(1)

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$\gamma:[0,1]\longrightarrow \mathbb{C}$ closed rectifiable curve $\underset{a\notin}{\&}\{\gamma\}$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} \in \mathbb{Z}$$

Proof: Reduce to smooth & define

$$g(t)=\int_0^t \frac{\gamma'(s)}{\gamma(s)-a}\,ds$$

We have $g(0)=0$ & $g'(t)=\dfrac{\gamma'(t)}{\gamma(t)-a}$, $0\le t\le 1$

Then $\dfrac{d}{dt}\left\{e^{-g}(\gamma-a)\right\} = e^{-g}\cdot\gamma' - g'e^{-g}(\gamma-a) =$

$$e^{-g}\left(\gamma' - \frac{\gamma'}{\gamma-a}\,\gamma-a\right)=0$$

Why does this trick work? Don't overthink it...

$$\Rightarrow e^{-g}(\gamma-a) = \text{constant}$$

Since $\gamma(0)=\gamma(1)$, $e^{-g(1)}=1 \Rightarrow g(1)=\underbrace{2\pi K}_{\text{integer}}$

If γ is a closed rectifiable curve in $\mathbb{C}$, then for $a\notin\{\gamma\}$

$$n(\gamma,a)=\frac{1}{2\pi i}\int_\gamma (z-a)^{-1}dz$$

index of γ with respect to the point a.

or a winding number

<u>Observation:</u> If γ & σ are closed rectifiable curves having the same initial points, then

a) $n(\gamma, a) = -n(-\gamma, a)$ for every $a \notin \{\gamma\}$

b) $n(\gamma + \sigma, a) = n(\gamma, a) + n(\sigma, a)$ for every $a \notin \{\gamma\} \cup \{\sigma\}$

<u>Theorem:</u> Let γ be a closed rectifiable curve in $\mathbb{C}$. Then $n(\gamma, a)$ is a constant belonging to a component of $G = \mathbb{C} - \{\gamma\}$. Also, $n(\gamma, a) = 0$ for $a \in$ unbounded component of G.

Proof: Let $f(a) = n(\gamma, a)$. If we can show that f is continuous, combined w/ the fact that $n(\gamma, a) \in \mathbb{Z}$, reduces the image of f to a single point, since continuous functions map connected sets to connected sets.

To establish continuity, fix $a \in G$ & take b w/

$$|a-b| < \delta < \tfrac{1}{2} d(a, \{\gamma\}), \text{ so}$$

$$|f(a) - f(b)| = \frac{1}{2\pi} \left| \int_\gamma \left[(z-a)^{-1} - (z-b)^{-1} \right] dz \right|$$

$$= \frac{1}{2\pi} \left| \int_\gamma \frac{(a-b)}{(z-a)(z-b)} dz \right| \leq \frac{|a-b|}{2\pi} \int_\gamma \frac{|dz|}{\underbrace{|z-a|}_{\geq r} \underbrace{|z-b|}_{\geq \frac{1}{2} r}} \leq$$

$$\frac{|a-b|}{2\pi} \cdot \frac{2}{r^2} V(\gamma) < \frac{2\delta}{\pi r^2} V(\gamma) \text{ \& we are done!}$$

The fact that $n(\gamma, a) = 0$ for $a \in$ unbounded component is immediate.

(3)

Cauchy's theorem: we are ready to move on from the disk to general domains.

Lemma: γ rectifiable & φ continuous on $\{\gamma\}$. For each $m \geq 1$, let

$$F_m(z) = \int_\gamma \varphi(w)(w-z)^{-m}\,dw \quad \text{for } z \notin \{\gamma\}.$$

Then each F_m is analytic on $\mathbb{C} - \{\gamma\}$ &

$$F_m'(z) = m F_{m+1}(z).$$

Proof: Continuity follows from

$$\frac{1}{(w-z)^m} - \frac{1}{(w-a)^m} = \left[\frac{1}{w-z} - \frac{1}{w-a}\right] \sum_{k=1}^{m} \frac{1}{(w-z)^{m-k}} \cdot \frac{1}{(w-a)^{k-1}} =$$

$$(z-a)\left[\frac{1}{(w-z)^m(w-a)} + \frac{1}{(w-z)^{m-1}(w-a)^2} + \cdots + \frac{1}{(w-z)(w-a)^m}\right]$$

& the result follows.

To establish differentiability,

$$\frac{F_m(z) - F_m(a)}{z-a} = \int_\gamma \frac{\varphi(w)(w-a)^{-1}}{(w-z)^m}\,dw + \cdots + \int_\gamma \frac{\varphi(w)(w-a)^{-m}}{w-z}\,dw$$

Since $a \notin \{\gamma\}$, $\varphi(w)(w-a)^{-k}$ is continuous on $\{\gamma\}$ for each k. Letting $z \to a$ yields

$F_m'(a) = m F_{m+1}(a)$ ✓

4

Cauchy Integral formula 1: $G \subseteq \mathbb{C}$ open, $f: G \to \mathbb{C}$ analytic

If γ is a closed rectifiable curve in G w/ $n(\gamma, w) = 0$ $\forall$ $w \in \mathbb{C} - G$,

then for $a \in G - \{\gamma\}$,

$$n(\gamma, a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz$$

Proof: Define $\varphi: G \to \mathbb{C}$ by

$\varphi(z, w) = [f(z) - f(w)]/(z-w)$ if $z \neq w$ &

$\varphi(z, z) = f'(z)$ $\Rightarrow$ φ continuous & $z \to \varphi(z, w)$ is analytic (homework)

Let $H = \{w \in \mathbb{C} : n(\gamma, w) = 0\}$, open since $n(\gamma, w)$ is continuous & integer valued.

$H \cup G = \mathbb{C}$ (by assumption)

Let $g(z) = \int_\gamma \varphi(z, w) dw$ if $z \in G$ &

$$g(z) = \int_\gamma (w - z)^{-1} f(w) dw \text{ if } z \in H.$$

If $z \in G \cap H$, then

$$\int_\gamma \varphi(z, w) dw = \int_\gamma \frac{f(w) - f(z)}{w - z} dw$$

⑤

$$= \int_\gamma \frac{f(w)}{w-z}\,dw - f(z)\,n(\gamma,z)\,2\pi i$$

$$= \int_\gamma \frac{f(w)}{w-z}\,dw, \text{ so } g \text{ is well-defined.}$$

By the lemma above, g is entire. By the previous theorem, H contains a neighborhood of ∞ in $\mathbb{C}_\infty$. But f is bounded on $\{\gamma\}$, so $\lim_{z\to\infty} (w-z)^{-1} = 0$ uniformly for w in $\{\gamma\}$,

and $\lim_{z\to\infty} g(z) = \lim_{z\to\infty} \int_\gamma \frac{f(w)}{w-z}\,dw = 0.$

$\Downarrow$

g is constant by Liouville! & so $g \equiv 0$.

So if $a \in G - \{\gamma\}$, then

$$0 = \int_\gamma \frac{f(z)-f(a)}{z-a}\,dz = \int_\gamma \frac{f(z)}{z-a}\,dz - f(a)\int_\gamma \frac{dz}{z-a}$$

✓

This is good, but more complicated domains require more elaborate setups

Cauchy Integral formula 2: G open in $\mathbb{C}$ & $f: G \to \mathbb{C}$ analytic. If $\gamma_1, \gamma_2, \ldots, \gamma_m$ are closed rectifiable curves in G ∋ $n(\gamma_1, w) + \ldots + n(\gamma_m, w) = 0$ $\forall\, w \in \mathbb{C} - G$,

then for $a \in G - \bigcup_{k=1}^{m} \{\gamma_k\}$,

$$f(a)\sum_{k=1}^{m} n(\gamma_k, a) = \sum_{k=1}^{m} \frac{1}{2\pi i}\int_{\gamma_k} \frac{f(z)}{z-a}\,dz$$

~ same proof ✓