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Math 173, Fall 2022, November 21

Linear functional: linear transformation from V to F, i.e. $f: V \to F$ linear such that $f(c\alpha + \beta) = c f(\alpha) + f(\beta)$

$$f(\underbrace{c}_{\text{scalar}}\underbrace{\alpha}_{\text{vector}} + \underbrace{\beta}_{\text{vector}})$$

Example: $V = F^n$

$f: V \to F$: $f(x_1, \dots, x_n) = a_1 x_1 + \dots + a_n x_n$, a_j's $\in F$

Note that $f(\underbrace{e_j}_{\text{standard basis}}) = a_j$

In general, $f(x_1, \dots, x_n) = f(\sum_j x_j e_j) = \sum_j x_j f(e_j)$, so every linear functional from $V = F^n$ to F is of the form described above.

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Another example:

A $n \times n$ matrix over F; $tr(A) \equiv a_{11} + \dots + a_{nn}$

$$tr(cA+B) = \sum_{i=1}^{n} cA_{ii} + B_{ii} =$$

$$c\sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii} = c\,tr(A) + tr(B), \text{ so}$$

linear!

Example: V = polynomials from F to F.

$\underbrace{L_t(p)}_{\text{polynomial}} = \underbrace{p(t)}_{\text{evaluation functional}}$, clearly linear.

Example: $C([a,b])$ = continuous functions on $[a,b]$

Define $L(g) = \underbrace{\int_a^b g(t)\,dt}_{\text{linear functional}}$

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By Theorem 5, $\dim V^* = \dim V$ ~ if V is finite dimensional

$\underset{\text{linear functionals on } V \ (L(V,F))}{\overset{|||}{}}$

Let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ basis for V.

We know (Theorem 1) that $\underline{\exists!}$ functional $f_i \ni \underline{f_i(\alpha_j) = \delta_{ij}}$

This generates n distinct linear functionals on V. They are linearly independent, for assume $f = \sum_{i=1}^{n} c_i f_i$

Then $f(\alpha_j) = \sum_{i=1}^{n} c_i f_i(\alpha_j) =$

$\sum_{i=1}^{n} c_i \delta_{ij} = \underline{c_j}$. Thus we have a basis for V^* !!!

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Definition: V/F vector space, $S \subseteq V$ subset

The annihilator of S is $S^0 \subset$ functionals in $V \ni$ $f(\alpha)=0 \ \forall \alpha \in S$

Observation: $S^0 \subset V^*$ whether or not $S \subset V$ is a subspace.

If $S = \{0\}$, $V^* = S^0$. If $S = V$, $S^0 = 0$-subspace.

Theorem 16: V/F finite dimensional, $W \subset V$ subspace.
Then $\dim W + \dim W^0 = \dim V$

Proof: Let $\dim(W) = k$; $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ basis for W.
Extend to basis of V: $\{\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_n\}$
Let $f_1, f_2, \ldots, f_n$ be the basis of V^* dual to this basis of V. We claim that $\{f_{k+1}, \ldots, f_n\}$ is a basis for the annihilator W^0.

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To see that $\underbrace{f_i}_{i \geq k+1} \in W^0$, observe that

$f_i(\alpha_j) = \delta_{ij} = 0$ if $i \geq k+1$, $j \leq k$, so

$f_i(\alpha) = 0$ if $\alpha =$ linear combo of

$\alpha_1, \ldots, \alpha_k$. These functionals are linearly independent, so we are left to check that they span W^0:

Let $f \in V^*$, so $f = \sum_{i=1}^{n} f(\alpha_i) f_i$.

If $f \in W^0$, $f(\alpha_i) = 0$ for $i \leq k$ and

$f = \sum_{i=k+1}^{n} f(\alpha_i) f_i$. This implies that if $\dim W = k$ and $\dim V = n$, then

$\dim W^0 = n - k$, and we ~~are~~ done.

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<u>Corollary</u>: If W is a k-dimensional subspace of an n-dimensional vector space V, then W is the intersection of $(n-k)$ hypersurfaces in V.

<u>Corollary</u>: If W_1 and W_2 are subspaces of a finite dimensional vector space, then $W_1 = W_2$ iff $W_1^0 = W_2^0$.

Proof: If $W_1 = W_2$, then $W_1^0 = W_2^0$.

If $W_1 \neq W_2$, then $\exists \alpha \in W_2 \ni \alpha \notin W_1$ (WLOG)

By the proof of Theorem 16, $\exists$ linear functional $f \ni f(\beta) = 0 \ \forall \beta \in W$, but $f(\alpha) \neq 0$. Then $f \in W_1^0$, but not in W_2^0, so $W_1^0 \neq W_2^0$.