Reading seminar on L^2-Betti numbers

Organised by Alexis Marchand
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One can define the *L^2-cohomology* of a group by replacing finite sums with L^2-summable sums in the usual definition of group cohomology. The resulting objects are Hilbert modules; their von Neumann dimensions are called the *L^2-Betti numbers* of the group.

Since their introduction around the 1970s, L^2-Betti numbers have created considerable interest for their applications to geometric group theory and algebraic topology, and are still a very active subject of research.

The aim of this reading seminar is to gather people interested in L^2-Betti numbers from various points of view (including, but not limited to, geometric group theory, operator algebras, algebraic topology, and dynamics), and learn about the established theory of L^2-Betti numbers as well as very recent research in the subject.

Schedule. The plan is to have talks of approximately one hour, with extra time for questions and discussions.

Week 1	Wed 26 Feb	10:30-12:00	理学部3号館108号室
Week 2	Wed 5 Mar	10:30-12:00	理学部3号館108号室
Week 3	Fri 14 Mar	10:30-12:00	理学部3号館305号室
Week 4	No seminar		
Week 5	Wed 26 Mar	10:30-12:00	理学部3号館108号室
Week 6	Wed 2 Apr	10:30-12:00	理学部3号館108号室
Week 7	Wed 9 Apr	10:30-12:00	理学部3号館108号室
Week 8	Wed 16 Apr	10:30-12:00	理学部3号館108号室
Week 9 onwards	TBC		

Participation in the seminar. Everyone (Master's and PhD students, postdocs, professors) is welcome to attend any or all of the talks! If you would like to receive announcements, please email me at

marchand.alexis.85h@st.kyoto-u.ac.jp

I am planning to give the first two talks to get the seminar started, but I will then need your help in volunteering to give talks. If you would like to give a talk on one of the topics listed below, please email me with your choice of topic and preferred date.

Tentative plan

The first three weeks will cover the fundamental theory as follows:

Week 1. *Introduction, and background on von Neumann dimension*

Motivation from geometric group theory and algebraic topology. Introduction to Hilbert modules and von Neumann dimension.

References: [1, Chapter 1], [2], [6, Chapter 1], [5, §1.1]

Speaker: Alexis Marchand

Week 2. *L^2-homology and L^2-Betti numbers*

Background on group cohomology, $K(G,1)$-spaces, etc. Definition of L^2-cohomology and L^2-Betti numbers. First examples of computations, and basic applications.

References: [1, (parts of) Chapters 3-4], [6, Chapter 2], [5, §1.2].

Speaker: Alexis Marchand

Week 3. *Lück's Approximation Theorem*

Statement and proof of Lück's Approximation Theorem.

References: [1, §5.1 to §5.3], [6, Chapter 3]

Speaker: Ryoya Arimoto

Week 4. No seminar

Week 5. *Relation with acylindrical hyperbolicity (part 1)*

Background on Gromov-hyperbolicity and acylindrical hyperbolicity. Statement and proof of Osin's Theorem on the acylindrical hyperbolicity of groups with positive first L^2-Betti number.

References: [7]

Speaker: Bingxue Tao

Week 6. *Relation with acylindrical hyperbolicity (part 2)*

Background on Gromov-hyperbolicity and acylindrical hyperbolicity. Statement and proof of Osin's Theorem on the acylindrical hyperbolicity of groups with positive first L^2-Betti number.

References: [7]

Speaker: Bingxue Tao

Week 7. *The measure-theoretic viewpoint*

Definition of rank-gradient, cost, and relation with L^2-Betti numbers.

References: [6, Chapter 4], [1, §5.5]

Speaker: Hiroto Nishikawa

Week 8. *L^2-torsion and group automorphisms*

Discussion on the L^2-torsion of certain mapping tori of group automorphisms, including surface group automorphisms and free group automorphisms.

References: [5, Chapter §7.4]

Speaker: Matteo Wei

The rest of the reading seminar will be based on independent topics. Speakers are free to choose among the following suggestions (biased on the interest of a geometric group theorist) or propose other topics (many ideas can be found in Lück's book [5]):

Topic 1. *Vanishing of L^2-Betti numbers*

Investigation of groups with vanishing conditions on their L^2-Betti numbers and relation with Euler characteristic.

References: [5, §7.1 and §7.2]

Speaker TBC

Topic 2. *The Atiyah and Kaplansky Conjectures*

Discussion of the Atiyah Conjecture on the rationality of L^2-Betti numbers and of the Kaplansky Conjecture on divisors of zeros, units, etc. in group rings. Known results and relations between the conjectures.

References: [1, §3.5], [4] and references therein

Speaker TBC

Topic 3. *L^2-Betti numbers of 3-manifolds*

Background on 3-manifolds and computations of their L^2-Betti numbers.

References: [5, Chapter 4]

Speaker TBC

Topic 4. *L^2-Betti numbers of symmetric spaces*

Background on symmetric spaces and computations of their L^2-Betti numbers.

References: [5, Chapter 5]

Speaker TBC

Topic 5. *Torsion invariants*

Discussion of torsion invariants, relations with L^2-Betti numbers, conjectures, and known results.

References: [1, Chapter 6]

Speaker TBC

Topic 6. *L^2-Betti numbers and fiberings*

Discussion of Kielak's Theorem on virtual fibering from vanishing of the first L^2-Betti number.

References: [3]

Speaker TBC

Hopefully new topics will arise from the talks, and everyone is welcome to suggest a topic at any point!

References

[1] Holger Kammeyer, *Introduction to ℓ^2-invariants*, Lecture Notes in Mathematics, vol. 2247, Springer, Cham, 2019. MR3971279

[2] ———, *ℓ^2-Betti numbers*, Bounded cohomology and simplicial volume, 2023, pp. 57–62. MR4496345

[3] Dawid Kielak, *Residually finite rationally solvable groups and virtual fibring*, J. Amer. Math. Soc. **33** (2020), no. 2, 451–486. MR4073866

[4] Peter A. Linnell, *The Atiyah conjecture*, Geometry, topology, and dynamics in negative curvature, 2016, pp. 198–220. MR3497261

[5] Wolfgang Lück, *L^2-invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 44, Springer-Verlag, Berlin, 2002. MR1926649

[6] Clara Löh, *Ergodic theoretic methods in group homology—a minicourse on L^2-Betti numbers in group theory*, SpringerBriefs in Mathematics, Springer, Cham, [2020] ©2020. MR4177798

[7] D. Osin, *On acylindrical hyperbolicity of groups with positive first ℓ^2-Betti number*, Bull. Lond. Math. Soc. **47** (2015), no. 5, 725–730. MR3403956

Please let me know if you have trouble accessing any of the references; I should be able to email you a PDF.