Math 173, Fall 2022 October 3

Definition: A vector space consists of

1. a field F of scalars
2. a set V of objects, called vectors
3. A rule $\oplus$, called addition, which takes $\alpha, \beta \in V$ to $\alpha + \beta$ in V $\ni$
   a) $\alpha + \beta = \beta + \alpha$
   b) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
   c) $\exists!\, 0$ in $V \ni \alpha + 0 = \alpha \quad \forall \alpha \in V$
   d) For each $\alpha \in V$, $\exists -\alpha \in V \ni \alpha + (-\alpha) = 0$.
4. A rule $\cdot$ called scalar multiplication $\ni$

$$\underbrace{c}_{\text{scalar}}, \underbrace{\alpha}_{\text{vector}} \longrightarrow c\alpha \text{ in } V \ni$$

   a) $1\alpha = \alpha \quad \forall \alpha \in V$
   b) $(c_1 c_2)\alpha = c_1(c_2\alpha)$
   c) $c(\alpha + \beta) = c\alpha + c\beta$
   d) $(c_1 + c_2)\alpha = c_1\alpha + c_2\alpha$

②

Example: $V = F^n = \{\alpha = (x_1, x_2, \dots, x_n) : x_i \in F\}$

field

& scalars from F ✓

Example: $m \times n$ matrices over F

field

Example: S = non-empty set F = field

$V = \{\text{functions from } S \to F\}$

Example: polynomial functions w/ coefficients from F ~ field

Harder example: $V = \{$ sequences of real numbers $\ni \sum_{j=1}^{\infty} a_j^2 < \infty$ w/ $F = \mathbb{R}$ ~ real numbers

If $\{a_j\}$, $\{b_j\}$ are sequences,

$$\{a_j\}_{j=1}^{\infty} + \{b_j\}_{j=1}^{\infty} = \{a_j + b_j\}_{j=1}^{\infty}$$

We must have $\{a_j + b_j\}_{j=1}^{\infty} \in V$

Why is that true?

$$\sum_{j=1}^{\infty}(a_j+b_j)^2 = \sum_{j=1}^{\infty} a_j^2 + \sum_{j=1}^{\infty} b_j^2 + 2\sum_{j=1}^{\infty} a_j b_j$$

$$\sum_{j=1}^{\infty} a_j b_j \overset{\text{Cauchy-Schwarz}}{\leq} \left(\sum_{j=1}^{\infty} a_j^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{\infty} b_j^2\right)^{\frac{1}{2}}$$

proof:

$$A = \left(\sum_{j=1}^{\infty} a_j^2\right)^{\frac{1}{2}}$$

$$B = \left(\sum_{j=1}^{\infty} b_j^2\right)^{\frac{1}{2}}$$

We must prove that

$$\sum_{j=1}^{\infty} \frac{a_j}{A}\frac{b_j}{B} \leq 1 \qquad (*)$$

Note that $xy \leq \frac{x^2+y^2}{2}$ (complete the square)

so $$\sum_{j=1}^{\infty} \frac{a_j}{A}\frac{b_j}{B} \leq \frac{\frac{1}{2}\sum_{j=1}^{\infty} a_j^2}{A^2} + \frac{\frac{1}{2}\sum_{j=1}^{\infty} b_j^2}{B^2} = 1$$

④

It follows that

$$\sum_j (a_j+b_j)^2 \le \sum_j a_j^2 + \sum b_j^2$$

$$+2\left(\sum a_j^2\right)^{\frac{1}{2}}\left(\sum b_j^2\right)^{\frac{1}{2}} =$$

$$\left(\left(\sum a_j^2\right)^{\frac{1}{2}} + \left(\sum b_j^2\right)^{\frac{1}{2}}\right)^2 < \infty$$

Definition: $\beta \in V$ is a linear combination of $\alpha_1, \dots, \alpha_n \in V$ if $\exists$ $c_1, c_2, \dots, c_n \in F$ $\ni$

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n = \sum_{j=1}^{n} c_j\alpha_j$$

Subspaces: V = vector space over F.

A subspace of V is a subset W of V $\ni$ W is itself a vector space over F w/ same operations

Example: $V = \mathbb{R}^2$

$$W = \{(t,0) : t \in \mathbb{R}\}$$

$$W = \{(t,t) : t \in \mathbb{R}\}$$

Theorem 1: A non-empty subset W of $\bar{V}$ is a subspace of $\bar{V}$ iff $c\alpha + \beta \in W$ for all $\alpha, \beta \in W$, $c \in F$.

Proof: W = non-empty subset of $\bar{V}$ $\ni$ $c\alpha + \beta \in W$ whenever $\alpha, \beta \in W$ & $c \in F$.

Since $W \neq \emptyset$, $\exists\ \rho \in W$ and hence $(-1)\rho + \rho = 0 \in W$.

It follows that $\underset{F}{c}\underset{W}{\alpha} = c\alpha + 0 = c\alpha \in W$.

In particular, $(-1)\alpha = -\alpha \in W$. Finally, if $\alpha, \beta \in W$, then $\alpha + \beta = 1 \cdot \alpha + \beta \in W$, so W is a subspace of $\bar{V}$.

Example: $\bar{V}$ = space of $n \times n$ matrices over $\mathbb{C}$ (complex numbers)

W = space of symmetric matrices, i.e matrices where $A_{ij} = A_{ji}$ $\forall i,j \le n$

6

Example: A = $m \times n$ matrix over $\mathbb{R}$.

Let V = set of $n \times 1$ matrices over $\mathbb{R}$

W = set of X ∋ $AX = O$

$\sim$ subspace of V.

<u>Lemma:</u> A = $m \times n$ matrix over F (field) and B, C are $n \times p$ matrices over F.

Then $A(dB+C) = d(AB) + AC$

for each d in F.

Proof: $[A(dB+C)]_{ij} = \sum_k A_{ik}(dB+C)_{kj}$

$= \sum_k (d A_{ik} B_{kj} + A_{ik} C_{kj})$

$= d \sum_k A_{ik} B_{kj} + \sum_k A_{ik} C_{kj}$

$= [dAB + AC]_{ij}$ ✓

Theorem 2: Let V be a vector space over F. Then the intersection of any number of subspaces of V is a subspace of V.

not so easy to visualize in complicated cases.

Proof: Let $\{W_\alpha\}$ be a collection of subspaces; define $W = \bigcap_\alpha W_\alpha$.

Each W_α contains 0, so $0 \in W$.

If $\beta_1, \beta_2 \in W$, $\beta_1, \beta_2 \in W_\alpha$ for each α,
so $c\beta_1 + \beta_2 \in W_\alpha$ for each α,
hence $c\beta_1 + \beta_2 \in W$.

We are done by Theorem 1.

Definition: Let $\underbrace{S \subset V}_{\text{subset}}$.

The subspace spanned by S is the intersection of all subspaces of V that contain S.

Example: $V = \mathbb{R}^2$, $S = \{(1,0), (0,1)\}$

Any subspace that contains all vectors of the form $x_1(1,0) + x_2(0,1) = (x_1, x_2)$ w/ arbitrary $x_1, x_2 \in \mathbb{R}$. In other words, $\operatorname{span}(S) = \mathbb{R}^2$.

8

Theorem 3: The subspace spanned by a non-empty subset S of the vector space is the set of all linear combinations of elements of S.

proof: Let $W = \text{span}(S)$. Then each $\alpha = x_1\alpha_1 + x_2\alpha_2 + \dots + x_m\alpha_m \in W$, so W contains all the linear combinations of elements of S. The set of linear combinations of elements of S (denoted by L) contains S and is non-empty. If $\alpha, \beta \in L$, then $\alpha = x_1\alpha_1 + \dots + x_m\alpha_m$, $\beta = y_1\beta_1 + \dots + y_n\beta_n$, so for each $c \in F$,

$\alpha + \beta$ is also a linear combination of elements of S, hence belongs to L.

Thus L is a subspace of V and any subspace which contains S contains L.

It follows that $\text{span}(S) = L$. ✓