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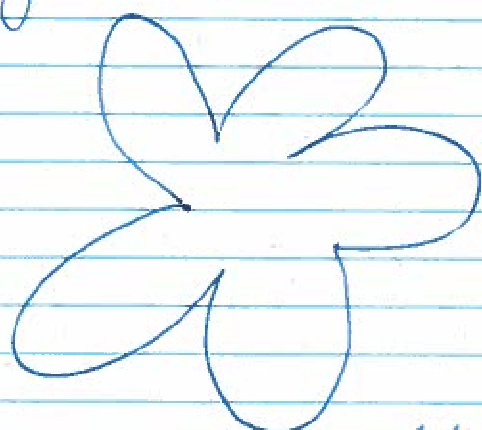
Definition: An open set G is simply connected if G is connected and every curve in G is homotopic to zero.

Cauchy's theorem (4th version) If G is simply connected then $\int_\gamma f = 0$ for every closed rectifiable curve and every analytic function f.

Example: Every star-shaped domain is simply connected.



Corollary: If G is simply connected and $f: G \longrightarrow \mathbb{C}$ analytic, in G then f has a primitive in G.

Proof: Fix $a \in G$ & let γ_1, γ_2 rectifiable curves in G from a to $z \in G$. Then

$$0 = \int_{\gamma_1 - \gamma_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f$$

by 4'th Cauchy theorem above.

goes from a to z along γ_1
& back along $-\gamma_2$.

Let $F(z) = \int_{\gamma_z} f$ from a to z Claim: F = primitive of f.

If $z_0 \in G$ & $r > 0 \ni B(z_0, r) \subset G$, let γ be a path from a to z.

For $z \in B(z_0, r)$, $\gamma_z = \gamma + [z_0, z]$, i.e. γ followed by a straight segment

It follows that $\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{[z_0, z]} f$

$\Longrightarrow F'(z_0) = f(z_0)$ by the proof of Morera's theorem.

It is natural to explore at this point whether one can define a reasonable branch of $\log f(z)$.

Corollary: Let G be simply connected and let $f: G \to \mathbb{C}$ analytic $\ni f(z) \neq 0$ for any $z \in G$. Then $\exists$ analytic function $g: G \to \mathbb{C} \ni f(z) = \exp g(z)$. If $z_0 \in G$ and $e^{w_0} = f(z_0)$, we may choose $g \ni g(z_0) = w_0$.

Proof: Since $f \neq 0$, $\frac{f'}{f}$ is analytic on G, so it must have a primitive g_1. If $h(z) = \exp g_1(z)$, h is analytic & never vanishes.

It follows that $\frac{f}{h}$ is analytic w/ derivative

$$\frac{h(z)f'(z) - h'(z)f(z)}{h^2(z)}$$

Since $h' = g_1' h = \frac{f'}{f} h \implies hf' - fh' = 0$

$\implies f/h$ is a constant.

In other words, $f(z) = c\exp g_1(z) = \exp(g_1(z) + c')$.

Let $g(z) = g_1(z) + c' + 2\pi i k$ for an appropriate k

& $g(z_0) = w_0$ (we can make sure of this)

$\implies$ result.

Definition: If G is an open set, then γ is homologous to zero, $\gamma \approx 0$, if $n(\gamma, w) = 0 \;\; \forall\, w \in \mathbb{C} - G$.

We proved in the previous lecture that $\gamma \sim 0 \implies \gamma \approx 0$.

Our next frontier is counting zeroes of analytic functions. This is one of the most natural applications of the Cauchy Integral Theorem.

Write $f(z) = (z-a_1)(z-a_2)\dots(z-a_m)g(z)$, $\underset{z \in G}{g(z) \neq 0}$

$$\frac{f'(z)}{f(z)} = \frac{1}{z-a_1} + \frac{1}{z-a_2} + \dots + \frac{1}{z-a_m} + \frac{g'(z)}{g(z)}$$

$z \neq a_1, a_2, \dots, a_m$

Theorem: G region; f analytic on G w/ zeroes $a_1, a_2, \ldots, a_m$ (repeated according to multiplicity). If γ is a closed rectifiable curve in G which does not pass through any point a_k and if $\gamma \approx 0$ then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{k=1}^{m} n(\gamma, a_k)$$

Proof: If $g(z) \neq 0$ for any $z \in G$, then $\frac{g'(z)}{g(z)}$ analytic in G; since $\gamma \approx 0$, Cauchy's theorem yields $\int_\gamma \frac{g'(z)}{g(z)}\, dz = 0$.

So we are done by the formula on the previous page.

Corollary: Let f, G & γ be as above except that $f(z) = \alpha$ at $a_1, a_2, \ldots, a_m$. Then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)-\alpha}\, dz = \sum_{k=1}^{m} n(\gamma, a_k)$$

Example: $\displaystyle\int_{|z|=2} \frac{(2z+1)}{z^2+z+1}\, dz = 4\pi i$

5

Theorem: Suppose that f is analytic in $B(a,R)$ and let $\alpha = f(a)$. If $f(z)-\alpha$ has a zero of order m at $z=a$, then $\exists\ \epsilon>0$ & $\delta>0$ $\ni$ if $|\zeta-\alpha|<\delta$, $f(z)=\zeta$ has exactly m simple roots in $B(a,\epsilon)$

roots of multiplicity $\underline{\underline{1}}$.

Proof: Since the zeroes of analytic functions are isolated, we can choose $\epsilon>0$ $\ni$ $\epsilon<\frac{1}{2}R$, $f(z)=\alpha$ has no solutions w/ $0<|z-a|<2\epsilon$ &

$$f'(z)\neq 0 \text{ if } 0<|z-a|<2\epsilon$$

(Note that if $m\geq 2$, $f'(a)=0$)

Let $\gamma(t)=a+\epsilon\exp(2\pi i t)$, $0\leq t\leq 1$ & define $\sigma=f\circ\gamma$.
We know that $\alpha\notin\{\sigma\}$, so $\exists\ \delta>0$ $\ni$

$$B(\alpha,\delta)\cap\{\sigma\}=\emptyset.$$

Thus $B(\alpha,\delta)$ is in the same component of $\mathbb{C}-\{\sigma\}$.
In other words, $|\alpha-\zeta|<\delta \Rightarrow n(\sigma,\alpha)=n(\sigma,\zeta)$

$$=\sum_{k=1}^{p} n(\gamma,z_k(\zeta)).$$

Since $n(\gamma,z)$ must be 0 or 1, there are exactly m solutions of the equation $f(z)=\zeta$ inside $B(a,\epsilon)$. Since $f'(z)\neq 0$ for $0<|z-a|<\epsilon$, each of these roots must be simple.

⑥

Open mapping theorem:

Let G be a region and suppose that f is non-constant analytic function on G. Then for any open set U in G, $f(U)$ is open.

Proof: Note that the previous theorem says, in particular, that $\exists\ \epsilon > 0$ & $\delta > 0 \ni B(a, \epsilon) \subset U$ &

$$f(B(a,\epsilon)) \supset B(\alpha, \delta).$$

In other words, we must show that for each $a \in U$, $\exists\ \delta > 0$ $\ni B(\alpha, \delta) \subset f(U)$, where $\alpha = f(a)$. Now just use the observation above to see that $\exists\ \epsilon > 0, \delta > 0$ w/

$$B(\alpha, \delta) \subset f(B(a, \epsilon)) \quad w/ \quad B(a,\epsilon) \subset U.$$

Corollary: Suppose that $f : G \to \mathbb{C}$ is 1-1, analytic and $f(G) = \Omega$. Then $f^{-1} : \Omega \to \mathbb{C}$ is analytic &

$$(f^{-1})'(\omega) = [f'(z)]^{-1} \quad w/ \quad \omega = f(z)$$

Proof: By the open mapping theorem, f^{-1} is continuous & Ω is open.

Since $z = f^{-1}(f(z))$, the result follows by calculus.