Representation Theory & Homological Algebra

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Contents

1 Generalities on modules

Notation 1.1. *Rings are assumed to be unitary and associative.*

1.1 Modules and algebras

Definition 1.2 (Modules)**.** *Let R be a ring. A* ***(left)*** R***-module*** *is an abelian group M together with a map $(x, m) \in R \times M \mapsto xm \in M$ s.t., for $x, x' \in R$ and $m, m' \in M$,*

- $x(m + m') = xm + xm'$,

- $1_R \cdot m = m$,
- $(x + x')\, m = xm + x'm$,
- $(xx')\, m = x\,(x'm)$.

We write $\mathbf{Mod}R$ *for the set of (left) R-modules. A **right R-module** is defined similarly.*

Remark 1.3. *Given a ring R, R^{op} is the ring R with inversed multiplication: xy in R^{op} is yx in R. Hence, a right R-module is a left R^{op}-module.*

Proposition 1.4. *Given an abelian group M, endowing M with the structure of a left (resp. right) R-module is equivalent to specifying a ring homomorphism $R \to \mathrm{End}(M)$ (resp. $\mathrm{End}(M)^{\mathrm{op}}$).*

Definition 1.5 (Bimodule)**.** *If R, R' are rings, then an **R-R'-bimodule** is an abelian group M that is both a left R-module and a right R'-module, s.t. $(xm)\,x' = x\,(mx')$ for $x \in R$, $m \in M$, $x' \in R'$.*

Notation 1.6. *From now on, k denotes a commutative ring (often $\mathbb{Z}$ or a field).*

Definition 1.7 (Algebras)**.** *A **k-algebra** is a k-module A that is also a ring, s.t. $\lambda(ab) = (\lambda a)\, b = a\,(\lambda b)$ for $\lambda \in k$, $a, b \in A$.*

Example 1.8. (i) *Any ring is a $\mathbb{Z}$-algebra.*

(ii) *Given a group G, its **algebra group** is the free k-module over G, with multiplication induced by the group law in G.*

Remark 1.9. (i) *Specifying a structure of k-algebra on a ring A is equivalent to specifying a ring homomorphism $k \to A$ whose image is contained in the centre of A.*

(ii) *If A is a k-algebra, then specifying a structure of A-module on a k-module M is equivalent to specifying a k-linear map $A \to \mathrm{End}_k(M)$.*

Proposition 1.10. *Let G be a group. Then representations of G (i.e. k-vector spaces V together with a group homomorphism $\rho : G \to GL(V)$) are equivalent to kG-modules.*

1.2 Homomorphism modules and tensor products

Notation 1.11. *From now on, A denotes a k-algebra (for example $A = kG$, where G is a group).*

Definition 1.12 (Homomorphism module)**.** *Given $M, N \in \mathbf{Mod}A$, we denote by $\mathrm{Hom}_A(M, N)$ the set of A-linear maps $M \to N$ and $\mathrm{End}_A(M) = \mathrm{Hom}_A(M, M)$.*

$\mathrm{Hom}_A(M, N)$ is naturally a k-module and $\mathrm{End}_A(M)$ is a k-algebra.

Proposition 1.13. *Let A, B, C be k-algebras. If ${}_AM_B$ is an A-B-bimodule, and ${}_AN_C$ is an A-C-bimodule, then $\mathrm{Hom}_A\left({}_AM_B, {}_AN_C\right)$ is naturally a B-C-bimodule, with action given by $(bfc)(m) = f(mb)c$.*

Proposition 1.14. (i) *Given $M \in \mathbf{Mod}A$, $\mathrm{Hom}_A(A, M) \cong M$ as A-modules.*

(ii) *$\mathrm{End}_A(A) \cong A^{\mathrm{op}}$ as k-algebras.*

Definition 1.15 (Duals)**.** *Let $M \in \mathbf{Mod}A$. We have two notions of **duals**:*

- $M^* = \mathrm{Hom}_k\,(M, k) \in \mathbf{Mod}A^{\mathrm{op}}$,
- $M^\vee = \mathrm{Hom}_A\,(M, A) \in \mathbf{Mod}A^{\mathrm{op}}$.

In particular, $A^* = \mathrm{Hom}_k\,(A, k) \in \mathbf{Mod}A^{\mathrm{op}}$.

Example 1.16 (Dual representation). *Assume that $A = kG$, and $\rho : G \to GL(V)$ is a f.d. representation of G, i.e. V is a kG-module with $\dim_k V < \infty$. Hence, V^* is a kG^{op}-module. But there is an isomorphism of k-algebras $kG \xrightarrow{\cong} kG^{\mathrm{op}}$ given by $g \mapsto g^{-1}$, so V^* is naturally a kG-module; the corresponding representation of G is $\rho^* : G \to GL(V^*)$ given by $\rho^*(g) \cdot \varphi = \varphi \circ \rho(g^{-1})$.*

Definition 1.17 (Restriction of scalars). *Let B be a sub-k-algebra of A. Then A is a right-B-module. Given $M \in \mathbf{Mod}A$, the B-module $\operatorname{Hom}_A({}_AA_B, {}_AM)$ is M seen as a B-module.*

In particular, if $H \leqslant G$ is a subgroup and $V \in \mathbf{Mod}kG$, we write $\operatorname{Res}^H_G(V) = \operatorname{Hom}_{kG}(kG, V) \in \mathbf{Mod}kH$.

Proposition 1.18. *Let M, M_1, M_2, N, N_1, N_2 be A-modules.*

(i) $\operatorname{Hom}_A(M_1 \oplus M_2, N) \cong \operatorname{Hom}_A(M_1, N) \oplus \operatorname{Hom}_A(M_2, N)$,

(ii) $\operatorname{Hom}_A(M, N_1 \oplus N_2) \cong \operatorname{Hom}_A(M, N_1) \oplus \operatorname{Hom}_A(M, N_2)$,

(iii) $\operatorname{End}_A(M_1 \oplus M_2) \cong \begin{pmatrix} \operatorname{End}_A(M_1) & \operatorname{Hom}_A(M_2, M_1) \\ \operatorname{Hom}_A(M_1, M_2) & \operatorname{End}_A(M_2) \end{pmatrix}$.

Definition 1.19 (Tensor product). *Let ${}_BM_A$ be a B-A-bimodule and ${}_AN_C$ be an A-C-bimodule. The **tensor product** ${}_BM \otimes_A N_C$ is the quotient of the free k-module generated by $m \otimes n$ (for $m \in M$, $n \in N$) by the submodule generated by:*

- $(m_1 + m_2) \otimes n - (m_1 \otimes n + m_2 \otimes n)$ *for* $m_1, m_2 \in M$ *and* $n \in N$,
- $m \otimes (n_1 + n_2) - (m \otimes n_2 + m \otimes n_2)$ *for* $m \in M$ *and* $n_1, n_2 \in N$,
- $(ma) \otimes n - m \otimes (an)$ *for* $m \in M$, $n \in N$ *and* $a \in A$.

${}_BM \otimes_A N_C$ is naturally a B-C-bimodule.

Proposition 1.20. (i) ${}_AA \otimes_A Y_C \cong_A Y_C$ *and* ${}_BX \otimes_A A_A \cong_B X_A$,

(ii) $(X \otimes_A Y) \otimes_B Z \cong X \otimes_A (Y \otimes_B Z)$,

(iii) *If A is commutative, then $X \otimes_A Y \cong Y \otimes_A X$.*

(iv) $(X_1 \oplus X_2) \otimes_A Y \cong (X_1 \otimes_A Y) \oplus (X_2 \otimes_A Y)$ *and similarly for* $X \otimes_A (Y_1 \oplus Y_2)$,

(v) *Given maps $f : X_A \to X'_A$ and $g :_A Y \to_A Y'$, there is a k-linear map $f \otimes g : X \otimes_A Y \to X' \otimes_A Y'$.*

Definition 1.21 (Extension of scalars). *Given a subalgebra $B \hookrightarrow A$, and a B-module M, $A \otimes_B M$ is naturally an A-module; it is M seen as an A-module.*

In particular, if $H \leqslant G$ is a subgroup and $W \in \mathbf{Mod}kH$, we write $\operatorname{Ind}^H_G(W) = kG \otimes_{kH} W \in \mathbf{Mod}kG$.

Remark 1.22. *If A, B are k-algebras, then $A \otimes_k B$ is naturally a k-algebra.*

An A-B-bimodule is the same as an $A \otimes_k B^{\mathrm{op}}$-module.

Theorem 1.23 (Adjunction Formula). *Let ${}_AX$ be an A-module, ${}_BY_A$ be a B-A-bimodule and ${}_BZ$ be a B-module. Then we have the following isomorphism of k-modules:*

$$\operatorname{Hom}_A({}_AX, \operatorname{Hom}_B({}_BY_A, {}_BZ)) \cong \operatorname{Hom}_B({}_BY \otimes_A X, {}_BZ).$$

Corollary 1.24. *Let $H \leqslant G$ be a subgroup. Let $V \in \mathbf{Mod}kG$ and $W \in \mathbf{Mod}kH$. Then*

$$\operatorname{Hom}_{kH}\left(W, \operatorname{Res}^G_H(V)\right) \cong \operatorname{Hom}_G\left(\operatorname{Ind}^G_H(W), V\right).$$

1.3 Finite-infinite modules

Proposition 1.25. (i) $\operatorname{Hom}_A\left(M, \prod_{j\in J} N_j\right) \cong \prod_{j\in J} \operatorname{Hom}_A(M, N_j)$,

(ii) $\operatorname{Hom}_A\left(\bigoplus_{i\in I} M_i, N\right) \cong \prod_{i\in I} \operatorname{Hom}_A(M_i, N)$.

Notation 1.26. *Given a set I, define:*

- $A^I = \prod_{i\in I} A$,
- $A^{(I)} = \bigoplus_{i\in I} A$.

Definition 1.27 (Free module)**.** *Given an A-module M, the following conditions are equivalent:*

(i) *M admits an A-basis, i.e. a family $(e_i)_{i\in I}$ in M that is both generating and linearly independent.*

(ii) *$M \cong A^{(I)}$ for some set I.*

*We say that M is **free** if it satisfies the above conditions.*

Remark 1.28. *It is false in general that if $A^{(I)} \cong A^{(J)}$ (as A-modules), then the sets I and J are in bijection. We may even have $A^m \cong A^n$ with $m \neq n$. However, the above implication holds if A is a finite-dimensional k-algebra.*

Remark 1.29. *Any module is a quotient of a free module, i.e. there is a surjective morphism $p : A^{(I)} \to M$.*

Reiterating the above operation with $\operatorname{Ker} p$, *we see that M is the cokernel of some map $A^{(J)} \to A^{(I)}$ of free modules.*

Definition 1.30 (Finite generation and finite presentation)**.** *Let M be an A-module.*

- *We say that M is **finitely generated (f.g.)** if there is a surjective morphism $A^n \to M$ for some $n \in \mathbb{N}$.*
- *We say that M is **finitely presented (f.p.)** if* $M \cong \operatorname{Coker}\left(A^m \xrightarrow{f} A^n\right)$ *for some map f and some $m, n \in \mathbb{N}$.*

Remark 1.31. *If A is a finite-dimensional k-algebra and k is a field, then an A-module M is finitely generated* iff *it is finite-dimensional over k* iff *it is finitely presented.*

Later, we will see that if A is noetherian, then M is f.g. iff *M is f.p.*

Also note that if $A = \mathbb{Z}$, then a subgroup of a f.g. abelian group is f.g.

2 Categories of modules

Notation 2.1. *Throughout this section, k is a commutative ring and A is a k-algebra.*

2.1 k-linear categories

Definition 2.2 (k-linear category)**.** *A category $\mathbf{C}$ is k-**linear** if* $\operatorname{Hom}_{\mathbf{C}}(X, Y)$ *has the structure of a k-module for all $X, Y \in \mathbf{C}$ s.t.*

- *The composition* $\circ : \operatorname{Hom}_{\mathbf{C}}(Y, Z) \times \operatorname{Hom}_{\mathbf{C}}(X, Y) \longrightarrow \operatorname{Hom}_{\mathbf{C}}(X, Z)$ *is k-linear,*
- *There is a zero object $0_{\mathbf{C}} \in \mathbf{C}$ satisfying* $\operatorname{Hom}_{\mathbf{C}}(X, 0_{\mathbf{C}}) = 0$ *and* $\operatorname{Hom}_{\mathbf{C}}(0_{\mathbf{C}}, Y) = 0$ *for all $X, Y \in \mathbf{C}$,*
- *There is $X \oplus Y \in \mathbf{C}$ for all $X, Y \in \mathbf{C}$, together with isomorphisms* $\operatorname{Hom}_{\mathbf{C}}(X \oplus Y, Z) \cong \operatorname{Hom}_{\mathbf{C}}(X, Z) \oplus \operatorname{Hom}_{\mathbf{C}}(Y, Z)$ *and* $\operatorname{Hom}_{\mathbf{C}}(Z, X \oplus Y) \cong \operatorname{Hom}_{\mathbf{C}}(Z, X) \oplus \operatorname{Hom}_{\mathbf{C}}(Z, Y)$.

In particular, a $\mathbb{Z}$-linear category is simply an ***additive category****.*

Definition 2.3 (*k*-linear functor)**.** *A functor $F : \mathbf{C}_1 \to \mathbf{C}_2$ is k-**linear** if the induced maps $F : \mathrm{Hom}_{\mathbf{C}_1}(X,Y) \to \mathrm{Hom}_{\mathbf{C}_2}(FX,FY)$ are k-linear, $F0_{\mathbf{C}_1} = 0_{\mathbf{C}_2}$, and for all $X, Y \in \mathbf{C}_1$, there is an isomorphism $F(X \oplus Y) \cong FX \oplus FY$ making the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Hom}\left(X \oplus Y, Z\right) \arrow[r, "F"] \arrow[drr, "\cong"] & \mathrm{Hom}\left(F\left(X \oplus Y\right), Z\right) \arrow[r, "\cong"] & & \mathrm{Hom}\left(FX \oplus FY, FZ\right) \arrow[d, "\cong"] \\
& & \mathrm{Hom}\left(X,Z\right) \oplus \mathrm{Hom}\left(Y,Z\right) \arrow[r, "F"] & \mathrm{Hom}\left(FX,FZ\right) \oplus \mathrm{Hom}\left(FY,FZ\right)
\end{tikzcd}$$

Example 2.4. *Let $B \hookrightarrow A$ be a subalgebra. Then the restriction functor $\mathbf{Mod}A \to \mathbf{Mod}B$ is k-linear.*

Proposition 2.5. *Let M be an A-B-bimodule.*

(i) $\mathrm{Hom}_A(M,-) : \mathbf{Mod}A \to \mathbf{Mod}B$ *is a covariant k-linear functor,*

(ii) $\mathrm{Hom}_A(-,M) : \mathbf{Mod}A \to \mathbf{Mod}B^{\mathrm{op}}$ *is a contravariant k-linear functor,*

(iii) $M \otimes (-) : \mathbf{Mod}B \to \mathbf{Mod}A$ *is a covariant k-linear functor,*

(iv) $(-) \otimes M : \mathbf{Mod}A^{\mathrm{op}} \to \mathbf{Mod}B^{\mathrm{op}}$ *is a covariant k-linear functor.*

Remark 2.6. *All isomorphisms of Section 1 are natural isomorphisms between k-linear functors, for instance:*

- *The isomorphism $\mathrm{Hom}_A(A,M) \cong M$ is functorial in M,*
- *The isomorphism $\mathrm{Hom}_A(X \otimes Y, Z) \cong \mathrm{Hom}_B(Y, \mathrm{Hom}_A(X,Z))$ is functorial in Y, Z.*

2.2 Short exact sequences

Proposition 2.7. *Let $f \in \mathrm{Hom}_A(X,Y)$. Then, in the category $\mathbf{Mod}A$, there are two short exact sequences*

$$0 \to K \to X \xrightarrow{p} I \to 0,$$
$$0 \to I \xrightarrow{i} X \to C \to 0.$$

such that $i \circ p = f$. Moreover, those exact sequences are unique up to isomorphism.

Note that this is not true in the category of finitely generated A-modules.

In fact, this is because $\mathbf{Mod}A$ is an ***abelian category****.*

Definition 2.8 (Splitting)**.** *A short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is said to* ***split*** *in $\mathbf{Mod}A$ if one of the following equivalent conditions is satisfied:*

(i) *There is an isomorphism $\varphi : Y \xrightarrow{\cong} X \oplus Z$ s.t. the following diagram commutes:*

$$\begin{tikzcd}
& & Y \arrow[d, "\varphi"] \arrow[dr, "g"] & & \\
0 \arrow[r] & X \arrow[ur, "f"] \arrow[dr, "i_X"'] & & Z \arrow[r] & 0 \\
& & X \oplus Z \arrow[ur, "p_Z"'] & &
\end{tikzcd}$$

(ii) *There is a morphism $\varphi : Y \to X \oplus Z$ s.t. the above diagram commutes.*

(iii) *f is a **section**, i.e. there exists $h : Y \to X$ s.t. $hf = 1_X$.*

(iv) *g is a **retraction**, i.e. there exists $h : Z \to Y$ s.t. $gh = 1_Z$.*

Proposition 2.9. **Mod**A *has pullbacks and pushforwards.*

Remark 2.10. *Consider a pullback square:*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ Y_2 \arrow[r] & Y \end{tikzcd}$$

Then there is a short exact sequence $0 \to X \to Y_1 \oplus Y_2 \to Y$.

Theorem 2.11. *Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in* **Mod**A *(or any abelian category).*

(i) *Given a morphism $Z' \xrightarrow{w} Z$, there is a commutative diagram with exact rows:*

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "="'] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d, "w"'] & 0 \\ 0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \end{tikzcd}$$

(ii) *Given a morphism $X \xrightarrow{u} X'$, there is a commutative diagram with exact rows:*

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d, "="'] & 0 \\ 0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

Proof. It suffices to prove (i) because (ii) is dual.

Form the pullback $Y \xleftarrow{v} Y' \xrightarrow{g'} Z'$ of $Y \xrightarrow{g} Z \xleftarrow{w} Z'$. First show that g' is epic. Then consider $K = \operatorname{Ker}\left(Y' \xrightarrow{g'} Z'\right)$. There is a natural arrow $K \to X$; it suffices to show that this is an isomorphism. Construct an inverse by considering the cone $Y \xleftarrow{f} X \xrightarrow{0} Z'$ over Z. □

2.3 Exact functors

Definition 2.12 (Exact functors)**.** *Let $F : \mathbf{C}_1 \to \mathbf{C}_2$ be a covariant (resp. contravariant) k-linear functor.*

- *We say that F is **left exact** if for any exact sequence $0 \to X \to Y \to Z$ (resp. $Z \to Y \to X \to 0$), the sequence $0 \to FX \to FY \to FZ$ is exact.*
- *We say that F is **right exact** if for any exact sequence $X \to Y \to Z \to 0$ (resp. $0 \to Z \to Y \to X$), the sequence $FX \to FY \to FZ \to 0$ is exact.*
- *We say that F is **exact** if it is both left and right exact.*

Hence, a covariant functor is exact if and only if it preserves short exact sequences.

Example 2.13. *The forgetful functors* $\mathbf{Mod}A \to \mathbf{Mod}k$ *and* $\mathbf{Mod}k \to \mathbf{Mod}\mathbb{Z}$ *are exact.*

Theorem 2.14. *Let M be an A-B-bimodule. Then*

(i) $\mathrm{Hom}_A(M,-) : \mathbf{Mod}A \to \mathbf{Mod}B$ *is left exact,*

(ii) $\mathrm{Hom}_A(-,M) : (\mathbf{Mod}A)^{\mathrm{op}} \to \mathbf{Mod}B^{\mathrm{op}}$ *is left exact,*

(iii) $M \otimes_B (-) : \mathbf{Mod}B \to \mathbf{Mod}A$ *is right exact,*

(iv) $(-) \otimes_A M : \mathbf{Mod}A^{\mathrm{op}} \to \mathbf{Mod}B^{\mathrm{op}}$ *is right exact.*

The key point is that none of the above functors are exact: their default of exactness is the starting point of cohomology.

Proof. (i) We prove a stronger statement: $0 \to X \to Y \to Z$ is exact if and only if for all A-B-bimodules M, the sequence

$$0 \to \mathrm{Hom}(M,X) \to \mathrm{Hom}(M,Y) \to \mathrm{Hom}(M,Z)$$

is exact. The statement ($\Rightarrow$) is easily proved; for the converse ($\Leftarrow$), take $M = X$, then $M = \mathrm{Ker}\, f$ and $M = \mathrm{Ker}\, g$.

(ii) is similar to (i).

(iii) Use the stronger statement proved in (i) together with the Adjunction Formula (Theorem 1.23).

(iv) is similar to (iii). □

Definition 2.15 (Projective, injective and flat modules)**.** *Let* $P, I, M \in \mathbf{Mod}A$.

- *P is **projective** if* $\mathrm{Hom}_A(P,-) : \mathbf{Mod}A \to \mathbf{Mod}k$ *is exact.*
  Equivalently, for any epimorphism $Y \twoheadrightarrow Z$, any map $P \to Z$ factors through Y.
- *I is **injective** if* $\mathrm{Hom}_A(-,I) : (\mathbf{Mod}A)^{\mathrm{op}} \to \mathbf{Mod}k$ *is exact.*
  Equivalently, for any monomorphism $X \rightarrowtail Y$, any map $X \to I$ factors through Y.
- *M is **flat** if* $(-) \otimes_A M : \mathbf{Mod}A^{\mathrm{op}} \to \mathbf{Mod}k$ *is exact.*

Example 2.16. *A is projective (in $\mathbf{Mod}A$) because $\mathrm{Hom}_A(A,-)$ is isomorphic to the forgetful functor $\mathbf{Mod}A \to \mathbf{Mod}k$, and it is flat because $(-) \otimes_A A$ is the forgetful functor $\mathbf{Mod}A^{\mathrm{op}} \to \mathbf{Mod}k$.*
However, A is not injective in general.

Lemma 2.17. *Let* $(M_i)_{i \in I} \in \mathbf{Mod}A$.

(i) $\bigoplus_i M_i$ *is projective* iff M_i *is projective for all* i.

(ii) $\prod_i M_i$ *is injective* iff M_i *is injective for all* i.

(iii) $\bigoplus_i M_i$ *is flat* iff M_i *is projective for all* i.

This is one of the reasons why, for instance, projective modules are preferred to free modules: because projectivity is stable under taking direct summands.

Theorem 2.18. *An A-module P is projective* iff *it is a direct summand of a free module.*

Proof. ($\Leftarrow$) This is true by Lemma 2.17, since free modules are projective because A is. ($\Rightarrow$) If P is a projective A-module, note that there exists an epimorphism $F \twoheadrightarrow P$, where F is a free A-module. Using the projectivity of P, show that this epimorphism is a retraction (i.e. it has a right inverse $P \rightarrowtail F$), and conclude that P is a direct summand of F. □

Corollary 2.19. *Free implies projective and projective implies flat.*

Example 2.20. *$\mathbb{Q}$ is flat but not projective over $\mathbb{Z}$.*

2.4 Existence of injective modules

Proposition 2.21. *If k is a field, then k is injective in* $\mathbf{Mod}k$.

Proof. This is due to the fact that if W is a subspace of a vector space V, then W is in fact a direct summand of V. $\square$

Notation 2.22. *For $M \in \mathbf{Mod}A$, we set $M^* = \mathrm{Hom}_k(M, k) \in \mathbf{Mod}A^{\mathrm{op}}$.*

Theorem 2.23. *Assume that k is a field. If $M \in \mathbf{Mod}A$ is projective, then $M^* \in \mathbf{Mod}A^{\mathrm{op}}$ is injective.*

Proof. Use the fact that M embeds into M^{**} to apply the projectivity of M. $\square$

Corollary 2.24. *If k is a field, then A^* is injective in $\mathbf{Mod}A$.*

Remark 2.25. *$\mathbb{Z}$ is not injective in $\mathbf{Mod}\mathbb{Z}$: there is no factorisation of $\mathbb{Z} \xrightarrow{\mathrm{id}_\mathbb{Z}} \mathbb{Z}$ through $\mathbb{Z} \rightarrowtail \mathbb{Q}$.*

Theorem 2.26 (Baer's Criterion)**.** *Let $I \in \mathbf{Mod}A$. Then I is injective* iff *for any monomorphism $X \rightarrowtail A$, any map $X \to I$ factors through A.*

Proof. See [1]. $\square$

Corollary 2.27. *$\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective in $\mathbf{Mod}\mathbb{Z}$.*

Proof. We use Baer's Criterion: consider a monomorphism $X \rightarrowtail \mathbb{Z}$ and a map $X \xrightarrow{f} I$. We may assume that $X = a\mathbb{Z}$ and $X \rightarrowtail \mathbb{Z}$ is the inclusion. Either $a = 0$, in which case $f = 0$. Or $a \neq 0$ and we can set $\widetilde{f} : \mathbb{Z} \to \mathbb{Q}$ by $n \mapsto \frac{n}{a} f(a)$. This $\widetilde{f}$ is then a factorisation of f through $\mathbb{Z}$. $\square$

Definition 2.28 (Pontryagin dual)**.** *Given $M \in \mathbf{Mod}A$, we define its **Pontryagin dual** by*

$$M^\wedge = \mathrm{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) \in \mathbf{Mod}\left(A^{\mathrm{op}}\right).$$

Lemma 2.29. *If $M \neq 0$, then $M^\wedge \neq 0$.*

Proof. Let $x \in M \smallsetminus \{0\}$. Define a morphism $(x) \to \mathbb{Q}/\mathbb{Z}$ by $x \mapsto \frac{1}{2}$ if x has infinite order, or $x \mapsto \frac{1}{n}$ if x has order n. Now we have a morphism $(x) \to \mathbb{Q}/\mathbb{Z}$, as well as a monomorphism $(x) \rightarrowtail M$. By injectivity of $\mathbb{Q}/\mathbb{Z}$, there is a factorisation $M \xrightarrow{\varphi} \mathbb{Q}/\mathbb{Z}$ of $(x) \to \mathbb{Q}/\mathbb{Z}$ through M. Thus, $\varphi(x) \neq 0$, so $\varphi \in M^\wedge \smallsetminus \{0\}$. $\square$

Proposition 2.30. *A morphism $X \to Y$ is injective* iff *$Y^\wedge \to X^\wedge$ is surjective.*

Proof. ($\Rightarrow$) This is due to the injectivity of $\mathbb{Q}/\mathbb{Z}$. ($\Leftarrow$) Consider the exact sequence

$$0 \to \mathrm{Ker}\, f \to X \xrightarrow{f} Y.$$

Since $\mathbb{Q}/\mathbb{Z}$ is injective, it induces an exact sequence

$$Y^\wedge \to X^\wedge \to (\mathrm{Ker}\, f)^\wedge \to 0.$$

But if $Y^\wedge \to X^\wedge$ is surjective, then we must have $(\mathrm{Ker}\, f)^\wedge = 0$, and therefore $\mathrm{Ker}\, f = 0$ by Lemma 2.29, so f is injective. $\square$

Theorem 2.31. *Let $M \in \mathbf{Mod}A^{\mathrm{op}}$. Then M is flat* iff *$M^\wedge$ is injective.*

Proof. Use the Adjunction Formula (Theorem 1.23) together with Proposition 2.30. $\square$

Corollary 2.32. *The A-module $A^\wedge$ is injective.*

3 Chain complexes

3.1 Graded modules

Definition 3.1 (Graded module)**.** *A* ***graded*** A***-module*** *is an* A*-module* M *together with a direct sum decomposition* $M = \bigoplus_{n\in\mathbb{Z}} M_n$*. A morphism between two graded* A*-modules* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *and* $N = \bigoplus_{n\in\mathbb{Z}} N_n$ *is a morphism of* A*-modules* $f : M \to N$ *s.t.* $f(M_n) \subseteq N_n$ *for all* $n \in \mathbb{Z}$*. We denote by* $\mathrm{Hom}_{\mathrm{gr}\,A}(M, N)$ *the set of such homomorphisms. This defines the category* $\mathbf{GMod}A$ *of graded* A*-modules.*

Remark 3.2. *Any* A*-module* M *can be seen as a graded* A*-module with* $M_0 = M$ *and* $M_n = 0$ *for* $n \neq 0$*. This defines a fully faithful functor* $\iota : \mathbf{Mod}A \to \mathbf{GMod}A$*. The graded module* $\iota(M)$ *is called* ***concentrated in degree zero****.*

Definition 3.3 (Homogeneous elements and degree)**.** *If* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *is a graded* A*-module, a nonzero element* $m \in M_n$ *will be called* ***homogeneous of degree*** n*. The* ***degree*** *of a homogeneous element will be denoted by* $\deg m$ *or* $|m|$*.*

Definition 3.4 (Graded submodule)**.** *A* ***graded submodule*** *of* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *is a sub-*A*-module* N *of* M *such that*

$$N = \sum_{n\in\mathbb{Z}} (N \cap M_n).$$

In particular, $N = \bigoplus_{n\in\mathbb{Z}} (N \cap M_n)$ *is also graded, and the inclusion* $N \hookrightarrow M$ *is a morphism of graded modules.*

In this case, the quotient M/N *has a structure of graded module given by* $M/N = \bigoplus_{n\in\mathbb{Z}} M_n/N$*, and the projection* $M \twoheadrightarrow M/N$ *is a morphism of graded modules.*

Example 3.5. *If* $f : M \to N$ *is a morphism of graded modules, then* $\mathrm{Ker}\, f$ *is a graded submodule of* M *and* $\mathrm{Im}\, f$ *is a graded submodule of* N*.*

Notation 3.6. *Given a graded* A*-module* $M = \bigoplus_{n\in\mathbb{Z}} M_n$*, its* ***shift*** $M[k]$ *is the graded* A*-module whose underlying* A*-module is* M*, and with* $M[k]_n = M_{k+n}$*. We denote by* $s_{M,k} : M \to M[k]$ *(or simply* s_M *when* $k = 1$*) the morphism of* A*-modules whose underlying map is* id_M*. Note that* $s_{M,k}$ *is not a morphism of graded* A*-modules.*

The shift induces a functor $(-)[k] : \mathbf{GMod}A \to \mathbf{GMod}A$*.*

Definition 3.7 (Internal Hom)**.** *Given graded* A*-modules* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *and* $N = \bigoplus_{n\in\mathbb{Z}} N_n$*, we denote by* $\mathcal{H}\mathrm{om}_{\mathrm{gr}\,A}(M, N)$ *the graded* $\mathbb{Z}$*-module given by*

$$\mathcal{H}\mathrm{om}_{\mathrm{gr}\,A}(M, N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{gr}\,A}(M, N[n]).$$

An element of $\mathrm{Hom}_{\mathrm{gr}\,A}(M, N[n])$ *will be called a* ***homogeneous morphism of degree*** n*.*

Definition 3.8 (Product and coproduct)**.** *Given a family* $(M^i)_{i\in I}$ *of graded modules, their* ***product*** $\prod_{i\in I} M^i$ *(resp.* ***coproduct*** $\coprod_{i\in I} M^i$*) is the graded module* $P = \bigoplus_{n\in\mathbb{Z}} P_n$ *(resp.* $C = \bigoplus_{n\in\mathbb{Z}} C_n$*) defined by* $P_n = \prod_{i\in I} M^i_n$ *(resp.* $P_n = \coprod_{i\in I} M^i_n$*).*

Note that $P \ncong \prod_{i\in I} M^i$ *as* A*-modules in general (because products and coproducts do not commute). However,* $C \cong \coprod_{i\in I} M^i$ *as* A*-modules.*

3.2 Complexes of modules

Definition 3.9 (Complex of modules)**.** *A* ***(homological) complex of*** A***-modules****, also called a* ***(homological) differential graded*** A***-module*** *is a graded* A*-module* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *together with a morphism of graded* A*-modules* $d_M : M \to M[-1]$*, called the* ***differential****, and satisfying* $d_M \circ d_M = 0$*.*

In other words, a complex of A-modules if a sequence

$$\cdots \xrightarrow{d_{n+2}} M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots,$$

s.t. $d_n \circ d_{n+1} = 0$ *for all* n.

A ***morphism*** *between two complexes of A-modules* (M, d_M) *and* (N, d_N) *is a morphism of graded A-modules* $f : M \to N$ *s.t.* $d_N \circ f = f \circ d_M$*. We denote by* $\mathrm{Hom}_{\mathrm{dg}\,A}(M, N)$ *the set of such morphisms. This defines the category* $\mathbf{DGMod}A$ *of complexes of A-modules.*

Definition 3.10 (Subcomplex)**.** *A* ***subcomplex*** *of a complex* (M, d_M) *is a graded submodule* N *of* M *s.t.* $d_M(N) \subseteq N$*. In this case, both* N *and* M/N *can be endowed with the structure of a complex of modules.*

Example 3.11. *Let* (M, d_M) *and* (N, d_N) *be two complexes. Then the graded module* $\mathcal{H}\mathrm{om}_{\mathrm{gr}\,A}(M, N)$ *can be endowed with a structure of complex given by*

$$d(f) = d_{N[k]} \circ f - f \circ d_M,$$

for $f \in \mathcal{H}\mathrm{om}_{\mathrm{gr}\,A}(M, N)_k = \mathrm{Hom}_{\mathrm{gr}\,A}(M, N[k])$.

Note that a morphism of complexes $(M, d_M) \to (N, d_N)$ *is precisely an element of* $\mathrm{Ker}\, d \subseteq \mathcal{H}\mathrm{om}_{\mathrm{gr}\,A}(M, N)$.

Definition 3.12 (Homology groups)**.** *Let* (M, d_M) *be a complex of modules. Then its* n***-th homology group*** *is the module*

$$H_n(M, d_M) = \mathrm{Ker}\, d_n / \mathrm{Im}\, d_{n+1}.$$

The graded module $H(M, d_M) = \bigoplus_{n \in \mathbb{Z}} H_n(M, d_M)$ *is called the* ***total homology group****.*

Proposition 3.13. *Given a morphism* $f : (M, d_M) \to (N, d_N)$ *of complexes, there is an induced morphism of graded modules* $H(f) : H(M, d_M) \to H(N, d_N)$*. This defines a functor*

$$H : \mathbf{DGMod}A \to \mathbf{GMod}A.$$

Definition 3.14 (Quasi-isomorphisms, exact complexes, etc.)**.**

- *A morphism of complexes* $f : (M, d_M) \to (N, d_N)$ *is called a* ***quasi-isomorphism*** *if* $H(f) : H(M, d_M) \to H(N, d_N)$ *is an isomorphism.*
- *A complex* (M, d_M) *is said to be* ***exact*** *(or* ***acyclic****) if* $H(M, d_M) = 0$*, i.e. the zero morphism* $(0, 0) \to (M, d_M)$ *is a quasi-isomorphism.*
- *A complex* (M, d_M) *is said to be* ***split*** *if there is a homogeneous element* $h \in \mathcal{H}\mathrm{om}_{\mathrm{gr}\,A}(M, M)_1$ *s.t.* $d_M = d_M \circ h \circ d_M$*. In other words, there is a family* $(h_n : M_n \to M_{n+1})$ *of morphisms of modules s.t.* $d_{n+1} \circ h_n \circ d_{n+1} = d_{n+1}$.
- *A complex is called* ***split exact*** *if it is split and exact.*

Remark 3.15. *Given a complex* (M, d_M) *and* $k \in \mathbb{Z}$*, its shift* $M[k]$ *can be equipped with the structure of a complex given by*

$$d_{M[k]} \circ s_{M,k}(m) = (-1)^k s_{M[-1],k}(d_M(m)).$$

This defines a functor $(-)[k] : \mathbf{DGMod}A \to \mathbf{DGMod}A$.

3.3 Snake Lemma and long exact sequences

Lemma 3.16 (Snake Lemma). *Suppose given a black commutative diagram of modules with exact rows:*

$$\begin{tikzcd}
 & \operatorname{Ker} f' \arrow[r] \arrow[d] & \operatorname{Ker} f \arrow[r] \arrow[d] & \operatorname{Ker} f'' \arrow[d] \arrow[dddll, "\partial"] & \\
 & M' \arrow[r, "i"] \arrow[d, "f'"'] & M \arrow[r, "p"] \arrow[d, "f"'] & M'' \arrow[r] \arrow[d, "f''"'] & 0 \\
0 \arrow[r] & N' \arrow[r, "j"] \arrow[d] & N \arrow[r, "q"] \arrow[d] & N'' \arrow[d] & \\
 & \operatorname{Coker} f' \arrow[r] & \operatorname{Coker} f \arrow[r] & \operatorname{Coker} f'' &
\end{tikzcd}$$

Then there exist red morphisms making the red sequence exact. In addition, we have the following facts:

(i) *The morphism ∂ is defined as follows: given $m'' \in \operatorname{Ker} f''$, there exists $m \in M$ s.t. $m'' = p(m)$; then $f(m) \in \operatorname{Ker} q = \operatorname{Im} j$, so there exists $n' \in N'$ s.t. $f(m) = j(n')$; we set $\partial(m'') = [n'] \in \operatorname{Coker} f'$.*

(ii) *If i is injective, then so is $\operatorname{Ker} f' \to \operatorname{Ker} f$.*

(iii) *If q is surjective, then so is $\operatorname{Coker} f \to \operatorname{Coker} f''$.*

(iv) *The morphism ∂ is natural, i.e. given two black commutative diagrams as above fitting into one three-dimensional commutative diagram, we get a commutative diagram of red exact sequences.*

Theorem 3.17. *Consider a short exact sequence of complexes of modules:*

$$0 \to (M', d') \xrightarrow{f} (M, d) \xrightarrow{g} (M'', d'') \to 0.$$

*Then there are morphisms $\partial_n : H_n(M'', d'') \to H_{n-1}(M', d')$ of modules (called the **connecting morphisms**), s.t. the sequence*

$$\cdots \to H_n(M', d') \xrightarrow{H_n(f)} H_n(M, d) \xrightarrow{H_n(g)} H_n(M'', d'') \xrightarrow{\partial_n} H_{n-1}(M', d') \to \cdots$$

is exact.

Moreover, the connecting morphisms are natural.

Proof. Apply the Snake Lemma to the following diagram:

$$\begin{tikzcd}
 & \operatorname{Coker} d'_{n+1} \arrow[r, "\overline{f}_n"] \arrow[d, "\overline{d}'_n"'] & \operatorname{Coker} d_{n+1} \arrow[r, "\overline{g}_n"] \arrow[d, "\overline{d}_n"'] & \operatorname{Coker} d''_{n+1} \arrow[r] \arrow[d, "\overline{d}''_n"'] & 0 \\
0 \arrow[r] & \operatorname{Ker} d'_{n-1} \arrow[r, "f_{n-1}"] & \ker d_{n-1} \arrow[r, "g_{n-1}"] & \operatorname{Ker} d''_{n-1} &
\end{tikzcd}$$

$\square$

Definition 3.18 (Homotopy). *Let $f, g : (M, d) \rightrightarrows (N, d')$ be two morphisms of complexes. A **homotopy** from f to g is a homogeneous element $h \in \mathcal{H}\mathrm{om}_{\mathrm{gr}\,A}(M, N)_1$ s.t. $d(h) = f - g$, or in other words, a family $(h_n : M_n \to N_{n+1})_{n \in \mathbb{Z}}$ of morphisms of modules s.t.*

$$d'_{n+1} \circ h_n + h_{n-1} \circ d_n = f_n - g_n.$$

We then say that f and g are ***homotopic****, and we write $f \sim g$. This defines an equivalence relation on* $\operatorname{Hom}_{\mathrm{dg}\,A}(M, N)$.

A ***homotopy inverse*** *of a morphism $f : (M, d) \to (N, d')$ is a morphism $g : (N, d') \to (M, d)$ satisfying $g \circ f \sim \mathrm{id}_M$ and $f \circ g \sim \mathrm{id}_N$. We then say that f and g are* ***homotopy equivalences****.*

Example 3.19. *A complex (M, d_M) is split exact* iff $\mathrm{id}_M \sim 0$.

Lemma 3.20. (i) *If two morphisms $f, g : (M, d) \rightrightarrows (N, d')$ are homotopic, then $H(f) = H(g)$.*

(ii) *If two complexes (M, d) and (N, d') are homotopy equivalent, then $H(M, d) \cong H(N, d')$.*

4 Derived functors

4.1 Projective and injective resolutions

Proposition 4.1. *Given an A-module M and a short exact sequence of A-modules*

$$0 \to N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0,$$

the following sequences of $\mathbb{Z}$-modules are exact:

(i) $0 \to \operatorname{Hom}_A(M, N') \xrightarrow{f_*} \operatorname{Hom}_A(M, N) \xrightarrow{g_*} \operatorname{Hom}_A(M, N'')$,

(ii) $0 \to \operatorname{Hom}_A(N'', M) \xrightarrow{g^*} \operatorname{Hom}_A(N, M) \xrightarrow{f^*} \operatorname{Hom}_A(N', M)$.

Definition 4.2 (Projective and injective modules)**.** *An A-module M is called* ***projective*** *(resp.* ***injective****) if the functor* $\operatorname{Hom}_A(M, \cdot)$ *(resp.* $\operatorname{Hom}_A(\cdot, M)$*) is right exact.*

Theorem 4.3. *An A-module M is projective* iff *it is a free summand of a free A-module.*

Proof. See Theorem 2.18. □

Lemma 4.4 (Enough projectives and injectives)**.** *Let M be an A-module.*

(i) *There exists a projective (and in fact, free) A-module F together with an epimorphism $F \twoheadrightarrow M$.*

(ii) *There exists an injective A-module I together with a monomorphism $M \rightarrowtail I$.*

Proof. (i) Consider the free module $F(M)$ on M together with the morphism $F(M) \to M$ given by $m \mapsto m$. (ii) Denote by $M^\wedge$ the Pontryagin dual of M (c.f. Definition 2.28) and consider $I(M) = \operatorname{Hom}_{\mathbb{Z}}\left(A, \prod_{f \in M^\wedge} \mathbb{Q}/\mathbb{Z}\right)$ together with the morphism $i_M : M \to I(M)$ given by $i_M(m)(a) = (f(am))_{f \in M^\wedge}$. □

Definition 4.5 (Resolution)**.** *Let M be an A-module. A* ***left resolution*** *(resp.* ***right resolution****) of M is a complex (P, d_P) (resp. (I, d_I)) of A-modules s.t.*

- *$P_n = 0$ for $n < 0$ (resp. $I_n = 0$ for $n > 0$),*
- *$H_n(P, d_P) = 0$ for $n > 0$ (resp. $H_n(I, d_I) = 0$ for $n < 0$),*
- *There is a surjective morphism $\varepsilon : P_0 \to M$, called the* ***augmentation****, and whose kernel is the image of $d_1 : P_1 \to P_0$ (resp. an injective morphism $\eta : M \to I_0$, called the* ***coaugmentation****, and whose image is the kernel of $d_0 : I_0 \to I_{-1}$).*

In other words, a left resolution is an exact complex

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0,$$

which shall be denoted by $\varepsilon : P_\bullet \to M$, and a right resolution is an exact complex

$$0 \to M \xrightarrow{\eta} I_0 \xrightarrow{d_0} I_{-1} \xrightarrow{d_{-1}} I_{-2} \xrightarrow{d_{-2}} \cdots ,$$

which shall be denoted by $\eta : M \to I_\bullet$.

We say that a resolution (P, d_P) is ***projective*** *(resp.* ***injective****) if P_n is projective (resp.* ***injective****) for all $n \in \mathbb{Z}$.*

Lemma 4.6 (Existence of projective and injective resolutions)**.** (i) *Given an A-module M, there is a projective resolution $\varepsilon_M : P(M) \to M$ and an injective resolution $\eta_M : M \to E(M)$.*

(ii) *If $f : M \to N$ is a morphism of A-modules, then there are morphisms of complexes $P(f) : P(M) \to P(N)$ and $E(f) : E(M) \to E(N)$ s.t. $\varepsilon_N \circ P(f)_0 = f \circ \varepsilon_M$ and $\eta_N \circ f = E(f)_0 \circ \eta_M$.*

This defines functors $P : \mathbf{Mod}A \to \mathbf{GMod}A$ and $E : \mathbf{Mod}A \to \mathbf{GMod}A$.

Proof. Apply Lemma 4.4 inductively for (i). Use the explicit forms of the given resolutions for (ii). $\square$

Lemma 4.7 (Comparison Lemma)**.** *Let $\varepsilon_P : P_\bullet \to M$ be a projective (not necessarily exact) complex with $P_n = 0$ for $n < 0$, and let $\varepsilon_Q : Q_\bullet \to N$ be a left (not necessarily projective) resolution of N.*

Given a morphism of A-modules $f : M \to N$, there is a morphism of complexes $F : (P, d_P) \to (Q, d_Q)$ s.t. $f \circ \varepsilon_P = \varepsilon_Q \circ F_0$. Moreover, F is unique up to homotopy.

In other words, we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "d_3"] & P_2 \arrow[r, "d_2"] \arrow[d, "F_2"'] & P_1 \arrow[r, "d_1"] \arrow[d, "F_1"'] & P_0 \arrow[r, "\varepsilon_P"] \arrow[d, "F_0"'] & M \arrow[d, "f"'] & \\
\cdots \arrow[r, "d_3"] & Q_2 \arrow[r, "d_2"] & Q_1 \arrow[r, "d_1"] & Q_0 \arrow[r, "\varepsilon_Q"] & N \arrow[r] & 0
\end{tikzcd}$$

Moreover, the dual result holds for injectives.

Proof. Construct the morphisms F_n inductively, using the projectivity of the P_n. $\square$

Lemma 4.8 (Horseshoe Lemma)**.** *Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence of A-modules. Let $\varepsilon' : P'_\bullet \to M'$ and $\varepsilon'' : P''_\bullet \to M''$ be projective resolutions. Then there is a projective resolution $\varepsilon : P_\bullet \to M$ s.t. $P_n = P'_n \oplus P''_n$ for all $n \in \mathbb{Z}$, and the following diagram commutes (and has exact rows and columns):*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & P'_2 \arrow[r] \arrow[d, "\binom{1}{0}"'] & P'_1 \arrow[r] \arrow[d, "\binom{1}{0}"'] & P'_0 \arrow[r, "\varepsilon'"] \arrow[d, "\binom{1}{0}"'] & M' \arrow[r] \arrow[d, "f"'] & 0 \\
\cdots \arrow[r] & P'_2 \oplus P''_2 \arrow[r] \arrow[d, "(0\ 1)"'] & P'_1 \oplus P''_1 \arrow[r] \arrow[d, "(0\ 1)"'] & P'_0 \oplus P''_0 \arrow[r, "\varepsilon"] \arrow[d, "(0\ 1)"'] & M \arrow[r] \arrow[d, "g"'] & 0 \\
\cdots \arrow[r] & P''_2 \arrow[r] \arrow[d] & P''_1 \arrow[r] \arrow[d] & P''_0 \arrow[r, "\varepsilon''"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Moreover, this construction is natural.

The dual result holds for injectives.

4.2 Universal δ-functors

Notation 4.9. *From now on, we consider two abelian categories $\mathbf{A}$ and $\mathbf{B}$.*

Definition 4.10 (Homological δ-functor)**.** *A **homological** (resp. **cohomological**) δ**-functor** $\mathbf{A} \to \mathbf{B}$ is a family of additive functors $(T_n : \mathbf{A} \to \mathbf{B})_{n\in\mathbb{Z}}$ (resp. $(T^n : \mathbf{A} \to \mathbf{B})_{n\in\mathbb{Z}}$) s.t. $T_n = 0$ (resp. $T^n = 0$) for all $n < 0$, together with morphisms $(\delta_n : T_n(M'') \to T_{n-1}(M'))_{n\in\mathbb{Z}}$ (resp. $(\delta^n : T^n(M'') \to T^{n+1}(M'))_{n\in\mathbb{Z}}$) for any short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathbf{A}$.*

We assume that, given a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "\alpha"'] & M \arrow[r, "g"] \arrow[d, "\beta"'] & M'' \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & N' \arrow[r, "h"] & N \arrow[r, "k"] & N'' \arrow[r] & 0
\end{tikzcd}$$

with exact rows in $\mathbf{A}$, the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & T_{n+1}(M'') \arrow[r, "\delta_{n+1}"] \arrow[d, "F_2"'] & T_n(M') \arrow[r, "T_n(f)"] \arrow[d, "F_0"] \arrow[l, phantom, "F_1"] & T_n(M) \arrow[r, "T_n(g)"] \arrow[d, "f"'] & T_n(M'') \arrow[r, "\delta_n"] \arrow[d] & \cdots \\
\cdots \arrow[r] & T_{n+1}(N'') \arrow[r, "\delta_{n+1}"] & T_n(N') \arrow[r, "T_n(h)"] & T_n(N) \arrow[r, "T_n(k)"] & T_n(N'') \arrow[r, "\delta_n"] & \cdots
\end{tikzcd}$$

is commutative with exact rows (and similarly in the cohomological case).

In particular, this implies that T_0 (resp. T^0) is right (resp. left) exact.

Example 4.11. *Let A be a commutative ring and $a \in A$. Let $T_0, T_1 : \mathbf{Mod}A \to \mathbf{Mod}A$ be the functors given by $T_0(M) = M/aM$ and $T_1(M) = \{m \in M,\ am = 0\}$, and let $T_n = 0$ for $n \notin \{0,1\}$. Then $(T_n : \mathbf{Mod}A \to \mathbf{Mod}A)_{n\in\mathbb{Z}}$ is a homological δ-functor.*

Definition 4.12 (Morphism of δ-functors)**.** *Let $T_n, U_n : \mathbf{A} \to \mathbf{B}$ be two homological δ-functors with connecting morphisms $\delta_{T,n}, \delta_{U,n}$. A **morphism** of homological δ-functors is a family of natural transformations $(t_n : T_n \to U_n)_{n\in\mathbb{Z}}$ s.t. given any short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathbf{A}$, the following diagram commutes:*

$$\begin{tikzcd}
T_n(M'') \arrow[r, "\delta_{T,n}"] \arrow[d, "t_n(M'')"'] & T_{n-1}(M') \arrow[d, "t_n(M')"] \\
U_n(M'') \arrow[r, "\delta_{U,n}"] & U_{n-1}(M')
\end{tikzcd}$$

The definition is similar for cohomological δ-functors.

Definition 4.13 (Universal δ-functor)**.** *A homological (resp. cohomological) δ-functor $(T_n)_{n\in\mathbb{Z}}$ (resp. $(T^n)_{n\in\mathbb{Z}}$) is called **universal** if given any homological (resp. cohomological) δ-functor $(U_n)_{n\in\mathbb{Z}}$ (resp. $(U^n)_{n\in\mathbb{Z}}$) and natural transformation $t_0 : U_0 \to T_0$ (resp. $t^0 : T^0 \to U^0$), there exists a unique morphism of homological (resp. cohomological) δ-functor $U_n \to T_n$ (resp. $T^n \to U^n$) extending t_0 (resp. t^0).*

Example 4.14. *The homological δ-functor of Example 4.11 is universal.*

Example 4.15. *Let $\mathbf{A}$ be the subcategory of $\mathbf{DGMod}A$ given by the complexes (M, d_M) of A-modules with $M_n = 0$ for $n < 0$. Then $(H_n : \mathbf{A} \to \mathbf{Mod}A)_{n\in\mathbb{Z}}$ is a universal δ-functor.*

Definition 4.16 (Erasable functor)**.** *An additive functor $F : \mathbf{A} \to \mathbf{B}$ of abelian categories is said to be **erasable** (resp. **coerasable**) if for every object $M \in \mathbf{A}$, there exists an epimorphism $\varepsilon : P \twoheadrightarrow M$ (resp. a monomorphism $\eta : M \rightarrowtail I$) s.t. $F(\varepsilon) = 0$ (resp. $F(\eta) = 0$).*

Proposition 4.17. *Let $T_n : \mathbf{A} \to \mathbf{B}$ (resp. $T^n : \mathbf{A} \to \mathbf{B}$) be a homological (resp. cohomological) δ-functor s.t. T_n is erasable (resp. T^n is coerasable) for all $n \geqslant 1$. Then $(T_n)_{n\in\mathbb{Z}}$ (resp. $(T^n)_{n\in\mathbb{Z}}$) is universal.*

Proof. We prove the proposition in the homological case. Let $(U_n)_{n\in\mathbb{Z}}$ be a homological δ-functor together with a natural transformation $t_0 : U_0 \to T_0$. Assume we have constructed natural transformations $t_i : U_i \to T_i$ for $0 \leqslant i < n$ for some n, satisfying the commutativity condition in the definition of morphisms of δ-functors. Given $N \in \mathbf{A}$, since T_n is erasable, there exists an epimorphism $g : P \twoheadrightarrow N$ s.t. $T_n(g) = 0$. Consider the short exact sequence $0 \to K \to P \to N \to 0$ in $\mathbf{A}$. We then have the commutative diagram

$$\begin{tikzcd}
 & U_n(N) \arrow[r, "\delta_{U,n}"] & U_{n-1}(K) \arrow[r, "U_{n-1}(i)"] \arrow[d, "t_{n-1}(K)"] & U_{n-1}(P) \arrow[d, "t_{n-1}(P)"] \\
T_n(P) \arrow[r, "T_n(g)"] & T_n(N) \arrow[r, "\delta_{T,n}"] & T_{n-1}(K) \arrow[r, "T_{n-1}(i)"] & T_{n-1}(P)
\end{tikzcd}$$

with exact rows. Chasing the diagram, we construct a morphism $t_n(N) : U_n(N) \to t_n(N)$ in $\mathbf{B}$, and we check that is is independent of the choice of $g : P \to N$. This gives a natural transformation $t_n : U_n \to T_n$. Then do all the verifications to check that we obtain a morphism of δ-functors. $\square$

4.3 Derived functors

Definition 4.18 (Enough projectives and injectives)**.** *We say that an abelian category $\mathbf{A}$ has **enough projectives** (resp. **injectives**) if for all objects $M \in \mathbf{A}$, there is a projective object $P \in \mathbf{A}$ (resp. an injective object $I \in \mathbf{A}$) together with an epimorphism $P \twoheadrightarrow M$ (resp. a monomorphism $M \rightarrowtail I$).*

Notation 4.19. *When considering left (resp. right) derived functors, we assume that $\mathbf{A}$ has enough projectives (resp. injectives).*

Definition 4.20 (Derived functors)**.** *Let $F : \mathbf{A} \to \mathbf{B}$ be an additive right exact (resp. left exact) functor. Let $\varepsilon_P : (P, d_P) \to M$ (resp. $\eta_I : M \to (I, d_I)$) be a projective (resp. injective) resolution of M in $\mathbf{A}$. We define the **left** (resp. **right**) **derived functors** at M as*

$$L_nF(M) = H_n\left(F(P), F\left(d_P\right)\right)$$

(resp. $R^nF(M) = H^n\left(F(I), F\left(d_I\right)\right)$). This is well-defined because two resolutions of M are homotopy equivalent by the Comparison Lemma (Lemma 4.7).

Remark 4.21. *Let $\mathbf{A} \xrightarrow{F} \mathbf{B} \xrightarrow{G} \mathbf{C}$ be additive functors.*

(i) *If $M \in \mathbf{A}$ is projective (resp. injective), then $L_nF(M) = 0$ (resp. $R^nF(M) = 0$) for all $n \geqslant 1$.*

(ii) *If F is right (resp. left) exact, then, then $L_0F \cong F$ (resp. $R^0F \cong F$).*

(iii) *if F is exact, then $L_nF = 0$ (resp. $R^nF = 0$) for $n \geqslant 1$.*

(iv) *If F is right (resp. left) exact and G is exact, then $L_n(G \circ F) \cong G \circ L_nF$ (resp. $R^n(G \circ F) \cong G \circ R^nF$) for all $n \geqslant 0$.*

Definition 4.22 (Acyclic objects and resolutions)**.** *Given an additive right (resp. left) exact functor $F : \mathbf{A} \to \mathbf{B}$, we say that an object $M \in \mathbf{A}$ is F-**acyclic** if $L_nF(M) = 0$ (resp. $R^nF(M) = 0$) for all $n \geqslant 1$.*

A projective (resp. injective) resolution is F-acyclic if all its objects are F-acyclic.

Lemma 4.23. *Let $F : \mathbf{A} \to \mathbf{B}$ be an additive right (resp. left) exact functor. Given a morphism $f : M \to N$ in $\mathbf{A}$, there exist natural morphisms $L_nF(f) : L_nF(M) \to L_nF(N)$ (resp. $R^nF(f) : R^nF(M) \to R^nF(N)$) for all $n \geqslant 0$ that makes L_nF (resp. R^nF) an additive functor.*

Theorem 4.24. *Let $F : \mathbf{A} \to \mathbf{B}$ be an additive right (resp. left) exact functor. Then the derived functors $L_n F : \mathbf{A} \to \mathbf{B}$ (resp. $R^n F : \mathbf{A} \to \mathbf{B}$) for $n \geqslant 0$ form a universal homological (resp. cohomological) δ-functor.*

Proof. This follows from the application of Theorem 3.17 to the short exact sequence given by the Horseshoe Lemma (Lemma 4.8). □

4.4 Tor and Ext groups

Definition 4.25 (Tor and Ext)**.** (i) *Let $M \in \mathbf{Mod} A^{\mathrm{op}}$. The functor $M \otimes (-) : \mathbf{Mod} A \to \mathbf{Mod}\mathbb{Z}$ is right exact, and we define the* ***torsion groups***

$$\mathrm{Tor}_n^A (M, N) = L_n (M \otimes (-)) (N).$$

In fact, $\mathrm{Tor}_n^A (M, N) \cong L_n ((-) \otimes N) (M)$ for all $n \geqslant 0$, where $(-) \otimes N : \mathbf{Mod} A^{\mathrm{op}} \to \mathbf{Mod}\mathbb{Z}$ is right exact.

(ii) *Let $M \in \mathbf{Mod} A$. The functor $\mathrm{Hom}_A (M, -) : \mathbf{Mod} A \to \mathbf{Mod}\mathbb{Z}$ is left exact, and we define the* ***extension groups***

$$\mathrm{Ext}_A^n (M, N) = R^n \mathrm{Hom}_A (M, -) (N).$$

In fact, $\mathrm{Ext}_A^n (M, N) \cong R^n \mathrm{Hom} (-, N) (M)$ for all $n \geqslant 0$, where $\mathrm{Hom} (-, N) : (\mathbf{Mod} A)^{\mathrm{op}} \to \mathbf{Mod}\mathbb{Z}$ is left exact.

References

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[2] C. A. Weibel. *An Introduction to Homological Algebra.*