Find two positive numbers whose product is 100 and whose sum is minimal.

Solution: $x =$ first number

$y =$ second number

$$\underbrace{xy = 100}_{\text{given}} \qquad \underbrace{x+y}_{\text{quantity to be minimized}}$$

$$x + y = x + \frac{100}{x}$$

$$y = \frac{100}{x}$$

Therefore, we have reduced the problem to minimizing the function $f(x) = x + \dfrac{100}{x}$.

$$\underbrace{f'(x) = 1 - \frac{100}{x^2} = 0}_{\text{local minimum}} \qquad x = 10 \quad y = 10$$

$$\underbrace{x + y = 20}_{\text{absolute minimum}}$$

(2)

$$\int \frac{dx}{x\sqrt{x^2-1}}$$

$u = \sqrt{x^2-1}$ $\quad du = \frac{1}{\cancel{2}}(x^2-1)^{-\frac{1}{2}} \cdot \cancel{2}x$

$$= \frac{x}{\sqrt{x^2-1}}\, dx \qquad x = \sqrt{u^2+1}$$

$$\int \frac{du}{x^2} = \int \frac{du}{u^2+1} =$$

$$\tan^{-1}(u) + C = \tan^{-1}\left(\sqrt{x^2-1}\right) + C$$

Clarification:

$u = \sqrt{x^2-1} \qquad u^2 = x^2 - 1$

$u^2 + 1 = x^2 \qquad x = \sqrt{u^2+1}$

(3)

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx$$

$x = 3\sin\theta \quad dx = 3\cos\theta\, d\theta$

$$\int \frac{9\sin^2\theta \cdot \cancel{3\cos\theta}\, d\theta}{\cancel{3\cos\theta}} =$$

$$9\int \sin^2\theta \, d\theta$$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$9\int \frac{1}{2}\,d\theta - \frac{9}{2}\int \cos(2\theta)\,d\theta$$

$$= \frac{9}{2}\theta - \frac{9}{4}\sin(2\theta) + C$$

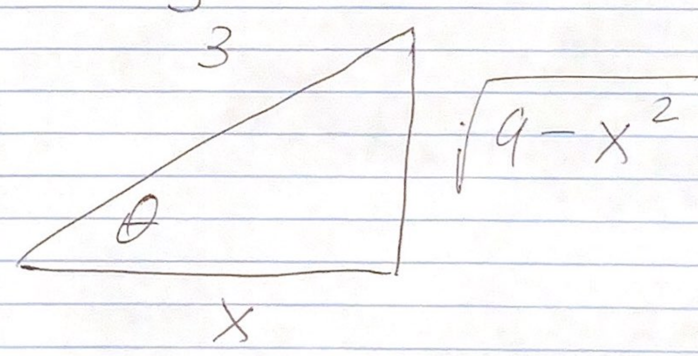
$x = 3\cos\theta$

$$\frac{9}{2}\underbrace{\theta}_{=} - \frac{9}{4}\sin(2\theta) + C$$

$$\frac{9}{2}\cos^{-1}\left(\frac{x}{3}\right) - \frac{9}{4}\sin(2\theta) + C$$

$$= \frac{9}{2}\cos^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\sin\theta\cos\theta + C$$

$$= \frac{9}{2}\cos^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\underbrace{\sin\theta}_{=\frac{\sqrt{9-x^2}}{3}}\frac{x}{3} + C$$



(5)

$$\int_0^1 \sqrt{x - x^2}\, dx$$

$$-x^2 + x = -(x^2 - x) =$$

$$-\left(\left(x-\tfrac{1}{2}\right)^2 - \tfrac{1}{4}\right) = \tfrac{1}{4} - \left(x - \tfrac{1}{2}\right)^2$$

$$x - \tfrac{1}{2} = \tfrac{1}{2}\sin\theta$$

$$dx = \tfrac{1}{2}\cos\theta\, d\theta$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\tfrac{1}{4} - \tfrac{1}{4}\sin^2\theta}\ \ \tfrac{1}{2}\cos\theta\, d\theta =$$

$$\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

(c)

$$\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\, d\theta + \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(2\theta)}{2}\, d\theta$$

$$\frac{1}{8}\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{\sin(2\theta)}{16}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{8}$$