Math 173, Fall 2022, December 14

Definition: A $n\times n$ matrix over the field F, a characteristic value of A is a scalar c in F $\ni$ $(A - c\underline{I})$ is singular.

The characteristic polynomial of A is

$$\det(x\underline{I} - A)$$

Lemma: Similar matrices have the same characteristic polynomial.

Proof: $B = P^{-1} A \cancel{B}$ $\hookrightarrow$

$$\det(x\underline{I} - B) = \det(x\underline{I} - P^{-1}AP)$$
$$= \det(P^{-1}(x\underline{I} - A)P)$$
$$= \det P^{-1} \det(x\underline{I} - A) \det P$$
$$= \det(x\underline{I} - A).$$

Therefore, the characteristic polynomial is independent of the basis and is a property of a linear transformation.

2

Example:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \det(x\underline{I} - A) = x^2+1$$

no real roots

$x = \pm i$

Example:

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix} \qquad \det(x\underline{I} - A) =$$

$$x^3 - 5x^2 + 8x - 4 = 0$$

$$= (x-1)(x-2)^2$$

$$A - \underline{I} = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix}$$

rank = 2

nullity = 1

$\hookrightarrow$ 1-dim characteristic space

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(3)

$$2x_1 + x_2 - x_3 = 0$$
$$2x_1 + 2x_2 - x_3 = 0$$

$x_2 = 0 \quad 2x_1 - x_3 = 0 \quad \{t, 0, 2t\}$

// null space

$$A - 2\underline{I} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & 2 & -2 \end{pmatrix}$$

rank = 2
nullity = 1

$$x_1 + x_2 - x_3 = 0$$
$$2x_1 \quad - x_3 = 0$$

$x_1 + x_2 - 2x_1 = 0 \quad x_2 - x_1 = 0 \quad x_3 = 2x_1$

$\underbrace{\{(t, t, 2t)\}}$ null space

Definition: $T \in L(V,V)$ finite dimensional

T is diagonalizable if if $\exists$ basis of $V \ni$ each vector $\in$ basis is a characteristic vector of T.

the resulting matrix is diagonal since

$$\overline{T\alpha_i} = c_i \alpha_i$$

Neither of the examples above is diagonalizable over $\mathbb{R}$.

Suppose that T is diagonalizable. Then $\exists$ β basis $\ni$

$$[T]_\beta = \begin{bmatrix} c_1 I_1 & & & \\ & c_2 I_2 & & 0 \\ & 0 & \ddots & \\ & & & c_k I_k \end{bmatrix}$$

It follows that $f = (x-c_1)^{d_1} \cdots (x-c_k)^{d_k}$

d_i = dimension of the characteristic space associated w/ $\underline{\underline{c_i}}$

Lemma: $T \in L(V,V)$; $c_1, \ldots, c_k$ char values; W_i char space associated w/ c_i. If $W = W_1 + \ldots + W_k$, then

$$\dim W = \dim W_1 + \ldots + \dim W_k$$

$\mathcal{B}_i$ ordered basis for W_i, then $(\mathcal{B}_1, \ldots, \mathcal{B}_k)$ ordered basis for W.

Proof: Suppose that $\beta_1 + \ldots + \beta_k = 0$, $\beta_i \in W_i$. Let f be a polynomial. Since $T\beta_i = c_i\beta_i$,

$$0 = f(T)0 = f(T)\beta_1 + \ldots + f(T)\beta_k$$

$$\underset{S}{=} f(c_1)\beta_1 + \ldots + f(c_k)\beta_k.$$

requires proof

Choose polys $f_1, \ldots, f_k \ni$

$$f_i(c_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Then $0 = f_i(T)0 = \sum_j \delta_{ij} \beta_j = \beta_i$.

Now let $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_K)$

$\mathcal{B}_i$ = a basis of W_i

$W = W_1 + \dots + W_K$ is spanned by $\mathcal{B}$

$\mathcal{B}$ linearly independent collection by above,

so we are done; except for that polynomial business we left for later.

Lemma: Suppose that $T\alpha = c\alpha$. Then

$f(T)\alpha = f(c)\alpha$ polynomial

Proof: $f(t) = a_0 + a_1 t + \dots + a_n t^n$

$$f(T)\alpha = a_0\alpha + a_1 T\alpha + a_2 T^2\alpha + \dots + a_n T^n\alpha$$

$$= a_0\alpha + a_1 c\alpha + a_2 c^2\alpha + \dots + a_n c^n\alpha$$

$$= f(c)\alpha$$

Theorem 2:

finite dimensional

Let $T \in L(V,V)$

$c_1, \dots, c_k$ distinct characteristic values of T and

W_i – null space of $(T - c_i I)$. TFAE

i) T is diagonalizable

ii) The characteristic poly for T is

$$f = (x-c_1)^{d_1} \dots (x-c_k)^{d_k}$$

and $\dim W_i = d_i$, $i = 1, \dots, k$.

iii) $\dim W_1 + \dots + \dim W_k = \dim V$

Proof: We have observed that i) $\hookrightarrow$ ii).

If f is as in ii), then $d_1 + \dots + d_k = \dim V$, so

ii) $\hookrightarrow$ iii). By the lemma, we have $V = W_1 + \dots + W_k$,

so characteristic vectors of T span V.

(8)

Matrix version:

$c_1, \ldots, c_K$ distinct char values of A in F

For each i; W_i = space of column matrices X of A $\ni (A - c_i I)X = 0$ w/ basis $\mathcal{B}_i$.

$(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_K)$ string together to form the sequence of columns of a matrix P

$P = [P_1\ P_2 \ldots\ P_K] = (\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_K)$

The matrix A is similar over F to a diagonal matrix iff P is a square matrix. When P is square, $P^{-1}AP$ is diagonal.