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Proposition: Let f be analytic in the disk $B(a,R)$ and suppose that γ is a rectifiable closed curve in $B(a,R)$. Then f has a primitive and hence

$$\int_\gamma f = 0.$$

Proof: We know that $f(z) = \sum a_n (z-a)^n$; $|z-a| < R$.

Let $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-a)^{n+1} =$

$$(z-a)\sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-a)^n$$

same radius of convergence (check)

So, we are done since $F'(z) = f(z)$ for $|z-a| < R$.

Definition: $f: G \xrightarrow{} \mathbb{C}$ (open) is analytic at $a \in G$ w/ $f(a) = 0$ has a zero of order or multiplicity $m \geq 1$ if $\exists\, g: G \to \mathbb{C}$ analytic $\ni$ $f(z) = (z-a)^m g(z)$, $g(a) \neq 0$.

Definition: An entire function is an analytic function in all of $\mathbb{C}$.

Proposition: If f is an entire function, then f has power series with infinite radius of convergence.

follows from the bounded case by expanding the radius

Liouville's theorem: If f is a bounded entire function, then f is constant.

Proof: By Cauchy's estimate,

$|f'(z)| \leq 1 \cdot \frac{M}{R}$ ← bound on $|f|$ for any R,

so $f'(z) \equiv 0 \Rightarrow f$ is constant.

Fundamental theorem of algebra: If $p(z)$ is a non-constant polynomial, then $\exists\, a \in \mathbb{C} \ni p(a) = 0$.

Proof: Suppose that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then

$[p(z)]^{-1}$ is entire.

But $[p(z)]^{-1}$ is bounded (why?), so $p(z) =$ constant. Contradiction!

Corollary: If $p(z)$ is a polynomial of degree n, then $p(z) = c(z-a_1)^{k_1} \ldots (z-a_m)^{k_m}$, w/ $k_1 + \ldots + k_m = n$

a_j's roots of p.

By above, given a non-constant polynomial $p(z)$ & $b \in \mathbb{C}$ $\exists\, a \in \mathbb{C} \ni p(a) = b$. This idea has limitations. For example, $e^z = 0$ never holds!

What is going on?

Theorem: Let G be a connected subset (open) and let $f: G \to \mathbb{C}$ be an analytic function. TFAE

a) $f \equiv 0$ b) $\exists\, a \in G \ni f^{(n)}(a) = 0 \quad \forall n \geq 0$

c) $\{z \in G : f(z) = 0\}$ has a limit point in G.

Proof: a) $\Rightarrow$ b), a) $\Rightarrow$ c) ☺

Suppose that c) holds. Let $a \in G$ limit point of

& $B(a,R) \subset G$ $\quad Z = \{z \in G : f(z) = 0\}$

Since f is continuous, $f(a) = 0$. Suppose that there is an integer $n \ni$

$$f(a) = f'(a) = \dots = f^{(n-1)}(a) = 0 \ \&$$
$$f^{(n)}(a) \neq 0.$$

We have $f(z) = \sum_{k=n}^{\infty} a_k (z-a)^k$ for $|z-a| < R$

Let $g(z) = \sum_{k=n}^{\infty} a_k (z-a)^{k-n} \Rightarrow g$ analytic in $B(a,R)$,

$f(z) = (z-a)^n g(z)$,

$a_n = g(a) \neq 0$

By continuity, $\exists\ 0 < r < R \ni g(z) \neq 0$ in $B(a,r)$.

But... since a is a limit point of Z, $\exists\, b$ w/ $f(b) = 0$ and $|a-b| < r \Rightarrow 0 = (b-a)^n g(b) \Rightarrow g(b) = 0$!

Contradiction!

Therefore c) $\Rightarrow$ b).

(4)

Let's now assume b). Let $A = \{z \in G : f^{(n)}(z) = 0 \ \forall n\}$ $\neq \emptyset$

A is closed by continuity.

Now suppose that $a \in A$ & $B(a,R) \subset G$. Then $f(z) = \sum a_n (z-a)^n$
$|z-a| < R$

$$a_n = \frac{1}{n!} f^{(n)}(a) = 0 \quad \forall n \geq 0$$

$$\Longrightarrow f(z) = 0 \ \forall z \in B(a,R)$$

$$\Longrightarrow B(a,R) \subset A$$

We conclude that $A = G$ & the proof is complete.

Corollary: If f & g are analytic on a region G, then $f \equiv g$ iff $\{z \in G : f(z) = g(z)\}$ has a limit point in G.

Corollary: If f is analytic on G open, $f \not\equiv 0$, then for each $a \in G$ w/ $f(a) = 0$, $\exists\, n \geq 1 \ni f(z) = (z-a)^n g(z)$ ← analytic
$g(a) \neq 0$.

Proof: Let n be the largest integer $\ni f^{(k)}(a) = 0$, $0 \leq k < n$ & define $g(z) = (z-a)^{-n} f(z)$, $z \neq a$ & $g(a) = \frac{f^{(n)}(a)}{n!}$.

Then g is analytic in $G - \{a\}$. To see that g is analytic in the neighborhood of a, just use the proof of c) ⟹ b) above.

Maximum modulus principle: If G is a region & $f: G \to \mathbb{C}$ is an analytic function & $\exists\, a \in G$ w/ $|f(a)| \geq |f(z)|$ $\forall z \in G$, then f is constant.

Proof: Let $\bar{B}(a,r) \subset G$, $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$

We have $f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-a}\, dw = \frac{1}{2\pi} \int_0^{2\pi} f(a+re^{it})\, dt$

$$\Rightarrow |f(a)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a+re^{it})|\, dt \leq |f(a)|$$

$$\Rightarrow 0 = \int \left[|f(a)| - |f(a+re^{it})| \right] dt$$

$$\Rightarrow |f(a)| = |f(a+re^{it})| \quad \forall t$$

We conclude that f maps any disk in $B(a,R)$ to the circle $|z| = |\alpha|$, $\alpha = f(a)$, which implies that f is constant on $B(a,R)$. why?

In particular, $f(z) = \alpha$ for $|z-a| < R \Rightarrow f \equiv \alpha$ by the previous corollary.

Proposition: If $\gamma: [0,1] \to \mathbb{C}$ closed rectifiable & $a \notin \{\gamma\}$, then $\frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$ is an integer.

Proof: Reduce to γ smooth using Lemma 1.19.

Let $g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s)-a}\, ds$; $g(0) = 0$ & $g(1) = \int_\gamma (z-a)^{-1} dz$.

⑥

$$g'(t) = \frac{\gamma'(t)}{\gamma(t) - a}, \quad 0 \le t \le 1$$

Then $\frac{d}{dt} e^{-g}(\gamma - a) = e^{-g}\gamma' - g' e^{-g}(\gamma - a)$

$$= e^{-g}\left(\gamma' - \gamma'(\gamma - a)^{-1}(\gamma - a)\right)$$

$$= 0 \Rightarrow e^{-g}(\gamma - a) = \text{const.}$$

$$e^{-g(0)}(\gamma(0) - a) = \gamma(0) - a = e^{-g(1)}(\gamma(1) - a)$$

So $\gamma(0) - a = e^{-g(1)}(\gamma(0) - a)$

$$\Rightarrow e^{-g(1)} = 1 \Rightarrow g(1) = 2\pi i k$$

Definition: If γ is a closed rectifiable curve in $\mathbb{C}$, then for $a \notin \{\gamma\}$, $n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma (z-a)^{-1}\, dz$

winding number of γ around $\underline{a}$.