GEOMETRY OF THE GAUSS MAP AND LATTICE POINTS IN CONVEX DOMAINS

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Abstract. Let Ω be a convex planar domain, with no curvature or regularity assumption on the boundary. Let $N_\theta(R) = card\{R\Omega_\theta \cap \mathbb{Z}^2\}$, where Ω_θ denotes the rotation of Ω by θ. We prove, up to a small logarithmic transgression, that $N_\theta(R) = |\Omega|R^2 + O(R^{\frac{2}{3}})$, for almost every rotation. We also obtain a refined result based on the fractal structure of the image of the boundary of Ω under the Gauss map.

Introduction

Let Ω be a bounded convex planar domain, and let $N(R) = card\{R\Omega \cap \mathbb{Z}^2\}$. It was observed by Gauss that $N(R) = |\Omega|R^2 + D(R)$, where $|D(R)| \lesssim R$, since the discrepancy $D(R)$ cannot be larger than the number of lattice points that live a distance at most $1/\sqrt{2}$ from the boundary of Ω. Here, and throughout the paper, $A \lesssim B$ means that there exists a uniform C, such that $A \leq CB$. Similarly, $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. For a general domain the estimate $|D(R)| \lesssim R$ cannot be improved, as can be seen by taking Ω to be a square with sides parallel to the axis or, more generally, a polygon with a side of rational slope. However,the purpose of this paper is to show that the remainder term is better for almost every rotation of the domain.

If the boundary of Ω has everywhere non-vanishing Gaussian curvature, better estimates for the remainder term are possible. It is a classical result of Hlawka and Herz that, in that case, $|D(R)| \lesssim R^{\frac{2}{3}}$ and an example due to Jarnik shows that without further assumptions, this result is best possible. See, for example, [L] and [J]. If the boundary is assumed to have a certain degree of smoothness, further improvements have been obtained, culminating (at the moment) in a result due to Huxley, see [Hu], which says that if the boundary of Ω is five times differentiable and has curvature bounded below by a fixed constant, then $|D(R)| \lesssim R^{\frac{46}{73}}$. This is also the current best result for the circle problem, for which the well

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1991 *Mathematics Subject Classification.* Primary 11H06, 42B10, Secondary 42B25.

Key words and phrases. Fourier transform, convex sets, lattice points, maximal function.

Research supported in part by the NSF grant DMS97-06825, INdAM, Integration Program No. 326.53 and RFFI No. 00-15-96-022.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

known conjecture is that $|D(R)| \lesssim R^{\frac{1}{2}+\varepsilon}$ and indeed it was observed by Hardy that one cannot do better than $R^{\frac{1}{2}}$ times an appropriate power of the logarithm.

We have noted that in general the trivial estimate, $|D(R)| \lesssim R$ cannot be improved without curvature assumption on the boundary of the domain. For example, it was proved by Randol in [R], that if Ω is given by the equation $x_1^m + x_2^m \leq 1$, $m > 2$, then $|D(R)| \lesssim R^{\frac{m-1}{m}}$, and $\frac{m-1}{m}$ cannot be replaced by any smaller number. On the other hand, Colin de Verdiere showed in [C] that if the boundary of Ω has finite order of contact with its tangent lines, then, for almost every rotation of Ω, the corresponding error term $|D(R,\theta)| \lesssim R^{\frac{2}{3}}$. This result was extended to a certain class of domains, where the order of contact is infinite, by the third author in [I]. This raises the obvious question of whether this result holds for an arbitrary convex planar domain. Up to a logarithmic transgression, we answer this question in the affirmative. This is the substance of our first result.

Theorem 0.1. *Let Ω be a convex domain, and let $\delta > \frac{1}{2}$. Define*

$$\mathcal{M}(\theta) = \sup_{R\geq 2} \log^{-\delta}(R) R^{-\frac{2}{3}} |D(R,\theta)|,$$

where $D(R,\theta)$ is the discrepancy corresponding to the domain Ω dilated by R and rotated by the angle $-\theta$. Then $\mathcal{M}(\theta) < \infty$ for almost every θ. More precisely $\mathcal{M} \in Weak - L^2(S^1)$, i.e.

$$|\{\theta \in S^1 : \mathcal{M}(\theta) > t\}| \lesssim t^{-2}.$$

Theorem 0.1 is begging to be generalized for the following reason. A result due to Skriganov, see [Sk], says that if Ω is a polygon then $|D(R,\theta)| \lesssim \log^{1+\varepsilon}(R)$, for any $\varepsilon > 0$, for almost every θ. There is much room between this result and $R^{\frac{2}{3}} \log^{\frac{1}{2}+\varepsilon}(R)$ we obtain above, and it makes one ask which geometric properties are in play here. We address this issue in the following way. At every point of a convex set there is the left and the right tangent. Therefore, at every point we have the left$(-)$ and the right$(+)$ normal. Let $\mathcal{N}^{\pm} : \partial\Omega \to S^1$ denote the Gauss maps, which take each point on the boundary of Ω to the right/left unit normal at that point. Our second main result is the following.

Theorem 0.2. *Let Ω be a convex domain. Let $\mathcal{N}^{\pm}$ be the Gauss maps defined above, and let $\mathcal{N}(\partial\Omega) = \mathcal{N}^+(\partial\Omega) \cup \mathcal{N}^-(\partial\Omega)$. Suppose there exists $0 \leq d \leq 1$ such that for any small ε,*

$$|\{\theta \in S^1 : dist(\theta, \mathcal{N}(\partial\Omega)) \leq \varepsilon\}| \lesssim \varepsilon^{1-d}. \tag{1}$$

Define

$$\mathcal{M}_d(\theta) = \sup_{R\geq 2} \log^{-\delta}(R) R^{-\frac{2d}{2d+1}} |D(R,\theta)|.$$

Then $\mathcal{M}_d \in L^1(S^1)$ if $d > 0$ and $\delta > 1$, or, if $d = 0$ and $\delta > 3$. In particular, $\mathcal{M}_d(\theta) < \infty$ for almost every θ.

The estimate (1) implies that the upper Minkowski dimension of $\mathcal{N}(\partial\Omega)$ is at most d. Conversely, if the upper Minkowski dimension of $\mathcal{N}(\partial\Omega)$ is d, then the estimate (1) holds, up to an arbitrarily small power of ε, i.e $|\{\theta \in S^1 : dist(\theta, \mathcal{N}(\partial\Omega)) \le \varepsilon\}| \lesssim \varepsilon^{1-d-\eta}$ for any $\eta > 0$.

The conclusion of Theorem 0.2 can be improved under additional assumptions. For example, see [BIT], if Ω is the convex hull of a subset of a circle, then one can replace the exponent $\frac{d}{d+1}$ in Lemma 2.1 below by $\frac{d}{2}$ and this changes $\frac{2d}{2d+1}$ in Theorem 0.2 to $\frac{2d}{d+2}$.

Theorem 0.2 is stated in terms of the estimate (1) for the sake of simplicity, but it could be restated somewhat more precisely in terms of the properties of the distribution function $|\{\theta \in S^1 : dist(\theta, \mathcal{N}(\partial\Omega)) \le \varepsilon\}|$. When $d = 0$ the condition (1) defines a polygon with finitely many sides but, as we said, in this case a better result is known. The case $d > 0$ includes polygon with infinitely many sides and also more complicated bodies. We provide some easy examples.

Example 1. *Our first example illustrates the case $d > 0$ of Theorem 0.2. Consider a polygon with infinitely many sides, where the slopes of the normals to the sides form a sequence $\{j^{-\alpha}\}_{j=1,2,\ldots}$. It is not hard to see that the upper Minkowski dimension of $\mathcal{N}(\partial\Omega)$ is $\frac{1}{1+\alpha}$ and also that (1) holds with $d = \frac{1}{1+\alpha}$.*

Example 2. *We now consider the case of a polygon with infinitely many sides, such that the slopes of the normals form a lacunary sequence, for example, $\{2^{-j}\}_{j=0,1,\ldots}$. In this case, the upper Minkowski dimension of $\mathcal{N}(\partial\Omega)$ is 0, whereas the estimate (1) does not hold with $d = 0$, though it holds for every $d > 0$. So, Theorem 0.2 says that for every positive ν, $\sup_{R\ge 2} R^{-\nu}|D(R,\theta)|$ is finite for almost every θ. However, as we foreshadowed above, we can do better if we work directly with the quantity $|\{\theta \in S^1 : dist(\theta, \mathcal{N}(\partial\Omega)) \le \varepsilon\}|$, instead of the condition (1). It is not difficult to see that, in this case,*

$$|\{\theta \in S^1 : dist(\theta, \mathcal{N}(\partial\Omega)) \le \varepsilon\}| \approx \varepsilon \log\left(\frac{1}{\varepsilon}\right). \tag{2}$$

The estimate (2) along with the proof of Theorem 0.2 yields the conclusion of Theorem 0.2 with $d = 0$ and $\delta > 4$.

We also note that Theorem 0.2 does not apply only to polygons, with finitely, or infinitely many sides. In fact, it is not difficult to construct examples of convex domains, where $\mathcal{N}(\partial\Omega)$ has upper Minkowski dimension $0 < d < 1$, which are not polygons. It is just a matter of constructing an appropriate increasing function, for example a Cantor-Lebesgue type function, which defines the tangent vector field.

As we have seen the quest for the best exponent in lattice point problems has a long history and it seems far from a definitive results. Also the exponent $\frac{2}{3}$ in Theorem 0.1 is probably not sharp and a natural conjecture is $\frac{1}{2} + \varepsilon$. This belief is supported by the fact that

$$\left(\int_{S^1}\int_{\mathbb{T}^2} |D(R,\theta,\tau)|^2 d\tau d\theta\right)^{\frac{1}{2}} \approx R^{\frac{1}{2}},$$

where $\mathbb{T}^2$ denotes the two-dimensional torus, and $D(R, \theta, \tau)$ denotes the discrepancy corresponding to the case where a convex domain Ω is rotated by $-\theta$ and translated by τ. See e.g. [R] or [BRT, Theorem 6.2].

Estimates for the Fourier transform

The main ingredient in the proof of Theorem 0.1 and Theorem 0.2 is the following maximal estimate for the Fourier transform of the characteristic function of Ω, which is interesting in its own right.

Theorem 0.3. *Let Ω be a convex domain. Then*

$$|\{\theta \in S^1 : \sup_{R\geq 0} R^{\frac{3}{2}} |\widehat{\chi}_\Omega(R\theta)| > t\}| \lesssim t^{-2}.$$

The above inequality means that the function $\sup_{R\geq 0} R^{\frac{3}{2}} |\widehat{\chi}_\Omega(R\theta)|$ belongs to $Weak - L^2(S^1)$. This should be compared with a result in [P] where it is proved that $R^{\frac{3}{2}} \widehat{\chi}_\Omega(R\theta)\}$ is in $L^2(S^1)$ with norms bounded by a constant independent of R. On the contrary the above maximal function does not necessary belong to $L^2(S^1)$, as the example of a square shows. Indeed

$$\widehat{\chi}_{[-\frac{1}{2},\frac{1}{2}]\times[-\frac{1}{2},\frac{1}{2}]}(R\theta) = \frac{\sin(\pi R\cos(\theta))}{\pi R\cos(\theta)} \frac{\sin(\pi R\sin(\theta))}{\pi R\sin(\theta)}$$

and therefore

$$\sup_{R\geq 0} R^{\frac{3}{2}} |\widehat{\chi}_{[-\frac{1}{2},\frac{1}{2}]\times[-\frac{1}{2},\frac{1}{2}]}(R\theta)| \approx |\cos(\theta)|^{-\frac{1}{2}} |\sin(\theta)|^{-\frac{1}{2}}.$$

When assuming $\partial\Omega$ analytic, the estimate in the above theorem is implied by a result obtained by Svensson in [Sv]. However, lack of any smoothness assumption, besides convexity, involves considerable difficulties.

In order to prove Theorem 0.2 we shall also use the following decay estimate, the proof of which is taken from [BIT].

Theorem 0.4. *Let Ω be a convex domain. Then if $\theta \notin \mathcal{N}(\partial\Omega)$,*

$$|\widehat{\chi}_\Omega(R\theta)| \lesssim R^{-2}(dist(\theta, \mathcal{N}(\partial\Omega))^{-1}.$$

This paper is organized as follows. In Section I, we prove Theorem 0.1 using Theorem 0.3 and the Poisson Summation Formula. In Section II, we prove Theorem 0.2 using an appropriate modification of Theorem 0.3, and Theorem 0.4. In Section III, we prove Theorem 0.3, and in Section IV we repeat the proof of Theorem 0.4 already contained in [BIT] .

Section I: Proof of Theorem 0.1

The proof of this theorem as long as the proof of Theorem 0.2 is a consequence of the techniques in [Hl] and [He] along with the maximal estimate for the Fourier transform in Theorem 0.3.

Let ψ be a smooth positive radial function of mass 1 supported in the unit disc centered at the origin, and let $\psi_\varepsilon(x) = \varepsilon^{-2}\psi(\varepsilon^{-1}x)$. Define

$$N(R,\theta,\varepsilon) = \sum_{k\neq(0,0)} \chi_{R\theta^{-1}\Omega} * \psi_\varepsilon(k),$$

$$D(R,\theta,\varepsilon) = N(R,\theta,\varepsilon) - R^2|\Omega|.$$

Lemma 1.1. *We have*

$$D(R,\theta,\varepsilon) = R^2 \sum_{k\neq(0,0)} \widehat{\chi}_\Omega(R\theta k)\widehat{\psi}(\varepsilon k).$$

Proof. This is the Poisson summation formula.

□

Lemma 1.2. *We have*

$$D(R-\varepsilon,\theta,\varepsilon) - (2R\varepsilon - \varepsilon^2)|\Omega| \leq D(R,\theta) \leq D(R+\varepsilon,\theta,\varepsilon) + (2R\varepsilon + \varepsilon^2)|\Omega|.$$

Proof. We may assume that Ω contains the origin. We have

$$\chi_{(R-\varepsilon)\theta^{-1}\Omega} * \psi_\varepsilon(k) \leq \chi_{R\theta^{-1}\Omega}(k) \leq \chi_{(R+\varepsilon)\theta^{-1}\Omega} * \psi_\varepsilon(k),$$

along with

$$N(R-\varepsilon,\theta,\varepsilon) \leq N(R,\theta) \leq N(R+\varepsilon,\theta,\varepsilon),$$

and the result follows.

□

Lemma 1.3. *We have*

$$|\{\theta \in S^1 : \sup_{2^j \leq R \leq 2^{j+1}} R^{-\frac{2}{3}}|D(R,\theta)| > t\}| \lesssim t^{-2}.$$

Proof. By Lemma 1.1 we have

$$\sup_{2^j \leq R \leq 2^{j+1}} R^{-\frac{2}{3}}|D(R,\theta,\varepsilon)| \leq 2^{-\frac{j}{6}} \sum_{k\neq(0,0)} |k|^{-\frac{3}{2}}|\widehat{\psi}(\varepsilon k)| \sup_{2^j \leq R \leq 2^{j+1}} |R\theta k|^{\frac{3}{2}}|\widehat{\chi}_\Omega(R\theta k)|.$$

By Theorem 0.3, the maximal function $\sup_{2^j \leq R \leq 2^{j+1}} |R\theta k|^{\frac{3}{2}}|\widehat{\chi}_\Omega(R\theta k)|$ is uniformly in $Weak - L^2$, which is a Banach space. Hence the sum is also in this space, with norm controlled by

$$2^{-\frac{j}{6}} \sum_{k\neq(0,0)} |k|^{-\frac{3}{2}}|\widehat{\psi}(\varepsilon k)| \lesssim 2^{-\frac{j}{6}}\varepsilon^{-\frac{1}{2}},$$

since $\widehat{\psi}(\xi)$ is rapidly decreasing. The result now follows from Lemma 1.2 by taking $\varepsilon = 2^{-\frac{j}{3}}$.

□

We are now ready to complete the proof of Theorem 0.1. Observe that

$$\sup_{R\geq 2} \log^{-\delta}(R) R^{-\frac{2}{3}} |D(R,\theta)| \lesssim \left\{ \sum_{j=1}^{\infty} j^{-2\delta} \sup_{2^j \leq R \leq 2^{j+1}} R^{-\frac{4}{3}} |D(R,\theta)|^2 \right\}^{\frac{1}{2}}.$$

The function $\sup_{2^j \leq R \leq 2^{j+1}} R^{-\frac{4}{3}} |D(R,\theta)|^2$ is uniformly in $Weak - L^1$, and can therefore be summed by the sequence $j^{-2\delta}$ if $2\delta > 1$. The conclusion of Theorem 0.1 follows.

Section II: Proof of Theorem 0.2

Lemma 2.1. *Under the assumptions of Theorem 0.2, if $d > 0$ we have*

$$\int_{S^1} \left[\sup_{2^j \leq R \leq 2^{j+1}} R^{2-\frac{d}{d+1}} |\widehat{\chi}_\Omega(R\theta)| \right] d\theta \lesssim 1.$$

If $d = 0$,

$$\int_{S^1} \left[\sup_{2^j \leq R \leq 2^{j+1}} R^{2} |\widehat{\chi}_\Omega(R\theta)| \right] d\theta \lesssim j.$$

Proof. Applying Theorem 0.3, we have

$$\begin{aligned}
&\int_{\{d(\theta,\mathcal{N}(\partial\Omega)) \leq 2^{-j/(d+1)}\}} \left[\sup_{2^j \leq R \leq 2^{j+1}} R^{2-\frac{d}{d+1}} |\widehat{\chi}_\Omega(R\theta)| \right] d\theta \\
&\lesssim 2^{j(\frac{1}{2}-\frac{d}{d+1})} \int_{\{d(\theta,\mathcal{N}(\partial\Omega)) \leq 2^{-j/(d+1)}\}} \left[\sup_{2^j \leq R \leq 2^{j+1}} R^{\frac{3}{2}} |\widehat{\chi}_\Omega(R\theta)| \right] d\theta \\
&\lesssim 2^{j(\frac{1}{2}-\frac{d}{d+1})} \int_0^{\infty} \min\Big\{ |\{d(\theta,\mathcal{N}(\partial\Omega)) \leq 2^{-\frac{j}{d+1}}\}|, t^{-2} \Big\} dt \\
&\lesssim 2^{j(\frac{1}{2}-\frac{d}{d+1})} |\{\theta \in S^1 : d(\theta,\mathcal{N}(\partial\Omega)) \leq 2^{-\frac{j}{d+1}}\}|^{\frac{1}{2}} \\
&\lesssim 1.
\end{aligned}$$

Moreover, by Theorem 0.4, when $d>0$ we have

$$\begin{aligned}
&\int_{\{d(\theta,\mathcal{N}(\partial\Omega))>2^{-j/(d+1)}\}}\left[\sup_{2^j\le R\le 2^{j+1}} R^{2-\frac{d}{d+1}}|\widehat{\chi}_\Omega(R\theta)|\right]d\theta\\
&\quad\lesssim 2^{-j\frac{d}{d+1}}\int_{\{d(\theta,\mathcal{N}(\partial\Omega))>2^{-j/(d+1)}\}}(d(\theta,\mathcal{N}(\partial\Omega)))^{-1}d\theta\\
&\quad\lesssim 2^{-j\frac{d}{d+1}}\sum_{h=0}^{\infty}\left(2^{h-\frac{j}{d+1}}\right)^{-1}|\{\theta\in S^1:2^{h-\frac{j}{d+1}}\le d(\theta,\mathcal{N}(\partial\Omega))\le 2^{h+1-\frac{j}{d+1}}\}|\\
&\quad\lesssim 2^{-j\frac{d}{d+1}}\sum_{h=0}^{\infty}\left(2^{h-\frac{j}{d+1}}\right)^{-1}\left(2^{h-\frac{j}{d+1}}\right)^{1-d}\\
&\quad\lesssim 1.
\end{aligned}$$

Observe that when $d=0$, it suffices to sum the series in the range $0\le h\lesssim j$. This completes the proof of Lemma 2.1.

□

Lemma 2.2. *Under the assumptions of Theorem 0.2, if $d>0$ we have*

$$\int_{S^1}\left[\sup_{2^j\le R\le 2^{j+1}} R^{-\frac{2d}{2d+1}}|D(R,\theta)|\right]d\theta\lesssim 1.$$

If $d=0$, we have

$$\int_{S^1}\left[\sup_{2^j\le R\le 2^{j+1}} |D(R,\theta)|\right]d\theta\lesssim j^2.$$

Proof. By Lemma 1.2 we have

$$\begin{aligned}
&\int_{S^1}\left[\sup_{2^j\le R\le 2^{j+1}} R^{-\frac{2d}{2d+1}}|D(R,\theta)|\right]d\theta\\
&\qquad\lesssim\int_{S^1}\left[\sup_{2^j\le R\le 2^{j+1}} R^{-\frac{2d}{2d+1}}|D(R,\theta,\varepsilon)|\right]d\theta+2^{\frac{j}{2d+1}}\varepsilon.
\end{aligned}$$

Assume that $d>0$. By Lemma 1.1 and Lemma 2.1 we have

$$\begin{aligned}
&\int_{S^1}\left[\sup_{2^j\le R\le 2^{j+1}} R^{-\frac{2d}{2d+1}}|D(R,\theta,\varepsilon)|\right]d\theta\\
&\quad\le 2^{-\frac{jd}{(d+1)(2d+1)}}\sum_{k\neq(0,0)}|k|^{\frac{d}{d+1}-2}|\widehat{\psi}(\varepsilon k)|\int_{S^1}\left[\sup_{2^j\le R\le 2^{j+1}}|R\theta k|^{2-\frac{d}{d+1}}|\widehat{\chi}_\Omega(R\theta k)|\right]d\theta\\
&\quad\lesssim\varepsilon^{-\frac{d}{d+1}}2^{-\frac{jd}{(d+1)(2d+1)}}.
\end{aligned}$$

Choosing $\varepsilon = 2^{-\frac{j}{2d+1}}$ yields the result. The proof in the case $d = 0$ is similar.

□

We are now ready to complete the proof of Theorem 0.2. We have

$$\int_{S^1} \left[\sup_{R \geq 2} \log^{-\delta}(R) R^{-\frac{2d}{2d+1}} |D(R,\theta)| \right] d\theta \lesssim \sum_{j=1}^{\infty} j^{-\delta} \int_{S^1} \left[\sup_{2^j \leq R \leq 2^{j+1}} R^{-\frac{2d}{2d+1}} |D(R,\theta)| \right] d\theta.$$

In view of Lemma 2.2, when $d > 0$ the series converges provided $\delta > 1$. When $d = 0$ one has to take $\delta > 3$. This completes the proof of Theorem 0.2.

Section III: Proof of Theorem 0.3

We start out by arguing that we may take Ω with a smooth boundary and everywhere non-vanishing curvature, so long as the constants in our arguments do not depend on curvature and smoothness. Indeed, suppose that $\sup_{R>0} R^{\frac{3}{2}} |\widehat{\chi}_\Omega(R\theta)|$ is not in *weak* $- L^2(S^1)$. Then, given $k > 0$, there exists $N > 0$, such that the *weak*$-L^2(S^1)$ norm of $\sup_{0<R<N} R^{\frac{3}{2}} |\widehat{\chi}_\Omega(R\theta)|$ is at least k. For every ε it is possible to approximate Ω by a convex domain Ω_ε, the boundary of which is smooth and has everywhere non-vanishing curvature, and $|\Omega - \Omega_\varepsilon| < \varepsilon$. Then, for $\varepsilon \leq N^{-\frac{3}{2}}$,

$$\sup_{0<R<N} R^{\frac{3}{2}} |\widehat{\chi}_{\Omega_\varepsilon}(R\theta)| \geq \sup_{0<R<N} R^{\frac{3}{2}} |\widehat{\chi}_\Omega(R\theta)| - N^{\frac{3}{2}} |\Omega - \Omega_\varepsilon| \geq k - 1.$$

Hence, from now on, we assume that Ω is a convex body having smooth boundary with strictly positive curvature.

We now need the following lemma (see [P] or Lemma 3.8 in [BRT]).

Lemma 3.1. *Let Ω be a convex bounded domain, let $S_\theta = \sup_{x \in \Omega} x \cdot \theta$ and denote by $\mu(\theta, \varepsilon)$ the length of the chord $C(\theta, \varepsilon) = \{x \in \Omega : x \cdot \theta = S_\theta - \varepsilon\}$. Then, for a constant c independent of smoothness and curvature, we have*

$$|\widehat{\chi}_\Omega(R\theta)| \leq \frac{c}{R} \left(\mu\left(\theta, \frac{1}{2R}\right) + \mu\left(-\theta, \frac{1}{2R}\right) \right).$$

Since, by the above lemma,

$$\sup_{R>0} R^{\frac{3}{2}} |\widehat{\chi}_\Omega(R\theta)| \lesssim \sup_{R>0} R^{-\frac{1}{2}} \mu\left(\theta, \frac{1}{2R}\right) + \sup_{R>0} R^{-\frac{1}{2}} \mu\left(-\theta, \frac{1}{2R}\right),$$

we are reduced to studying the maximal function

$$\mu^*(\theta) = \sup_{\varepsilon>0} \frac{1}{\sqrt{\varepsilon}} \mu(\theta, \varepsilon).$$

Observe that $C(\theta, 0)$ is a single point, which we denote by $z(\theta)$. Let us choose a direction θ_o and denote by $\lambda(\theta)$ the arc-length on $\partial\Omega$ between $z(\theta_o)$ and $z(\theta)$. Let

$$\lambda_+^*(\theta) = \sup_{\psi>0} \frac{\lambda(\theta+\psi) - \lambda(\theta)}{\psi}, \qquad \lambda_-^*(\theta) = \sup_{\psi>0} \frac{\lambda(\theta) - \lambda(\theta-\psi)}{\psi}.$$

We have the following estimate.

Lemma 3.2. $[\mu^*(\theta)]^2 \le 2\lambda_+^*(\theta) + 2\lambda_-^*(\theta)$.

Proof. The normal at the point $z(\theta)$ determines, on the chord $C(\theta, \varepsilon)$ or possibly on its continuation, two segments of length $\mu_+(\theta, \varepsilon)$ and $\mu_-(\theta, \varepsilon)$. Therefore the maximal function $\mu^*(\theta)$ is dominated by the sum $\mu_+^*(\theta) + \mu_-^*(\theta)$. Now observe that the computation of $\sup_{\varepsilon>0} \frac{1}{\sqrt{\varepsilon}} \mu_\pm(\theta, \varepsilon)$ involves only values of ε for which $\mu_\pm(\theta, \varepsilon)$ increases. Let us consider $\mu_+^*(\theta)$. We may assume that the boundary of Ω is locally a graph of a smooth function $f(x)$, defined on an interval $[0, a]$, with $f(0) = f'(0^+) = 0$. We have

$$[\mu_+^*(\theta)]^2 = \sup_{\varepsilon>0} \frac{(\mu_+(\theta, \varepsilon))^2}{\varepsilon} \le \sup_{0<x<a} \frac{x^2}{f(x)}.$$

By the mean value theorem,

$$\begin{aligned} \sup_{0<x<a} \frac{x^2}{f(x)} &\le \sup_{0<z<a} \frac{2z}{f'(z)} \\ &\le \sup_{0<z<a} \frac{2}{f'(z)} \int_0^z \sqrt{1 + (f'(t))^2}\, dt \\ &= \sup_{0<\psi<f'(a)} \frac{2}{\psi} \int_0^{(f')^{-1}(\psi)} \sqrt{1 + (f'(t))^2}\, dt \\ &= 2 \sup_{0<\psi<f'(a)} \frac{\lambda(\psi+\theta) - \lambda(\theta)}{\psi}. \end{aligned}$$

□

Theorem 0.3 now follows from the classical Hardy-Littlewood maximal theorem, which we state in the following form (see I.3 in [RN]).

Lemma 3.3. *Let λ be and increasing bounded function on the interval $[a, b]$. Then for every $t > 0$,*

$$\left| \left\{ \theta : \sup_{\psi>0} \frac{\lambda(\theta+\psi) - \lambda(\theta)}{\psi} > t \right\} \right| \le \frac{\lambda(b) - \lambda(a)}{t}.$$

Section IV: Proof of Theorem 0.4

By Lemma 3.1 we have

$$|\widehat{\chi}_\Omega(R\theta)| \le \frac{c}{R}\left(\mu\left(\theta, \frac{1}{2R}\right) + \mu\left(-\theta, \frac{1}{2R}\right)\right).$$

Without loss of generality we may assume that $\theta = -\frac{\pi}{2}$. We may also assume that the boundary of Ω passes through the origin and Ω lies in the upper half plane. In a neighborhood of the origin the boundary of Ω is described by a convex function, say $y = \phi(x)$, satisfying $\phi(x) \ge 0$, and $\phi(0) = 0$. Let $\phi'(0^-)$ and $\phi'(0^+)$ denote the left and the right derivatives at the origin. Since we assume that $\theta \notin \mathcal{N}(\partial\Omega)$, we must have $\phi'(0^-) < 0 < \phi'(0^+)$. By convexity, Ω is contained in the set

$$\{(x, y) \in \mathbb{R}^2 : x \ge 0, y > \phi'(0^+)x\} \cup \{(x, y) \in \mathbb{R}^2 : x \le 0, y > \phi'(0^-)x\}.$$

It follows that

$$\begin{aligned}\mu\left(\theta, \frac{1}{R}\right) &\le \frac{1}{R\phi'(0^+)} - \frac{1}{R\phi'(0^-)} \\ &\le \frac{2}{R\min\{\phi'(0^+), -\phi'(0^-)\}}\end{aligned}$$

and the proof follows since

$$\min\{\phi'(0^+), |\phi'(0^-)|\} \approx dist(\theta, \mathcal{N}(\partial\Omega)).$$

□

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