# FALCONER CONJECTURE IN THE PLANE FOR RANDOM METRICS

S. Hofmann and A. Iosevich

December 19, 2002

ABSTRACT. The Falconer conjecture says that if a compact planar set has Hausdorff dimension $> 1$, then the Euclidean distance set $\Delta(E) = \{|x - y| : x, y \in E\}$ has positive Lebesgue measure. In this paper we prove, under the same assumptions, that for almost every ellipse $K$, $\Delta_K(E) = \{||x - y||_K : x, y \in E\}$ has positive Lebesgue measure, where $|| \cdot ||_K$ is the norm induced by an ellipse $K$. Equivalently, we prove that if a compact planar set has Hausdorff dimension $> 1$, then $\Delta(TE)$ has positive Lebesgue measure for almost every transformations $T$ with bounded positive eigenvalues. We also use this result to deduce a version of the Erdos Distance Conjecture in the plane.

## INTRODUCTION

Let $E \subset [0,1]^d$, $d \geq 2$. The celebrated Falconer conjecture says that if the Hausdorff dimension of $E$ exceeds $\frac{d}{2}$, then the distance set $\Delta(E) = \{|x - y| : x, y \in E\}$ has positive Lebesgue measure.

The initial result in this direction was proved by Falconer ([Falconer86]) who showed that $\Delta(E)$ has positive Lebesgue measure if the Hausdorff dimension of $E$ exceeds $\frac{d+1}{2}$. This result was later improved in all dimensions by Bourgain ([Bourgain94]). The best known result in the plane is due to Tom Wolff who proved that $\Delta(E)$ has positive Lebesgue measure provided that the Hausdorff dimension of $E$ is greater than $\frac{4}{3}$.

The purpose of this paper is to prove that the conclusion of the Falconer conjecture holds for almost every linear perturbation of the Euclidean metric. More precisely, let

$$(0.1) \qquad \Delta_K(E) = \{||x - y||_K : x, y \in E\},$$

where $K$ is a symmetric bounded convex set in $\mathbb{R}^2$ and $|| \cdot ||_K$ is the distance induced by $K$. Our main result is the following.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

**Theorem 0.1.** *Let $E \subset [0,1]^2$ be a set of Hausdorff dimension greater than 1. Let $K_{a,\phi}$ denote the ellipse with eccentricities $a_1, a_2$, $1 \leq a_j \leq 2$, rotated by the angle $\phi$, and let $\Delta_{a,\phi}(E)$ denote the corresponding distance set. Then $\mathbb{E}(m(\Delta_{a,\phi}(E))) > 0$, where the expectation is taken with respect to the uniform distribution on $[1,2] \times [1,2] \times [0,\pi]$, and $m$ denotes the one-dimensional Lebesgue measure.*

The sharpness of Theorem 0.1 is demonstrated by a modification of a construction due to Falconer ([Falconer86]). Let $0 < s \leq 2$. Let $q_1, q_2, \ldots, q_i \ldots$ be a sequence of positive integers such that $q_{i+1} \geq q_i^i$. Let $E_i = \{x \in \mathbb{R}^2 : 0 \leq x_j \leq 1, |x_j - p_j/q_i| \leq q_i^{-\frac{2}{s}}$ for some integers $p_j$, $j = 1, 2\}$. It is not hard to see (see e.g. [Wolff02]) that the Hausdorff dimension of $E = \cap_{i=1}^{\infty} E_i$ is $s$. Also, $\Delta(E) \subset \cap_{i=1}^{\infty} \Delta(E_i)$.

Let $P_i = \{p = (p_1, p_2) : 0 \leq p_j \leq q_i\}$. Let $\Delta_{a,\phi}^{(0,0)}(P_i) = \{||p||_{a,\phi} : p \in P_i\}$. It is immediate that $\#\Delta_{a,\phi}^{(0,0)}(P_i) \leq (q_i + 1)^2$. By translation invariance it follows that $\#\Delta_{a,\phi}(P_i) \leq (q+1)^2$. We conclude that $\Delta_{a,\phi}(E_i)$ is contained in at most $(q_i + 1)^2$ intervals of length $\approx q_i^{-\frac{2}{s}}$. It follows that the Hausdorff dimension of $\Delta(E)$ is $\leq s$. Thus if $s < 1$, $\Delta_{a,\phi}(E)$ has Lebesgue measure 0 for every $a, \phi$.

Observe that in the above example it is not necessary for $p_i$s to be integers. It is quite sufficient for $P_i$ to be sufficiently dense, and separated in the sense that there exists $c > 0$ such that $|p - p'| \geq c$, $p, p' \in P_i$, $p \neq p'$. This observation allows us to use Theorem 0.1 to deduce a version of the Erdos Distance Conjecture. See e.g. [PaAg95] for a thorough description of the Erdos Distance Problem and related concepts.

The classical Erdos Distance Problem is to obtain a lower bound for $\Delta(S)$, where $S$ is a finite subset of $\mathbb{R}^2$. Erdos Distance Conjecture says that for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $\#\Delta(S) \geq C_\epsilon(\#S)^{1-\epsilon}$. A slightly weaker version of the Erdos Distance can be stated as follows.

**Asymptotic version of the Erdos Distance Conjecure.** *Let $A$ be a separated subset of $\mathbb{R}^2$. Suppose that $A$ is actually a Delone set, which means that there exists a universal constant $C > 0$ such that the intersection of $A$ with any cube of side-length $R$ contains $\geq CR^2$ elements of $A$. Then for any $\epsilon > 0$ there exists a positive constant $C'_\epsilon > 0$ such that*

$$(0.1) \qquad \#\Delta(A \cap [-R, R]^2) \geq C'_\epsilon R^{2-\epsilon}.$$

Using the above counter-example used to establish sharpness of Theorem 0.1 we can prove the following random variant of the asymptotic version of the Erdos Distance Conjecture. See [IoLa2003] for a systematic application of this mechanism to non-Euclidean distances in $\mathbb{R}^d$.

**Corollary 0.2.** *Let $A$ be as in the statement of the asymptotic version of the Erdos Distance Conjecture. Then for any $\epsilon > 0$, there exists $C'_\epsilon > 0$ such that*

$$(0.2) \qquad \#\Delta_{a,\phi}(A \cap [-R, R]^2) \geq C'_\epsilon R^{2-\epsilon}.$$

2

*for almost every* $(a, \phi) \in [1, 2]^2 \times [0, \pi]$.

To prove Corollary 0.2, let $E_i, E$, and $\{q_i\}$ be defined as in the counter-example above with $P_i = A \cap [0, q_i]^2$. Suppose that $\#\Delta_{a,\phi}(P_i) \leq Cq_i^{2-\epsilon}$ for some $\epsilon > 0$ for a sequence of *is* going to infinity. Then $\Delta_{a,\phi}(E_i)$ can be covered by $\leq Cq_i^{2-\epsilon}$ intervals of length $\approx q^{-\frac{2}{s}}$. It follows that the Hausdorff dimension of $\Delta_{a,\phi}(E)$ is $\leq s - \frac{s\epsilon}{2}$. Now let $s = 1 + \delta$, $\delta > 0$. We conclude that the Hausdorff dimension of $\Delta_{a,\phi}(E)$ is $\leq 1 + \delta - \frac{(1+\delta)\epsilon}{2} < 1$ if $\delta$ is sufficiently small. This is a contradiction because Theorem 0.1 implies that $\Delta_{a,\phi}(E)$ has positive Lebesgue measure for almost every $(a, \phi)$.

*Remark 1.* Another way of stating Theorem 0.1 is to say that if $E \subset [0, 1]^2$ be a set of Hausdorff dimension greater than 1, then for almost every linear transformation $T$ of the form rotation followed by an anisotropic dilation, $\Delta(TE)$ has positive Lebesgue measure.

*Remark 2.* The proof of Theorem 0.1 below will show that the conclusion of Theorem 0.1 still holds if in the definition of $\Delta_{a,\phi}$, the Euclidean circle is replaced by any smooth curve with everywhere non-vanishing curvature. In other words, Theorem 0.1 does not just hold for perturbations of the Euclidean metric, but also for perturbations of any metric whose unit circle is smooth and has non-vanishing curvature.

*Remark 3.* In principle, an appropriate variant of Theorem 0.1 should hold in the context of two-dimensional Riemannian manifolds. We shall address this issue in a subsequent paper.

*Remark 4.* It is worth noting that large classes of two-dimensional sets of Hausdorff dimension $\alpha > 1$ for which the Falconer conjecture holds can be constructed using more complicated probabilistic schemes. For example, let $A$ be a compact subset of the real line of Hausdorff dimension $\frac{1}{2} < \frac{\alpha}{2} < 1$, and $W(t)$ an almost surely continuous version of the real Wiener process. A theorem due to J.P. Kahane (see e.g. [Kahane68]) says that $E = W(A)$ is almost surely a Salem set of dimension $\alpha > 1$, which means that $E$ is equipped with a Borel measure $\mu$ such that $|\hat{\mu}(\xi)| \leq C(1 + |\xi|)^{-\frac{\alpha}{2}}$. Using Theorem 1.1 below one easily deduces that $\Delta(E)$ has positive Lebesgue measure, so Falconer conjecture holds for this class of fractal sets.

*Remark 5.* In particular, the proof of Corollary 0.2 shows that Falconer Distance Conjecture implies the asymptotic Erdos Distance conjecture. It would be nice to prove that the Falconer conjecture in fact implies the standard Erdos Distance Conjecture. This amounts to eliminating the well-distributivity assumption on $A$ in the statement of Corollary 0.2 and replacing $C'_\epsilon R^{2-\epsilon}$ on the right hand side of (0.2) by $C'_\epsilon (\#A \cap [-R, R]^2)^{1-\epsilon}$.

METHOD OF PROOF OF THEOREM 0.1

We use a modification of the following result due to Mattila ([Mattila87]).

**Theorem 1.1.** *Let $E \subset [0,1]^2$ with a Borel measure $\mu$. Suppose that*

$$(1.1) \qquad \int_1^\infty \left( \int_{S^1} |\widehat{\mu}(t\omega)|^2 d\omega \right)^2 t \, dt < \infty.$$

*Then $\Delta(E)$ has positive Lebesgue measure. (Here and throughout $\Delta(E) = \Delta_K(E)$ with $K$ a unit disk).*

In fact, the argument used to prove Theorem 0.1 combined with a standard stationary phase argument (see e.g. Theorem 0.4 below) yields the following slightly more general result:

**Theorem 1.2.** *Let $E \subset [0,1]^2$ with Borel measure $\mu$. Let $K$ be a bounded convex set such that $\partial K$ is smooth and has everywhere non-vanishing curvature. Suppose that*

$$(1.2) \qquad \int_1^\infty \left( \int_{\partial K} |\widehat{\mu}(t\omega_K)|^2 d\omega_K \right)^2 t \, dt < \infty,$$

*where $d\omega_K$ denotes the Lebesgue measure on $\partial K$, the boundary of $K$.*
*Let $K^* = \{\xi : \sup_{x \in K} x \cdot \xi \leq 1\}$, the convex set dual to $K$. Then $\Delta_{K^*}(E)$ has positive Lebesgue measure.*

We shall give a proof of Theorem 1.2 at the end of this paper for the sake of completeness.
In view of Theorem 1.1 and Theorem 1.2, Theorem 0.1 follows from the following estimate

**Theorem 1.3.** *Let $E \subset [0,1]^2$ with Borel measure $\mu$. Suppose that the Hausdorff measure of $E$ is greater than 1. Then*

$$(1.3) \qquad \int_0^\pi \int \int_1^\infty \left( \int_{S^1} |\widehat{\mu}(t\omega_{a,\phi})|^2 d\omega \right)^2 t \, dt \psi(a) \, da \, d\phi < \infty,$$

*where $\omega_{a,\phi}$ is the standard parameterization of the ellipse with eccentricities $a_1, a_2$, rotated by $\phi$, and $\psi$ is a smooth cutoff function identically equal to 1 in $[1,2]^2$ and vanishing outside $[1/2, 4]^2$.*

The result due to Wolff mentioned above was proved by showing that under the assumptions of Theorem 0.1,

$$(1.4) \qquad \int_{S^1} |\widehat{\mu}(t\omega)|^2 d\omega \lesssim t^{-\frac{\alpha}{2}},$$

and an example due to Sjolin ([Sjolin93]) shows that this estimate cannot, in general, be improved. This means that the proof of Theorem 0.3 must heavily rely on averaging in $t$, $a$, and $\phi$.

Throughout the paper we shall make use of the following version of the method of stationary phase. See e.g. [Sogge93], Theorem 1.2.1.

4

**Theorem 1.4.** *Let $S$ be a convex smooth hyper-surface in $\mathbb{R}^d$ with everywhere non-vanishing Gaussian curvature and $d\mu$ a $C_0^\infty$ measure on $S$. Then*

$$(1.5) \qquad |\widehat{d\mu}(\xi)| \lesssim |\xi|^{-\frac{d-1}{2}}.$$

*Moreover, suppose that $\Gamma \in \mathbb{R}^d \setminus (0,\dots,0)$ is the cone consisting of all vectors $\xi$ normal to $S$ at some point $x$ in a fixed relatively compact neighborhood of support of $d\mu$. Then*

$$(1.6) \qquad \left| \left( \frac{\partial}{\partial \xi} \right)^\alpha \widehat{d\mu}(\xi) \right| = O((1+|\xi|)^{-N}) \ \forall \ N, \ \text{if } \xi \notin \Gamma,$$

*and*

$$(1.7) \qquad \widehat{d\mu}(\xi) = \sum_{j=1}^{2} e^{-2\pi i x_j \cdot \xi} a_j(\xi), \ \text{if } \xi \in \Gamma,$$

*where the finite sum is taken over the points $x_j \in \mathcal{N}$ having $\xi$ as a normal and*

$$(1.8) \qquad \left| \left( \frac{\partial}{\partial \xi} \right)^\alpha a_j(\xi) \right| \leq C_\alpha (1+|\xi|)^{-\frac{d-1}{2}-|\alpha|}.$$

**Notation:** Throughout this paper, $a \lesssim b$ means that there exists a positive constant $C$ such that $a \leq Cb$. Similarly, $a \lessapprox b$, with respect to a parameter $s$, means that given $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $a \leq C_\epsilon s^\epsilon b$.

<center>Proof of Theorem 1.3</center>

Let $\beta \in C_0^\infty$ be the usual Littlewood-Paley cutoff, i.e $\beta$ is supported in $[1/2, 4]$, $\beta \equiv 1$ in $[1, 2]$, and $\sum_j \beta(2^{-n} \cdot) \equiv 1$. Let $\omega_{a,-\phi} = \rho_\phi^{-1} \delta_a \omega$, where $\rho_\phi$ denotes the rotation by the angle $\phi$, and $\delta_a(x) = (a_1 x_1, a_2 x_2)$. Let $\mu$ be a probability measure on $E$ such that $\mu(\{x \in E : |x - y| \leq r\}) \leq Cr^\alpha$, $r > 0$, where $\alpha$ is the Hausdorff dimension of $E$. For the existence of such a measure, see, for example, Proposition 8.2 in [Wolff02]. Define

$$I_n = \int\int \left( \int_{S^1} |\widehat{\mu}(t\omega_{a,-\phi})|^2 d\omega \right)^2 t\beta(2^{-n}t)dt\psi(a)dad\phi$$

$$= \int\int\int\int |\widehat{\mu}(t\omega_{a,-\phi})|^2 |\widehat{\mu}(t\omega'_{a,-\phi})|^2 d\sigma(\omega)d\sigma(\omega')t\beta(2^{-n}t)dt\psi(a)dad\phi$$

$$(2.1) \quad = \int\int\int\int\int e^{2\pi i((x-y)\cdot t\omega'_{a,-\phi}+(x'-y')\cdot t\omega'_{a,-\phi})} d\mu^* d\sigma(\omega)d\sigma(\omega')t\beta(2^{-n}t)dt\psi(a)dad\phi,$$

<center>5</center>

where

$$(2.2) \qquad\qquad d\mu^* = d\mu(x)d\mu(y)d\mu(x')d\mu(y'),$$

and $d\sigma$ denotes a $C_0^\infty$ measure on the sphere. Using a partition of unity we see that it is enough to consider this situation.

Integrating in $\omega$ and $\omega'$ first, we get

$$(2.3) \qquad \int\int\int\int \widehat{\sigma}(t(x-y)_{a,\phi})\widehat{\sigma}(t(x'-y')_{a,\phi})d\mu^*\beta(2^{-n}t)dt\psi(a)dad\phi,$$

where $\sigma$ is $C_0^\infty$ measure on $S^1$ as above, and $x_a = (a_1(x_1\cos(\phi) - x_2\sin(\phi)), a_2(x_1\sin(\phi) + x_2\cos(\phi)))$.

**Case 1:** $2^n|x-y| \lesssim 1$ **and** $2^n|x'-y'| \lesssim 1$. Then for any $\epsilon > 0$, (1.3) is bounded by

$$(2.4) \qquad \int |x-y|^{-\alpha+\epsilon}|x'-y'|^{-\alpha+\epsilon}t^{-2\alpha+2\epsilon}d\mu(x)d\mu(y)d\mu(x')d\mu(y')tdt < \infty$$

as desired if $\epsilon$ is sufficiently small.

**Case 2:** $2^n|x-y| >> 1$ **and** $2^n|x'-y'| >> 1$. Observe that the symbol of order $0$ resulting from pulling $(t|x-y|)^{-\frac{1}{2}}$ from the symbol $a_j$ given by Theorem 1.4 can be incorporated into the smooth cut-off $\beta$ without effecting the size or the support of $\beta$ or its derivatives. We shall suppress (harmless) dependence of $\beta$ on $x, y, x', y', a, \phi$ in what follows. Using this observation and (1.7) above, we see that (2.3) can be written as a sum of terms of the form

$$(2.5) \quad \int\int\int\int e^{2\pi i t(|x-y|_{a,\phi} - |x'-y'|_{a,\phi})}t^{-1}|x-y|_{a,\phi}^{-\frac{1}{2}}\,|x'-y'|_{a,\phi}^{-\frac{1}{2}}\,td\mu^*\beta(2^{-n}t)dt\psi(a)dad\phi.$$

We must also consider the term where the phase function is $2\pi i t(|x-y|_{a,\phi} + |x'-y'|_{a,\phi})$, but this case is very easy. Let $\eta$ be a small parameter to be determined later. We have

$$\int\int\int\int e^{2\pi i t(|x-y|_{a,\phi} + |x'-y'|_{a,\phi})}t^{-1}|x-y|_{a,\phi}^{-\frac{1}{2}}\,|x'-y'|_{a,\phi}^{-\frac{1}{2}}\,td\mu^*\beta(2^{-n}t)dt\psi(a)dad\phi$$

$$= 2^n\int_{\{(x,y,x',y'):||x-y|+|x'-y'||\lesssim 2^{-n(1-\eta)}\}}|x-y|^{-\frac{1}{2}}|x'-y'|^{-\frac{1}{2}}d\mu^* + O(2^n 2^{-n\eta N})$$

$$\lesssim 2^n\int |x-y|^{-\alpha+\epsilon}2^{-n(1-\eta)(\alpha-\frac{1}{2}-\epsilon)}|x'-y'|^{-\alpha+\epsilon}2^{-n(1-\eta)(\alpha-\frac{1}{2}-\epsilon)}d\mu^*$$

$$(2.6) \qquad\qquad\qquad \lesssim 2^{2n(1-\eta)}2^{-2n(1-\eta)\alpha}2^{2n(1-\eta)\epsilon}$$

6

which sums if $\epsilon < \alpha - 1$ and $\eta < 1$. The second line of (2.6) follows from the first using the fact, which follows easily by integration by parts, that the Fourier transform of a smooth compactly supported function decays rapidly at infinity.

We now turn our attention to (2.5). Integrating in $t$ first we get

$$(2.7) \qquad 2^n \int \int \int \widehat{\beta}(2^n(|x-y|_{a,\phi} - |x'-y'|_{a,\phi}))|x-y|_{a,\phi}^{-\frac{1}{2}}|x'-y'|_{a,\phi}^{-\frac{1}{2}}\psi(a)dad\phi d\mu^*.$$

Localizing to the sets where $2^{-k} \leq |x-y|_{a,\phi} \leq 2^{-k+1}$, $2^{-k'} \leq |x'-y'|_{a,\phi} \leq 2^{-k'+1}$, we obtain

$$(2.8) \qquad I_{n,k,k'} \approx 2^n 2^{\frac{k}{2}} 2^{\frac{k'}{2}} \int \int \int |\widehat{\beta}(2^n(|x-y|_{a,\phi} - |x'-y'|_{a,\phi}))|\psi(a)dad\phi d\mu^*$$

Let $x - y = |x-y|e^{iA}$, $x' - y' = |x'-y'|e^{iB}$. We now decompose $x-y$ and $x'-y'$ into sectors of aperture $\delta$ to be determined. Let $S_\delta^{j,k}$ denote the "rectangle" formed by intersection the annulus $\{z : 2^{-k} \leq |z| \leq 2^{-k+1}\}$ and the angular sector $\{z : j\delta \leq A \leq (j+1)\delta\}$. Define $S_\delta^{j',k'}$ analogously.

Let

$$I_{n,k,k'}^{j,j',\delta} = 2^n 2^{\frac{k}{2}} 2^{\frac{k'}{2}} \int_{S_\delta^{j,k} \otimes S_\delta^{j',k'}} \int \int |\widehat{\beta}(2^n(|x-y|_{a,\phi} - |x'-y'|_{a,\phi}))|\psi(a)dad\phi d\mu^*$$

$$= 2^{n+\frac{k}{2}+\frac{k'}{2}} \int_{\{x-y\in S_\delta^{j,k};x'-y'\in S_\delta^{j',k'};||x-y|_{a,\phi}-|x'-y'|_{a,\phi}|\lesssim 2^{-n(1-\eta)}\}} \int \int \psi(a)dad\phi d\mu^*$$

$$(2.9) \qquad\qquad\qquad\qquad + O(2^n 2^{\frac{k}{2}} 2^{\frac{k'}{2}} 2^{-n\eta N}),$$

where $\eta > 0$ is a small parameter to be chosen later.

Observe that for $j, j'$ fixed, we have that $x - y$ is in a $\delta 2^{-k}$ by $2^{-n(1-\eta)}$ rectangle and $x' - y'$ is in a $\delta 2^{-k'}$ by $2^{-n(1-\eta)}$ rectangle. Also observe that if both $k, k' \gtrsim n(1-\eta)$, then we have a simple estimate analogous to the one in Case 1 if $\eta$ is chosen to be sufficiently small and $\delta \approx 1$. Otherwise, if at least one of $k, k' << n(1-\eta)$, then they both are, and, moreover, $k \approx k'$. Therefore, in what follows we may assume that we are in the latter situation, so that the double index appearing above may now be replaced by the single index $k$.

Now, multiplying both sides by $||x-y|_{a,\phi} + |x'-y'|_{a,\phi}|$ and computing the area of the resulting set, we see that

$$(2.10) \quad |\{(a_1,a_2) : ||x-y|_{a,\phi} - |x'-y'|_{a,\phi}| \lesssim 2^{-n(1-\eta)}\}| \lesssim 2^{-(n(1-\eta)-k)}[|A-B||A+\phi|]^{-1}$$

where, without loss of generality, $A \geq B$, so that $j - j' \geq 1$ unless $j = j'$. We also take $A, B, \phi$ to be small and positive. (The other cases follow by the same argument). It follows that if $j \neq j'$,

$$(2.11) \qquad\qquad I_{n,k}^{j,j',\delta} \lesssim 2^{n+k} \int_S \int \max\left\{1, \frac{2^{-(n(1-\eta)-k)}}{|j-j'|\delta|\delta j + \phi|}\right\} d\phi d\mu^*.$$

7

where integration is over the set $S$ where $x - y \in S_\delta^{j,k}$, $x' - y' \in S_\delta^{j',k'}$ (recall that $k \approx k'$). Let $\delta \approx 2^{-n(1-\eta)+k}$. Then for $j'$ and $x - y$ fixed, we have that $x' - y'$ is located in a ball of radius $\approx 2^{-n(1-\eta)}$. We have

$$I_{n,k}^{j,j'} \lesssim 2^{n+k} \int_S \int \max \left\{ 1, \ \frac{1}{|j - j'||\delta j + \phi|} \right\} d\phi d\mu^*$$

$$(2.12) \qquad \qquad \lesssim 2^{n+k} \frac{1}{|j - j'|} \int_S d\mu(x) d\mu(y) d\mu(x') d\mu(y').$$

We must estimate

$$(2.13) \qquad \qquad \sum_{k,n} 2^{n+k} \sum_{j,j'} \frac{1}{|j - j'|} \int_S d\mu(x) d\mu(y) d\mu(x') d\mu(y').$$

Let $l = j - j'$. We get

$$(2.14) \qquad \qquad \sum_{k,n} 2^{n+k} \sum_l \frac{1}{l} \sum_{j'} \int_S d\mu(x) d\mu(y) d\mu(x') d\mu(y').$$

For a fixed $x - y$ and a sector given indexed by $j$, $x' - y'$ is contained in a ball of radius $C2^{-n(1-\eta)}$ since $\delta 2^{-k} \approx 2^{-n(1-\eta)}$. Fixing $y'$ and integrating in $x'$ we get $2^{-n(1-\eta)\alpha}$ since $\mu$ is $\alpha$-dimensional. Taking the union over all the sectors indexed by $j'$, we have $x - y$ in the annulus of width $2^{-k}$. Fixing $y$ and integrating in $x$, we pick up $C2^{-k\alpha}$. It follows that (2.14) is bounded by a constant multiple of

$$(2.15) \qquad \qquad \sum_{k,n} 2^{n+k} \sum_l \frac{1}{l} 2^{-n(1-\eta)\alpha} 2^{-k\alpha} \lesssim 1$$

if $(1 - \eta)\alpha > 1$, since $l$ runs up to $C2^{n(1-\eta)-k}$, the number of sectors.

If $j = j'$, (2.14) takes the form

$$(2.16) \qquad \qquad \sum_{k,n} 2^{n+k} \sum_{j'} \int_S d\mu(x) d\mu(y) d\mu(x') d\mu(y')$$

which is bounded by the same argument.

**Case 3:** $2^n |x - y| >> 1$ **and** $2^n |x' - y'| \lesssim 1$. This case basically vacuous, which can be seen as follows. We have

$$\int \int \int \int e^{2\pi i t(|x-y|_{a,\phi} - |x'-y'|_{a,\phi})} t^{-\frac{1}{2}} |x - y|_{a,\phi}^{-\frac{1}{2}} \, t d\mu^* \beta(2^{-n}t) dt \psi(a) da d\phi$$

8

$$= 2^n 2^{\frac{n}{2}} \int_{\{2^n |x'-y'| \lesssim 1\}} \int \int \widehat{\beta}(2^n(|x-y|_{a,\phi} - |x'-y'|_{a,\phi}))|x-y|_{a,\phi}^{-\frac{1}{2}} \psi(a) da d\phi d\mu^*$$

$$\lesssim 2^n 2^{\frac{n}{2}} \int_{\{2^n |x'-y'| \lesssim 1\}} \int \int \widehat{\beta}(2^n(|x-y|_{a,\phi}))|x-y|_{a,\phi}^{-\frac{1}{2}} \psi(a) da d\phi d\mu^*$$

$$(2.17) \qquad \lesssim 2^n 2^{\frac{n}{2}} 2^{-n\alpha} \int \int \widehat{\beta}(2^n(|x-y|))|x-y|^{-\frac{1}{2}} d\mu(x) d\mu(y)$$

Localizing to the sets where $2^{-k} \leq |x-y| \leq 2^{-k+1}$, we obtain

$$(2.18) \qquad 2^n 2^{\frac{n}{2}} 2^{-n\alpha} 2^{\frac{k}{2}} \int \int \widehat{\beta}(2^n(|x-y|)) \, d\mu(x) d\mu(y) \lesssim 2^n 2^{\frac{n}{2}} 2^{-n\alpha} 2^{\frac{k}{2}} 2^{-n\alpha}$$

which sums since $k << n$. This completes the proof of Theorem 0.3, and, consequently, the proof of Theorem 0.1.

<center>PROOF OF THEOREM 1.2</center>

Define the measure $\nu_0$ by

$$(3.1) \qquad \int f d\nu_0 = \int f(||x-y||_{K^*}) d\mu(x) d\mu(y).$$

Let

$$(3.2) \qquad d\nu(s) = e^{i\frac{\pi}{4}} s^{-\frac{1}{2}} d\nu_0(s) + e^{-i\frac{\pi}{4}} |s|^{-\frac{1}{2}} d\nu_0(-s).$$

Since $\nu_0$ is supported on $\Delta_{K^*}(E)$, $\nu$ is supported on $\Delta_{K^*}(E) \cup -\Delta_{K^*}(E)$. We have

$$(3.3) \qquad \int_{\partial K} |\widehat{\mu}(t\omega_K)|^2 d\omega_K = \int \widehat{\sigma}_t * \mu d\mu,$$

where $\sigma$ is the measure on $\partial K$.

Using a variant of Theorem 1.4 (see e.g. [Herz62]), we see that

$$(3.4) \qquad \widehat{\sigma}_t(x) = 2(t\rho^*(x))^{-\frac{1}{2}} \cos\left(2\pi\left(t\rho^*(x) - \frac{1}{8}\right)\right) + O((t|x|)^{-\frac{3}{2}}),$$

where

$$(3.5) \qquad \rho^*(x) = \sup_{y \in \partial K} x \cdot y.$$

<center>9</center>

In other words, $\rho^*(x) = ||x||_{K^*}$.

By definition,

$$\hat{\nu}(k) = e^{i\frac{\pi}{4}} \int ||x - y||_{K^*}^{-\frac{1}{2}} e^{-2\pi i k||x-y||_{K^*}} d\mu(x)d\mu(y)$$

$$+ e^{-i\frac{\pi}{4}} \int ||x - y||_{K^*}^{-\frac{1}{2}} e^{2\pi i k||x-y||_{K^*}} d\mu(x)d\mu(y)$$

(3.6)
$$= 2 \int ||x - y||_{K^*}^{-\frac{1}{2}} \cos\left(2\pi\left(|k|\rho^*(x - y) - \frac{1}{8}\right)\right) d\mu(x)d\mu(y).$$

By (3.3),

$$\int |\widehat{\mu}(k\omega_K)|^2 d\omega_K = |k|^{-\frac{1}{2}} \int 2||x - y||_{K^*}^{-\frac{1}{2}} \cos\left(2\pi\left(|k|\rho^*(x - y) - \frac{1}{8}\right)\right) d\mu(x)d\mu(y)$$

$$+ O\left(\int_{|x-y|\geq|k|^{-1}} (|k||x - y|)^{-\frac{3}{2}} d\mu(x)d\mu(y)\right)$$

$$+ O\left(\int_{|x-y|\leq|k|^{-1}} (|k||x - y|)^{-\frac{1}{2}} d\mu(x)d\mu(y)\right)$$

$$= |k|^{-\frac{1}{2}} \int 2||x - y||_{K^*}^{-\frac{1}{2}} \cos\left(2\pi\left(|k|\rho^*(x - y) - \frac{1}{8}\right)\right) d\mu(x)d\mu(y)$$

(3.7)
$$+ O\left(\int (|k||x - y|)^{-\frac{1}{2}} d\mu(x)d\mu(y)\right)$$

for any $\alpha \in [1/2, 3/2]$. It follows that

(3.8)
$$\hat{\nu}(k) = |k|^{\frac{1}{2}} \int |\widehat{\mu}(k\omega)|^2 d\omega + O(|k|^{\frac{1}{2}-\alpha} I_\alpha(\mu)).$$

Since the error term is clearly in $L^2(|k| \geq 1)$, we see that $\hat{\nu} \in L^2(|k| \geq 1)$ if and only if $|k|^{\frac{1}{2}} \int |\widehat{\mu}(k\omega_K)|^2 d\omega_K \in L^2(|k| \geq 1)$. This precisely what Theorem 1.2 asserts.

10

## References

[Bourgain94]    J. Bourgain, *Hausdorff dimension and distance sets*, Israel J. Math. **87** (1994), 193-201.

[Falconer86]    K. J. Falconer, *On the Hausdorff dimensions of distance sets*, Mathematika **32** (1986), 206-212.

[Herz62]    C. Herz, *Fourier transforms related to convex sets*, Ann. of Math. **75** (1962), 81-92.

[IoLa2003]    A. Iosevich and I. Laba, *K-distance sets and Falconer conjecture*, (in preparation) (2002).

[Mattila87]    P. Mattila, *Spherical averages of Fourier transforms of measures with finite energy: dimensions of intersections and distance sets*, Mathematika **34** (1987), 207-228.

[PA95]    J. Pach and and P. Agarwal, *Combinatorial Geometry*, Wiley-Interscience Series (1995).

[Sogge93]    C. D. Sogge, *Fourier integrals in classical analysis*, Cambridge University Press (1993).

[Sjolin93]    P. Sjolin, *Estimates of spherical averages of Fourier transforms and dimensions of sets*, Mathematika **40** (1993), 322-330.

[Wolff02]    T. Wolff, *Lectures in Harmonic Analysis*, California Institute of Technology Class Lectures Notes (revised by I. Laba) (2002).

S. Hofmann, University of Missouri  email: hofmann @ math.missouri.edu

A. Iosevich, University of Missouri  email: iosevich @ math.missouri.edu