta < \pi$

unique

Define $f(re^{i\theta}) = \log(r) + i\theta$

Is it continuous? Check...

Is the branch we just constructed (if continuity holds up) analytic? This is where we go next.

Corollary: A branch of the logarithm function is analytic and its derivative is z^{-1}.

Principal branch of the logarithm: branch defined above on $\mathbb{C} - \{z \in \mathbb{R} : z \leq 0\}$.

We will now see how to define branches of different functions by bootstrapping the logarithm.