MATH 238: EXAM #1

ALEX IOSEVICH

Problem #1: Let A be a finite subset of a square of side-length R in the plane. Suppose that smallest distance between any two points in A is $\sqrt{R}$. Prove that there exists a fixed positive constant C such that

$$\#A \leq CR.$$

Suggestion: Do not try to come up with the best possible constant C. Go for "good enough" :). I still want an explicit constant though...

Problem #2: Prove that if E is a finite subset of $\mathbb{R}^2$ of size n, then there exists a fixed positive constant C such that

$$\#\Delta(E) \geq Cn^{\frac{2}{3}}.$$

Problem #3: Let M be an n by n matrix of 1s and 0s with the following property. Suppose that in a given row, 1s occur in the jth and j'th slot, for some $j \neq j'$. Then 1s may occur in the jth and j'th slot in at most 7 other rows. Prove that there exists a fixed positive constant C such that the number of 1s in M does not exceed $Cn^{\frac{3}{2}}$.

Problem #4: i) State (do not prove) the Szemeredi-Trotter incidence theorem.

ii) Let E be a finite subset of $\mathbb{R}^2$ of size n. Let

$$\mathcal{A}(E) = \{area(x, y, \vec{0}) : x, y \in E\},$$

where $area(x, y, z)$ denotes the area of the triangle in the plane with vertices x, y, z, and $\vec{0} = (0, 0)$ is the origin.

Prove that there exists a fixed positive constant C such that

$$\#\mathcal{A}(E) \geq Cn^{\frac{2}{3}}.$$

Hint: Write down the expression for $area(x, y, \vec{0})$ in terms of x and y, set it equal to some value $t \neq 0$, fix x, and ask yourself what geometric object is described by the resulting equation. Then use part i) to argue that the number of pairs $(x, y) \in E \times E$ that determine a fixed area, is bounded above by something appropriate. The pigeon-hole principle should allow you to finish the proof at this point.

Problem #5: Let E be a finite subset of $\mathbb{R}^4$. Let $x = (x_1, x_2, x_3, x_4)$ and define $\pi_1(x) = (x_2, x_3, x_4), \pi_2(x) = (x_1, x_3, x_4), \pi_3(x) = (x_1, x_2, x_4), \pi_4(x) = (x_1, x_2, x_3)$.

Prove that

$$(\#E)^3 \leq \#\pi_1(E) \cdot \#\pi_2(E) \cdot \#\pi_3(E) \cdot \#\pi_4(E).$$

Hint: This is not the only way to approach the problem, but you may, if you wish, use Holder's inequality without proof, i.e if $a_i, b_i \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\sum_i a_i b_i \leq \left(\sum_i a_i^p \right)^{\frac{1}{p}} \cdot \left(\sum_i b_i^{p'} \right)^{\frac{1}{p'}}.$$