# Calculus Review

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What is calculus? Let us not go into it... Instead, let us review a few basic ideas and formulas that will be if of use to us. The derivative typically arises a a slope of the tangent line. For example, if $y = f(x)$ is a graph in the plane, then the slope of the tangent line at $x_0$ is

$$slope = f'(x_0) = \frac{dy}{dx}(x_0),$$

and the equation of the tangent line at $x_0$ is

$$y = f(x_0) + f'(x_0)(x - x_0).$$

An important and historically difficult point to remember is that the slope of the tangent line is always $\frac{dy}{dx}$, regardless of the coordinate system we happen to be using. For example, suppose that the curve is given by the equation

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)),$$

then at the time $t = t_0$,

$$slope = \frac{dy}{dx} = \frac{\gamma_2'(t_0)}{\gamma_1'(t_0)}.$$

Why is this? Well, what does it mean for $\gamma(t) = (\gamma_1(t), \gamma_2(t))$? It means, in the rectangular, $(x, y)$-coordinates, that

$$x = \gamma_1(t) \quad y = \gamma_2(t).$$

Consequently,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\gamma_2'(t)}{\gamma_1'(t)},$$

as claimed above.

What if the curve is given in polar coordinates by the equation

$$r = f(\theta)?$$

Well, we are brimming with confidence from our previous calculation, so we are not afraid. Since the curve is polar, presumably it is generated using polar coordinates, so

$$x = r\cos(\theta) \quad y = r\sin(\theta),$$

and, since $r = f(\theta)$,

$$x = f(\theta)\cos(\theta) \quad y = f(\theta)\sin(\theta).$$

By the same argument as above,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}.$$

Now that we are tossing derivatives around tomatoes in a pasta sauce, let us recall us their properties.

Product Rule: $(fg)' = f'g + fg'$. For example, if $f(x) = x\sin(x)$,

$$f'(x) = \sin(x) + x\cos(x),$$

since

$$(\sin(x))' = \cos(x).$$

While we are at it,

$$(\cos(x))' = -\sin(x) \quad (\tan(x))' = \sec^2(x) \quad (\sec(x))' = \sec(x)\tan(x).$$

Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$. For example, if $f(x) = \frac{e^x}{x}$,

$$f'(x) = \frac{xe^x - e^x}{x^2}$$

since

$$(e^x)' = e^x.$$

Recall that if $a > 0$, $a^x = e^{x\log(a)}$, so

$$(a^x)' = (e^{x\log(a)})' = \log(a)e^{x\log(a)} = \log(a)a^x.$$

Basic Integration: Let $f$ be a nice function. Then

$$\int_a^b f(x)dx = F(b) - F(a),$$

where $F'(x) = f(x)$. This is sometimes known as the Fundamental Theorem of Calculus. For example,

$$\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3},$$

and, indeed,

$$\left(\frac{x^3}{3}\right)' = x^2.$$

Integration by parts: If $f$ and $g$ are not too obscene,

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx,$$

or

$$\int udv = uv - \int vdu.$$

For example,

$$\int x\cos(x)dx = \int x(\sin(x))'dx = x\sin(x) - \int \sin(x)dx - x\sin(x) + \cos(x),$$

since

$$(\cos(x))' = -\sin(x).$$

Trigonometric substitution: Suppose we have the integral of the form

$$\int \frac{dx}{1+x^2}.$$

What do we do? Well, recall that

$$\sin^2(x) = \cos^2(x) = 1,$$

so dividing both sides by $\cos^2(x)$ yields

$$\tan^2(x) + 1 = \sec^2(x).$$

With this in mind, let $\tan(\theta) = x$. Then $x^2 + 1 = \sec^2(\theta)$ and $dx = \sec^2(\theta)d\theta$. It follows that

$$\int \frac{dx}{1+x^2} = \int \frac{\sec^2(\theta)d\theta}{\sec^2(\theta)} = \theta + C.$$

What is $\theta$? Well, since $x = \tan(\theta)$, it follows that $\theta = \tan^{-1}(x)$, so

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C.$$