Exam # 2: Math 1500

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Write your solutions on the clear white paper provided by the examiner. Clearly indicate which problem you are solving on a given piece of paper and do not write a solution to more than one problem on a single piece of paper. Show all your work. Good luck!

Problem 1. Let
$$f(x) = 5x^3 - x^2 - 7.$$
Show that there exists a real number c such that $f(c) = 2$.

Solution: By a direct calculation, $f(0) = -7$ ad $f(2) = 29$. It follows by the Intermediate Value Theorem that there exists $c \in (0, 2)$ such that $f(c) = 0$.

Problem 2. Let
$$g(x) = \sin(x).$$
Compute $g^{(18)}\left(\frac{\pi}{2}\right)$.

Solution: We have $g'(x) = \cos(x), g''(x) = -\sin(x), g'''(x) = -\cos(x)$, and $g^{(4)}(x) = \sin(x)$, which is where we started. This means that $g^{(16)}(x) = g^{(12)}(x) = g^{(8)}(x) = g^{(4)}(x) = \sin(x)$. We conclude that $g^{(17)}(x) = \cos(x)$ and thus $g^{(18)}(x) = -\sin(x)$.

Problem 3. A refrigerator is thrown upwards with the initial speed of 5 meters per second from the roof of a 50 meter high dormitory. Determine the highest point reached by the refrigerator and the time when this point is reached. Use the value of $g = 10$ meters per second squared for acceleration due to gravity.

Derive all the equations from scratch! Simply writing down the equation you end up using without explanation will not result in much credit.

Solution: Acceleration is due to gravity, so $a(t) = -10$. Since $a(t) = v'(t)$, $v(t) = -10t+5$, where 5 arises because 5 meters per second is the initial speed. Since $x'(t) = v(t)$, we conclude that $x(t) = -5t^2 + 5t + 50$. The highest point is reached when $v(t) = 0$ which is when $-10t + 5 = 0$, which occurs at $t = \frac{1}{2}$. To compute the highest point, we just plug this value into the equation for $x(t)$, obtaining $-\frac{5}{4} + \frac{5}{2} + 50$.

Problem 4. Consider the equation

$$x^4 + x^2y^2 + y^4 = 3.$$

(A) Let $x = 1$. Find all the possible values of y such that $(1, y)$ lies on the curve above.
(B) Find the equation of the tangent line at all the points discovered in part (A).

Solution: Plugging in $x = 1$, we get $y + 4 + y^2 + 1 = 3$, so $y^4 + y^2 - 2 = 0$. Factoring we get $(y^2 + 2)(y^2 - 1) = 0$, so $y = \pm 1$.

Differentiating implicitly, we get

$$4x^3 + 2xy^2 + 2x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0.$$

Plugging in $x = 1, y = 1$, we get $\frac{dy}{dx} = -1$. Plugging in $x = 1, y = -1$, we get $\frac{dy}{dx} = 1$. In the former case, the equation of the tangent line is given by $\frac{y-1}{x-1} = -1$ and in the latter case by $\frac{y+1}{x-1} = 1$.

Problem 5. Let

$$h(x) = \begin{cases} \frac{\sqrt{x}-3}{x^2-8x-9}, & x > 9 \\ c, & x \leq 9. \end{cases}$$

Determine the value of c that makes this function continuous.

Solution: Since we are dealing with a rational function, by a theorem from class, $h(x)$ is continuous except possibly at $x = 9$. In order to make it continuous at $x = 9$ we need to make sure that $\lim_{x\to 9} h(x) = c$. We have

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{x^2-8x-9} = \lim_{x\to 9} \frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)(x+1)}$$

$$= \lim_{x\to 9} \frac{1}{(\sqrt{x}+3)(x+1)} = \frac{1}{60}.$$

It follows that setting $c = \frac{1}{60}$ makes $h(x)$ continuous.