ON AN ARGUMENT OF SHKREDOV ON TWO-DIMENSIONAL CORNERS

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ABSTRACT. Let $\mathbb{F}_2^n$ be the finite field of cardinality 2^n. For large n, any subset $A \subset \mathbb{F}_2^n \times \mathbb{F}_2^n$ of cardinality

$$|A| \gtrsim 4^n \left(\frac{\log\log n}{\log n}\right)^4$$

must contain three points $\{(x,y), (x+d,y), (x,y+d)\}$ for $x, y, d \in \mathbb{F}_2^n$ and $d \neq 0$. Our argument is a further simplification of an argument of Shkredov [12], building upon the simplifications of Ben Green [9].

1. Main Theorem

We are interested in extensions of Roth's theorem on arithmetic progressions in dense sets of integers to a two-dimensional, finite-field setting. Specifically, for a finite group G define the quantity $r_\angle(G)$ to be the cardinality of the largest subset of $G \times G$ containing no *corner*. A corner is triple of points of the form, $((x,y),(x+d,y),(x,y+d))$ with $d \neq 0$.

While this concept is most interesting in the context of the groups $G = \mathbb{Z}_N$, it already makes sense–and is substantial–in the context of finite fields. In this paper, we only consider the case of $G = \mathbb{F}_2^n$. Here and throughout this paper we write $N = 2^n = |\mathbb{F}_2^n|$.

1.1. **Theorem.** *We have* $r_\angle(\mathbb{F}_2^n) \ll N^2 (\frac{\log\log\log N}{\log\log N})^4$.

This bound is an improvement, in the setting of $\mathbb{F}_2^n$, of the bounds provided by Shkredov [12, 13], and as simplified by Ben Green [9, 10]. In fact, we are able to simplify the argument to the point that further improvements would seem to require additional concepts and methodology.

Our theorem is an example of the quantitative bounds on questions of arithmetic combinatorics. We refer the reader to the papers of Gowers [5], and surveys by T. Tao [15] and Ben Green [7, 9] for more history.

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Erdős and Graham raised the question of quantitative bounds for $r_\angle$, and this question was raised again by Gowers [5]. Ajtai and Szemerédi [1] first proved that $r_\angle(\mathbb{Z}_N) = o(N^2)$. Furstenberg and Katznelson [3,4] gave a far reaching extension, though their method of proof does not in and of itself permit explicit bounds. Solymosi [14], and V. Vu [16] provided such bounds, although of a weak nature.

I. Shkredov [12] provided the first 'reasonable' bounds. We are using his ingenious argument, as explained and simplified by Ben Green [9] in the finite field setting. In particular Green showed that one could achieve an estimate in which $r_\angle(N)/N^2$ decreased like $(\log\log N)^{-c}$ where c could be taken to be $1/21$. We find some additional simplifications, and sharpen some inequalities to obtain our Theorem. We will investigate the implications of our approach in the higher dimensional setting in a future paper.

We comment that 'the finite field thesis' holds that questions of this type should first be studied in the context of finite fields. This is because one can implement many of the tools of analysis, e.g. convolution and Fourier transform, in that setting. In addition, one has the powerful concept of being able to pass to appropriate affine subspaces. Moreover, there are a range of methods that one can use to 'lift' the finite field argument to $\mathbb{Z}_N$. See papers by Bourgain [2], Green and Tao [8] and Shkredov [13] for more information.

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1.1. **Outline of Proof.**

Ben Green has provided a comprehensive outline of the method of proof [9], so we will be somewhat brief.

Let $A \subset \mathbb{F}_2^n \times \mathbb{F}_2^n$. It is natural to count the expected number of corners that A has. This expectation will be approximately what it should be if the 'Box norms' of A are small. These norms are explicitly given in (2.2), and are a two dimensional analog of the 'combinatorial square' norms that play such a prominent role in the proof of Roth's Theorem [8,9,11]. It is therefore a certain measure of 'uniformity.'

Thus, this norm being large is an 'obstruction to uniformity.' (See [15].) But, in contrast to the case of Roth's theorem, this obstruction to uniformity has no clear arithmetic consequence. It does imply, however, that the set A has an increased density on a sublattice. That is, there is a subset $X \subset \mathbb{F}_2^n$ of relatively large density, for which $A \cap X \times \mathbb{F}_2^n$ has a larger density. See Lemma 3.5. This is a point where our argument is much easier than the arguments of Shkredov [12] and Green [9]. Our Lemma is an consequence of a simple probabilistic fact, and permits us to pass to a sublattice by refining only *one coordinate.* This fact permits other technical simplifications in the proof.

It was a significant insight of Shkredov [12] that (1) one could, after an additional argument, assume that X had an arithmetic structure, namely that it was *uniform* in the sense of (2.1), see Lemma 3.8; and (2) the Box norm could still be used as an 'obstruction to uniformity' if X were uniform. This is the content of the Lemma 3.1.

If there is an obstruction to uniformity, an increment in the density of A can be found. One can find the obstruction to uniformity only a finite number of times, else the density of A would exceed one. Thus at some point, there would be no obstruction to uniformity, and so A would have a corner. All of the details of the proof are below.

2. Preliminaries and Definitions

We will let $H \subset \mathbb{F}_2^n$ denote a subspace and $X \subset H$ denote a generic subset. We adopt the notations of probability and expectation with respect to the counting measure on H, as well as $X \times H$. This is a standard part of the subject, following Green and Tao [6].

Write *the density of X in H* as

$$\delta_X := \mathbb{P}(X \mid H) = \tfrac{|X|}{|H|}\,.$$

Throughout this paper we will not only be concerned with the density of this subset X, but also with how 'uniformly distributed' it is in H. A quantification of this quality comes in the following definition:

$$\|X\|_{\mathrm{Uni}} := \sup_{\xi \neq 0} \tfrac{|\hat{X}(\xi)|}{|H|}\,. \tag{2.1}$$

If $\|X\|_{\mathrm{Uni}} \leq \eta$ then we say that X is $\eta - uniform$. Here $\hat{X}$ represents the the Fourier transform of X which is defined as follows, in which $\omega = \sqrt{-1}$ is a second root of unity:

$$\hat{g}(\xi) := \sum_{x \in H} g(x)\omega^{x \cdot \xi}.$$

Now, let $A \subset X \times H$ and write *the density of A* as

$$\delta := \mathbb{P}(A \mid X \times H).$$

We define the *balanced function* of A to be the function supported on $X \times H$ as

$$f(x, y) := A(x, y) - \delta X \times H.$$

Further, for a function $f : X \times H \longrightarrow \mathbb{C}$, define the following norms:

$$\begin{aligned}
\|f\|_{\Box,X} &:= \big[\mathbb{E}_{\substack{x,x' \in X \\ y,y' \in H}} f(x,y)f(x',y)f(x,y')f(x',y')\big]^{1/4} \\
\|f\|_{\tilde{\Box},X} &:= \big[\mathbb{E}_{\substack{x,x' \in X \\ d,d' \in H}} f(x,x+d)f(x',x'+d)f(x,x+d')f(x',x'+d')\big]^{1/4}
\end{aligned} \tag{2.2}$$

The first norm averages 'cross correlations' of f over all boxes in $X \times H$. The second norm is of the same nature, but the choice of basis is different. As we are in the setting of $\mathbb{F}_2^n$, and that we only pass to the sublattice in the second coordinate, these two norms are the same.

$$\begin{aligned}\|f\|_{\square,X}^2 &= \mathbb{E}_{x\in X}|\mathbb{E}_{y\in\mathbb{F}_2^n} f(x,y)|^2 \\ &= \mathbb{E}_{x\in X}|\mathbb{E}_{d\in\mathbb{F}_2^n} f(x,x+d)|^2 \\ &= \|f\|_{\tilde{\square},X}^2 \,.\end{aligned}$$

(In the general $\mathbb{F}_p^n$ case, these norms are not the same.)

3. Primary Lemmata

Our first lemma is a generalized von Neumann estimate, a term coined by Green and Tao [6,8,9]. It gives us sufficient conditions from which to conclude that A has a corner.

3.1. **Lemma** (Generalized von Neumann). *Suppose that $A \subset X \times H$ with $\mathbb{P}(A \mid X \times H) = \delta$, and that we have the three inequalities*

$$\delta_X \delta^2 N^2 > C \,, \tag{3.2}$$

$$\|X\|_{\mathrm{Uni}} \le (\delta\delta_X)^C \,, \tag{3.3}$$

$$\|f\|_{\square} \le \kappa\delta^{5/4} \,. \tag{3.4}$$

Then A has a corner.

Here, and throughout this paper, C represents a large absolute constant. The exact value of C does not impact impact the qualitative nature of our estimate, so we do not seek to specify an optimal value for it. Also $0 < c, \kappa, \kappa' < 1$ are small fixed constants, which plays a role similar to C.

Our second lemma tells us that if the conditions of our first lemma are not satisfied then we can find a sublattice on which A has increased density.

3.5. **Lemma** (Density Increment). *Suppose that $A \subset X \times H$ with $\mathbb{P}(A \mid X \times H) = \delta$, that f is the balanced function of A on $X \times H$, and that*

$$\|f\|_{\square,X} > \kappa\delta^{5/4}$$

then there exists $X' \subset X$ such that these two conditions hold.

$$\mathbb{P}(A \mid X' \times H) \ge \delta + \tfrac{\kappa}{2}\delta^{5/4} \,, \tag{3.6}$$

$$|X'| \ge \kappa'\delta^{5/2}|X|. \tag{3.7}$$

Our third lemma is taken from a note of Ben Green [9]. It tells us that we can find a *uniform* sublattice on which A has increased density which is important since it is a required premise in applying the generalized von Neumann lemma.

3.8. **Lemma** (Uniformizing a Sublattice). *Suppose that (a) $A \subset X' \times H$ with $\mathbb{P}(A \mid X' \times H) = \delta + c\delta^{5/4}$ (b) $\dim(H) > 10(\delta\delta_X)^{-C}$ and (c) that X' is not $(\delta\delta_{X'})^C$uniform. Then there exists a translate of A, $\operatorname{tr}(A)$, $X'' \subset X'$ and H', a subspace of H, so that*

$$\|X''\|_{\mathrm{Uni}} \le (\delta\delta_{X''})^C, \tag{3.9}$$

$$\mathbb{P}(\operatorname{tr}(A) \mid X'' \times H') \ge \delta + \tfrac{c}{2}\delta^{5/4}, \tag{3.10}$$

$$\dim(H') \ge \dim(H) - \delta^{-C'}. \tag{3.11}$$

$$|X''| \ge \delta_{X'}|H'| \tag{3.12}$$

4. Proof of Theorem 1.1

Combining lemmas 3.1 through 3.8 of the previous section yields the proof of theorem 1.1. Since the proof is by recursion, we describe the conditional loop needed for the proof.

Proof. Initialize $X = \mathbb{F}_2^n$, $H = \mathbb{F}_2^n$, and $\delta_X \leftarrow 1$. Fix a set A_0 with density δ_0 in $\mathbb{F}_2^n \times \mathbb{F}_2^n$. Initialize $A \leftarrow A_0$ and $\delta \leftarrow P(A \mid X \times H)$. Now we will iteratively apply the following steps in the following order.

(1) If $\|f\|_{\square,X}\|f\|_{\tilde{\square},X} > \kappa\delta^{5/2}$, apply lemma 3.5.
(2) If X' is not $(\delta\delta_{X'})^C$ uniform, apply lemma 3.8.
(3) Update variables:

$$X \leftarrow X'', \quad H \leftarrow H', \quad \delta_X \leftarrow \mathbb{P}(X'' \mid H')$$
$$A \leftarrow \operatorname{tr}(A), \quad \delta \leftarrow P(A \mid X \times H).$$

(4) Observe that the density of the incremented A on the sublattice has increased by at least $\kappa\delta_0^{5/4}$. Also, the incremented density, δ_X, has decreased by no more than $\kappa(\delta\delta)^{5/2}$.

Once this loop stops, Lemma 3.1 applies and we conclude that A has a corner. This iteration must stop in $\lesssim \delta_0^{-1/4}$ iterates, else the density of A on the sublattice would exceed one. Thus we need to be able to apply Lemmas 3.5 and 3.8 $\lesssim \delta_0^{-1/4}$ times. In order to do that, both X and H must be sufficiently large at each stage of the loop.

This requirement places several lower bounds on $N = 2^n$. The most stringent of these comes from the loss of dimensions in (3.11). Note that before the loop terminates, we can have δ_X as small as

$$\delta_X \geq (\kappa\delta_0)^{(\kappa\delta_0)^{-1/4}}.$$

In order to apply Lemma 3.8 at that stage, we need

$$N > \delta_0^{C''} 2^{(C\delta_0)^{C\delta_0^{-1/4}}}.$$

From this condition we get the bound stated in the theorem. □

5. Proof of Lemmata

5.1. Proof of Lemma 3.1. Define

$$\mathrm{T}(f,g,h) = \mathbb{E}_{x,d,y\in H} f(x,y)g(x+d,y)h(x,y+d). \tag{5.1}$$

Thus, $\mathrm{T}(A,A,A)$ is the expected number of corners in A.

In lemma 3.1 it is important that we have that X is sufficiently uniform. The following proposition of Green and Shkredov respectively from [10] and [12] illustrate the leverage we gain from this premise.

5.2. Proposition. *Let $X \subset H$ and denote the density of X in H by δ_X. Suppose X is η–uniform. Then*

$$\Bigl[\mathbb{E}_{d\in H}\Bigl|\mathbb{E}_{y\in H} X(d-y)G(y) - \delta_X^2 \mathbb{E}_{y\in H} G(y)\Bigr|^2\Bigr]^{1/2} \leq \eta^{1/2}\bigl[\mathbb{E}_y G(y)^2\bigr]^{1/2} \tag{5.3}$$

for any function G.

Notice that we are comparing a convolution to it's zero Fourier mode. The Proposition follows from Plancherel, and the definition of uniformity.

Proof of Lemma 3.1. Since $\mathrm{T}(A,A,A)$ is the expected number of corners, we show that it is at least a fixed small multiple of $\delta_X^2\delta^3$. By assumption (3.2), $C\delta_X^2\delta^3N^3 > \delta\delta_X N$. The left hand side is the expected number of corners in A, while the right is the number of trivial corners in A–that is the number of points in A. Thus A is seen to have a corner.

Throughout the proof, it is convenient to make the substitution $d \to x+y+d$ in the expression for $\mathrm{T}(f,g,h)$, thus

$$\mathrm{T}(f,g,h) = \mathbb{E}_{x,y,d} f(x,y)g(y+d,y)h(x,x+d).$$

For the sake of simplicity we set $S = X \times H$. We make the substitution $A = f + \delta S$ to get

$$\mathrm{T}(A,A,A) = \delta^3\,\mathrm{T}(S,S,S) \tag{5.4}$$

$$+\,\delta^2\,\mathrm{T}(f,S,S) + \delta^2\,\mathrm{T}(S,f,S) + \delta^2\,\mathrm{T}(S,S,f) \tag{5.5}$$

$$+\,\delta\,\mathrm{T}(f,f,S) + \delta\,\mathrm{T}(S,f,f) + \delta\,\mathrm{T}(f,S,f) \tag{5.6}$$

$$+\operatorname{T}(f,f,f)\,. \tag{5.7}$$

We have grouped the terms according to the number of f's that appear.

The main term is (5.4). Proposition 5.2 implies that $\delta^3\operatorname{T}(S,S,S)\geq\frac{1}{2}\delta_X\delta^3$.

All three terms in (5.5) are approximately zero, but we have to use uniformity in X. For instance, for the last term, we appeal to (5.3) and (3.3) to see that

$$\begin{aligned}\delta^2\big|\operatorname{T}(S,S,f)\big| &= \delta^2\big|\mathbb{E}_{x,y,d}X(y+d)f(x,x+d)\big| \\ &= O((\delta\delta_X)^{C/2})\end{aligned} \tag{5.8}$$

for C sufficiently large.

Two of the three terms in (5.6) are positive, and the third is positive, assuming uniformity. For instance,

$$\operatorname{T}(f,f,S)=\mathbb{E}_{x,y,d}f(x,y)f(x+d,y)=\mathbb{E}_y\big|\mathbb{E}_xf(x,y)\big|^2\geq 0\,. \tag{5.9}$$

Likewise, $\operatorname{T}(S,f,f)\geq 0$. The term is $\operatorname{T}(f,S,f)$ requires, however, uniformity. Observe that by (5.3) and (3.3),

$$\mathbb{E}_d\big|\mathbb{E}_yf(x,y)X(y+d)\big|^2\leq(\delta\delta_X)^C,$$

uniformly in x. Therefore,

$$\operatorname{T}(f,S,f)=\delta_X\mathbb{E}_x\big|\mathbb{E}_yf(x,y)\big|^2+O((\delta\delta_X)^C).$$

Thus, under our assumption, we can assume that $\delta\operatorname{T}(f,S,f)\geq-(\delta\delta_X)^C\geq-\frac{1}{6}\delta^3\delta_X^2$.

The last term to consider is $\operatorname{T}(f,f,f)$. It is this term that we bound by the box norm. Specifically,

$$\big|\operatorname{T}(f,f,f)\big|\leq\delta_X^2\delta^{1/2}\|f\|_\Box^2 \tag{5.10}$$

Using the hypothesis on the Box norm, (3.4), and the equations (5.8) and (5.9) will prove the Lemma.

Apply Cauchy Schwartz once, in the variables y,d, to get

$$\begin{aligned}|\operatorname{T}(f,f,f)| &\leq \big(\mathbb{E}_{y,d\in H}|f(y+d,y)|^2\big)^{1/2}\cdot\operatorname{U} \\ \operatorname{U} &:= \big(\mathbb{E}_{x,x',y,d\in H}X(y+d)f(x,y)f(x',y)\cdot f(x,x+d)f(x',x+d)\big)^{1/2}\end{aligned}$$

The first term on the right is no more than $(\delta\delta_X)^{1/2}$ since we are taking the average over all of H. Note that in the definition of U, we have inserted the term $X(y+d)$, which arises from $A(y+d,y)$.[1]

[1]Without this term, we would not get the right powers of δ_X in our estimates.

As for the second term, apply Proposition 5.2 to replace $X(y+d)$ by δ_X and then Cauchy Schwartz again in the variables x, x' to get $\mathrm{U} \le 2\delta_X^{1/2}\,\mathrm{U}_1 \cdot \mathrm{U}_2$ where

$$\mathrm{U}_1 = \bigl(\mathbb{E}_{x,x'\in H}|\mathbb{E}_{d\in H} f(x,x+d)f(x',x'+d)|^2\bigr)^{1/4} = \delta_X^{1/2}\|f\|_{\widetilde{\Box},X}\,,$$
$$\mathrm{U}_2 = \bigl(\mathbb{E}_{x,x'\in H}|\mathbb{E}_{y\in H} f(x,y)f(x',y)|^2\bigr)^{1/4} \le 2\delta_X^{1/2}\|f\|_{\Box,X}$$

The second inequality holds after another application of uniformity. Combining these bounds yields (5.10). Our proof is complete.

□

In the proof of Lemma 3.5 we will use the following simple proposition, whose proof is left to the reader. It states that a random variable, bounded in L^∞ norm by one, with mean zero, and standard deviation σ, must be at least a constant multiple of σ on a set of probability proportional to the variance.

5.11**. Proposition.** *Let Z be a random variable with $-1 \le Z \le 1$, $\mathbb{E}Z = 0$, and $\mathbb{E}Z^2 = \sigma^2$. Then, $\mathbb{P}(Z > \sigma/4) \ge \sigma^2/4$.*

Proof of Lemma 3.5. Set$Z = \mathbb{E}_{y\in\mathbb{F}_2^n} f(x,y)$. We see that this is a random variable of mean zero, $-1 \le Z \le 1$, and by assumption, of variance at least $\sqrt{\kappa}\delta^{5/4}$. Apply Proposition 5.11 to see that there is an X' satisfying (3.7) for which we have

$$\mathbb{E}_{y\in\mathbb{F}_2^n} f(x,y) \ge \frac{\sqrt{\kappa}}{4}\delta^{5/4}.$$

Since $f = A - \delta S$ is the balanced function, this implies the density increment condition (3.6).

□

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