HW 2 Soln

1.
$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, \quad h \in \mathbb{C}$$

$$= \lim_{h \to 0} \frac{(z+h)(\bar{z}+\bar{h}) - z\bar{z}}{h}$$

$$= \lim_{h \to 0} \left(\bar{z} + \bar{h} + \frac{\bar{h}}{h} \cdot z \right)$$

① $h \in \mathbb{R}$

Then $f'(z) = \bar{z} + z$

② $h \in i\mathbb{R}$

Then $f'(z) = \bar{z} - z$

If $f'(z)$ exists, then $\bar{z} + z = \bar{z} - z \Rightarrow z = 0$ $\boxtimes$

2. Assume $a = \limsup a_n < \infty$ (the case $= \infty$ is obvious)

$\exists \{a_{n_k}\} \to a$ and thus $\{b_{n_k}\} \to b$ automatically

$\therefore a_{n_k} b_{n_k} \to ab$

$\Rightarrow \limsup a_n b_n \geq \limsup a_{n_k} b_{n_k} = ab$

On the other hand,

from previous part, we have

$$\limsup a_n b_n \geq ab = \limsup a_n \cdot \limsup b_n .$$

Hence $$ab = \limsup a_n \cdot b \geq (\limsup a_n b_n)\left(\limsup \frac{1}{b_n}\right) \cdot b$$
$$= \limsup a_n b_n$$ $\square$

~~8.~~
9.

$$|z_n - z| \geq \big|\, |z_n| - |z| \,\big| = |r_n - r| \quad \Rightarrow \quad r_n \to r$$

Since $\lim r_n e^{i\theta_n} = re^{i\theta}$

$$\Rightarrow \quad \lim e^{i\theta_n} = e^{i\theta}$$

$$\Rightarrow \quad \mathrm{Arg}(e^{i\theta_n}) \to \mathrm{Arg}(e^{i\theta})$$ as arg is cont.

and $\theta_n, \theta \in (-\pi, \pi)$.

$$\Rightarrow \quad \theta_n \to \theta .$$

$\square$

10. For $z \in G$, $h^* \in \mathbb{C}$ s.t $z + h^* \in G$, $h^* \neq 0$,

Consider

$$\lim_{h^* \to 0} \frac{h(z+h^*) - h(z)}{h^*} = h'(z) \quad \text{as } h \text{ analytic on } G.$$

$$= \lim_{h^* \to 0} \frac{g \circ f(z+h^*) - g \circ f(z)}{f(z+h^*) - f(z)} \cdot \frac{f(z+h^*) - f(z)}{h^*}$$

$$= g'(f(z)) \cdot f'(z) \quad \text{using } f \text{ cont.}$$

$$\Rightarrow \quad f'(z) = \frac{h'(z)}{g'(f(z))}$$

$f'(z)$ being cont. as h', $g'(f)$ both cont.

$\Rightarrow$ f analytic. $\boxtimes$

11. f being a branch

$$\Rightarrow \quad z = e^{f(z)}$$

The cases when $n \in \mathbb{Z}^+$ and $n \in \mathbb{Z}^-$ are symmetric.

WLOG if $n \in \mathbb{Z}^+$, $z^n = \underbrace{e^{f(z)} \cdots e^{f(z)}}_{n \text{ times}} = e^{nf(z)}$

If $n=0$, trivial. $\boxed{4}$

13. Let $S \triangleq \{ f : G \to \mathbb{C} \mid z = f(z)^n, \ \forall z \in G \}$

$S \neq \phi$ as $e^{\frac{\log z}{n}} \in S$.

Then $\forall f \in S$, $f^n(z) = \left(e^{\frac{\log z}{n}}\right)^n = z$

$\Rightarrow \dfrac{f(z)}{e^{(\log z)/n}} = \alpha$, α being nth root of unity.

$\Rightarrow f(z) = e^{(\log z)/n} \cdot e^{\frac{2\pi i k}{n}}$ for $k = 0, 1, \cdots, n-1$

17. $\sqrt{1-z} = |1-z|^{\frac{1}{2}} e^{i(\arg(1-z) + 2k\pi)/2}$

$k=0$
$\Rightarrow$ principal branch : $|1-z|^{\frac{1}{2}} e^{\frac{i \arg(1-z)}{2}}$

This is defined on $G = \mathbb{C} - \{ z \mid z \geq 1 \}$.

and $-\pi < \arg(1-z) < \pi$

Additional:

1. By Cauchy - Riemann Equation, we have

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

We also have

$$\begin{cases} a u_x + b v_x = 0 \\ a u_y + b v_y = 0 \end{cases}$$. Combining 4 equations, we

get $u_x = u_y = v_x = v_y = 0$

$$\Rightarrow \begin{cases} u = c_1 \\ v = c_2 \end{cases}, \quad c_1, c_2 \in \mathbb{C}$$

$$\Rightarrow f = c_1 + i c_2, \text{ where } a c_1 + b c_2 + c = 0 \quad \boxtimes$$

2. $f(z)$ differentiable at z_0 $\Rightarrow$ Cauchy-Riemann holds at the point.

$$\Rightarrow \begin{cases} \left.\frac{\partial u}{\partial x}\right|_{z_0} = \left.\frac{\partial v}{\partial y}\right|_{z_0} \\ \left.\frac{\partial u}{\partial y}\right|_{z_0} = -\left.\frac{\partial v}{\partial x}\right|_{z_0} \end{cases} \Rightarrow \left.\frac{\partial f}{\partial \bar{z}}\right|_{z_0} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)_{z_0} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)_{z_0}\right] = 0$$

$\boxtimes$