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Laurent Series: $f$ analytic in $\operatorname{ann}(a, R_1, R_2)$

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$$

convergence absolute & uniform in $\operatorname{ann}(a, r_1, r_2)$

$R_1 < r_1 < r_2 < R_2$

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{n+1}}\, dz \quad (*)$$

The series is unique.

$$\int_{|z-a| = r}, \quad R_1 < r < R_2.$$

Proof: If $R_1 < r_1 < r_2 < R_2$ & $\gamma_1, \gamma_2$ are the circles $|z-a| = r_1$, $|z-a| = r_2$, respectively, then $\gamma_1 \sim \gamma_2$ in $\operatorname{ann}(a, R_1, R_2)$

Cauchy $\Rightarrow$ $g$ analytic, $\int_{\gamma_1} g = \int_{\gamma_2} g$

In particular, $(*)$ is independent of $r$, so for each $n$, $a_n$ is constant.

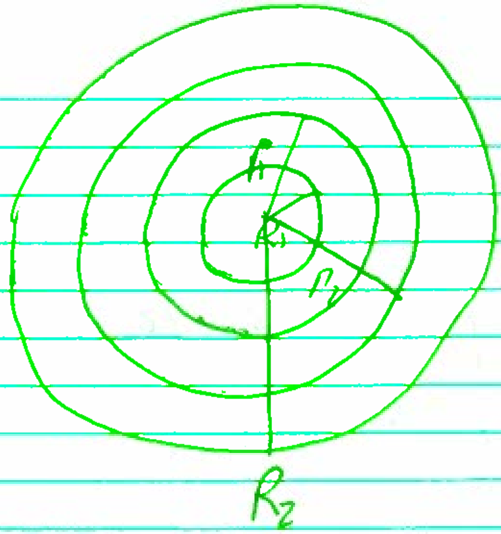
Observe that any $f_2 : B(a, R_2) \to \mathbb{C}$ is given by

$$f_2(z) = \frac{1}{2\pi i} \int_{|w-a| = r_2} \frac{f(w)}{w-z}\, dw$$

w/ $|z-a| < r_2$

$R_1 < r_2 < R_2$

is a well-defined function and analytic in $B(a, r_2)$



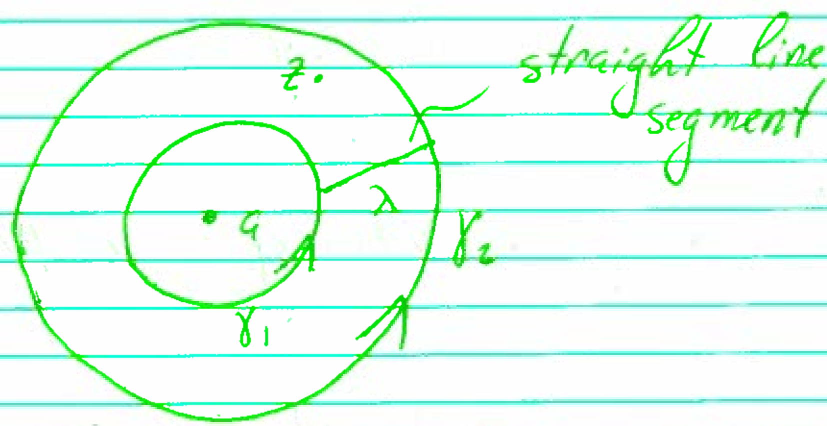

Similarly, if $G = \{z : |z-a| > r_1\}$

then $f_1 : G \to \mathbb{C}$, given by

$$f_1(z) = -\frac{1}{2\pi i}\int_{|w-a|=r_1} \frac{f(w)}{w-z}\,dw \qquad |z-a| > r_1,$$

$$R_1 < r_1 < R_2$$

is analytic in $G$.




If $R_1 < |z-a| < R_2$, let $r_1, r_2 \ni R_1 < r_1 < |z-a| < r_2 < R_2$.

Let $\gamma_1(t) = a + r_1 e^{it}$, $\gamma_2(t) = a + r_2 e^{it}$, $0 \le t \le 2\pi$

Since $\gamma_1 \sim \gamma_2$ in $\operatorname{ann}(a, R_1, R_2)$,

$\gamma = \gamma_2 - \lambda - \gamma_1 + \lambda$ is homotopic to zero.

(3)

Observe that $n(\gamma_2, z) = 1$ & $n(\gamma_1, z) = 0$, so

$$\text{Cauchy} \implies f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z}\, dw$$

$$= \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)\, dw}{w - z} - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w - z}\, dw$$

$$= f_1(z) + f_2(z)$$

now expand into power series

$f_2(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ w/ usual formula for $\underline{\underline{a_n}}$

Let $g(z) = f_1\left(a + \frac{1}{z}\right)$ for $0 < |z| < \frac{1}{R_1}$

essential singularity

It is not hard to see that this singularity is removable!

If $r > R_1$, let $\rho(z) = d(z, C)$ $\{|w-a| = r\}$

Let $M = \max\{|f(w)| : w \in C\}$. Then for $|z-a| > r$,

$$|f_1(z)| \le \frac{Mr}{\rho(z)} \xRightarrow{\text{since } \rho(z) \to \infty} \lim_{z \to 0} g(z)$$

$$= \lim_{z \to 0} f_1\left(a + \frac{1}{z}\right) = 0.$$

This means that if we set $g(0) = 0$, $g$ is analytic in $B\left(0, \frac{1}{R_1}\right)$.

If we expand $g(t) = \sum_{n=1}^{\infty} B_n t^n$, a bit of algebra yields

$$f_1(z) = \underbrace{\sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}}_{\text{w/ usual formula}}$$

The uniqueness is automatic since convergence is absolute & uniform, and

$$a_n = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-a)^{n+1}}\,dz.$$

<u>Classification:</u> Let $z=a$ be an isolated singularity of $f$ and let $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$ be its Laurent expansion in $\operatorname{ann}(a, 0, R)$

Then a) $\{a\}$ is removable iff $a_n = 0$ for $n \le -1$.

b) $\{a\}$ is a pole of order $m$ iff

$a_{-m} \ne 0$ & $a_n = 0$ for $n \le -(m+1)$.

c) $\{a\}$ is essential in the remaining cases.

Proof: part a) Let $g(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \Longrightarrow \underbrace{\text{analytic}}_{\text{in } B(a,R)}$

& agrees w/ $f$ is a punctured disk.

b) By the same argument,

$(z-a)^m f(z)$ has a removable singularity

$\Longrightarrow$ pole of order $m$.

Casorati-Weierstrass: If $f$ has an essential singularity at $z=a$, then for every $\delta > 0$,

$$\overline{\{f[\text{ann}(a,0,\delta)]\}} = \mathbb{C}$$

Proof: We must show that if $c, \epsilon$ are given, then for each $\delta > 0$ we can find a $z$ w/ $|z-a| < \delta$ & $|f(z)-c| < \epsilon$.

Assume that this is false. Then $\exists\, c \in \mathbb{C}$ & $\epsilon > 0 \ni$

$|f(z)-c| \geq \epsilon \quad \forall z \in G = \text{ann}(a,0,\delta)$. It follows that

$\lim_{z \to a} |z-a|^{-1}|f(z)-c| = \infty \implies (z-a)^{-1}(f(z)-c)$ has a pole at $z=a$.

If $m$ is the order of this pole,

$$|z-a|^{m+1}|f(z)-c| = 0 \implies$$

$$|z-a|^{m+1}|f(z)| \leq |z-a|^{m+1}|f(z)-c| + |z-a|^{m+1}|c|$$

$$\implies \lim_{z \to a} |z-a|^{m+1}|f(z)| = 0 \text{ since } m \geq 1.$$

This implies that $f(z)(z-a)^m$ has a removable singularity at $z=a$, which contradicts the hypothesis.

We are now ready for the calculus of residues!

Residue theorem: Let $f$ be analytic in the region $G$ except for the isolated singularities $a_1, \ldots, a_m$. If $\gamma$ is a rectifiable curve that does not pass through any of the points $a_k$ & if $\gamma \approx 0$ in $G$,

then $$\frac{1}{2\pi i}\int_\gamma f = \sum_{k=1}^{m} n(\gamma; a_k)\underbrace{\text{Res}(f; a_k)}_{\text{coefficient } a_{-1} \text{ in Laurent expansion}}$$

⑥

Proof: Let $m_k = n(\gamma, a_k)$, $1 \le k \le m$

& choose positive numbers $r_1, r_2, \ldots, r_m \ni B(a_k, r_k) \subseteq G$, $B(a_k; r_k) \cap B(a_{k'}; r_{k'}) = \emptyset$ & none of them intersect $\gamma$.

Let $\gamma_k(t) = a_k + r_k e^{-2\pi i m_k t}$, $0 \le t \le 1$. Then

$$n(\gamma, a_j) + \sum_{k=1}^{m} n(\gamma_k, a_j) = 0$$

Since $f$ is analytic in $G - \{a_1, a_2, \ldots, a_m\}$,

$$0 = \int_\gamma f + \sum_{k=1}^{m} \int_{\gamma_k} f$$

If $f(z) = \sum_{-\infty}^{\infty} b_n (z - a_k)^n$ → converges uniformly on $\partial B(a_k, r_k)$

$$\Rightarrow \int_{\gamma_k} f = \sum_{-\infty}^{\infty} b_n \underbrace{\int_{\gamma_k} (z-a)^n dz}_{= 0 \text{ if } n \neq -1}$$

$$b_{-1} \int_{\gamma_k} (z - a_k)^{-1} dz = 2\pi i \, n(\gamma_k, a_k) \operatorname{Res}(f, a_k)$$

& we are done.