(1)

Proposition: Let $f: G \to \mathbb{C}$ analytic & suppose that $B(a,r) \subset G$, $r>0$. If $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$, then

$$2\pi i\, f(z) = \int_\gamma \frac{f(w)}{w-z}\, dw \quad \text{for } |z-a|<r.$$

eventually to be proved for general γ.

Proof: We may assume that $a=0$ & $r=1$.

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{is})\, e^{is}}{e^{is}-z}\, ds$$

definition; not much else to be done

We must show that

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{f(e^{is})\, e^{is}}{e^{is}-z} - f(z) \right) ds$$

Let $\varphi(s,t) = \dfrac{f(z+t(e^{is}-z))\, e^{is}}{e^{is}-z} - f(z)$

$0 \le t \le 1$, $0 \le s \le 2\pi$

well-defined and continuously differentiable

Let $g(t) = \int_0^{2\pi} \varphi(s,t)\, ds$

We must show that $g(1)=0$. Strategy is to show that $g(0)=0$ & proving that g is constant.

(2)

$$g(0) = \int_0^{2\pi} \varphi(s,0)\,ds = \int_0^{2\pi} \left(\frac{f(z)e^{is}}{e^{is}-z} - f(z) \right) ds$$

$$= f(z) \int_0^{2\pi} \frac{e^{is}}{e^{is}-z}\,ds - 2\pi f(z) = 0$$

since $\int_0^{2\pi} \frac{e^{is}}{e^{is}-z}\,dz = 2\pi$ ✓

by the previous lecture

$$g'(t) = \int_0^{2\pi} \frac{\partial\varphi}{\partial t}(s,t)\,dt$$

Note that $\frac{\partial \varphi}{\partial t}(s,t) = e^{is} f'(z + t(e^{is}-z))$

Let $\Phi(s) = -it^{-1} f(z+t(e^{is}-z))$ is a primitive of $\frac{\partial\varphi}{\partial t}$.

It follows that $g'(t) = \Phi(2\pi) - \Phi(0) = 0$

$\Rightarrow g$ is constant since g is continuous.

This completes the proof.

Superficially, we are ready to show that $f(z)$ has a power series expansion. Indeed,

$$\frac{1}{w-z} = \frac{1}{w-a} \frac{1}{1-\frac{z-a}{w-a}} = \frac{1}{w-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n$$

since $|z-a| < r = |w-a|$.

Now multiply both sides by $\frac{f(w)}{2\pi i}$ & integrate.

But what about the right hand side?

We need to understand how integrals & infinite sums can be interchanged.

Lemma: Let γ be a rectifiable curve in $\mathbb{C}$ and suppose that F_n, F continuous w/ $F_n \xrightarrow[unif]{} F$, on $\{\gamma\}$. Then

$$\int_\gamma F = \lim \int_\gamma F_n$$

Proof: Let $\epsilon > 0$ be given. Then $\exists\, N \ni$

$|F - F_n| < \frac{\epsilon}{V(\gamma)}$. It follows that

$$\left|\int_\gamma F - \int_\gamma F_n\right| = \left|\int_\gamma (F - F_n)\right| \leq \int_\gamma |F(w) - F_n(w)|\,|dw| \leq \epsilon,$$

if $n \geq N$ ✓

Now we are ready for power series.

Theorem: Let f be analytic in $B(a,R)$. Then

$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ for $|z-a| < R$, where

$a_n = \frac{1}{n!} f^{(n)}(a)$ w/ radius of convergence $\geq R$.

Proof: Let $0 < r < R \Rightarrow \bar{B}(a,r) \subset B(a,r)$. If $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw \quad \text{for } |z-a| < r.$$

Furthermore, if $w \in \{\gamma\}$,

$$\frac{|f(w)|\,|z-a|^n}{|w-a|^{n+1}} \leq \frac{M}{r}\left(\frac{|z-a|}{r}\right)^n \quad \text{w/ } M = \max\{|f(w)| : |w-a| = r\}.$$

(4)

Since $\frac{|z-a|}{r} < 1$, $\sum f(w) \frac{(z-a)^n}{(w-a)^{n+1}}$ converges uniformly for $w \in \{\gamma\}$

putting everything together,

$$f(z) = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}}\, dw \right] (z-a)^n$$

& we have convergence if $|z-a| < r$.

Let $a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}}\, dw$. But we already know that $a_n = \frac{1}{n!} f^{(n)}(a)$ independent of r!

We conclude that $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ for $|z-a| < r$

& we are done since $r < R$ is arbitrary.

Corollary: If $f: G \to \mathbb{C}$ analytic & $a \in G$, then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ for $|z-a| < R$, where $R = d(a, \partial G)$.

Corollary: If $f: G \to \mathbb{C}$ analytic, then f is infinitely differentiable

Corollary: If $f: G \to \mathbb{C}$ is analytic & $\overline{B}(a,r) \subset G$, then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}}\, dw,$$

where $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$ ✓

Cauchy estimate: Let f be analytic in $B(a,R)$ and suppose $|f(z)| \leq M \quad \forall\ z \in B(a,R)$. Then

$$|f^{(n)}(a)| \leq \frac{n!\,M}{R^n}.$$

Proof: By the corollary above,

$$|f^{(n)}(a)| \leq \frac{n!}{2\pi} \left| \int_\gamma \frac{f(w)}{(w-a)^{n+1}}\,dw \right|$$

$$\leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} \cdot 2\pi r = \frac{n!\,M}{r^n}$$ & we are done since $r < R$ is arbitrary.

it is time to extend our results to more general curves

Proposition: Let f be analytic in the disk $B(a,R)$ and suppose that γ is a closed rectifiable curve in $B(a,R)$. Then f has a primitive and hence

$$\int_\gamma f = 0.$$

Proof: Showing that f has a primitive is enough by before. We know that $f(z) = \sum a_n (z-a)^n$ for $|z-a| < R$. Let

$$F(z) = \sum_{n=0}^{\infty} \left(\frac{a_n}{n+1}\right)(z-a)^{n+1} = (z-a)\sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-a)^n$$

same radius of convergence as $(n+1)^{\frac{1}{n}} \longrightarrow 1$

It is clear that $F'(z) = f(z)$ and we are done.

Zeroes of analytic functions:

Basic point: z^m is analytic, but $e^{-\frac{1}{z^2}}$ is not.

Definition: If $f: G \longrightarrow \mathbb{C}$ is analytic and $a \in G$ satisfies $f(a) = 0$, then a is a zero of multiplicity $m \geq 1$ if there is an analytic function $g: G \longrightarrow \mathbb{C}$ $\ni$

$$f(z) = (z-a)^m g(z), \text{ where } g(a) \neq 0.$$