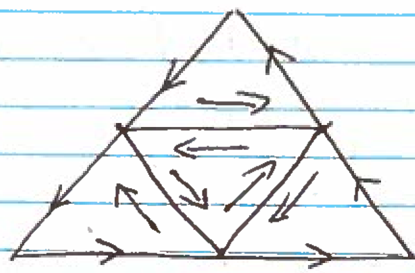
Mar 25, 2019

Theorem: Let G be an open set & let $f: G \to \mathbb{C}$ be a differentiable function. Then f is analytic on G.

Proof: We must prove that f' is continuous on each open disk $\subset G$, so assume that G = open disk.



$$\underbrace{\int_T f}_{\text{big triangle}} = \sum_{j=1}^{4} \underbrace{\int_{T^{(j)}} f}_{\text{small triangles}}$$

$$\exists\, T' \ni \left|\int_{T^{(1)}} f\right| \geq \left|\int_{T_j} f\right|, \quad j = 1, 2, 3, 4$$

$$\ell(T_j) = \tfrac{1}{2}\ell(T), \quad \operatorname{diam} T_j = \tfrac{1}{2} \operatorname{diam} T$$

$$\& \quad \left|\int_T f\right| \leq 4\left|\int_{T^{(1)}} f\right|.$$

Perform the same procedure on $T^{(1)}$, obtaining $T^{(2)}, \ldots$ $\{T^{(n)}\}$ w/ interior $\Delta^{(n)}$

$$\Delta^{(1)} \supset \Delta^{(2)} \supset \ldots \supset \Delta^{(n)} \supset \ldots$$

$$\left|\int_{T^{(n)}} f\right| \leq 4\left|\int_{T^{(n+1)}} f\right|; \quad \ell(T^{n+1}) = \tfrac{1}{2}\ell(T^n)$$

&

$$\operatorname{diam}\left(\Delta^{(n+1)}\right) = \tfrac{1}{2} \operatorname{diam} \Delta^{(n)}$$

(2)

It follows that $\left|\int_T f\right| \le 4^n \left|\int_{T^{(n)}} f\right|$

$$\ell(T^{(n)}) = \frac{1}{2^n}\ell, \quad \ell = \ell(T)$$

$$\operatorname{diam} \Delta^{(n)} = \frac{1}{2^n} d, \quad d = \operatorname{diam}(\Delta)$$

By Cantor's theorem, $\bigcap_{n=1}^{\infty} \Delta^{(n)}$ consists of one point, z_0

Given $\epsilon > 0$, find $\delta > 0 \ni B(z_0, \delta) \subset G$ &

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \epsilon |z - z_0| \quad \text{whenever } |z - z_0| < \delta.$$

Choose $n \ni \operatorname{diam} \Delta^{(n)} = \left(\frac{1}{2}\right)^n d < \delta$. Since $z_0 \in \Delta^{(n)}$,

$$\Delta^{(n)} \subset B(z_0, \delta).$$

By Cauchy, $\int_{T^{(n)}} dz = \int_{T^{(n)}} z\, dz = 0$, so

$$\left|\int_{T^{(n)}} f\, dz\right| = \left|\int_{T^{(n)}} [f(z) - f(z_0) - f'(z_0)(z - z_0)]\, dz\right|$$

$$\le \epsilon \int_{T^{(n)}} |z - z_0|\, |dz|$$

$$\le \epsilon \operatorname{diam}(\Delta^{(n)})\, \ell(T^{(n)})$$

$$= \epsilon\, d\, \ell \left(\frac{1}{4}\right)^n$$

We conclude that $\left|\int_T f\right| \le \epsilon \cdot 4^n d\, \ell \left(\frac{1}{4^n}\right)$

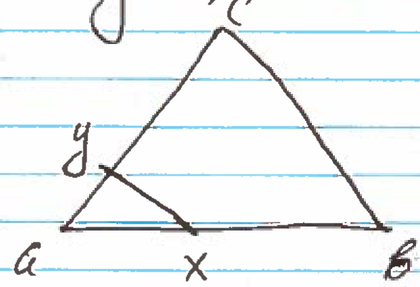
$\Longrightarrow \int_T f = 0$ & we are done by Morera. $= \underbrace{\epsilon}_{\text{arbitrary}} \underbrace{d\ell}_{\text{fixed}}$ ~ fixed

④

We proceed by Morera, as usual. Let $T \subset B(a,R)$
triangle w/ inside Δ

If $a \notin \Delta$, we are done.

If a is a vertex of T, write $T = [a, b, c, a]$



$T_1 = [a, x, y, a]$

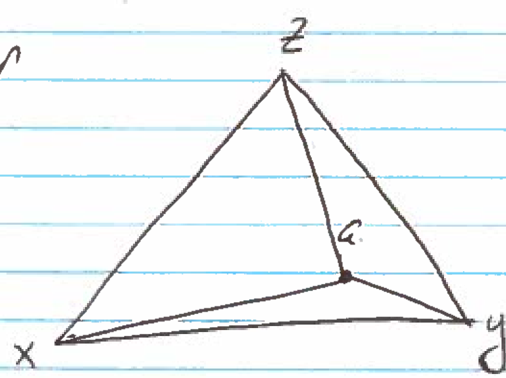
$\rho = [x, b, c, y, x]$, $\quad \int_T g = \int_{T_1} g + \int_\rho g = \int_{T_1} g$

since $\rho \sim 0$

Since g is continuous & $g(a) = 0$, x & y can be chosen
$\ni |g(z)| \leq \frac{\epsilon}{\ell}$ for any $z \in T_1$

$\Rightarrow \left| \int_{T_1} g \right| \leq \epsilon \Rightarrow \int_T g = 0.$

If $a \in \Delta$, consider



$T_1 = [x, y, a, x]$, $T_2 = [y, z, a, y]$, $T_3 = [z, x, a, z)$

Definition: f has an isolated singularity at $z=a$ if there is an $R>0$ $\ni$ f is analytic in $B(a,R) - \{a\}$, but not in $B(a,R)$.

$\{a\}$ is a removable singularity if $\exists\, g: B(a,R) \to \mathbb{C}$ analytic w/
$f = g \quad 0<|z-a|<R$.

Examples: $\underbrace{\frac{\sin z}{z}}_{\text{removable}}, \underbrace{\frac{1}{z}, e^{\frac{1}{z}}}_{\text{not removable}}$

Theorem: If f has an isolated singularity at $\{a\}$ then $\{a\}$ is a removable singularity iff
$$\lim_{z\to a} (z-a) f(z) = 0.$$

Proof: Suppose f is analytic in $\{0<|z-a|<R\}$ & define
$$g(z) = (z-a) f(z), \quad z \neq a$$
$$g(a) = 0.$$

Suppose that $\lim_{z\to a} (z-a) g(z) = 0 \Rightarrow g$ is continuous.

Imagine that we can show that g is analytic. Then
$g(z) = (z-a) \underbrace{h(z)}_{\text{analytic}}$ & we have a removable singularity.

By above, $\int_{T_j} f = 0$, so we are done.

Converse? Suppose that $\lim_{z\to a} (z-a) f(z) \neq 0$.

Then f does not have a removable singularity at a. Why?

Definition: If $z=a$ is an isolated singularity of f, then a is a pole of f if

$$\lim_{z\to a} |f(z)| = \infty.$$

If an isolated singularity is neither a pole nor removable, it is called an essential singularity.

Proposition: If G is a region, with a in G and if f is analytic on $G - \{a\}$ with a pole at $z=a$, then there is a positive integer m and an analytic function $g: G \to \mathbb{C}$ $\ni$

$$f(z) = \frac{g(z)}{(z-a)^m}$$

the smallest such m is called the order of the pole.

Series expansions around poles: Let f have a pole of order m at $z=a$, let $f(z) = \frac{g(z)}{(z-a)^m}$ analytic

$$g(z) = A_m + A_{m-1}(z-a) + \dots A_1 (z-a)^{m-1} + (z-a)^m \sum_{k=0}^{\infty} a_k (z-a)^k,$$

ie $f(z) = \frac{A_m}{(z-a)^m} + \dots + \frac{A_1}{z-a} + g_1(z)$ analytic $\quad A_m \neq 0$

⑥

Definition: If f has a pole of order m at $z=a$ and f satisfies

$$f(z) = \frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{z-a} + g_1(z),$$ then

is called the singular part of $z=a$.

we will work hard to generalize this notion as much as we can.

Definition: If $\{z_n : n = 0, \pm 1, \pm 2, \dots\}$ is a doubly infinite sequence of complex numbers, $\sum_{n=-\infty}^{\infty} z_n$ is absolutely convergent if both $\sum_{n=0}^{\infty} z_n$ & $\sum_{n=1}^{\infty} z_{-n}$ are absolutely convergent.

For functions, $\sum_{-\infty}^{\infty} u_n(s)$ is absolutely convergent if $\sum_{n=0}^{\infty} u_n$ & $\sum_{n=1}^{\infty} u_{-n}$ converge uniformly.