(1)

$$f(x) = \frac{1}{\sqrt{x-1}}$$

$$\int_1^2 \frac{1}{\sqrt{x-1}}\,dx$$ cannot be computed directly because we are dividing by 0 at $x=1$

$\epsilon > 0$

$$\lim_{\epsilon \to 0} \int_{1+\epsilon}^{2} \frac{dx}{\sqrt{x-1}} =$$

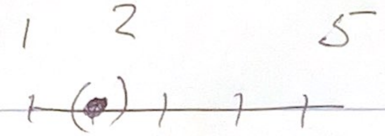
$$\lim_{\epsilon \to 0} 2\sqrt{x-1}\Big|_{1+\epsilon}^{2} =$$

$$\lim_{\epsilon \to 0} 2 - 2\sqrt{\epsilon} = 2$$

The improper integral converges, and is equal to 2.

(2)

$$f(x) = \frac{1}{x-2}$$



$$\int_1^5 f(x)\,dx$$ cannot be computed directly because we are dividing by 0 at $x=2$.

$\epsilon \to 0$

$$\lim_{\epsilon \to 0} \int_1^{2-\epsilon} \frac{1}{x-2}\,dx + \int_{2+\epsilon}^{5} \frac{1}{x-2}\,dx$$

$$= \lim_{\epsilon \to 0} \ln(|x-2|)\Big|_1^{2-\epsilon} + \ln(|x-2|)\Big|_{2+\epsilon}^{5}$$

$$= \lim_{\epsilon \to 0} \ln(\epsilon) - \ln(1) = -\infty$$ diverges

$$+ \lim_{\epsilon \to 0} \ln(3) - \ln(\epsilon) = +\infty$$ diverges

③

p = positive real number

$\int_0^1 \frac{dx}{x^p} =$ cannot be computed directly because we are dividing by 0 at $x=0$.

$\epsilon > 0$

$$\lim_{\epsilon \to 0} \int_\epsilon^1 x^{-p}\,dx = \begin{cases} \frac{x^{-p+1}}{1-p}, & \text{if } p \neq 1 \\ \ln(|x|), & \text{if } p = 1 \end{cases}$$

Case 1: $p \neq 1$

We get $\lim_{\epsilon \to 0} \left. \frac{x^{-p+1}}{1-p} \right|_\epsilon^1 = \frac{1}{1-p}\left(1 - \epsilon^{1-p}\right)$ as $\epsilon \to 0$

$$= \begin{cases} \frac{1}{1-p}, & \text{if } p < 1 \\ -\infty, & \text{if } p > 1 \end{cases}$$

~~[illegible]~~

(4)

If, $p \neq 1$,

$\int_0^1 \frac{1}{x^p}\,dx$ converges if $p<1$
and diverges if $p>1$.

Case 2: $p=1$

$$\lim_{\epsilon \to 0} \int_\epsilon^1 \frac{1}{x}\,dx = \lim_{\epsilon \to 0} \ln(|x|)\Big|_\epsilon^1$$

$$= \underbrace{\ln(1)}_{=0} - \ln(\epsilon) = \infty$$

diverges

(5)

Consider the integral

$$\int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$$

Determine if this improper integral converges. If it does, compute the value of the integral.

$\epsilon > 0$

$$\lim_{\epsilon \to 0} \int_{1+\epsilon}^{3} \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{\epsilon \to 0} 3(x-1)^{\frac{1}{3}} \Big|_{1+\epsilon}^{3}$$

$$= \lim_{\epsilon \to 0} 3 \cdot 2^{\frac{1}{3}} - \underset{\substack{\downarrow \\ 0}}{3\epsilon^{\frac{1}{3}}} = 3\sqrt[3]{2}$$

Clarification:

Anti-derivative of $(x-1)^{-\frac{2}{3}}$ is

$$\frac{(x-1)^{-\frac{2}{3}+1}}{\frac{1}{3}} = 3(x-1)^{\frac{1}{3}}$$

(6)

Consider

$$\int_1^\infty \frac{dx}{x^2}$$ cannot be evaluated directly because ∞ is involved

Choose N, a large positive number, and consider

$$\lim_{N \to \infty} \int_1^N \frac{dx}{x^2} =$$

$$\lim_{N \to \infty} -\frac{1}{x} \Big|_1^N = \lim_{N \to \infty} -\frac{1}{N} + \frac{1}{1}$$

$$= \underline{1}$$

So, the integral converges, and is equal to $\underline{\underline{1}}$.

⑦

Consider

$$\int_5^\infty \frac{dx}{\sqrt{x}}$$ cannot be evaluated directly since ∞ is involved.

$$\lim_{N\to\infty} \int_5^N \frac{dx}{\sqrt{x}} =$$

$$\lim_{N\to\infty} 2\sqrt{x}\Big|_5^N = \lim_{N\to\infty} 2\sqrt{N} - 2\sqrt{5}$$

$$= \infty,$$

so the integral diverges.

In general, $(p \geq 0)$

$\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$

and diverges if $p \leq 1$

(8)

Consider

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$ problems at $x=\infty$ & $x=-\infty$

M, N large positive numbers

$$\lim_{M\to\infty} \int_{-M}^{0} \frac{dx}{1+x^2} + \lim_{N\to\infty} \int_{0}^{N} \frac{dx}{1+x^2}$$

$$= \lim_{M\to\infty} \tan^{-1}(x) \Big|_{-M}^{0} + \lim_{N\to\infty} \tan^{-1}(x) \Big|_{0}^{N}$$

$$= \lim_{M\to\infty} \overset{0}{\overset{\shortparallel}{\tan^{-1}(0)}} - \tan^{-1}(-M)$$

$$+ \lim_{N\to\infty} \tan^{-1}(N) - \overset{=0}{\tan^{-1}(0)}$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

converges

(9)

Consider

$$\int_{-\infty}^{0} e^{x}\,dx$$

Choose N, a large positive number.

$$\lim_{N\to\infty}\int_{-N}^{0} e^{x}\,dx = \lim_{N\to\infty} e^{x}\Big|_{-N}^{0} =$$

$$\lim_{N\to\infty} e^{0} - e^{-N} =$$

$$\lim_{N\to\infty} 1 - e^{-N} = 1 - 0 = \underline{1}$$

converges and equal to $\underline{1}$.

(10)

$$\left(-e^{-x}\right)' = -\left(-e^{-x}\right) = e^{-x}$$

Consider

$$\int_5^{\infty} xe^{-x}\,dx$$

Choose N large positive number

$$\lim_{N\to\infty} \int_5^N \underbrace{x}_{f(x)}\underbrace{e^{-x}}_{g'(x)}\,dx$$

$$\int_5^N f(x)g'(x)\,dx = f(x)g(x)\Big|_5^N - \int_5^N f'(x)g(x)\,dx$$

$$\lim_{N\to\infty} -xe^{-x}\Big|_5^N + \int_5^N e^{-x}\,dx$$

$$= \lim_{N\to\infty} -Ne^{-N} + 5e^{-5} + e^{-x}\Big|_5^N$$

(11)

$$\lim_{N \to \infty} -Ne^{-N} + 5e^{-5} + e^{-N} - e^{-5}$$

$$= 4e^{-5} + 0 + \lim_{N \to \infty} -Ne^{-N}$$

$$= 4e^{-5} - \lim_{N \to \infty} \frac{N}{e^N}$$

$$= 4e^{-5} - \lim_{N \to \infty} \frac{1}{e^N}$$

$$= 4e^{-5} - 0 = 4e^{-5}$$

converges and equals to $4e^{-5}$.

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$$\int_5^\infty \frac{dx}{1+x^2} =$$

$$\lim_{N\to\infty} \int_5^N \frac{dx}{1+x^2} =$$

$$\lim_{N\to\infty} \tan^{-1}(x) \Big|_5^N =$$

$$\lim_{N\to\infty} \tan^{-1}(N) - \tan^{-1}(5)$$

$$= \frac{\pi}{2} - \tan^{-1}(5)$$

converges and equals to

$$\frac{\pi}{2} - \tan^{-1}(5)$$