①

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Definition: $[a,b]$ interval, $f:[a,b]\to\mathbb{R}$ convex if for any $x_1, x_2 \in [a,b]$

$$f(tx_2+(1-t)x_1) \le t f(x_2) + (1-t) f(x_1), \quad 0\le t\le 1.$$

$A \subseteq \mathbb{C}$ convex if for any $z, w \in A$, $(1-t)z + tw \in A$

$t \in [0,1]$.

<u>Proposition:</u> A function $f:[a,b]\to\mathbb{R}$ is convex iff for any $x_1, \dots, x_n$ in $[a,b]$ & $t_1 \dots t_n \in \mathbb{R}$, $t_j \ge 0$,

$$\sum_{k=1}^{n} t_k = 1,$$

$$f\left(\sum_{k=1}^{n} t_k x_k\right) \le \sum_{k=1}^{n} t_k f(x_k)$$

A set $A \subset \mathbb{C}$ is convex iff for any $z_1, \dots z_n \in A$ & $\sum_{k=1}^{n} t_k = 1$ $t_k \in \mathbb{R}$ $t_k \ge 0$,

$$\sum_{k=1}^{n} t_k z_k \in A.$$

simple induction on the definition

What is the connection between convexity of the set & convexity of the function?

Proposition: A differentiable function f on $[a,b]$ is convex iff $f' \nearrow$.

Proof: Assume that f is convex. Let $a \le x < y \le b$ & suppose that $0 < t < 1$. Since $0 < (1-t)x + ty - x = t(y-x)$,

$$\frac{f((1-t)x+ty) - f(x)}{t(y-x)} \le \frac{f(y)-f(x)}{y-x}$$

by the definition of convexity (subtract & divide)

Now let $t \to 0$:

$$f'(x) \le \frac{f(y)-f(x)}{y-x} \quad (*)$$

$0 > (1-t)x + ty - y = (1-t)(x-y)$ & let $t \to 1$ to obtain $f'(y) \ge \frac{f(y)-f(x)}{y-x}$. $(**)$

Combine $(*)$ & $(**)$ to see that $f' \nearrow$.

Now suppose that $f' \nearrow$ & $x < u < y$. By the mean value theorem for differentiation, we can find r, s w/ $x < r < u < s < y$ $\ni$

$$f'(r) = \frac{f(u)-f(x)}{u-x} \qquad f'(s) = \frac{f(y)-f(u)}{y-u}$$

Since $f'(r) \le f'(s)$,

$$\frac{f(u)-f(x)}{u-x} \le \frac{f(y)-f(u)}{y-u}$$

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Let $u = (1-t)x + ty$, $0<t<1$

$$\frac{f(u)-f(x)}{t(y-x)} \leq \frac{f(y)-f(u)}{y-u}$$

$$\Longrightarrow (1-t)[f(u)-f(x)] \leq t[f(y)-f(u)]$$

so $(1-t)[f((1-t)x+ty)-f(x)] \leq t f(y)$
$- t(f((1-t)x+ty)$

So $f((1-t)x+ty) \leq (1-t)f(x) + tf(y)$

Proposition: A function $f:[a,b] \Longrightarrow \mathbb{R}$ is convex iff
$A = \{(x,y): a \leq x \leq b \text{ \& } f(x) \leq y\}$ is convex

Proof: Suppose $f:[a,b] \longrightarrow \mathbb{R}$ convex
& (x_1,y_1), (x_2,y_2) be points in A. If $0 \leq t \leq 1$,
then $f(tx_2 + (1-t)x_1) \leq tf(x_2) + (1-t)f(x_1) \leq ty_2 + (1-t)y_1$
It follows that $t(x_2,y_2) + (1-t)(x_1,y_1)$
$= (tx_2 + (1-t)x_1,\ ty_2 + (1-t)y_1) \in A \Longrightarrow A$ convex

Now suppose that A is convex & let $x_1, x_2 \in [a,b]$
Then $(tx_2 + (1-t)x_1,\ tf(x_2) + (1-t)f(x_1)) \in A$, $0 \leq t \leq 1$
by convexity
$\Longrightarrow f((1-t)x_1 + tf(x_2)) \leq (1-t)f(x_1) + tf(x_2)$
by definition of A, so
f is convex.

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Theorem: $a < b$ $\quad G = \{x+iy : a < x < b\}$

Suppose that $f : \overline{G} \to \mathbb{C}$ is continuous and f is analytic in G. Define $M : [a,b] \to \mathbb{R}$ by

$$M(x) = \sup\{|f(x+iy)| : -\infty < y < \infty\},$$

and $|f(z)| < B$ $\forall z \in G$, then $\log M(x)$ is a convex function.

Proof: It is enough to show that for

$$a \le x < u < y \le b,$$

$$(y-x)\log M(u) \le (y-u)\log M(x) + (u-x)\log M(y)$$

you will prove this on the homework

Exponentiate & obtain

$$M(u)^{y-x} \le M(x)^{y-u}\, M(y)^{u-x}$$

for $a \le x < u < y \le b$.

We need the following technical observation:

Lemma: Let f, G be as above & suppose that $|f(z)| \le 1$ for $z \in \partial G$. Then $|f(z)| \le 1$ $\forall z \in G$.

Proof: For each $\epsilon > 0$, let $g_\epsilon(z) = [1 + \epsilon(z-a)]^{-1}$, $z \in \overline{G}$

Then for $z = x+iy$ in $\overline{G}$, $|g_\epsilon(z)| \le |\mathrm{Re}[1+\epsilon(z-a)]|^{-1}$

$$= |1+\epsilon(x-a)|^{-1} \le 1.$$

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$\Longrightarrow$ for $z \in \partial G$, $|f(z) g_\epsilon(z)| \leq 1$.

Since $|f(z)| \leq B$ in G,

$$|f(z) g_\epsilon(z)| \leq B\,|1+\epsilon(z-a)|^{-1} \leq B\,[\epsilon\,|\mathrm{Im}(z)|]^{-1} \quad (*)$$

So if $R = \{x+iy : a \leq x \leq b,\ |y| \leq B/\epsilon\}$

$\Longrightarrow |f(z) g_\epsilon(z)| \leq 1$ for $z \in \partial R$.

By the maximum modulus principle,

$$|f(z) g_\epsilon(z)| \leq 1 \text{ for } z \in R$$

But if $|\mathrm{Im}(z)| > \frac{B}{\epsilon}$, $(*) \Longrightarrow |f(z) g_\epsilon(z)| \leq 1 \Longrightarrow$ for all $z \in G$,

$$|f(z)| \leq |1+\epsilon(z-a)|$$

& let $\epsilon \to 0$.

We are now ready to prove the theorem. As we noted above, it is enough to show that

$$M(u)^{b-a} \leq M(a)^{b-u}\, M(b)^{u-a} \qquad a < u < b.$$

Observe that if $A > 0$, $A^z = e^{z\log(A)}$ is an entire function, which has no zeroes.

So, $g(z) = M(a)^{b-z/b-a}\, M(b)^{z-a/b-a}$

is entire and never vanishes

Moreover,

$$|g(z)| = \underset{\underset{0}{\#}}{M(a)}^{b-z/b-a}\; \underset{\underset{0}{\#}}{M(b)}^{z-a/b-a}$$

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It follows that g^{-1} must be bounded on $\overline{G}$.

Observe that $|g(a+iy)| = M(a)$ & $|g(b+iy)| = M(b)$

It follows that $|f(z)/g(z)| \leq 1$, $z \in G$

& f/g satisfies the assumptions of the lemma.

Hence $|f(z)| \leq |g(z)|$, $z \in G$ $\Rightarrow$

$$M(u) \leq M(a)^{b-u/b-a} \, M(b)^{u-a/b-a},$$

which is what we want.