HW8 Solution.

Pg 121-123

1)

a) $\frac{\sqrt{3}}{6}\pi$ b) $-\frac{\pi}{2}$ c) $\frac{\pi a^2}{1-a^2}$ d) $\frac{\pi a}{(\sqrt{a^2-1})^3}$

2) Omitted

(3) Let $g(z) = \frac{f(z)}{(z-a)(z-b)}$

By Residue's Theorem

$$\int_\gamma \frac{f(z)}{(z-a)(z-b)}\,dz = 2\pi i\left[Res(g;a) + Res(g;b)\right]$$

$$= 2\pi i \frac{f(a)-f(b)}{a-b}$$

Supposed f is bounded by M, we have

$$\lim_{R\to\infty}\left|\int_\gamma g(z)\,dz\right| = \lim_{R\to\infty}\left|\int_0^{2\pi} \frac{f(Re^{it})\,iRe^{it}\,dt}{(Re^{it}-a)(Re^{it}-b)}\right|$$

$$\leq \lim_{R\to\infty}\int_0^{2\pi}\left|\frac{MR}{(R-a)(R-b)}\right| \to 0$$

$\Rightarrow f(a) = f(b)$, $\forall\, a,b \in B(0,R)$ $\boxed{//}$

Pg 126

1) Similar to Thm 3.4

$$\frac{f'}{f}(z) = \sum_{k=1}^{n} \frac{1}{z - z_k} - \sum_{j=1}^{m} \frac{1}{z - p_j} + \frac{h'(z)}{h(z)}$$

where h is analytic and doesn't vanish in G.

Then we have $g\frac{h'}{h}$ is analytic, too.

So $\frac{1}{2\pi i}\int g\frac{f'}{f}(z)\,dz$

$$= \frac{1}{2\pi i}\sum_{k=1}^{n}\int_\gamma \frac{g}{z - z_k}\,dz - \frac{1}{2\pi i}\sum_{j=1}^{m}\int_\gamma \frac{g}{z - p_j}\,dz$$

$$+ \frac{1}{2\pi i}\int_\gamma g\frac{h'}{h}\,dz$$

$$= \sum_{k=1}^{n} g(z_i)\, n(\gamma; z_k) - \sum_{j=1}^{m} g(p_j)\, n(\gamma; p_j) + 0$$

$\boxtimes$

5) Zeros: Use Thm 3.7

Poles: Consider $\frac{1}{f}$ whose zeroes are poles of f.

7~~6~~) Let f and g be meromorphic on G with no zeros or poles on γ (rectifiable), and $\gamma \approx 0$ in G. Suppose

$|f+g| < |f|+|g|$ on γ.

Define Z_f, Z_g to be sums of winding # at zero for f, g, Similarly P_f, P_g for that at poles.

Then $Z_f - P_f = Z_g - P_g$

Proof: Observe that $\frac{f(z)}{g(z)} \in \mathbb{C} - [0,\infty)$

otherwise $\left|\frac{f}{g}+1\right| = \left|\frac{f}{g}\right|+1$, Contradiction.

Check that $\log \frac{f}{g}$ is a primitive for $\frac{(f/g)'}{(f/g)}$ in a nbhd of γ

So $$0 = \frac{1}{2\pi i} \int_\gamma \frac{(f/g)'}{f/g} \, dz$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{f'}{f} - \frac{g'}{g} \, dz$$

$$= (Z_f - P_f) - (Z_g - P_g) \quad \text{as } \gamma \approx 0$$