MATH 238: HOMEWORK #3 DUE MONDAY, 10/03/16

ALEX IOSEVICH

Problem #1: Construct a finite set $E \subset \mathbb{R}^4$ such that
$$\#\{(x,y) \in E \times E : |x-y| = 1\} \geq \frac{(\#E)^2}{10}.$$

Problem #2: Do Exercise 5.1 on page 62.

Problem #3: Let $E, F \subset \mathbb{R}^4$ with $\#E = \#F = N$. Given $x \in \mathbb{R}^4$, write $x = (x', x'')$, where $x' = (x_1, x_2)$, $x'' = (x_3, x_4)$. Let
$$B(E,F) = \{(|x'-y'|, |x''-y''|) : x \in E, y \in F\}.$$

i) Prove that $\#B(E,E) \geq C\sqrt{N}$. Can you do better?

ii) Construct E and F, both of size N arbitrarily large, such that $\#B(E,F) = 1$.

Problem #4: Let $\Omega = \mathbb{Z}^3 \cap [0, n-1]^3$ and let $f : A \to \{0,1\}$ with the property that if $\{x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}$ is a subset of Ω forming vertices of a box parallel to the coordinate axes, then $f(x^j) = 0$ for at least one $j \in \{1, 2, \dots, 8\}$.

Prove that
$$\#\{x \in \Omega : f(x) = 1\} \leq Cn^{\alpha}$$
for some $\alpha < 3$, where C is a uniform constant. You have seen shadows of this animal before.

Hint: Consider $\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} I_{i,jk}$, where I_{ijk} takes on values 0 or 1. When something like this came up before, we used Cauchy-Schwartz. Give it a go, but you will need to be careful and persistent.

Problem #5: Do Exercise 4.8 on page 43.