Wednesday, January 23

$\sum_{n=0}^{\infty} a_n$ converges to $z \in \mathbb{C}$ iff

for every $\epsilon > 0$ $\exists N \ni \left| \sum_{n=0}^{m} a_n - z \right| < \epsilon$

whenever $m \geq N$.

$\sum_{n=0}^{\infty}$ converges absolutely if $\sum_{n=0}^{\infty} |a_n|$ converges.

Proposition: If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Proof: The key idea must be the triangle inequality. Let $\epsilon > 0$ be given and define $z_n = a_0 + a_1 + \ldots + a_n$. Since $\sum |a_n|$ converges, $\exists N \ni \sum_{n=N}^{\infty} |a_n| < \epsilon$. Thus if $m > k \geq N$,

$$|z_m - z_k| = \left| \sum_{n=k+1}^{m} a_n \right| \underbrace{\leq}_{\text{triangle inequality}} \sum_{n=k+1}^{m} |a_n| \leq \sum_{n=N}^{\infty} |a_n| < \epsilon.$$

This proves that $\{z_m\}$ is a Cauchy sequence, so $\exists z \in \mathbb{C} \ni z_n \to z$. This completes the proof. ■

Recall that $\liminf a_n = \lim_{n\to\infty} [\inf\{a_n, a_{n+1}, \ldots\}]$

$\limsup a_n = \lim_{n\to\infty} [\sup\{a_n, a_{n+1}, \ldots\}]$

possible quiz problem: why does $\overline{\lim}$ & $\underline{\lim}$ always exist?

A power series about a is an infinite series of the form $\sum_{n=0}^{\infty} a_n (z-a)^n$. The simplest meaningful case is

$$a=0 \quad a_n \equiv 1.$$

$$1+z+\dots+z^n = \frac{1-z^{n+1}}{1-z} \longrightarrow \frac{1}{1-z} \quad \text{if } |z|<1$$

diverges in $|z|>1$; interesting if $|z|=1$.

Theorem: Define $\frac{1}{R} = \overline{\lim} |a_n|^{\frac{1}{n}}$

i) If $|z-a|<R$, the series converges absolutely

ii) If $|z-a|>R$, the terms become unbounded, so the series diverges

iii) If $0<r<R$ then the series converges uniformly on $\{z: |z-a| \le r\}$. Moreover, R is the unique number satisfying i) & ii).

Proof: Let $a=0$. If $|z|<R$, $\exists\, r \ni |z|<r<R$. It follows that $\exists\, N \ni |a_n|^{\frac{1}{n}} < \frac{1}{r}$ $\forall\, n \ge N$. It follows that $|a_n| < r^{-n}$ so $|a_n z^n| < \left(\frac{|z|}{r}\right)^n$. This implies that $\left| \sum_{n=N}^{\infty} a_n z^n \right| \le \sum_{n=N}^{\infty} \left(\frac{|z|}{r}\right)^n$, which converges absolutely for each $|z|<R$.

Suppose that $r<R$ and choose $\rho \ni r<\rho<R$. Let N be such that

$$|a_n| < \frac{1}{\rho^n} \quad \forall\, n \ge N. \text{ Then if } |z| \le r,$$

$|a_n z^n| \le \left(\frac{r}{\rho}\right)^n$ and $\frac{r}{\rho} < 1$. This implies uniform convergence on $\{z: |z| \le r\}$ by Weierstrass.

To prove iii) let $|z|>R$ and choose r w/ $|z|>r>R$.
Hence $\frac{1}{r}<\frac{1}{R}$. This implies that for infinitely many n,
$\frac{1}{r}<|a_n|^{\frac{1}{n}} \Rightarrow |a_n z^n| > \left|\frac{|z|}{r}\right|^n$ for those n's & since $\frac{|z|}{r}>1$, the terms become unbounded.

We call R "the radius of convergence" of the power series.

Proposition: If $\sum a_n(z-a)^n$ is a given power series with a radius of convergence R, then $R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ if this limit exists.

Proof: What you should be thinking is roughly the following:

$$\frac{1}{R}=\overline{\lim}\,|a_n|^{\frac{1}{n}} \Rightarrow \underbrace{\frac{1}{R}\sim|a_n|^{\frac{1}{n}}}_{\text{very (deliberately) imprecise}}$$

$$\Rightarrow \frac{1}{R^n}\sim|a_n| \text{ \& } \frac{1}{R^{n+1}}\sim|a_{n+1}|, \text{ so}$$

$$\left|\frac{a_{n+1}}{a_n}\right|\sim\frac{1}{R^{n+1}}\Big/\frac{1}{R^n}=\frac{1}{R}, \text{ or } \left|\frac{a_n}{a_{n+1}}\right|\sim R,$$

but this is not a proof.

Real proof: Let $a=0$ and set $\alpha=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$. Let $|z|<r<\alpha$ and find $N \ni r<\left|\frac{a_n}{a_{n+1}}\right|\ \forall\, n\geq N$. Then if $B=|a_N|r^N$,
then $|a_{N+1}|r^{N+1}=|a_{N+1}|\cdot r\cdot r^N<|a_N|r^N=B$;
$|a_{N+2}|r^{N+2}=|a_{N+1}|\cdot r\cdot r^{N+1}<|a_{N+1}|r^{N+1}<B$ ...
$|a_n||r|^n\leq B \quad \forall\, n\geq N.$

Then $|a_n z^n| \leq |a_n r^n| \frac{|z|^n}{r^n} \leq B \frac{|z|^n}{r^n} \quad \forall\, n \geq N$

Since $|z| < r$, $\sum a_n z^n$ converges. Since $r < \alpha$ is arbitrary, $\alpha \leq R$.

In the opposite direction, if $|z| > r > \alpha$, then

$|a_n| < r|a_{n+1}|$ if $n \geq N$.

This implies that $|a_n r^n| \geq B = |a_N r^N|$ for $n \geq N$.

This gives $|a_n z^n| \geq B \frac{|z|^n}{|r|^n} \xrightarrow[n \to \infty]{} \infty \implies$ divergence,

so $R \leq \alpha$, yielding $R = \alpha$.

Define $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ ~ radius of convergence $= \infty$

Proposition: $\sum a_n$, $\sum b_n$ absolutely convergent. Define $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Then $\sum c_n$ is absolutely convergent with sum $\sum a_n \cdot \sum b_n$

Brief open discussion about how to prove this

HW 1.1: $a_0, a_1, \ldots,$ sequence in $\mathbb{C}$ $a_0 = a_1 = 1$ and $\sum_{n=0}^{\infty} |a_{n+1} - a_n| < \infty$. Prove that the radius of convergence of $\sum a_n z^n$ is ≥ 1.

Proposition: Let $\sum a_n (z-a)^n$ & $\sum b_n (z-a)^n$ w/ radius of convergence $\geq r > 0$. Put $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, then $\sum (a_n + b_n)(z-a)^n$ & $\sum c_n (z-a)^n$ have radius of convergence is $\geq r$ &

$\sum c_n (z-a)^n = \sum a_n (z-a)^n \cdot \sum b_n (z-a)^n$ & $\sum (a_n + b_n)(z-a)^n$ splits.

G open in $\mathbb{C}$ & $f: G \to \mathbb{C}$ then f is differentiable at $a \in G$ if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists,}$$

denoted by $f'(a)$.

Proposition: If $f: G \to \mathbb{C}$ is differentiable at $a \in G$ then f is continuous at a.

Proof: $$\lim_{z \to a} |f(z) - f(a)| = \left[\lim_{z \to a} \frac{|f(z) - f(a)|}{|z-a|}\right] \cdot \left[\lim_{z \to a} |z-a|\right]$$

$$= |f'(a)| \cdot 0 = 0.$$

Definition: If f is continuously differentiable on G, it is called analytic on G.

Chain rule: f, g analytic on G & Ω respectively and suppose $f(G) \subset \Omega$. Then $g \circ f$ is analytic on G and

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z) \quad \forall z \in G.$$

Proof: Fix $z_0 \in G$ and choose $r > 0 \ni B(z_0, r) \subset G$. We must show that if $0 < |h_n| < r$ and $\lim h_n = 0$ then

$$\lim \frac{g(f(z_0+h)) - g(f(z_0))}{h_n} \text{ exists and is}$$

equal to $g'(f(z_0)) f'(z_0)$.

Why is this sufficient? Think about reformulation of continuity in terms of sequences.

First, suppose that $f(z_0) \neq f(z_0+h_n)$ $\forall n$

Then $\frac{g\circ f(z_0+h_n) - g\circ f(z_0)}{h_n} = \frac{g(f(z_0+h_n)) - g(f(z_0))}{f(z_0+h_n) - f(z_0)} \frac{f(z_0+h_n)-f(z_0)}{h_n}$

$\longrightarrow g'(f(z_0))f'(z_0)$

Now suppose that $f(z_0) = f(z_0+h_n)$ for infinitely many n.

Then $\{h_n\} = \{k_n\} \cup \{l_n\}$ $\ni$ $f(z_0) \neq f(z_0+k_n)$ &

$f(z_0) = f(z_0+l_n)$ $\forall n$.

It follows that $f'(z_0) = \lim_{n\to\infty} \frac{f(z_0+l_n)-f(z_0)}{l_n} = 0$ &

$\lim \frac{g\circ f(z_0+l_n) - g\circ f(z_0)}{l_n} = 0.$

By the previous case,

$\lim \frac{g\circ f(z_0+k_n) - g\circ f(z_0)}{k_n} = g'(f(z_0))f'(z_0) = 0.$

Therefore, $\lim_{n\to\infty} \frac{g\circ f(z_0+h_n) - g\circ f(z_0)}{h_n} = 0 = g'(f(z_0))f'(z_0)$

The general case follows easily.

So far, everything is more or less like in the real case.
But this is about to change.