L^2-Betti numbers

Reading seminar

Kyoto University

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Contents

Week 1 – Background on von Neumann dimension

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1.1 Motivation

The goal of what follows is to develop a good *equivariant* homology theory for actions $G \curvearrowright X$ of groups on topological spaces. The usual singular chain complex $C_*^{\mathrm{sing}}(X;\mathbb{C})$ and singular homology $H_*(X;\mathbb{C})$ inherit a G-action, so they have the structure of $\mathbb{C}G$-modules. However, the group G is typically infinite and we do not have a good notion of dimension for modules over $\mathbb{C}G$. This is why we will work in an L^2 setting.

We will introduce a homology theory $H_*^{(2)}(G \curvearrowright X)$, together with associated Betti numbers $b_*^{(2)}(G \curvearrowright X)$. They will be well-defined when X is a G-CW-complex under a certain finiteness property. Examples of properties they will satisfy include the following:

- (Homotopy invariance) If there is a G-equivariant homotopy equivalence $X \xrightarrow{\sim} X'$, then
$$b_*^{(2)}(G \curvearrowright X) = b_*^{(2)}(G \curvearrowright X').$$
- (Dimension) If X is a G-CW-complex of dimension $\leq n$, then
$$b_p^{(2)}(G \curvearrowright X) = 0 \quad \text{for } p > n.$$
- (Euler characteristic) If X is a free G-CW-complex and the quotient $G\backslash X$ is a finite CW-complex, then
$$\chi(G\backslash X) = \sum_{p=0}^{\infty} (-1)^p b_p^{(2)}(G \curvearrowright X).$$
- (Finite-index subgroups) If H is a finite-index subgroup of G, then
$$b_*^{(2)}(H \curvearrowright X) = [G : H] \cdot b_*^{(2)}(G \curvearrowright X).$$

In the first talk, we will introduce the relevant notions around Hilbert modules and von Neumann dimension that will allow us to define L^2-Betti numbers.

1.2 Hilbert G-modules

We fix a countable group G. We will work with $\mathbb{C}$-coefficients throughout.

Definition 1.1. The *group ring* of G over $\mathbb{C}$ is the $\mathbb{C}$-algebra $\mathbb{C}G$ (or $\mathbb{C}[G]$), with underlying $\mathbb{C}$-vector space
$$\mathbb{C}G \coloneqq \bigoplus_{g \in G} \mathbb{C}g,$$
with multiplication defined on the basis vectors by $g \cdot h = gh$.

Example 1.2.

- $\mathbb{C}[\mathbb{Z}] = \mathbb{C}[t, t^{-1}]$ is the ring of Laurent polynomials over $\mathbb{C}$.
- For $n \in \mathbb{N}_{\geq 1}$, $\mathbb{C}[\mathbb{Z}/n] = \mathbb{C}[t]/(t^n - 1)$.

The group ring $\mathbb{C}G$ can be equipped with a natural inner product $\langle \cdot, \cdot \rangle$ defined by

$$\left\langle \sum_{g\in G} a_g g, \sum_{g\in G} b_g g \right\rangle \coloneqq \sum_{g\in G} \bar{a}_g b_g$$

The completion of $\mathbb{C}G$ with respect to $\langle \cdot, \cdot \rangle$ is a complex Hilbert space, which we denote by $\ell^2 G$; it can also be defined as the $\mathbb{C}$-vector space of ℓ^2-summable functions $G \to \mathbb{C}$.

Note that $\ell^2 G$ has the structure of a $\mathbb{C}G$-module, with action given by

$$h \cdot \sum_{g\in G} a_g g \coloneqq \sum_{g\in G} a_{gh} g.$$

Example 1.3.

- If G is finite, then $\ell^2 G = \mathbb{C}G$.
- If $G = \mathbb{Z}$, then Fourier analysis shows that there is an isomorphism $\ell^2 G \cong L^2([-\pi, \pi], \mathbb{C})$ given by

$$\sum_{n\in\mathbb{Z}} a_n t^n \longmapsto \left(x \mapsto \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} a_n e^{inx} \right).$$

Since the group G is assumed to be countable, the Hilbert space $\ell^2 G$ is separable.

Definition 1.4. A *Hilbert G-module* is a complex Hilbert space V with a $\mathbb{C}$-linear isometric (left) G-action such that there is an isometric G-embedding

$$V \hookrightarrow \left(\ell^2 G\right)^n$$

for some $n \in \mathbb{N}_{\geq 1}$.

A *morphism* between two Hilbert G-modules V and W is a G-equivariant bounded $\mathbb{C}$-linear map $V \to W$.

Our homology groups will be Hilbert G-modules; our main task will be to define a notion of dimension for such modules.

1.3 Background on von Neumann algebras

Let $\mathcal{H}$ be a complex Hilbert space. Then the space $B(\mathcal{H})$ of bounded linear operators $\mathcal{H} \to \mathcal{H}$ is a $\mathbb{C}$-algebra, with multiplication given by composition.

Recall that, given $u \in B(\mathcal{H})$, there is a unique $u^* \in B(\mathcal{H})$ — called the *adjoint* of f — such that, for all $x, y \in \mathcal{H}$,

$$\langle u(x), y \rangle = \langle x, u^*(y) \rangle .$$

(This follows from the Riesz Representation Theorem applied to the linear form $\langle u(\cdot), y \rangle$ for fixed $y \in \mathcal{H}$.) Hence, $\cdot^*$ defines an involution on $B(\mathcal{H})$; this turns the latter into a $*$-algebra.

There are several topologies that one can define on $B(\mathcal{H})$:

- The *norm topology*, given by

$$u_n \xrightarrow{\|\cdot\|} u \overset{\text{def}}{\Longleftrightarrow} \|u_n - u\| \to 0,$$

- The *strong topology*, given by

$$u_n \xrightarrow{s} u \overset{\text{def}}{\Longleftrightarrow} \forall x \in \mathcal{H},\ \|u_n(x) - u(x)\| \to 0,$$

- The *weak topology*, given by

$$u_n \xrightarrow{w} u \overset{\text{def}}{\Longleftrightarrow} \forall x, y \in \mathcal{H},\ \langle u_n(x), y \rangle \to \langle u(x), y \rangle .$$

Definition 1.5. A *von Neumann algebra* is a unital weakly closed $*$-subalgebra of $B(\mathcal{H})$ for some complex Hilbert space $\mathcal{H}$.

Given a subset $S \subseteq B(\mathcal{H})$, its *commutant* is defined by

$$S' := \{u \in B(\mathcal{H}) \mid \forall s \in S,\ us = su\} .$$

The *bicommutant* of S is simply $S'' := (S')'$.

The following theorem is a fundamental structural result for von Neumann algebras:

Theorem 1.6 (von Neumann Bicommutant Theorem)**.** *Let $\mathcal{H}$ be a complex Hilbert space and let $A \subseteq B(\mathcal{H})$ be a unital $*$-subalgebra of $B(\mathcal{H})$. Then the following are equivalent:*

(i) *$A'' = A$.*

(ii) *A is strongly closed.*

(iii) *A is weakly closed.*

For abelian von Neumann algebras, we also have the following:

Theorem 1.7. *Let N be an abelian von Neumann algebra on a separable Hilbert space. Then there is a standard Borel σ-finite measure space (X, μ) such that N is isomorphic to the von Neumann algebra $L^\infty(X, \mu) \subseteq L^2(X, \mu)$.*

1.4 The group von Neumann algebra and its trace

We come back to the setup of §1.2: G is a countable group and we are considering the Hilbert space $\ell^2 G$. As above, we denote by $B(\ell^2 G)$ the $\mathbb{C}$-algebra of bounded linear operators $\ell^2 G \to \ell^2 G$.

Observe that there are two embeddings

$$\lambda, \rho : \mathbb{C}G \hookrightarrow B\left(\ell^2 G\right)$$

given by the respective actions of $\mathbb{C}G$ on $\ell^2 G$ by left and right multiplication.

Proposition/Definition 1.8. *The following subsets of $B(\ell^2 G)$ are all equal:*

(i) *The weak closure of $\rho(\mathbb{C}G)$,*

(ii) *The strong closure of $\rho(\mathbb{C}G)$,*

(iii) *The bicommutant of $\rho(\mathbb{C}G)$,*

(iv) *The set of $u \in B(\ell^2 G)$ that are* left *$\mathbb{C}G$-equivariant.*

This set is called the (right) group von Neumann algebra *of G, and denoted by* $\mathrm{N}G$.

Proof. The equalities (i) = (ii) = (iii) follow from the Bicommutant Theorem (1.6).

We first show that (ii) $\subseteq$ (iv). It is clear that $\rho(\mathbb{C}G) \subseteq$ (iv), so it suffices to prove that (iv) is strongly closed. Let $(u_n)_{n\geq 1}$ be a sequence of left $\mathbb{C}G$-equivariant bounded linear operators on $\ell^2 G$, converging to $u \in B(\ell^2 G)$. For all $a \in \mathbb{C}G$ and $x \in \ell^2 G$, we have

$$a \cdot u(x) = a \cdot \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} (a \cdot u_n(x)) = \lim_{n\to\infty} u_n(ax) = u(ax),$$

so u is also left $\mathbb{C}G$-equivariant. This proves that (iv) is sequentially closed in the strong topology. The same proof, after replacing sequences with nets, shows that (iv) is strongly closed.

Next, we will show that (iv) $\subseteq$ (iii). Observe that

$$\lambda(\mathbb{C}G)' = \rho(\mathbb{C}G).$$

Thus, $\rho(\mathbb{C}G)' = \lambda(\mathbb{C}G)'' = \overline{\lambda(\mathbb{C}G)}$ by the Bicommutant Theorem (1.6) (for the weak or the strong closure). Now let $u \in$ (iv). By definition, u commutes with every element of $\lambda(\mathbb{C}G)$, and therefore with every element of $\overline{\lambda(\mathbb{C}G)} = \rho(\mathbb{C}G)'$ by continuity. In other words, $u \in \rho(\mathbb{C}G)''$ as wanted. ☐

In order to define a notion of dimension for Hilbert G-modules, the basic idea is that, in a finite-dimensional Hilbert space, the dimension of a subspace is equal to the trace of the orthogonal projection onto that subspace.

Our next step is therefore to equip $\mathrm{N}G$ with a trace.

Definition 1.9. The *trace* on $\mathrm{N}G$ is the map $\mathrm{tr}_G : \mathrm{N}G \to \mathbb{C}$ given by

$$\mathrm{tr}_G : a \mapsto \langle e, a(e) \rangle ,$$

where $e \in \mathbb{C}G \subseteq \ell^2 G$ is the atomic function at the identity $e \in G$.

Proposition 1.10. *The following properties hold for all $a, b \in \mathrm{N}G$:*

(i) (Trace property) $\mathrm{tr}_G(a \circ b) = \mathrm{tr}_G(b \circ a)$.

(ii) (Faithfulness) $\mathrm{tr}_G (a^* \circ a) = 0$ *if and only if* $a = 0$.

(iii) (Positivity) *Suppose that $a \geq 0$, in the sense that $\forall x \in \ell^2 G$, $\langle x, a(x) \rangle \geq 0$. Then $\mathrm{tr}_G(a) \geq 0$.*

Proof. (i) Note that, for $a = \sum_g a_g g \in \mathbb{C}G$, we have $\mathrm{tr}_G(a) = a_e$. Moreover, for $a, b \in \mathbb{C}G$, the composition $a \circ b$ acts on $\ell^2 G$ as the product ba (because $\mathbb{C}G$ acts on $\ell^2 G$ by *right* multiplication!), so $\mathrm{tr}_G(a \circ b)$ is equal to the coefficient of e in ba:

$$\mathrm{tr}_G(a \circ b) = \sum_{\substack{g,h \in G \\ gh = e}} b_g a_h .$$

This is symmetric in a and b, and hence equal to $\mathrm{tr}_G(b \circ a)$. This proves the trace property for $a, b \in \mathbb{C}G$, which extends by continuity to $\mathrm{N}G$.

(ii) Let $a \in \mathrm{N}G$ with $\mathrm{tr}_G (a^* \circ a) = 0$. Then

$$0 = \langle e, a^* \circ a(e) \rangle = \langle a(e), a(e) \rangle ,$$

so $a(e) = 0$. By G-equivariance, we have $a(g) = g \cdot a(e) = 0$ for all $g \in G$. It follows by linearity that a is 0 on $\mathbb{C}G$, and by continuity that a is 0 on $\mathrm{N}G$.

(iii) This is clear. $\square$

Given a matrix $A \in M_{n \times n} (\mathrm{N}G)$, we define

$$\mathrm{tr}_G(A) := \sum_{j=1}^{n} \mathrm{tr}_G (A_{jj}) .$$

Usual linear algebra shows that this trace also satisfies Proposition 1.10.

Now any bounded left G-equivariant map $(\ell^2 G)^n \to (\ell^2 G)^n$ is represented by a matrix in $M_{n \times n} (\mathrm{N}G)$ and hence has a trace.

1.5 Von Neumann dimension

Let G be a countable group.

Proposition/Definition 1.11. *Let V be a Hilbert G-module. The* von Neumann-G-dimension *of V is defined by*

$$\dim_{\mathrm{N}G} V \coloneqq \operatorname{tr}_G(p),$$

where $i : V \hookrightarrow (\ell^2 G)^n$ is a choice of isometric G-embedding for some $n \in \mathbb{N}_{\geq 1}$ and $p : (\ell^2 G)^n \to (\ell^2 G)^n$ is the orthogonal projection onto the closed subspace $i(V)$.

This is independent of the choice of i, and $\dim_{\mathrm{N}G} V \in \mathbb{R}_{\geq 0}$.

Proof. Let $j : V \hookrightarrow (\ell^2 G)^m$ be another isometric G-embedding, with $m \in \mathbb{N}_{\geq 1}$, and let $q : (\ell^2 G)^m \to (\ell^2 G)^m$ be the orthogonal projection onto $j(V)$.

Define a map $u : (\ell^2 G)^n \to (\ell^2 G)^m$ by $u_{|\operatorname{Im} i} \coloneqq j \circ i^{-1}$ and $u_{|(\operatorname{Im} i)^\perp} \coloneqq 0$. By construction, $j = u \circ i$; it follows that $q = p \circ u^*$. Hence,

$$\operatorname{tr}_G(q) = \operatorname{tr}_G(u \circ q) = \operatorname{tr}_G(u \circ p \circ u^*) = \operatorname{tr}_G(p \circ u^* \circ u) = \operatorname{tr}_G(p \circ p) = \operatorname{tr}_G(p).$$

To see that $\dim_{\mathrm{N}G} V \in \mathbb{R}_{\geq 0}$, note that p is a positive operator, so the diagonal entries of its matrix are also positive operators; the result follows from positivity of the trace. □

We give two examples of computations of von Neummann dimensions.

Example 1.12 (Finite groups)*.* If G is a finite group, then $\mathbb{C}G = \ell^2 G = \mathrm{N}G$. A Hilbert G-module V is finite-dimension over $\mathbb{C}$ and satisfies

$$\dim_{\mathrm{N}G} V = \frac{1}{|G|} \dim_{\mathbb{C}} V.$$

Example 1.13 ($\mathbb{Z}$)*.* If $G = \mathbb{Z}$, then $\ell^2 G \cong L^2([-\pi, \pi], \mathbb{C})$ (see Example 1.3), and

$$\mathrm{N}G \cong L^\infty([-\pi, \pi], \mathbb{C}),$$

with the action of $\mathrm{N}G$ on $\ell^2 G$ given by pointwise multiplication.

Under this isomorphism, $\operatorname{tr}_G : L^\infty([-\pi, \pi], \mathbb{C})$ is given by

$$\operatorname{tr}_G : f \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} f \, d\lambda.$$

Now let $A \subseteq [-\pi, \pi]$ be a measurable set, and consider

$$V \coloneqq \left\{ f \cdot \chi_A \,\middle|\, f \in L^2([-\pi, \pi], \mathbb{C}) \right\} \subseteq L^2([-\pi, \pi], \mathbb{C}) \cong \ell^2 G.$$

This is a Hilbert-G-module (embedding into $\ell^2 G$). The orthogonal projection onto A is represented by the matrix $(\chi_A) \in M_{1\times 1}(\mathrm{N}G)$. Therefore,

$$\dim_{\mathrm{N}G} V = \operatorname{tr}_G(\chi_A) = \frac{1}{2\pi}\lambda(A).$$

In particular, every number in $[0,1]$ occurs as a von Neumann dimension!

We finish with some basic properties of the von Neumann dimension.

Proposition 1.14. *The von Neumann dimension has the following properties.*

(i) (Normalisation) $\dim_{\mathrm{N}G} \ell^2 G = 1$.

(ii) (Faithfulness) *For every Hilbert G-module V, we have* $\dim_{\mathrm{N}G} V = 0$ *if and only if* $V = 0$.

(iii) (Weak isomorphism invariance) *If $f : V \to W$ is a morphism of Hilbert G-modules with* $\operatorname{Ker} f = 0$ *and* $\overline{\operatorname{Im} f} = W$*, then* $\dim_{\mathrm{N}G} V = \dim_{\mathrm{N}G} W$.

(iv) (Additivity) *Assume that the sequence of Hilbert G-modules*

$$0 \to V_1 \xrightarrow{i} V_2 \xrightarrow{\pi} V_3 \to 0$$

is weakly exact, in the sense that i is injective, $\overline{\operatorname{Im} i} = \operatorname{Ker} \pi$*, and* $\overline{\operatorname{Im} \pi} = V_3$*. Then*

$$\dim_{\mathrm{N}G} V_2 = \dim_{\mathrm{N}G} V_1 + \dim_{\mathrm{N}G} V_3.$$

(v) (Multiplicativity) *Let H be another countable group. Let V be a Hilbert G-module and W a Hilbert H-module. Then the completed tensor product $V \bar{\otimes}_{\mathbb{C}} W$ is a Hilbert $G \times H$-module, and*

$$\dim_{\mathrm{N}(G\times H)} (V \bar{\otimes}_{\mathbb{C}} W) = \dim_{\mathrm{N}G} V \cdot \dim_{\mathrm{N}H} W.$$

(vi) (Restriction) *Let V be a Hilbert G-module and let $H \leq G$ be a finite-index subgroup. Then V is naturally a Hilbert H-module, and*

$$\dim_{\mathrm{N}H} \operatorname{Res}^G_H V = [G : H] \cdot \dim_{\mathrm{N}G} V.$$

Sketch of proof. (i) This is clear (taking $\ell^2 G \hookrightarrow (\ell^2 G)^1$ and $p = \mathrm{id}$).

(ii) This follows from faithfulness of the von Neumann trace (1.10).

(iii) This is a consequence of *polar decomposition*: the map f can be written as $f = u \circ p$, where u is a partial isometry and p is a positive operator with $\operatorname{Ker} u = \operatorname{Ker} p$. In this case, f is injective, so $\operatorname{Ker} u = \operatorname{Ker} p = 0$; moreover, u has closed image and so $\operatorname{Im} u = \overline{\operatorname{Im} u} = \overline{\operatorname{Im} f} = W$. Hence, u is an isometry, which is G-equivariant by uniqueness of the polar decomposition.

(iv) Note first that $\dim_{\mathrm{N}G}$ is additive with respect to direct sums, and define a weak isomorphism $V \to \overline{\operatorname{Im} i} \oplus V_3$ by $x \mapsto p(x) \oplus \pi(x)$, where $p : V \to \overline{\operatorname{Im} i}$ is the orthogonal projection.

(v) The key fact is that there is an isomorphism $\ell^2 (G \times H) \cong \ell^2 G \bar{\otimes}_{\mathbb{C}} \ell^2 H$ of Hilbert $G \times H$-modules. $\square$

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