August 31, 2022 Math 173

What is a field?

Start w/ integers: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

$\mathbb{Z}$

$a, b \in \mathbb{Z} \hookrightarrow a+b,\ a \cdot b \in \mathbb{Z}$

$a \cdot \underline{1} = \underline{1} \cdot a = a \quad \forall a \in \mathbb{Z}$

Problem: If $a \neq 1$, one cannot find $b \in \mathbb{Z}$ $\ni a \cdot b = \underline{1}$, i.e no multiplicative inverse!

If we want multiplicative inverses, consider

$$\mathbb{Q} = \left\{ \frac{a}{b},\ b \neq 0,\ a, b \in \mathbb{Z} \right\}$$

rational numbers

Now multiplicative inverses exist, additive inverses are still fine, and, as we shall see in a moment, $\mathbb{Q}$ is a field.

Another problem: while $\mathbb{Q}$ is a field, sequences of elements of $\mathbb{Q}$ do not necessarily converge to elements of $\mathbb{Q}$!

Exercise: Let $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$, $x_1 = 2$

Prove that $x_n \to \sqrt{2}$ as $n \to \infty$, i.e

$$\lim_{n\to\infty} x_n = \sqrt{2}.$$

Please note that you must prove that the limit exists AND that the limit equals $\sqrt{2}$.

If we take all possible limits of rational sequences, we get the reals, denoted by $\mathbb{R}$. We shall see that $\mathbb{R}$ is a field and $\mathbb{Q}$ is a subfield

to be defined

The reals do not constitute the end of the story. Consider the equation

$$x^2+1=0 \quad (*)$$

Since $x^2 \geq 0$, $x^2+1>0$, so $(*)$ has no solutions over $\mathbb{R}$. This leads us to complex numbers, obtained by taking pairs of real number (a,b) and defining

Field axioms: F is a set w/ operations $(+,\cdot)$

i) Addition is commutative,

$$x+y=y+x \quad \forall x,y \in F$$

ii) Addition is associative

$$x+(y+z)=(x+y)+z \quad \forall x,y,z \in F$$

iii) $\exists!$ 0 (zero) in F $\ni$ $x+0=x$

iv) For each $x \in F$ $\exists$ $-x \in F$ $\ni$ $x+(-x)=0$,

v) Multiplication is commutative,

$$xy=yx \quad \forall x,y \in F$$

vi) Multiplication is associative,

$$x(yz) = (xy)z \quad \forall x, y, z \in F$$

vii) $\exists!\ 1 \in F \ni x \cdot 1 = x \quad \forall x \in F$

viii) To each $x \neq 0$ in F there corresponds x^{-1} (or $\frac{1}{x}$) in F $\ni x \cdot x^{-1} = 1$.

ix) Multiplication distributes over addition:

$$x(y+z) = xy + xz \quad \forall x, y, z \in F$$

Subfield F' of F is a subset of F which is a field under the same operations.

The key question in linear algebra:

F field. Given a collection of elements of F $\{A_{ij}\}_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ & elements $y_1, y_2, \dots, y_m \in F$

find $x_1, x_2, \dots, x_n \ni$

$$A_{11} x_1 + A_{12} x_2 + \dots + A_{1n} x_n = y_1$$
$$A_{21} x_1 + A_{22} x_2 + \dots + A_{2n} x_n = y_2 \qquad (*)$$
$$\vdots$$
$$A_{m1} x_1 + A_{m2} x_2 + \dots + A_{mn} x_n = y_m$$

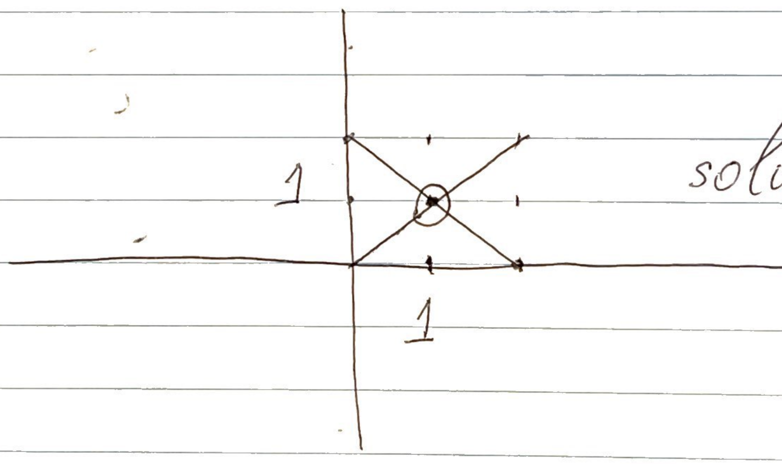
A system of m linear equations in n unknowns.

Then n-tuple $(x_1, \dots, x_n)$ which satisfies $(*)$ is called a solution of the system.

If $y_1 = y_2 = \dots = y_m = 0$, the system is called homogeneous

A basic example:

$$\begin{cases} x_1 + x_2 = 2 \\ x_1 - x_2 = 0 \end{cases}$$ over the real numbers, i.e $F = \mathbb{R}$



solution:
$x_1 = 1$
$x_2 = 1$

An equally basic example:

$$\left.\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + 2x_2 &= 2 \end{aligned}\right\}$$ the second equation is a multiple of the first, so no additional information is provided.

Yet another example:

$$\left.\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 + 2x_2 &= 5 \end{aligned}\right\}$$ the second equation is a multiple of the first, but the multiples do not match.

$2x_1 + 2x_2 = 5$, so

$x_1 + x_2 = \frac{5}{2}$ and, since $x_1 + x_2 = 2$,

$2 = \frac{5}{2}$, which is not true.

So there are no solutions to the system above.

Linear combinations: Multiply the j'th equation by c_j and add:

$$\begin{aligned} &(c_1 A_{11} + c_2 A_{21} + \ldots + c_m A_{m1})x_1 \\ + &(c_1 A_{12} + c_2 A_{22} + \ldots + c_m A_{m2})x_2 \\ + \ldots &(c_1 A_{1n} + c_2 A_{2n} + \ldots + c_m A_{mn})x_n = \\ &c_1 y_1 + \ldots + c_m y_m \end{aligned}$$

linear combination

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The key idea behind solving systems of equations is to replace the original equations by suitable linear combinations.