Math 173, Fall 2022, November 2

Theorem: V, W, Z vector spaces over F;

$T: V \to W$ $\quad U: W \to Z$

linear $\quad$ linear

$$V \xrightarrow{T} W \xrightarrow{U} Z$$

Then $(UT)(\alpha) = U(T(\alpha))$

is a linear transformation from $V \to Z$.

Proof: $UT(c\alpha + \beta) = U(T(c\alpha + \beta))$

$= U(cT\alpha + T\beta)$

$= c\,U(T\alpha) + U(T\beta)$

$= c(UT)(\alpha) + (UT)\beta$ ✓

<u>Definition:</u> A linear operator on V is a linear transformation from V to V.

Lemma: V vector space over F; U, T_1, T_2 linear operators on V; $c \in F$.

Then

straight from the definitions

a) $IU = UI = U$

b) $U(T_1 + T_2) = UT_1 + UT_2$;

$(T_1 + T_2)U = T_1 U + T_2 U$

c) $c(UT_1) = (cU)T_1 + U(cT_1)$

Example: A $m \times n$ over F

$$T(X) = AX \quad T: F^{n\times 1} \longrightarrow F^{m\times 1}$$

$$B = p\times m \text{ matrix over } F$$

$$U(Y) = BY \quad U: F^{m\times 1} \longrightarrow F^{p\times 1}$$

$$(UT)(X) = U(TX) = U(AX)$$

$$= B(AX) = (BA)X$$

matrix multiplication

Another Key example:

$\mathcal{B} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$

ordered basis for V

Let $E^{p,q}(\alpha_i) = \delta_{iq}\alpha_p$

basis of linear operators on V

$$E^{p,q} E^{r,s}(\alpha_i) = E^{p,q}(\delta_{is}\alpha_r) =$$

$$\delta_{is} E^{p,q}(\alpha_r) =$$

$$\delta_{is}\delta_{rq}\alpha_p$$

So, $E^{p,q}E^{r,s}(\ ) = \begin{cases} 0 & \text{if } r \neq q \\ E^{p,s} & \text{if } r = q \end{cases}$

Let T be a linear operator on V. Then

$$TU = \left(\sum_p \sum_q A_{pq} E^{p,q}\right)\left(\sum_r \sum_s B_{rs} E^{r,s}\right)$$

$$= \sum_p \sum_q \sum_r \sum_s A_{pq} B_{rs} E^{p,q} E^{r,s}$$

$$= \sum_p \sum_s \left(\sum_r A_{pr} B_{rs} \right) E^{p,s}$$

$q=r$

$$= \sum_p \sum_s (AB)_{ps} E^{p,s}$$

matrix multiplication

Theorem 7: V, W vector spaces over F,

$T: V \to W$ linear 1-1 & onto

If T is invertible, T^{-1} is a linear transformation from W to V.

Proof: Consider

$$T^{-1}(c\beta_1 + \beta_2) \stackrel{?}{=} cT^{-1}\beta_1 + T^{-1}\beta_2$$

Let $\alpha_i = T^{-1}\beta_i$. Since T is linear,

$$T(c\alpha_1 + \alpha_2) = cT\alpha_1 + T\alpha_2$$

$$= c\beta_1 + \beta_2 \text{, so}$$

$$T^{-1}(c\beta_1 + \beta_2) = c\alpha_1 + \alpha_2 = cT^{-1}\beta_1 + T^{-1}\beta_2.$$

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Note: T is linear implies that

$T\alpha = T\beta$ iff $T(\alpha-\beta) = 0$.

<u>Definition:</u> T is non-singular if $T\gamma = 0$ implies that $\gamma = 0$.

Observe that T is 1-1 iff T is non-singular.

Theorem 8: $\underbrace{T : V \to W}_{\text{linear}}$

Then T is non-singular iff T maps linearly independent sets to linearly independent sets.

Proof: T non-singular; S linearly independent

Let $\alpha_1, \alpha_2, \dots, \alpha_K \in S$. We claim that

$\{T\alpha_1, T\alpha_2, \dots, T\alpha_K\}$ are linearly independent.

Suppose otherwise. Then

$$c_1 T\alpha_1 + \dots + c_K T\alpha_K = 0$$

$$\| $$

$T(c_1\alpha_1 + \dots + c_K\alpha_K)$. This implies that

$c_1\alpha_1 + \dots + c_k\alpha_k = 0$ since T is 1-1,

and it follows that $c_1 = c_2 = \dots = c_k = 0$,

as desired.

Conversely, suppose that T: independent $\to$ independent

Let α be a non-zero vector $\ni T\alpha = 0$. But

then T maps independent set $\{\alpha\}$ to a

dependent set $\{0\}$. Contradiction!

Example: V polynomials over $\mathbb{R}$

$Tf(x) = f'(x)$. Then T maps constants

to $0 \hookrightarrow$ not invertible!

Theorem 9: V, W vector spaces over F, $\dim V = \dim W$.

If $T : V \to W$ linear TFAE

i) T is invertible

ii) T is non-singular

iii) T is onto.

Proof: $n = \dim V = \dim W$

We have $\operatorname{rank}(T) + \operatorname{nullity}(T) = n$

T is non-singular iff $\operatorname{nullity}(T) = 0$, ie $\operatorname{rank}(T) = n$. It follows that T is non-singular iff $T(V) = W$. ✓