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Math 173 September 12, 2022

Definition: An $m \times n$ matrix $R$ is called row-reduced if:

a) The first non-zero entry in each row of $R$ is equal to 1.

b) Each column of $R$ which contains the leading non-zero entry of some row has all its other entries 0.

Row-reduced example: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Not row-reduced example:

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

Question: Is every matrix equivalent to a row-reduced matrix?

②

Theorem: Every $m \times n$ matrix over a field $F$ is row-equivalent to a row reduced matrix.

Proof: Consider the first $k$ ($k \leq m$) rows of your matrix and assume, inductively, that they have been row reduced such that the resulting $k \times n$ matrix is a row-reduced matrix. If all the entries of the next row are 0, there is nothing more to do there, otherwise identify the first non-zero entry of that row. If this entry is underneath the leading entry of one of the previous rows, turn it into 0 by applying R2. If there is another entry in this row sitting underneath a leading entry of another row, turn it 0 by applying R2 w/ respect to that other row and note that none of the previous entries are affected. Proceeding in this way completes the proof.

Fun question: What is the maximum number of steps required to row reduce an $m \times n$ matrix by the method above?

Questions of this type were considered in Math 150, albeit in a different context.

③

you probably noticed that row-reduced matrices are not unique in the sense that there is more than way to row-reduce a matrix. For example, both

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ \& } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

are row-reduced and can be obtained from one another via R3.

We want to pin things down a bit more.

Definition: An $m \times n$ matrix $R$ is called a row-reduced echelon matrix if:

a) $R$ is row-reduced

b) The zero rows are on the bottom

c) If rows $1, \ldots, r$ are the non-zero rows and the leading non-zero entry is in the column $k_i$, $(i = 1, 2, \ldots, r)$, then $k_1 < k_2 < \ldots < k_r$.

④

Equivalent formulations are often helpful:

Either every entry in $R$ (row-reduced echelon matrix) is $0$, or there exists a positive integer $r$, $1 \leq r \leq m$, and $r$ positive integers $k_1, k_2, \dots, k_r$ w/ $1 \leq k_i \leq n$ and

a) $R_{ij} = 0$ for $i > r$, and $R_{ij} = 0$ if $j < k_i$

b) $R_{ik_j} = \delta_{ij}$, $1 \leq i \leq r$, $1 \leq j \leq r$

c) $k_1 < k_2 < \dots < k_r$.

Non-trivial example:

$$\begin{pmatrix} 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Theorem 5: Every $m \times n$ matrix is row-equivalent to a row-reduced echelon matrix.

Proof: Swap finitely many rows in Theorem 4.

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Let's go back to the previous example and compute the solution set to

$$RX = 0,$$

$$R = \begin{pmatrix} 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{aligned} x_2 - 3x_3 + \frac{x_5}{2} &= 0 \\ x_4 + 2x_5 &= 0 \end{aligned} \right\}$$

Strategy: Let $x_1 = u_1$, $x_3 = u_2$, $x_5 = u_3$

We get $\left\{ \begin{aligned} x_2 &= 3u_2 - \frac{u_3}{2} \\ x_4 &= -2u_3 \end{aligned} \right\}$

Set $u_1, u_2, u_3 =$ whatever you want and solve for $x_2$ & $x_4$.

In general, we get

$$x_{k_1} + \sum_{j=1}^{n-r} C_{1j} u_j = 0$$

$$\vdots$$

$$x_{k_r} + \sum_{j=1}^{n-r} C_{rj} u_j = 0$$

~ set $u_j$'s to be whatever and solve for $x_{k_j}$'s.

Observation: If $r<n$, then $RX=0$ has a non-trivial solution, i.e one where not every $x_j$ is $0$.

Do you see why?

Theorem 6: If $A$ is an $m \times n$ matrix and $m<n$, then the homogeneous system $AX=0$ has a non-trivial solution.

Proof: See the remark above.

Theorem 7: If $A$ is an $n \times n$ (square) matrix, then $A$ is row equivalent to the $n \times n$ identity matrix iff

$AX = 0$ has only trivial solutions.

Proof: If $A$ is row-equivalent to $I$ (identity), $AX = 0$ and $IX = 0$ have the same solutions.

Conversely, suppose that $AX = 0$ has only the trivial solution $X = 0$. Let $R$ be the $n \times n$ row-reduced eschelon matrix row-equivalent to $A$, and let $r$ be the number of non-zero rows of $R$. If $r < n$, $RX = 0$ has non-trivial solutions (why?). Therefore, $r = n$ and the reduced row eschelon scheme guarantees that $R$ is the identity matrix.