HW5

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9)

$$\int_0^{2\pi} \frac{1}{e^{int}} d(e^{int}) = \int_\gamma \frac{1}{z} dz \quad \text{as } \gamma$$

being smooth and continuous.

$$\Rightarrow \quad = \int_0^{2\pi} e^{-int} \cdot (ine^{int} dt) = 2\pi n i$$

12)

$$\int_\gamma \frac{e^{iz}}{z} dz = \int_0^{\pi} \frac{e^{ire^{it}}}{re^{it}} d(re^{it})$$

as γ Smooth and Cont.

$$= i\int_0^{\pi} e^{ire^{it}} dt$$

Triangle ineq.

$$\Rightarrow \quad |I(r)| \le \int_0^{\pi} |e^{ire^{it}}| dt = \int_0^{\pi} e^{-r\sin t} dt$$

$$= 2\int_0^{\frac{\pi}{2}} e^{-r\sin t} dt \le 2\int_0^{\frac{\pi}{2}} e^{-r\cdot\frac{2t}{\pi}} dt$$

$$= 2\cdot\left[\frac{e^{-\frac{2r}{\pi}t}}{-\frac{2r}{\pi}}\right]_0^{\frac{\pi}{2}} \to 0 \quad \left(\text{using } \frac{\sin t}{t} \ge \frac{2}{\pi} \text{ on } [0, \tfrac{\pi}{2}]\right.$$

$\boxed{4}$

22) $F_1' = F_2' = f$

Define $F = F_1 - F_2$

then $F'(z) = 0 \Rightarrow F(z) = c$ on G

since G open and connected.

$\Rightarrow F_1 = c + F_2$ $\boxed{\checkmark}$

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4)

a) WLOG let $a = 0$

Define $S_k = \sum_{n=0}^{k} a_n$ then

$$\sum_{n=0}^{N} a_n r^n = a_0 + \sum_{n=1}^{N} (S_n - S_{n-1}) r^n$$

$$= (1-r) \sum_{n=0}^{N} S_n r^n$$

Since $S_n \to A$, let this N be

the # s.t. $\forall\ n > N$, $|S_n - A| < \frac{\epsilon}{2}$

for arbitrary ϵ fixed.

Then

$$\left| \sum a_n r^n - A \right|$$

$$= \left| \sum a_n r^n - (1-r) \sum A r^n \right|$$

$$= \left| (1-r) \sum_{n=0}^{N} (S_n - A) r^n + (1-r) \sum_{>N} (S_n - A) r^n \right|$$

$$\leq \left| (1-r) \sum_{n=0}^{N} (S_n - A) r^n \right| + \frac{\epsilon}{2}$$

when $r \to 1^-$, $\lim_{r \to 1^-} \left| (1-r) \sum_{n=0}^{N} (S_n - A) r^n \right| = 0$

Hence $\lim_{r \to 1^-} \sum a_n r^n = A$

b)

$$\log(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad \text{has RoC} = 1$$

By a)

$$\log 2 = \lim_{x \to 1^-} \log(1+x) = 1 - \frac{1}{2} + \frac{1}{3} \cdots$$

$\square$

7)

a) $\int_\gamma \frac{e^{iz}}{z^2}\,dz$

Let $f(z) = e^{iz}$

$$f'(0) = \frac{1!}{2\pi i}\int_\gamma \frac{f(z)}{(z-0)^2}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{e^{iz}}{z^2}\,dz$$

$$\Rightarrow \int_\gamma \frac{e^{iz}}{z^2}\,dz = -2\pi$$

b)

Let $f(z) = 1$ then

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz$$

$$\Rightarrow \int_\gamma \frac{1}{z-a}\,dz = 2\pi i$$

c) let $f(z) = \sin z$

then $f''(0) = \frac{2!}{2\pi i} \int_\gamma \frac{f(z)}{(z-0)^3} dz$

$\Rightarrow \int_\gamma \frac{\sin z}{z^3} dz = 0$

d) let $f(z) = \frac{z-1}{z^n} \log z$

then $f(1) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-1} dz$

$\Rightarrow \int_\gamma \frac{\log z}{z^n} dz = 0$

9)

a) let $f(z) = e^z - e^{-z}$

$f^{(n-1)}(0) = \frac{(n-1)!}{2\pi i} \int_\gamma \frac{f(z)}{z^n} dz$

$\Rightarrow \int_\gamma \frac{e^z - e^{-z}}{z^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(0) = \begin{cases} \frac{4\pi i}{(n-1)!}, & n \text{ even} \\ 0, & \text{odd} \end{cases}$

b) Let $f = 1$

$$f^{(n-1)}\left(\tfrac{1}{2}\right) = \frac{(n-1)!}{2\pi i} \int_\gamma \frac{1}{(z-\frac{1}{2})^n} dz$$

$$\Rightarrow \int_\gamma \frac{1}{(z-\frac{1}{2})^n} dz = 0$$

c) $$\int_\gamma \frac{dz}{z^2+1} = \frac{1}{2i}\left[\int_\gamma \frac{1}{z-i} dz - \int_\gamma \frac{1}{z+i} dz \right]$$

Let $f(z) = 1$

$$f(i) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-i} dz = 1 \Rightarrow \int_\gamma \frac{1}{z-i} dz = 2\pi i$$

$$f(-i) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z+i} dz = 1 \Rightarrow \int_\gamma \frac{1}{z+i} dz = 2\pi i$$

$$\Rightarrow \int_\gamma \frac{dz}{z^2+1} = 0$$

d) $f(z) = \sin z$

$$f(0) = \frac{1}{2\pi i} \int_\gamma \frac{\sin z}{z-0} dz = 0$$

$$\Rightarrow \int_\gamma \frac{\sin z}{z} dz = 0$$

e) Put $f(z) = z^{\frac{1}{m}}$

$$f^{(m-1)}(1) = \frac{(m-1)!}{2\pi i} \int_\gamma \frac{z^{\frac{1}{m}}}{(z-1)^m} dz$$

Calculate $f^{(m-1)}(z) = \prod_{i=0}^{m-2} \left(\frac{1}{m} - i\right)$

$$\Rightarrow \int_\gamma \frac{z^{\frac{1}{m}}}{(z-1)^m} dz = \frac{2\pi i}{(m-1)!} \prod_{i=0}^{m-2} \left(\frac{1}{m} - i\right)$$

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4) Easy to see $e^{z+a} = e^z e^a$ when $z \in \mathbb{C}$, $a \in \mathbb{R}$

Let $f(a) = e^{z+a}$. $g(a) = e^z e^a$

Then $\mathbb{R} \subseteq \{a \in \mathbb{C} \mid f(a) = g(a)\}$

By Corollary one can extend $a \in \mathbb{R}$ to $a \in \mathbb{C}$ as $\mathbb{R}$ obviously contains limit points of $\mathbb{C}$. $\boxed{}$

5) Let $f(a) = \cos(a+b)$
$g(a) = \cos a \cos b - \sin a \sin b$.

Then $\mathbb{R} \subseteq \{a \in \mathbb{C} \mid f(a) = \cancel{f(a)}\ g(a)\}$

Similar argument shows that

$f(a) = g(a)$ for $a \in \mathbb{C}$, $b \in \mathbb{R}$.

Apply argument again now regarding b, one is able to reach the conclusion.

$\boxed{\text{Q}}$