MATH471: EXAM #1

ALEX IOSEVICH

"God has given man no sharper spur to victory than contempt of defeat"

– Hannibal Barca

Problem #1: Define what it means for a set $A \subset \mathbb{R}$ to have measure 0. Define what it means for a set $A \subset \mathbb{R}$ to be measurable. Prove that sets of measure 0 are measurable.

Problem #2: Let E be a measurable set of finite outer measure. Then for each $\epsilon > 0$, there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^n$ for which if $O = \cup_{k=1}^n I_k$, then

$$m^*(E \sim O) + m^*(O \sim E) < \epsilon.$$

Problem #3: i) Determine if the set E below has Lebesgue measure 0 and prove your assertion. If you use a result from the book, you must state it precisely.

$$E = \left\{ x \in [0,1] : \left| x - \frac{a}{q} \right| \leq \frac{1}{q^2 \log^2(q)} \text{ for infinitely many } (a,q) \in \mathbb{N} \times \mathbb{N} \right\}.$$

ii) Determine the Lebesgue measure of the set

$$F = \left\{ x \in [0,1] : \left| x - \frac{a}{q} \right| \leq \frac{1}{q^2} \text{ for infinitely many } (a,q) \in \mathbb{N} \times \mathbb{N} \right\}$$

and prove your assertion.

Problem #4: Prove that any **unbounded** set E of real numbers with positive outer measure contains a subset that fails to be measurable. If you use the fact that bounded sets have this property, you must prove it also.

Problem #5: Let $f \geq 0$ be a step function on $[0,1]$, i.e there exists

$$0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1$$

such that $f(x) = c_1$ on $[a_0, a_1)$, $f(x) = c_2$ on $[a_1, a_2)$, $\dots$, $f(x) = c_n$ on $[a_{n-1}, a_n]$. Define

$$||f||_2 = \left(\sum_{k=1}^{n} c_k^2 (a_k - a_{k-1}) \right)^{\frac{1}{2}}.$$

Prove that for $\lambda > 0$,

$$m\left(\{x \in [0,1] : f(x) > \lambda ||f||_2\}\right) \leq \frac{1}{\lambda^2}.$$

Note: We have not defined integration in this class, so you may not use it. But summation is allowed :).