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A mapping of the form $S(z) = \frac{az+b}{cz+d}$ is called a linear fractional transformation.

If a, b, c, d satisfy $ad - bc \neq 0$, then $S(z)$ is called a Mobius transformation.

$S^{-1}(z) = \frac{dz-b}{-cz+a}$, $S^{-1} \circ S = S \circ S^{-1} = \text{identity}$

Mobius

Note that $S(z) = \frac{(\lambda a)z + \lambda b}{(\lambda c)z + \lambda d}$, so the coefficients are not unique.

$z \to z+a$ translation $\quad z \to e^{i\theta} z$ rotation

$z \to az$ dilation $\quad z \to z^{-1}$ inversion

Proposition: If S is a Mobius transformation, then S is a composition of a translation, dilation, and ~~an~~ inversion.

Proof: Let $c = 0$. Hence $S(z) = \frac{a}{d} z + \frac{b}{d}$, so if $S_2(z) = z + \frac{b}{d}$, $S_1(z) = \frac{a}{d} z$, then $S_2 \circ S_1 = S$ ✓

If $c \neq 0$, let $S_1(z) = z + \frac{d}{c}$, $S_2(z) = \frac{1}{z}$, $S_3(z) = \frac{bc-ad}{c^2} z$

$S_4(z) = z + \frac{a}{c}$. Then $S = S_4 \circ S_3 \circ S_2 \circ S_1$.

How did somebody know to do this? Experimentation...

Suppose $S(z) = z$ (fixed points are useful)

Then $az + b = z(cz+d)$

$cz^2 + dz - a\cancel{z} - b = 0$

$= cz^2 + (d-a)z - b = 0.$

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Solving the quadratic yields at most two values for $S(z)=z$. Indeed, if

$$cz^2+(d-a)z-b=0,$$

$$c\left(z^2+\frac{d-a}{c}z-\frac{b}{c}\right)=0 \quad \text{unless} \quad c=0.$$

If $c=0$, we are left w/ $(d-a)z=b$, i.e

$$z=\frac{b}{d-a}, \text{ unless } d=a$$

If $d=a$, then $b=0$ & we get $S(z)=z$. Returning to $c\neq 0$,

$$z^2+\frac{d-a}{c}z-\frac{b}{c}=\left(z+\frac{d-a}{2c}\right)^2-\frac{(d-a)^2}{4c^2}-\frac{b}{c}=0,$$

$$\text{i.e } \left(z+\frac{d-a}{2c}\right)^2=\frac{(d-a)^2}{4c^2}+\frac{b}{c}, \text{ which}$$

yields at most two solutions.

Let S be a Mobius transformation w/ $\alpha=S(a)$, $\beta=S(b)$, $\gamma=S(c)$ (a,b,c distinct). Suppose that T satisfies the same property. Then $T^{-1}\circ S$ has a,b,c as fixed points, so $S=T$.

It follows that a Mobius map is determined by its action on any three points in $\mathbb{C}_\infty$.

Let $z_2,z_3,z_4\in\mathbb{C}_\infty$. If $z_2,z_3,z_4\in\mathbb{C}$,

$$S(z)=\left(\frac{z-z_3}{z-z_4}\right)\Big/\left(\frac{z_2-z_3}{z_2-z_4}\right)$$

$S(z) = \dfrac{z - z_3}{z - z_4}$ if $z_2 = \infty$

$S(z) = \dfrac{z_2 - z_4}{z - z_4}$ if $z_3 = \infty$ $\Longrightarrow$ $S(z_2) = 1$

$S(z_3) = 0$

$S(z) = \dfrac{z - z_3}{z_2 - z_3}$ if $z_4 = \infty$ $\quad S(z_4) = \infty$.

Definition: If $z_1 \in \mathbb{C}_\infty$ then $(z_1, z_2, z_3, z_4) \equiv$ image of z_1 under the unique Mobius transformation that takes $z_2 \to 1$, $z_3 \to 0$, $z_4 \to \infty$

called cross-ratio

Example: $z_2 = 0 \quad z_3 = 1 \quad z_4 = i$

$$S(z) = \left(\frac{z-1}{z-i}\right)\left(\frac{-1}{-i}\right) = -i\left(\frac{z-1}{z-i}\right)$$

Example: $(z_2, z_2, z_3, z_4) = 1$, $(z, 1, 0, \infty) = z$.

Proposition: If z_2, z_3, z_4 are distinct and T is Mobius, then

$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$ for any $\underline{\underline{z_1}}$.

Proof: Let $Sz = (z, z_2, z_3, z_4) \longrightarrow$ Mobius

If $M = ST^{-1}$, then $M(Tz_2) = ST^{-1}(Tz_2) = Sz_2 = 1$

Similarly, $M(Tz_3) = 0$ & $M(Tz_4) = \infty$. It follows that $ST^{-1}z = (z, Tz_2, Tz_3, Tz_4)$ $\forall z \in \mathbb{C}_\infty$

If $z = Tz_1$, we see that

common theme!

$Sz_1 = (Tz_1, Tz_2, Tz_3, Tz_4)$

$\shortparallel$

(z_1, z_2, z_3, z_4) & we are done.

Proposition: If z_2, z_3, z_4 are distinct points in $\mathbb{C}_\infty$ & w_2, w_3, w_4 are also distinct points in $\mathbb{C}_\infty$, then $\exists$ unique S Mobius $\ni$

$$Sz_2 = \omega_2, \quad Sz_3 = \omega_3, \quad Sz_4 = \omega_4$$

Proof: Let $Tz = (z, z_2, z_3, z_4)$, $Mz = (z, \omega_2, \omega_3, \omega_4)$ and put $S = M^{-1}T$

yet again!

The point is that $Sz_2 = \omega_2$, $Sz_3 = \omega_3$, $Sz_4 = \omega_4$

by construction!

If R is another such map, $R^{-1} \circ S$ has three fixed points, so $\underline{R = S}$.

Three points determine a circle. A circle in $\mathbb{C}_\infty$ passing through ∞ is a straight line in $\mathbb{C}$. Please review this if necessary.

Proposition: Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}_\infty$. Then $(z_1, z_2, z_3, z_4) \in \mathbb{R}$ iff z_j's lie on a circle.

Proof: Let $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be defined by $Sz = (z, z_2, z_3, z_4)$. Then $S^{-1}(\mathbb{R}) = \{ z : (z, z_2, z_3, z_4) \in \mathbb{R} \}$.

Therefore, it is enough to show that the image of $\mathbb{R}_\infty$ under a Mobius transformation is a circle. We proceed very directly.

Let $Sz = \dfrac{az+b}{cz+d}$, $z = x \in \mathbb{R}$ $\quad \omega = S^{-1}x \neq \infty$

$$\Longrightarrow S(\omega) = \overline{S(\omega)}$$

⑤

It follows that

$$\frac{a\omega+b}{c\omega+d} = \frac{\bar{a}\bar{\omega}+\bar{b}}{\bar{c}\bar{\omega}+\bar{d}} \implies$$

$$(a\bar{c}-\bar{a}c)|\omega|^2 + (a\bar{d}-\bar{b}c)\omega + (b\bar{c}-d\bar{a})\bar{\omega} + (b\bar{d}-\bar{b}d) = 0$$

Case 1: $a\bar{c} \in \mathbb{R} \implies a\bar{c}-\bar{a}c = 0$

Let $\alpha = 2(a\bar{d}-\bar{b}c)$; $\beta = i(b\bar{d}-\bar{b}d)$

$$\frac{\alpha}{2}\omega - \frac{\bar{\alpha}}{2}\bar{\omega} + \frac{\beta}{i} = 0$$

$$\frac{i}{2}\left(\alpha\omega - \overline{\alpha\omega}\right) + \beta = 0$$

$$\implies 0 = \operatorname{Im}(\alpha\omega) - \underbrace{\beta}_{\text{real}} = \operatorname{Im}(\alpha\omega - \beta)$$

This implies that $\omega \in$ line w/ α, β fixed.

If $a\bar{c} \notin \mathbb{R}$ (Case 2): we get

$$|\omega|^2 + \bar{\gamma}\omega + \gamma\bar{\omega} - \delta = 0 \quad \text{for some } \gamma, \delta \in \mathbb{R}$$

$$\implies |\omega+\gamma| = \left(|\gamma|^2+\delta\right)^{\frac{1}{2}}$$

$$= \left|\frac{ad-bc}{\bar{a}c-a\bar{c}}\right| \neq 0$$

& we have a circle!

Theorem: A Mobius transformation takes circles onto circles

Proof: Let Γ be any circle in $\mathbb{C}_\infty$ & let S be Mobius.

Let z_2, z_3, z_4 distinct on Γ & define $\omega_j = Sz_j$, $j = 2,3,4$.

$\Longrightarrow$ ω_j's determine a circle Γ'

Claim: $\Gamma' = S(\Gamma)$. By above,

$$(z, z_2, z_3, z_4) = (Sz, \omega_2, \omega_3, \omega_4).$$

If $z \in \Gamma$, then $LHS \in \mathbb{R} \Longrightarrow RHS \in \mathbb{R}$

$\Longrightarrow Sz \in \Gamma'$ ✓