

# Reading seminar on $L^2$ -Betti numbers

Organised by Alexis Marchand

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One can define the  $L^2$ -cohomology of a group by replacing finite sums with  $L^2$ -summable sums in the usual definition of group cohomology. The resulting objects are Hilbert modules; their von Neumann dimensions are called the  $L^2$ -Betti numbers of the group.

Since their introduction around the 1970s,  $L^2$ -Betti numbers have created considerable interest for their applications to geometric group theory and algebraic topology, and are still a very active subject of research.

The aim of this reading seminar is to gather people interested in  $L^2$ -Betti numbers from various points of view (including, but not limited to, geometric group theory, operator algebras, algebraic topology, and dynamics), and learn about the established theory of  $L^2$ -Betti numbers as well as very recent research in the subject.

**Schedule.** The plan is to have talks of approximately one hour, with extra time for questions and discussions.

Week 1	Wed 26 Feb	10:30-12:00	理学部3号館108号室
Week 2	Wed 5 Mar	10:30-12:00	理学部3号館108号室
Week 3	Fri 14 Mar	10:30-12:00	理学部3号館305号室
Week 4	No seminar		
Week 5	Wed 26 Mar	10:30-12:00	理学部3号館108号室
Week 6	Wed 2 Apr	10:30-12:00	理学部3号館108号室
Week 7	Wed 9 Apr	10:30-12:00	理学部3号館108号室
Week 8	Wed 16 Apr	10:30-12:00	理学部3号館108号室
Week 9 onwards	TBC		

**Participation in the seminar.** Everyone (Master's and PhD students, postdocs, professors) is welcome to attend any or all of the talks! If you would like to receive announcements, please email me at

[marchand.alexis.85h@st.kyoto-u.ac.jp](mailto:marchand.alexis.85h@st.kyoto-u.ac.jp)

I am planning to give the first two talks to get the seminar started, but I will then need your help in volunteering to give talks. If you would like to give a talk on one of the topics listed below, please email me with your choice of topic and preferred date.

## Tentative plan

The first three weeks will cover the fundamental theory as follows:

**Week 1.** *Introduction, and background on von Neumann dimension*

Motivation from geometric group theory and algebraic topology. Introduction to Hilbert modules and von Neumann dimension.

References: [1, Chapter 1], [2], [6, Chapter 1], [5, §1.1]

Speaker: Alexis Marchand

**Week 2.**  *$L^2$ -homology and  $L^2$ -Betti numbers*

Background on group cohomology,  $K(G, 1)$ -spaces, etc. Definition of  $L^2$ -cohomology and  $L^2$ -Betti numbers. First examples of computations, and basic applications.

References: [1, (parts of) Chapters 3-4], [6, Chapter 2], [5, §1.2].

Speaker: Alexis Marchand

**Week 3.** *Lück's Approximation Theorem*

Statement and proof of Lück's Approximation Theorem.

References: [1, §5.1 to §5.3], [6, Chapter 3]

Speaker: Ryoya Arimoto

**Week 4.** No seminar

**Week 5.** *Relation with acylindrical hyperbolicity*

Background on Gromov-hyperbolicity and acylindrical hyperbolicity. Statement and proof of Osin's Theorem on the acylindrical hyperbolicity of groups with positive first  $L^2$ -Betti number.

References: [7]

Speaker: Bingxue Tao

The rest of the reading seminar will be based on independent topics. Speakers are free to choose among the following suggestions (biased on the interest of a geometric group theorist) or propose other topics (many ideas can be found in Lück's book [5]):

**Topic 1.** *Vanishing of  $L^2$ -Betti numbers*

Investigation of groups with vanishing conditions on their  $L^2$ -Betti numbers and relation with Euler characteristic.

References: [5, §7.1 and §7.2]

Speaker TBC

**Topic 2.** *The measure-theoretic viewpoint*

Definition of rank-gradient, cost, and relation with  $L^2$ -Betti numbers.

References: [6, Chapter 4], [1, §5.5]

Speaker TBC

**Topic 3.** *The Atiyah and Kaplansky Conjectures*

Discussion of the Atiyah Conjecture on the rationality of  $L^2$ -Betti numbers and of the Kaplansky Conjecture on divisors of zeros, units, etc. in group rings. Known results and relations between the conjectures.

References: [1, §3.5], [4] and references therein

Speaker TBC

**Topic 4.**  *$L^2$ -Betti numbers of 3-manifolds*

Background on 3-manifolds and computations of their  $L^2$ -Betti numbers.

References: [5, Chapter 4]

Speaker TBC

**Topic 5.**  *$L^2$ -Betti numbers of symmetric spaces*

Background on symmetric spaces and computations of their  $L^2$ -Betti numbers.

References: [5, Chapter 5]

Speaker TBC

**Topic 6.**  *$L^2$ -torsion and group automorphisms*

Discussion on the  $L^2$ -torsion of certain mapping tori of group automorphisms, including surface group automorphisms and free group automorphisms.

References: [5, Chapter §7.4]

Speaker TBC

**Topic 7.** *Torsion invariants*

Discussion of torsion invariants, relations with  $L^2$ -Betti numbers, conjectures, and known results.

References: [1, Chapter 6]

Speaker TBC

**Topic 8.**  *$L^2$ -Betti numbers and fiberings*

Discussion of Kielak's Theorem on virtual fibering from vanishing of the first  $L^2$ -Betti number.

References: [3]

Speaker TBC

Hopefully new topics will arise from the talks, and everyone is welcome to suggest a topic at any point!

## References

- [1] Holger Kammeyer, *Introduction to  $\ell^2$ -invariants*, Lecture Notes in Mathematics, vol. 2247, Springer, Cham, 2019. MR3971279
- [2] ———,  *$\ell^2$ -Betti numbers*, Bounded cohomology and simplicial volume, 2023, pp. 57–62. MR4496345
- [3] Dawid Kielak, *Residually finite rationally solvable groups and virtual fibering*, J. Amer. Math. Soc. **33** (2020), no. 2, 451–486. MR4073866
- [4] Peter A. Linnell, *The Atiyah conjecture*, Geometry, topology, and dynamics in negative curvature, 2016, pp. 198–220. MR3497261
- [5] Wolfgang Lück,  *$L^2$ -invariants: theory and applications to geometry and  $K$ -theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 44, Springer-Verlag, Berlin, 2002. MR1926649
- [6] Clara Löh, *Ergodic theoretic methods in group homology—a minicourse on  $L^2$ -Betti numbers in group theory*, SpringerBriefs in Mathematics, Springer, Cham, [2020] ©2020. MR4177798
- [7] D. Osin, *On acylindrical hyperbolicity of groups with positive first  $\ell^2$ -Betti number*, Bull. Lond. Math. Soc. **47** (2015), no. 5, 725–730. MR3403956

Please let me know if you have trouble accessing any of the references; I should be able to email you a PDF.