L^2-Betti numbers

Reading seminar

Kyoto University

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Contents

Week 1 – Background on von Neumann dimension

Speaker: Alexis Marchand.
References: [1, Chapter 2], [2], [4, Chapter 1], [3, §1.1].

1.1 Motivation

The goal of what follows is to develop a good *equivariant* homology theory for actions $G \curvearrowright X$ of groups on topological spaces. The usual singular chain complex $C_*^{\mathrm{sing}}(X;\mathbb{C})$ and singular homology $H_*(X;\mathbb{C})$ inherit a G-action, so they have the structure of $\mathbb{C}G$-modules. However, the group G is typically infinite and we do not have a good notion of dimension for modules over $\mathbb{C}G$. This is why we will work in an L^2 setting.

We will introduce a homology theory $H_*^{(2)}(G \curvearrowright X)$, together with associated Betti numbers $b_*^{(2)}(G \curvearrowright X)$. They will be well-defined when X is a G-CW-complex under a certain finiteness property.

In the first talk, we introduce the relevant notions around Hilbert modules and von Neumann dimension that will allow us to define L^2-Betti numbers.

1.2 Hilbert G-modules

We fix a countable group G. We will work with $\mathbb{C}$-coefficients throughout.

Definition 1.1. The *group ring* of G over $\mathbb{C}$ is the $\mathbb{C}$-algebra $\mathbb{C}G$ (or $\mathbb{C}[G]$), with underlying $\mathbb{C}$-vector space

$$\mathbb{C}G \coloneqq \bigoplus_{g \in G} \mathbb{C}g,$$

with multiplication defined on the basis vectors by $g \cdot h = gh$.

Example 1.2.

- $\mathbb{C}[\mathbb{Z}] = \mathbb{C}\left[t, t^{-1}\right]$ is the ring of Laurent polynomials over $\mathbb{C}$.
- For $n \in \mathbb{N}_{\geq 1}$, $\mathbb{C}\left[\mathbb{Z}/n\right] = \mathbb{C}[t]/\left(t^n - 1\right)$.

The group ring $\mathbb{C}G$ can be equipped with a natural inner product $\langle \cdot, \cdot \rangle$ defined by

$$\left\langle \sum_{g \in G} a_g g, \sum_{g \in G} b_g g \right\rangle \coloneqq \sum_{g \in G} \bar{a}_g b_g$$

The completion of $\mathbb{C}G$ with respect to $\langle \cdot, \cdot \rangle$ is a complex Hilbert space, which we denote by $\ell^2 G$; it can also be defined as the $\mathbb{C}$-vector space of ℓ^2-summable functions $G \to \mathbb{C}$.

Note that $\ell^2 G$ has the structure of a $\mathbb{C}G$-module, with action given by

$$h \cdot \sum_{g \in G} a_g g \coloneqq \sum_{g \in G} a_{gh} g.$$

Example 1.3.

- If G is finite, then $\ell^2 G = \mathbb{C}G$.
- If $G = \mathbb{Z}$, by Fourier analysis, there is an isomorphism $\ell^2 G \cong L^2\left([-\pi, \pi], \mathbb{C}\right)$ given by

$$\sum_{n \in \mathbb{Z}} a_n t^n \longmapsto \left(x \mapsto \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} a_n e^{inx} \right).$$

Since the group G is assumed to be countable, the Hilbert space $\ell^2 G$ is separable.

Definition 1.4. A *Hilbert G-module* is a complex Hilbert space V with a $\mathbb{C}$-linear isometric (left) G-action such that there is an isometric G-embedding

$$V \hookrightarrow \left(\ell^2 G\right)^n$$

for some $n \in \mathbb{N}_{\geq 1}$.

A *morphism* between two Hilbert G-modules V and W is a G-equivariant bounded $\mathbb{C}$-linear map $V \to W$.

Our homology groups will be Hilbert G-modules; our main task will be to define a notion of dimension for such modules.

1.3 Background on von Neumann algebras

Let $\mathcal{H}$ be a complex Hilbert space. Then the space $B(\mathcal{H})$ of bounded linear operators $\mathcal{H} \to \mathcal{H}$ is a $\mathbb{C}$-algebra, with multiplication given by composition.

Recall that, given $u \in B(\mathcal{H})$, there is a unique $u^* \in B(\mathcal{H})$ — called the *adjoint* of f — such that, for all $x, y \in \mathcal{H}$,

$$\langle u(x), y \rangle = \langle x, u^*(y) \rangle .$$

(This follows from the Riesz Representation Theorem applied to the linear form $\langle u(\cdot), y \rangle$ for fixed $y \in \mathcal{H}$.) Hence, $\cdot^*$ defines an involution on $B(\mathcal{H})$; this turns the latter into a $*$-algebra.

There are several topologies that one can define on $B(\mathcal{H})$:

- The *norm topology*, given by
$$u_n \xrightarrow{\|\cdot\|} u \overset{\text{def}}{\iff} \|u_n - u\| \to 0,$$
- The *strong topology*, given by
$$u_n \xrightarrow{s} u \overset{\text{def}}{\iff} \forall x \in \mathcal{H},\ \|u_n(x) - u(x)\| \to 0,$$
- The *weak topology*, given by
$$u_n \xrightarrow{w} u \overset{\text{def}}{\iff} \forall x, y \in \mathcal{H},\ \langle u_n(x), y \rangle \to \langle u(x), y \rangle .$$

Definition 1.5. A *von Neumann algebra* is a unital weakly closed $*$-subalgebra of $B(\mathcal{H})$ for some complex Hilbert space $\mathcal{H}$.

Given a subset $S \subseteq B(\mathcal{H})$, its *commutant* is defined by

$$S' \coloneqq \{u \in B(\mathcal{H}) \mid \forall s \in S,\ us = su\} .$$

The *bicommutant* of S is simply $S'' \coloneqq (S')'$.

The following theorem is a fundamental structural result for von Neumann algebras:

Theorem 1.6 (von Neumann Bicommutant Theorem)**.** *Let $\mathcal{H}$ be a complex Hilbert space and let $A \subseteq B(\mathcal{H})$ be a unital $*$-subalgebra of $B(\mathcal{H})$. Then the following are equivalent:*

(i) $A'' = A$.

(ii) *A is strongly closed.*

(iii) *A is weakly closed.*

1.4 The group von Neumann algebra and its trace

We come back to the setup of §1.2: G is a countable group and we are considering the Hilbert space ℓ^2G. As above, we denote by $B(\ell^2G)$ the $\mathbb{C}$-algebra of bounded linear operators $\ell^2G \to \ell^2G$.

Observe that there are two embeddings

$$\lambda, \rho : \mathbb{C}G \hookrightarrow B\left(\ell^2G\right)$$

given by the respective actions of $\mathbb{C}G$ on ℓ^2G by left and right multiplication.

Proposition/Definition 1.7. *The following subsets of $B(\ell^2G)$ are all equal:*

(i) *The weak closure of $\rho(\mathbb{C}G)$,*

(ii) *The strong closure of $\rho(\mathbb{C}G)$,*

(iii) *The bicommutant of $\rho(\mathbb{C}G)$,*

(iv) *The set of $u \in B(\ell^2G)$ that are* left *$\mathbb{C}G$-equivariant, i.e. $\lambda(\mathbb{C}G)'$.*

This set is called the (right) group von Neumann algebra *of G, and denoted by* $\mathcal{N}G$*.*

Proof. The equalities (i) = (ii) = (iii) follow from the Bicommutant Theorem (1.6).

We first show that (ii) $\subseteq$ (iv). It is clear that $\rho(\mathbb{C}G) \subseteq$ (iv), so it suffices to prove that (iv) is strongly closed. Let $(u_n)_{n\geq 1}$ be a sequence of left $\mathbb{C}G$-equivariant bounded linear operators on ℓ^2G, converging to $u \in B(\ell^2G)$. For all $a \in \mathbb{C}G$ and $x \in \ell^2G$, we have

$$a \cdot u(x) = a \cdot \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} (a \cdot u_n(x)) = \lim_{n\to\infty} u_n(ax) = u(ax),$$

so u is also left $\mathbb{C}G$-equivariant. This proves that (iv) is sequentially closed in the strong topology. The same proof, after replacing sequences with nets, shows that (iv) is strongly closed.

Conversely, we show that (iv) $\subseteq$ (iii)[1]. We consider the operator $J : \ell^2G \to \ell^2G$ defined by

$$J : \sum_{g\in G} a_g g \mapsto \sum_{g\in G} \bar{a}_{g^{-1}} g.$$

Claim. (i) $J \circ J = \mathrm{id}$.

(ii) *$J \circ \lambda(x) \circ J = \rho(Jx)$ for all $x \in \mathbb{C}G$. In particular, $J\lambda(\mathbb{C}G)J = \rho(\mathbb{C}G)$.*

(iii) *For all $u \in \lambda(\mathbb{C}G)'$, we have $J \circ u(e) = u^*(e)$.*

Proof of the claim. (i) This is clear.

[1] Thanks are due to Hiroto Nishikawa for explaining this part of the proof.

(ii) The equality follows from a simple computation.

(iii) A computation shows that, for all $x, y \in \mathbb{C}G$,

$$\langle Jx, y\rangle = \langle e, xy\rangle = \langle e, yx\rangle = \langle Jy, x\rangle \,. \tag{$*$}$$

Note that ($*$) also holds for $x \in \mathbb{C}G$ and $y \in \ell^2 G$ by density. Now take $u \in \lambda(\mathbb{C}G)'$. Using ($*$), we have for all $x \in \mathbb{C}G$,

$$\langle J \circ u(e), x\rangle = \langle e, \lambda(x) \circ u(e)\rangle = \langle e, u \circ \lambda(x)(e)\rangle = \langle u^*(e), \lambda(x)(e)\rangle = \langle u^*(e), x\rangle \,.$$

Hence, the linear forms $\langle J \circ u(e), -\rangle$ and $\langle u^*(e), -\rangle$ agree on $\mathbb{C}G$ and therefore on $\ell^2 G$ by density; it follows that $J \circ u(e) = u^*(e)$. □

Using the above, we are now ready to show that $\lambda(\mathbb{C}G)' \subseteq \rho(\mathbb{C}G)''$; this will prove the inclusion (iv) $\subseteq$ (iii). Proving that $\lambda(\mathbb{C}G)' \subseteq \rho(\mathbb{C}G)''$ amounts to showing that every $u \in \lambda(\mathbb{C}G)'$ and $v \in \rho(\mathbb{C}G)'$ commute. But by (ii) of the claim, we have

$$\rho(\mathbb{C}G)' = (J\lambda(\mathbb{C}G)\, J)' = J\lambda(\mathbb{C}G)'\, J.$$

Hence, we write $v = JwJ$ with $w \in \lambda(\mathbb{C}G)'$. Let $x \in \ell^2 G$. Using repeatedly (iii) of the claim, together with the facts that $\rho(\mathbb{C}G) \subseteq \lambda(\mathbb{C}G)'$ and that $\left(\lambda(\mathbb{C}G)'\right)^* = \lambda(\mathbb{C}G)'$, we obtain

$$\begin{aligned} u(JwJ)x &= uJwJ\rho(x)e = uJw\rho(x)^*e = u\rho(x)w^*e, \\ (JwJ)ux &= JwJu\rho(x)e = Jw\rho(x)^*u^*e = u\rho(x)w^*e, \end{aligned}$$

proving that $u(JwJ) = (JwJ)u$ as wanted. □

In order to define a notion of dimension for Hilbert G-modules, the basic idea is that, in a finite-dimensional Hilbert space, the dimension of a subspace is equal to the trace of the orthogonal projection onto that subspace.

Our next step is therefore to equip $\mathrm{N}G$ with a trace.

Definition 1.8. The *trace* on $\mathrm{N}G$ is the map $\mathrm{tr}_G : \mathrm{N}G \to \mathbb{C}$ given by

$$\mathrm{tr}_G : a \mapsto \langle e, a(e)\rangle \,,$$

where $e \in \mathbb{C}G \subseteq \ell^2 G$ is the atomic function at the identity $e \in G$.

Proposition 1.9. *The following properties hold for all $a, b \in \mathrm{N}G$:*

(i) (Trace property) $\mathrm{tr}_G(a \circ b) = \mathrm{tr}_G(b \circ a)$.

(ii) (Faithfulness) $\mathrm{tr}_G(a^* \circ a) = 0$ *if and only if* $a = 0$.

(iii) (Positivity) *Suppose that* $a \geq 0$*, in the sense that* $\forall x \in \ell^2 G$, $\langle x, a(x)\rangle \geq 0$*. Then* $\operatorname{tr}_G(a) \geq 0$.

Proof. (i) Note that, for $a = \sum_g a_g g \in \mathbb{C}G$, we have $\operatorname{tr}_G(a) = a_e$. Moreover, for $a, b \in \mathbb{C}G$, the composition $a \circ b$ acts on $\ell^2 G$ as the product ba (because $\mathbb{C}G$ acts on $\ell^2 G$ by *right* multiplication!), so $\operatorname{tr}_G(a \circ b)$ is equal to the coefficient of e in ba:

$$\operatorname{tr}_G(a \circ b) = \sum_{\substack{g,h \in G \\ gh = e}} b_g a_h.$$

This is symmetric in a and b, and hence equal to $\operatorname{tr}_G(b \circ a)$. This proves the trace property for $a, b \in \mathbb{C}G$, which extends by continuity to $\mathrm{N}G$.

(ii) Let $a \in \mathrm{N}G$ with $\operatorname{tr}_G(a^* \circ a) = 0$. Then

$$0 = \langle e, a^* \circ a(e)\rangle = \langle a(e), a(e)\rangle,$$

so $a(e) = 0$. By G-equivariance, we have $a(g) = g \cdot a(e) = 0$ for all $g \in G$. It follows by linearity that a is 0 on $\mathbb{C}G$, and by continuity that a is 0 on $\mathrm{N}G$.

(iii) This is clear. ◻

Given a matrix $A \in M_{n \times n}(\mathrm{N}G)$, we define

$$\operatorname{tr}_G(A) := \sum_{j=1}^{n} \operatorname{tr}_G(A_{jj}).$$

Usual linear algebra shows that this trace also satisfies Proposition 1.9.

Now any bounded left G-equivariant map $(\ell^2 G)^n \to (\ell^2 G)^n$ is represented by a matrix in $M_{n \times n}(\mathrm{N}G)$ and hence has a trace.

1.5 Von Neumann dimension

Let G be a countable group.

Proposition/Definition 1.10. *Let V be a Hilbert G-module. The* von Neumann-G-dimension *of V is defined by*

$$\dim_{\mathrm{N}G} V := \operatorname{tr}_G(p),$$

where $i : V \hookrightarrow (\ell^2 G)^n$ is a choice of isometric G-embedding for some $n \in \mathbb{N}_{\geq 1}$ and $p : (\ell^2 G)^n \to (\ell^2 G)^n$ is the orthogonal projection onto the closed subspace $i(V)$.

This is independent of the choice of i, and $\dim_{\mathrm{N}G} V \in \mathbb{R}_{\geq 0}$.

Proof. Let $j : V \hookrightarrow (\ell^2 G)^m$ be another isometric G-embedding, with $m \in \mathbb{N}_{\geq 1}$, and let $q : (\ell^2 G)^m \to (\ell^2 G)^m$ be the orthogonal projection onto $j(V)$.

Define a map $u : (\ell^2 G)^n \to (\ell^2 G)^m$ by $u_{|\operatorname{Im} i} := j \circ i^{-1}$ and $u_{|(\operatorname{Im} i)^\perp} := 0$. By construction, $j = u \circ i$; it follows that $q = p \circ u^*$. Hence,

$$\operatorname{tr}_G(q) = \operatorname{tr}_G(u \circ q) = \operatorname{tr}_G(u \circ p \circ u^*) = \operatorname{tr}_G(p \circ u^* \circ u) = \operatorname{tr}_G(p \circ p) = \operatorname{tr}_G(p).$$

To see that $\dim_{\mathrm{N}G} V \in \mathbb{R}_{\geq 0}$, note that p is a positive operator, so the diagonal entries of its matrix are also positive operators; the result follows from positivity of the trace. □

We give two examples of computations of von Neummann dimensions.

Example 1.11 (Finite groups). If G is a finite group, then $\mathbb{C}G = \ell^2 G = \mathrm{N}G$. A Hilbert G-module V is finite-dimension over $\mathbb{C}$ and satisfies

$$\dim_{\mathrm{N}G} V = \frac{1}{|G|} \dim_{\mathbb{C}} V.$$

Example 1.12 ($\mathbb{Z}$). If $G = \mathbb{Z}$, then $\ell^2 G \cong L^2([-\pi, \pi], \mathbb{C})$ (see Example 1.3), and

$$\mathrm{N}G \cong L^\infty([-\pi, \pi], \mathbb{C}),$$

with the action of $\mathrm{N}G$ on $\ell^2 G$ given by pointwise multiplication.

Under this isomorphism, $\operatorname{tr}_G : L^\infty([-\pi, \pi], \mathbb{C})$ is given by

$$\operatorname{tr}_G : f \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} f \, d\lambda.$$

Now let $A \subseteq [-\pi, \pi]$ be a measurable set, and consider

$$V := \left\{ f \cdot \chi_A \,\middle|\, f \in L^2([-\pi, \pi], \mathbb{C}) \right\} \subseteq L^2([-\pi, \pi], \mathbb{C}) \cong \ell^2 G.$$

This is a Hilbert-G-module (embedding into $\ell^2 G$). The orthogonal projection onto A is represented by the matrix $(\chi_A) \in M_{1 \times 1}(\mathrm{N}G)$. Therefore,

$$\dim_{\mathrm{N}G} V = \operatorname{tr}_G(\chi_A) = \frac{1}{2\pi} \lambda(A).$$

In particular, every number in $[0, 1]$ occurs as a von Neumann dimension!

We finish with some basic properties of the von Neumann dimension.

Proposition 1.13. *The von Neumann dimension has the following properties.*

(i) (Normalisation) $\dim_{\mathrm{N}G} \ell^2 G = 1$.

(ii) (Faithfulness) *For every Hilbert G-module V, we have* $\dim_{\mathrm{N}G} V = 0$ *if and only if* $V = 0$.

(iii) (Weak isomorphism invariance) *If $f : V \to W$ is a morphism of Hilbert G-modules with* $\operatorname{Ker} f = 0$ *and* $\overline{\operatorname{Im} f} = W$*, then* $\dim_{\mathrm{N}G} V = \dim_{\mathrm{N}G} W$*.*

(iv) (Additivity) *Assume that the sequence of Hilbert G-modules*

$$0 \to V_1 \xrightarrow{i} V_2 \xrightarrow{\pi} V_3 \to 0$$

is weakly exact, in the sense that i is injective, $\overline{\operatorname{Im} i} = \operatorname{Ker} \pi$*, and* $\overline{\operatorname{Im} \pi} = V_3$*. Then*

$$\dim_{\mathrm{N}G} V_2 = \dim_{\mathrm{N}G} V_1 + \dim_{\mathrm{N}G} V_3.$$

(v) (Multiplicativity) *Let H be another countable group. Let V be a Hilbert G-module and W a Hilbert H-module. Then the completed tensor product $V \bar{\otimes}_{\mathbb{C}} W$ is a Hilbert $G \times H$-module, and*

$$\dim_{\mathrm{N}(G \times H)} (V \bar{\otimes}_{\mathbb{C}} W) = \dim_{\mathrm{N}G} V \cdot \dim_{\mathrm{N}H} W.$$

(vi) (Restriction) *Let V be a Hilbert G-module and let $H \leq G$ be a finite-index subgroup. Then V is naturally a Hilbert H-module, and*

$$\dim_{\mathrm{N}H} \operatorname{Res}_H^G V = [G : H] \cdot \dim_{\mathrm{N}G} V.$$

Sketch of proof. (i) This is clear (taking $\ell^2 G \hookrightarrow (\ell^2 G)^1$ and $p = \mathrm{id}$).

(ii) This follows from faithfulness of the von Neumann trace (1.9).

(iii) This is a consequence of *polar decomposition*: the map f can be written as $f = u \circ p$, where u is a partial isometry and p is a positive operator with $\operatorname{Ker} u = \operatorname{Ker} p$. In this case, f is injective, so $\operatorname{Ker} u = \operatorname{Ker} p = 0$; moreover, u has closed image and so $\operatorname{Im} u = \overline{\operatorname{Im} u} = \overline{\operatorname{Im} f} = W$. Hence, u is an isometry, which is G-equivariant by uniqueness of the polar decomposition.

(iv) Note first that $\dim_{\mathrm{N}G}$ is additive with respect to direct sums, and define a weak isomorphism $V \to \overline{\operatorname{Im} i} \oplus V_3$ by $x \mapsto p(x) \oplus \pi(x)$, where $p : V \to \overline{\operatorname{Im} i}$ is the orthogonal projection.

(v) The key fact is that there is an isomorphism $\ell^2 (G \times H) \cong \ell^2 G \bar{\otimes}_{\mathbb{C}} \ell^2 H$ of Hilbert $G \times H$-modules. □

Week 2 – L^2-cohomology and L^2-Betti numbers

Speaker: Alexis Marchand.
References: [1, (parts of) Chapters 3-4], [4, Chapter 2], [3, §1.2].

2.1 Eilenberg–MacLane space and finiteness properties

Let G be a discrete group. We will study the (co)homology of G via its Eilenberg–MacLane space.

Definition 2.1. An *Eilenberg–MacLane space* for G — or $K(G,1)$ space — is a connected aspherical CW-complex X with $\pi_1 X = G$.

Up to homotopy equivalence, a $K(G,1)$ space is unique.

In order to construct a $K(G,1)$ space, we start with a (possibly infinite) presentation of G, we build its *presentation complex* (with one 0-cell, one 1-cell for each generator, one 2-cell for each relation), and we successively add higher dimensional cells to kill all the homotopy groups. The resulting CW-complex is a $K(G,1)$.

Remark 2.2. Let G be a countable group and let X be a $K(G,1)$ space. The *group homology* of G can be defined as

$$H_*(G) \coloneqq H_*(X),$$

where $H^*(X)$ denotes the (singular/cellular) homology of X.

Given a $K(G,1)$ space X, there is a free cellular action of G on the universal cover $\tilde{X}$; we say that $\tilde{X}$ is a *free G-CW-complex.*

We will define the L^2-Betti numbers of a general free G-CW-complex Y. The L^2-Betti numbers of G will then be defined as those of $\tilde{X}$, where X is a $K(G,1)$ space.

We will need certain finiteness properties.

Definition 2.3. Let Y be a free G-CW-complex. We say that Y has type

- F_n ($n \geq 0$) if Y has a finite number of orbits of n-cells,
- F_∞ if Y is of type F_n for all $n \geq 0$.

We say that G is of type F_n or F_∞, if the G-CW-complex $\tilde{X}$ (for X a $K(G,1)$ space) is of type F_n or F_∞ respectively.

Remark 2.4. (i) We have $\mathrm{F}_\infty \Rightarrow \cdots \Rightarrow \mathrm{F}_{n+1} \Rightarrow \mathrm{F}_n \Rightarrow \cdots \Rightarrow \mathrm{F}_0$. All those implications are strict.

(ii) Every group is of type F_0.

(iii) A group is of type F_1 if and only if it is finitely generated.

(iv) A group is of type F_2 if and only if it is finitely presented.

2.2 Definition of L^2-Betti numbers

We now define the L^2-Betti numbers of a free G-CW-complex Y of type F_∞.

Let $C_*^{\mathrm{cell}}(Y)$ be the cellular chain complex of Y over $\mathbb{C}$: for each degree $n \in \mathbb{N}_{\geq 0}$, $C_n^{\mathrm{cell}}(Y)$ is the $\mathbb{C}$-vector space with basis the set of n-cells of Y. The action $G \curvearrowright Y$ induces $G \curvearrowright C_*^{\mathrm{cell}}(Y)$, which gives $C_*^{\mathrm{cell}}(Y)$ the structure of a chain complex over $\mathbb{C}G$. The L^2-*cellular chain complex* of Y is defined by

$$C_*^{(2)}(G \curvearrowright Y) := \ell^2 G \otimes_{\mathbb{C}G} C_*^{\mathrm{cell}}(Y),$$

where $\ell^2 G$ is equipped with the action of $\mathbb{C}G$ by multiplication on the right. The L^2-boundary maps are defined by

$$\partial_n^{(2)} := \mathrm{id}_{\ell^2 G} \otimes \partial_n^{\mathrm{cell}} : C_n^{(2)}(G \curvearrowright Y) \to C_{n-1}^{(2)}(G \curvearrowright Y).$$

This makes $C_*^{(2)}(G \curvearrowright Y)$ a chain complex.

Definition 2.5. The L^2-*homology* of a free G-CW-complex Y is defined by

$$H_n^{(2)}(G \curvearrowright Y) := \operatorname{Ker} \partial_n^{(2)} / \overline{\operatorname{Im} \partial_{n+1}^{(2)}}.$$

Proposition 2.6. *If $G \curvearrowright Y$ is of type F_∞, then $H_n^{(2)}(G \curvearrowright Y)$ is a Hilbert G-module.*

Proof. Fix $n \geq 0$. The n-th chain group $C_n^{(2)}(G \curvearrowright Y)$ can be described as follows. Pick a collection $\{\sigma_i\}_{i \in I}$ of n-cells of Y whose orbits are disjoint and cover the n-skeleton of Y. The set I can be chosen finite since $G \curvearrowright Y$ is of type F_∞.

We have

$$C_n^{\mathrm{cell}}(Y) = \bigoplus_{i \in I} \bigoplus_{g \in G} \mathbb{C}(g \cdot \sigma_i) = \bigoplus_{i \in I} \mathbb{C}G[\sigma_i],$$

and therefore

$$C_n^{(2)}(G \curvearrowright Y) = \bigoplus_{i \in I} \ell^2 G[\sigma_i].$$

It is now clear that $C_n^{(2)}(G \curvearrowright Y)$ is a Hilbert G-module (with an embedding into $(\ell^2 G)^{|I|}$). Moreover, the L^2-boundary maps $\partial_n^{(2)}$ are morphisms of Hilbert G-modules.

Hence, the result is a consequence of the general fact that, if $\varphi : V \to W$ is a morphism of Hilbert G-modules, then $\operatorname{Ker} \varphi$ and $W / \overline{\operatorname{Im} \varphi}$ are Hilbert G-modules. $\square$

Definition 2.7. Let Y be a free G-CW-complex of type F_∞. For $n \in \mathbb{N}_{\geq 0}$, the n-th L^2-*Betti number* of $G \curvearrowright Y$ is

$$b_n^{(2)}(G \curvearrowright Y) := \dim_{NG} H_n^{(2)}(G \curvearrowright Y).$$

In order to define the L^2-Betti numbers of a group, we must make sure that L^2-Betti numbers are invariant under homotopy equivalence, so that they do not depend on the choice of a $K(G,1)$ space.

Proposition 2.8. *Let Y_1, Y_2 be free G-CW-complexes. If $f : Y_1 \to Y_2$ is a G-equivariant homotopy equivalence, then for all $n \in \mathbb{N}_{\geq 0}$,*

$$b_n^{(2)}(G \curvearrowright Y_1) = b_n^{(2)}(G \curvearrowright Y_2).$$

Proof. The map f induces a $\mathbb{C}G$-chain homotopy equivalence

$$f_* : C_*^{\text{cell}}(Y_1) \xrightarrow{\simeq} C_*^{\text{cell}}(Y_2),$$

which then induces a chain homotopy equivalence in the category of Hilbert G-modules

$$C_*^{(2)}(G \curvearrowright Y_1) \xrightarrow{\simeq} C_2^{(2)}(G \curvearrowright Y_2). \qquad \square$$

Definition 2.9. Let G be a group of type F_∞. For $n \in \mathbb{N}_{\geq 0}$, the n-th *L^2-Betti number* of G is

$$b_n^{(2)}(G) \coloneqq b_n^{(2)}\left(G \curvearrowright \tilde{X}\right),$$

where $\tilde{X}$ is the universal cover of a $K(G,1)$ space.

It follows from Proposition 2.8 and the uniqueness of $K(G,1)$ spaces up to homotopy that $b_n^{(2)}(G)$ does not depend on the choice of a $K(G,1)$ space.

From now on, we will focus on L^2-Betti numbers of groups.

Remark 2.10. (i) An alternative approach would have been to start with a projective resolution of $\mathbb{C}$ by $\mathbb{C}G$-modules, and then to apply $\ell^2 G \otimes_{\mathbb{C}G} -$.

(ii) One could also have defined the L^2-cochain complex of $G \curvearrowright Y$ by

$$C^*_{(2)}(G \curvearrowright Y) \coloneqq \mathrm{Hom}_{\mathbb{C}G}\left(C_*^{\text{cell}}(Y), \ell^2 G\right),$$

and take $H^*_{(2)}(G \curvearrowright Y)$ to be the cohomology of this cochain complex. In fact, this leads to isomorphisms of Hilbert G-modules

$$H^*_{(2)}(G \curvearrowright Y) \cong H_*^{(2)}(G \curvearrowright Y),$$

so that homological and cohomological L^2-Betti numbers are equal.

2.3 Basic properties

We start by computing $b_0^{(2)}$.

Proposition 2.11. *Let G be a group of type* F_∞*. Then*

$$b_0^{(2)}(G) = \frac{1}{|G|},$$

with the convention $1/\infty = 0$*.*

Proof. We construct a $K(G,1)$ space as in §2.1, and obtain isomorphisms

$$C_0^{(2)}\left(G \curvearrowright \tilde{X}\right) \cong \ell^2 G \qquad \text{and} \qquad C_1^{(2)}\left(G \curvearrowright \tilde{X}\right) \cong \bigoplus_{s \in S} \ell^2 G[s],$$

where S is a generating set for G, and the boundary map is given by $\partial_1^{(2)}(s) = s - e$. Hence,

$$H_0^{(2)}\left(G \curvearrowright \tilde{X}\right) = \ell^2 G / \overline{\langle x - gx \mid x \in \ell^2 G,\ g \in G\rangle_{\mathbb{C}}}.$$

- If G is finite, then $\ell^2 G = \mathbb{C}G$ and $H_0^{(2)}\left(G \curvearrowright \tilde{X}\right) = \mathbb{C}$ (with G acting trivially), so

$$b_0^{(2)}(G) = \dim_{NG} \mathbb{C} = \frac{1}{|G|}.$$

- If G is infinite, we will show that $H_0^{(2)}\left(G \curvearrowright \tilde{X}\right) = 0$, or equivalently that the dual of $H_0^{(2)}\left(G \curvearrowright \tilde{X}\right)$ is trivial. This amounts to proving that, if $f : \ell^2 G \to \mathbb{C}$ is $\mathbb{C}$-linear, bounded, and zero on $\langle x - gx \mid x \in \ell^2 G,\ g \in G\rangle_{\mathbb{C}}$ (i.e. f is left-G-invariant), then $f = 0$. As G is infinite and countable, we can enumerate $G = \{g_n\}_{n \geq 1}$, and consider $x = \sum_n \frac{1}{n} g_n \in \ell^2 G$. We have

$$f(x) = \sum_{n \geq 1} \frac{1}{n} f(g_n) = \sum_{n \geq 1} \frac{1}{n} f(e).$$

  Therefore, $f(e) = 0$, so $f(g) = 0$ for all $g \in G$ since f is G-invariant, and $f = 0$ by linearity and continuity. □

We now give basic properties that will be useful for computations of L^2-Betti numbers.

Proposition 2.12. *Let G and H be two groups of type* F_∞ *and $n \geq 0$.*

(i) (Dimension) *If G has a $K(G,1)$ space of dimension $\leq n-1$, then*

$$b_n^{(2)}(G) = 0.$$

(ii) (Restriction) *If H is a finite-index subgroup of G, then*

$$b_n^{(2)}(H) = [G : H] \cdot b_n^{(2)}(G)$$

(iii) (Künneth formula)

$$b_n^{(2)}(G \times H) = \sum_{j=0}^{n} b_j^{(2)}(G) \cdot b_{n-j}^{(2)}(H).$$

(iv) (Additivity)

$$b_1^{(2)}(G * H) = b_1^{(2)}(G) + b_1^{(2)}(H) + 1 - b_0^{(2)}(G) - b_0^{(2)}(H)$$

and moreover, if $n \geq 2$,

$$b_n^{(2)}(G * H) = b_n^{(2)}(G) + b_n^{(2)}(H).$$

(v) (Poincaré duality) *If G has a $K(G,1)$ space which is an orientable closed connected manifold of dimension d, then*

$$b_n^{(2)}(G) = b_{d-n}^{(2)}(G).$$

(vi) (Euler characteristic) *If G has a $K(G,1)$ space with a finite number of cells, then*

$$\chi(G) = \sum_{n \geq 0} (-1)^n b_n^{(2)}(G).$$

Proof. (i) If X is a $K(G,1)$ space of dimension $\leq n-1$, then $C_n^{(2)}\left(G \curvearrowright \tilde{X}\right) = 0$ and $H_n^{(2)}\left(G \curvearrowright \tilde{X}\right) = 0$.

(ii) Let X be a $K(G,1)$ space, and let $X_H \to X$ be the covering associated to the subgroup $H \leq G$. Hence, X_H is a $K(H,1)$ and $\tilde{X}$ is the common universal cover of X and X_H. Therefore, $C_*^{(2)}\left(H \curvearrowright \tilde{X}\right)$ is obtained from $C_*^{(2)}\left(G \curvearrowright \tilde{X}\right)$ by applying the restriction functor Res_H^G. The result now follows from Proposition 1.13(vi).

(iii) If X is $K(G,1)$ space and Y is a $K(H,1)$ space, then $X \times Y$ is a $K(G \times H, 1)$ space. The rest of the proof is similar to that of the usual Künneth formula, using 1.13(v).

(iv) If X is a $K(G,1)$ space and Y is a $K(H,1)$ space, then $X \vee Y$ is a $K(G * H, 1)$. We then use a Mayer–Vietoris-type argument.

(v) This uses a Poincaré duality "twisted" by the action of G, and the fact that L^2-Betti numbers can also be computed in terms of cohomology (see Remark 2.10(ii)).

(vi) The main ingredient is an "L^2-rank-nullity theorem" : if $\varphi : V \to W$ is a morphism of Hilbert G-modules, then

$$\dim_{NG}(V) - \dim_{NG}(W) = \dim_{NG}(\operatorname{Ker}\varphi) - \dim_{NG}\left(W/\overline{\operatorname{Im}\varphi}\right).$$

It follows that, for $n \geq 0$,

$$b_n^{(2)}(G) = \dim_{NG}\left(\operatorname{Ker}\partial_n^{(2)}\right) + \dim_{NG}\left(\operatorname{Ker}\partial_{n+1}^{(2)}\right) - \dim_{NG}\left(C_{n+1}^{(2)}\left(G \curvearrowright \tilde{X}\right)\right).$$

Therefore,

$$\sum_{n\geq 0}(-1)^n b_n^{(2)}(G) = \sum_{n\geq 0}(-1)^n \dim_{NG}\left(C_n^{(2)}\left(G \curvearrowright \tilde{X}\right)\right).$$

But $\dim_{NG}\left(C_n^{(2)}\left(G \curvearrowright \tilde{X}\right)\right)$ is the number of n-cells of X, so the above sum is equal to $\chi(X) = \chi(G)$. □

Remark 2.13. Among the properties listed in Proposition 2.12, items (i), (iii), (iv), (v) et (vi) are also true for usual Betti numbers (defined by $b_n(G) := \dim_{\mathbb{C}} H_n(G)$). So far, the only property that is specific to the L^2 world is the *restriction formula* (ii).

2.4 Some examples

We now give explicit computations of L^2-Betti numbers in a few simple cases.

Example 2.14 (Finite groups). Let G be a finite group. Then

$$b_n^{(2)}(G) = \begin{cases} \frac{1}{|G|} & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}.$$

Proof. Note that the trivial group $\{1\}$ has index $|G|$ in G; its L^2-Betti numbers are

$$b_n^{(2)}(\{1\}) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}.$$

(Indeed, the trivial group has a $K(G,1)$ space of dimension 0, and the case $n = 0$ comes from Proposition 2.11.) Now the result follows from the restriction formula (2.12(ii)). □

Example 2.15 ($\mathbb{Z}$).

$$b_n^{(2)}(\mathbb{Z}) = 0 \text{ for all } n \in \mathbb{N}_{\geq 0}.$$

Proof. The circle S^1 is a $K(\mathbb{Z},1)$ space. Since $\dim S^1 = 1$, we have $b_n^{(2)}(\mathbb{Z}) = 0$ for $n \geq 2$ (by 2.12.(i)). Moreover, $b_0^{(2)}(\mathbb{Z}) = 0$ since $\mathbb{Z}$ is infinite (2.11). We can then compute $b_1^{(2)}(\mathbb{Z})$ in several different manners:

- *Explicit computation.* There is a cellular structure on S^1 with one 0-cell and one 1-cell. Therefore, $C_0^{(2)}(\mathbb{Z} \curvearrowright S^1) = \ell^2\mathbb{Z}$, and $C_1^{(2)}(\mathbb{Z} \curvearrowright S^1) = \ell^2\mathbb{Z}$, and $C_n^{(2)}(\mathbb{Z} \curvearrowright S^1) = 0$ for $n \geq 2$. Denoting by t a generator of $\mathbb{Z}$, the boundary map $\partial_1^{(2)}$ is given by

$$\partial_1^{(2)}(x) = (t-1)x.$$

Hence, we see that $H_1^{(2)}(\mathbb{Z} \curvearrowright S^1) = \operatorname{Ker} \partial_1^{(2)} = 0$.

- *Euler characteristic.* By Proposition 2.12.(vi), we have

$$-b_1^{(2)}(\mathbb{Z}) = \chi(\mathbb{Z}) = 0.$$

- *Finite-index subgroups.* For all $d \geq 1$, the group $\mathbb{Z}$ contains an index-d subgroup isomorphic to $\mathbb{Z}$, so the restriction formula (2.12.(ii)) yields

$$b_n^{(2)}(\mathbb{Z}) = d \cdot b_n^{(2)}(\mathbb{Z}).$$

It follows that $b_n^{(2)}(\mathbb{Z}) = 0$ for all $n \geq 0$. □

Example 2.16 (Free groups). Let F_r be the free group of rank $r \geq 1$. Then

$$b_n^{(2)}(F_r) = \begin{cases} 0 & \text{if } n = 0 \\ r-1 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}.$$

Proof. We have $b_0^{(2)}(F_r) = 0$ since F_r is infinite (2.11). Moreover, $b_n^{(2)}(F_r) = 0$ for $n \geq 2$ because F_r has a $K(G,1)$ space of dimension 1 (2.12.(i)). Here are two different computations of $b_1^{(2)}(F_r)$:

- By additivity of L^2-Betti numbers (2.12.(iv)), we have

$$b_1^{(2)}(F_r) = b_1^{(2)}(\mathbb{Z}) + b_1^{(2)}(F_{r-1}) + 1 - b_0^{(2)}(\mathbb{Z}) - b_0^{(2)}(F_{r-1}) = b_1^{(2)}(F_{r-1}) + 1.$$

We conclude by induction using $b_1^{(2)}(F_1) = b_1^{(2)}(\mathbb{Z}) = 0$ (2.15).

- By considering the Euler characteristic (2.12.(iv)), we have

$$-b_1^{(2)}(F_r) = \chi(F_r) = 1 - r.$$ □

Remark 2.17. The wedge of two circles $S^1 \vee S^1$ is a $K(F_2, 1)$ space; its universal cover is the degree-4 regular tree T. Example 2.16 shows that $b_1^{(2)}(F_r) = 1$. Figure 1 shows an explicit 1-cycle in $C_1^{(2)}(F_2 \curvearrowright T)$.

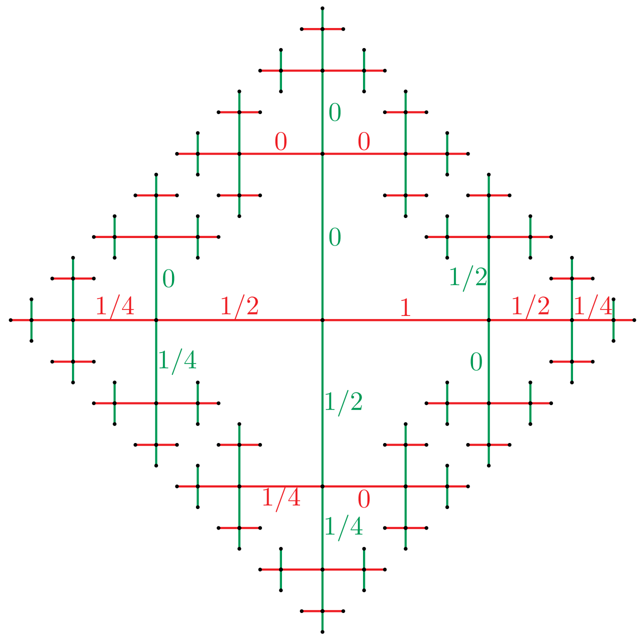

Figure 1: An L^2-1-cycle for $F_2 \curvearrowright T$.

Example 2.18 (Surface groups). Let Σ_g be the orientable closed connected genus-g surface, for $g \geq 1$. Then

$$b_n^{(2)}(\pi_1 \Sigma_g) = \begin{cases} 0 & \text{if } n = 0 \\ 2(g-1) & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases} .$$

Proof. The group $\pi_1 \Sigma_g$ is infinite, so $b_0^{(2)}(\pi_1 \Sigma_g) = 0$ (2.11), and Σ_g is a $K(G, 1)$ space of dimension 2, so $b_n^{(2)}(\pi_1 \Sigma_g) = 0$ for $n \geq 3$ (2.12.(i)). Moreover, Poincaré duality (2.12.(v)) implies that

$$b_2^{(2)}(\pi_1 \Sigma_g) = b_0^{(2)}(\pi_1 \Sigma_g) = 0.$$

Finally, $-b_1^{(2)}(\pi_1 \Sigma_g) = \chi(\Sigma_g) = 2 - 2g$ (2.12.(iv)). □

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