1

Math 265H August 31, 2022

$\mathbb{Q}$ = field of rational numbers

$= \left\{ \frac{a}{b} : b \neq 0 \quad a, b \in \mathbb{Z} \right\}$

$\mathbb{Z}$ = integers

One gateway to analysis is the observation that the equation

$x^2 - 2 = 0$ has no rational solutions

Proof: Suppose that $x = \frac{a}{b}$, w/ a & b relatively prime $\ni x^2 = 2$.

Then $\left(\frac{a}{b}\right)^2 = 2$, so $a^2 = 2b^2$.

This implies that a^2 is even. Since a^2 is even, a is even also (why?), which means that a^2 is divisible by 4.

This means that b^2 is even, which implies that b is even. We conclude that a and b are not relatively prime, so we have a contradiction and the theorem is proved.

Theorem: Let $A = \{p \in \mathbb{Q} : p^2 < 2\}$

$B = \{p \in \mathbb{Q} : p^2 > 2\}$

Then A contains no largest number and B contains no smallest number.

Proof: Let $q = p - \frac{p^2 - 2}{p+2} = \frac{2p+2}{p+2}$, (i)

so $q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2}$ (ii)

If $p \in A$, $p^2 - 2 < 0$, so (i) $\Rightarrow q > p$

and (ii) $\Rightarrow q^2 < 2$

$\Rightarrow q \in A$

If $p \in B$, then $p^2 - 2 > 0$, so (i) $\Rightarrow 0 < q < p$

& (ii) $\Rightarrow q^2 > 2 \Rightarrow q \in B$.

The proof is complete.

A set, $x \in A$

x is an element of A

$\emptyset$ = empty set

$A \subset B$

A is contained in B

$A \subset B$ & $B \subset A \iff A = B$

Order: An order on a set S is a relation, denoted by $<$ $\ni$

i) $x < y$, $x = y$, or $y < x$

one of these is true

ii) $x, y, z \in S$, if $x < y$ & $y < z$,

then $x < z$.

<u>Upper</u> <u>Bound:</u> S ordered $\quad E \subset S$

If $\exists$ $\beta \in S \ni x \leq \beta$ for every $x \in E$,

there exists

we say that E is bounded above, β = upper bound.

4

Lower bounds are defined the same way.

<u>Least</u> <u>upper</u> <u>bound</u>:

S ordered set, $E \subset S$, E bounded above.

Suppose that $\exists\, \alpha \in S$ w/ following properties:

i) α is an upper bound of E

ii) If $\gamma < \alpha$, then γ <u>is</u> <u>not</u> an upper bound of E.

The α = least upper bound of E.

$$\underbrace{\alpha = \sup(E)}_{\text{notation}}$$

The greatest lower bound (infinum)

$\alpha = \inf E$ is defined the same way w/ respect to the lower bound.

Example: $\underbrace{\left\{ \frac{1}{n} : n = 1, 2, \quad \right\}}_{=\; E \,\subseteq\, \mathbb{R}}$

Claim: $0 = \inf(E)$

Proof: $0 < \frac{1}{n}$ for any $n = 1, 2, \ldots$,
so 0 is <u>a</u> lower bound.

Let $\gamma > 0$. Choose a positive integer n
$\ni \frac{1}{n} < \gamma$. This is accomplished
by taking $n > \frac{1}{\gamma}$, which is
always possible since integers
are not bounded above.

We conclude that $0 = \inf(E)$, as
claimed!

Another example: $A = \{p \in \mathbb{Q} : p^2 > 2\}$

We already saw that A does not have the ~~least~~ greatest lower bound in $\mathbb{Q}$.

think this through!!!

Definition: An ordered set S is said to have the least-upper-bound property if:

i) If $E \subset S$, $E \neq \emptyset$, E bounded above, then $\sup(E)$ exists in S.

Theorem: S ordered w/ least upper bound property, $B \subset S$, $B \neq \emptyset$, B bound below.

Let L = set of all lower bounds of B.

Then $\alpha = \sup L$ exists in S, and

$$\alpha = \inf B.$$

In particular, $\inf B$ exists in S.

⑦

Proof:

B bounded below $\Rightarrow L \neq \emptyset$

Every $x \in B$ is an upper bound for L (why?)

$\Rightarrow L$ is bounded above

$\Rightarrow \alpha = \sup L$ exists in S.

If $\gamma < \alpha$, then γ is not an upper bound for L, so $\gamma \notin B$.

It follows that $\alpha \leq x$ for every $x \in B$,

i.e. $\underline{\underline{\alpha \in L}}$.

If $\alpha < \beta$, $\beta \notin L$ since α is an upper bound for L.

Therefore, $\alpha \in L$, but $\beta \notin L$ if $\beta > \alpha$.

By definition, $\alpha = \inf B$, as desired.