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Math 173, October 12, 2022

Theorem: Let V be a vector space spanned by a finite set of vectors $\beta_1, \beta_2, \ldots, \beta_m$. Then any independent set of vectors in V is finite and contains no more than m elements.

Proof: It is enough to show that every subset S of V w/ more than m elements is dependent. Let S be such a set.

Consider $\alpha_1, \alpha_2, \ldots, \alpha_n \in S$, $n > m$

By assumption, $\alpha_j = \sum_{i=1}^{m} \underbrace{A_{ij}}_{\text{for some } A_{ij}\text{s}} \beta_i$

For any scalars $x_1, x_2, \ldots, x_n$,

$$x_1\alpha_1 + \cdots + x_n\alpha_n = \sum_{j=1}^{n} x_j \alpha_j$$

$$= \sum_{j=1}^{n} x_j \sum_{i=1}^{m} A_{ij}\beta_i = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_j A_{ij}\right)\beta_i$$

$\{A_{ij}\}_{\substack{1\le i\le m\\ 1\le j\le n}}$ is an $m\times n$ matrix,

so $A\cancel{X}=0$ has a non-trivial solution $(x_1,\dots,x_n)$ Theorem 6, Chapter 1

It follows that $\exists\, x_1,\dots,x_n$ not all $0 \ni$

$x_1\alpha_1+\dots+x_n\alpha_n=0 \Longleftrightarrow S$ is linearly dependent $\blacksquare$

Corollary: If V is a finite-dimensional vector space, then any two bases of V have the same number of elements

Proof: Let $\{\beta_1,\dots,\beta_m\}$ denote a basis of V. By Theorem 4, every basis has $\le m$ elements. By the same argument (why?) every basis has $\ge m$ elements and we are done.

③

Definition: V = vector space

$\dim(V)$ = # elements in a basis of V

Corollary: V = finite dimensional vector space, and let $n = \dim V$. Then

a) Any subset of V which contains more than n vectors is linearly dependent.

b) No subset of V w/ fewer than n vectors can span V.

Examples: $V = F^n \hookrightarrow \dim(V) = n$

$V = m \times n$ matrices over F; $\dim(V) = mn$

discuss w/ concrete cases!

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(4)

Lemma: S linearly independent subset of V. Suppose that $\beta \in V$ is not in span(S). Then the set obtained by adjoining β to S is linearly independent.

Proof: Suppose that $\alpha_1, \dots, \alpha_m \in S$ distinct such that $c_1\alpha_1 + c_2\alpha_2 + \dots + c_m\alpha_m + b\beta = 0$.

Then $b = 0$, since otherwise

$$\beta = \left(\frac{-c_1}{b}\right)\alpha_1 + \dots + \left(\frac{-c_m}{b}\right)\alpha_m$$

$\hookrightarrow$ contradiction!

This forces $c_1\alpha_1 + c_2\alpha_2 + \dots + c_m\alpha_m = 0$

$\hookrightarrow c_i \equiv 0 \ \forall i$ since S is linearly independent.

Theorem 5: If $\underset{\text{subspace}}{W \subset V}$, every linearly independent subset of W is finite and is a part of a basis of V.

Proof: If $S_0 \subset W$ linearly independent

$\Rightarrow S_0 \subset V$ linearly independent

$\Rightarrow$ size of $S_0 \leq \dim(V)$

Let's extend S_0 to a basis as follows:

If $\mathrm{span}(S_0) = V$, we are done.

If not, find $\beta_1 \notin \mathrm{span}(S_0)$ and consider $S_1 = S_0 \cup \{\beta_1\}$ independent.

Continue in this way and obtain

$$S_m = S_0 \cup \{\beta_1, \dots, \beta_m\}$$

Once the size of S_m reaches $\dim(V)$, we are done.

Corollary: If W is a proper subspace of V finite dimensional, then W is finite-dimensional, and $\dim(W) < \dim(V)$

Proof: Assume $\alpha \neq 0$ is in W.
We just proved that $\exists$ basis of W containing α and having $\leq \dim(V)$ elements in total.
It follows that $\dim(W) \leq \dim(V)$.
Since W is a proper subspace, $\exists \beta \in V \ni \beta \notin W$, so basis of $W \cup \{\beta\}$ is linearly independent.
We conclude that $\dim(W) < \dim(V)$.