20) ① Suffice to show the base case $n=2$, i.e

$\log z_1 z_2 = \log z_1 + \log z_2$, which reduces to showing

$\arg z_1 z_2 = \arg z_1 + \arg z_2$.

Let $\begin{cases} z_1 = r_1 e^{i\theta_1} \\ z_2 = r_2 e^{i\theta_2} \end{cases}$. From the restrictions, we have

$\Rightarrow \quad \theta_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \arg z_1 z_2 = \theta_1 + \theta_2 \mod (-\pi, \pi) = \theta_1 + \theta_2$

Hence $\arg z_1 + \arg z_2 = \theta_1 + \theta_2 = \cancel{\arg z_1 + z} \arg z_1 z_2$.

Then an induction leads to our conclusion.

② If the restrictions are removed.

Consider $\begin{cases} z_1 = e^{i\frac{\pi}{3}} \\ z_2 = e^{i\frac{2\pi}{3}} \end{cases}$, we see that $\operatorname{Re} z_1 < 0$

$\log z_1 z_2$ is not even defined as a principal branch.

21) Suppose f is a bol on G.

then f has to be continuous.

Let g be principal bol, then

$$f = g + 2\pi k i \quad , \quad k \in \mathbb{Z}.$$

$$= \log|z| + i \arg z + 2\pi k i \quad . \quad \arg z \in (-\pi, \pi)$$

Consider any $z^* \in \mathbb{R}^-$ (negative real line)

One can check

$$\lim_{z \to (z^*)^{\uparrow}} f(z) = \log|z^*| + i\pi + 2\pi k i$$

$$\lim_{z \to (z^*)^{\downarrow}} f(z) = \log|z^*| + i(-\pi) + 2\pi k i$$

(Note : $z \to (z^*)^{\uparrow}$ means z approaching z^* from the upper half complex plane)

They should agree due to continuity of f, but they don't.

Hence the conclusion.

$\square$

1) Let $z = a + ib \Rightarrow \begin{cases} a < 0 \\ |b| < \pi \end{cases}$

$\Rightarrow \quad e^z = e^a e^{ib}$

$$\begin{cases} e^a < e^0 = 1 \\ -\pi < \arg e^z = b < \pi \\ e^z \neq 0 \end{cases}$$

$\Rightarrow \quad \text{Image} = \{z \mid |z| < 1\} - \{z \leq 0\}$

2) Let $z = a + ib \Rightarrow \begin{cases} a \in \mathbb{R} \\ -\frac{\pi}{2} < b < \frac{\pi}{2} \end{cases}$

$\Rightarrow \quad |z| = e^a > 0$

$-\frac{\pi}{2} < \arg z = b < \frac{\pi}{2}$

$\Rightarrow$ Image: right half Complex plane

i.e. $\{z \mid \operatorname{Re} z > 0\}$.

4) Let $z = re^{i\theta}$

$f(z) = z^n$ and $g(z) = z^{\frac{1}{n}}$

① $f(z) = z^n = r^n e^{in\theta}$

~~If $r=1$, then~~ z^n will have n preimages with arguments being $\theta + \frac{2\pi k i}{n}$ for $k = 0, 1, \cdots, n-1$

So the map is not 1-1 but surjective.

② $g(z) = z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\frac{\theta}{n}}$ is bijective as easy to check.

6) easy.