Math 174 Homework 1 Solutions

Pages 4-5

Problem 3: Prove that $|x-y| \le |x| + |y|$. When does equality hold?

- Note that $(|x|+|y|)^2 = |x|^2 + |y|^2 + 2|x|\cdot|y| \geq |x|^2 + |y|^2 + 2xy = (x+y)^2 = |x+y|^2$
  $\Rightarrow |x|+|y| \geq |x+y|$
- $|x|+|y| = |x+y| \Rightarrow (|x|+|y|)^2 = (|x+y|)^2 \Rightarrow |x|^2 + |y|^2 + 2|x|\cdot|y| = x^2+y^2+2xy$

$\Rightarrow$ Equality holds iff $|xy| = xy$ $\Rightarrow$ if $x = \lambda y$ for $\lambda \le 0$.

Problem 6: Let f and g be integrable on $[a,b]$ (This is a special case of Hölder's inequality)

a) Prove that $\left|\int_a^b f\cdot g\right| \le \left(\int_a^b f^2\right)^{1/2} \cdot \left(\int_a^b g^2\right)^{1/2}$

- First, note that when f, g are continuous, $\left(\int_a^b f^2\right)^{1/2} = 0 \Rightarrow f \equiv 0$ and the answer follows trivially. If f is not continuous, it can only be nonzero on a measure zero set (a set of isolated points) in which case the inequality still follows trivially. Thus, we will assume $\left(\int_a^b f^2\right)^{1/2}$ and $\left(\int_a^b g^2\right)^{1/2} \neq 0$.

- Next, we have $(x-y)^2 = x^2+y^2-2xy > 0 \Rightarrow x^2+y^2 \geq 2xy$. Set

$$x = \frac{|f|}{\left(\int_a^b f^2\right)^{1/2}} \qquad y = \frac{|g|}{\left(\int_a^b g^2\right)^{1/2}}$$

Then

$$2 \cdot \frac{|f|}{\left(\int_a^b f^2\right)^{1/2}} \cdot \frac{|g|}{\left(\int_a^b g^2\right)^{1/2}} \le \frac{|f|^2}{\int_a^b f^2} + \frac{|g|^2}{\int_a^b g^2}$$

- Next, take the integral from a to b of both sides. Remember $\int_a^b f^2$ is just a constant. We have

$$* \quad \frac{\cancel{2}}{\left(\int_a^b f^2\right)^{1/2} \cdot \left(\int_a^b g^2\right)^{1/2}} \cdot \int_a^b |f|\cdot|g| \le (1+1) = \cancel{2} \Rightarrow \int_a^b |f|\cdot|g| \le \left(\int_a^b f^2\right)^{1/2} \cdot \left(\int_a^b g^2\right)^{1/2}$$

- Lastly, the triangle inequality tells us that $\left|\int_a^b f\cdot g\right| \le \int_a^b |f|\cdot|g|$ so we are done. ✓

b) If equality holds, must $f = \lambda g$?

- No. As remarked earlier, f may $\equiv 0$ on $[a,b]$ while $g(x) = \begin{cases} 0, & x \neq x_0 \\ 1, & x = x_0 \end{cases}$ for some $x_0 \in [a,b]$ we will still have zero on both sides.

What if f and g are continuous?

- Well, $x^2+y^2 = 2xy \Rightarrow x^2+y^2-2xy = 0 \Rightarrow (x-y)^2 = 0 \Rightarrow x = y$. Going back to our proof, we can see from $*$ that equality holds iff $x = y$ or in otherwords, if

$$\frac{|f|}{\left(\int_a^b f^2\right)^{1/2}} = \frac{|g|}{\left(\int_a^b g^2\right)^{1/2}}$$

In which case cleary $f = \lambda g$ with $\lambda = \pm \dfrac{\left(\int_a^b f^2\right)^{1/2}}{\left(\int_a^b g^2\right)^{1/2}}$ ✓

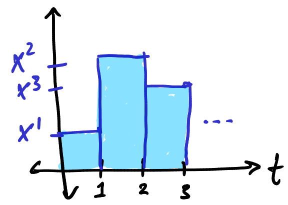
c) Show that theorem 1-1 (2) is a special case of (a).

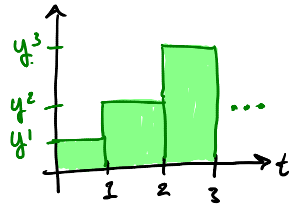
Theorem 1-1 (2) says: $\left|\sum_{i=1}^n x^i y^i\right| \le |x|\cdot|y|$ + equality holds iff $x = \lambda y$., $x, y \in \mathbb{R}^n$

a) Says: $\left|\int_a^b f\cdot g\right| \le \left(\int_a^b f^2\right)^{1/2} \cdot \left(\int_a^b g^2\right)^{1/2}$

Let $x = (x^1, x^2, \dots, x^n)$ and $y = (y^1, y^2, \dots, y^n)$. Then define $f(t) = x^i$ for $i = \lceil t \rceil$ and $g(t) = y^i$ for $i = \lceil t \rceil$. In other words $x(t)$ = the ith componen of x when $t \in [i, i+1)$.

Thus we have two step functions:





Since the steps each have a width of 1, the area of each block is just x^i or y^i.

$\Rightarrow \left(\int_0^n f(t)^2 dt\right)^{1/2} = \left(\sum_{i=1}^n (x^i)^2\right)^{1/2} = |x|$

+ similarly with $|y|$.

Of course $\left|\int_0^n f\cdot g\right| = \left|\sum_{i=0}^n x^i y^i\right|$. Thus part a) $\Rightarrow \left|\sum_{i=1}^n x^i y^i\right| \le |x|\cdot|y|$ ✓

Problem 7:

a) Prove that a linear transformation T is norm-preserving iff it is inner-product preserving.

- First assume $\langle Tx, Ty\rangle = \langle x, y\rangle$. Then $\langle Tx, Tx\rangle = \langle x, x\rangle \Rightarrow |Tx|^2 = |x|^2 \Rightarrow |Tx| = |x|$, so T also preserves norms.
- Next, assume $|Tx| = |x|$ + show $\langle Tx, Ty\rangle = \langle x, y\rangle$.

(This kind of thing is often called "polarization")

Notice that $|Tx+Ty|^2 - |Tx-Ty|^2 = \cancel{Tx^2} + \cancel{Ty^2} + 2Tx\cdot Ty - \cancel{Tx^2} - \cancel{Ty^2} + 2Tx\cdot Ty = 4Tx\cdot Ty$

$\Rightarrow \langle Tx, Ty\rangle = \frac{1}{4}|Tx+Ty|^2 - \frac{1}{4}|Tx-Ty|^2 = \frac{1}{4}|T(x+y)|^2 - \frac{1}{4}|T(x-y)|^2 = \frac{1}{4}|x+y|^2 - \frac{1}{4}|x-y|^2$

$= \frac{1}{4}\left[x^2+y^2+2x\cdot y - x^2 - y^2 + 2x\cdot y\right] = \langle x, y\rangle$ ✓

b) Prove that such a linear transformation is 1-1 w/ T^{-1} also 1-1.

- To show that T is injective, it is enough to show $\ker T = \{0\}$. Clearly $0 \in \ker T$. Suppose $x \ne 0 \in \ker T$. Then $|Tx| = |x| = 0$. Contradiction. $\Rightarrow$ T is injective.
- The rank-nullity theorem states that $\mathrm{Rank}(T) + \dim(\ker T) = \dim \mathbb{R}^n$ $\Rightarrow \mathrm{Rank}\, T = n \Rightarrow$ T is also surjective. Thus for every $y \in \mathbb{R}^n$, $\exists\, x$ such that $Tx = y$. In which case $x = T^{-1}y \Rightarrow |Tx| = |x| = |T^{-1}y| = |y| \Rightarrow T^{-1}$ also preserves norms + is therefore also injective for the same reasons T is. ✓

Problem 8: If $x, y \in \mathbb{R}^n$ are nonzero, $\angle(x,y) = \cos^{-1}\left(\frac{\langle x, y\rangle}{|x|\cdot|y|}\right)$

a) Prove that if T is norm-preserving it is also angle-preserving.

Well as in problem 7, if T is norm-preserving it also preserves inner products. So we have:

$$\angle(x,y) = \cos^{-1}\left(\frac{\langle x,y\rangle}{|x|\cdot|y|}\right) = \cos^{-1}\left(\frac{\langle Tx, Ty\rangle}{|Tx|\cdot|Ty|}\right) = \angle(Tx, Ty) \quad ✓$$

b) If there is a basis $x_1, \dots, x_n$ of $\mathbb{R}^n$ and numbers $\lambda_1, \dots, \lambda_n$ such that $Tx_i = \lambda_i x_i$, prove that T preserves angles iff all λ_i are equal.

- Suppose all the λ_i are equal to λ. Then $Tx = \lambda x \;\; \forall x \in \mathbb{R}^n$

$$\angle(Tx,Ty) = \cos^{-1}\left(\frac{\langle Tx,Ty\rangle}{|Tx|\cdot|Ty|}\right) = \cos^{-1}\left(\frac{\langle \lambda x,\lambda y\rangle}{|\lambda x|\cdot|\lambda y|}\right) = \cos^{-1}\left(\frac{\cancel{\lambda^2}\langle x,y\rangle}{\cancel{|\lambda|^2}|x|\cdot|y|}\right) = \angle(x,y) \Rightarrow T \text{ preserves angles.}$$

- Now suppose T preserves angles and $Tx_i = \lambda_i x_i$. Suppose $n \geq 2$ and choose x_i, x_j such that $\lambda_i \neq \lambda_j$. Then $Tx_i = \lambda_i x_i$ and $Tx_j = \lambda_j x_j$ are linearly independent vectors. (I.e. the line between their ends does not pass through the origin).

$\Rightarrow$ We may form two triangles: $\Delta_1 = \Delta(x_i, x_j, x_i - x_j)$ and $\Delta_2 = T\Delta_1 = \Delta(Tx_i, Tx_j, Tx_i - Tx_j)$

Since T preserves angles by assumption, $\Delta_1 \sim \Delta_2 \Rightarrow$ their side lengths have equal pairwise proportions. $\Rightarrow$ In particular, $\frac{x_i}{x_j} = \frac{Tx_i}{Tx_j} = \frac{\lambda_i x_i}{\lambda_j x_j} \Rightarrow \lambda_i = \lambda_j$. Since this holds for arbitrary $i \neq j$, all the λ_i's must be equal.

c) What are all the angle-preserving maps $T: \mathbb{R}^n \to \mathbb{R}^n$?

- Suppose T preserves angles and let $\{e_1, \dots, e_n\}$ be an orthonormal basis for $\mathbb{R}^n$. Then the set $\{f_i\} = \{Te_1, \dots, Te_n\}$ is an orthogonal basis. Geometrically, we can already see that T must represent a scaling: magnifying or shrinking the volume elements, perhaps with some reflections.
- More concretely, T must be represented by a matrix whose columns $= f_i$ are orthogonal.
- Next, suppose T is an orthogonal matrix. Then by linear algebra we know

$|Tv| = (Tv)^T(Tv) = v^T T^T T v = v^T v = \langle v,v\rangle = |v|$ (since $T^{-1} = T^T$) $\Rightarrow$ All orthogonal matrizes preserve lengths + hence also angles. $\Rightarrow$ Set of angle-preserving linear maps on $\mathbb{R}^n = O_n(\mathbb{R})$.

Problem 9: If $0 \leq \theta < \pi$, let $T: \mathbb{R}^2 \to \mathbb{R}^2$ have the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Show that T is angle-preserving and if $x \neq 0$, then $\angle(x, Tx) = \theta$.

- Let $v = (x,y)$. Then $Tv = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

$$\left.\begin{aligned} (x')^2 &= x^2\cos^2\theta + y^2\sin^2\theta + \cancel{2xy\cos\theta\sin\theta} \\ + \; (y')^2 &= x^2\sin^2\theta + y^2\cos^2\theta - \cancel{2xy\cos\theta\sin\theta} \end{aligned}\right\} \; |Tv| = |v| \Rightarrow T \text{ preserves norms + hence also angles.}$$

$$x^2 + y^2 = |v|^2$$

- We also have $\langle x, Tx\rangle = x^2\cos\theta + xy\sin\theta - xy\sin\theta + y^2\cos\theta = |v|^2\cos\theta$

$$\Rightarrow \angle(v,Tv) = \cos^{-1}\left(\frac{\langle v,Tv\rangle}{|v|\cdot|Tv|}\right) = \cos^{-1}\left(\frac{\cancel{|v|^2}\cos\theta}{\cancel{|v|^2}}\right) = \theta \quad \checkmark$$

Problem 10: If $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, show that there is a number M such that $|T(h)| \le M \cdot |h|$ for $h \in \mathbb{R}^m$. Hint: Estimate $|T(h)|$ in terms of $|h|$ & the entries in the matrix of T.

- T is an $n \times m$ matrix. Let $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$

$$\begin{array}{c} v_1 = \\ v_2 = \\ \\ v_n = \end{array}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & \ddots & & \vdots \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} v_1 \cdot x \\ v_2 \cdot x \\ \vdots \\ v_n \cdot x \end{pmatrix} = Tx \implies |Tx|^2 = (v_1 \cdot x)^2 + (v_2 \cdot x)^2 + \cdots + (v_n \cdot x)^2$$

$$= (a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1m}x_m)^2 + (a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m)^2 + \cdots$$

$$= (a_{11}^2 x_1^2 + a_{12}^2 x_2^2 + \overset{1}{\cdots} + a_{1m}^2 x_m^2) + (a_{21}^2 x_1^2 + \overset{2}{a_{21}^2} x_2^2 + \cdots + a_{2m}^2 x_m^2) + \cdots +$$

$$(a_{n1}^2 x_1^2 + \overset{m}{\cdots} + a_{nm}^2 x_m^2) + (\text{some other terms}).$$

Each of the lower order terms is clearly $\le A$.

- Choose $A = \max\{\text{first } m \text{ terms}\}$. Then

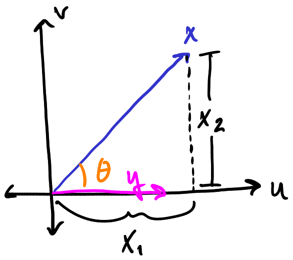
$\implies |Tx|^2 \le (\text{first } m \text{ terms}) + kA$

- Now we may factor out $\sum x_i^2$ by rearranging everything. What is left is a constant depending only on the $\{a_i\}$. ✓

Problem 13: Prove that if x & y are perpendicular ($\langle x,y\rangle = 0$) then $|x+y|^2 = |x|^2 + |y|^2$

Lemma: $\langle x,y\rangle = |x|\cdot|y|\cdot\cos\theta$, $\theta = \angle(x,y)$

Proof: Due to the fact that rotation matrices preserve lengths & angles, it is enough to consider vectors in $\mathbb{R}^2$ of the following form: $x = \langle x_1, x_2\rangle$, $y = \langle y_1, 0\rangle$. (If your vectors in $\mathbb{R}^n$ are not of this form, rotate them so that they both lie in a coordinate plane with y on a coordinate axis. We have:

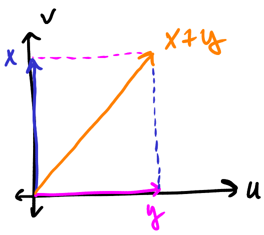


- By definition $\langle x,y\rangle = x_1 y_1$
- On the other hand, $|x| = (x_1^2 + x_2^2)^{1/2}$, $|y| = y_1$, and $\cos\theta = \frac{x_1}{|x|}$

$\implies |x|\cdot|y|\cdot\cos\theta = |x|\cdot y_1 \cdot \frac{x_1}{|x|} = y_1 x_1 = \langle x,y\rangle$ ✓

- Therefore for $x, y \neq 0$, $\langle x,y\rangle = |x|\cdot|y|\cdot\cos\theta = 0 \implies \cos\theta = 0 \implies \theta = \pi/2 + n\pi$, $n \in \mathbb{Z}$

$\implies$ the vectors x & y form a right triangle with side lengths $|x|$ & $|y|$. The vector $x+y$ has the same magnitude as the diagonal of this triangle. Rotated into a coordinate plane, this follows easily by inspection. ✓



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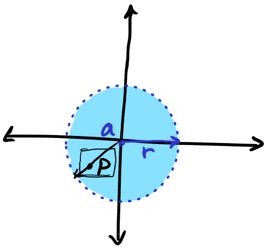
Problem 15: Prove that $S = \{x \in \mathbb{R}^n : |x-a| < r\}$ is open.

• We will use the definitions given in Spivak: A set $U \subset \mathbb{R}^n$ is called open if for each $x \in U$ there is an open rectangle A such that $x \in A \subset U$. ($A = (a_1,b_1) \times \cdots \times (a_n,b_n)$).

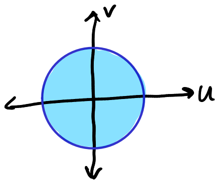


• Suppose that S is not open. Then $\exists\, p \in S$ such that p is not contained in an open rectangle $\subset S$. Without loss of generality, we may assume $a = 0$. Let $\vec{p} = (p_1, p_2, \dots, p_n)$. Define $\vec{d} = (d_1, d_2, \dots, d_n)$ such that $v = (p_1+d_1, \dots, p_n+d_n)$ has length $= r$. (also $\angle(p,d) = 0$). Then clearly $d \in S$ since $|v|^2 = |p|^2 + |d|^2 + 2\langle p,d\rangle = r^2 \Rightarrow |d| < r$.

• Define an open rectangle by the points $P_k = (P_1 \square d_1, P_2 \square d_2, \dots, P_n \square d_n)$. Either a plus or minus sign goes in the box. The collection $\{P_k\}$ 2^n elements and thus forms a cube of dimension n. Clearly it contains p and is contained in S since each vector P_k has $|P_k| < r$. $\Rightarrow S$ is open. ✓

Problem 16: Find the interior, exterior, and boundary of all the sets.

• $\{x \in \mathbb{R}^n : |x| \leq 1\}$



Interior: $\{x \in \mathbb{R}^n : |x| < 1\}$

Exterior: $\{x \in \mathbb{R}^n : |x| > 1\}$

Boundary: $\{x \in \mathbb{R}^n : |x| = 1\}$

• $\{x \in \mathbb{R}^n : |x| = 1\}$ This set $= \partial$ of previous set. $\Rightarrow$ int is empty. Boundary = itself.

Exterior $= \{x \in \mathbb{R}^n : |x| < 1\} \cup \{x \in \mathbb{R}^n : |x| > 1\}$.

• $\{x \in \mathbb{R}^n : \text{each } x^i \in \mathbb{Q}\} = S$

Interior: $\emptyset$ because any open rectangle contains both rational and irrational points

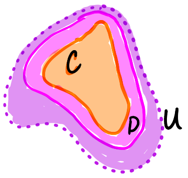
Exterior: $\emptyset$ same reason. S is a <u>dense</u> set

Boundary: $\partial S = S$

Problem 19: If A is a closed set that contains every rational number $r \in [0,1]$ show that $[0,1] \subset A$.

• Numbers are either rational or irrational. Suppose $x \in [0,1]$ is irrational. Then it may be approximated by rational numbers. So every neighborhood of x contains a number $r \in A$ + a number not in A (i.e. x). $\Rightarrow x \in \partial A \Rightarrow x \in A$ since closed sets must contain their boundary. (Maybe you guys should prove that closed sets contain their boundary. See the beginning of page 7. The answer is essentially there).

Problem 22: If U is open and $C \subset U$ is compact, show that there is a compact set D such that $C \subset \text{int}\, D$ and $D \subset U$.



- Let $d = \min\{|x-y| \mid x \in C, y \in \partial U\}$
- Let $O = \{U_x \mid U_x$ is an open ball of radius $d/2$ and $x \in U_x\}$
- Let $D = \bigcup_{x \in C} \overline{U_x}$ be the union of the closures of U_x. Then D is closed + bounded + compact with $C \subset \text{int}\, D$.

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Problem 24: Prove that $f: A \to \mathbb{R}^m$ is continuous at a iff each f^i is.

- f is continuous $\Leftrightarrow \lim_{x \to a} f(x) = f(a)$
- $\lim_{x \to a} f(x) = f(a) \Leftrightarrow \lim_{x \to a} f^i(x) = f^i(a) \;\; \forall i \Leftrightarrow$ each f^i is continuous. ✓

Problem 25: Prove that a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is continuous. Hint: Use 1-10.

Problem 1-10 tells us there $\exists$ M such that $|T(h)| \le |h| \cdot M \Rightarrow$ if $x \in \mathbb{R}^n$, we have

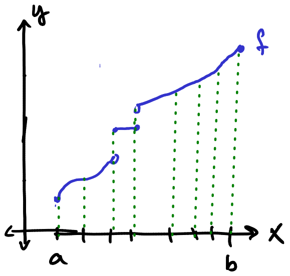
$$|Th - Tx| = |T(h-x)| \le |h-x| \cdot M$$

$\Rightarrow$ For every $\varepsilon > 0$, let $\delta = \varepsilon/M$. Then $|Th - Tx| < \varepsilon$ for h, x, $0 < |h-x| < \delta = \varepsilon/M$

$\Rightarrow \lim_{h \to x} Th = Tx \Rightarrow T$ is continuous ✓

Problem 30: Let $f: [a,b] \to \mathbb{R}$ be an increasing function. If $x_1, \dots, x_n \in [a,b]$ are distinct, show that $\sum_{i=1}^{n} o(f, x_i) < f(b) - f(a)$

Recall: $o(f,a)$ is the "oscillation" of f at $a = \lim_{\delta \to 0} [M(a,f,\delta) - m(a,f,\delta)]$ where

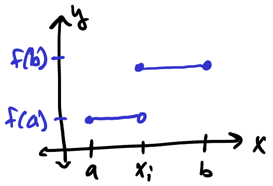


$M(a,f,\delta) = \sup\{f(x) \mid x \in A$ and $|x-a| < \delta\}$

$m(a,f,\delta) = \inf\{f(x) \mid x \in A$ and $|x-a| < \delta\}$

- This question is kind of obvious by inspection. If a function is increasing, $f(x_i) < f(x_j)$, $x_i < x_j$. Thus if f is continuous, $o(f, x_i) = 0$ since $M \to m$ as $\delta \to 0$ $\forall p \in [a,b]$. If f is discontinuous, will simply be the size of the "gap" left by the discontinuity. At a discontinuity x_i, $o(f, x_i) = \lim_{x \to x_i^-} f(x) - \lim_{x \to x_i^+} f(x) = f(x_i)^- - f(x_i)^+$

- Suppose $\sum_{i=1}^{n} o(f, x_i) > f(b) - f(a)$. Then $\exists$ x_i with $f(x_i)^- - f(x_i)^+ > f(b) - f(a)$. Since $f(a)$ is the smallest value, this implies $f(x_i)^- - f(x_i)^+ > f(b) - f(x_i)^+ \Rightarrow f(x_i)^- > f(b) \Rightarrow$ contradiction.
- Suppose $\sum_{i=1}^{n} o(f, x_i) = f(b) - f(a)$. This could happen only if we have a function like:



Then $o(f,p) = 0$ for $p \in [a,b]$, $p \ne x_i$ and $o(f, x_i) = f(b) - f(a)$.

It is possible that Spivak meant "strictly increasing" instead of just increasing.

✓