

MATH 238: HOMEWORK #1 DUE MONDAY, 10/19/16

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Problem #1: Prove that number of incidences between n points m lines in the plane is

$$\leq C (n + m + n\sqrt{m} + m\sqrt{n}),$$

where C is a positive constant independent of n, m .

Food for thought: What is the most general collection of geometric shapes you can think of such that if you replace the lines above by these shapes, you obtain the same bound as above, possibly with a different constant C ?

Problem #2: We say that a set K in \mathbb{R}^d is convex if given $x, y \in K$, the line segment connecting x and y is entirely contained in K . We say that K is strictly convex if the line segment connecting x and y does not intersect the boundary of K , except possibly at x and y .

Given a strictly convex bounded set K in \mathbb{R}^d , $d \geq 2$, and a finite point set E , define

$$\Delta_K(E) = \{ \|x - y\|_K : x, y \in E \},$$

where

$$\|x\|_K = \inf \{ t > 0 : x \in tK \},$$

where

$$tK = \{ ty : y \in K \}.$$

i) Prove that if $E \subset \mathbb{R}^2$ is a finite set, then there exists $0 < C_K < \infty$, independent of the size of E , such that

$$\#\Delta_K(E) \geq C_K (\#E)^{\frac{1}{2}}.$$

ii) Define $\|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_d|\}$. Find a bounded convex set K_∞ such that $\|x\|_\infty = \|x\|_{K_\infty}$. Prove that if E is a finite point set in \mathbb{R}^2 , there exists C , independent of the size of E such that $\#\Delta_{K_\infty}(E) \geq C(\#E)^{\frac{1}{2}}$.

Food for thought: If $E \subset \mathbb{R}^2$ is a finite set, can you prove that there exists $0 < C_K < \infty$, independent of the size of E , such that

$$\#\Delta_K(E) \geq C_K (\#E)^{\frac{1}{d}}?$$

Problem #3: Let P be a finite set of points in the plane and let L denote a finite set of circles of radius 1. Suppose that $\#P = \#L = n$. Let $I(P, L)$ denote

the total number of incidences between the points in P and the circles in L . Prove that

$$I(P, L) \leq Cn^{\frac{3}{2}}$$

for some constant $C > 0$ independent of n .

Food for thought: Can you handle the case when circles are replaced by translates of K -circles, defined by

$$\{x \in \mathbb{R}^2 : \|x\|_K = 1\},$$

where K is strictly convex and $\|x\|_K$ is defined above?

Problem #4: We say that $A \subset \mathbb{R}^d$, $d \geq 2$, is a Delone set if there exist $C, c > 0$ such that:

- $|x - y| \geq c > 0$ for all $x, y \in A$.
- Every cube of side-length C contains at least one point of E .

i) Prove that there exist $C_1, C_2 > 0$ such that

$$C_1 R^d \leq \#(A \cap [-R, R]^d) \leq C_2 R^d.$$

ii) In the case $d = 2$, use the Moser method to show that there exists $C' > 0$ such that

$$\#\Delta(E) \geq C'(\#E)^{\frac{3}{4}}.$$

iii) What is the best you can do in higher dimensions?

Problem #5: Show that for any sufficiently large positive integer n , there exists a point set P in the plane, and a set L of translates of the parabola $\{(t, t^2) : t \in \mathbb{R}\}$, satisfying the following properties:

- $\#P \approx \#L$.
- $I(P, L) \approx (\#P)^{\frac{4}{3}}$.

Food for thought: Can you come up with a construction of this type if parabolas are replaced by circles of a fixed radius?

Problem #6: Problem 1.3 from the book.

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