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Math 173, Fall 2022, December 12, 2022

D alternating n-linear on $n \times n$ matrices over F.

$\alpha_1, \alpha_2, \ldots, \alpha_n$ rows

$\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ - standard basis

$$\alpha_i = \sum_{j=1}^{n} A(i,j)\,\epsilon_j, \quad 1 \leq i \leq n$$

$$D(A) = D\Big(\sum_j A(1,j)\,\epsilon_j, \alpha_2, \ldots, \alpha_n\Big)$$

$$= \sum_j A(1,j)\, D(\epsilon_j, \alpha_2, \ldots, \alpha_n)$$

$$D(\epsilon_j, \alpha_2, \ldots, \alpha_n) = \sum_k A(2,k)\, D(\epsilon_j, \epsilon_k, \alpha_3, \ldots, \alpha_n)$$

So $$D(A) = \sum_{j,k} A(1,j)A(2,k)\, D(\epsilon_j, \epsilon_k, \alpha_3, \ldots, \alpha_n)$$

$$= \sum_{k_1, k_2, \ldots, k_n} A(1,k_1)A(2,k_2)\ldots A(n,k_n)\, D(\epsilon_{k_1}, \epsilon_{k_2}, \ldots, \epsilon_{k_n})$$

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Note that $D(\epsilon_{k_1}, \dots, \epsilon_{k_n}) = 0$

if any two indices are equal,

so we are reduced to considering permutations

of $(k_1, k_2, \dots, k_n)$.

\# of permutions $= n!$

A permutation is a 1-1 function from

$\{1, 2, \dots, n\}$ to itself.

Putting everything together,

$$D(A) = \sum_{\sigma} A(1, \sigma 1)\, A(2, \sigma 2) \dots A(n, \sigma n)\, D(\epsilon_{\sigma 1}, \dots, \epsilon_{\sigma n})$$

all possible permutations

Next step: $D(\epsilon_{\sigma 1}, \dots, \epsilon_{\sigma n}) = \pm D(\epsilon_1, \dots, \epsilon_n)$

depends only on $\underline{\sigma}$

If D is the determinant function,

$D(\epsilon_{\sigma 1}, \dots, \epsilon_{\sigma n}) = (-1)^m$, where

$m = \#$ of interchanges needed to change $\begin{pmatrix} \epsilon_{\sigma 1} \\ \vdots \\ \epsilon_{\sigma n} \end{pmatrix}$ to $\begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{pmatrix} = I_n$.

Fact to be established in a moment:

If σ is a permutation on n letters, one can pass from $\{1, 2, \dots, n\}$ to $\{\sigma 1, \dots, \sigma n\}$ by interchanging pairs. The number of interchanges is either even or odd.

Def: $\operatorname{sgn} \sigma = \begin{cases} 1, & \text{if } \sigma \text{ even} \\ -1, & \text{if } \sigma \text{ odd} \end{cases}$

If we assume this for a moment,

$\det(A) = \sum_{\sigma} (\operatorname{sgn} \sigma) A(1, \sigma 1) \dots A(n, \sigma n)$ & we are done!

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We now prove the claim:

Suppose that $(1, 2, \dots, n) \to (\sigma 1, \dots, \sigma n)$ in m steps.
(interchanges of pairs)

$D(\epsilon_{\sigma 1}, \dots, \epsilon_{\sigma n}) = (-1)^m$, as we showed above,
so m can be odd or even, but not both!

Theorem: A, B $n \times n$ matrices over F ~ field.
Then $\det(AB) = \det(A) \cdot \det(B)$

Proof: Let $D(A) = \det(AB)$ — B fixed

$\alpha_1, \alpha_2, \dots, \alpha_n$ — rows of A

$D(\alpha_1, \dots, \alpha_n) = \det(\alpha_1 B, \dots, \underbrace{\alpha_n B}_{1 \times n \text{ matrix!}})$

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Since $(c\alpha_i + \alpha_i')B = c\alpha_i B + \alpha_i' B$,

and det is n-linear, we see that D is n-linear.

If $\alpha_i = \alpha_j$, $\alpha_i B = \alpha_j B$, so

$D(\alpha_1, \ldots, \alpha_n) = 0$ since D is alternating.

It follows that $D(A) = (\det A)\, D(I)$

But, $D(I) = \det(IB) = \det(B)$, so

$\det(AB) = D(A) = (\det A)(\det B)$ ✓

Corollary: $\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$

(6)

Definition: V/F $\quad T: V \longrightarrow V$

$\underbrace{F}_{\text{field}}$

A characteristic value of T is a scalar c in F $\ni$ $\exists$ non-zero vector $\alpha \in V$ w/ $T\alpha = c\alpha$. ~~The~~ vector α is then called the <u>characteristic</u> <u>vector</u> with characteristic value c.

The collection of all $\alpha \ni T\alpha = c\alpha$ is called the characteristic space associated w/ c.

Theorem 1: T is a linear operator on a finite dimensional space V and c is a scalar.

TFAE:

i) c is a characteristic value of T

ii) $T - c\underline{I}$ is singular

iii) $\det(T - c\underline{I}) = 0$.

The only missing piece here is the fact that $\det(T) = 0$ iff T is not 1-1. We shall now back track a bit and take care of this fact comprehensively.

By uniqueness, $\underbrace{\det(A)}_{n\times n \text{ matrix}} = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det(A(i|j))$

$(-1)^{i+j} \det A(i|j)$ is called a co-factor.

Let C_{ij}. Then

$$\det(A) = \sum_{i=1}^{n} A_{ij} C_{ij}$$

Observe that $\sum_{i=1}^{n} A_{ik} C_{ij} = 0$ if $k \neq j$.

To see this, replace the j'th column of A by its k'th column, and call this matrix B.

Then $0 = \det(B) = \sum_{i=1}^{n} (-1)^{i+j} B_{ij} \det B(i|j)$

$$= \sum_{i=1}^{n} (-1)^{i+j} A_{ik} \det A(i|j)$$

$$= \sum_{i=1}^{n} A_{ik} C_{ij}.$$

It follows that

$$\sum_{i=1}^{n} A_{ik} C_{ij} = \delta_{jk} \det(A)$$

The transpose matrix of the matrix of co-factors of A is called the classical adjoint of A, i.e. $(\operatorname{adj} A)_{ij} = C_{ji} = (-1)^{i+j} \det A(j|i)$.

In other words,

$$(\operatorname{adj} A) A = (\det A) \cdot \underline{I}$$

But what about $A (\operatorname{adj} A)$?

$$A^t(i|j) = A(j|i)^t$$

Therefore, $(-1)^{i+j} \det A^t(i|j) = (-1)^{i+j} \det A(j|i)$

since $\det M = \det M^t$ (page 156-157 – read it!)

It follows that $\operatorname{adj}(A^t) = (\operatorname{adj} A)^t$

Since $(\operatorname{adj} A)A = (\det A)\underline{I}$,

$$(\operatorname{adj} A^t)A^t = (\det A^t)\underline{I} = (\det A)\cdot\underline{I}$$

Transposing,

$$A\cdot(\operatorname{adj} A^t)^t = (\det A)\underline{I}, \text{ ie}$$

$$A\cdot \operatorname{adj} A = (\det A)\underline{I}, \text{ as desired!}$$

Back to characteristic values!

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\det(x\underline{I} - A) = \det\begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix} = x^2+1, \text{ so}$$

<u>no</u> characteristic values

over $\mathbb{R}$