

①

January 30, 2019

Proposition: If G is open and connected, and $f: G \rightarrow \mathbb{C}$ is differentiable with $f'(z) = 0 \quad \forall z \in G$, then f is constant.

Proof: Fix $z_0 \in G$ and let $w_0 = f(z_0)$. Put

$A = \{z \in G : f(z) = w_0\} \rightarrow$ we will show that A is both open and closed.
(Do you remember topology? If not, crack open Munkres)

Let $z \in G$ & let $z_n \rightarrow z$, $z_n \in A$. Since $f(z_n) = w_0$ for each $n \geq 1$, and f is continuous, we get $f(z) = w_0$, or $z \in A$.

This proves that A is closed in G .

Now fix $a \in A$, and let $\epsilon > 0$ be such that $B(a, \epsilon) \subset G$. If $z \in B(a, \epsilon)$, set

key property of continuity.

$g(t) = f(tz + (1-t)a)$, $0 \leq t \leq 1$. Then

$$\frac{g(t) - g(s)}{t - s} = \frac{g(t) - g(s)}{(t-s)z + (s-t)a} \cdot \frac{(t-s)z + (s-t)a}{t-s}$$

Letting $t \rightarrow s$, the chain rule yields

$$\lim_{t \rightarrow s} \frac{g(t) - g(s)}{t - s} = f'(sz + (1-s)a) \cdot (z - a) = 0.$$

We just proved that $g'(s) = 0$ for $0 \leq s \leq 1$, which implies that g is constant. Hence

$$f(z) = g(1) = g(0) = f(a) = w_0.$$

In other words, $B(a, \epsilon) \subset A$, so A is open.

Since A is open and closed, $A = G$.

③

Going back to $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ real coefficients

This implies that $e^{\bar{z}} = \overline{e^z}$. In particular, if θ is real,
 $|e^{i\theta}|^2 = e^{i\theta} e^{-i\theta} = e^0 = 1$. In general, $|e^z|^2 = e^z e^{\bar{z}} = e^{z+\bar{z}} = e^{2\operatorname{Re}(z)}$
 $|e^z| = e^{\operatorname{Re}(z)}$

We just proved that $|e^z| = e^{\operatorname{Re}(z)}$.

We are ready to define $\cos z, \sin z$:

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots + (-1)^n \frac{z^{2n}}{(2n)!} + \dots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots + (-1)^n \frac{z^{2n-1}}{(2n-1)!} + \dots$$

radius of convergence = ∞ , as before

It is not difficult to see that

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

instantly

$$\implies \cos^2 z + \sin^2 z = 1 \quad \text{and}$$

$$e^{iz} = \cos z + i \sin z$$

It follows that $z = |z|e^{i\theta}$, $\theta = \arg(z)$.

Also, $|e^z| = e^{\operatorname{Re}(z)}$ & $\arg e^z = \operatorname{Im}(z)$.

So far, the properties are quite analogous to the real case, but this is about to end. The key difference is the notion of periodicity, which is where we turn our attention.

⑤

What is not immediately obvious is whether the same k works for each $z \in G$. To understand what is going on, let

$$h(z) = \frac{f(z) - g(z)}{2\pi i} \longrightarrow \text{continuous \& takes on integer values}$$

Since G is connected, $h(G)$ is connected, hence

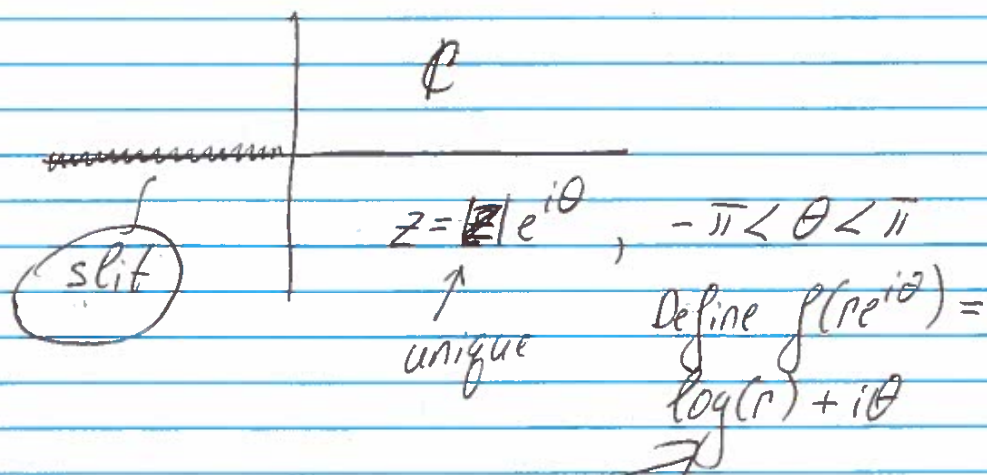
$$\exists k \in \mathbb{Z} \ni f(z) + 2\pi ki = g(z).$$

In summary, we get the following result.

Proposition: If $G \subset \mathbb{C}$ is open and connected and f is a branch of $\log(z)$ on G , then the totality of branches of $\log(z)$ are the functions $f(z) + 2\pi ki, k \in \mathbb{Z}$.

Characterizations are useful, but what do these branches actually look like?

$$\text{Let } G = \mathbb{C} - \{z \in \mathbb{R} : z \leq 0\}$$



Is it continuous? Check...

Is the branch we just constructed (if continuity holds up) analytic? This is where we go next.

(7)

Corollary: A branch of the logarithm function is analytic and its derivative is z^{-1} .

Principal branch of the logarithm: branch defined above on $\mathbb{C} - \{z \in \mathbb{R} : z \leq 0\}$.

We will now see how to define branches of different functions by bootstrapping the logarithm.