ps #3) We can write the systems as two matrices

$$\begin{bmatrix} -1 & 1 & 4 \\ 1 & 3 & 8 \\ \frac{1}{2} & 1 & \frac{5}{2} \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

we will proceed by manipulating the first matrix

$$\begin{bmatrix} -1 & 1 & 4 \\ 1 & 3 & 8 \\ \frac{1}{2} & 1 & \frac{5}{2} \end{bmatrix} \begin{array}{l} \xrightarrow{R_2+R_1} \\ \\ \xrightarrow{2\cdot R_3} \end{array} \begin{bmatrix} 0 & 4 & 12 \\ 1 & 3 & 8 \\ 1 & 2 & 5 \end{bmatrix} \begin{array}{l} \xrightarrow{R_1/4} \\ \xrightarrow{R_2-R_3} \\ \\ \end{array} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 3 \\ 1 & 2 & 5 \end{bmatrix} \begin{array}{l} \\ \to R_2-R_1 \\ \to R_3-2R_1 \end{array}$$

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{array}{l} \to R_3 \\ \to R_1 \\ \to R_2 \end{array} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Then the first system can be transformed into the second and so they are equivalent.

Then $x_2 + 3x_3 = \frac{R_2+R_1}{4} = \frac{1}{4}(-x_1 + x_2 + 4x_3) + \frac{1}{4}(x_1 + 3x_2 + 8x_3)$

$x_1 - x_3 = 2R_3 - 2\left(\frac{R_2+R_1}{4}\right) =$

$= 2\left(\frac{1}{2}x_1 + x_2 + \frac{5}{2}x_3\right) - \frac{1}{2}(-x_1 + x_2 + 4x_3) - \frac{1}{2}(x_1 + 3x_2 + 8x_3)$

page 5. Question 4.

Are the following two systems of linear equations equivalent?
If so, express each equation in each system as a linear combination of the equations in the other system.

To see if two systems are equivalent, we must compare the sets of solutions to these systems.

System A: 1) $2x_1 + (-1+i)x_2 + 0x_3 + 1x_4 = 0$

2) $0x_1 + 3x_2 + (-2i)x_3 + 5x_4 = 0$

System B: $(1+\frac{i}{2})x_1 + 8x_2 - ix_3 - x_4 = 0$

$\frac{2}{3}x_1 + (-\frac{1}{2})x_2 + x_3 + 7x_4 = 0$

Because System B looks much harder to solve, we will simply solve System A and check that all of its solutions are also solutions of System B.

We will write this system in the form $AX = 0$ for simplicity as we manipulate the equations.

$$\begin{bmatrix} 2 & -1+i & 0 & 1 \\ 0 & 3 & -2i & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1+i & 0 & 1 \\ 0 & 3 & -2i & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 \to R_2 \cdot \frac{1}{3}]{R_1 \to R_1 \cdot \frac{1}{2}} \begin{bmatrix} 1 & \frac{-1+i}{2} & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{2}{3}i & \frac{5}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - \left(\frac{-1+i}{2}\right)R_2} \begin{bmatrix} 1 & 0 & \frac{-1-i}{3} & \frac{4}{3} - \frac{5}{6}i \\ 0 & 1 & -\frac{2}{3}i & \frac{5}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Our system can be written as:

$$x_1 + 0x_2 + \left(\frac{-1-i}{3}\right)x_3 + \left(\frac{4}{3} - \frac{5}{6}i\right)x_4 = 0$$
$$0x_1 + x_2 + \left(-\frac{2}{3}i\right)x_3 + \left(\frac{5}{3}\right)x_4 = 0$$

- If we specify that we want $x_3 = a$ and $x_4 = b$ for some $a, b \in \mathbb{C}$, we can solve this system:

$$x_1 + 0 + \frac{-1-i}{3}a + \left(\frac{4}{3} - \frac{5}{6}i\right)b = 0$$

$$0 + x_2 + \left(-\frac{2}{3}i\right)a + \left(5/3\right)b = 0$$

$$\Rightarrow \quad \begin{aligned} x_1 &= \left(\tfrac{1+i}{3}\right)a + \left(-\tfrac{4}{3} + \tfrac{5}{6}i\right)b \\ x_2 &= \left(\tfrac{2}{3}i\right)a + \left(-5/3\right)b \\ x_3 &= a \\ x_4 &= b \end{aligned}$$

Will be a solution for each choice of $a, b \in \mathbb{C}$.

Now, we will check if these solutions also satisfy System B.

- We will begin by plugging this solution into the second equation of System B.

$$\frac{2}{3}x_1 + \left(-\frac{1}{2}\right)x_2 + x_3 + 7x_4 = 0$$

$$\Rightarrow \frac{2}{3}\left(\frac{1+i}{3}a - \left(\frac{4}{3} - \frac{5}{6}i\right)b\right) - \frac{1}{2}\left(\frac{2}{3}i\,a - \frac{5}{3}b\right) + a + 7b = 0$$

$$\Rightarrow \frac{2+2i}{9}a + \frac{-16+10i}{18}b - \frac{i}{3}a + \frac{5}{6}b + a + 7b = 0$$

$$\Rightarrow \left(\left(\frac{2}{9}+1\right) + \left(\frac{2i}{9} - \frac{i}{3}\right)\right)a + \left(\left(7 + \frac{5}{6} - \frac{16}{18}\right) + \frac{10}{18}i\right)b = 0$$

$$\Rightarrow \left(\frac{11}{9} - \frac{1}{9}i\right)a + \left(\frac{125}{18} + \frac{5}{9}i\right)b = 0$$

This is not true for arbitrary values of a and b, so the solution sets are not the same, and the systems are not equivalent.

1 pg.5 question 6

First, we prove that any system of homogeneous linear equations with 2 unknowns is equivalent to a system of at most 2 equations in 2 unknowns. Intuitively, you should think this is true because the process of solving linear systems involves reducing the number of variables by one for each equation, so two equations is already enough to solve for all the unknowns and the rest of the equations must be redundant.

Let $\{a_i x + b_i y\}_i$ be a system of homogeneous linear equations in two unknowns. If all the equations are 0, then we are done since it is equivalent to a system with ≤ 2 linear equations. Pick the first nonzero linear equation otherwise, denoted as $a_n x + b_n y = 0$. Compare the next nonzero linear equation $a_m x + b_m y = 0$ to the equation $a_n x + b_n y = 0$. If the equations are multiples of each other, then $a_m x + b_m y = 0$ can be reduced to a zero equation by simply subtracting the multiple of $a_n x + b_n y = 0$ that it equals. If all equations are of this form, then the system is equivalent to a system with 1 linear equation, which is equivalent to a system of ≤ 2 linear equations. If they are not multiples of each other, you can multiply $a_n x + b_n y = 0$ by $\frac{a_m}{a_n}$ for $a_n \neq 0$ and subtract it from $a_m x + b_m y = 0$ to get $(\frac{b_m a_m}{a_n} - b_n)y = 0$ and this implies $y = 0$ as the constant cannot be zero since the two chosen linear equations were not multiples of each other. If $a_n = 0$, then we have $b_n y = 0$, which implies $y = 0$ as $b_n \neq 0$ (since the equation was chosen to be nonzero). Either way, we have that $y = 0$ for any possible solution of this linear equation. Plugging that value into one of the nonzero equations with the x coefficient being nonzero (which exists as not every equation is a multiple of one equation) and you also get that $x = 0$. So this system is equivalent to a system of two homogeneous equations in two unknowns where the equations are not multiples of each other, showing every system of linear equations in 2 unknowns is equivalent to a system of ≤ 2 linear equations in 2 unknowns.

From this, you can classify the solution sets of systems of linear equations in two unknowns as being 3 categories: If they have two equations which are not multiples of each other, then the only solution is a single point $(0, 0)$ in the plane. If they only have one equation and the others are all multiples of that one equation, then the solution sets are equivalent to the solution set of a system of one linear equation in two unknowns, which is a line in $\mathbb{C}^2$ (why is this necessarily true? Think about it to yourself). If the system has no nonzero equation, it has solutions being all points in $\mathbb{C}^2$. Sanity check: why do all the solutions necessarily lie in $\mathbb{C}^2$?

From here, it is clear how the problem is answered. If the solution sets are both a single point $(0, 0)$, then the two systems are both equivalent to (in matrix form), the row reduced identity matrix. If the two systems share a line of solutions, then the systems are (in matrix form) a single nonzero row which represents the line in slope intercept form, and are equivalent since lines are equal in slope intercept form if and only if they are scalar multiples of each other. If both systems have solution set being all of $\mathbb{C}^2$, then both must have all equations equal to 0 and thus be equivalent.

pS #7) Let $S \subseteq \mathbb{C}$ be a subfield of the complex numbers

By S a field, it contains the multiplicative identity 1 and the additive identity 0.

Claim $\mathbb{N} \subseteq S$. Proceed by induction.
Note $1 \in S$. If $k \in \mathbb{N}$ and $k \in S$, by S a subfield S closed under addition so $k+1 \in S$. Then by Induction, $\mathbb{N} \subseteq S$

By S a field, $\forall x \in S$, $-x \in S$. Then $-\mathbb{N}, \mathbb{N}, \{0\} \subseteq S \Rightarrow$ $\mathbb{Z} \subseteq S$.

Let $\frac{p}{q} \subseteq \mathbb{R}$. Note $\frac{p}{q} = p \cdot q^{-1}$ with $p, q \in \mathbb{Z}$. By S a field, $\forall x \in S$, $x^{-1} \in S$. By $\mathbb{Z} \in S$, $p, q \in \mathbb{Z}$, $p, q^{-1} \in S$.

By S a field, S closed under multiplication $\Rightarrow$ $pq^{-1} \in S \Rightarrow \frac{p}{q} \in S$. Then $\forall x \in \mathbb{R}$, $x \in S \Rightarrow \mathbb{R} \subseteq S$.

p10 #2) We shall proceed with the row redactions

$$\begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix} \xrightarrow[R_2-2R_3]{R_1-3R_3} \begin{bmatrix} 0 & 8 & 2 \\ 0 & 7 & 1 \\ 1 & -3 & 0 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 7 & 1 \\ 1 & -3 & 0 \end{bmatrix} \xrightarrow[R_3+3R_1]{R_2-7R_1}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & -6 \\ 1 & 0 & 3 \end{bmatrix} \xrightarrow{R_2/-6} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 3 \end{bmatrix} \xrightarrow[R_3-3R_2]{R_1-R_2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[R_1]{R_3}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_2]{R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $AX=0 \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{matrix} \Leftrightarrow X=0$

Page 10 #4

4) $\begin{bmatrix} i & -(1+i) & 0 \\ 1 & -2 & 1 \\ 1 & 2i & -1 \end{bmatrix} \begin{matrix} \to R_1 - iR_3 \\ \to R_2 - R_3 \\ \end{matrix} \begin{bmatrix} 0 & -i+1 & i \\ 0 & -2-2i & 2 \\ 1 & 2i & -1 \end{bmatrix} \xrightarrow{R_1 \cdot \frac{1+i}{2}} \begin{bmatrix} 0 & 1 & \frac{i-1}{2} \\ 0 & -2-2i & 2 \\ 1 & 2i & -1 \end{bmatrix} \begin{matrix} \xrightarrow{R_2 + (2+2i)R_1} \\ \xrightarrow{R_3 - 2iR_1} \end{matrix}$ ps

$\begin{bmatrix} 0 & 1 & \frac{i-1}{2} \\ 0 & 0 & 0 \\ 1 & 0 & i-2 \end{bmatrix} \begin{matrix} \xrightarrow{R_3} \\ \xrightarrow{R_1} \end{matrix} \begin{bmatrix} 1 & 0 & i-2 \\ 0 & 0 & 0 \\ 0 & 1 & \frac{i-1}{2} \end{bmatrix} \begin{matrix} \xrightarrow{R_3} \\ \xrightarrow{R_2} \end{matrix} \begin{bmatrix} 1 & 0 & i-2 \\ 0 & 1 & \frac{i-1}{2} \\ 0 & 0 & 0 \end{bmatrix}$

This is a row reduced form and so we are done

Page 10 #5

5) Begin with matrix 1

$\begin{bmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{bmatrix} \begin{matrix} \to R_2 - \frac{a}{2}R_1 \\ \to R_3 - \frac{b}{2}R_1 \end{matrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & c & 3 \end{bmatrix} \begin{matrix} \xrightarrow{R_1/2} \\ \xrightarrow{R^2 \cdot -1} \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 3 \end{bmatrix} \xrightarrow{R_3 - cR_2}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{R_3/3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Then $AX=0$ iff $X=0$

We will manipulate matrix 2

$\begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{bmatrix} \begin{matrix} \xrightarrow{R_2+2R_1} \\ \xrightarrow{R_3-R_1} \end{matrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{bmatrix} \begin{matrix} \xrightarrow{R_1 - \frac{1}{2}R_2} \\ \to R_3 - R_2 \end{matrix} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2/2} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$

Then $\begin{bmatrix} \frac{-1}{2} \\ \frac{-3}{2} \\ 1 \end{bmatrix}$ is a solution to matrix 2 but not to matrix 1.

p 10 #6) Case 1) $a \neq 0$.

Then $a=1, c=0$.

Case 1a) $d \neq 0$

Then $d=1, b=0$. But $a+d=2 \neq 0$ so we have a contradiction

Case 1b) $d=0$

Then $b=-a-c-d=-1$ so $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

Case 2) $a=0$.

Case 2a) $c \neq 0$. Then $c=1$

Case 2aa) $b \neq 0$. Then $b=1, d=0$. but $a+b+c+d=2 \neq 0$ Contra

Case 2ab) $b=0$. Then $b=0, a=0, d=-1$ so $A = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$

Case 2b) $c=0$.

Case 2ba) $b \neq 0$. Then $b=1, d=c=a=0$. Then $a+b+c+d=1 \neq 0$ Contra

Case 2bb) $b=0$. Then $d=-a-b-c=0$. Then $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Thus A must be $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$, or $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Page 11. Question 7:

Prove that the interchange of two rows of a matrix can be accomplished by a finite sequence of elementary row operations of the other two types.

- Suppose we have an $n \times m$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$

- We will call each row B_k where k is the row number.

- $A = \begin{bmatrix} \cdots B_1 \cdots \\ \cdots B_2 \cdots \\ \vdots \\ \cdots B_n \cdots \end{bmatrix}$

- If we want to interchange rows ℓ and k, we will follow these steps:

$$\begin{array}{c} 1 \\ \vdots \\ \ell \\ \vdots \\ k \\ \vdots \\ n \end{array} \begin{bmatrix} B_1 \\ \vdots \\ B_\ell \\ \vdots \\ B_k \\ \vdots \\ B_n \end{bmatrix} \xrightarrow{R_\ell \to R_\ell - R_k} \begin{bmatrix} B_1 \\ \vdots \\ B_\ell - B_k \\ \vdots \\ B_k \\ \vdots \\ B_n \end{bmatrix} \xrightarrow{R_k \to R_\ell + R_k} \begin{bmatrix} B_1 \\ \vdots \\ B_\ell - B_k \\ \vdots \\ B_\ell \\ \vdots \\ B_n \end{bmatrix} \xrightarrow{R_\ell \to R_\ell - R_k} \begin{bmatrix} B_1 \\ \vdots \\ -B_k \\ \vdots \\ B_\ell \\ \vdots \\ B_n \end{bmatrix}$$

$$\xrightarrow{R_\ell \to -1(R_\ell)} \begin{bmatrix} B_1 \\ \vdots \\ B_k \\ \vdots \\ B_\ell \\ \vdots \\ B_n \end{bmatrix} \quad \square$$

This procedure carried out on a 2×2 is as follows:

$$\begin{bmatrix} a_{11} & b_{12} \\ c & d \end{bmatrix} \to \begin{bmatrix} a-c & b-d \\ c & d \end{bmatrix} \to \begin{bmatrix} a-c & b-d \\ a & d \end{bmatrix} \to \begin{bmatrix} -c & -d \\ a & b \end{bmatrix}$$

$$\to \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

2 pg. 11 question 8

First of all, note the solution of this problem is extremely similar to the solution for problem number 6 on page 5, as both of them involve classifying solutions to linear equations of two unknowns. This one tells you at the beginning that there are only two equations though.

a) In relation to the previous problem, this is classifying solutions to "trivial" systems, or zero matrices, which have solutions over the entire field. This is immediate as 0 multiplied by any value is 0, so putting in any pair always outputs 0. This is a simple numerical calculation of adding and multiplying 0.

b) In relation to the previous problem, this is classifying solutions to systems which have two equations that are not multiples of each other. If $ad - bc \neq 0$, then you know that one of a, c must be nonzero. WLOG, assume $a \neq 0$ since if not, interchange the rows and relabel as it does not affect the solution set. Then you can take the row with a, b and multiply it by $\frac{c}{a}$ before subtracting the row with c, d by it, which gives a row with $0, d - \frac{bc}{a}$. As $ad - bc \neq 0$ and $a \neq 0$, we know $d - \frac{bc}{a} \neq 0$ and this means solutions are of the form $(x, 0)$ for x values to be determined. If $b = 0$, we are done as it shows solutions are of the form $(0, 0)$, and if not, we can use the row with $0, d - \frac{bc}{a}$ to change the value of the entry b to 0. This shows all solutions to such matrices (which are just linear equations) is only $(0, 0)$.

c) In relation to the previous problem, this is classifying solutions to systems which have only one equation up to multiplication of constants. If $ad - bc = 0$, and assume WLOG that $a = 0$. Then either $b = 0$, which means we have a row of zeroes, or $c = 0$, which means we have a column of zeroes, and as one entry is nonzero, the matrix will reduce to have a row of zeroes. If none of them are 0 then we can write $d = \lambda b$, which is always possible, and to preserve equality we write $c = \lambda a$. This also allows us to reduce to a row of zeroes as subtracting λ multiplied by one row from the other will do the job. Once the matrix has a row of zeroes, the solution to the matrix equation is just the linear equation of the last nonzero row, which gives a line of solutions as noted in the previous problem.

p15 #2) $A = \begin{bmatrix} 1 & -i \\ 2 & 2 \\ i & 1+i \end{bmatrix} \xrightarrow[R_3 - iR_1]{R_2 - 2r_1} \begin{bmatrix} 1 & -i \\ 0 & 2-2i \\ 0 & 2+i \end{bmatrix} \xrightarrow[R_3/2+i]{R_2/2-2i} \begin{bmatrix} 1 & -i \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_3 - R_2]{R_1 + iR_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$

If $AX = 0$, $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = 0,\ x_2 = 0$.
Then $X = 0$.

15 #7) $\left[\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{array}\right] \xrightarrow{R_1 - 2R_2,\ R_3 - 2R_2,\ R_4 - R_1} \left[\begin{array}{ccccc|c} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & -2 & 4 & 3 & 1 & -2 \\ 0 & 4 & 4 & -2 & -1 & 7 \\ 0 & -3 & -3 & 3 & 1 & -7 \end{array}\right] \xrightarrow{R_2 + 2R_1,\ R_3 - 4R_1,\ R_4 + 3R_1}$

$\left[\begin{array}{ccccc|c} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & 0 & 6 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} \rightarrow R \\ \rightarrow R_2 - R_4 \\ \rightarrow R_3 + R_4 \\ \end{array} \left[\begin{array}{ccccc|c} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & 0 & 6 & 1 & 0 & 3 \\ 0 & 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \rightarrow R_3/2$

$\left[\begin{array}{ccccc|c} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & 0 & 6 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} \rightarrow R_1 + R_3 \\ \rightarrow R_2 - R_3 \\ \\ \end{array} \left[\begin{array}{ccccc|c} 0 & 1 & 1 & 0 & 0 & 3 \\ 1 & 0 & 6 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right].$

Then $x_2 = 3 - x_3$, $x_1 = 2 - 6x_1$, $x_4 = 1$, $x_5 = -1$.
Let $x_3 = \lambda \in \mathbb{R}$ so we have solutions of form

p15 #7
$$\left[\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{array}\right] \begin{array}{c} \xrightarrow{R_1-2R_2} \\ \xrightarrow{R_3-2R_2} \\ \xrightarrow{R_4-R_2} \end{array} \left[\begin{array}{ccccc|c} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 4 & 4 & -4 & -1 & 7 \\ 0 & -3 & -3 & 3 & 1 & -5 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{R_2-R_4} \\ \xrightarrow{R_3+R_4} \end{array} \left[\begin{array}{ccccc|c} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & -2 & -4 & 3 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_2+2R_1}$$

$$\left[\begin{array}{ccccc|c} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & 0 & -2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$ Let $x_3 = \lambda_1, x_4 = \lambda_2$.

Then solutions are of form
$(2+2\lambda_1-\lambda_2, 2-\lambda_1+\lambda_2, \lambda_1, \lambda_2, 1)$.

Page 16. Question 3:

Describe explicitly all row-reduced echelon matrices (2×2):

2×2 matrices can be used to represent systems of 2 equations with 2 unknowns. Specifically, for a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ we can write } \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Or, more succinctly, $AX = Y$ to represent ~~solutions to~~ the inhomogeneous system:

$$a x_1 + b x_2 = y_1$$
$$c x_1 + b x_2 = y_2.$$

Our types of row reduced echelon matrices will represent the different types of sets of solutions that we can have to these systems.

The types of row-reduced echelon matrices are as follows (with their types of solutions to $AX=0$):

i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; a specific value for x_1 and a specific value for x_2 : $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

ii) $\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}$; We will have an infinite set of solutions. ~~of the form~~ For $y_1=y_2=0$, we will have solutions of the form: $\lambda \cdot \begin{bmatrix} a \\ -1 \end{bmatrix}$ $\forall \lambda \in \mathbb{C}$ (or whatever our field is).

iii) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; We will have an infinite set of solutions of the form: $\lambda \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\forall \lambda \in \mathbb{C}$ (or whatever our field is)

iiii) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; We will have that any pair of values for x_1 & x_2 will be a solution (infinitely many solutions). will be of the form: $\begin{bmatrix} a \\ b \end{bmatrix}$ $\forall a,b \in \mathbb{C}$ (or whatever field we are in) p16 #8) Let $Y = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Then we need to reduce $A|Y$

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & a \\ 2 & 1 & 1 & b \\ 1 & -3 & 0 & c \end{array}\right] \xrightarrow[R_2-2R_1]{R_1-3R_3} \left[\begin{array}{ccc|c} 0 & 8 & 2 & a-3c \\ 0 & 7 & 1 & b-2c \\ 1 & -3 & 0 & c \end{array}\right] \xrightarrow{R_1/8} \left[\begin{array}{ccc|c} 0 & 1 & \frac{1}{4} & \frac{a-3c}{8} \\ 0 & 7 & 1 & b-2c \\ 1 & -3 & 0 & c \end{array}\right] \xrightarrow[R_3+3R_1]{R_2-7R_1}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & \frac{1}{4} & \frac{a-3c}{8} \\ 0 & 0 & \frac{-3}{4} & \frac{-7a+8b-14c}{8} \\ 1 & 0 & \frac{3}{4} & \frac{3a-c}{8} \end{array}\right] \xrightarrow[R_3+R_2]{R_1+\frac{R_2}{3}} \left[\begin{array}{ccc|c} 0 & 1 & 0 & \frac{-4a+8b-28c}{24} \\ 0 & 0 & \frac{-3}{4} & \frac{-7a+8b-19c}{8} \\ 1 & 0 & 0 & \frac{-4a+8b-18c}{8} \end{array}\right] \xrightarrow{R_2 \cdot \frac{-4}{3}}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & \frac{-a+2b-7c}{6} \\ 0 & 0 & 1 & \frac{7a+8b-19c}{6} \\ 1 & 0 & 0 & \frac{-2a+4b-9c}{4} \end{array}\right].$$

Then $x_2 = \frac{-a+2b-7c}{6}$
$x_3 = \frac{7a+8b-19c}{6}$
$x_1 = \frac{-2a+4b-9c}{4}$

so $\forall Y$ a solution exists.

Page 21 Question 2:

Let

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix}$$

Directly verify that $A^2B = A(AB)$.

$$A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 2 & -1 & 1 \\ 5 & -2 & 3 \\ 6 & -3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{bmatrix}$$

$$A(AB) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix}$$

$\Rightarrow \quad A^2B = A(AB) \quad \square$

Page 21 Question 2:

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Directly verify that $A^2B = A(AB)$.

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$$A(AB) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 8 & 0 \\ 10 & -2 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ 20 & -4 \\ 25 & -5 \end{bmatrix}$$

$\Rightarrow A^2B = A(AB)$ $\square$

Page 21. Question 5:

Let

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ -4 & 4 \end{bmatrix}$$

Is there a matrix C s.t. $CA = B$?

If such a matrix does exist, it will necessarily be 2×3:
As such, we will let

$$C = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}.$$

For $CA = B$, we will need that:

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -4 & 4 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} a + 2b + c &= 3 \\ -a + 2b &= 1 \\ d + 2e + f &= -4 \\ -d + 2e &= 4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 4b + c &= 4 \\ 2a + c &= 2 \\ a - 2b &= -1 \\ 2d + f &= -8 \\ 4e + f &= 0 \\ d - 2e &= -4 \end{aligned}$$

[redacted]

Let $a = 0 \Rightarrow$ $c = 2$, $b = 1/2$ (can freely choose ~~ae~~ a and d)

let $d = 1 \Rightarrow$ $e = 5/2$, $f = -10$

Note: Could have restricted any of $\{a, b, c\}$ and any of $\{d, e, f\}$

$$\Rightarrow \begin{bmatrix} 0 & 1/2 & 2 \\ 1 & 5/2 & -10 \end{bmatrix}$$

Check that it works:

$$\begin{bmatrix} 0 & 1/2 & 2 \\ 1 & 5/2 & -10 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -4 & 4 \end{bmatrix} \quad \checkmark$$

$\square$

3 pg. 21 Problem 7

We know $AB = I$. Consider $(AB)A = A(BA) = A$ and $B(AB) = (BA)B = B$. At this point in time, it is enough to prove the uniqueness of the identity as a matrix which has the property $CI = IC = C$ for any $2x2$ matrix C. This is a straightforward computation using whatever entries you use for C, but it is simple enough. At this point, suppose two matrices have the same property as the identity, so $\exists I'$ s.t. $CI' = I'C = C$ for any $2x2$ matrix C. Then $II' = I = I'$ and uniqueness is proved. So BA is a matrix which satisfies properties of the identity, so it must be the identity matrix and $BA = I$

This proof is a relatively standard proof you use to show for arbitrary inverses, they commute. If you just computed the way to the solution, that is fine. But remember this sort of proof because it will come up later on in higher level math classes you take.