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Corollary: If $A$ is a $n \times n$ matrix and if a sequence of elementary row operations reduces $A$ to the identity, then the same row operations applied to $I$ yield $A^{-1}$.

Proof: $I = \underbrace{E_k E_{k-1} \dots E_1}_{\text{elementary row operations}} A$

Therefore $E_k E_{k-1} \dots E_1 I \cdot A$

$= E_k E_{k-1} \dots E_1 A = I$

Similarly, $A \cdot E_k E_{k-1} \dots E_1 I =$

$A \cdot E_k E_{k-1} \dots E_1 A \cdot A^{-1} =$

$A \cdot I \cdot A^{-1} = I$ ✓

Example:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \xrightarrow{R2} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \xrightarrow{R2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

row2 → row2 + (−1) row(1)

row1 → row1 + (−1) row2

Computing the inverse:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

row2 → row2 + (−1) row1

row1 → row1 + (−1) row2

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

③

<u>Corollary</u>: $A, B$ $m \times n$ matrices. Then $B$ is row-equivalent to $A$ iff $B = PA$ where $P$ is an invertible $m \times m$ matrix.

Proof: If $B = \underset{\text{invertible } m\times m}{P}A$,

$\underbrace{P = E_1 E_2 \dots E_k}_{\text{elementary matrices}}$ by Theorem 12 and we are done.

If $B$ row equivalent to $A$; let $e_1, e_2, \dots e_k$ be the elementary row operations, and let

$E_i = e(I)$ be the corresponding matrices $m \times m$

Then $B = E_k E_{k-1} \dots E_1 A$

and we are done.

④

Theorem 13: For an $n \times n$ matrix, the following are equivalent:

i) $A$ is invertible

ii) The homogeneous system $AX=0$ has only the trivial solution $X=0$.

iii) $AX=Y$ has a solution $X$ for each $n \times 1$ matrix $Y$.

Proof: By Theorem 7, ii) equivalent to the fact that $A$ is row-equivalent to the identity matrix, so i) and ii) are equivalent by Theorem 12.

If $A$ is invertible, $AX=Y$ leads to

$$X = A^{-1}Y.$$

Conversely, suppose that $AX=Y$ has a solution for each given $Y$. Let $R$ = row-reduced echelon matrix equivalent to $A$. We claim that $R=I$, which is the same as showing $R$ has no zero rows.

Let $E = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

⑤

If the system $RX = E$ can be solved for $X$, the last row of $R$ cannot be $0$.

We have $R = PA$. Thus $RX = E$ iff

invertible

$PAX = E$, i.e

$AX = P^{-1}E$.

has a solution according to iii).

Corollary: A square matrix w/ either left or right inverse is invertible.

Proof: Let $A$ be an $n \times n$ matrix. Suppose $A$ has a left inverse, ie $BA = I$. Then $AX = 0$ has only the trivial solution since $X = IX = B(AX)$. Therefore, $A$ is invertible.

Now suppose that $A$ has a right inverse $C$, i.e. $AC = I$. Then $C$ has a left inverse, so $C$ is invertible. It follows that $A = C^{-1}$, so $A$ is invertible w/ inverse $C$.

**Corollary:** $A = A_1 A_2 \ldots A_k$ $A_i$ $n \times n$

Then $A$ is invertible iff each $A_i$ is invertible.

**Proof:** We have shown that the product of two invertible matrices is invertible, so if each $A_i$ is invertible, so is the product (induction).

Now suppose that $A$ is invertible. Let $X$ be a $n \times 1$ matrix $\ni$ $A_k X = 0$. Then

$$AX = (A_1 \ldots A_{k-1}) A_k X = 0.$$

Since $A$ is invertible, $X = 0 \hookrightarrow A_k$ is invertible. But now

$$A_1 \ldots A_{k-1} = A A_k^{-1}$$

is invertible, so we proceed by induction.