DISTANCE SETS

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Chapter I: Erdos distance problem for general metrics

In this chapter we begin discussing a beautiful problem introduced by Paul Erdos ([Erdos45]). The basic idea is to determine the smallest number of distinct distances determined by a set of points in a Euclidean space. In order to state the problem precisely, we need the following definitions.

Definition 1.1. We say that a set $K \subset \mathbb{R}^d$ is bounded if it is contained in a ball of finite radius.

Definition 1.2. We say that a set $K \subset \mathbb{R}^d$ is symmetric with respect to the origin if $-x \in K$ whenever $x \in K$.

Definition 1.3. We say that a set $K \subset \mathbb{R}^d$ is convex if the line segment connecting points x and y is contained in K whenever x and y are both contained in K.

Definition 1.4. We say that $||\cdot||_K$ is the norm induced by a bounded convex set K, symmetric with respect to the origin, if $||x||_K = \inf\{t : tx \in K\}$.

Definition 1.5. The cardinality of a finite set $S \subset \mathbb{R}^d$ is the number of points it contains.

We are now ready to formulate the Erdos Distance Problem (EDP). Let S be a finite subset of $\mathbb{R}^d$, $d \geq 2$. Let

$$\Delta_K(S) = \#\{||x-y||_K : x, y \in S\}, \tag{1.1}$$

where $||\cdot||_K$ is the norm induced by a bounded convex set K, symmetric with respect to the origin. The question we ask is, what is the smallest possible cardinality of $\Delta_K(S)$,

$$g_K(n,d) = \inf_{\#S=n} \#\Delta_K(S)? \tag{1.2}$$

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Lemma 1.1. *There exists a bounded symmetric convex set K such that*

$$g_K(n,d) \le n^{\frac{1}{d}}. \tag{1.3}$$

Indeed, let $S = [0, n^{\frac{1}{d}}]^d \cap \mathbb{Z}^d$, and there is no harm in assuming that $n^{\frac{1}{d}}$ is an integer. Let $K = [-1,1]^d$. Since the difference of two vectors in $\mathbb{Z}^d$ is a vector in $\mathbb{Z}^d$, $\#\Delta_K(S) = \#\{||x||_K : x \in S\}$. Observe that $||x||_K = \max\{x_1, \dots, x_d\}$, where x_j is the jth component of the vector $x \in S$. It follows that $\#\{||x||_K : x \in S\} \le n^{\frac{1}{d}}$, as claimed.

Next we shall see that without further assumptions on K, Lemma 1.3 gives the best possible estimate for $g_K(n,d)$. More previsely,

Theorem 1.2. *(*[Erdos45]*) Without any further assumptions on K,*

$$g_K(n,d) \ge C(n-1)^{\frac{1}{d}}. \tag{1.4}$$

The proof is by induction on the dimension. We first prove the result in two dimensions. Fix any point of S and observe that the remaining $n-1$ points of S are located on t circles centered at that point. What t is a bit of a mystery, but it is an integer between 1 and $n-1$. Let N denote the largest number of points of S among those circles. It follows that

$$Nt \ge n-1. \tag{1.5}$$

Consider the circle containing N points of S. At least $\frac{N}{2}$ of these points must lie on the same semi-circle. If we fix one of these points, and measure distances to the other points on the same semi-circle, we see that the number of distinct distances is at least $\frac{N}{2}$. It follows that in two dimensions,

$$\#\Delta_K(S) \ge \max\left\{\frac{N}{2}, \frac{n-1}{N}\right\}. \tag{1.6}$$

It follows that

$$\#\Delta_K(S) \ge \sqrt{\frac{n-1}{2}}. \tag{1.7}$$

This completes the proof of the two-dimensional case of Theorem 1.2. We now observe that the proof above works just as well if we replace $\mathbb{R}^2$ by the boundary of a bounded convex set in $\mathbb{R}^3$. This allows us to proceed by induction. Suppose that Theorem 0.2 holds on $\mathbb{R}^{d-1}$ and also on the boundary of a bounded convex set in $\mathbb{R}^d$. We must now prove that Theorem 0.2 holds in $\mathbb{R}^d$. Fix a point of S and observe that the remaining $n-1$ points are located on t spheres centered at that point. As before, the positive integer t is a mystery to be resolved. Let N again denote the largest number of points of S among those spheres.

As before, (1.5) holds. By the induction hypothesis, the sphere with N points determines at least $C_d N^{\frac{1}{d-1}}$ distances for some $C_d > 0$. It follows that

$$\#\Delta_K(S) \geq \max\left\{C_d N^{\frac{1}{d-1}}, \frac{n-1}{N}\right\}. \tag{1.8}$$

It follows that

$$\#\Delta_K(S) \geq C_d^{\frac{d-1}{d}} (n-1)^{\frac{1}{d}}, \tag{1.9}$$

and the proof of Theorem 0.2 is complete.

Before concluding this chapter, let us summarize what happened. We posed the Erdos Distance Problem for metrics that come from bounded convex sets. We saw that the number of distances determined by n points in d dimensions is at least a constant multiple of $n^{\frac{1}{d}}$, and we saw that for at least one metric, the one that comes from $K = [-1.1]^d$, this result is best possible.

Where do we go from here? In order to get a hint, let us re-examine the proof of Lemma 1.1. We saw that if $K = [-1,1]^d$, then $\#\Delta_K([0, n^{\frac{1}{d}}]^d \cap \mathbb{Z}^d) = n^{\frac{1}{d}}$. On the other hand, suppose that $K = \{x \in \mathbb{R}^d : |x| \leq 1\}$. Then $\#\Delta_K([0, n^{\frac{1}{d}}]^d \cap \mathbb{Z}^d) = \#\{(x_1, \dots, x_d) : x_1^2 + x_2^2 + \cdots + x_d^2 \leq m; m = 1, 2, \dots, n^{\frac{2}{d}}\}$. We shall see that the size of this set is very close $n^{\frac{2}{d}}$ in all dimensions. This leads one to the following conjecture due to Paul Erdos:

Erdos Distance Conjecture. *With the notation above, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that*

$$g_{B_d}(n,d) = \inf_{\#S=n} \Delta_{B_d}(S) \geq C_\epsilon n^{\frac{2}{d}-\epsilon}, \tag{1.10}$$

where B_d is the unit ball in $\mathbb{R}^d$.

In plain language, Erdos' conjecture says that the number of standard Euclidean distances determined by n points is nearly a constant multiple of $n^{\frac{2}{d}}$. If anything like this conjecture is true, and we are to understand why, we will need to grasp the essential geometric difference between the metric that comes from the cube $[-1,1]^d$, and the one that comes from the unit ball B_d.

Exercises

Exercise 1.1. *Explicitly compute the constant in front of $n^{\frac{1}{d}}$ in Theorem 0.2. Is the resulting constant best possible in the sense that there exists a set of n points and a convex set K such that this constant is achieved?*

Exercise 1.2. *A general positive-definite metric on* $\mathbb{R}^d$ *is a function* $\rho : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *that satisfies the following aximoms:*

(1) $\rho(x, y) \geq 0$ *and* $\rho(x, y) = 0$ *if and only if* $x = y$.
(2) $\rho(x, y) = \rho(y, x)$.
(3) $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$.

Does Theorem 0.2 hold for such general metrics? Prove it or give a counter-example.

Exercise 1.3. *Prove that* $g_{B_d}(n, d) \leq Cn^{\frac{2}{d}}$ *for some* $C > 0$*. In fact, prove that* $g_K(n, d) \leq Cn^{\frac{2}{d}}$ *for any* K.

Exercise 1.4. *Give an alternative proof of Theorem 0.2 in the case* $K = B_d$ *as follows. Let* S *be a set of cardinality* n *in* $\mathbb{R}^2$*. Number these points from* 1 *to* n*. Draw a circle of radius* R *around each point and number these circles from* 1 *to* n*. Consider an* n *by* n *matrix with entries* I_{ij} *such that* $I_{ij} = 1$ *if the* i*th points lies on the* j*th sphere, and* 0 *otherwise.*

Our first step is to prove that $I = \sum_{i,j} I_{ij} \leq Cn^{\frac{3}{2}}$ *for some* $C > 0$*. Apply Cauchy-Scwartz, expand the* 2*ond power, and think about what it means for* $I_{ij}I_{ij'} = 1$ *for many values of* i *given fixed* (j, j').

Now observe that you have shown that a single distance cannot repeat more than $Cn^{\frac{3}{2}}$ *times. Since the total number of distances (possibly repeating) is* $\frac{n(n-1)}{2}$*, conclude that the number of distinct distances is at least* $Cn^{\frac{1}{2}}$ *for some* $C > 0$*. Now apply induction on the dimension as in the proof of Theorem 1.2.*

Can you carry out the same argument with B_d *replaced by an arbitrary bounded symmetric convex set? If so, carry out the details, otherwise explain why the proof of Theorem 1.2 given above seems to give us more flexibility.*

Chapter II: Incidence theorems and applications to distance sets

In Chapter I we saw that there exists a bounded symmetric convex set K such that $g_K(n, d) \leq Cn^{\frac{1}{d}}$. We also proved that $g_K(n, d) \geq Cn^{\frac{1}{d}}$ for any bounded symmetric convex set K. However, the dicussion at the end of that chapter suggests that this estimate may improve if we make appropriate assumption on K. In this chapter we shall see that this is indeed the case. Before we proceed to state the precise version of this idea, let us explore further some of the ideas introduced in the previous chapter.

Definition 2.1. We say that the pair (p, l) is an incidence of the point p and the curve l if p is contained in l.

In Exercise 1.4 we gave an alternate proof of Theorem 1.2 in the case $K = B_d$. We recall that the basic idea was to show that a single distance cannot repeat more than $Cn^{\frac{1}{2}}$ times in two dimensions and then use induction on the dimension. The argument requires one to observe that the intersection of 2 circles with the same radius and different centers contains at most two points. The method can be restated in the language of incidences as follows.

Lemma 2.1. *Let S denote a set of n points in $\mathbb{R}^2$. Let L denote the set of n curves in $\mathbb{R}^2$ satisfying the condition that the intersection of any 2 of these curves contains at most M points. Then then number of incidences among the points of S and curves in L is at most $C_M n^{\frac{3}{2}}$.*

A reasonable question to ask at this point is whether the conclusion of Lemma 2.1 is sharp. The answer turns out to be no in two dimensions, as demonstrated by the following celebrated result due to Szemeredi and Trotter.

Theorem 2.2. *Let S be a set of n points on $\mathbb{R}^2$ or the boundary of a convex bounded set in $\mathbb{R}^3$. Let L be a set of lines in $\mathbb{R}^2$, or curves on $\mathbb{R}^2$ (or the boundary of a convex bounded set in $\mathbb{R}^3$ satisfying the following axioms:*

(1) *The intersection of any two curves in L consists of at most α points.*
(2) *No more than β curves in L pass through any pair of points of S.*

Then the number of incidences among the points of S and curves in L is at most $C(n + m + (nm)^{\frac{2}{3}}(\alpha\beta)^{\frac{1}{3}})$.

Moreover, this result is sharp, at least in the case $\alpha = \beta = 1$, in the sense that for every n, m, there exists a set S with n points, and the set L with m lines (or curves) in $\mathbb{R}^2$ satisfying the conditions above such that the number of incidences is comparable to $n + m + (nm)^{\frac{2}{3}}$. (See Exercise 2.1 below).

Using Theorem 2.1 and induction on the dimension, we shall prove the following result.

Definition 2.2. A translate of a convex body K in $\mathbb{R}^d$ is the set $x + K$ for some $x \in \mathbb{R}^d$. A dilate of a convex body K is the set λK for some $\lambda \in \mathbb{R}^+$.

Theorem 2.3. *Let K be a bounded symmetric convex set in $\mathbb{R}^d$ such that the intersection of any d translates of arbitrary dilates of ∂K contains at most 2 points. Then*

$$g_K(n, d) \geq Cn^{\frac{1}{d-\frac{1}{2}}}. \tag{2.1}$$

We assume Theorem 2.2 for a moment and prove Theorem 2.3 in the two-dimensional case. The extension to higher dimensions is outlined in Exercise 2.2.

Let S be a set of n points in $\mathbb{R}^2$ (or a boundary of a convex set in $\mathbb{R}^3$. Let L be the set $\{x + \partial RK : x \in S\}$. By Theorem 2.2, the number of incidences between points in S and curves in L is at most $Cn^{\frac{4}{3}}$. It follows that a single distance cannot repeat more than $Cn^{\frac{4}{3}}$ times, so the number of distinct distances is at least $Cn^{\frac{2}{3}}$.

Before we turn to the proof of Theorem 2.2, we observe that so far our approach to estimating the number of distinct distances has been to show that a single distance cannot repeat very many times. In fact, the following stronger version of the Erdos Distance Conjecture has been posed.

Definition 2.3. We say that a convex curve in $\mathbb{R}^2$ is strictly convex if it does not contain any straight line segments.

Single Distance Conjecture (SDC). *Let S be a subset of $\mathbb{R}^2$ containing n points. Then for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that*

$$\#\{(x,y) \in S \times S : |x-y| = R\} \leq C_\epsilon n^{1+\epsilon}, \tag{2.2}$$

where $|\cdot|$ denotes the standard Euclidean distance.

We shall see that SDC is false if we replace the Euclidean distance by an arbitrary distance that comes from a bounded symmetric convex set whose boundary has everywhere non-vanishing curvature. See Exercise 2.3.

The best known result in the direction of solving the Single Distance Conjecture is the one implied by Theorem 2.2, namely that the left hand side of (2.2) is bounded by $Cn^{\frac{4}{3}}$. This implies, as we note above, that the number of Euclidean distances determined by n points in the plane is at least $Cn^{\frac{2}{3}}$. We shall see that this latter estimate has been strengthened considerably, while the estimate on the number of repetitions of a single distance has so far resisted efforts to improve it.

We now turn to the proof of Theorem 2.2.

Definition 2.4. A graph G with n vertiices and e edges is a set G of cardinality n equipped with a map $E : G \times G \to \{0,1\}$. If $E(x,y) = 1$, we say that x and y are connected by an edge.

Definition 2.5. A planar drawing of a graph G is a drawing in the plane where elements of G are represented by points (called vertices) in the plane, and edges, connecting pairs of points, are represented by piece-wise C^1 curves.

Definition 2.6. A crossing of a planar drawing of a graph is an intersection of two edges not at a vertex. The crossing number of a graph G (denoted by $cr(G)$) is the minimum of crossings over all the planar drawings of G.

Definition 2.7. We say that a graph G is planar if it can be drawn in the plane without any crossings.

Definition 2.8. We say that a graph G is connected if one can reach every vertex from any other vertex by following edges in some planar drawing of G.

Definition 2.9. We say that a graph G is simple if every pair of vertices is connected by at most 1 edge.

Definition 2.10. The maximum edge multiplicity of a graph with finitely many vertices and edges is the positive integer m such that every pair of vertices is connected by at most m vertices.

We shall deduce Theorem 2.2 from the following graph theoretic estimate.

Lemma 2.4. *Let G be a connected graph with n vertices and e edges. Suppose that $e \geq 4n$. Suppose that the maximum edge multiplicity of G is M. Then*

$$cr(G) \geq C\frac{e^3}{Mn^2}. \tag{2.3}$$

We shall prove Lemma 2.4 in the case $M = 1$. The reader is then asked to recover the general case in Exercise 2.3 below.

We need the following fact whose proof is outlined in the Exercise 2.4. Let H be a planar graph with n vertices and e edges. Then

$$e \leq 3n - 6. \tag{2.4}$$

It follows that if G is any connected graph, then

$$cr(G) \geq e - 3n. \tag{2.5}$$

If the last claim is not transparent, play the following game with an "oracle". You give the oracle a connected graph and ask him if it is planar. The oracle responds by asking you if (2.4) holds since it is after all a criterion for planarity. If the answer is no, there must be a crossing which can only be removed by removing an edge. This game must be played until the graph is planar, at which point at least $e - 3n$ edges have been removed.

Now let H be a random graph constructed by choosing each vertex of G (here G is from the statement of Lemma 2.4) with probability p. Keep an edge if both vertices are kept. It follows that the expected number of vertices is np, the expected number of edges is ep^2, and the expected value of the crossing number of H is at most $cr(G)p^4$. Since expectation is linear, it follows from (2.5) that

$$cr(G) \geq \frac{e}{p^2} - \frac{3n}{p^3}. \tag{2.6}$$

Lemma 2.4 follows by choosing $p = \frac{e}{4n}$, as we may since $e \geq 4n$ by assumption. For those not familiar with expectations, just think of the expected value of the number of vertices in H, for instance, as the average number of vertices in H over all possible outcomes of the coin flip with probability p. More precisely, for each possible subgraph of G, count the number of vertices and multiply the resulting number by the probability that this given subgraph arises if the vertices are kept with probability p. The sum of all these numbers is the expected value of the number of vertices in H. The expected number of edges and the expected value of the crossing number is computed in the same way.

We are now ready to prove Theorem 2.2, which we do in the case $\alpha = \beta = 1$. The reader is asked to carry out the details of the proof of the general case in Exercise 2.3 below. Let the elements of S be the vertices of G. Connect two vertices by and edge if the corresponding points are consecutive on some curve in L. Then $e = I - m$ (think about it...). If $e < 4n$, then $I < m + 4n$ and we are done. If $e \geq 4n$, then Lemma 2.4 kicks in and we have $cr(G) \geq C\frac{e^3}{n^2} \geq C'\frac{(I-m)^3}{n^2}$. On the other hand, an incidence cannot in our graph unless two lines from our collection intersect. It follows that $cr(G) \leq m^2$. Combining the two estimates yields Theorem 2.2 in the case $\alpha = \beta = 1$.

Exercise 2.1. *Prove that Theorem 2.2 is sharp. In the case $n = m$, suppose that $n = 4k^3$ for a positive integer k. Let $S = \{(i, j) : 0 \le i \le k-1; 0 \le j \le 4k^2 - 1\}$. Let L be the set of lines with equation $y = ax + b$, $a = 0, 1, \ldots, 2k-1$, $b = 0, 1, \ldots, 2k^2 - 1$. Check that there is n points and n lines, and the number of incidences is $\approx n^{\frac{4}{3}}$. Generalize this example to arbitrary n and m.*

Exercise 2.2. *Carry out the details of the induction on dimension in the proof of Theorem 2.3. You should end up with the recursion $\frac{1}{\alpha_d} - \frac{1}{\alpha_{d-1}} = 1$, where the number of distinct distances in d dimensions determined by N points is at least N^{α_d}.*

Exercise 2.3. *Carry out the details of proof of Lemma 2.4 for the general multiplicty m. Use this result to deduce the general version of Theorem 2.2*

Exercise 2.4. *Construct a convex curve Γ, of diameter $\approx N^3$, symmetric with respect to the origin, with everywhere non-vanishing curvature, such that $\#\Gamma \cap \mathbb{Z}^2 \approx N^2$. Let K denote the interior of Γ. Show that there exists a set S of cardinality n such that $\#\{(x, y) \in S \times S : ||x - y||_K = 1\} \approx n^{\frac{4}{3}}$. Conclude that the Single Distance Conjecture is in general false for convex sets K whose boundary has everywhere non-vanishing Gaussian curvature.*

Exercise 2.5. *Prove* (2.4), *i.e that if H is a connected planar graph with n vertices and e edges, then $e \le 3n - 6$. First prove by induction that $n - e + f = 2$, where f is the number of faces. Here a face is defined as a connected component of a drawing of H, viewed as a subset of the two-dimensional sphere. Then prove that $3f \le 2e$. Conclude that $e \le 3n - 6$, as advertised.*

Exercise 2.6. *Let K be a convex domain in $\mathbb{R}^d$, $d \ge 2$. We say that K is spectral if $L^2(K)$ has an orthogonal basis of the form $\{e(x \cdot a)\}_{a \in A}$, with $\int_K e(x \cdot (a - a'))dx = 0$ for $a \ne a'$, where A is a subset of $\mathbb{R}^d$ and $e(t) = e^{2\pi i t}$.*

Prove that B_d is not spectral by following the following outline. Let A be the putative spectrum of B_d. First prove that A is separated, i.e there exists $c > 0$ such that $|a - a'| \ge c$ for all $a \ne a'$. Then prove that A is well-distributed in the sense that there exists $r_0 > 0$ such that every cube of side-length r_0 contains at least one point from A. Now consider a ball of radius R in $\mathbb{R}^d$. By well-distributivity this ball contains $\approx R^d$ points of A. Denote this subset of A by A_R. On the other hand $\int_{B_d} e(x \cdot (a - a'))dx = C_d |a - a'|^{-\frac{d-2}{2}} J_{\frac{d}{2}}(2\pi |a - a'|)$, where $J_{\frac{d}{2}}$ denotes the Bessel function of order $\frac{d}{2}$. Use properties of Bessel functions and this formula to see that there exists an absolute constant $C > 0$ such that $\#\Delta_{B_d}(A_R) \le CR$. Use Theorem 2.3 to derive a contradiction. Can you derive a contradiction by using Theorem 1.2 instead? Calculate all the constants carefully before you answer.

Exercise 2.7. *Let P be a convex polygon in the plane whose vertices have integer coordinates. Use the Szemeredi-Trotter incidence theorem to prove that the number of vertices of P is bounded from below by a constant times the cube root of the area of P.*

Chapter III: In the realm of the Euclidean metric?

Let us briefly summarize what happened in the previous two chapters. We first proved that for any distance induced by a bounded symmetric convex set K, $g_K(n,d) \geq Cn^{\frac{1}{d}}$ and one cannot in general do better. However, in the following chapter we saw that if the boundary of K is sufficiently well "curved" in the sense of intersection properties, then the estimate for $g_K(n,d)$ improves considerably.

In this chapter we shall obtain better estimates on $g_K(n,d)$ under the assumption that $K = B_d$. At the end of the chapter we shall take a small step towards extending this result to other metrics by introducing some basic technology from algebraic geometry.

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