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Math 173, Fall 2022, October 19

$\mathcal{B}$ basis $\hookrightarrow [\alpha]_{\mathcal{B}}$ = coordinate matrix of $\alpha$ relative to the basis $\mathcal{B}$

What happens when bases change?

$V$ – $n$-dimensional;

$\underbrace{\mathcal{B} = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \quad \mathcal{B}' = \{\alpha_1', \alpha_2', \dots, \alpha_n'\}}_{\text{ordered bases}}$

$$\alpha_j' = \sum_{i=1}^{n} \underbrace{P_{ij}}_{\text{unique}} \alpha_i, \quad 1 \le j \le n$$

Suppose that $x_1', x_2', \dots, x_n'$ are coordinates of $\alpha$ in $\mathcal{B}'$. Then

$$\alpha = x_1'\alpha_1' + \dots + x_n'\alpha_n'$$

$$= \sum_{j=1}^{n} x_j'\alpha_j' = \sum_{j=1}^{n} x_j' \sum_{i=1}^{n} P_{ij}\alpha_i$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} (P_{ij}x_j')\alpha_i = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} P_{ij}x_j'\right)\alpha_i$$

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It follows that

$$\alpha = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} P_{ij} x_j' \right) \alpha_i \text{, hence}$$

$$x_i = \sum_{j=1}^{n} P_{ij} x_j' \text{, i.e } X = P X' \text{, or}$$

$$\underbrace{X' = P^{-1} X}_{\text{change of coordinates}}$$

In other words, $[\alpha]_\beta = P [\alpha]_{\beta'}$

$$[\alpha]_{\beta'} = P^{-1} [\alpha]_\beta$$

Theorem 7: $V$ = $n$-dimensional over $F$ ~ field; $\beta, \beta'$ two ordered bases. Then $\exists!$ invertible, $n \times n$ matrix $P$ w/ entries in $F$ $\ni$

$$[\alpha]_\beta = P [\alpha]_{\beta'}$$
$$[\alpha]_{\beta'} = P^{-1} [\alpha]_\beta \quad \text{for every vector } \alpha \in V$$

The columns of $P$ are given by

$$P_j = [\alpha_j']_\beta \text{, } j = 1, 2, \ldots, n$$

Theorem 8: Suppose that $P$ is $n \times n$ invertible matrix over $F$. Let $V$ be an $n$-dimensional vector space over $F$, and let $\mathcal{B}$ be an ordered basis of $V$. Then $\exists!$ ordered basis $\mathcal{B}'$ of $V \ni$

$$[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'} \quad [\alpha]_{\mathcal{B}'} = P^{-1}[\alpha]_{\mathcal{B}}$$

Proof: Let $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$

Then if $\mathcal{B}' = \{\alpha_1', \alpha_2', \dots, \alpha_n'\}$ is an ordered basis for which $[\alpha_j]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$, then

$$\alpha_j' = \sum_{i=1}^{n} P_{ij}\alpha_i. \quad (*)$$

To complete the proof, we need to prove that $\alpha_j'$'s given by $(*)$ form a basis.

Let $Q = P^{-1}$, then

$$\sum_j Q_{jK}\alpha_j' = \sum_j Q_{jK}\sum_i P_{ij}\alpha_i$$

$$= \sum_j \sum_i P_{ij} Q_{jK}\, \alpha_i = \sum_i \left( \sum_j P_{ij} Q_{jK} \right) \alpha_i$$

$$= \alpha_K$$

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It follows that the subspace spanned by $\beta' = \{\alpha_1', \alpha_2', \dots, \alpha_n'\}$, and hence equals $V$. So $\beta'$ is a basis and the claimed formulas hold.

Example: $\beta = \{\underset{=}{\overset{\alpha_1}{(1,0)}}, \underset{=}{\overset{\alpha_2}{(0,1)}}\}$

$$\beta' = \{\underbrace{(1,0)}_{\alpha_1'}, \underbrace{(1,1)}_{\alpha_2'}\}$$

$$\alpha_1' = \sum_{i=1}^{2} P_{i1}\alpha_i \qquad \alpha_2' = \sum_{i=1}^{2} P_{i2}\alpha_i$$

$$(1,0) = P_{11}(1,0) + P_{21}(0,1)$$

$$(1,1) = P_{12}(1,0) + P_{22}(0,1)$$

It follows that $P_{11} = 1$, $P_{21} = 0$
$P_{12} = 1$ $P_{22} = 1$

We have $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(5)

Let's test it:

$\beta = \{(1,0), (0,1)\}$

Let $\alpha = (1,2)$. Then $[\alpha]_\beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Then $[\alpha]_{\beta'} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Indeed, $(1,2) = (-1)(1,0) + 2(1,1)$

$= (1,2)$ ✓

<u>Please compute many more!</u>

Putting all the row-equivalence notions together

$A$ $m \times n$ matrix over $F$

$\alpha_1, \ldots, \alpha_m$ row vectors in $F^n$

$\alpha_i = (A_{i1}, A_{i2}, \ldots, A_{in})$

row space $(A)$ = span $\{\alpha_i, i = 1, 2, \ldots, n\}$

row rank $(A)$ = dimension of the row space of $A$.

If $P = K \times m$ matrix over $F$, then
$B = PA$ is a $K \times n$ matrix whose row vectors $\beta_1, \ldots, \beta_K$

are given by $\beta_i = P_{i1}\alpha_1 + \ldots + P_{im}\alpha_m$

It follows that row space $(B) \subset$ row space $(A)$.

⑦

If $P$ is $m \times m$ invertible, then $B$ is row equivalent to $A$, so $A = P^{-1}B$ implies that row space $(A)$ = row space $(B)$

Theorem 9. Row equivalent matrices have the same row space.