HW4 Solution

8.

$\Rightarrow$) If $\mathbb{R}_\infty = T(\mathbb{R}_\infty)$, $\exists$ $z_1, z_2, z_3 \in \mathbb{R}_\infty$ such that

$$Tz_1 = \frac{az_1+b}{cz_1+d} = 1$$

$$\begin{cases} Tz_2 = \dfrac{az_2+b}{cz_2+d} = 0 \\ \\ Tz_3 = \dfrac{az_3+b}{cz_3+d} = \infty \end{cases}$$

$$\Rightarrow \begin{cases} z_1 = \dfrac{d-b}{a-c} \\ z_2 = -\dfrac{b}{a} \\ z_3 = -\dfrac{d}{c} \end{cases}$$

$$\Rightarrow \frac{b}{a}, \frac{d}{c} \in \mathbb{R}_\infty$$

From $z_1 = \dfrac{d-b}{a-c}$ ~~we can solve for~~

$\cancel{\dfrac{c}{a} =}$ Let $\dfrac{c}{a} = r$, then

$$z_1 = \frac{d-b}{a-ar} = \frac{\frac{d}{a} - \frac{b}{a}}{1-r}$$

$$\Rightarrow z_1 = \frac{\frac{d}{c}\cdot\frac{c}{a} - \frac{b}{a}}{1-r} = \frac{\frac{d}{c}r - \frac{b}{a}}{1-r}$$

$$\Rightarrow \frac{c}{a} = r = \frac{z_1 + \frac{b}{a}}{z_1 + \frac{d}{c}} \in \mathbb{R}_\infty.$$

Hence $\frac{d}{a} = \frac{d}{c} \cdot \frac{c}{a} \in \mathbb{R}_\infty$

Now $Tz = \dfrac{a z + b}{cz + d}$

$$= \frac{z + \frac{b}{a}}{\frac{c}{a} z + \frac{d}{a}}$$

Since $1, \frac{b}{a}, \frac{c}{a}, \frac{d}{a} \in \mathbb{R}_\infty$ as proven above,

we have $\Rightarrow)$

$\Leftarrow)$ Obvious. $\boxed{/}$

9. If $z\bar{z} = 1$, then

$|Tz| = 1$

$\Leftrightarrow \left| \dfrac{az+b}{cz+d} \right| = 1$

$\Leftrightarrow \dfrac{(az+b)(\bar{a}\bar{z}+\bar{b})}{(cz+d)(\bar{c}\bar{z}+\bar{d})} = 1$

$\Leftrightarrow (|a|^2 - |c|^2)(|z|^2) + (a\bar{b} - c\bar{d})z + (\bar{a}b - \bar{c}d)\bar{z}$

$+ (|b|^2 - |d|^2) = 0$

Then $\begin{cases} |a|^2+|b|^2 = |c|^2+|d|^2 & ① \\ a\bar{b} - c\bar{d} = 0 & ② \end{cases}$ would be

sufficient conditions.

Moreover, if $a = r\bar{d}$

then from ② we have $c = r\bar{b}$

plug into ① we have

$$|r|^2|d|^2 + |b|^2 = |r|^2|b|^2 + |d|^2$$

We can let $|r| = 1$

Then $Tz = \dfrac{az+b}{cz+d} = \dfrac{r\bar{d}z+b}{r\bar{b}z+d}$

for $|r| = 1$

One can verify that $|Tz| = 1$ as long as $|z| = 1$.

$\square$

17) WLOG, suppose the circle centered at origin, radius being r.

Then $\forall z \in G$, $|f(z)| = r$

Let $f = u + iv$

By Cauchy-Riemann Eq,

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad (1)$$

$|f(z)| = r$

$\Rightarrow \quad u^2 + v^2 = r^2$

$$\Rightarrow \begin{cases} u u_x + v v_x = 0 \\ u u_y + v v_y = 0 \end{cases} \quad (2)$$

Combining (1) (2) one can solve for

$u_x = u_y = v_x = v_y = 0$

Hence f is constant.

19 a) $\log Mz = \cancel{\frac{\log z - ia}{\log z - ib}} \quad \log \frac{z-ia}{z-ib}$

let $z = x+iy$ then

$$Mz = \frac{z-ia}{z-ib} = \frac{x^2+(y-a)(y-b)}{x^2+(y-b)^2} - i\,\frac{x(y-b)-x(y-a)}{x^2+(y-b)^2}$$

Since log is not defined on $\{z \le 0\}$.

if $\begin{cases} x^2+(y-a)(y-b) \le 0 \\ x(y-b) - x(y-a) = 0 \end{cases}$

we have $\begin{cases} x = 0 \\ y \in [a,b] \end{cases}$

So $\log Mz$ is not defined on $z = ic$, $c \in [a,b]$.

For $\operatorname{Im} \log \frac{z-ia}{z-ib} = \operatorname{Arg} Mz$, $x = \operatorname{Re} z > 0$

Since $\operatorname{Im} Mz = -\frac{x(y-b) - x(y-a)}{x^2+(y-b)^2}$

$$= \frac{x(a-b)}{x^2+(y-b)^2} > 0$$

We see that Mz is on upper half

Complex plane.

Hence $h(z) \in (0, \pi)$

b) $\log(z - ic) = \log[x + i(y - c)]$

is well defined for $x = \operatorname{Re} z > 0$, $\forall c \in \mathbb{R}$

$|\operatorname{Im} \log(z - ic)| = |\operatorname{Arg}(z - ic)| < \frac{\pi}{2}$ since $x > 0$

c)

Let $z - ia = r_a e^{i\theta_a}$ $\quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$z - ib = r_b e^{i\theta_b}$

$\Rightarrow \operatorname{Im}(\log(z - ia) - \log(z - ib))$

$= \theta_a - \theta_b$

$\operatorname{Im} \log Mz = \operatorname{Im} \log \frac{r_a}{r_b} e^{i(\theta_a - \theta_b)} = \theta_a - \theta_b$

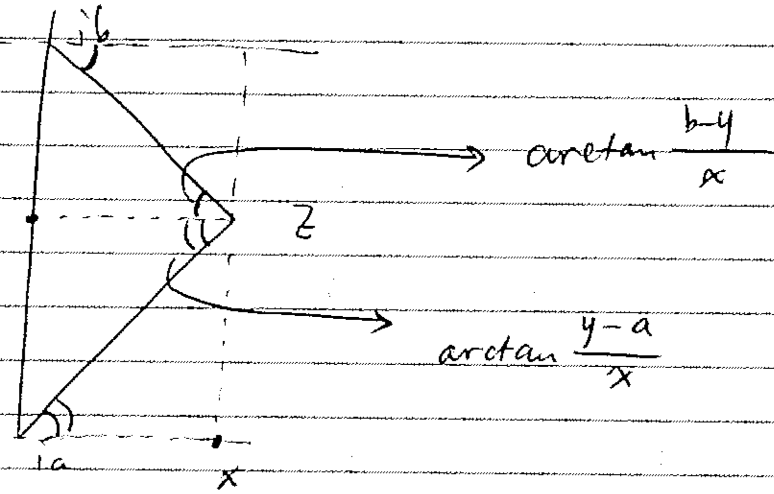
since $\theta_a - \theta_b \in (-\pi, \pi)$

we have the ~~inequ~~ equality.

d) —

e) —

f)



$$h(\cancel{x+iy}) = \mathrm{Im}[\log(z-ia) - \log(z-ib)]$$

$$= \mathrm{Arg}(z-ia) - \mathrm{Arg}(z-ib)$$

= angle between $z-ia$, $z-ib$

Since $T \neq I$

24) ① T has 1 fixed pt :

Say $Tz_1 = Sz_1 = z_1$

let M be a transf that $M(z_1) = \infty$

Then $MTM^{-1}(\infty) = MTz_1 = \infty$

also $\cancel{MSz_1} = MSM^{-1}(\infty) = \infty$

$\Rightarrow$ $MSM^{-1}z = z + a$ by 22b
$MTM^{-1}z = z + b$

$\Rightarrow$ $(MSM^{-1})(MTM^{-1}) = (MTM^{-1})(MSM^{-1}) = z+a+b$

$\Rightarrow$ $ST = TS$

② two fixed pts :

say z_1, z_2

let M be a transf s.t

$M(z_1) = 0$, $M(z_2) = \infty$

One verifies

$MSM^{-1}(0) = 0$, $MSM^{-1}(\infty) = \infty$
$MTM^{-1}(0) = 0$, $MTM^{-1}(\infty) = \infty$

Hence $MSM^{-1}z = az$ by 22 a)
$MTM^{-1}z = bz$

$\Rightarrow (MSM^{-1})(MTM^{-1})z = (MTM^{-1})(MSM^{-1})z$
$= abz$

$\Rightarrow ST = TS$. $\boxed{}$

1) Let $P = \{t_0 = a, t_1, \dots, t_n = b\}$

$$v(\gamma; P) = \sum_{i=0}^{n-1} |\gamma(t_{i+1}) - \gamma(t_i)|$$

$$= \sum_{i=0}^{n-1} \gamma(t_{i+1}) - \gamma(t_i) \quad , \gamma \text{ non-dec}$$

$$= \gamma(t_n) - \gamma(t_0) = \gamma(b) - \gamma(a)$$

which is a constant

Hence $V(\gamma) = \gamma(b) - \gamma(a)$

6) γ Lipschitz

$\Rightarrow \exists\ C > 0$ s.t $|\gamma(x) - \gamma(y)| \le C|x-y|$

Let $P = \{t_0 = a, t_1, \cdots t_n = b\}$

then $$v(\gamma; P) = \sum_{i=0}^{n-1} |\gamma(t_{i+1}) - \gamma(t_i)|$$

$$\le \sum_{i=0}^{n-1} C|t_{i+1} - t_i|$$

$$\le C(b-a) \qquad \boxed{4}$$

7) Apparently γ is cont. on $(0,1]$

And $\left|\lim_{t\to 0} \gamma(t)\right| = \lim_{t\to 0} |t|\left|1 + i\sin\frac{1}{t}\right| = 0$

So γ is cont. on $[0,1]$

hence a path.

Not rectifiable:

Let $P = \left\{t_0 = 0,\ t_i = \frac{2}{(2i+1)\pi},\ t_n = 1\right\}$

Then $$v(\gamma; P) = \sum_{k=0}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)|$$

$$\geqslant \sum_{k=1}^{n-2} \left| \gamma(t_{k+1}) - \gamma(t_k) \right|$$

$$= \sum_{k=1}^{n-2} \left| (t_{k+1} - t_k) + i \left(t_{k+1} \sin \frac{(2k+3)\pi}{2} - t_k \sin \frac{(2k+1)\pi}{2} \right) \right|$$

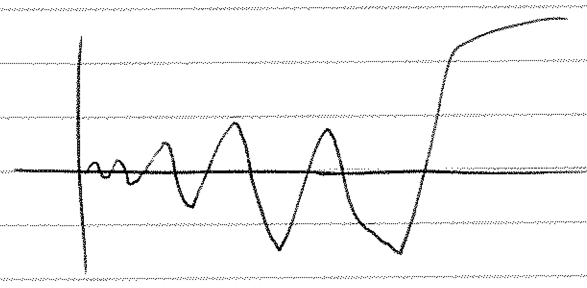
$$\geqslant \sum_{k=1}^{n-2} \left| t_{k+1} \sin \frac{(2k+3)\pi}{2} - t_k \sin \frac{(2k+1)\pi}{2} \right|$$

$$= \sum_{k=1}^{n-2} \left| t_{k+1} + t_k \right|$$

$$= \sum_{k=1}^{n-2} \frac{2}{(2k+3)\pi} + \frac{2}{(2k+1)\pi}$$

$$\geqslant \sum_{k=1}^{n-2} \frac{2}{6k\pi} \cdot 2 = \frac{2}{3\pi} \sum_{k=1}^{n-2} \frac{1}{k} \to \infty$$

as $n \to \infty$.



$\square$