L^2-Betti numbers

Reading seminar

Kyoto University

Spring 2025

Contents

Week 1 – Background on von Neumann dimension

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References: [1, Chapter 2], [2], [4, Chapter 1], [3, §1.1].

1.1 Motivation

The goal of what follows is to develop a good *equivariant* homology theory for actions $G \curvearrowright X$ of groups on topological spaces. The usual singular chain complex $C_*^{\mathrm{sing}}(X;\mathbb{C})$ and singular homology $H_*(X;\mathbb{C})$ inherit a G-action, so they have the structure of $\mathbb{C}G$-modules. However, the group G is typically infinite and we do not have a good notion of dimension for modules over $\mathbb{C}G$. This is why we will work in an L^2 setting.

We will introduce a homology theory $H_*^{(2)}(G \curvearrowright X)$, together with associated Betti numbers $b_*^{(2)}(G \curvearrowright X)$. They will be well-defined when X is a G-CW-complex under a certain finiteness property.

In the first talk, we introduce the relevant notions around Hilbert modules and von Neumann dimension that will allow us to define L^2-Betti numbers.

1.2 Hilbert G-modules

We fix a countable group G. We will work with $\mathbb{C}$-coefficients throughout.

Definition 1.1. The *group ring* of G over $\mathbb{C}$ is the $\mathbb{C}$-algebra $\mathbb{C}G$ (or $\mathbb{C}[G]$), with underlying $\mathbb{C}$-vector space

$$\mathbb{C}G \coloneqq \bigoplus_{g\in G} \mathbb{C}g,$$

with multiplication defined on the basis vectors by $g \cdot h = gh$.

Example 1.2.

- $\mathbb{C}[\mathbb{Z}] = \mathbb{C}\left[t, t^{-1}\right]$ is the ring of Laurent polynomials over $\mathbb{C}$.
- For $n \in \mathbb{N}_{\geq 1}$, $\mathbb{C}\left[\mathbb{Z}/n\right] = \mathbb{C}[t]/\left(t^n - 1\right)$.

The group ring $\mathbb{C}G$ can be equipped with a natural inner product $\langle \cdot, \cdot \rangle$ defined by

$$\left\langle \sum_{g\in G} a_g g, \sum_{g\in G} b_g g \right\rangle \coloneqq \sum_{g\in G} \bar{a}_g b_g$$

The completion of $\mathbb{C}G$ with respect to $\langle \cdot, \cdot \rangle$ is a complex Hilbert space, which we denote by $\ell^2 G$; it can also be defined as the $\mathbb{C}$-vector space of ℓ^2-summable functions $G \to \mathbb{C}$.

Note that $\ell^2 G$ has the structure of a $\mathbb{C}G$-module, with action given by

$$h \cdot \sum_{g\in G} a_g g \coloneqq \sum_{g\in G} a_{gh} g.$$

Example 1.3.

- If G is finite, then $\ell^2 G = \mathbb{C}G$.
- If $G = \mathbb{Z}$, by Fourier analysis, there is an isomorphism $\ell^2 G \cong L^2\left([-\pi, \pi], \mathbb{C}\right)$ given by

$$\sum_{n\in\mathbb{Z}} a_n t^n \longmapsto \left(x \mapsto \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} a_n e^{inx} \right).$$

Since the group G is assumed to be countable, the Hilbert space $\ell^2 G$ is separable.

Definition 1.4. A *Hilbert G-module* is a complex Hilbert space V with a $\mathbb{C}$-linear isometric (left) G-action such that there is an isometric G-embedding

$$V \hookrightarrow \left(\ell^2 G\right)^n$$

for some $n \in \mathbb{N}_{\geq 1}$.

A *morphism* between two Hilbert G-modules V and W is a G-equivariant bounded $\mathbb{C}$-linear map $V \to W$.

Our homology groups will be Hilbert G-modules; our main task will be to define a notion of dimension for such modules.

1.3 Background on von Neumann algebras

Let $\mathcal{H}$ be a complex Hilbert space. Then the space $B(\mathcal{H})$ of bounded linear operators $\mathcal{H} \to \mathcal{H}$ is a $\mathbb{C}$-algebra, with multiplication given by composition.

Recall that, given $u \in B(\mathcal{H})$, there is a unique $u^* \in B(\mathcal{H})$ — called the *adjoint* of f — such that, for all $x, y \in \mathcal{H}$,

$$\langle u(x), y \rangle = \langle x, u^*(y) \rangle .$$

(This follows from the Riesz Representation Theorem applied to the linear form $\langle u(\cdot), y \rangle$ for fixed $y \in \mathcal{H}$.) Hence, $\cdot^*$ defines an involution on $B(\mathcal{H})$; this turns the latter into a $*$-algebra.

There are several topologies that one can define on $B(\mathcal{H})$:

- The *norm topology*, given by

$$u_n \xrightarrow{\|\cdot\|} u \overset{\text{def}}{\iff} \|u_n - u\| \to 0,$$

- The *strong topology*, given by

$$u_n \xrightarrow{s} u \overset{\text{def}}{\iff} \forall x \in \mathcal{H},\ \|u_n(x) - u(x)\| \to 0,$$

- The *weak topology*, given by

$$u_n \xrightarrow{w} u \overset{\text{def}}{\iff} \forall x, y \in \mathcal{H},\ \langle u_n(x), y \rangle \to \langle u(x), y \rangle .$$

Definition 1.5. A *von Neumann algebra* is a unital weakly closed $*$-subalgebra of $B(\mathcal{H})$ for some complex Hilbert space $\mathcal{H}$.

Given a subset $S \subseteq B(\mathcal{H})$, its *commutant* is defined by

$$S' \coloneqq \{u \in B(\mathcal{H}) \mid \forall s \in S,\ us = su\} .$$

The *bicommutant* of S is simply $S'' \coloneqq (S')'$.

The following theorem is a fundamental structural result for von Neumann algebras:

Theorem 1.6 (von Neumann Bicommutant Theorem)**.** *Let $\mathcal{H}$ be a complex Hilbert space and let $A \subseteq B(\mathcal{H})$ be a unital $*$-subalgebra of $B(\mathcal{H})$. Then the following are equivalent:*

(i) $A'' = A$.

(ii) *A is strongly closed.*

(iii) *A is weakly closed.*

1.4 The group von Neumann algebra and its trace

We come back to the setup of §1.2: G is a countable group and we are considering the Hilbert space $\ell^2 G$. As above, we denote by $B\left(\ell^2 G\right)$ the $\mathbb{C}$-algebra of bounded linear operators $\ell^2 G \to \ell^2 G$.

Observe that there are two embeddings

$$\lambda, \rho : \mathbb{C}G \hookrightarrow B\left(\ell^2 G\right)$$

given by the respective actions of $\mathbb{C}G$ on $\ell^2 G$ by left and right multiplication.

Proposition/Definition 1.7. *The following subsets of $B\left(\ell^2 G\right)$ are all equal:*

(i) *The weak closure of $\rho(\mathbb{C}G)$,*

(ii) *The strong closure of $\rho(\mathbb{C}G)$,*

(iii) *The bicommutant of $\rho(\mathbb{C}G)$,*

(iv) *The set of $u \in B\left(\ell^2 G\right)$ that are* left *$\mathbb{C}G$-equivariant, i.e. $\lambda\left(\mathbb{C}G\right)'$.*

This set is called the (right) group von Neumann algebra *of G, and denoted by* $\mathrm{N}G$.

Proof. The equalities (i) = (ii) = (iii) follow from the Bicommutant Theorem (1.6).

We first show that (ii) $\subseteq$ (iv). It is clear that $\rho(\mathbb{C}G) \subseteq$ (iv), so it suffices to prove that (iv) is strongly closed. Let $(u_n)_{n\geq 1}$ be a sequence of left $\mathbb{C}G$-equivariant bounded linear operators on $\ell^2 G$, converging to $u \in B\left(\ell^2 G\right)$. For all $a \in \mathbb{C}G$ and $x \in \ell^2 G$, we have

$$a \cdot u(x) = a \cdot \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} \left(a \cdot u_n(x)\right) = \lim_{n\to\infty} u_n(ax) = u(ax),$$

so u is also left $\mathbb{C}G$-equivariant. This proves that (iv) is sequentially closed in the strong topology. The same proof, after replacing sequences with nets, shows that (iv) is strongly closed.

Conversely, we show that (iv) $\subseteq$ (iii)[1]. We consider the operator $J : \ell^2 G \to \ell^2 G$ defined by

$$J : \sum_{g\in G} a_g g \mapsto \sum_{g\in G} \bar{a}_{g^{-1}} g.$$

Claim. (i) $J \circ J = \mathrm{id}$.

(ii) *$J \circ \lambda(x) \circ J = \rho\left(Jx\right)$ for all $x \in \mathbb{C}G$. In particular, $J\lambda\left(\mathbb{C}G\right)J = \rho\left(\mathbb{C}G\right)$.*

(iii) *For all $u \in \lambda(\mathbb{C}G)'$, we have $J \circ u(e) = u^*(e)$.*

Proof of the claim. (i) This is clear.

[1] Thanks are due to Hiroto Nishikawa for explaining this part of the proof.

(ii) The equality follows from a simple computation.

(iii) A computation shows that, for all $x, y \in \mathbb{C}G$,

$$\langle Jx, y\rangle = \langle e, xy\rangle = \langle e, yx\rangle = \langle Jy, x\rangle \,. \tag{$*$}$$

Note that ($*$) also holds for $x \in \mathbb{C}G$ and $y \in \ell^2 G$ by density. Now take $u \in \lambda(\mathbb{C}G)'$. Using ($*$), we have for all $x \in \mathbb{C}G$,

$$\langle J \circ u(e), x\rangle = \langle e, \lambda(x) \circ u(e)\rangle = \langle e, u \circ \lambda(x)(e)\rangle = \langle u^*(e), \lambda(x)(e)\rangle = \langle u^*(e), x\rangle \,.$$

Hence, the linear forms $\langle J \circ u(e), -\rangle$ and $\langle u^*(e), -\rangle$ agree on $\mathbb{C}G$ and therefore on $\ell^2 G$ by density; it follows that $J \circ u(e) = u^*(e)$. □

Using the above, we are now ready to show that $\lambda(\mathbb{C}G)' \subseteq \rho(\mathbb{C}G)''$; this will prove the inclusion (iv) $\subseteq$ (iii). Proving that $\lambda(\mathbb{C}G)' \subseteq \rho(\mathbb{C}G)''$ amounts to showing that every $u \in \lambda(\mathbb{C}G)'$ and $v \in \rho(\mathbb{C}G)'$ commute. But by (ii) of the claim, we have

$$\rho(\mathbb{C}G)' = (J\lambda(\mathbb{C}G)\, J)' = J\lambda(\mathbb{C}G)'\, J.$$

Hence, we write $v = JwJ$ with $w \in \lambda(\mathbb{C}G)'$. Let $x \in \ell^2 G$. Using repeatedly (iii) of the claim, together with the facts that $\rho(\mathbb{C}G) \subseteq \lambda(\mathbb{C}G)'$ and that $\left(\lambda(\mathbb{C}G)'\right)^* = \lambda(\mathbb{C}G)'$, we obtain

$$\begin{aligned} u(JwJ)x &= uJwJ\rho(x)e = uJw\rho(x)^*e = u\rho(x)w^*e, \\ (JwJ)ux &= JwJu\rho(x)e = Jw\rho(x)^*u^*e = u\rho(x)w^*e, \end{aligned}$$

proving that $u(JwJ) = (JwJ)u$ as wanted. □

In order to define a notion of dimension for Hilbert G-modules, the basic idea is that, in a finite-dimensional Hilbert space, the dimension of a subspace is equal to the trace of the orthogonal projection onto that subspace.

Our next step is therefore to equip $\mathrm{N}G$ with a trace.

Definition 1.8. The *trace* on $\mathrm{N}G$ is the map $\mathrm{tr}_G : \mathrm{N}G \to \mathbb{C}$ given by

$$\mathrm{tr}_G : a \mapsto \langle e, a(e)\rangle \,,$$

where $e \in \mathbb{C}G \subseteq \ell^2 G$ is the atomic function at the identity $e \in G$.

Proposition 1.9. *The following properties hold for all $a, b \in \mathrm{N}G$:*

(i) (Trace property) $\mathrm{tr}_G(a \circ b) = \mathrm{tr}_G(b \circ a)$.

(ii) (Faithfulness) $\mathrm{tr}_G(a^* \circ a) = 0$ *if and only if* $a = 0$.

(iii) (Positivity) *Suppose that* $a \geq 0$*, in the sense that* $\forall x \in \ell^2 G$, $\langle x, a(x) \rangle \geq 0$*. Then* $\operatorname{tr}_G(a) \geq 0$.

Proof. (i) Note that, for $a = \sum_g a_g g \in \mathbb{C}G$, we have $\operatorname{tr}_G(a) = a_e$. Moreover, for $a, b \in \mathbb{C}G$, the composition $a \circ b$ acts on $\ell^2 G$ as the product ba (because $\mathbb{C}G$ acts on $\ell^2 G$ by *right* multiplication!), so $\operatorname{tr}_G(a \circ b)$ is equal to the coefficient of e in ba:

$$\operatorname{tr}_G(a \circ b) = \sum_{\substack{g,h \in G \\ gh = e}} b_g a_h.$$

This is symmetric in a and b, and hence equal to $\operatorname{tr}_G(b \circ a)$. This proves the trace property for $a, b \in \mathbb{C}G$, which extends by continuity to $\mathrm{N}G$.

(ii) Let $a \in \mathrm{N}G$ with $\operatorname{tr}_G(a^* \circ a) = 0$. Then

$$0 = \langle e, a^* \circ a(e) \rangle = \langle a(e), a(e) \rangle,$$

so $a(e) = 0$. By G-equivariance, we have $a(g) = g \cdot a(e) = 0$ for all $g \in G$. It follows by linearity that a is 0 on $\mathbb{C}G$, and by continuity that a is 0 on $\mathrm{N}G$.

(iii) This is clear. □

Given a matrix $A \in M_{n \times n}(\mathrm{N}G)$, we define

$$\operatorname{tr}_G(A) := \sum_{j=1}^{n} \operatorname{tr}_G(A_{jj}).$$

Usual linear algebra shows that this trace also satisfies Proposition 1.9.

Now any bounded left G-equivariant map $(\ell^2 G)^n \to (\ell^2 G)^n$ is represented by a matrix in $M_{n \times n}(\mathrm{N}G)$ and hence has a trace.

1.5 Von Neumann dimension

Let G be a countable group.

Proposition/Definition 1.10. *Let* V *be a Hilbert* G*-module. The* von Neumann-G-dimension *of* V *is defined by*

$$\dim_{\mathrm{N}G} V := \operatorname{tr}_G(p),$$

where $i : V \hookrightarrow (\ell^2 G)^n$ *is a choice of isometric* G*-embedding for some* $n \in \mathbb{N}_{\geq 1}$ *and* $p : (\ell^2 G)^n \to (\ell^2 G)^n$ *is the orthogonal projection onto the closed subspace* $i(V)$*.*

This is independent of the choice of i*, and* $\dim_{\mathrm{N}G} V \in \mathbb{R}_{\geq 0}$*.*

Proof. Let $j : V \hookrightarrow (\ell^2 G)^m$ be another isometric G-embedding, with $m \in \mathbb{N}_{\geq 1}$, and let $q : (\ell^2 G)^m \to (\ell^2 G)^m$ be the orthogonal projection onto $j(V)$.

Define a map $u : (\ell^2 G)^n \to (\ell^2 G)^m$ by $u_{|\operatorname{Im} i} := j \circ i^{-1}$ and $u_{|(\operatorname{Im} i)^\perp} := 0$. By construction, $j = u \circ i$; it follows that $q = p \circ u^*$. Hence,

$$\operatorname{tr}_G(q) = \operatorname{tr}_G(u \circ q) = \operatorname{tr}_G(u \circ p \circ u^*) = \operatorname{tr}_G(p \circ u^* \circ u) = \operatorname{tr}_G(p \circ p) = \operatorname{tr}_G(p).$$

To see that $\dim_{\mathrm{N}G} V \in \mathbb{R}_{\geq 0}$, note that p is a positive operator, so the diagonal entries of its matrix are also positive operators; the result follows from positivity of the trace. □

We give two examples of computations of von Neummann dimensions.

Example 1.11 (Finite groups). If G is a finite group, then $\mathbb{C}G = \ell^2 G = \mathrm{N}G$. A Hilbert G-module V is finite-dimension over $\mathbb{C}$ and satisfies

$$\dim_{\mathrm{N}G} V = \frac{1}{|G|} \dim_{\mathbb{C}} V.$$

Example 1.12 ($\mathbb{Z}$). If $G = \mathbb{Z}$, then $\ell^2 G \cong L^2([-\pi, \pi], \mathbb{C})$ (see Example 1.3), and

$$\mathrm{N}G \cong L^\infty([-\pi, \pi], \mathbb{C}),$$

with the action of $\mathrm{N}G$ on $\ell^2 G$ given by pointwise multiplication.

Under this isomorphism, $\operatorname{tr}_G : L^\infty([-\pi, \pi], \mathbb{C})$ is given by

$$\operatorname{tr}_G : f \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} f \, d\lambda.$$

Now let $A \subseteq [-\pi, \pi]$ be a measurable set, and consider

$$V := \left\{ f \cdot \chi_A \,\middle|\, f \in L^2([-\pi, \pi], \mathbb{C}) \right\} \subseteq L^2([-\pi, \pi], \mathbb{C}) \cong \ell^2 G.$$

This is a Hilbert-G-module (embedding into $\ell^2 G$). The orthogonal projection onto A is represented by the matrix $(\chi_A) \in M_{1 \times 1}(\mathrm{N}G)$. Therefore,

$$\dim_{\mathrm{N}G} V = \operatorname{tr}_G(\chi_A) = \frac{1}{2\pi} \lambda(A).$$

In particular, every number in $[0, 1]$ occurs as a von Neumann dimension!

We finish with some basic properties of the von Neumann dimension.

Proposition 1.13. *The von Neumann dimension has the following properties.*

(i) (Normalisation) $\dim_{\mathrm{N}G} \ell^2 G = 1$.

(ii) (Faithfulness) *For every Hilbert G-module V, we have* $\dim_{\mathrm{N}G} V = 0$ *if and only if* $V = 0$.

(iii) (Weak isomorphism invariance) *If $f : V \to W$ is a morphism of Hilbert G-modules with* $\operatorname{Ker} f = 0$ *and* $\overline{\operatorname{Im} f} = W$*, then* $\dim_{\mathcal{N}G} V = \dim_{\mathcal{N}G} W$.

(iv) (Additivity) *Assume that the sequence of Hilbert G-modules*

$$0 \to V_1 \xrightarrow{i} V_2 \xrightarrow{\pi} V_3 \to 0$$

is weakly exact, in the sense that i is injective, $\overline{\operatorname{Im} i} = \operatorname{Ker} \pi$*, and* $\overline{\operatorname{Im} \pi} = V_3$*. Then*

$$\dim_{\mathcal{N}G} V_2 = \dim_{\mathcal{N}G} V_1 + \dim_{\mathcal{N}G} V_3.$$

(v) (Multiplicativity) *Let H be another countable group. Let V be a Hilbert G-module and W a Hilbert H-module. Then the completed tensor product $V \bar{\otimes}_{\mathbb{C}} W$ is a Hilbert $G \times H$-module, and*

$$\dim_{\mathcal{N}(G\times H)} (V \bar{\otimes}_{\mathbb{C}} W) = \dim_{\mathcal{N}G} V \cdot \dim_{\mathcal{N}H} W.$$

(vi) (Restriction) *Let V be a Hilbert G-module and let $H \leq G$ be a finite-index subgroup. Then V is naturally a Hilbert H-module, and*

$$\dim_{\mathcal{N}H} \operatorname{Res}_H^G V = [G : H] \cdot \dim_{\mathcal{N}G} V.$$

Sketch of proof. (i) This is clear (taking $\ell^2 G \hookrightarrow (\ell^2 G)^1$ and $p = \mathrm{id}$).

(ii) This follows from faithfulness of the von Neumann trace (1.9).

(iii) This is a consequence of *polar decomposition*: the map f can be written as $f = u \circ p$, where u is a partial isometry and p is a positive operator with $\operatorname{Ker} u = \operatorname{Ker} p$. In this case, f is injective, so $\operatorname{Ker} u = \operatorname{Ker} p = 0$; moreover, u has closed image and so $\operatorname{Im} u = \overline{\operatorname{Im} u} = \overline{\operatorname{Im} f} = W$. Hence, u is an isometry, which is G-equivariant by uniqueness of the polar decomposition.

(iv) Note first that $\dim_{\mathcal{N}G}$ is additive with respect to direct sums, and define a weak isomorphism $V \to \overline{\operatorname{Im} i} \oplus V_3$ by $x \mapsto p(x) \oplus \pi(x)$, where $p : V \to \overline{\operatorname{Im} i}$ is the orthogonal projection.

(v) The key fact is that there is an isomorphism $\ell^2 (G \times H) \cong \ell^2 G \bar{\otimes}_{\mathbb{C}} \ell^2 H$ of Hilbert $G \times H$-modules. ☐

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