Fourier analysis and geometric combinatorics

Alex Iosevich[*]

February 23, 2004

Abstract

This article is based on the series of lectures on the interaction of Fourier analysis and geometric combinatorics delivered by the author in Padova at the Minicorsi di Analisi Matematica in June, 2002.

This paper is based on the lectures the author gave in Padova at the Minicorsi di Analisi Matematica in June, 2002. The author wishes to thank the organizers, the participants, and the fellow lecturers for many interesting and useful remarks. The author also wishes to thank Georgiy Arutyunyants, Leonardo Colzani, Julia Garibaldi, Derrick Hart, and Bill McClain for many useful comments and suggestions about the content and style of the paper.

The main theme of this paper is an old and beautiful subject of geometric combinatorics. We will not even attempt to cover anything resembling a significant slice of this broad and influential discipline. See, for example, [23] for a thorough description of this subject. The purpose of this article is to describe Szekely's ([29]) beautiful and elementary proof of the Szemeredi-Trotter incidence theorem ([27]), a result that found a tremendous number of applications in combinatorics, analysis, and analytic number theory. We shall describe some of the consequences of this seminal result and its interaction with problems and techniques of Fourier analysis and additive number theory.

Definition 0.1. An incidence of a point and a line is a pair (p, l), where p is a point, l is a line, and p lies on l.

Theorem 0.2 (Szemeredi-Trotter). *Let I denote the number of incidences of a set of n points and m lines (or m strictly convex closed curves). Then*

$$I \lesssim n + m + (nm)^{\frac{2}{3}}, \tag{0.1}$$

where here and throughout the paper, $A \lesssim B$ means that there exists a positive constant C such that $A \leq CB$.

Quite often in applications, one uses the following "weighted" version the the Szemeredi-Trotter theorem due to L. Szekely ([29]).

Theorem 0.1'. *Given a set of n points and m simple (no self-intersections) curves in the plane, such that any two curves intersect in at most α points and any two points belong to at most β curves, the number of incidences is at most $C(\alpha\beta)^{\frac{1}{3}}(nm)^{\frac{2}{3}} + m + 5\beta n$.*

The probabilistic proof of Theorem 0.1 (due to Szekely) given below, can be modified (as it is done in [29], Theorem 8) to yield Theorem 0.1'. We outline these modifications at the end of Section I below where we also briefly describe some of the applications of weighted incidence theory to the theory of diophantine equations.

Corollary 0.3. *Let S be a subset of $\mathbb{R}^2$ of cardinality n. Let $\Delta(S) = \{|x-y| : x, y \in S\}$, where $|\cdot|$ denotes the Euclidean norm. Then*

$$\#\Delta(S) \gtrsim n^{\frac{2}{3}}. \tag{0.2}$$

This estimate is not sharp. It is conjectured to hold with the exponent 1 in place of $\frac{2}{3}$. For the best known exponents to date (around .86), see [24] and [25]. However, Corollary 0.2 is still quite useful as we shall see in the final section of this paper.

Corollary 0.4. *Let A be a subset of $\mathbb{R}$ of cardinality n. Then either $A+A = \{a + a' : a, a' \in A\}$ or $A \cdot A = \{aa' : a, a' \in A\}$ has cardinality $\gtrsim n^{\frac{5}{4}}$.*

This estimate has been recently improved in a number of ways by several authors. See, for example, [4] and references contained therein.

1 Proof of Theorem 0.1, Corollary 0.2 and Corollary 0.3

We shall deduce Theorem 0.1 from the following graph theoretic result due to Ajtai et al ([2]), and, independently, to Leighton. Note that for the purposes of this paper, a pair of vertices in a graph can be connected by at most one edge.

Definition 1.1. The crossing number of a graph, $cr(G)$, is the minimal number of crossings over all the possible drawings of this graph in the plane. A crossing is an intersection of two edges not at a vertex.

Definition 1.2. We say that a graph G is planar if there exists a drawing of G in the plane without any crossings.

Theorem 1.3. *Let G be a graph with n vertices and e edges. Suppose that $e \geq 4n$. Then*

$$cr(G) \gtrsim \frac{e^3}{n^2}. \tag{1.1}$$

Before proving Theorem 1.1, we show how it implies Theorem 0.1. Take the points in the statement of the Theorem as vertices of a graph. Connect two vertices with an edge if the two corresponding points are consecutive on some line. It follows that

$$e = I - m. \tag{1.2}$$

If $e < 4n$ we get $I < 4n + m$, which is fine with us. If $e \geq 4n$, we invoke Theorem 1.1 to see that

$$cr(G) \gtrsim \frac{e^3}{n^2} = \frac{(I-m)^3}{n^2}. \tag{1.3}$$

Combining (1.3) with the obvious estimate $cr(G) \leq m^2$, we complete the proof of Theorem 0.1. Observe that strictly speaking, we have only proved Theorem 0.1 for lines and points. In order to extend the argument to translates of the same strictly convex curve, one needs to replace (1.2) with an (easy) estimate $e \gtrsim I$.

We now turn our attention to the proof of Theorem 1.1. Let G be a planar graph with n vertices, e edges, and f faces. Euler's formula (proved by induction) says that

$$n - e + f = 2. \tag{1.4}$$

Combined with the observation that $3f \leq 2e$, we see that in such a planar graph

$$e \leq 3n - 6. \tag{1.5}$$

It follows that if G is any graph, then

$$cr(G) \geq e - 3n. \tag{1.6}$$

We now convert this linear estimate into the estimate we want by randomization. More precisely, let G be as in the statement of Theorem 1.1 and let H be a random subgraph of G formed by choosing each vertex with probability p to be chosen later. Naturally, we keep an edge if and only if both vertices survive the random selection. Let $\mathbb{E}()$ denote the usual expected value. An easy computation yields

$$\mathbb{E}(vertices) = np, \tag{1.7}$$
$$\mathbb{E}(edges) = ep^2, \tag{1.8}$$
$$\mathbb{E}(crossing\ number\ of\ H) \leq p^4 cr(G). \tag{1.9}$$

Observe that the inequality in (1.9) is due to the fact that the number of avoidable crossings in G may decrease once a smaller random subset is extracted.

It follows by linearity of expectation that

$$cr(G) \geq \frac{e}{p^2} - \frac{3n}{p^3}. \tag{1.10}$$

Choosing $p = \frac{4n}{e}$ we complete the proof of Theorem 1.1, and consequently of Theorem 0.1. Theorem 0.1' can be proved in a similar fashion. First one shows that under the assumption that any two vertices are connected by at most β edges, the conclusion of Theorem 1.1 becomes $cr(G) \gtrsim \frac{e^3}{\beta n^2}$. Then, in the application of this estimate to incidences, the upper bound on $cr(G)$

4

is no longer m^2, but rather $m^2\alpha$. Combining this with the trivial estimates yields the conclusion of Theorem 0.1'.

Further developments in weighted incidence theory have recently led to the following result proved in the case $d = 2$ by S. Konyagin ([21]), and by Iosevich, Rudnev and Ten ([19]) for $d > 2$.

Theorem 1.4. *Let $\{b_j\}_{j=1}^N$ denote a strictly convex sequence of real numbers (in the sense that vectors (j, b_j) lie on a strictly convex curve). Then the number of solutions of the equation*

$$b_{i_1} + \cdots + b_{i_d} = b_{j_1} + \cdots + b_{j_d} \tag{1.11}$$

is

$$\lesssim N^{2d-2+2^{-d}}. \tag{1.12}$$

Taking $b_j = j^2$, for example, shows that $N^{2d-2+2^{-d}}$ in (1.12) cannot be replaced by anything smaller than N^{2d-2}. We conjecture that $O(N^{2d-2})$ is the right estimate, up to logarithms, for any strictly convex sequence $\{b_j\}$.

2   Proof of Corollary 0.2

Draw a circle of fixed radius around each point in S. By Theorem 0.1, the number of incidences is $\lesssim n^{\frac{4}{3}}$. This means that a single distance cannot repeat more than $\approx n^{\frac{4}{3}}$ times. It follows that there must be at least $\approx n^{\frac{2}{3}}$ distinct distances since the total number of distances is $\approx n^2$. In other words, we just proved that $\Delta(S) \gtrsim n^{\frac{2}{3}}$ as promised.

2.1   Proof of Corollary 0.3

The choice of lines and points is less obvious here. Let $P = (A+A) \times (A \cdot A)$. Let L be the set of lines of the form $\{(ax, a' + x) : a, a' \in A\}$. We have

$$\#P = \#(A + A) \times \#(A \cdot A), \tag{2.1}$$
$$\#L = n^2, \tag{2.2}$$

while the number of incidences is clearly $n \times n^2 = n^3$. It follows that

$$n^3 \lesssim (\#P)^{\frac{2}{3}} n^{\frac{4}{3}}, \tag{2.3}$$

which means that

$$\#P \gtrsim n^{\frac{5}{2}}. \tag{2.4}$$

It follows that either $\#(A + A)$ or $\#(A \cdot A)$ exceeds a constant multiple of $n^{\frac{5}{4}}$. This completes the proof of Corollary 0.3.

3 Application to Fourier analysis

Definition 3.1. We say that a domain $\Omega \subset \mathbb{R}^d$ is spectral if $L^2(\Omega)$ has an orthogonal basis of the form $\{e^{2\pi i x \cdot a}\}_{a \in A}$.

The following result is due to Fuglede ([9]). It was also proved in higher dimensions by Iosevich, Katz and Pedersen ([13]).

Theorem 3.2. A disc, $D = \{x \in \mathbb{R}^2 : |x| \leq r\}$, is not spectral.

Proof of Theorem 2.1. Let A denote a putative spectrum. We need the following basic lemmas:

Lemma 3.3. *A is separated in the sense that there exists $c > 0$ such that $|a - a'| \geq c$ for all $a, a' \in A$.*

Lemma 3.4. *There exists $s > 0$ such that any square of side-length s contains at least one element of A.*

For a sharper version of Lemma 2.3 see [17].

The proof of Lemma 2.2 is straightforward. Orthogonality implies that

$$\int_D e^{2\pi i x \cdot (a - a')} dx = 0, \tag{3.1}$$

whenever $a \neq a' \in A$. Since $\int_D dx = 2\pi r$ and the function $\int_D e^{2\pi i x \cdot \xi} dx$ is continuous, the left hand side of (2.1) would have to be strictly positive if $|a - a'|$ were small enough. This implies that $|a - a'|$ can never be smaller than a positive constant depending on r.

The proof of Lemma 2.3 is a bit more interesting. By Bessel's inequality we have

$$\sum_A |\widehat{\chi}_D(\xi + a)|^2 \equiv |D|^2, \tag{3.2}$$

for almost every $\xi \in \mathbb{R}^d$, since the left hand side is a sum of squares of Fourier coefficients of the exponential with the frequency ξ with respect to the putative orthogonal basis $\{e^{2\pi i x \cdot a}\}_{a \in A}$. We have

$$\sum_{A_\xi} |\widehat{\chi}_D(a)|^2 = \sum_{A_\xi \cap Q_s} + \sum_{A_\xi \cap Q_s^c} = I + II, \tag{3.3}$$

where $A_\xi = A - \xi$ and Q_s is a square of side-length s centered at the origin.

We invoke the following basic fact. See, for example, [26]. We have

$$|\widehat{\chi}_D(\xi)| \lesssim |\xi|^{-\frac{3}{2}}. \tag{3.4}$$

It follows that

$$II \lesssim \sum_{A_\xi \cap Q_s^c} |a|^{-3} \lesssim s^{-1}. \tag{3.5}$$

Choosing s big enough so that $s^{-1} << |D|^2$, we see that $I \neq 0$, and, consequently, that $A_\xi \cap Q_s$ is not empty. This completes the proof of Lemma 2.3.

We are now ready to complete the proof of Theorem 2.1. Intersect A with a large disc of radius R. By Lemma 2.2 and Lemma 2.3, this disc contains $\approx R^2$ points of A. We need another basic fact about $\widehat{\chi}_D(\xi)$, that it is radial, and in fact equals, up to a constant, to $|\xi|^{-1} J_1(2\pi|\xi|)$, where J_1 is the Bessel function of order 1. We also need to know that zeros of Bessel functions are separated in the sense of Lemma 2.2. This fact is contained in any text on special functions. See also [30].

With this information in tow, recall that orthogonality implies that $|a - a'|$ is a zero of J_1. Since the largest distance in the disc of radius R is $2R$ and zeros of J_1 are separated, we see that the total number of distinct distances between the elements of A in the disc or radius R is at most $\approx R$. This is a contradiction since Corollary 0.2 says that R^2 points determine at least $R^{\frac{4}{3}}$ distinct distances. This completes the proof of Theorem 2.1. $\square$

It turns out that not only does $L^2(D)$ not possess an orthogonal basis of exponentials, the numbers of exponentials orthogonal with respect to D is in fact finite. This is a theorem due to Fuglede ([10]) which was extended to all sufficiently smooth well-curved symmetric convex domains by Iosevich and Rudnev ([18]). The latter paper is based on the generalization of the following beautiful geometric principle due to Erdos ([6]).

Theorem 3.5 (Erdos integer distance principle). *Let S be an infinite subset of $\mathbb{R}^d$ such that the distance between any pair of points in S is an integer. Then S is a subset of a line.*

More recently, the author's student, Georgiy Arutyunyants ([1]) found an interesting approach for obtaining a good explicit bound on the number of orthogonal exponentials with respect to the unit ball $B_d \subset \mathbb{R}^d$. His idea is based on the following beautiful observation. Erdos ([6]) noticed that given any positive integer N, one can construct a set of cardinality N in the plane such that all the pair-wise distances are integers. This fact and the prominence of the Erdos integer distance principle in the aforementioned paper of Iosevich and Rudnev proved to be a serious obstacle in obtaining a good upper bound on the number of orthogonal exponentials. However, Georgiy observed that in the orthogonality problem the distances do not only need to be integers, but in fact odd integers. He proceeded to prove that it is impossible construct a set in the plane of cardinality greater than three (with corresponding bounds in higher dimensions) such that all the pair-wise distances are odd integers. Georgiy is currently working on an asymptotic version of this combinatorial principle. This should enable him to obtain an explicit upper bound on the number of orthogonal exponentials with respect to the unit ball.

4   Applications to convex geometry

The following result is due to Andrews ([3]).

Theorem 4.1. *Let Q be a convex polygon with n integer vertices. Then $n \lesssim |Q|^{\frac{1}{3}}$.*

4.1   Proof of Theorem 3.1

Let $\mathcal{C}$ denote a strictly convex curve running through the vertices of Q. Let Ω denote the convex domain bounded by $\mathcal{C}$. Let L denote the set of strictly convex curves obtained by translating $\mathcal{C}$ by every lattice point inside Ω. Let P denote the set of lattice points contained in the union of all those translates. By Theorem 0.1 the number incidences between the elements of

P and elements of L is $\lesssim |\Omega|^{\frac{4}{3}}$ since $\#L \approx \#P \approx |\Omega|$. Since each translate of $\mathcal{C}$ contains exactly the same number of lattice points,

$$\#\mathcal{C} \cap \mathbb{Z}^2 \lesssim \frac{|\Omega|^{\frac{4}{3}}}{|\Omega|} = |\Omega|^{\frac{1}{3}}. \tag{4.1}$$

This completes the proof of Theorem 3.1. Observe that proof implies the following (easier) estimate.

Lemma 4.2. *Let Γ be a closed strictly convex curve in the plane. Then*

$$\#\{R\Gamma \cap \mathbb{Z}^2\} \lesssim R^{\frac{2}{3}}. \tag{4.2}$$

What sort of an incidence theorem would be required to prove a more general version of this result?

Definition 4.3. We say that $A \subset \mathbb{R}^d$ is well-distributed if the conclusions of Lemma 2.2 and Lemma 2.3 hold for A.

Let A be a well-distributed set, and let A_R denote the intersection of A and the ball of radius R centered at the origin. Observe that $\#A_R \approx R^d$. Let U be a strictly convex hyper-surface contained in the unit ball. Suppose we had a theorem which said that the number of incidences between A_{5R} and a family of hyper-surfaces $\{RU + x\}_{x \in A_R}$ is $\lesssim R^{d\alpha}$. Repeating the argument above, we would arrive at the conclusion that if P is a convex polyhedron with N lattice vertices, then

$$|P| \gtrsim N^{\frac{1}{\alpha-1}}. \tag{4.3}$$

However, a higher dimensional version of the aforementioned theorem of Andrews says that

$$|P| \gtrsim N^{\frac{d+1}{d-1}}. \tag{4.4}$$

This leads us to conjecture that the putative incidence theorem described above should hold with $\alpha = 2 - \frac{2}{d+1}$, which was recently proved in [16] under additional smoothness assumptions. This result is sharp in view of (3.3) and the following result due to Barany and Larman ([5]).

Theorem 4.4. *The number of vertices of P_R, the convex hull of the lattice points contained in the ball of radius $R \gg 1$ centered at the origin is $\approx R^{d\frac{d-1}{d+1}}$.*

9

5 Higher dimensions

Theorem 5.1. *If $R > 0$ is sufficiently large, then*

$$\#(\Delta(A \cap [-R, R]^d) \gtrsim R^{2-\frac{1}{d}}. \tag{5.1}$$

This result was recently proved in a more general setting, using different methods, by Solymosi and Vu ([28]).

Corollary 5.2. *The ball $B_d = \{x : |x| \leq 1\}$ is not spectral in any dimension greater than 1.*

Corollary 4.2 follows from Theorem 4.1 in the same way as Theorem 2.1 follows from Corollary 0.2. Lemma 2.2 and Lemma 2.3 go through without change except that in $\mathbb{R}^d$,

$$|\widehat{\chi}_{B_d}(\xi)| \lesssim |\xi|^{-\frac{d+1}{2}}, \tag{5.2}$$

$\widehat{\chi}_{B_d}(\xi)$ is a constant multiple of

$$|\xi|^{-\frac{d}{2}} J_{\frac{d}{2}}(2\pi|\xi|), \tag{5.3}$$

and the zeroes of $J_{\frac{d}{2}}$ are still separated.

See [26], [30] and/or any text on special functions for the details.

We are left to prove Theorem 4.1. Since A is well-distributed, there is $s > 0$ such that every cube of side-length s contains at least one point of A. Fix a reference cube of side-length s and consider a row of consecutive cubes in each of the coordinate directions with respect to the reference cube. Chose a point of A in the 10th cube in each coordinate direction. Name those points $P_1, P_2, \ldots, P_d$. Let O denote the center of the reference cube. Construct a system of annuli centered at O of width Md, with the first annulus of radius $\approx R$. Construct $\approx R$ such annuli.

It follows from the assumption that A is well distributed that each constructed annulus $\mathcal{A}$ has $\approx R^{d-1}$ points of A. Let

$$\cup_{i=1}^d \{|x - P_i| : x \in \mathcal{A}\} = \{d_1, \ldots, d_k\}. \tag{5.4}$$

Let

$$A_j^l = \{x \in \mathcal{A} \cap A : |x - P_l| = d_j\}. \tag{5.5}$$

10

It is not hard to see that

$$A_j^l = \cup_{1 \le j_m \le k} \cup_{m=1}^{d-1} A_j^l \cap_{l' \ne l} A_{j_m}^{l'}. \tag{5.6}$$

Taking unions of both sides in j and counting, we see that

$$R^{d-1} \lesssim k^d. \tag{5.7}$$

This follows from the fact, which follows by a direct calculation, that the intersection of d spheres in question consists of at most two points. Taking d'th roots and using the fact that we have $\approx R$ annuli with $\approx R^{d-1}$ point of A, we conclude that

$$\#\Delta(A \cap [-R, R]^d) \gtrsim R^{1+\frac{d-1}{d}} = R^{2-\frac{1}{d}}, \tag{5.8}$$

as desired.

Observe that while the intersection claim made above is not difficult to verify for spheres, the situation becomes much more complicated for boundaries of general convex bodies, even under smoothness and curvature assumptions. This issue is partially addressed in [11].

Another point of view on distance set problems was recently pursued by Iosevich and Laba ([14]) and, independently, by Kolountzakis ([22]).

Theorem 5.3. *([14] for $d = 2$ and [22] for $d > 2$). Let A be a well-distributed subset of $\mathbb{R}^d$, $d \ge 2$. Let K be a symmetric bounded convex set. Then $\Delta_K(A)$ is separated only if K is a polyhedron with finitely many vertices.*

The more difficult question of which polyhedra can result in separated distance sets is partially addressed in both aforementioned papers, but the question is, in general, unresolved.

6 Some comments on finite fields

In this section we consider incidence theorems in the context of finite fields. More precisely, let F_q denote the finite field of q elements. Let F_q^d denote the d-dimensional vector space over F_q. A line in F_q^d is a set of points $\{x + tv : t \in F_q\}$ where $x \in F_q^d$ and $v \in F_q^d \setminus (0, \ldots, 0)$. A hyperplane in F_q^d is a set of points $(x_1, \ldots, x_d)$ satisfying the equation $A_1 x_1 + \cdots + A_d x_d = D$, where $A_1, \ldots, A_d, D \in F_q$ and not all A_j's are 0.

It is clear that without further assumptions, the number of incidences between n hyper-planes and n points is $\approx n^2$ and no better, since we can take all n planes to be rotates of the same plane about a line where all the points are located. We shall remove this "difficulty" by operating under the following non-degeneracy assumption.

Definition 6.1. We say that a family of hyperplanes in F_q^d is non-degenerate if the intersection of any d (or fewer) of the hyper-planes in the family contains at most one point.

The main result of this section is the following:

Theorem 6.2. *Suppose that a family $\mathcal{F}$ of n hyper-planes in F_q^d is non-degenerate. Let $\mathcal{P}$ denote a family of n points in F_q^d. The the number of incidences between the elements of $\mathcal{F}$ and $\mathcal{P}$ is $\lesssim n^{2-\frac{1}{d}}$. Moreover, this estimate is sharp.*

We prove sharpness first. Let $\mathcal{F}$ denote the set of all the hyper-planes in F_q^d and $\mathcal{P}$ denote the set of all the points in F_q^d. It is clear that $\#\mathcal{F} \approx \mathcal{P} \approx q^d$. On the other hand, the number of incidences is simply the number of hyper-planes times the number of points on each hyper-planes, which is $\approx q^{2d-1}$. Since $q^{2d-1} = (q^d)^{2-\frac{1}{d}}$, the sharpness of the Theorem 5.1 is proved.

We now prove the positive result. Consider an n by n matrix whose (i,j) entry if 1 if i'th point lies on j's line, and 0 otherwise. The non-degeneracy condition implies that this matrix does not contain a d by 2 submatrix consisting of 1's. Using Holder's inequality we see that the number of incidences,

$$I = \sum_{i,j} I_{ij} \leq \left(\sum_i \left(\sum_j I_{ij} \right)^d \right)^{\frac{1}{d}} \times n^{\frac{d-1}{d}} \tag{6.1}$$

$$= \left(\sum_i \sum_{j_1,\dots,j_d} I_{ij_1} \dots I_{ij_d} \right)^{\frac{1}{d}} \times n^{\frac{d-1}{d}} \lesssim n \times n^{\frac{d-1}{d}} = n^{2-\frac{1}{d}}, \tag{6.2}$$

because when j_k's are distinct, $I_{ij_1} \dots I_{ij_d}$ can be non-zero for at most one value of i due to the non-degeneracy assumption. If j_k's are not distinct, we win for the same reason. This completes the proof of Theorem 5.1.

Why should the finite field case be different from the Euclidean case? The proof of Szemeredi-Trotter theorem given above suggests that main difference

12

may be the notion of order. In the proof of Szemeredi-Trotter we used the fact that points on a line may be ordered. However, no such notion exists in a finite field. Nevertheless, Tom Wolff conjectured that if q is a prime, then there exists $\epsilon > 0$ such that the number of incidences between n points and n lines in F_q^2 should not exceed $n^{\frac{3}{2}-\epsilon}$ for $n \approx q$. This fact has recently been proved by Bourgain, Katz, and Tao ([4]).

7   A Fourier approach

In this sections we briefly outline how some results in geometric combinatorics can be obtained using Fourier analysis. For a more complete description, see, for example, [12], [15], and [16].

We could take a more direct approach, but we take advantage of this opportunity to introduce the following beautiful problem in geometric measure theory.

Falconer Distance Conjecture. *Let $E \subset [0,1]^d$, $d \geq 2$. Suppose that the Hausdorff dimension of E is greater than $\frac{d}{2}$. Then $\Delta(E) = \{|x-y| : x, y \in E\}$ has positive Lebesgue measure.*

We shall not discuss the history and other particulars of the Falconer Distance Problem in this paper. See, for example, [31] and references contained therein for a thorough description of the problem and related machinery. The main thrust of this section is to show that any non-trivial theorem about the Falconer Distance Conjecture can be used to deduce a corresponding "discrete" result about distance sets of well-distributed subsets of $\mathbb{R}^d$.

Theorem 7.1. *Let K be a bounded convex set in $\mathbb{R}^d$, $d \geq 2$, symmetric with respect to the origin. Suppose that the Lebesgue measure of $\Delta_K(E)$ is positive whenever the Hausdorff dimension of $E \subset [0,1]^d$ is greater than s_0, with $0 < s_0 < d$. Let A be a well-distributed subset of $\mathbb{R}^d$. Then $\#\Delta_K(A \cap [-R,R]^d) \gtrsim R^{\frac{d}{s_0}}$.*

The following result is essentially proved in [8].

Theorem 7.2. *Let $E \subset [0,1]^d$, $d \geq 2$, of Hausdorff dimension greater than $\frac{d+1}{2}$. Suppose that K is a bounded convex set, symmetric with respect to the origin, with a smooth boundary and everywhere non-vanishing Gaussian curvature. Then the Lebesgue measure of $\Delta_K(E)$ is positive.*

Theorem 6.1 and 6.2 combine to yield the following "discrete" theorem.

Theorem 7.3. *Let A be a well-distributed subset of $\mathbb{R}^d$, $d \geq 2$. Suppose that K is a bounded convex set, symmetric with respect to the origin, with a smooth boundary and everywhere non-vanishing Gaussian curvature. Then $\#\Delta_K(A \cap [-R, R]^d) \gtrapprox R^{2 - \frac{2}{d+1}}$.*

Observe that while this result is not as strong as the one given by Theorem 4.1, it is more flexible since it does not require K to be the Euclidean ball.

Proof of Theorem 6.1. Let $q_1 = 2$ and choose integers $q_{i+1} > q_i^i$. Let

$$E_i = \{x \in [0,1]^d : |x_k - p_k/q_i| \leq q_i^{-\frac{d}{s}} \text{ for some } p = (p_1, \ldots, p_d) \in A \cap [0, q_i]^d\}. \tag{7.1}$$

Let $E = \cap E_i$. It follows from the proof of Theorem 8.15 in [7] that the Hausdorff dimension of E is s. Suppose that there exists an infinite subsequence of q_is such that $\#\Delta_K(A \cap [0, q_i]^d) \lesssim q_i^{\beta}$ for some $\beta > 0$. Then we can cover $\Delta_K(E_i)$ by $\lesssim q_i^{\beta}$ intervals of length $\approx q_i^{-\frac{d}{s}}$. If $\beta < \frac{d}{s}$, $|\Delta_K(E_i)| \to 0$ as $i \to \infty$. It follows that $\Delta_K(E)$ has Lebesgue measure 0. However, by assumption, $\Delta_K(E)$ is positive if $s > s_0$. The conclusion follows. $\square$

The proof of Theorem 6.1 suggests that one may be able to make further progress on the Erdos Distance Conjecture for well-distributed sets using Fourier methods by studying the Falconer Distance Conjecture for special sets constructed in the previous paragraph.

References

[1] G. Arutyunyants An effective upper bound on the number of orthogonal exponentials on the unit disk (in preparation) 2004

[2] M. Ajtai, V. Chvatal, M. Newborn, and E. Szemeredi Crossing-free subgraphs North-Holland Math Studies 60 1986

[3] G. Andrews A lower bound for the volume of strictly convex bodies with many boundary lattice points Trans. Amer. Math. Soc. 106 1963 270-279

[4] J. Bourgain, N. Katz, and T. Tao A sum-product estimate in finite fields, and applications (preprint) 2003

[5] I. Barany and D. Larman The convex hull of the integer points in a large ball 1999 Math. Ann. 312 167-181

[6] P. Erdos Integral distances Bull. Amer. Math. Soc. 51 996 1945

[7] K. J. Falconer The geometry of fractal sets Cambridge University Press 1985

[8] K. J. Falconer On the Hausdorff dimensions of distance sets Mathematika 32 206-212 1986

[9] B. Fuglede Commuting self-adjoint partial differential operators and a group theoretic problem J. Func. Anal. 16 1974 101-121

[10] B. Fuglede Orthogonal exponentials on the ball Expo. Math. 19 267-272 2001

[11] M. Ganichev, A. Iosevich, and I. Laba Generic intersections of boundaries of convex bodies and Fourier analysis (in preparation) 2004

[12] S. Hofmann and A. Iosevich Circular averages and Falconer/Erdos distance conjecture in the plane for random metrics 2003 Proc. Amer. Math. Soc. (accepted for publication)

[13] A. Iosevich, N. Katz, and S. Pedersen Fourier bases and a distance problem of Erdos Math. Res. Lett. 6 1999 251-255

[14] A. Iosevich and I. Laba Distance sets of well-distributed planar sets (accepted for publication by the Journal of Discrete and Computational Geometry) 2003

[15] A. Iosevich and I. Laba K-distance, Falconer conjecture, and discrete analogs 2003 (submitted for publication)

[16] A. Iosevich, I. Laba, and M. Rudnev Incidence theorems via Fourier analysis (in preparation) 2004

[17] A. Iosevich and S. Pedersen How large are the spectral gaps? Pacific J. Math. 192 2000

[18] A. Iosevich and M. Rudnev Combinatorial approach to orthogonal exponentials Inter. Math. Res. Notices 2003 49 1-12

[19] A. Iosevich, M. Rudnev and V. Ten Combinatorial complexity of convex sequences (submitted for publication) 2003

[20] A. Iosevich, E. Sawyer, and A. Seeger Bounds for the mean square lattice point discrepancy. II: Convex domains in the plane (preprint) 2003

[21] S. Konyagin An estimate of the L^1-norm of an exponential sum (in Russian) The theory of approximation of functions and operators 2000 88-89

[22] M. Kolountzakis Distance sets corresponding to convex bodies GAFA (to appear) 2003

[23] P. Agarwal and J. Pach Combinatorial geometry A Wiley-Interscience publication 1995

[24] J. Solymosi and Cs. D. Toth Distinct distances in the plane Discrete Comp. Geom. (Misha Sharir birthday issue) 25 2001 629-634

[25] J. Solymosi, G. Tardos, and C. D. Toth Distinct distances in the plane 25 2001 629-634

[26] E. M. Stein Harmonic Analysis 1993 Princeton University Press

[27] E. Szemeredi and W. Trotter Extremal problems in discrete geometry Combinatorica 3 1983 381-392

[28] J. Solymosi and V. Vu Distinct Distances in High Dimensional Homogeneous Set Contemporary Mathematics Series of the AMS (to appear) 2003

[29] L. Szekely Crossing numbers and hard Erdos problems in discrete geometry 1997 6 353-358 Combinatorics, probability and computing

[30] E. M. Stein and G. Weiss Introduction to Fourier analysis on Euclidean spaces 1971 Princeton University Press

[31] T. Wolff Lectures in Harmonic Analysis AMS University Lecture Series 29 2003