(1)

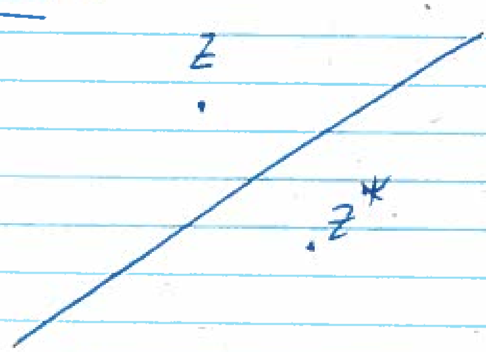
February 11, 2019

Let Γ be a circle through z_2, z_3, z_4. The points z & z^* are symmetric with respect to Γ if

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}.$$

You are going to prove in your homework assignment that symmetry is independent of the choice of z_2, z_3, z_4.

It is not difficult to see that z is symmetric to itself if $z \in \Gamma$ (check!)



$\Gamma \sim$ line

yes?? yes...

Let $z_4 = \infty \implies$ definition of symmetry takes the form

$$\frac{z^* - z_3}{z_2 - z_3} = \frac{\bar{z} - \bar{z}_3}{\bar{z}_2 - \bar{z}_3} \implies |z^* - z_3| = |z - z_3|$$

$\implies z$ & z^* are equidistant from Γ.

Furthermore,

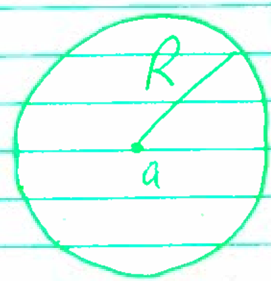
$$\operatorname{Im} \frac{z^* - z_3}{z_2 - z_3} = \operatorname{Im} \frac{\bar{z} - \bar{z}_3}{\bar{z}_2 - \bar{z}_3} = -\operatorname{Im} \frac{z - z_3}{z_2 - z_3}$$

$\implies z$ & z^* lie in different half-planes unless they are both on Γ.

determined by Γ

$\implies [z, z^*] \perp \Gamma$.

Circles are useful also ☺. Let $\Gamma = \{z : |z-a| = R\}$
$0 < R < \infty$



$\sim z_2, z_3, z_4 \in \Gamma$

Now,

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}$$

$$= \overline{(z-a, z_2-a, z_3-a, z_4-a)} \quad \text{(why?)}$$

$$= \left(\bar{z}-\bar{a}, \frac{R^2}{z_2-a}, \frac{R^2}{z_3-a}, \frac{R^2}{z_4-a}\right) = \left(\frac{R^2}{\bar{z}-\bar{a}}, z_2-a, z_3-a, z_4-a\right)$$

$$= \left(\frac{R^2}{\bar{z}-\bar{a}}+a, z_2, z_3, z_4\right)$$

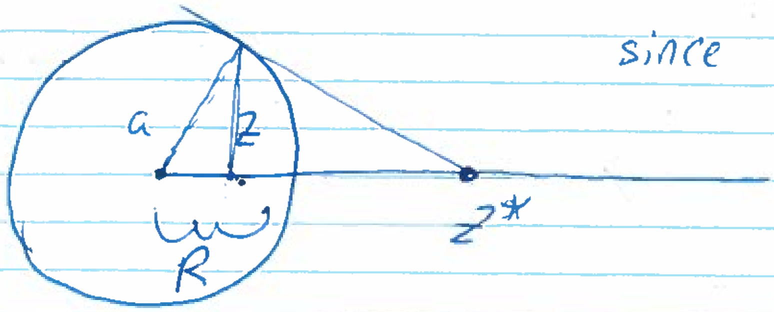
$$\Longrightarrow z^* = a + R^2(\bar{z}-\bar{a})^{-1} \quad \text{or} \quad (z^*-a)\overline{(z-a)} = R^2$$

$$\Longrightarrow \frac{z^*-a}{z-a} = \frac{R^2}{|z-a|^2} > 0$$

$$\Longrightarrow z^* \in \text{ray } \{a+t(z-a) : 0<t<\infty\}$$

from a to z.

③



since $|z-a|\cdot|z^*-a| = R^2$

work out the geometry!

$\Longrightarrow$ a & ∞ are symmetric with respect to Γ.

Symmetry principle: If $\underset{\text{Mobius}}{T} : \underset{\text{circles}}{\Gamma_1 \longrightarrow \Gamma_2}$

& z, z^* symmetric w/ respect to $\Gamma_1 \Longrightarrow$

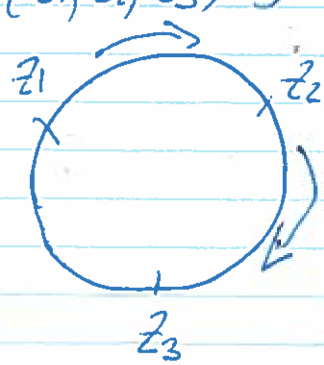
Tz, Tz^* " " " Γ_2.

Proof: Let $z_2, z_3, z_4 \in \Gamma_1 \Longrightarrow$ if z & z^* are symmetric w/ respect to Γ_1, then

$$(Tz^*, Tz_2, Tz_3, Tz_4) = (z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)} \text{ by Prop 3.8}$$

$$= \overline{(Tz, Tz_2, Tz_3, Tz_4)} \text{ \& we are done}$$

If Γ is a circle then an orientation for Γ is an ordered triple of points $(z_1, z_2, z_3) \ni$ each z_j is in Γ.



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Let $\Gamma = \mathbb{R}$ $z_1, z_2, z_3 \in \mathbb{R}$ $Tz = (z, z_1, z_2, z_3) = \frac{az+b}{cz+d}$

$T(\mathbb{R}_\infty) = \mathbb{R}_\infty$, so a, b, c, d can be chosen in $\mathbb{R}$ (prove it)

Hence $Tz = \frac{az+b}{cz+d} = \frac{az+b}{|cz+d|^2} \, \overline{cz+d}$

$= \frac{1}{|cz+d|^2} \cdot \left[ac|z|^2 + bd + bc\bar{z} + adz\right]$

It follows that $\operatorname{Im}(z, z_1, z_2, z_3) = \frac{(ad-bc)}{|cz+d|^2} \operatorname{Im}(z)$

It follows that $\{z : \operatorname{Im}(z, z_1, z_2, z_3) > 0\}$ is either the upper or lower half plane depending on whether $ad-bc$ (determinant) is positive or negative.

Now let Γ be arbitrary w/ $z_1, z_2, z_3 \in \Gamma$. Take S Mobius, then $\{z : \operatorname{Im}(z, z_1, z_2, z_3) > 0\} = \{z : \operatorname{Im}(Sz, Sz_1, Sz_2, Sz_3) > 0\}$

$= S^{-1}\{z : \operatorname{Im}(z, Sz_1, Sz_2, Sz_3) > 0\}$,

so choosing $S : \Gamma \to \mathbb{R}_\infty$, then

$\{z : \operatorname{Im}(z, z_1, z_2, z_3)\} = S^{-1}(\text{upper})$ or $S^{-1}(\text{lower})$

(5)

If (z_1, z_2, z_3) is an orientation of Γ, then the RHS of Γ is $\{z : \operatorname{Im}(z, z_1, z_2, z_3) > 0\}$

The following is immediate:

<u>Orientation principle</u>: Γ_1, Γ_2 circles $\mathbb{C}_\infty$

$\underbrace{T}_{\text{Mobius}} : \Gamma_1 \to \Gamma_2$ & (z_1, z_2, z_3) orientation of Γ_1

Then $T : \text{RHS}(\Gamma_1) \to \text{RHS}(\Gamma_2)$

w/ respect to (Tz_1, Tz_2, Tz_3) orientation of Γ_2.