MATH 238: HOMEWORK #7 DUE MONDAY, 11/7/2016

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Problem #1: Let $E = A \times B$, where $A, B \subset \mathbb{Z}_p$, p prime, $p \equiv 3 \mod 4$. Suppose that $\#A \geq cq^{s_A}$ and $\#B \geq cq^{s_B}$. If $s_A + s_B + \max\{s_A, s_B\} > 2$, then

$$\#\Delta(E) \geq C(c)p.$$

Hint: Use the same method as the one we used to obtain the $\frac{4}{3}$ exponent in class and think about what the product structure $A \times B$ buys you.

Problem #2: Let $E \subset \mathbb{Z}_p^d$, $d \geq 2$, p prime. Let

$$P = \{x \in \mathbb{Z}_p^d : x_d = x_1^2 + x_2^2 + \cdots + x_{d-1}^2\}.$$

Prove that if $\#E > p^{\frac{d+1}{2}}$ and p is sufficiently large, then $(E - E) \cap P \neq \emptyset$, where

$$E - E = \{x - y : x \in E; y \in E\}.$$

Hint: Consider $\#\{(x, y) \in E \times E : x - y \in P\} = \sum_{x,y} E(x)E(y)P(x - y)$ and proceed as in the distance set argument, except that the monster you are going to encounter is less scary than then the Kloosterman beast.

Problem #3: Let $E, F \subset \mathbb{Z}_p^2$, p prime, $p \equiv 3 \mod 4$. Prove that if $\#E \cdot \#F \geq cp^{\frac{8}{3}}$, then $\#\Delta(E, F) \geq C(c)p$, where

$$\Delta(E, F) = \{||x - y|| : x \in E; y \in F\}.$$

Problem #4: Given $E \subset \mathbb{Z}_p^2$, p prime, $p \equiv 3 \mod 4$, let $T_2(E)$ denote the set of congruence classes of triangles determined by E. More precisely, $T_2(E)$ is the set of equivalences classes of triples in $E \times E \times E$ such that $(x, y, z) \sim (x', y', z')$ if and only if

$$||x - y|| = ||x' - y''||, ||x - z|| = ||x' - z'||, ||y - z|| = ||y' - z'||,$$

where x, y, z are distinct.

Prove that if $\#E \geq cp^{\frac{8}{5}}$, then $\#T_2(E) \geq C(c)p^3$. If you can beat $\frac{8}{5}$, I will be even happier...

Hint: Follow the proof of the $\frac{4}{3}$ exponent for the distance set problem, making the necessary adjustments along the way.

Problem #5**:** Suppose that if $u, v, u', v' \in S_t = \{x \in \mathbb{Z}_p^2 : ||x|| = t\}$, $t \neq 0$, p prime, $p \equiv 3 \mod 4$. Prove that if

$$u + v = u' + v';\ u \neq -v,$$

then $u = u', v = v'$ or $u = v', v = u'$.