INTRODUCTION TO THE BESICOVITCH/KAKEYA CONJECTURE: PART I

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Abstract. This is a set of notes on the Besicovitch/Kakeya conjecture written for the summer school for mathematically talented high-school students to be held in August 2004. The notes are dedicated to Tom Wolff whose legacy continues to tower over the study of the Besicovitch/Kakeya conjecture and many other important problems in mathematics to this day.

Introduction

This is a set of notes written for the summer program in combinatorial mathematics to be held at the University of Missouri-Columbia in August 2004. The program will last a total of five days, with approximately 1.5 hours of lectures and 1.5 hours of problem solving each day. The purpose of this program and these notes is to introduce a motivated high-school students to one of the most far-reaching and beautiful problems of modern mathematics- the Besicovitch/Kakeya conjecture which related the size of sets in the Euclidean space with the number unit line segments in different directions. Due to the technical nature of the full-fledged Besicovitch/Kakeya conjecture and its connection to problems in analysis, partial differential equations and analytic number theory, no effort is made to present an exhaustive and rigorous view of the subject. Instead, the students are given a glimpse of this sophisticated problem in a simple "discrete" setting, where the most salient features of the Kakeya conjecture can already be seen.

The program was initially designed for two weeks and consequently, these notes contain way too much material to be covered in one week. It is my hope that the participants will choose to continue reading and exploring these notes in particular and the subject matter in general after the program is over. I also plan to reintroduce some of the material in these notes in the process of teaching Math 395, the Problem Solving class, during the Fall semester.

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In the first part of these notes we present the basics of the Besicovitch/Kakeya problem that does not require anything beyond the Cauchy-Schwartz in equality (described and illustrated in Section 2 below) and basic counting. In the second part of the notes slicing arguments, discrete Fourier transform, discrete restriction theory, and more advanced arithmetic arguments are introduced.

The student is expected to work hard on these notes. This is not bed time reading, nor is it a fantasy novel. You must have a pen and plenty of paper handy, and expect to fill up about ten pages of calculations for every page you read. Mathematics is not a spectator sport, so create in addition to reading and computing. Every time you see a theorem or a calculation, try to formulate a new one. Every time you see a proof, try to find a better one. And most importantly, have fun!

Acknowledgements. Much of the material in these notes is either inspired by or taken directly from a set of notes, entitled "On Besicovitch sets", by Ben Green, a survey article by Tom Wolff, entitled "Recent work connected with the Kakeya problem" ([W99]), class notes by Terry Tao on the Besicovitch/Kakeya that can be found at http://www.math.ucla.edu/ ˜ tao, a survey of the Besicovitch/Kakeya problem by Izabella Laba, which can be found on her website at http://www.math.ubc.ca/ ˜ ilaba, and a semi-expository article by A. Iosevich, entitled "Geometric measure theory and Fourier analysis" ([I04]).

Basic setup

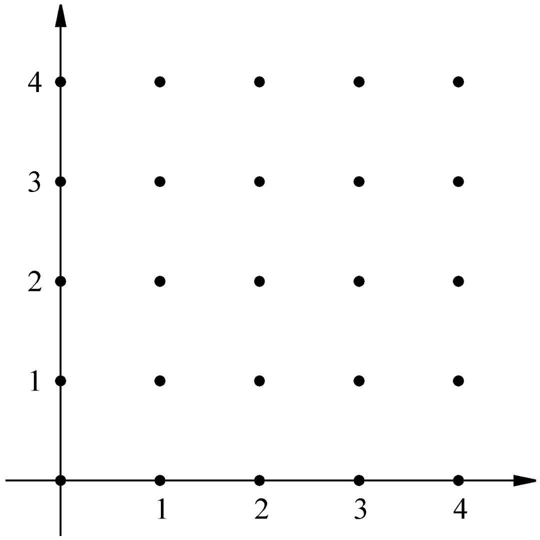
Let q be a positive integer, prime in the sense that an integer a divides q if and only if $a = 1$ or $a = q$. Define $\mathbb{F}_q$ to be the set $\{0, 1, 2, \ldots, q-1\}$ with the rule that addition and multiplication is taken modulo q. What this means is that if $a \in \mathbb{F}_q$ and $b \in \mathbb{F}_q$, $a + b$ (in the world of $\mathbb{F}_q$) is obtained by adding a and b in the standard way and computing the remainder after division by q. Similarly, to compute $a \cdot b$, we multiply a and b in the standard way and again compute the remainder after the division by q.

Example 1.1. *Let $q = 5$. Then $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$. Suppose that we want to multiply 2 and 4 in the world of $\mathbb{F}_5$. Well, $2 \cdot 4 = 8$ in the sense of regular multiplication. If we divide 8 by 5, the remainder is 3. Thus $2 \cdot 4 = 3$ in the world of $\mathbb{F}_5$.*

Now let's compute $2 + 4$. In the sense of regular addition, this equals 6. The remainder of division of 6 by 5 is 1. Thus $2 + 4 = 1$ in the world of $\mathbb{F}_5$.

After this point, we shall stop saying "in the world of $\mathbb{F}_q$". We shall simply perform our addition and multiplication according to the rules we just described and illustrated.

We now introduce a "grid" $\mathbb{F}_q^d$, defined as a set of d-tuples $(a_1, a_2, \ldots, a_d)$, such that a_j is an element of $\mathbb{F}_q$.



The grid $\mathbb{F}_5^2$.

Example 1.2. *The set* $\mathbb{F}_3^2$ *consists of* 9 *pairs:* $(0,0)$, $(0,1)$, $(0,2)$, $(1,0)$, $(1,1)$, $(1,2)$, $(2,0)$, $(2,1)$, *and* $(2,2)$.

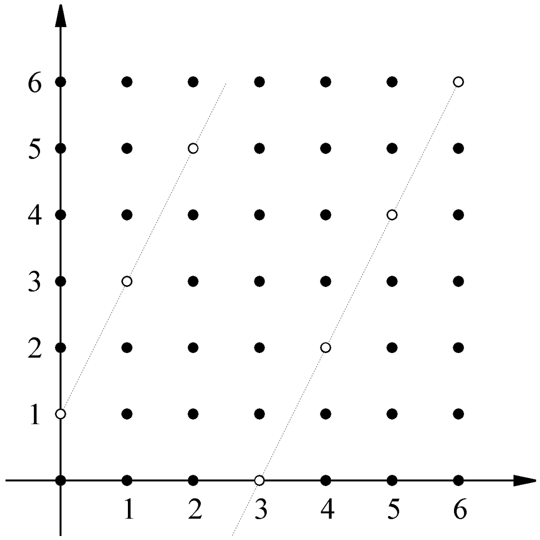
The set $\mathbb{F}_3^3$ *consists of* 27 *triples. Write them out!*

Exercise 1.1. *Show that* $\mathbb{F}_q^d$, d *a positive integer, consists of* q^d *elements.*

We now introduce the notion of a line in $\mathbb{F}_q^d$. Let $x \in \mathbb{F}_q^d$ and let $v \in \mathbb{F}_q^d$ with the additional restriction that $v \neq (0,0,\ldots,0)$. Let

$$L(x,v) = \{x + tv : t = 0,1,\ldots,q-1\}. \tag{1.1}$$

Example 1.3. *Let* $x = (0,1)$ *and* $v = (1,1)$ *and let* $q = 3$. *Then* $L(x,v)$ *is the "line" consisting of points* $(0,1)$, $(1,2)$, *and* $(2,0)$. *Why is that? Well, by definition,* $L((0,1),(1,1)) = \{(0,1) + t(1,1) : t = 0,1,2\}$. *If* $t = 0$, *we get* $(0,1)$ *easily enough. If* $t = 1$, *we get* $(1,2)$, *no problem. If* $t = 2$, *we get* $(0+2\cdot 1, 1+2\cdot 1) = (2,0)$ *since* $1+2\cdot 1 = 3$ *which is* 0 *in the world of* $\mathbb{F}_3$.



A line in $\mathbb{F}_7^2$.

Exercise 1.2. *Let $q = 3$. Show that $L((0,1),(1,1))$ is the same line as $L((0,1),(2,2))$ and $L((1,2),(1,1))$. What is going on here?*

Exercise 1.3. *We have seen in the previous exercise that given $x \in \mathbb{F}_q^d$, there may exist $v \neq v' \neq (0,\dots,0)$ such that $L(x,v)$ and $L(x,v')$ are the same line. You probably observed that this problem takes place if and only if $v = \lambda v'$ for some $\lambda \in \mathbb{F}_q$. (If you did not observe this, please verify it right away). Let V be a subset of $\mathbb{F}_q^d$ with the following properties:*

(1) *If $v \in V$, $v \neq (0,\dots,0)$.*
(2) *If $v \in V$ and $v' \in V$, there does not exist $\lambda \in \mathbb{F}_q$ such that $v = \lambda v'$.*

Suppose that V is maximal in the sense that it is impossible to add even more element to V without violating one of the properties stated above. (Note that without this restriction we may simply take V to consist of a single point, say, $(1,0,\dots,0)$).

Compute $\#V$, the number of elements of V.

In the standard (Euclidean) space, two different lines either do not intersect at all, or intersect at a single point. We shall now verify that the same is true of lines in $\mathbb{F}_q^d$.

Exercise 1.4. *Two different lines $L(x,v)$ and $L(x',v')$ in $\mathbb{F}_q^d$ either do not intersect at all or intersect at a single point. If $d = 2$ prove that two distinct lines $L(x,v)$ and $L(x',v')$ do not intersect if and only if there exists $\lambda \in \mathbb{F}_q$ such that $v = \lambda v'$.*

We are now ready to state the main problem to be studied in these notes.

Besicovitch/Kakeya conjecture. *Let $K \subset \mathbb{F}_q^d$, $d \geq 2$, such that for every $v \in \mathbb{F}_q^d$ with $v \neq (0,\dots,0)$, there exists $x \in \mathbb{F}_q^d$ so that $L(x,v) \subset K$. Then there exists $C > 0$, independent of q, such that*

$$\#K \geq Cq^d. \tag{1.2}$$

To put it simply, the Besicovitch/Kakeya conjecture says that if K contains a line with every possible slope, then this set occupies a positive proportion of the points in $\mathbb{F}_q^d$. In short,

$$\text{MANY SLOPES} \rightarrow \text{MANY POINTS} \tag{1.3}$$

The Besicovitch/Kakeya conjecture is solved in two dimensions. However, in higher dimensions, it far from being resolved. For example, the best known result in three dimensions (see [KLT00] is that

$$\#K \geq Cq^{\frac{5}{2}+10^{-10}}. \tag{1.4}$$

One of the motivations behind this set of notes is to convince you to dive head first into this mysterious problem which is not terribly likely to be completely solved any time soon.

Before we start proving results pertaining to the Besicovitch/Kakeya conjecture, we shall develop some preliminary concepts that will serve to build up the necessary technique and intuition for the results that come later.

Cauchy-Schwartz inequality and some simple consequences[1]

In this section we shall follow a procedure often considered nasty, but the one I hope to convince you to appreciate. We shall work backwards, discovering concepts as we go along, instead of stating them ahead of time. Let a and b denote two real numbers. Then

$$(a-b)^2 \geq 0. \tag{2.1}$$

This statement is so vacuous, you are probably wondering why I am telling you this. Nevertheless, expand the left hand side of (2.1). We get

$$a^2 - 2ab + b^2 \geq 0, \tag{2.2}$$

which implies that

$$ab \leq \frac{a^2+b^2}{2}. \tag{2.3}$$

Now consider

$$A_N = \sum_{k=1}^{N} a_k = a_1 + \cdots + a_N, \ B_N = \sum_{k=1}^{N} b_k = b_1 + \cdots + b_N, \tag{2.4}$$

[1]Some people call this inequality the Cauchy-Schwartz-Buniakowski inequality. I have a suspicion that this inequality was known and relatively widely used long before any of the three individuals in question was born. I decided to stick with the "Cauchy-Schwartz" usage primarily out of habit.

where $a_1, \ldots, a_N$, and $b_1, \ldots, b_N$ are real numbers. Let

$$X_N = \left(\sum_{k=1}^{N} a_k^2\right)^{\frac{1}{2}}, \; Y_N = \left(\sum_{k=1}^{N} b_k^2\right)^{\frac{1}{2}}. \tag{2.5}$$

Our goal is to take advantage of (2.3). Let's take a look at

$$\sum_{k=1}^{N} a_k b_k = X_N Y_N \sum_{k=1}^{N} \frac{a_k}{X_N} \cdot \frac{b_k}{Y_N}$$

$$\leq X_N Y_N \sum_{k=1}^{N} \left[\frac{1}{2}\left(\frac{a_k}{X_N}\right)^2 + \frac{1}{2}\left(\frac{b_k}{Y_N}\right)^2\right]. \tag{2.6}$$

Exercise 2.1. *Explain using complete English sentences how* (2.6) *follows from* (2.3).

Exercise 2.2. *Explain why if C is a constant, then $\sum_{k=1}^{N} C a_k = C \sum_{k=1}^{N} a_k$.*

Exercise 2.3. *Explain why $\sum_{k=1}^{N}(a_k + b_k) = \sum_{k=1}^{N} a_k + \sum_{k=1}^{N} b_k$.*

We now use Exercise 2.2 and 2.3 to rewrite (2.6) in the form

$$X_N Y_N \frac{1}{2} \frac{1}{X_N^2} \sum_{k=1} a_k^2 + X_N Y_N \frac{1}{2} \frac{1}{Y_N^2} \sum_{k=1}^{N} b_k^2$$

$$= X_N Y_N \frac{1}{2} \frac{1}{X_N^2} X_N^2 + X_N Y_N \frac{1}{2} \frac{1}{Y_N^2} Y_N^2$$

$$= \frac{1}{2} X_N Y_N + \frac{1}{2} X_N Y_N = X_N Y_N. \tag{2.7}$$

Putting everything together, we have shown that

$$\sum_{k=1}^{N} a_k b_k \leq \left(\sum_{k=1}^{N} a_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{N} b_k^2\right)^{\frac{1}{2}}. \tag{2.8}$$

This is known as the Cauchy-Schwartz inequality.

Exercise 2.4. *(quite difficult if you do not know calculus) Let $1 < p < \infty$ and define the exponent p' by the equation $\frac{1}{p} + \frac{1}{p'} = 1$. Then*

$$\sum_{k=1}^{N} a_k b_k \leq \left(\sum_{k=1}^{N} |a_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{N} |b_k|^{p'} \right)^{\frac{1}{p'}}. \tag{2.9}$$

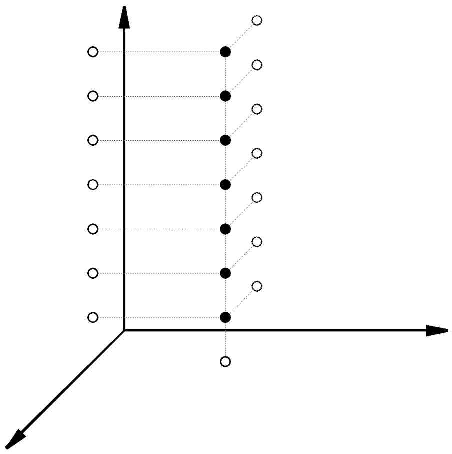
Observe that (2.9) reduces to (2.8) if $p = 2$. Hint: prove that $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$ and proceed as in the case $p = 2$. One way to prove this inequality is to set $a^p = e^x$ and $b^{p'} = e^y$ (why are we allowed to do that?). Let $\frac{1}{p} = t$ and observe that $0 \leq t \leq 1$. We are then reduced to showing that for any real valued x, y and $t \in [0,1]$, $e^{tx+(1-t)y} \leq te^x + (1-t)e^y$. Let $f(t) = e^{tx+(1-t)y}$ and $g(t) = te^x + (1-t)e^y$. Observe that $f(0) = g(0) = e^y$ and $f(1) = g(1) = e^x$. Can you complete the argument?

Box inequality. Let's now try to see what Cauchy-Schwartz (C-S) inequality is good for. Let S_N be a finite set of N points in $\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_j \text{ is a real number}\}$, the three-dimensional Euclidean space. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and define

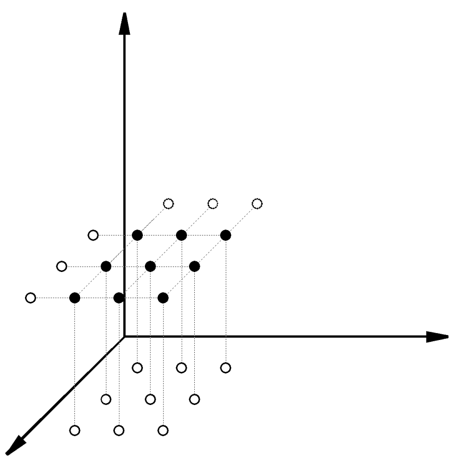
$$\pi_1(x) = (x_2, x_3), \ \pi_2(x) = (x_1, x_3), \ \text{and } \pi_3(x) = (x_1, x_2). \tag{2.10}$$

The question we ask is the following. We are assuming that $\#S_N = N$. What can we say about the size of $\pi_1(S_N), \pi_2(S_N)$, and $\pi_3(S_N)$? Before we do anything remotely complicated, let's make up some silly looking examples and see what we can learn from them.

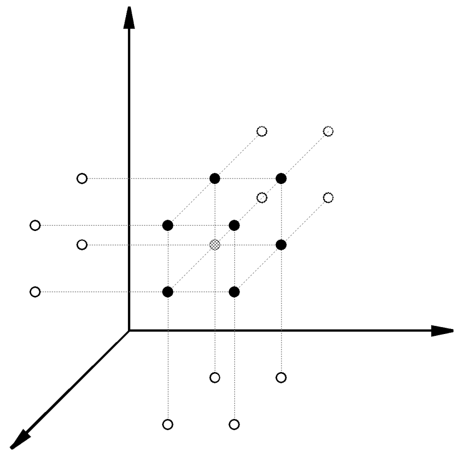
Let $S_N = \{(0, 0, k) : k \text{ integer } k = 0, 1, \ldots, N-1\}$. This set clearly has N elements. What is $\pi_3(S_N)$ in this case. It is precisely the set $\{(0, 0)\}$, a set consisting of one element. However, $\pi_2(S_N)$ and $\pi_1(S_N)$ are both $\{(0, k) : k = 0, 1, \ldots, N-1\}$, sets consisting of N elements. In summary, one of the projections is really small and the others are as large as they can be.



Let's be a bit more even handed. Let $S_N = \{(k,l,0) : k,l \text{ integers } 1 \le k \le \sqrt{N}, 1 \le l \le \sqrt{N}\}$, where $\sqrt{N}$ is an integer. Again $\#S_N = N$. What do projections look like? Well, S_N is already in the (x_1,x_2)-plane, so $\pi_3(S_N) = \{(k,l) : k,l \text{ integers } 1 \le k \le \sqrt{N}, 1 \le l \le \sqrt{N}\}$. It follows that $\#\pi_3(S_N) = N$. On the other hand, $\pi_2(S_N) = \{(k,0) : k \text{ integer } 1 \le k \le \sqrt{N}\}$, and $\pi_1(S_N) = \{(l,0) : l \text{ integer } 1 \le l \le \sqrt{N}\}$, both containing $\sqrt{N}$ elements. Again we see that it is difficult for all the projections to be small.



Let's think about our examples so far from a geometric point of view. The first example is "one-dimensional" since the points all lie on a line. The second example is "two-dimensional" since the points lie on a plane. Let's now build a truly "three-dimensional" example with as much symmetry as possible. Let $S_N = \{(k,l,m) : k,l,m \text{ integers } 1 \le k,l,m \le N^{\frac{1}{3}}\}$, where $N^{\frac{1}{3}}$ is an integer. Again, $\#S_N = N$, as required. The projections this time all look the same. We have $\pi_1(S_N) = \{(l,m) : l,m \text{ integers } 1 \le l,m \le N^{\frac{1}{3}}\}$, a set of size $N^{\frac{2}{3}}$, and the same is true of $\#\pi_2(S_N)$ and $\#\pi_3(S_N)$.



Let's summarize what happened. In the case when all the projections have the same size, each projection has $N^{\frac{2}{3}}$ elements. We will see in a moment that for any S_N, one of the projections must of size at least $N^{\frac{2}{3}}$. We will see here and later in these notes that C-S inequality is very useful in showing that the "symmetric" case is "optimal", whatever that means in a given instance.

Before starting a more detailed investigation, consider the two dimensional case. Take a set of N points in $\mathbb{R}^2$ and consider projections onto x_1-axis and x_2-axis, respectively. Can we prove by a simple geometric argument that one of these projections must contain at least $C\sqrt{N}$ points? Well, let S_N be the set in question. Observe that $\chi_{S_N}(x_1,x_2) \le \chi_{\pi_1(S_N)}(x_2) \cdot \chi_{\pi_2(S_N)}(x_1)$ (see Exercise 2.5 below). It follows that $\sum_{x_1,x_2} \chi_{S_N}(x_1,x_2) = \sum_{x_1,x_2} \chi_{\pi_1(S_N)}(x_2) \cdot \chi_{\pi_2(S_N)}(x_1) = \sum_{x_1} \chi_{\pi_2(S_N)}(x_1) \cdot \sum_{x_2} \chi_{\pi_1(S_N)}(x_2)$ (why?). It follows that $N = \#S_N \le \#\pi_1(S_N) \cdot \#\pi_2(S_N) \le (\max_{j=1,2} \#\pi_j(S_N))^2$. We conclude that indeed $\max_{j=1,2} \pi_j(S_N) \ge \sqrt{N}$ as promised.

The point of considering the two-dimensional case is that while it does not entail any of the interesting complexities of the higher dimensional situation, it is based on the same intuition. Indeed, let us think about the two-dimensional case from a slightly different point of view. Suppose for a moment that $\#\pi_1(S_N)$ is smaller than $c\sqrt{N}$, where c is a constant. Then S_N must consist of at most $c\sqrt{N}$ columns, by definition of π_1. On the other hand, the total number of points in all of those columns is N, by assumption. It follows that one of these columns has more than $\frac{N}{c\sqrt{N}}$ points. We conclude, by setting $c = 1$, that either $\#\pi_1(S_N) \ge \sqrt{N}$, or $\#\pi_2(S_N) \ge \sqrt{N}$, since the latter is precisely what it means for a column to have more than $\sqrt{N}$ points. This gives an "alternate" argument for the two-dimensional case. Observe that the argument given in the previous paragraph is at leat superficially mechanical, while the argument we just went over is visual and conceptual. Are the arguments really different, however? As an informal exercise, cut through the mechanical non-sense of the first argument and explain why it is the same as the second one.

The three dimensional case is not going to fall quite so easily. To see this, let us try to run the argument of the previous paragraph. Suppose that $\#\pi_1(S_N) < N^{\frac{2}{3}}$. This means that S_N consists of fewer than $N^{\frac{2}{3}}$ columns of points over the (x_2, x_3)-axis. Since the total number of points is N, this tells us that one of the columns has more than $N^{\frac{1}{3}}$ points. This is not enough, however, and more careful analysis is needed. The proof can be completed this way with some work and I urge you to try! We will take a slightly different road below in order to illustrate what a beautiful bookkeeping tool the C-S inequality often is.

To start our analysis of the three dimensional case we need the following basic definition. Let S be any set. Define $\chi_S(x) = 1$ if $x \in S$ and 0 otherwise.

Exercise 2.5. *Let S_N be as above. Then*

$$\chi_{S_N}(x) \leq \chi_{\pi_1(S_N)}(x_2, x_3)\chi_{\pi_2(S_N)}(x_1, x_3)\chi_{\pi_3(S_N)}(x_1, x_2). \tag{2.11}$$

With Exercise 2.5 in tow, we write

$$\begin{aligned}
N = \#S_N = \sum_x \chi_{S_N}(x) &\leq \sum_x \chi_{\pi_1(S_N)}(x_2, x_3)\chi_{\pi_2(S_N)}(x_1, x_3)\chi_{\pi_3(S_N)}(x_1, x_2) \\
&= \sum_{x_1,x_2} \chi_{\pi_3(S_N)}(x_1, x_2) \sum_{x_3} \chi_{\pi_1(S_N)}(x_2, x_3)\chi_{\pi_2(S_N)}(x_1, x_3) \\
&\leq \left(\sum_{x_1,x_2} \chi^2_{\pi_3(S_N)}(x_1, x_2) \right)^{\frac{1}{2}} \left(\sum_{x_1,x_2} \left(\sum_{x_3} \chi_{\pi_1(S_N)}(x_2, x_3)\chi_{\pi_2(S_N)}(x_1, x_3) \right)^2 \right)^{\frac{1}{2}} \\
&= I \times II. \qquad (2.12)
\end{aligned}$$

Now,

$$I = \left(\sum_{x_1,x_2} \chi^2_{\pi_3(S_N)}(x_1, x_2) \right)^{\frac{1}{2}} = \left(\sum_{x_1,x_2} \chi_{\pi_3(S_N)}(x_1, x_2) \right)^{\frac{1}{2}} = (\#\pi_3(S_N))^{\frac{1}{2}}. \tag{2.13}$$

On the other hand,

$$II^2 = \sum_{x_1,x_2} \left(\sum_{x_3} \chi_{\pi_1(S_N)}(x_2, x_3)\chi_{\pi_2(S_N)}(x_1, x_3) \right)^2$$

$$= \sum_{x_1,x_2} \sum_{x_3} \sum_{x_3'} \chi_{\pi_1(S_N)}(x_2,x_3)\chi_{\pi_2(S_N)}(x_1,x_3)\chi_{\pi_1(S_N)}(x_2,x_3')\chi_{\pi_2(S_N)}(x_1,x_3')$$

$$\leq \sum_{x_1,x_2} \sum_{x_3} \sum_{x_3'} \chi_{\pi_1(S_N)}(x_2,x_3)\chi_{\pi_2(S_N)}(x_1,x_3')$$

$$= \sum_{x_2,x_3} \chi_{\pi_1(S_N)}(x_2,x_3) \sum_{x_1,x_3'} \chi_{\pi_2(S_N)}(x_1,x_3') = \#\pi_1(S_N)\cdot\#\pi_2(S_N). \tag{2.14}$$

Putting everything together, we have proved that

$$\#S_N \leq \sqrt{\#\pi_1(S_N)}\sqrt{\#\pi_2(S_N)}\sqrt{\#\pi_3(S_N)}. \tag{2.15}$$

Exercise 2.6. *Verify each step above. Where was C-S inequality used? Why does* $\chi^2_{\pi_j(S_N)}(x) = \chi_{\pi_j(S_N)}(x)$*?*

The product of three positive numbers certainly does not exceed the largest of these numbers raised to the power of three. It follows from this and (2.15) that

$$N = \#S_N \leq \max_{j=1,2,3} (\#\pi_1(S_N))^{\frac{3}{2}}. \tag{2.16}$$

We conclude by raising both sides to the power of $\frac{2}{3}$ that

$$\# \max_{j=1,2,3} \pi_j(S_N) \geq N^{\frac{2}{3}} \tag{2.17}$$

as claimed.

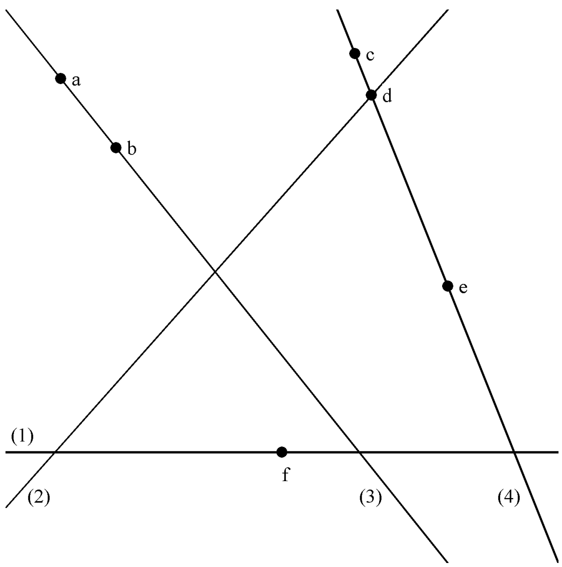
Exercise 2.7. *Let* Ω *be a convex set in* $\mathbb{R}^3$*. This means that for any pair of points* $x,y\in\Omega$*, the line segment connecting* x *and* y *is entirely contained in* Ω*. Prove that* $vol(\Omega) \leq \sqrt{area(\pi_1(\Omega))}\cdot\sqrt{area(\pi_2(\Omega))}\cdot\sqrt{area(\pi_3(\Omega))}$.

If you can't prove this exactly, can you at least prove using (2.15) *and its proof that* $\max_{j=1,2,3} area(\pi_j(\Omega)) \geq (vol(\Omega))^{\frac{2}{3}}$*? This would say that a convex object of large volume has at least one large coordinate shadow. Using politically incorrect language this can be restated as saying that if an elephant is overweight, there must be a way to place a mirror to make this obvious...*

Exercise 2.8. *(Project question) Generalize* (2.15)*. What do I mean, you ask... Replace three dimensions by* d *dimensions. Replace projections onto two-dimensional coordinate planes by projections onto* k*-dimensional coordinate planes, with* $1\leq k\leq d-1$*. Finally, replace the right hand side of* (2.15) *by what it should be...*

Incidences and matrices. Consider a set of n lines and n points in the plane. Define an incidence to be a pair (p, l), where p is one of the points in our point set, l is one of the lines in our set of lines, and p lies on l. Let $I(n)$ denote the total number of incidences determined by a given set of n points and a given set of n lines. In order to avoid needless headaches we assume that every point in our point set lies on at least one line in our set of lines, and every line in our line set contains at least one point in our point set.

How large can $I(n)$ be? Well, it is clear that $I(n) \leq n^2$. This observation is not terribly valuable, however, since $I(n)$ cannot possibly be this large! I mean, how can every line contain every point, and every point lie on every line?! You might retort that maybe, just maybe, it is possible for about $n/10$ lines to contain about $n/100$ points each, and for each of those points to be contain in about $n/1000$ of those lines. We shall see that nothing like that can happen.



Our main tools in this endeavor are matrices and the C-S inequality. Recall that a N by N matrix A is an array with n rows and n columns. The elements of A are designated by a_{ij}, where i determines the row and j determines the column. Let's define A as follows. Enumerate the n points in our point set from 1 to n, and do the same for lines in our set of lines. Let $a_{ij} = 1$ if the i'th point lies on the j'th line, and 0 otherwise. Observe that if j and j' are fixed, with $j \neq j'$,

$$a_{ij} \cdot a_{ij'} = 1 \tag{2.18}$$

for at most one value of i. This is because $a_{ij} \cdot a_{ij'} = 1$ if and only if $a_{ij} = 1$ and $a_{ij'} = 1$. This means that the i'th point is on the j'th lines and also on the j'th line. Intersection of two distinct lines is either empty or consists of exactly one point. It follows that indeed the equality in (2.18) can hold for at most one i.

We are now ready for action. What is $I(n)$? It is nothing more than the total number of 1s in A! Since A consists of only 1s and 0s,

$$I(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \right) \cdot 1$$

$$\le \left(\sum_{i=1}^{n} 1 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \right)^2 \right)^{\frac{1}{2}} = \sqrt{n} \cdot \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \right)^2 \right)^{\frac{1}{2}}. \tag{2.19}$$

Now,

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \right)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j'=1}^{n} a_{ij} a_{ij'}$$

$$= \sum_{i=1}^{n} \sum_{1 \le j, j' \le n; j \ne j'} a_{ij} a_{ij'} + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 = apple + orange. \tag{2.20}$$

To estimate apple we use (2.18). Indeed, since $a_{ij} \cdot a_{ij'} = 1$ for at most one i,

$$apple \le \#\{(j, j') : 1 \le j, j' \le n; j \ne j'\} = n^2 - n. \tag{2.21}$$

Exercise 2.9. *Write out the details of the equality on the right hand side of* (2.21)*.*

On the other hand,

$$orange \le \#\{(i, j) : 1 \le i, j \le n\} = n^2. \tag{2.22}$$

Putting everything together and using the fact that $n^2 - n \le n^2$, we see that

$$I(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \le \sqrt{2} \cdot n^{\frac{3}{2}}. \tag{2.23}$$

We conclude that the number of incidences between n points and n lines in the plane is at most $\sqrt{2} n^{\frac{3}{2}}$. Can this estimate be improved? Sure it can... The sharp answer is $I(n) \le C n^{\frac{4}{3}}$, where C is a fixed positive constant. This is the celebrated Szemeredi-Trotter incidence theorem ([ST83]) and it is sharp in the sense that one can construct a set of n lines and n points such that the number of incidences is approximately $n^{\frac{4}{3}}$, up to a constant. The proof of this result will appear in the second part of these notes.

Exercise 2.10. *Show that the estimate $I(n) \leq Cn^{\frac{3}{2}}$ we just obtained for points and lines in the plane is best possible for points and lines in $\mathbb{F}_q^2$. Hint: Take as your point set all the points in $\mathbb{F}_q^2$ and take as your line set all the lines in $\mathbb{F}_q^2$.*

Exercise 2.11. *Let S_N be a subset of the plane with N elements. Define $\Delta(S_N) = \{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2} : x=(x_1,x_2) \in S_N, y=(y_1,y_2) \in S_N\}$. Use* (2.23) *to show that $\#\Delta(S_N) \geq C\sqrt{N}$ for some constant C independent of N.*

Can you do better? The conjectured answer is that $\#\Delta(S_N) \geq C\frac{N}{\sqrt{\log(N)}}$. The best known result to date, due to Katz and Tardos ([KT04]*), based on the previous result due to Solymosi and Toth (*[SoT01]*) is $\#\Delta(S_N) \geq CN^{\beta}$, where $\beta \approx .86$. (See* [KT04]*).*

What about higher dimensions? If $S_N \subset \mathbb{R}^d$ of size N, prove that $\#\Delta(S_N) \geq CN^{\frac{1}{d}}$. Can you do better? The conjectured answer here is $\#\Delta(S_N) \geq CN^{\frac{2}{d}}$ in dimensions three and higher. Do you see where the exponent $\frac{2}{d}$ is coming from? Hint: Let $S_N = \{n = (n_1,\dots,n_d) : n_j \in \mathbb{Z}; 1 \leq n_j \leq N^{\frac{1}{d}}\}$.

Exercise 2.12. *Show that the number of incidences between n points and n two-dimensional planes in $\mathbb{R}^3$ can be n^2. Suppose that we further insist that the intersection of any three planes in our collection contains at most one point. Prove that the number of incidences is $\leq Cn^{\frac{5}{3}}$.*

More generally, prove that if we have n points and n $d-1$-dimensional planes in $\mathbb{R}^d$, then the number of incidences can be n^2. Show that the number of incidences is $\leq Cn^{2-\frac{1}{d}}$ if we further insist that the intersection of any d planes from our collection intersect at at most one point.

Exercise 2.13. *Prove that n points and n spheres of the same radius in $\mathbb{R}^d$, $d \geq 4$, can have n^2 incidences. Use the techniques of the chapter that when $d=2$ the number of incidences is $\leq Cn^{\frac{3}{2}}$. What can you say about the case $d=3$?*

Besicovitch/Kakeya conjecture in two dimensions

In this section we verify (1.2) in the case $d=2$. What you should be asking yourselves at every step, is where are we using the peculiarities of the two-dimensional space, and why this approach should be harder in higher dimensions.

We have a set $K \subset \mathbb{F}_q^2$ which contains a line in every direction. This means that there exist lines $L_1, L_2, \dots, L_{q+1}$ entirely contained in K with the additional property that any pair of these lines intersects at exactly one point. How do we know this? An obnoxious answer is that you verified exactly this in Exercise 1.3 and 1.4. Let's discuss it again, however. Consider $L(x,v)$ in two dimensions. How many choices are there for v? Well, $v=(v_1,v_2)$, so there are q^2-1 choices, since $v=(0,0)$ is forbidden. On the other hand, multiplying v by $\lambda \in \mathbb{F}_q$ leads to the same line. How many λs are there? Since it makes no sense to use $\lambda = 0$, there are $q-1$ relevant λs. It follows that K indeed contains $\frac{q^2-1}{q-1} = q+1$ lines with different "slopes". By Exercise 1.4 (not very difficult) each pair of such lines intersects at exactly one point.

Before we get on with the precise calculations, let's try to understand why the Besicovitch/Kakeya conjecture should be true in two-dimensions. As we have just seen, K contains $q+1$ lines of different "slopes". Choose one of these lines and call it the stem. The other q lines intersect this stem forming a sort of a hairbrush. Since two of these lines intersect at exactly one point, it is pretty clear that the total number of points in K is at least $(q+1)\cdot\frac{q}{2}=\frac{q(q+1)}{2}$. Of course, we need to make this argument precise, which is what we are about to do.

All the tools are now in place. Let $K'=\cup_{i=1}^{q+1}L_i$. Since $K'\subset K$, it suffices to prove that $\#K'\geq Cq^2$. Let $\chi_{L_i}(x)=1$ if $x\in L_i$ and 0 otherwise. We must somehow take advantage of the fact that we have $q+1$ lines with each pair intersecting at exactly one point. How do we "encode" intersections? Well,

$$\sum_{x\in K'}\chi_{L_i}(x)\chi_{L_j}(x)=\#\{x\in K':x\in L_i \text{ and } x\in L_j\}=\#(L_i\cap L_j), \tag{3.1}$$

since $L_i\subset K'$ and $L_j\subset K'$ by assumption.

With this observation in tow, consider

$$\sum_{x\in K'}\left[\chi_{L_1}(x)+\cdots+\chi_{L_{q+1}}(x)\right]^2=\sum_{x\in K'}\sum_{i=1}^{q+1}\sum_{j=1}^{q+1}\chi_{L_i}(x)\chi_{L_j}(x)$$

$$=\sum_{i=1}^{q+1}\sum_{j=1}^{q+1}\#(L_i\cap L_j)=2q(q+1), \tag{3.2}$$

where to obtain the first line we used (3.1) and to compute the second line we used the fact that $\#L_i\cap L_j=1$ if $i\neq j$.

All this is very nice, but we need to somehow get a hold on $\#K'$. This is where Section 2 comes in handy. The left hand side of the first line of (3.2) is a sum of something squared. This immediately :) reminds us of the Cauchy-Schwartz inequality! Indeed, C-S tells us that

$$\left(\sum_{x\in K'}\left[\chi_{L_1}(x)+\cdots+\chi_{L_{q+1}}(x)\right]\cdot 1\right)^2\leq \#K'\cdot\sum_{x\in K'}\left[\chi_{L_1}(x)+\cdots+\chi_{L_{q+1}}(x)\right]^2. \tag{3.3}$$

Plugging in (3.2) we see that

$$\left(\sum_{x\in K'}\left[\chi_{L_1}(x)+\cdots+\chi_{L_{q+1}}(x)\right]\right)^2\leq \#K'\cdot 2q(q+1), \tag{3.4}$$

or, equivalently,

$$\#K' \geq \frac{\left(\sum_{x \in K'} [\chi_{L_1}(x) + \cdots + \chi_{L_{q+1}}(x)]\right)^2}{2q(q+1)}. \tag{3.5}$$

We seem to be getting somewhere provided we can evaluate the numerator of (3.5). We have

$$\sum_{x \in K'} [\chi_{L_1}(x) + \cdots + \chi_{L_{q+1}}(x)] = \sum_{x \in K'} \sum_{j=1}^{q+1} \chi_{L_j}(x) = \sum_{j=1}^{q+1} \sum_{x \in K'} \chi_{L_j}(x). \tag{3.6}$$

Since $L_j \subset K'$,

$$\sum_{x \in K'} \chi_{L_j}(x) = \#L_j = q. \tag{3.7}$$

We conclude that

$$\left(\sum_{x \in K'} [\chi_{L_1}(x) + \cdots + \chi_{L_{q+1}}(x)] \right)^2 = q^2(q+1)^2. \tag{3.8}$$

Plugging this into (3.5) yields

$$\#K' \geq \frac{q(q+1)}{2}, \tag{3.9}$$

as promised.

The result we just presented was first proved in $\mathbb{R}^2$ (whatever that means :)) by R. Davies in 1971 ([D71]), though the proof above is much closer to the one given for a related problem by A. Cordoba ([C77]).

This seems to be the end of the story in two dimensions. Unfortunately, (or rather fortunately) mathematicians always find a way to complicate things. The following exercises give a taste of things to come in Part II of these notes where the level of fun (and pain) will get wretched up another notch.

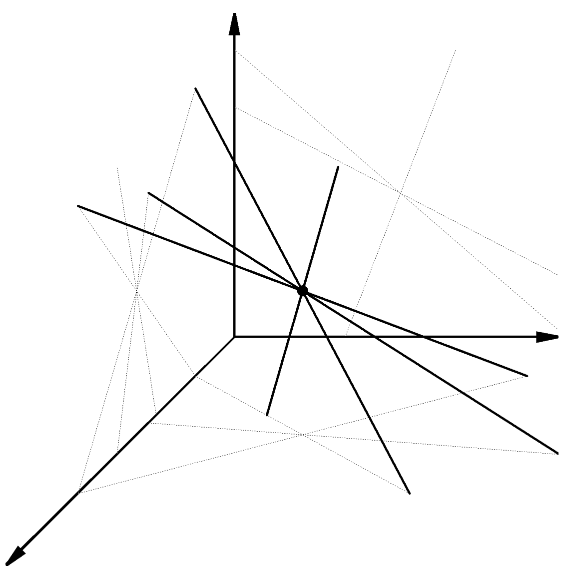
Exercise 3.1. *Find the smallest possible Besicovitch/Kakeya subset of $\mathbb{F}_q^2$. We know that it contains at least $\frac{q(q+1)}{2}$ elements. Get as close to this number as you can. Hint: consider $S = \{(x_1, x_2) \in \mathbb{F}_q^2 : x_1 + x_2^2$ is a square in $\mathbb{F}_q\}$. (A number s is a square in $\mathbb{F}_q$ if there exists $u \in \mathbb{F}_q$ such that $s = u^2$ in the world of $\mathbb{F}_q$).*

Exercise 3.2. *Let $0 < \alpha < 1$. Suppose that we only assume that K is a subset of $\mathbb{F}_q^2$ with the property that for every $v \neq (0,0)$, $v \in \mathbb{F}_q^2$, there exists an $x \in \mathbb{F}_q^2$ such that more than q^α points of $L(x,v)$ are contained in K'. What can you say about $\#K$? Once you obtain an answer, try to determine whether your estimate is "reasonable". More precisely, for various values of $\alpha < 1$, experiment with constructions of subset of $\mathbb{F}_q^2$ satisfying the required properties. This type of a formulation of the Kakeya problem is due to Hillel Furstenberg. See* [KT01] *and the references contained therein.*

Higher dimensional space is very annoying. It is no longer true that two lines are either "parallel" or intersect at a single point. It is quite easy for two lines to simply be in "parallel" planes which makes the structure of Besicovitch/Kakeya sets much harder to understand.

Bourgain's bush argument (late 80s). In this section we abandon our policy of systematically referencing the results we present. Instead, we refer the reader to Tom Wolff's beautiful survey article ([W99]) where all the relevant references are present.

Let's start with the following simple observation. Let K be a Besicovitch/Kakeya set in $\mathbb{F}_q^d$. How many lines must this set contain? Well, if have completed Exercise 1.3, we know that the answer is $\approx q^{d-1}$ (I am being obnoxious again...). Let's see why that is. Consider a line $L(x,v)$. We have $q^d - 1$ choices for v since $v = (0, \dots, 0)$ is not allowed. As before, v and $v' = \lambda v$, $\lambda \in \mathbb{F}_q$, $\lambda \neq 0$, lead to the same line. It follows that the number of distinct lines in K is at least $\frac{q^d-1}{q-1} \geq \frac{q^{d-1}}{2}$.



Bourgain's bush

Suppose that $\#K \leq \frac{q^{\frac{d+1}{2}}}{4}$. Then at least one point of K must lie on at least

$$L = \frac{q^{\frac{d-1}{2}}}{2} \tag{4.1}$$

lines entirely contained in K. To see this first observe the numbers of pairs (p, l), where $p \in K$, l is a line contained in K, and p lies on l, is at least $q \cdot \frac{q^{d-1}}{2} = \frac{q^d}{2}$ by the argument in the previous paragraph. By assumption, the number of points in K is $\leq \frac{q^{\frac{d+1}{2}}}{4}$. Then (4.1) follows since

$$\frac{q^d}{2} \leq \# \cup_p \cup_l \{(p,l) : p \in l\} \leq \frac{q^{\frac{d+1}{2}}}{4} \max_p \#\{(p,l) : p \in l\}, \tag{4.2}$$

where p is a point in K and l is a line in K. It follows that

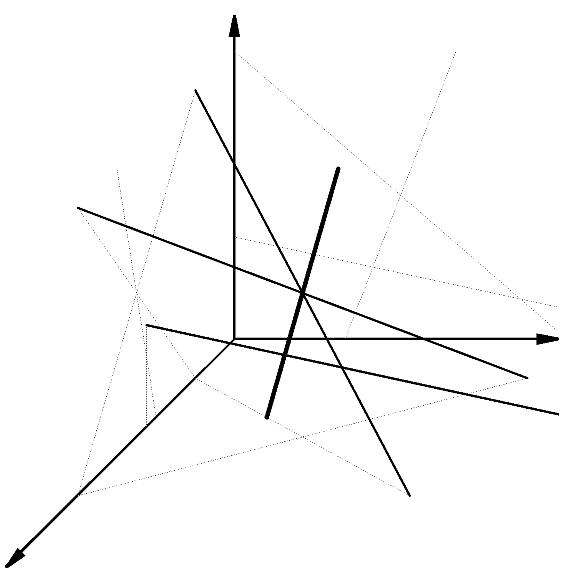
$$\max_p \#\{(p,l) : p \in l\} \geq \frac{q^{\frac{d-1}{2}}}{4}, \tag{4.3}$$

as advertised. What we just proved is that there exists a point $p_0 \in K$ which belongs to at least $\frac{q^{\frac{d-1}{2}}}{4}$ lines in K. Since each of these lines contains $q-1$ points aside from p_0,

$$\#K \geq 1 + L(q-1) \geq \frac{q^{\frac{d+1}{2}}}{4}. \tag{4.4}$$

Thus we have shown that a Besicovitch/Kakeya sets in $\mathbb{F}_q^d$ are $\frac{d+1}{2}$ "dimensional". This is horribly unsatisfactory since our goal is d, not $\frac{d+1}{2}$, and we can already do better than $\frac{d+1}{2}$ when $d = 2$. We did take an important step in the right direction, though, as the techniques we just developed will come in handy in a moment.

Wolff's hairbrush argument (mid 90s). What was the essence of the bush argument? If lines do not intersect much, we win because there are points all over the place. If lines do intersect, we look for places where lots of lines intersect in the same place. We call such a happy meeting place a bush. What we did above is first argued that there must exist a fairly large bush. We then estimated the number of points in this bush and obtained the estimate (4.41). As cute as this argument is, it is hopelessly naive if we are to get anywhere close to the full Besicovitch/Kakeya conjecture.



Wolff's hairbrush

Our next step in the direction of fame and glory (don't get too excited) is the hairbrush construction. Let K be a Besicovitch/Kakeya set and suppose that $\#K \leq q^{\frac{d+2}{2}}$. We repeat the argument we used in the first line of (3.2). We know by above that K contains at least $\frac{q^{d-1}}{2}$ lines with distinct "slopes". Let K' be the union of these lines. Reusing the proof of the two-dimensional Besicovitch/Kakeya conjecture, we have

$$q^2k^2 = \left(\sum_{x \in K'} [\chi_{L_1}(x) + \cdots + \chi_{L_k}(x)] \right)^2 \leq \#K' \cdot \sum_{x \in K'} [\chi_{L_1}(x) + \cdots + \chi_{L_{q+1}}(x)]^2$$

$=$

$$= \#K' \cdot \sum_{x \in K'} \sum_{i=1}^{k} \sum_{j=1}^{k} \chi_{L_i}(x)\chi_{L_j}(x) = \#K' \cdot \sum_{i=1}^{k} \sum_{j=1}^{k} \#(L_i \cap L_j), \tag{4.5}$$

where k is the number of lines (which by above is $\geq \frac{q^{d-1}}{2}$).

Exercise 4.1. *Why is the first line in* (4.5) *true? Did we use an inequality with a name in the second line? Which one?*

Since we have assumed that $\#K \leq q^{\frac{d+2}{2}}$, we also have $\#K' \leq q^{\frac{d+2}{2}}$. Plugging this into (4.5) we get

$$\sum_{i=1}^{k} \sum_{j=1}^{k} \#(L_i \cap L_j) \geq \frac{q^{\frac{3d-2}{2}}}{2}. \tag{4.6}$$

It follows that there exists $i = i_0$ such that

$$\sum_{1 \leq j \leq k; j \neq i_0} \#(L_j \cap L_{i_0}) \geq \frac{\frac{q^{\frac{3d-2}{2}}}{2}}{\frac{q^{d-1}}{2}} - q \geq \frac{q^{\frac{d}{2}}}{2}. \tag{4.7}$$

Exercise 4.2. *How did we go from* (4.6) *to* (4.7)*?*

We just proved that there exists a line L_{i_0}, called the base of the hairbrush, such that at least $m = \frac{q^{\frac{d}{2}}}{2}$ other lines contained in K' intersect it. We call this collection of these m lines the hairbrush, denoted by H.

Let Π_j denote the two-plane determined by L_{i_0} and L_j. Suppose that Π_j contains $n_j \geq 1$ lines from the hairbrush. We want to estimate $\#(\Pi_j \cap H)$ from below. Since Π_j is a two-dimensional plane, we should be able to use Section 3. Unfortunately, in that section we only learned to deal with sets containing approximately q lines with different slopes. In this case we have n_j lines, which may be smaller than q. This predicament forces us to rewrite the argument in Section 3 for the purpose at hand. Let $L_1, \dots, L_{n_j}$ be the lines in the hairbrush (after possibly doing some relabelling) that are contained in $\Pi_j \cap H$. We have

$$q^2(n_j+1)^2 = \left(\sum_{x \in \Pi_j \cap H} [\chi_{L_1}(x) + \cdots + \chi_{L_{n_j}}(x)] \right)^2$$

$$\leq \#(\Pi_j \cap H) \sum_{x \in \Pi_j \cap H} [\chi_{L_1}(x) + \cdots + \chi_{L_{n_j}}(x)]^2$$

$$= \#(\Pi_j \cap H) \sum_{i=1}^{n_j+1} \sum_{i'=1}^{n_j+1} \#(L_i \cap L_{i'}) = \#(\Pi_j \cap H) \cdot ((n_j+1)q + (n_j+1)(n_j))$$

$$\leq 2 \cdot \#(\Pi_j \cap H) \cdot (n_j+1)q. \tag{4.8}$$

It follows that

$$\#(\Pi_j \cap H) \geq \frac{1}{2}(n_j+1)q \geq \frac{qn_j}{4}. \tag{4.9}$$

Exercise 4.3. *Why does n_j+1 appear all over the place in* (4.8) *instead of n_j? Hint: Don't forget the base of the hairbrush...*

We are almost done since

$$\#K \geq \#H \geq \frac{q}{4} \sum_{j=1}^{t} n_j = \frac{qm}{4} \geq \frac{q^{\frac{d+2}{2}}}{8}. \tag{4.10}$$

We just proved that if K is a Besicovitch/Kakeya set in $\mathbb{F}_q^d$, then $\#K \geq \frac{q^{\frac{d+2}{2}}}{8}$. This is not quite the Besicovitch/Kakeya conjecture, but we are getting closer!

We conclude these notes with an exercise which may well be a gateway to further progress on the Besicovitch/Kakeya conjecture.

Exercise 4.4. *Does a hairbrush in the argument above need to contain a line of every "slope"? Given an explicit examples proving that it does not. Now suppose that you have a Besicovitch/Kakeya set containing a hairbrush containing a line with every possible "slope". Prove that $\#K \geq Cq^d$. Prove that same conclusion follows if instead of assuming that the hairbrush contains a line with every possible "slope", it only contains $\geq cq^{d-1}$ lines with different "slopes".*

Can you construct and example of a Besicovitch/Kakeya set K such that no hairbrush contains $\geq cq^{d-1}$ lines with different "slopes"?

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