L^2-Betti numbers

Reading seminar

Kyoto University

Spring 2025

Contents

Notes by Alexis Marchand.

Talk 1 – Background on the von Neumann dimension

Alexis Marchand

References: [14, Chapter 2], [15], [19, Chapter 1], [18, §1.1].

1.1 Motivation

The goal of what follows is to develop a good *equivariant* homology theory for actions $G \curvearrowright X$ of groups on topological spaces. The usual singular chain complex $C_*^{\mathrm{sing}}(X;\mathbb{C})$ and singular homology $H_*(X;\mathbb{C})$ inherit a G-action, so they have the structure of $\mathbb{C}G$-modules. However, the group G is typically infinite and we do not have a good notion of dimension for modules over $\mathbb{C}G$. This is why we will work in an L^2 setting.

We will introduce a homology theory $H_*^{(2)}(G \curvearrowright X)$, together with associated Betti numbers $b_*^{(2)}(G \curvearrowright X)$. They will be well-defined when X is a G-CW-complex under certain finiteness conditions.

In the first talk, we introduce the relevant notions around Hilbert modules and von Neumann dimension that will allow us to define L^2-Betti numbers.

1.2 Hilbert G-modules

We fix a countable group G. We will work with $\mathbb{C}$-coefficients throughout.

Definition 1.1. The *group ring* of G over $\mathbb{C}$ is the $\mathbb{C}$-algebra $\mathbb{C}G$ (or $\mathbb{C}[G]$), with underlying $\mathbb{C}$-vector space

$$\mathbb{C}G \coloneqq \bigoplus_{g\in G} \mathbb{C}g,$$

with multiplication defined on the basis vectors by $g \cdot h = gh$.

Example 1.2.

- $\mathbb{C}[\mathbb{Z}] = \mathbb{C}\left[t, t^{-1}\right]$ is the ring of Laurent polynomials over $\mathbb{C}$.
- For $n \in \mathbb{N}_{\geq 1}$, $\mathbb{C}\left[\mathbb{Z}/n\right] = \mathbb{C}[t]/\left(t^n - 1\right)$.

The group ring $\mathbb{C}G$ can be equipped with a natural inner product $\langle\cdot,\cdot\rangle$ defined by

$$\left\langle \sum_{g\in G} a_g g, \sum_{g\in G} b_g g \right\rangle \coloneqq \sum_{g\in G} \bar{a}_g b_g$$

The completion of $\mathbb{C}G$ with respect to $\langle\cdot,\cdot\rangle$ is a complex Hilbert space, which we denote by $\ell^2 G$; it can also be defined as the $\mathbb{C}$-vector space of ℓ^2-summable functions $G \to \mathbb{C}$.

Note that $\ell^2 G$ has the structure of a $\mathbb{C}G$-module, with action given by

$$h \cdot \sum_{g\in G} a_g g \coloneqq \sum_{g\in G} a_{gh} g.$$

Example 1.3.

- If G is finite, then $\ell^2 G = \mathbb{C}G$.
- If $G = \mathbb{Z}$, by Fourier analysis, there is an isomorphism $\ell^2 G \cong L^2([-\pi, \pi], \mathbb{C})$ given by
$$\sum_{n \in \mathbb{Z}} a_n t^n \longmapsto \left(x \mapsto \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} a_n e^{inx} \right).$$

Since the group G is assumed to be countable, the Hilbert space $\ell^2 G$ is separable.

Definition 1.4. A *Hilbert G-module* is a complex Hilbert space V with a $\mathbb{C}$-linear isometric (left) G-action such that there is an isometric G-embedding
$$V \hookrightarrow \left(\ell^2 G\right)^n$$
for some $n \in \mathbb{N}_{\geq 1}$.

A *morphism* between two Hilbert G-modules V and W is a G-equivariant bounded $\mathbb{C}$-linear map $V \to W$.

Our homology groups will be Hilbert G-modules; our main task will be to define a notion of dimension for such modules.

1.3 Background on von Neumann algebras

Let $\mathcal{H}$ be a complex Hilbert space. Then the space $B(\mathcal{H})$ of bounded linear operators $\mathcal{H} \to \mathcal{H}$ is a $\mathbb{C}$-algebra, with multiplication given by composition.

Recall that, given $u \in B(\mathcal{H})$, there is a unique $u^* \in B(\mathcal{H})$ — called the *adjoint* of f — such that, for all $x, y \in \mathcal{H}$,
$$\langle u(x), y \rangle = \langle x, u^*(y) \rangle .$$
(This follows from the Riesz Representation Theorem applied to the linear form $\langle u(\cdot), y \rangle$ for fixed $y \in \mathcal{H}$.) Hence, $\cdot^*$ defines an involution on $B(\mathcal{H})$; this turns the latter into a $*$-algebra.

There are several topologies that one can define on $B(\mathcal{H})$:

- The *norm topology* (or topology of uniform convergence), given by
$$u_n \xrightarrow{\|\cdot\|} u \overset{\text{def}}{\iff} \|u_n - u\| \to 0,$$
- The *strong topology* (or topology of pointwise convergence), given by
$$u_n \xrightarrow{s} u \overset{\text{def}}{\iff} \forall x \in \mathcal{H},\ \|u_n(x) - u(x)\| \to 0,$$
- The *weak topology*, given by
$$u_n \xrightarrow{w} u \overset{\text{def}}{\iff} \forall x, y \in \mathcal{H},\ \langle u_n(x), y \rangle \to \langle u(x), y \rangle .$$

Definition 1.5. A *von Neumann algebra* is a unital weakly closed $*$-subalgebra of $B(\mathcal{H})$ for some complex Hilbert space $\mathcal{H}$.

Given a subset $S \subseteq B(\mathcal{H})$, its *commutant* is defined by

$$S' \coloneqq \{u \in B(\mathcal{H}) \mid \forall s \in S,\ us = su\}.$$

The *bicommutant* of S is simply $S'' \coloneqq (S')'$.

The following theorem is a fundamental structural result for von Neumann algebras:

Theorem 1.6 (von Neumann Bicommutant Theorem)**.** *Let $\mathcal{H}$ be a complex Hilbert space and let $A \subseteq B(\mathcal{H})$ be a unital $*$-subalgebra of $B(\mathcal{H})$. Then the following are equivalent:*

(i) $A'' = A$.

(ii) *A is strongly closed.*

(iii) *A is weakly closed.*

1.4 The group von Neumann algebra

We come back to the setup of §1.2: G is a countable group and we are considering the Hilbert space ℓ^2G. As above, we denote by $B(\ell^2G)$ the $\mathbb{C}$-algebra of bounded linear operators $\ell^2G \to \ell^2G$.

Observe that there are two embeddings

$$\lambda, \rho : \mathbb{C}G \hookrightarrow B\left(\ell^2G\right)$$

given by the respective actions of $\mathbb{C}G$ on ℓ^2G by left and right multiplication.

Proposition/Definition 1.7. *The following subsets of $B(\ell^2G)$ are all equal:*

(i) *The weak closure of $\rho(\mathbb{C}G)$,*

(ii) *The strong closure of $\rho(\mathbb{C}G)$,*

(iii) *The bicommutant of $\rho(\mathbb{C}G)$,*

(iv) *The set of $u \in B(\ell^2G)$ that are* left *$\mathbb{C}G$-equivariant, i.e. $\lambda(\mathbb{C}G)'$.*

This set is called the (right) group von Neumann algebra *of G, and denoted by* $\mathcal{N}G$.

Proof. The equalities (i) = (ii) = (iii) follow from the Bicommutant Theorem (1.6).

We first show that (ii) $\subseteq$ (iv). It is clear that $\rho(\mathbb{C}G) \subseteq$ (iv), so it suffices to prove that (iv) is strongly closed. Let $(u_n)_{n\geq 1}$ be a sequence of left $\mathbb{C}G$-equivariant bounded linear operators on ℓ^2G, converging to $u \in B(\ell^2G)$. For all $a \in \mathbb{C}G$ and $x \in \ell^2G$, we have

$$a \cdot u(x) = a \cdot \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} (a \cdot u_n(x)) = \lim_{n\to\infty} u_n(ax) = u(ax),$$

so u is also left $\mathbb{C}G$-equivariant. This proves that (iv) is sequentially closed in the strong topology. The same proof, after replacing sequences with nets, shows that (iv) is strongly closed.

Conversely, we show that (iv) $\subseteq$ (iii). We consider the operator $J : \ell^2 G \to \ell^2 G$ defined by

$$J : \sum_{g\in G} a_g g \mapsto \sum_{g\in G} \bar{a}_{g^{-1}} g.$$

Claim. (i) $J \circ J = \mathrm{id}$.

(ii) *$J \circ \lambda(x) \circ J = \rho(Jx)$ for all $x \in \mathbb{C}G$. In particular, $J\lambda(\mathbb{C}G) J = \rho(\mathbb{C}G)$.*

(iii) *For all $u \in \lambda(\mathbb{C}G)'$, we have $J \circ u(e) = u^*(e)$.*

Proof of the claim. (i) This is clear.

(ii) The equality follows from a simple computation.

(iii) A computation shows that, for all $x, y \in \mathbb{C}G$,

$$\langle Jx, y\rangle = \langle e, xy\rangle = \langle e, yx\rangle = \langle Jy, x\rangle . \tag{$*$}$$

Note that ($*$) also holds for $x \in \mathbb{C}G$ and $y \in \ell^2 G$ by density. Now take $u \in \lambda(\mathbb{C}G)'$. Using ($*$), we have for all $x \in \mathbb{C}G$,

$$\langle J \circ u(e), x\rangle = \langle e, \lambda(x) \circ u(e)\rangle = \langle e, u \circ \lambda(x)(e)\rangle = \langle u^*(e), \lambda(x)(e)\rangle = \langle u^*(e), x\rangle .$$

Hence, the linear forms $\langle J \circ u(e), -\rangle$ and $\langle u^*(e), -\rangle$ agree on $\mathbb{C}G$ and therefore on $\ell^2 G$ by density; it follows that $J \circ u(e) = u^*(e)$. □

Using the above, we are now ready to show that $\lambda(\mathbb{C}G)' \subseteq \rho(\mathbb{C}G)''$; this will prove the inclusion (iv) $\subseteq$ (iii). Proving that $\lambda(\mathbb{C}G)' \subseteq \rho(\mathbb{C}G)''$ amounts to showing that every $u \in \lambda(\mathbb{C}G)'$ and $v \in \rho(\mathbb{C}G)'$ commute. But by (ii) of the claim, we have

$$\rho(\mathbb{C}G)' = (J\lambda(\mathbb{C}G) J)' = J\lambda(\mathbb{C}G)' J.$$

Hence, we write $v = JwJ$ with $w \in \lambda(\mathbb{C}G)'$. Let $x \in \ell^2 G$. Using repeatedly (iii) of the claim, together with the facts that $\rho(\mathbb{C}G) \subseteq \lambda(\mathbb{C}G)'$ and that $\left(\lambda(\mathbb{C}G)'\right)^* = \lambda(\mathbb{C}G)'$, we obtain

$$\begin{aligned} u(JwJ)x &= uJwJ\rho(x)e = uJw\rho(x)^*e = u\rho(x)w^*e, \\ (JwJ)ux &= JwJu\rho(x)e = Jw\rho(x)^*u^*e = u\rho(x)w^*e, \end{aligned}$$

proving that $u(JwJ) = (JwJ)u$ as wanted. □

Thanks are due to Hiroto Nishikawa for explaining the proof of (iv) $\subseteq$ (iii).

1.5 The trace on $\mathrm{N}G$

In order to define a notion of dimension for Hilbert G-modules, the basic idea is that, in a finite-dimensional Hilbert space, the dimension of a subspace is equal to the trace of the orthogonal projection onto that subspace.

Our next step is therefore to equip $\mathrm{N}G$ with a trace.

Definition 1.8. The *trace* on $\mathrm{N}G$ is the map $\mathrm{tr}_G : \mathrm{N}G \to \mathbb{C}$ given by

$$\mathrm{tr}_G : a \mapsto \langle e, a(e) \rangle ,$$

where $e \in \mathbb{C}G \subseteq \ell^2 G$ is the atomic function at the identity $e \in G$.

Proposition 1.9. *The following properties hold for all $a, b \in \mathrm{N}G$:*

(i) (Trace property) $\mathrm{tr}_G(a \circ b) = \mathrm{tr}_G(b \circ a)$.

(ii) (Faithfulness) $\mathrm{tr}_G (a^* \circ a) = 0$ *if and only if* $a = 0$.

(iii) (Positivity) *Suppose that $a \geq 0$, in the sense that $\forall x \in \ell^2 G$, $\langle x, a(x) \rangle \geq 0$. Then* $\mathrm{tr}_G(a) \geq 0$.

Proof. (i) Note that, for $a = \sum_g a_g g \in \mathbb{C}G$, we have $\mathrm{tr}_G(a) = a_e$. Moreover, for $a, b \in \mathbb{C}G$, the composition $a \circ b$ acts on $\ell^2 G$ as the product ba (because $\mathbb{C}G$ acts on $\ell^2 G$ by *right* multiplication!), so $\mathrm{tr}_G(a \circ b)$ is equal to the coefficient of e in ba:

$$\mathrm{tr}_G(a \circ b) = \sum_{\substack{g,h \in G \\ gh = e}} b_g a_h .$$

This is symmetric in a and b, and hence equal to $\mathrm{tr}_G(b \circ a)$. This proves the trace property for $a, b \in \mathbb{C}G$, which extends by continuity to $\mathrm{N}G$.

(ii) Let $a \in \mathrm{N}G$ with $\mathrm{tr}_G (a^* \circ a) = 0$. Then

$$0 = \langle e, a^* \circ a(e) \rangle = \langle a(e), a(e) \rangle ,$$

so $a(e) = 0$. By G-equivariance, we have $a(g) = g \cdot a(e) = 0$ for all $g \in G$. It follows by linearity that a is 0 on $\mathbb{C}G$, and by continuity that a is 0 on $\mathrm{N}G$.

(iii) This is clear. $\square$

Given a matrix $A \in M_{n \times n}(\mathrm{N}G)$, we define

$$\mathrm{tr}_G(A) \coloneqq \sum_{j=1}^{n} \mathrm{tr}_G (A_{jj}) .$$

Usual linear algebra shows that this trace also satisfies Proposition 1.9.

Now any bounded left G-equivariant map $(\ell^2 G)^n \to (\ell^2 G)^n$ is represented by a matrix in $M_{n \times n}(\mathrm{N}G)$ and hence has a trace.

1.6 The von Neumann dimension

Let G be a countable group.

Proposition/Definition 1.10. *Let V be a Hilbert G-module. The* von Neumann-G-dimension *of V is defined by*

$$\dim_{NG} V := \operatorname{tr}_G(p),$$

where $i : V \hookrightarrow (\ell^2 G)^n$ is a choice of isometric G-embedding for some $n \in \mathbb{N}_{\geq 1}$ and $p : (\ell^2 G)^n \to (\ell^2 G)^n$ is the orthogonal projection onto the closed subspace $i(V)$.

This is independent of the choice of i, and $\dim_{NG} V \in \mathbb{R}_{\geq 0}$.

Proof. Let $j : V \hookrightarrow (\ell^2 G)^m$ be another isometric G-embedding, with $m \in \mathbb{N}_{\geq 1}$, and let $q : (\ell^2 G)^m \to (\ell^2 G)^m$ be the orthogonal projection onto $j(V)$.

Define a map $u : (\ell^2 G)^n \to (\ell^2 G)^m$ by $u_{|\operatorname{Im} i} := j \circ i^{-1}$ and $u_{|(\operatorname{Im} i)^\perp} := 0$. By construction, $j = u \circ i$; it follows that $q = p \circ u^*$. Hence,

$$\operatorname{tr}_G(q) = \operatorname{tr}_G(u \circ q) = \operatorname{tr}_G(u \circ p \circ u^*) = \operatorname{tr}_G(p \circ u^* \circ u) = \operatorname{tr}_G(p \circ p) = \operatorname{tr}_G(p).$$

To see that $\dim_{NG} V \in \mathbb{R}_{\geq 0}$, note that p is a positive operator, so the diagonal entries of its matrix are also positive operators; the result follows from positivity of the trace. □

We give two examples of computations of von Neummann dimensions.

Example 1.11 (Finite groups)*.* If G is a finite group, then $\mathbb{C}G = \ell^2 G = NG$. A Hilbert G-module V has finite-dimension over $\mathbb{C}$ and satisfies

$$\dim_{NG} V = \frac{1}{|G|} \dim_{\mathbb{C}} V.$$

Example 1.12 ($\mathbb{Z}$)*.* If $G = \mathbb{Z}$, then $\ell^2 G \cong L^2([-\pi, \pi], \mathbb{C})$ (see Example 1.3), and

$$NG \cong L^\infty([-\pi, \pi], \mathbb{C}),$$

with the action of NG on $\ell^2 G$ given by pointwise multiplication.

Under this isomorphism, $\operatorname{tr}_G : L^\infty([-\pi, \pi], \mathbb{C})$ is given by

$$\operatorname{tr}_G : f \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} f \, d\lambda.$$

Now let $A \subseteq [-\pi, \pi]$ be a measurable set, and consider

$$V := \left\{ f \cdot \chi_A \,\middle|\, f \in L^2([-\pi, \pi], \mathbb{C}) \right\} \subseteq L^2([-\pi, \pi], \mathbb{C}) \cong \ell^2 G.$$

This is a Hilbert-G-module (embedding into $\ell^2 G$). The orthogonal projection onto A is represented by the matrix $(\chi_A) \in M_{1\times 1}(NG)$. Therefore,

$$\dim_{NG} V = \operatorname{tr}_G(\chi_A) = \frac{1}{2\pi} \lambda(A).$$

In particular, every number in $[0, 1]$ occurs as a von Neumann dimension!

We finish with some basic properties of the von Neumann dimension.

Proposition 1.13. *The von Neumann dimension has the following properties.*

(i) (Normalisation) $\dim_{\mathrm{N}G} \ell^2 G = 1$.

(ii) (Faithfulness) *For every Hilbert G-module V, we have $\dim_{\mathrm{N}G} V = 0$ if and only if $V = 0$.*

(iii) (Weak isomorphism invariance) *If $f : V \to W$ is a morphism of Hilbert G-modules with* $\operatorname{Ker} f = 0$ *and* $\overline{\operatorname{Im} f} = W$*, then* $\dim_{\mathrm{N}G} V = \dim_{\mathrm{N}G} W$*.*

(iv) (Additivity) *Assume that the sequence of Hilbert G-modules*

$$0 \to V_1 \xrightarrow{i} V_2 \xrightarrow{\pi} V_3 \to 0$$

is weakly exact, in the sense that i is injective, $\overline{\operatorname{Im} i} = \operatorname{Ker} \pi$*, and* $\overline{\operatorname{Im} \pi} = V_3$*. Then*

$$\dim_{\mathrm{N}G} V_2 = \dim_{\mathrm{N}G} V_1 + \dim_{\mathrm{N}G} V_3.$$

(v) (Multiplicativity) *Let H be another countable group. Let V be a Hilbert G-module and W a Hilbert H-module. Then the completed tensor product $V \bar{\otimes}_{\mathbb{C}} W$ is a Hilbert $G \times H$-module, and*

$$\dim_{\mathrm{N}(G\times H)} (V \bar{\otimes}_{\mathbb{C}} W) = \dim_{\mathrm{N}G} V \cdot \dim_{\mathrm{N}H} W.$$

(vi) (Restriction) *Let V be a Hilbert G-module and let $H \leq G$ be a finite-index subgroup. Then V is naturally a Hilbert H-module, and*

$$\dim_{\mathrm{N}H} \operatorname{Res}^G_H V = [G : H] \cdot \dim_{\mathrm{N}G} V.$$

Sketch of proof. (i) This is clear (taking $\ell^2 G \hookrightarrow (\ell^2 G)^1$ and $p = \mathrm{id}$).

(ii) This follows from faithfulness of the von Neumann trace (1.9).

(iii) This is a consequence of *polar decomposition*: the map f can be written as $f = u \circ p$, where u is a partial isometry and p is a positive operator with $\operatorname{Ker} u = \operatorname{Ker} p$. In this case, f is injective, so $\operatorname{Ker} u = \operatorname{Ker} p = 0$; moreover, u has closed image and so $\operatorname{Im} u = \overline{\operatorname{Im} u} = \overline{\operatorname{Im} f} = W$. Hence, u is an isometry, which is G-equivariant by uniqueness of the polar decomposition.

(iv) Note first that $\dim_{\mathrm{N}G}$ is additive with respect to direct sums, and define a weak isomorphism $V \to \overline{\operatorname{Im} i} \oplus V_3$ by $x \mapsto p(x) \oplus \pi(x)$, where $p : V \to \overline{\operatorname{Im} i}$ is the orthogonal projection.

(v) The key fact is that there is an isomorphism $\ell^2 (G \times H) \cong \ell^2 G \bar{\otimes}_{\mathbb{C}} \ell^2 H$ of Hilbert $G \times H$-modules. □

Talk 2 – L^2-homology and L^2-Betti numbers

Alexis Marchand

References: [14, (parts of) Chapters 3-4], [19, Chapter 2], [18, §1.2].

2.1 Eilenberg–MacLane space and finiteness properties

Let G be a discrete group. We will study the (co)homology of G via its Eilenberg–MacLane space.

Definition 2.1. An *Eilenberg–MacLane space* for G — or $K(G,1)$ space — is a connected aspherical CW-complex X with $\pi_1 X = G$.

Up to homotopy equivalence, a $K(G,1)$ space is unique.

In order to construct a $K(G,1)$ space, we start with a (possibly infinite) presentation of G, we build its *presentation complex* (with one 0-cell, one 1-cell for each generator, one 2-cell for each relation), and we successively add higher-dimensional cells to kill all the homotopy groups. The resulting CW-complex is a $K(G,1)$ space.

Remark 2.2. Let G be a countable group and let X be a $K(G,1)$ space. The *group homology* of G can be defined as

$$H_*(G) \coloneqq H_*(X),$$

where $H_*(X)$ denotes the (singular/cellular) homology of X.

Given a $K(G,1)$ space X, there is a free cellular action of G on the universal cover $\tilde{X}$; we say that $\tilde{X}$ is a *free G-CW-complex.*

We will define the L^2-Betti numbers of a general free G-CW-complex Y. The L^2-Betti numbers of G will then be defined as those of $\tilde{X}$, where X is a $K(G,1)$ space.

We will need certain finiteness properties.

Definition 2.3. Let Y be a free G-CW-complex. We say that Y has type

- F_n ($n \geq 0$) if Y has a finite number of orbits of n-cells,
- F_∞ if Y is of type F_n for all $n \geq 0$.

We say that G is of type F_n or F_∞, if the G-CW-complex $\tilde{X}$ (for X a $K(G,1)$ space) is of type F_n or F_∞ respectively.

Remark 2.4. (i) We have $\mathrm{F}_\infty \Rightarrow \cdots \Rightarrow \mathrm{F}_{n+1} \Rightarrow \mathrm{F}_n \Rightarrow \cdots \Rightarrow \mathrm{F}_0$. All those implications are strict.

(ii) Every group is of type F_0.

(iii) A group is of type F_1 if and only if it is finitely generated.

(iv) A group is of type F_2 if and only if it is finitely presented.

2.2 Definition of L^2-Betti numbers

We now define the L^2-Betti numbers of a free G-CW-complex Y of type F_∞.

Let $C_*^{\mathrm{cell}}(Y)$ be the cellular chain complex of Y over $\mathbb{C}$: for each degree $n \in \mathbb{N}_{\geq 0}$, $C_n^{\mathrm{cell}}(Y)$ is the $\mathbb{C}$-vector space with basis the set of n-cells of Y. The action $G \curvearrowright Y$ induces $G \curvearrowright C_*^{\mathrm{cell}}(Y)$, which gives $C_*^{\mathrm{cell}}(Y)$ the structure of a chain complex over $\mathbb{C}G$. The L^2*-cellular chain complex* of Y is defined by

$$C_*^{(2)}(G \curvearrowright Y) \coloneqq \ell^2 G \otimes_{\mathbb{C}G} C_*^{\mathrm{cell}}(Y),$$

where $\ell^2 G$ is equipped with the action of $\mathbb{C}G$ by multiplication on the right. The L^2-boundary maps are defined by

$$\partial_n^{(2)} \coloneqq \mathrm{id}_{\ell^2 G} \otimes \partial_n^{\mathrm{cell}} : C_n^{(2)}(G \curvearrowright Y) \to C_{n-1}^{(2)}(G \curvearrowright Y).$$

This makes $C_*^{(2)}(G \curvearrowright Y)$ a chain complex.

Definition 2.5. The L^2*-homology* of a free G-CW-complex Y is defined by

$$H_n^{(2)}(G \curvearrowright Y) \coloneqq \operatorname{Ker} \partial_n^{(2)} / \overline{\operatorname{Im} \partial_{n+1}^{(2)}}.$$

Proposition 2.6. *If $G \curvearrowright Y$ is of type F_∞, then $H_n^{(2)}(G \curvearrowright Y)$ is a Hilbert G-module.*

Proof. Fix $n \geq 0$. The n-th chain group $C_n^{(2)}(G \curvearrowright Y)$ can be described as follows. Pick a collection $\{\sigma_i\}_{i \in I}$ of n-cells of Y whose orbits are disjoint and cover the n-skeleton of Y. The set I can be chosen finite since $G \curvearrowright Y$ is of type F_∞.

We have

$$C_n^{\mathrm{cell}}(Y) = \bigoplus_{i \in I} \bigoplus_{g \in G} \mathbb{C}(g \cdot \sigma_i) = \bigoplus_{i \in I} \mathbb{C}G[\sigma_i],$$

and therefore

$$C_n^{(2)}(G \curvearrowright Y) = \bigoplus_{i \in I} \ell^2 G[\sigma_i].$$

It is now clear that $C_n^{(2)}(G \curvearrowright Y)$ is a Hilbert G-module (with an embedding into $(\ell^2 G)^{|I|}$). Moreover, the L^2-boundary maps $\partial_n^{(2)}$ are morphisms of Hilbert G-modules.

Hence, the result is a consequence of the general fact that, if $\varphi : V \to W$ is a morphism of Hilbert G-modules, then $\operatorname{Ker} \varphi$ and $W / \overline{\operatorname{Im} \varphi}$ are Hilbert G-modules. $\square$

Definition 2.7. Let Y be a free G-CW-complex of type F_∞. For $n \in \mathbb{N}_{\geq 0}$, the n-th L^2*-Betti number* of $G \curvearrowright Y$ is

$$b_n^{(2)}(G \curvearrowright Y) \coloneqq \dim_{NG} H_n^{(2)}(G \curvearrowright Y).$$

In order to define the L^2-Betti numbers of a group, we must make sure that L^2-Betti numbers are invariant under homotopy equivalence, so that they do not depend on the choice of a $K(G,1)$ space.

Proposition 2.8. *Let Y_1, Y_2 be free G-CW-complexes. If $f : Y_1 \to Y_2$ is a G-equivariant homotopy equivalence, then for all $n \in \mathbb{N}_{\geq 0}$,*

$$b_n^{(2)}(G \curvearrowright Y_1) = b_n^{(2)}(G \curvearrowright Y_2).$$

Proof. The map f induces a $\mathbb{C}G$-chain homotopy equivalence

$$f_* : C_*^{\text{cell}}(Y_1) \xrightarrow{\simeq} C_*^{\text{cell}}(Y_2),$$

which then induces a chain homotopy equivalence in the category of Hilbert G-modules

$$C_*^{(2)}(G \curvearrowright Y_1) \xrightarrow{\simeq} C_*^{(2)}(G \curvearrowright Y_2). \qquad \square$$

Definition 2.9. Let G be a group of type F_∞. For $n \in \mathbb{N}_{\geq 0}$, the n-th L^2*-Betti number* of G is

$$b_n^{(2)}(G) \coloneqq b_n^{(2)}\left(G \curvearrowright \tilde{X}\right),$$

where $\tilde{X}$ is the universal cover of a $K(G,1)$ space.

It follows from Proposition 2.8 and the uniqueness of $K(G,1)$ spaces up to homotopy that $b_n^{(2)}(G)$ does not depend on the choice of a $K(G,1)$ space.

From now on, we will focus on L^2-Betti numbers of groups.

Remark 2.10. (i) An alternative approach would have been to start with a projective resolution of $\mathbb{C}$ by $\mathbb{C}G$-modules, and then to apply $\ell^2 G \otimes_{\mathbb{C}G} -$.

(ii) One could also have defined the L^2-cochain complex of $G \curvearrowright Y$ by

$$C^*_{(2)}(G \curvearrowright Y) \coloneqq \operatorname{Hom}_{\mathbb{C}G}\left(C_*^{\text{cell}}(Y), \ell^2 G\right),$$

and take $H^*_{(2)}(G \curvearrowright Y)$ to be the cohomology of this cochain complex. In fact, this leads to isomorphisms of Hilbert G-modules

$$H^*_{(2)}(G \curvearrowright Y) \cong H_*^{(2)}(G \curvearrowright Y),$$

so that homological and cohomological L^2-Betti numbers are equal.

2.3 Basic properties

We start by computing $b_0^{(2)}$.

Proposition 2.11. *Let G be a group of type F_∞. Then*

$$b_0^{(2)}(G) = \frac{1}{|G|},$$

with the convention $1/\infty = 0$.

Proof. We construct a $K(G,1)$ space as in §2.1, and obtain isomorphisms

$$C_0^{(2)}\left(G \curvearrowright \tilde{X}\right) \cong \ell^2 G \qquad \text{and} \qquad C_1^{(2)}\left(G \curvearrowright \tilde{X}\right) \cong \bigoplus_{s \in S} \ell^2 G[s],$$

where S is a generating set for G, and the boundary map is given by $\partial_1^{(2)}(s) = s - e$. Hence,

$$H_0^{(2)}\left(G \curvearrowright \tilde{X}\right) = \ell^2 G / \overline{\langle x - gx \mid x \in \ell^2 G,\ g \in G\rangle_{\mathbb{C}}}.$$

- If G is finite, then $\ell^2 G = \mathbb{C}G$ and $H_0^{(2)}\left(G \curvearrowright \tilde{X}\right) = \mathbb{C}$ (with G acting trivially), so

$$b_0^{(2)}(G) = \dim_{NG} \mathbb{C} = \frac{1}{|G|}.$$

- If G is infinite, we will show that $H_0^{(2)}\left(G \curvearrowright \tilde{X}\right) = 0$, or equivalently that the dual of $H_0^{(2)}\left(G \curvearrowright \tilde{X}\right)$ is trivial. This amounts to proving that, if $f : \ell^2 G \to \mathbb{C}$ is $\mathbb{C}$-linear, bounded, and zero on $\langle x - gx \mid x \in \ell^2 G,\ g \in G\rangle_{\mathbb{C}}$ (i.e. f is left-G-invariant), then $f = 0$. As G is infinite and countable, we can enumerate $G = \{g_n\}_{n \geq 1}$, and consider $x = \sum_n \frac{1}{n} g_n \in \ell^2 G$. We have

$$f(x) = \sum_{n \geq 1} \frac{1}{n} f\left(g_n\right) = \sum_{n \geq 1} \frac{1}{n} f(e).$$

  Therefore, $f(e) = 0$, so $f(g) = 0$ for all $g \in G$ since f is G-invariant, and $f = 0$ by linearity and continuity. □

We now give basic properties that will be useful for computations of L^2-Betti numbers.

Proposition 2.12. *Let G and H be two groups of type F_∞ and $n \geq 0$.*

(i) (Dimension) *If G has a $K(G,1)$ space of dimension $\leq n-1$, then*

$$b_n^{(2)}(G) = 0.$$

(ii) (Restriction) *If H is a finite-index subgroup of G, then*

$$b_n^{(2)}(H) = [G : H] \cdot b_n^{(2)}(G)$$

(iii) (Künneth formula)

$$b_n^{(2)}(G \times H) = \sum_{j=0}^{n} b_j^{(2)}(G) \cdot b_{n-j}^{(2)}(H).$$

(iv) (Additivity)

$$b_1^{(2)}(G * H) = b_1^{(2)}(G) + b_1^{(2)}(H) + 1 - b_0^{(2)}(G) - b_0^{(2)}(H)$$

and moreover, if $n \geq 2$,

$$b_n^{(2)}(G * H) = b_n^{(2)}(G) + b_n^{(2)}(H).$$

(v) (Poincaré duality) *If G has a $K(G,1)$ space which is an orientable closed connected manifold of dimension d, then*

$$b_n^{(2)}(G) = b_{d-n}^{(2)}(G).$$

(vi) (Euler characteristic) *If G has a $K(G,1)$ space with a finite number of cells, then*

$$\chi(G) = \sum_{n \geq 0} (-1)^n b_n^{(2)}(G).$$

Proof. (i) If X is a $K(G,1)$ space of dimension $\leq n-1$, then $C_n^{(2)}\left(G \curvearrowright \tilde{X}\right) = 0$ and $H_n^{(2)}\left(G \curvearrowright \tilde{X}\right) = 0$.

(ii) Let X be a $K(G,1)$ space, and let $X_H \to X$ be the covering associated to the subgroup $H \leq G$. Hence, X_H is a $K(H,1)$ and $\tilde{X}$ is the common universal cover of X and X_H. Therefore, $C_*^{(2)}\left(H \curvearrowright \tilde{X}\right)$ is obtained from $C_*^{(2)}\left(G \curvearrowright \tilde{X}\right)$ by applying the restriction functor Res_H^G. The result now follows from Proposition 1.13(vi).

(iii) If X is $K(G,1)$ space and Y is a $K(H,1)$ space, then $X \times Y$ is a $K(G \times H, 1)$ space. The rest of the proof is similar to that of the usual Künneth formula, using 1.13(v).

(iv) If X is a $K(G,1)$ space and Y is a $K(H,1)$ space, then $X \vee Y$ is a $K(G * H, 1)$. We then use a Mayer–Vietoris-type argument.

(v) This uses a Poincaré duality "twisted" by the action of G, and the fact that L^2-Betti numbers can also be computed in terms of cohomology (see Remark 2.10(ii)).

(vi) The main ingredient is an "L^2-rank-nullity theorem" : if $\varphi : V \to W$ is a morphism of Hilbert G-modules, then

$$\dim_{NG}(V) - \dim_{NG}(W) = \dim_{NG}(\operatorname{Ker}\varphi) - \dim_{NG}\left(W/\overline{\operatorname{Im}\varphi}\right).$$

It follows that, for $n \geq 0$,

$$b_n^{(2)}(G) = \dim_{NG}\left(\operatorname{Ker}\partial_n^{(2)}\right) + \dim_{NG}\left(\operatorname{Ker}\partial_{n+1}^{(2)}\right) - \dim_{NG}\left(C_{n+1}^{(2)}\left(G \curvearrowright \tilde{X}\right)\right).$$

Therefore,

$$\sum_{n\geq 0}(-1)^n b_n^{(2)}(G) = \sum_{n\geq 0}(-1)^n \dim_{NG}\left(C_n^{(2)}\left(G \curvearrowright \tilde{X}\right)\right).$$

But $\dim_{NG}\left(C_n^{(2)}\left(G \curvearrowright \tilde{X}\right)\right)$ is the number of n-cells of X, so the above sum is equal to $\chi(X) = \chi(G)$. □

Remark 2.13. Among the properties listed in Proposition 2.12, items (i), (iii), (iv), (v) and (vi) are also true for usual Betti numbers (defined by $b_n(G) \coloneqq \dim_{\mathbb{C}} H_n(G)$). So far, the only property that is specific to the L^2 world is the *restriction formula* (ii).

2.4 Some examples

We now give explicit computations of L^2-Betti numbers in a few simple cases.

Example 2.14 (Finite groups). Let G be a finite group. Then

$$b_n^{(2)}(G) = \begin{cases} \frac{1}{|G|} & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}.$$

Proof. Note that the trivial group $\{1\}$ has index $|G|$ in G; its L^2-Betti numbers are

$$b_n^{(2)}(\{1\}) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}.$$

(Indeed, the trivial group has a $K(G,1)$ space of dimension 0, and the case $n = 0$ comes from Proposition 2.11.) Now the result follows from the restriction formula (2.12(ii)). □

Example 2.15 ($\mathbb{Z}$).

$$b_n^{(2)}(\mathbb{Z}) = 0 \text{ for all } n \in \mathbb{N}_{\geq 0}.$$

Proof. The circle S^1 is a $K(\mathbb{Z}, 1)$ space. Since $\dim S^1 = 1$, we have $b_n^{(2)}(\mathbb{Z}) = 0$ for $n \geq 2$ (by 2.12.(i)). Moreover, $b_0^{(2)}(\mathbb{Z}) = 0$ since $\mathbb{Z}$ is infinite (2.11). We can then compute $b_1^{(2)}(\mathbb{Z})$ in several different manners:

- *Explicit computation.* There is a cellular structure on S^1 with one 0-cell and one 1-cell. Therefore, $C_0^{(2)}(\mathbb{Z} \curvearrowright S^1) = \ell^2\mathbb{Z}$, and $C_1^{(2)}(\mathbb{Z} \curvearrowright S^1) = \ell^2\mathbb{Z}$, and $C_n^{(2)}(\mathbb{Z} \curvearrowright S^1) = 0$ for $n \geq 2$. Denoting by t a generator of $\mathbb{Z}$, the boundary map $\partial_1^{(2)}$ is given by

$$\partial_1^{(2)}(x) = (t-1)x.$$

Hence, we see that $H_1^{(2)}(\mathbb{Z} \curvearrowright S^1) = \operatorname{Ker} \partial_1^{(2)} = 0$.

- *Euler characteristic.* By Proposition 2.12.(vi), we have

$$-b_1^{(2)}(\mathbb{Z}) = \chi(\mathbb{Z}) = 0.$$

- *Finite-index subgroups.* For all $d \geq 1$, the group $\mathbb{Z}$ contains an index-d subgroup isomorphic to $\mathbb{Z}$, so the restriction formula (2.12.(ii)) yields

$$b_n^{(2)}(\mathbb{Z}) = d \cdot b_n^{(2)}(\mathbb{Z}).$$

It follows that $b_n^{(2)}(\mathbb{Z}) = 0$ for all $n \geq 0$. □

Example 2.16 (Free groups). Let F_r be the free group of rank $r \geq 1$. Then

$$b_n^{(2)}(F_r) = \begin{cases} 0 & \text{if } n = 0 \\ r-1 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}.$$

Proof. We have $b_0^{(2)}(F_r) = 0$ since F_r is infinite (2.11). Moreover, $b_n^{(2)}(F_r) = 0$ for $n \geq 2$ because F_r has a $K(G,1)$ space of dimension 1 (2.12.(i)). Here are two different computations of $b_1^{(2)}(F_r)$:

- By additivity of L^2-Betti numbers (2.12.(iv)), we have

$$b_1^{(2)}(F_r) = b_1^{(2)}(\mathbb{Z}) + b_1^{(2)}(F_{r-1}) + 1 - b_0^{(2)}(\mathbb{Z}) - b_0^{(2)}(F_{r-1}) = b_1^{(2)}(F_{r-1}) + 1.$$

We conclude by induction using $b_1^{(2)}(F_1) = b_1^{(2)}(\mathbb{Z}) = 0$ (2.15).

- By considering the Euler characteristic (2.12.(iv)), we have

$$-b_1^{(2)}(F_r) = \chi(F_r) = 1 - r.$$ □

Remark 2.17. The wedge of two circles $S^1 \vee S^1$ is a $K(F_2, 1)$ space; its universal cover is the degree-4 regular tree T. Example 2.16 shows that $b_1^{(2)}(F_r) = 1$. Figure 1 shows an explicit 1-cycle in $C_1^{(2)}(F_2 \curvearrowright T)$. Note however that this 1-cycle does not generate $H_1^{(2)}(F_2 \curvearrowright T)$ in any sense even though the latter is of von Neumann dimension 1.

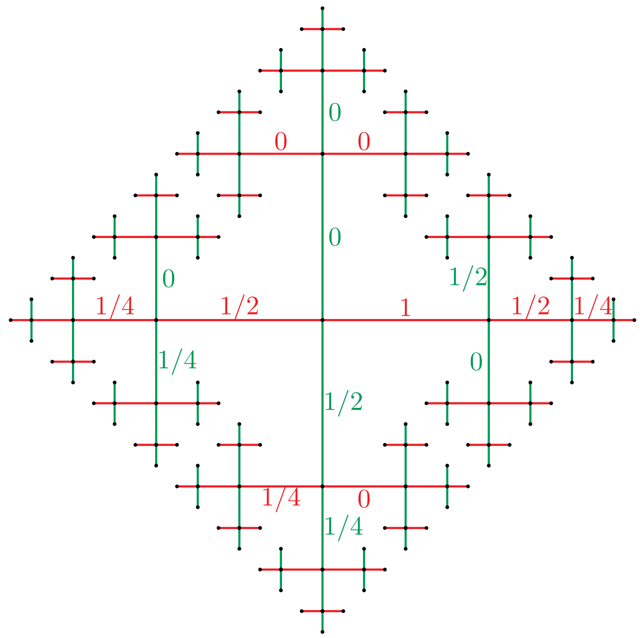


Figure 1: An L^2-1-cycle for $F_2 \curvearrowright T$.

Example 2.18 (Surface groups). Let Σ_g be the orientable closed connected genus-g surface, for $g \geq 1$. Then

$$b_n^{(2)}(\pi_1 \Sigma_g) = \begin{cases} 0 & \text{if } n = 0 \\ 2(g-1) & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}.$$

Proof. The group $\pi_1 \Sigma_g$ is infinite, so $b_0^{(2)}(\pi_1 \Sigma_g) = 0$ (2.11), and Σ_g is a $K(G,1)$ space of dimension 2, so $b_n^{(2)}(\pi_1 \Sigma_g) = 0$ for $n \geq 3$ (2.12.(i)). Moreover, Poincaré duality (2.12.(v)) implies that

$$b_2^{(2)}(\pi_1 \Sigma_g) = b_0^{(2)}(\pi_1 \Sigma_g) = 0.$$

Finally, $-b_1^{(2)}(\pi_1 \Sigma_g) = \chi(\Sigma_g) = 2 - 2g$ (2.12.(iv)). $\square$

Talk 3 – Lück's Approximation Theorem

Ryoya Arimoto

References: [14, §5.1 to §5.3], [19, Chapter 3].

3.1 Statement of the theorem

We start with the following basic observation.

Obervation 3.1. Let G be a group and let $N \trianglelefteq G$ be a finite-index normal subgroup of G. If Y is a free G-CW-complex of type F_∞, then for all $n \in \mathbb{N}_{\geq 0}$, we have

$$b_n^{(2)}\left(G/N \curvearrowright Y/N\right) = \frac{b_n(Y/N)}{[G:N]}.$$

Proof. We have

$$\begin{aligned} b_n^{(2)}\left(G/N \curvearrowright Y/N\right) &= \dim_{\mathbb{N}(G/N)} H_n^{(2)}\left(G/N \curvearrowright Y/N\right) && \text{by definition of } b_n^{(2)} \\ &= \frac{1}{[G:N]} \dim_{\mathbb{C}} H_n\left(Y/N\right) && \text{because } G/N \text{ is finite (see 1.11)} \\ &= \frac{b_n\left(Y/N\right)}{[G:N]} && \text{by definition of } b_n. \;\square \end{aligned}$$

Our goal is to show that, if G is "approximated" by finite-index normal subgroups $(N_i)_i$, then the L^2-Betti numbers $b_n^{(2)}\left(G/N_i \curvearrowright Y/N_i\right)$ approximate $b_n^{(2)}(G \curvearrowright Y)$. We will use the notion of residual finiteness to make precise the idea of a group being approximated by finite-index normal subgroups.

Definition 3.2. A group G is *residually finite* if there are finite-index normal subgroups $(N_i \trianglelefteq G)_{i\geq 1}$ with $N_1 \geq N_2 \geq \cdots$ and $\bigcap_i N_i = \{1\}$. The sequence of normal subgroups $(N_i)_{i\geq 1}$ is then called a *residual chain.*

Equivalently, for every $g \in G \setminus \{1\}$, there is a finite group H and a morphism $\pi : G \to H$ with $\pi(g) \neq 1$.

Example 3.3. The following groups are all residually finite:

- Finite groups,
- Finitely generated free groups,
- Finitely generated subgroups of $GL_n(k)$ (Mal'cev).

On the other hand, infinite simple groups or groups with no finite quotient are not residually finite.

We will prove the following theorem.

Theorem 3.4 (Lück's Approximation Theorem)**.** *Let G be a residually finite group with residual chain $(N_i \trianglelefteq G)_{i\geq 1}$, and let Y be a free G-CW-complex of type F_∞. Then, for all $n \geq 0$,*

$$b_n^{(2)}(G \curvearrowright Y) = \lim_{i\to\infty} \frac{b_n(Y/N_i)}{[G : N_i]}.$$

Corollary 3.5. *Let G be residually finite, of type F_∞, with residual chain $(N_i \trianglelefteq G)_{i\geq 1}$. Then, for all $n \geq 0$,*

$$b_n^{(2)}(G) = \lim_{i\to\infty} \frac{b_n(N_i)}{[G : N_i]}.$$

Proof. It suffices to apply Theorem 3.4 with Y the universal cover of a $K(G,1)$ space. □

3.2 The L^2-Laplacian and its operator norm

From now on, Y is a free G-CW-complex of type F_∞. The degree $n \in \mathbb{N}_{\geq 0}$ is fixed. Our goal is to prove Theorem 3.4.

The degree-n *L^2-Laplacian* on the L^2-chain complex $C_*^{(2)}(Y)$ is defined by

$$\Delta_n^{(2)} \coloneqq \left(\partial_n^{(2)}\right)^* \partial_n^{(2)} + \partial_{n+1}^{(2)} \left(\partial_{n+1}^{(2)}\right)^* : C_n^{(2)}(Y) \to C_n^{(2)}(Y).$$

Proposition 3.6. $\operatorname{Ker} \Delta_n^{(2)} = H_n^{(2)}(G \curvearrowright Y)$.

Proof. Note that

$$H_n^{(2)}(G \curvearrowright Y) = \operatorname{Ker} \partial_n^{(2)} \cap \left(\overline{\operatorname{Im} \partial_{n+1}^{(2)}}\right)^\perp = \operatorname{Ker} \partial_n^{(2)} \cap \operatorname{Ker} \left(\partial_{n+1}^{(2)}\right)^*.$$

The inclusion $(\supseteq)$ follows, while the reverse inclusion $(\subseteq)$ follows from the equality

$$\left\langle \xi, \Delta_n^{(2)} \xi \right\rangle = \left\| \partial_n^{(2)} \xi \right\|^2 + \left\| \left(\partial_{n+1}^{(2)}\right)^* \xi \right\|^2.$$ □

Recall that we can identify $C_n^{(2)}(Y) \cong \left(\ell^2 G\right)^{k_n}$, where k_n is the number of orbits of n-cells of Y under the action of G. Under this identification, $\Delta_n^{(2)}$ corresponds to right multiplication with a self-adjoint matrix $A \in M_{k_n \times k_n}(\mathbb{Z}G)$, with coefficients

$$A_{ij} \coloneqq \Delta_n^{(2)} \left(\delta_e^i\right)_j \in \mathbb{Z}G.$$

Proposition 3.7. *Given $k \geq 1$ and $A \in M_{k\times k}(\mathbb{Z}G)$, the linear operator*

$$(- \times A) : \left(\ell^2 G\right)^k \to \left(\ell^2 G\right)^k$$

is bounded.

Proof. Let $1 \leq i, j \leq n$. For $\xi_i \in \ell^2 G \subseteq_i \left(\ell^2 G\right)^k$, we have

$$\left\|\xi_i A_{ij}\right\|_2 = \left\|\sum_{g \in G} A_{ij}(g) \xi_i g\right\|_2 \leq \left\|A_{ij}\right\|_1 \cdot \left\|\xi_i\right\|_2,$$

where $A_{ij}(g)$ denotes the g-coefficient of $A_{ij} \in \mathbb{Z}G$.

Therefore, for $\xi = (\xi_i)_{1 \leq i \leq k} \in \left(\ell^2 G\right)^k$,

$$\begin{aligned}
\|\xi A\|_2^2 &= \sum_j \left\|\sum_i \xi_i A_{ij}\right\|_2^2 \\
&\leq \sum_j \left(\sum_i \left\|A_{ij}\right\|_1 \cdot \left\|\xi_i\right\|_2\right)^2 \\
&\leq k^2 \left(\max_{i,j} \left\|A_{ij}\right\|_1^2\right) \|\xi\|_2^2 \qquad \text{by Cauchy-Schwarz.}
\end{aligned}$$

□

Now let $(N_i \trianglelefteq G)_{i \geq 1}$ be a residual chain. For each $i \geq 1$, there is an analogously-defined L^2-Laplacian

$$\Delta_{n,i}^{(2)} : C_n^{(2)}\left(Y/N_i\right) \to C_n^{(2)}\left(Y/N_i\right).$$

We have $C_n^{(2)}\left(Y/N_i\right) \cong \left(\ell^2(G/N_i)\right)^{k_n}$, and $\Delta_{n,i}$ corresponds to right multiplication with a self-adjoint matrix $A_{n,i} \in M_{k_n \times k_n}\left(\mathbb{Z}\left[G/N_i\right]\right)$ obtained from A_n by applying the projection $\mathbb{Z}G \to \mathbb{Z}\left[G/N_i\right]$ to the entries.

In particular, there is a constant $C \geq 1$ such that

$$\|- \times A_n\| \leq C \qquad \text{and} \qquad \forall i \geq 1,\ \|- \times A_{n,i}\| \leq C. \tag{$*$}$$

3.3 Spectral measures of the L^2-Laplacians

Background 3.8. Let $\mathcal{H}$ be a complex Hilbert space and let $u \in B(\mathcal{H})$ be a bounded linear operator on $\mathcal{H}$. Given a polynomial $f \in \mathbb{R}[t]$, we can define $f(u)$. By continuity, we can then define $f(u)$ for f a continuous function $\mathbb{R} \to \mathbb{R}$. We then extend to $f(u)$ for f Borel $\mathbb{R} \to \mathbb{R}$. The *spectral measure* associated to u is given by

$$E(A) \coloneqq \chi_A(u) \in B(\mathcal{H})$$

for $A \subseteq \mathbb{R}$.

We will only use the following fact.

Fact 3.9. *Let $\mathcal{H}$ be a complex Hilbert space and $u \in B(\mathcal{H})$. For every $\lambda \in \mathbb{R}$, $E(\{\lambda\})$ is the orthogonal projection onto* $\operatorname{Ker}(u - \lambda \operatorname{id})$.

We consider the spectral measures

- $E_n : 2^{\mathbb{R}} \to B\left(\left(\ell^2 G\right)^{k_n}\right)$ of $\Delta_n^{(2)}$, and
- $E_{n,i} : 2^{\mathbb{R}} \to B\left(\left(\ell^2(G/N_i)\right)^{k_n}\right)$ of $\Delta_{n,i}^{(2)}$, for each $i \geq 1$.

Next, define measures

- $\mu \coloneqq \left\langle \delta_e^{\oplus k_n}, E_n(-)\delta_e^{\oplus k_n} \right\rangle$, and
- $\mu_i \coloneqq \left\langle \delta_e^{\oplus k_n}, E_{n,i}(-)\delta_e^{\oplus k_n} \right\rangle$ for $i \geq 1$.

The measures μ and μ_i (for $i \geq 1$) are all supported on $[0, c]$, with the same total measure $\mu([0,c]) = \mu_i([0,c]) = k_n$.

Obervation 3.10. $\mu(\{0\}) = b_n^{(2)}(G \curvearrowright Y)$, and $\mu_i(\{0\}) = b_n^{(2)}(G/N_i \curvearrowright Y/N_i)$ for each $i \geq 1$.

Proof. We have

$$\begin{aligned} \mu(\{0\}) &= \mathrm{tr}_{\mathrm{N}G}\left(\text{orthogonal projection onto Ker}\, \Delta_n^{(2)}\right) && \text{see Fact 3.9} \\ &= b_n^{(2)}(G \curvearrowright Y) && \text{by Proposition 3.6.} \end{aligned}$$

The computation of $\mu_i(\{0\})$ is similar. □

Hence, proving Theorem 3.4 amounts to proving that

$$\mu_i(\{0\}) \xrightarrow[i\to\infty]{} \mu(\{0\}). \tag{†}$$

This is our new goal.

3.4 Weak convergence of the spectral measures

Lemma 3.11. *For all continuous functions $f \in \mathcal{C}([0,c])$, we have*

$$\mu_i(f) \xrightarrow[i\to\infty]{} \mu(f).$$

Proof. By density of polynomial functions in $\mathcal{C}([0,c])$, it suffices to show that, for all $m \geq 0$, $\mu_i(x^m) \xrightarrow[i\to\infty]{} \mu(x^m)$. Fix $m \geq 0$ and choose $i \geq 1$ large enough so that, whenever $g \in G$ has a nonzero coefficient in any of the diagonal entries of A^m, then $g \notin N_i$. For such i, we have

$$\mu_i(x^m) = \left\langle \delta_e^{\oplus k_n}, \delta_e^{\oplus k_n} A_{n,i}^m \right\rangle = \left\langle \delta_e^{\oplus k_n}, \delta_e^{\oplus k_n} A_n^m \right\rangle = \mu(x^m).$$ □

Oberwation 3.12. Weak convergence of the spectral measures (i.e. Lemma 3.11) implies that

$$\limsup_{i\to\infty} \mu_i(\{0\}) \le \mu(\{0\}),$$

and, for all $\lambda \in (0, c)$,

$$\liminf_{i\to\infty} \mu_i([0,\lambda)) \ge \mu([0,\lambda)).$$

This shows half of (†).

Proof. For $\varepsilon > 0$, let $h_\varepsilon : \mathbb{R} \to \mathbb{R}$ be a piecewise affine function with $h_{\varepsilon|(-\infty,0]} = 1$, $h_{\varepsilon|[\varepsilon,+\infty)} = 0$, and $h_{\varepsilon|[0,\varepsilon]}$ affine. Lemma 3.11 implies that

$$\mu_i(\{0\}) \le \mu_i(h_\varepsilon) \xrightarrow[i\to\infty]{} \mu(h_\varepsilon) \le \mu([0,\varepsilon]).$$

This shows that $\limsup_i \mu_i(\{0\}) \le \mu([0,\varepsilon])$. Since this holds for all $\varepsilon > 0$, this gives $\limsup_i \mu_i(\{0\}) \le \mu(\{0\})$. The other inequality is similar. □

3.5 Convergence at zero

To complete the proof of (†), we will need finer estimates.

Lemma 3.13. *For all $i \ge 1$, $\varepsilon > 0$, $\lambda \in (0,1)$, we have*

$$\mu_i((0,\lambda)) \le \frac{k_n \log C}{|\log \lambda|},$$

where $C \ge 1$ is the constant of $(*)$.

Proof. Fix $i \ge 1$, $\varepsilon > 0$, $\lambda \in (0,1)$. Recall that $\left(\ell^2(G/N_i)\right)^{k_n} \cong \mathbb{C}^{k_n[G:N_i]}$ as Hilbert spaces. Set $d_{n,i} \coloneqq k_n[G:N_i]$. We have a matrix $A_{n,i} \in M_{k_n\times k_n}(\mathbb{Z}[G/N_i])$, which we now view as an element of $M_{d_{n,i}\times d_{n,i}}(\mathbb{Z})$. The matrix $A_{n,i}$ is self-adjoint, and we consider its spectrum

$$\sigma(A_{n,i}) = \{\lambda_1 < \cdots < \lambda_s\} \subseteq \mathbb{R}_{>0};$$

we denote by m_j the multiplicity of λ_j for each $1 \le j \le s$.

For each $1 \le j \le s$, we have (see Fact 3.9)

$$\begin{aligned}\mu_i(\{\lambda_j\}) &= \dim_{\mathrm{N}(G/N_i)} \operatorname{Ker}(\Delta_{n,i} - \lambda_j \operatorname{id})\\ &= \frac{1}{[G:N_i]} \dim_{\mathbb{C}} \operatorname{Ker}(\Delta_{n,i} - \lambda_j \operatorname{id})\\ &= \frac{m_j}{[G:N_i]}.\end{aligned}$$

Set $t \coloneqq \max\{j \in \{1,\ldots,s\} \mid \lambda_j < \lambda\}$. Hence

$$\mu_i\left((0,\lambda)\right) = \sum_{j=1}^{t} \frac{m_j}{[G:N_i]}. \tag{‡}$$

Now denote by p the characteristic polynomial of $A_{n,i}$; it has integral coefficients since $A_{n,i} \in M_{d_{n,i}\times d_{n,i}}(\mathbb{Z})$. We have

$$\frac{p(x)}{(-x)^{d_{n,i}-m_1-\cdots-m_s}} = (\lambda_1 - x)^{m_1}\cdots(\lambda_s - x)^{m_s} \xrightarrow[x\to 0]{} \lambda_1^{m_1}\cdots\lambda_s^{m_s}.$$

But since p has integral coefficients, the above limit must be ≥ 1 (as it is equal to the last nonzero coefficient of p, up to a sign), i.e. $\lambda_1^{m_1}\cdots\lambda_s^{m_s} \geq 1$. For $j \leq t$, we have $\lambda_j \leq \lambda$ by definition, while for $j > t$, we have $\lambda_j \leq \|- \times A_{n,i}\| \leq C$. Therefore,

$$\begin{aligned} 1 &\leq \lambda_1^{m_1}\cdots\lambda_s^{m_s} \\ &\leq \left(\lambda_1^{m_1}\cdots\lambda_t^{m_t}\right)\left(\lambda_{t+1}^{m_{t+1}}\cdots\lambda_s^{m_s}\right) \\ &\leq \lambda^{m_1+\cdots+m_t} C^{d_{n,i}}. \end{aligned}$$

Taking logarithms (and remembering that $\log\lambda < 0$) gives

$$\sum_{j=1}^{t} \frac{m_j}{[G:N_i]} \leq -\frac{k_n \log C}{\log\lambda}.$$

But (‡) says that $\mu_i\left((0,\lambda)\right) = \sum_{j=1}^{t} \frac{m_j}{[G:N_i]}$, so we have the desired inequality. $\square$

It follows from Lemma 3.13 that, for all $i \geq 1$,

$$\mu_i\left(\{0\}\right) \geq \mu_i\left([0,\lambda)\right) - \frac{k_n \log C}{|\log\lambda|},$$

and therefore

$$\liminf_{i\to\infty} \mu_i\left(\{0\}\right) \geq \lim_{\lambda\to 0} \mu\left([0,\lambda)\right) = \mu\left(\{0\}\right).$$

Together with Observation 3.12, this shows that

$$\mu_i\left(\{0\}\right) \xrightarrow[i\to\infty]{} \mu\left(\{0\}\right),$$

concluding the proof of (†), and hence of Theorem 3.4. $\square$

Talk 4 – HNN-splittings from positivity of the first L^2-Betti number

Bingxue Tao

References: [22].

Remark 4.1. So far, we have assumed that our groups were of type F_∞. Under the weaker assumption that our group G is of type F_n, we can still define $b_k^{(2)}(G)$ for all $k < n$, with the same definition. All the theory that we have covered so far remains valid in this context.

In fact, one can also define L^2-Betti numbers for general groups, using a notion of *extended von Neumann dimension* (see [18, Chapter 6] and [14, Chapter 4]). However, Lück's Approximation Theorem (3.4) does not hold in such generality.

The next two talks will be concerned with the first L^2-Betti number, and we will only assume that our groups are of type F_2, i.e. finitely presented.

4.1 First L^2-Betti number of s-normal subgroups

Let G be a finitely presented group. Observe that, if H is a finite-index subgroup of G, then it follows from Proposition 2.12(ii) that

$$b_1^{(2)}(H) = [G : H]\, b_1^{(2)}(G) \geq b_1^{(2)}(G).$$

We would like to understand how far this generalises beyond the assumption that H has finite index in G.

Definition 4.2. A subgroup $H \leq G$ is *s-normal* if

$$\forall g \in G, \left|H \cap gHg^{-1}\right| = \infty.$$

For instance, infinite-index normal subgroups are s-normal.

Theorem 4.3 (Peterson–Thom [24])**.** *If $H \leq G$ is an s-normal, finitely presented subgroup of a finitely presented group, then*

$$b_1^{(2)}(H) \geq b_1^{(2)}(G).$$

Corollary 4.4. *Let $H \leq G$ be an infinite-index, s-normal, finitely presented subgroup of a finitely presented group. Suppose in addition that G has finite-index subgroups G' containing H of arbitrarily large index.*

Then either $b_1^{(2)}(H) = \infty$ or $b_1^{(2)}(G) = 0$.

Proof. For each $k \in \mathbb{N}$, let $G' \leq G$ be a finite-index subgroup such that $G' \geq H$ and $[G : G'] \geq k$. The subgroup H is s-normal in G' too, so Theorem 4.3 gives

$$b_1^{(2)}(H) \geq b_1^{(2)}(G') = [G : G']\, b_1^{(2)}(G) \geq k \cdot b_1^{(2)}(G).$$

Since k can be taken arbitrarily large, the result follows. $\square$

Remark 4.5. In Corollary 4.4, the assumption on the G''s can in fact be removed [24].

4.2 HNN-splittings with non-s-normal edge groups

Our next goal is to prove the following.

Theorem 4.6 (Osin [22])**.** *Let G be a finitely presented group. Suppose that G is* indicable *(i.e. there exists an epimorphism $G \twoheadrightarrow \mathbb{Z}$), and that $b_1^{(2)}(G) > 0$.*

Then G splits as an HNN-extension $A*_{C\sim D}$, where A, C, D are finitely generated groups, $C, D \neq A$, and C, D are not s-normal in G.

We start by showing that, without the assumption on $b_1^{(2)}(G)$, we can already obtain an HNN-splitting where the edge groups might be s-normal.

Lemma 4.7. *Let G be a finitely presented, indicable group.*

Then G splits as an HNN-extension $A*_{C\sim D}$, where A, C, D are finitely generated groups.

Proof. By indicability, we can pick an epimorphism $\varepsilon : G \twoheadrightarrow \mathbb{Z}$. Choose an element $t \in G$ with $\varepsilon(t) = 1$; one can then choose elements $a_1, \ldots, a_k \in \operatorname{Ker} \varepsilon$ so that G admits a finite presentation

$$G = \langle t, a_1, \ldots, a_k \mid r_1, \ldots, r_m \rangle .$$

Observe that, for each $j \in \{1, \ldots, m\}$, the t-exponent sum of r_j is equal to 0. Consider

$$N \coloneqq \max \left\{ n \in \mathbb{N}_{\geq 0} \mid \exists j \in \{1, \ldots, m\},\, t^{\pm n} \text{ is a subword of } r_j \right\}.$$

Now each r_j can be rewritten as a product of words of the form $b_{i,n} \coloneqq t^n a_i t^{-n}$ and their inverses, with $i \in \{1, \ldots, k\}$ and $n \in \{0, \ldots, N\}$. By Tietze transformation, this gives

$$G = \langle t, \mathcal{B} \mid s_1, \ldots, s_m, \mathcal{T} \rangle ,$$

where $\mathcal{B} \coloneqq \{b_{i,n} \mid i \in \{1, \ldots, k\},\, n \in \{0, \ldots, N\}\}$, each s_j is a word over $\mathcal{B}$ which maps to r_j under $b_{i,n} \mapsto t^n a_i t^{-n}$, and

$$\mathcal{T} \coloneqq \left\{ t b_{i,n} t^{-1} b_{i,n+1}^{-1} \,\middle|\, i \in \{1, \ldots, k\},\, n \in \{0, \ldots, N-1\} \right\}.$$

It follows that $G = A*_{C\sim D}$, where $A = \langle \mathcal{B} \mid s_1, \ldots, s_m \rangle$, the subgroups C and D are $C \coloneqq \langle b_{i,n} \mid i \in \{1, \ldots, k\},\, n \in \{0, \ldots, N-1\} \rangle$, $D \coloneqq \langle b_{i,n} \mid i \in \{1, \ldots, k\},\, n \in \{1, \ldots, N\} \rangle$, with isomorphism $C \xrightarrow{\cong} D$ given by $b_{i,n} \mapsto b_{i,n+1}$. $\square$

Proof of Theorem 4.6. By Lemma 4.7, there is an HNN-splitting $G = A*_{C\sim D}$, with A, C, D finitely generated. Let t denote the stable letter of the HNN-extension. Let $\varepsilon : G \twoheadrightarrow \mathbb{Z}$ be an epimorphism such that $\varepsilon(A) = 0$ and $\varepsilon(t) = 1$. For each $n \in \mathbb{N}_{\geq 1}$, consider the normal subgroup

$$K_n \coloneqq \varepsilon^{-1}(n\mathbb{Z}) \trianglelefteq G.$$

Since $G/K_n \cong \mathbb{Z}/n\mathbb{Z}$, we have $[G : K_n] = n$. Moreover, $b_1^{(2)}(G) > 0$ by assumption, so

$$b_1^{(2)}(K_n) = [G : K_n]\, b_1^{(2)}(G) \xrightarrow[n\to\infty]{} \infty. \tag{$*$}$$

We now show that C, D are proper subgroups of A, and that they are not s-normal in G.

Properness of C, D in A. Assume for contradiction that $A = C$. It then follows from the normal form of elements in an HNN-extension that every $g \in G$ can be written as $g = t^{-\ell} a t^m$, with $\ell, m \in \mathbb{N}_{\geq 0}$ and $a \in A$. In particular, $G = \langle t \rangle A \langle t \rangle$, and moreover, $K_n = \langle A, t^n \rangle$. If d_A is the minimal number of generators of A, then $d_A + 1$ is an upper bound on the number of generators of K_n, which is uniform in n. It follows that

$$b_1^{(2)}(K_n) \leq (d_A + 1) - 1,$$

contradicting ($*$).

Non-s-normality of C, D in G. Since C is finitely generated, we have $b_1^{(2)}(C) < \infty$. But $b_1^{(2)}(G) > 0$ by assumption, so C cannot be s-normal in G by Corollary 4.4. □

Remark 4.8. (i) Finitely presented residually finite groups G with $b_1^{(2)}(G) > 0$ are virtually indicable.

(ii) Finitely presented groups of deficiency at least 2 satisfy the assumptions of Theorem 4.6.

Proof. (i) It follows from Lück's Approximation Theorem (3.4) that G admits a finite index subgroup H with $b_1(H) > 0$; hence, $H^1(H; \mathbb{Z}) \neq 0$ and H is indicable.

(ii) Let G be a finitely presented group; we recall that the *deficiency* of G is

$$\operatorname{def}(G) \coloneqq \max\{(|S| - |R|) \mid G \cong \langle S \mid R \rangle,\ |S| < \infty,\ |R| < \infty\},$$

and we assume that $\operatorname{def}(G) \geq 2$.

For indicability, note that, given a presentation $G \cong \langle S \mid R \rangle$, constructing a morphism $G \to \mathbb{Z}$ amounts to solving a linear system of $|R|$ equations in $|S|$ unknowns; if $|S| - |R| \geq 1$, then a non-trivial solution exists by linear algebra.

For positivity of $b_1^{(2)}(G)$, it is a general fact that

$$b_1^{(2)}(G) \geq \operatorname{def}(G) - 1.$$

Indeed, given a finite presentation $\langle S \mid R \rangle$ of G, let X be the associated Cayley 2-complex. Then we can compute

$$\chi(X) = 1 - |S| + |R|\,,$$

and also, by Proposition 2.12(vi),

$$\chi(X) = b_0^{(2)}\left(G \curvearrowright X\right) - b_1^{(2)}\left(G \curvearrowright X\right) + b_2^{(2)}\left(G \curvearrowright X\right).$$

But $b_0^{(2)}\left(G \curvearrowright X\right)$ because G is infinite, $b_1^{(2)}\left(G \curvearrowright X\right) = b_1^{(2)}(G)$ because we can add higher-dimensional cells to X to obtain a $K(G,1)$ space without changing the 2-skeleton, and $b_2^{(2)}\left(G \curvearrowright X\right) \geq 0$. It follows that

$$b_1^{(2)}(G) \geq |S| - |R| - 1,$$

and so $b_1^{(2)}(G) \geq \operatorname{def}(G) - 1$ after taking the maximum over all finite presentations of G. $\square$

Talk 5 – Acylindrical hyperbolicity of groups with positive first L^2-Betti number

Bingxue Tao

References: [22].

5.1 Hyperbolic and relatively hyperbolic groups

We start by recalling the definition of hyperbolic groups.

Definition 5.1. Given $\delta \geq 0$, a geodesic metric space X is *δ-hyperbolic* if, for any geodesic segments α, β, γ forming a triangle in X, α is contained in the δ-neighbourhood of $\beta \cup \gamma$.

A finitely generated group G is *hyperbolic* if its Cayley graph $\mathrm{Cay}(G, S)$ is δ-hyperbolic for some $\delta \geq 0$ and for some (and hence for any) finite generating set S.

Hyperbolic groups are known to be of type F_∞ (via the Rips complex).

Example 5.2. The following groups are hyperbolic:

- Finite-rank free groups,
- Fundamental groups of closed hyperbolic manifolds.

Next, we move on to relative hyperbolicity. The following definition is not complete; we leave out some technical assumptions that will not play a role in this text.

Definition 5.3. A group G is *hyperbolic relative to* a finite collection $\mathbb{P} = \{P_i\}_{1 \leq i \leq n}$ of subgroups if there is a finite subset $S \subseteq G$ such that $G = \langle P_1, \ldots, P_n, S \rangle$, and the Cayley graph $\mathrm{Cay}\left(G, S \cup \bigcup_i P_i\right)$ is δ-hyperbolic for some $\delta \geq 0$ (and some extra assumptions).

Example 5.4. The following pairs $(G, \mathbb{P})$ are relatively hyperbolic:

- $(G, \{1\})$, where G is a hyperbolic group,
- $(H *_F K, \{H, K\})$, where $|F| < \infty$,
- $(A*_{C \sim D}, \{A\})$, where $|C| = |D| < \infty$,
- $(\pi_1 M, \mathbb{P})$, where M is a complete finite-volume hyperbolic manifold, and $\mathbb{P}$ is the collection of cusp groups of M.

5.2 Acylindrically hyperbolic groups

Finally, we introduce acylindrical hyperbolicity, which we will be most concerned with.

Definition 5.5. Let G be a discrete group acting by isometries on a metric space X. For $\varepsilon > 0$ and $P \subseteq X$, the *pointwise ε-stabiliser* of P is

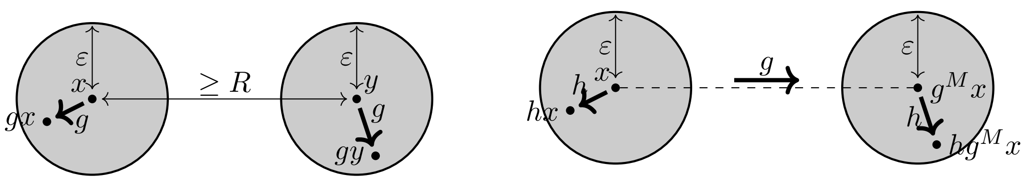
$$\mathrm{PStab}_G^{\varepsilon}(P) \coloneqq \{g \in G \mid \forall p \in P,\ d(gp, p) \leq \varepsilon\}\,.$$

We say that the action $G \curvearrowright P$ is *acylindrical* (see Figure 2a) if

$$\forall \varepsilon \geq 0,\ \exists R, N > 0,\ \forall x, y \in X,\ d(x, y) \geq R \Rightarrow \left|\mathrm{PStab}_G^{\varepsilon}\left(\{x, y\}\right)\right| \leq N. \tag{Acyl}$$

We say that an element $g \in G$ is *weakly properly discontinuous*, or *WPD* (see Figure 2b) if

$$\forall x \in X,\ \forall \varepsilon \geq 0,\ \exists M \in \mathbb{N},\ \left|\mathrm{PStab}_G^{\varepsilon}\left(\left\{g^M x, x\right\}\right)\right| < \infty. \tag{WPD}$$



(a) $g \in \mathrm{PStab}_G^{\varepsilon}\left(\{x, y\}\right)$.

(b) $h \in \mathrm{PStab}_G^{\varepsilon}\left(\{g^M x, x\}\right)$.

Figure 2: Illustrations of the definitions of acylindrical hyperbolicty (Acyl) and weak proper discontinuity (WPD).

Remark 5.6. If $G \curvearrowright X$ properly discontinuously, then every $g \in G$ is weakly properly discontinuous.

Definition 5.7. A group G is *acylindrically hyperbolic* if G admits an acylindrical action on a δ-hyperbolic space (for some $\delta \geq 0$) such that the action is *non-elementary* in the sense that $\left|\overline{G \cdot o} \cap \partial X\right| \geq 3$ for some $o \in X$.

Theorem 5.8 (Osin [23])**.** *Given a group G, the following assertions are equivalent:*

(i) *G is acylindrically hyperbolic.*

(ii) *G admits an action on a δ-hyperbolic space with at least one WPD element g, which in addition is* hyperbolic *in the sense that*

$$\lim_{n \to \infty} d\left(g^n o, o\right) > 0$$

for some $o \in X$.

Remark 5.9. In Theorem 5.8, the actions of (i) and (ii) might be on different spaces.

Example 5.10. The following are examples of acylindrically hyperbolic groups, or explicit acylindrical actions on hyperbolic spaces.

- If G is hyperbolic relative to a collection of proper subgroups $\mathbb{P} = \{P_i\}_{1 \leq i \leq n}$, then the action of G on $\mathrm{Cay}\left(G, S \cup \bigcup_i P_i\right)$ is acylindrical.
- Given $g \geq 2$, the action of $\mathrm{MCG}\left(\Sigma_g\right)$ on the curve complex $\mathcal{C}\left(\Sigma_g\right)$ is WPD (Bestvina–Fujiwara [3]) and in fact acylindrical (Bowditch [4]).
- For $n \geq 2$, $\mathrm{Out}\left(F_n\right)$ is acylindrically hyperbolic (Bestvina–Feighn [2]).

Next, we give a few properties that acylindrically hyperbolic groups enjoy.

Proposition 5.11. *Let G be an acylindrically hyperbolic group.*

(Dahmani–Guirardel–Osin [6]) *G is* SQ-universal*: any countable group embeds into some quotient of G.*

(Bestvina–Fujiwara [3], Frigerio–Pozzetti–Sisto [9]) $\dim_{\mathbb{R}} H_b^n\left(G; \mathbb{R}\right) = \infty$ *for* $n = 2, 3$.

Assume in addition that G has no nontrivial finite normal subgroup.

(Dahmani–Guirardel–Osin [6]) *The reduced C^*-algebra of G is simple and has unique trace.*

(Hull–Osin [13]) *G does not satisfy any nontrivial mixed identity.*

5.3 Acylindrical hyperbolicity from $b_1^{(2)}$

Our goal is to prove the following.

Theorem 5.12 (Osin [22])**.** *If G is finitely presented, indicable, with $b_1^{(2)}(G) > 0$, then G is acylindrically hyperbolic.*

In order to prove Theorem 5.12, we will use the HNN-splitting of G given by Theorem 4.6. We will need to understand when a group acting on a tree is acylindrically hyperbolic; this will be given by the following.

Theorem 5.13 (Minasyan–Osin [20])**.** *Let G be a group which is not virtually cyclic, acting minimally on a tree T. The following are equivalent:*

(i) *G contains a hyperbolic WPD element for its action on T.*

(ii) *G has no fixed point in ∂T, and there are vertices u, v of T such that*

$$\left|\mathrm{PStab}_G^0\left(\{u, v\}\right)\right| < \infty.$$

Essentially, Theorem 5.13 says that, for a group acting on a tree, instead of having to consider $\mathrm{PStab}_G^{\varepsilon}(\{u,v\})$ for all $\varepsilon \geq 0$ as in Definition 5.5, it suffices to look at $\varepsilon = 0$.

Proof of Theorem 5.12. Theorem 4.6 says that G splits as an HNN-extension $A*_{C\sim D}$, where A, C, D are finitely generated groups, $C, D \neq A$, and C, D are not s-normal in G. By Bass–Serre Theory, there is a minimal action of G on a tree T with one orbit of vertices (stabilised by conjugates of A), and one orbit of edges (stabilised by conjugates of $C \cong D$). The fact that $A \neq C, D$ implies that G does not have a fixed point in ∂T. The fact that C is not s-normal in G implies that there are vertices u, v of T such that $\left|\mathrm{PStab}_G^0(\{u,v\})\right| < \infty$.

Now it follows from the Minasyan–Osin Theorem (5.13) that G contains a hyperbolic WPD element. Finally, it follows from Theorem 5.8 that G is acylindrically hyperbolic. ☐

Question 5.14 (Osin)**.** Is every finitely presented group G with $b_1^{(2)}(G) > 0$ acylindrically hyperbolic?

Remark 5.15. (i) There are computations of L^2-Betti numbers for certain groups acting on trees [5].

(ii) In the reverse direction to Question 5.14, the first L^2-Betti number of acylindrically hyperbolic groups is mysterious in general. Here are a few examples of groups with vanishing $b_1^{(2)}$; some of those are acylindrically hyperbolic, so the converse to Question 5.14 cannot be true.

(Kida [16]) $b_1^{(2)}(\mathrm{MCG}(\Sigma_g)) = 0$ for $g \geq 2$ (in fact, $b_n^{(2)}(\mathrm{MCG}(\Sigma_g)) \neq 0$ only if $n = 3g - 3$).

(Gaboriau–Noûs [12]) $b_1^{(2)}(\mathrm{Out}(F_n)) = 0$ (but $b_{2n-3}^{(2)}(\mathrm{Out}(F_n)) > 0$).

(Lück [17], Gaboriau [10]) $b_1^{(2)}(K \rtimes \mathbb{Z}) = 0$ for every finitely generated group K.

(Davis–Leary [7]) $b_1^{(2)}(G) = 0$ if G is an Artin group with connected defining graph.

Talk 6 – The measure-theoretic viewpoint

Hiroto Nishikawa

References: [19, Chapter 4], [14, §5.5].

In this talk, we will consider a group acting freely on a probability space X. Associated to such an action, there is an equivalence relation $\mathcal{R}$ on the underlying set of X. We will define analogues of von Neumann dimension, L^2-Betti numbers, and rank, in this setting.

6.1 Free p.m.p. actions and their associated equivalence relations

Throughout, (X, μ) is a probability space that we assume to be *standard*, i.e. Borel-isomorphic to $[0, 1]$. For example, every non-countable Polish space is standard.

Definition 6.1. A *free probability-measure-preserving action* (or *free p.m.p. action*) $G \curvearrowright (X, \mu)$ is an action $G \curvearrowright X$ of an infinite countable group G such that

(PMP) $\mu(gA) = \mu(A)$ for every $g \in G$ and for every measurable subset $A \subseteq X$, and

(Free) $\mathrm{Stab}_G(x) = \{1\}$ for μ-almost every $x \in X$.

Example 6.2. The following are examples of free p.m.p. actions:

- The *Bernoulli shift* $G \curvearrowright \{0, 1\}^G$, given G an infinite group,
- The action $\Gamma \curvearrowright G/\Lambda$, given two lattices $\Gamma, \Lambda \leq G$,
- The action $\Gamma \curvearrowright G$, given a dense subgroup Γ of a compact group G.

Given a free p.m.p. action $G \curvearrowright (X, \mu)$, its *orbit relation* is

$$\mathcal{R}(G \curvearrowright X) \coloneqq \{(gx, x) \in X \times X \mid x \in X,\ g \in G\}.$$

The subset $\mathcal{R}(G \curvearrowright X) \subseteq X \times X$ is Borel-isomorphic to $G \times X$. We equip $\mathcal{R}(G \curvearrowright X)$ with the pullback ν of the measure $\# \otimes \mu$ on $G \times X$, where $\#$ is the counting measure on G.

Orbit relations are examples of the following.

Definition 6.3. A *standard equivalence relation* on X is an equivalence relation $\mathcal{R} \subseteq X \times X$ such that

- $\mathcal{R}$ is a measurable subset of $X \times X$, and
- Every $\mathcal{R}$-equivalence class in X is countable.

Orbit relations of free p.m.p. actions are standard, and in fact, every standard equivalence relation arises in this way (Feldman–Moore [8]). We define a measure ν on a standard equivalence relation $\mathcal{R} \subseteq X \times X$ by setting

$$\nu(A) \coloneqq \int_X \#\left(A \cap (\{x\} \times X)\right) \, \mathrm{d}\mu(x) \tag{$*$}$$

for every measurable subset $A \subseteq \mathcal{R}$. When $\mathcal{R} = \mathcal{R}\left(G \curvearrowright X\right)$, this coincides with the pullback of the measure $\# \otimes \mu$ on $G \times X$.

6.2 L^2-Betti numbers of an equivalence relation

We now fix a standard equivalence relation $\mathcal{R} \subseteq X \times X$, equipped with the measure ν defined by ($*$). Our goal is to define notions of von Neumann algebras and L^2-Betti numbers for $\mathcal{R}$.

Our base ring is now $L^\infty X$ rather than $\mathbb{C}$. We set

$$\begin{aligned} \mathbb{C}\mathcal{R} \coloneqq \{ f \in L^\infty \mathcal{R} \mid \exists N \in \mathbb{N},\ \forall x \in X,& \\ \#\left\{ y \in X \mid (x,y) \in \mathcal{R},\ f(x,y) \neq 0,\ f(y,x) \neq 0 \right\} \leq N \right\}.& \end{aligned}$$

Given $f, g \in L^\infty \mathcal{R}$, we define their *convolution* by

$$(fg) : (x,z) \in \mathcal{R} \mapsto \sum_{\substack{y \in X \\ (x,y) \in \mathcal{R} \\ (y,z) \in \mathcal{R}}} f(x,y) g(y,z) \in \mathbb{C}.$$

Note that, if $f \in \mathbb{C}\mathcal{R}$ and $g \in L^2\mathcal{R}$, then $fg, gf \in L^2\mathcal{R}$; hence, $\mathbb{C}\mathcal{R}$ acts on $L^2\mathcal{R}$ by left and right multiplication.

Definition 6.4. The *von Neumann algebra* $\mathrm{N}\mathcal{R}$ is the weak closure of $\mathbb{C}\mathcal{R}$ in $B\left(L^2\mathcal{R}\right)$ with respect to the right convolution action. It is equipped with a *trace* $\mathrm{tr}_\mathcal{R}$ defined by

$$\mathrm{tr}_\mathcal{R} : a \in \mathrm{N}\mathcal{R} \mapsto \left\langle \chi_\Delta, a\left(\chi_\Delta\right) \right\rangle \in \mathbb{C},$$

where $\Delta \coloneqq \{(x,x) \mid x \in X\} \subseteq \mathcal{R}$.

Using this trace on $\mathrm{N}\mathcal{R}$, we can define the dimension of a finitely generated projective $\mathrm{N}\mathcal{R}$-module P via

$$\mathrm{pdim}_{\mathrm{N}\mathcal{R}} P \coloneqq \mathrm{tr}_\mathcal{R}(p),$$

where $p : (\mathrm{N}\mathcal{R})^n \to (\mathrm{N}\mathcal{R})^n$ is an orthogonal projection with $\mathrm{Im}\, p \cong P$ as $\mathrm{N}\mathcal{R}$-modules. For a general $\mathrm{N}\mathcal{R}$-module V, we set

$$\begin{aligned} \dim_{\mathrm{N}\mathcal{R}} V \coloneqq \sup \left\{ \mathrm{pdim}_{\mathrm{N}\mathcal{R}} P \mid P \text{ finitely generated, projective, } \mathrm{N}\mathcal{R}\text{-submodule of } V \right\}& \\ \in [0, \infty].& \end{aligned}$$

We are now ready to define the L^2-Betti numbers of an equivalence relation. Contrary to §2, we will define them via a projective resolution rather than using the classifying space.

Recall that a *projective resolution* V_* of $\mathbb{C}$ over $\mathbb{C}G$ is an exact sequence

$$\cdots \to V_n \to \cdots \to V_1 \to V_0 \to \mathbb{C} \to 0,$$

where each V_i is a projective $\mathbb{C}G$-module. For instance, if X is a $K(G,1)$ space, then $C_*^{\mathrm{cell}}\left(\tilde{X};\mathbb{C}\right)$ gives a projective resolution of $\mathbb{C}$ over $\mathbb{C}G$. We then set

$$\operatorname{Tor}_*^{\mathbb{C}G}\left(\ell^2 G, \mathbb{C}\right) \coloneqq H_*\left(\ell^2 G \otimes_{\mathbb{C}G} V_*\right).$$

For standard equivalence relations, we use analogue notions, replacing $\mathbb{C}$ with $L^\infty\mathcal{R}$.

Definition 6.5. If $\mathcal{R}$ is a standard equivalence relation on X, we define its n-th L^2-Betti number (for $n \in \mathbb{N}_{\geq 0}$) by

$$b_n^{(2)}(\mathcal{R}) \coloneqq \dim_{\mathrm{N}\mathcal{R}} \operatorname{Tor}_n^{\mathbb{C}\mathcal{R}}\left(\mathrm{N}\mathcal{R}, L^\infty\mathcal{R}\right) \in [0,\infty].$$

6.3 Gaboriau's Theorem

It turns out that the L^2-Betti numbers of an orbit relation recover that of the acting group.

Theorem 6.6 (Gaboriau [11])**.** *Let $G \curvearrowright (X,\mu)$ be a free p.m.p. action and $\mathcal{R} \coloneqq \mathcal{R}\left(G \curvearrowright X\right)$. Then for all $n \in \mathbb{N}_{\geq 0}$,*

$$b_n^{(2)}\left(\mathcal{R}\right) = b_n^{(2)}\left(G\right).$$

Sketch of proof. We have

$$\begin{aligned} b_n^{(2)}(G) &= \dim_{\mathrm{N}G} \operatorname{Tor}_n^{\mathbb{C}G}\left(\mathrm{N}G, \mathbb{C}\right) && \text{see the end of §6.2} \\ &= \dim_{\mathrm{N}\mathcal{R}} \operatorname{Tor}_n^{\mathbb{C}G}\left(\mathrm{N}\mathcal{R}, \mathbb{C}\right) && \text{by taking } \mathrm{N}\mathcal{R} \otimes_{\mathrm{N}G} - \\ &= \dim_{\mathrm{N}\mathcal{R}} \operatorname{Tor}_n^{R_0}\left(\mathrm{N}\mathcal{R}, L^\infty X\right) && \text{by taking } R_0 \otimes_{\mathbb{C}G} \\ &= b_n^{(2)}\left(\mathcal{R}\right) && \text{by Definition 6.5,} \end{aligned}$$

where we set $R_0 \coloneqq L^\infty X \rtimes G \subseteq \mathbb{C}\mathcal{R}$; as $\mathbb{C}G$-modules, there is an isomorphism $R_0 \cong L^\infty X \otimes_{\mathbb{C}} \mathbb{C}G$. See [19, Theorem 4.2.5] for more details. ☐

6.4 Applications to orbit equivalence

Gaboriau's Theorem (6.6) has striking applications to orbit equivalence, which we now discuss.

Definition 6.7.
- Two free p.m.p. actions $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ are *orbit equivalent* if there is a Borel isomorphism $f : X' \to Y'$ between co-null subsets $X' \subseteq X$ and $Y' \subseteq Y$ such that

$$\forall x \in X',\ f\left(Gx \cap X'\right) = Hf(x) \cap Y'. \tag{†}$$

- Two free p.m.p. actions $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ are *stably orbit equivalent* if there is a Borel isomorphism $f : X' \to Y'$ between measurable subsets $X' \subseteq X$ and $Y' \subseteq Y$ with positive measure and such that (†) holds as well as

$$\frac{1}{\mu(X')} f_* \mu_{|X'} = \frac{1}{\nu\left(Y'\right)} \nu_{|Y'}.$$

  In this case, the *index* of the stable orbit equivalence is $\nu\left(Y'\right) / \mu\left(X'\right)$.

Two groups G and H are *(stably) orbit equivalent* if they admit (stably) orbit equivalent free p.m.p. actions.

Example 6.8.
- Commensurable groups are stably orbit equivalent.
- Two lattices Γ, Λ in a locally compact group G are stably orbit equivalent with index depending on $\mathrm{vol}(G/\Gamma)$ and $\mathrm{vol}(G/\Lambda)$.

Obervation 6.9. If $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ are orbit equivalent, then

$$\mathcal{R}\left(G \curvearrowright X\right) \cong \mathcal{R}\left(H \curvearrowright Y\right).$$

Now Theorem 6.6 implies that L^2-Betti numbers are invariants of orbit equivalence.

Corollary 6.10 (Invariance of L^2-Betti numbers under orbit equivalence)**.** *If two infinite countable groups G and H are orbit equivalent, then for all $n \in \mathbb{N}_{\geq 0}$,*

$$b_n^{(2)}(G) = b_n^{(2)}(H).$$

In turn, Corollary 6.10 has further applications to both computations of L^2-Betti numbers as well as orbit equivalence rigidity problems.

Corollary 6.11. *If G is amenable, then $b_n^{(2)}(G) = 0$ for all $n \in \mathbb{N}_{\geq 0}$.*

Proof. Amenable groups are orbit equivalent to $\mathbb{Z}$ by the Ornstein–Weiss Theorem [21], so this follows from Example 2.15. □

Corollary 6.12. *If $r \neq s$, then the free groups F_r and F_s are not orbit equivalent.*

Proof. Example 2.16 shows that $b_1^{(2)}\left(F_r\right) \neq b_1^{(2)}\left(F_s\right)$. □

Theorem 6.6 also gives the behaviour of L^2-Betti numbers under stable orbit equivalence.

Corollary 6.13. *If two infinite countable groups G and H are stably orbit equivalent with index C, then for all $n \in \mathbb{N}_{\geq 0}$,*

$$b_n^{(2)}(G) = C \cdot b_n^{(2)}(H).$$

6.5 Cost

We now introduce a new invariant, which can be thought of as an analogue of the rank for equivalence relations.

Recall that the *rank* of a group G is

$$d(G) \coloneqq \inf \{\#S \mid S \subseteq G,\ \langle S\rangle = G\}.$$

If $\mathcal{R} \subseteq X \times X$ is a standard equivalence relation on (X,μ), and $\Phi \subseteq \mathcal{R}$, we denote by $\langle\Phi\rangle$ the minimal equivalence relation on X containing Φ:

$$\langle\Phi\rangle \coloneqq \bigcup_{n=0}^{\infty} \Phi^n.$$

We say that Φ is a *graphing* of $\mathcal{R}$ if $\langle\Phi\rangle = \mathcal{R}$.

Definition 6.14. The *cost* of a standard equivalence relation $\mathcal{R}$ on (X,μ) is

$$\operatorname{cost}(\mathcal{R}) \coloneqq \inf \{\nu(\Phi) \mid \Phi \text{ is a graphing of } \mathcal{R}\},$$

where $\nu \coloneqq \mu \otimes \mu$ is the product measure on $X \times X$.

Proposition 6.15. *Let $G \curvearrowright (X,\mu)$ be a free p.m.p. action and $\mathcal{R} \coloneqq \mathcal{R}(G \curvearrowright X)$. Then*

$$\operatorname{cost}(\mathcal{R}) \leq d(G).$$

Proof. Given a generating set S of G, we can construct a graphing of $\mathcal{R}$ via

$$\Phi_S \coloneqq \bigcup_{s\in S} \{(sx,x) \mid x \in X\}.$$

The set $\Gamma_s \coloneqq \{(sx,x) \mid x \in X\}$ is the graph of s. This gives

$$\operatorname{cost}(\mathcal{R}) \leq \nu(\Phi_S) = \sum_{s\in S} \nu(\Gamma_s) = \#S \cdot \mu(X) = \#S,$$

which implies the result by taking the infimum over S. $\square$

For groups, we have the following relation between the rank and the first L^2-Betti number.

Proposition 6.16. $b_1^{(2)}(G) \leq d(G) - 1$.

Proof. Let S be a generating set for G, and consider the map $\bigoplus_{s\in S} \mathbb{C}G[s] \to \mathbb{C}G$ given by $[s] \mapsto (1-s)$ for each $s \in S$. Then

$$\bigoplus_{s\in S} \mathbb{C}G[s] \to \mathbb{C}G \to \mathbb{C} \to 0$$

is exact, so it gives a partial resolution of $\mathbb{C}$ over $\mathbb{C}G$, which can be used to compute $b_1^{(2)}(G)$ as explained in §6.2. Doing so yields $b_1^{(2)}(G) \leq \#S - 1$. $\square$

The following is an analogue of Proposition 6.16 for the cost.

Theorem 6.17 (Gaboriau [11])**.** *Let $G \curvearrowright (X,\mu)$ be a free p.m.p. action and $\mathcal{R} \coloneqq \mathcal{R}(G \curvearrowright X)$. Then*

$$b_1^{(2)}(G) \leq \operatorname{cost}(\mathcal{R}) - 1.$$

Sketch of proof. Let Φ be a graphing of $\mathcal{R}$. Analogously to the proof of Proposition 6.16, the idea is to use Φ to construct a projective resolution

$$V \to R_0 \to L^\infty X \to 0,$$

where $R_0 \coloneqq L^\infty X \rtimes G \subseteq \mathbb{C}\mathcal{R}$ as in the proof of Theorem 6.6. The most natural analogue of the projective resolution used in the proof of Proposition 6.16 doesn't quite work, so one needs to take a correction. See [19, Theorem 4.3.10] for more details. □

To summarise, given $G \curvearrowright (X,\mu)$ a free p.m.p. action with orbit relation $\mathcal{R}$, we have the chain of inequalities

$$b_1^{(2)}(G) \leq \operatorname{cost}(\mathcal{R}) - 1 \leq d(G) - 1. \tag{‡}$$

Gaboriau conjectured that the first inequality is in fact an equality.

Conjecture 6.18 (Gaboriau)**.** *Under the assumptions of Theorem 6.17,*

$$b_1^{(2)}(G) = \operatorname{cost}(\mathcal{R}) - 1.$$

For instance, the conjecture holds for free groups.

Example 6.19. Let $F_r \curvearrowright (X,\mu)$ be a free p.m.p. action of the rank-r free group F_r, with orbit relation $\mathcal{R}$. Then

$$b_1^{(2)}(F_r) = \operatorname{cost}(\mathcal{R}) - 1.$$

Proof. Note that $b_1^{(2)}(F_r) = r - 1 = d(F) - 1$ (see Example 2.16). The result then follows from the inequalities (‡). □

6.6 Rank gradient

Recall from Lück's Approximation Theorem (3.4) that if G is a residually finite group with residual chain $(N_i \trianglelefteq G)_{i\geq 1}$, then

$$b_1^{(2)}(G) = \lim_{i\to\infty} \frac{b_1(N_i)}{[G : N_i]}.$$

In view of the inequalities $b_1^{(2)}(G) \leq d(G) - 1$ (6.16) and $b_1(N_i) \leq d(N_i)$, it is natural to consider the following invariant.

Definition 6.20. The *rank gradient* of G with respect to a residual chain $N_* = (N_i \trianglelefteq G)_{i \geq 1}$ is

$$\operatorname{rg}(G, N_*) \coloneqq \lim_{i \to \infty} \frac{d(N_i) - 1}{[G : N_i]}.$$

Remark 6.21. In Definition 6.20, the limit on the right-hand side exists because the sequence $((d(N_i) - 1)/[G : N_i])_{i \geq 1}$ is non-increasing.

Proof. Fix $i \geq 1$. There exists a surjective morphism

$$\pi_i : F_{d(N_i)} \twoheadrightarrow N_i.$$

We have $N_{i+1} \trianglelefteq N_i$ with $[N_i : N_{i+1}] = [G : N_{i+1}]/[G : N_i]$; hence if $K \coloneqq \pi_i^{-1}(N_{i+1})$, then $K \trianglelefteq F_{d(N_i)}$ and $\left[F_{d(N_i)} : K\right] \leq [G : N_{i+1}]/[G : N_i]$. By the Nielsen–Schreier Theorem, K is a free group and

$$\operatorname{rk} K - 1 = \left[F_{d(N_i)} : K\right](d_i - 1) \leq \frac{[G : N_{i+1}]}{[G : N_i]}(d(N_i) - 1).$$

But p_i restricts to a surjective morphism $K \twoheadrightarrow N_{i+1}$, so $d(N_{i+1}) \leq \operatorname{rk} K$, which gives

$$\frac{d(N_{i+1}) - 1}{[G : N_{i+1}]} \leq \frac{d(N_i) - 1}{[G : N_i]}. \qquad \square$$

Remark 6.22. It is not known whether or not $\operatorname{rg}(G, N_*)$ depends on N_*.

Obervation 6.23. If G is a residually finite group of type F_∞, with a residual chain N_*, then there is an inequality

$$b_1^{(2)}(G) \leq \operatorname{rg}(G, N_*).$$

Proof. For each $i \geq 1$, we have $b_1(N_i) \leq d(N_i)$ (because a generating set for N_i gives a generating set for $H_1(N_i; \mathbb{Z}) \cong N_i / [N_i, N_i]$), so the inequality follows from Lück's Approximation Theorem (3.4). $\square$

Hence, the rank gradient and the cost both give upper bounds for the first L^2-Betti number (see Theorem 6.17 and Observation 6.23). The following theorem, originally motivated by 3-dimensional topology, shows that these upper bounds are equal.

Theorem 6.24 (Abért–Nikolov [1])**.** *Let G be a residually finite group of type* F_∞*, with a residual chain N_*. Let $\hat{G} \coloneqq \varprojlim_i G/N_i$ be the profinite completion of G, and note that there is a free p.m.p. action $G \curvearrowright \hat{G}$. Then*

$$\operatorname{rg}(G, N_*) = \operatorname{cost}\left(\mathcal{R}\left(G \curvearrowright \hat{G}\right)\right) - 1.$$

Proof of ($\geq$). Write $X \coloneqq \hat{G}$, with probability measure μ, and $\mathcal{R} \coloneqq \mathcal{R}\left(G \curvearrowright X\right)$. For each $i \geq 1$, consider $\pi_i : \hat{G} \to G/N_i$. Let $A_i \coloneqq \operatorname{Ker} \pi_i$, so that

$$\mu\left(A_i\right) = \frac{1}{[G : N_i]}.$$

Let $\underline{s} = \{s_1, \ldots, s_d\}$ be a generating set for N_i and let $\underline{t} = \{t_1, \ldots, t_c\}$ be a set of coset representatives for N_i in G. There are translation maps

$$\sigma_j \coloneqq (s_j \cdot -) : A_i \to A_i \quad \text{and} \quad \tau_k \coloneqq (t_k \cdot -) : A \to t_k A.$$

In fact, $\Phi \coloneqq \{\sigma_j\}_{1 \leq j \leq d} \cup \{\tau_k\}_{1 \leq k \leq c}$ is a graphing for $\mathcal{R}$. This yields

$$\begin{aligned} \operatorname{cost}\left(\mathcal{R}\right) \leq \nu\left(\Phi\right) = d \cdot \mu\left(A_i\right) + c \cdot \mu\left(A_i\right) &= \frac{d\left(N_i\right)}{[G : N_i]} + \frac{[G : N_i]}{[G : N_i]} \\ &= \frac{d\left(N_i\right)}{[G : N_i]} + 1 \xrightarrow[i \to \infty]{} \operatorname{rg}\left(G, N_*\right) + 1. \end{aligned}$$

$\square$

Talk 7 – L^2-torsion and mapping tori

Matteo Wei

References: [18, §7.4].

7.1 L^2-torsion

In order to define L^2-torsion, we will consider a certain determinant of the boundary maps of the L^2-chain complex.

Definition 7.1. Let $f : U \to V$ be a morphism of Hilbert G-modules. The *Fuglede–Kadison determinant* of f is

$$\det_{\mathcal{N}G}(f) \coloneqq \exp\left(\int_0^\infty (\log \lambda)\, \mathrm{d}F_f(\lambda)\right),$$

where F_f is the spectral density function of f, and $\mathrm{d}F_f$ is the associated measure on the Borel σ-algebra of $\mathbb{R}$, given by $\mathrm{d}F_f\left((a,b]\right) = F_f(b) - F_f(a)$.

Contrary to the classical determinant, the Fuglede–Kadison determinant only sees the positive eigenvalues.

Example 7.2. (i) $\det_{\mathcal{N}G}(0) = 1$.

(ii) Assume that G is finite, so that f is a linear G-equivariant map of finite-dimensional unitary G-representations. Let $\lambda_1, \ldots, \lambda_r$ be the positive eigenvalues of f^*f. Then

$$\det_{\mathcal{N}G}(f) = \left(\prod_{i=1}^r \lambda_i\right)^{1/(2|G|)}.$$

We will say that

- a morphism f of Hilbert G-modules is of *determinant class* if $\det_{\mathcal{N}G}(f) \neq 0$,
- a chain complex C_* of Hilbert G-modules is of *determinant class* if each boundary map $\partial_n : C_n \to C_{n-1}$ is of determinant class, and
- a group G is of $(\det \geq 1)$ *class* if for every $m, n \in \mathbb{N}_{\geq 1}$ and for every $A \in M_{m\times n}(\mathbb{Z}G)$, the morphism $A : \left(\ell^2 G\right)^n \to \left(\ell^2 G\right)^m$ has $\det_{\mathcal{N}G}(A) \geq 1$.

Note that if a group G is of $(\det \geq 1)$ class, and if Y is a free G-CW-complex of type F_∞, then the L^2-chain complex $C_*^{(2)}(G \curvearrowright Y)$ is of determinant class.

Conjecture 7.3. *All groups are of* $(\det \geq 1)$ *class.*

Throughout this section, we will only work with groups that are of $(\det \geq 1)$ class.

Definition 7.4. Let C_* be a chain complex of Hilbert-G-modules which we assume to be of determinant class. The L^2*-torsion* of C_* is

$$\rho\left(C_*\right) \coloneqq -\sum_{p=0}^{\infty}(-1)^p \log\left(\det_{\mathrm{N}G}\left(\partial_p\right)\right).$$

We say that a chain complex C_* of Hilbert-G-modules is L^2*-acyclic* if its L^2-homology is trivial. We say that C_* is det-L^2*-acyclic* if C_* is of determinant class and L^2-acyclic.

7.2 Cellular L^2-torsion

Let G be a group of $(\det \geq 1)$ class, and let X be a free G-CW-complex of type F_∞. We say that X is L^2*-acyclic* if $C_*^{(2)}(G \curvearrowright X)$ is L^2-acyclic (which is equivalent to it being det-L^2-acyclic since G is of $(\det \geq 1)$ class).

Definition 7.5. Provided that X is L^2-acyclic, we define the *cellular* L^2*-torsion* of X over $\mathrm{N}G$ by

$$\rho_{CW}^{(2)}\left(X; \mathrm{N}G\right) \coloneqq \rho\left(C_*^{(2)}\left(G \curvearrowright X\right)\right).$$

The next proposition gives some basic properties of cellular L^2-torsion.

Proposition 7.6. *Throughout, X, Y, X_0, X_1, X_2 are free G-CW-complexes of type F_∞, unless otherwise stated. All maps between them are assumed to be G-equivariant.*

(i) (Homotopy invariance) *Let $f : X \to Y$ be a G-equivariant homotopy equivalence. Assume that X and Y are L^2-acyclic and that G has vanishing Whitehead group (see* [18, §3.1.2]*). Then*

$$\rho_{CW}^{(2)}\left(X; \mathrm{N}G\right) = \rho_{CW}^{(2)}\left(Y; \mathrm{N}G\right).$$

(ii) (Sum formula) *Consider the following pushout square:*

$$\begin{array}{ccc} X_0 & \xrightarrow{j_1} & X_1 \\ \Big\downarrow{\scriptstyle j_2} & & \Big\downarrow{\scriptstyle i_1} \\ X_2 & \xrightarrow{i_2} & X \end{array}$$

If X_0, X_1, X_2 are L^2-acyclic, then so is X, and

$$\rho_{CW}^{(2)}\left(X; \mathrm{N}G\right) = \rho_{CW}^{(2)}\left(X_1; \mathrm{N}G\right) + \rho_{CW}^{(2)}\left(X_2; \mathrm{N}G\right) - \rho_{CW}^{(2)}\left(X_0; \mathrm{N}G\right).$$

(iii) (Product formula) *Let X be a free G-CW-complex, Y a free H-CW-complex, both of type F_∞, with X L^2-acyclic. Then the free $(G \times H)$-CW-complex $X \times Y$ is L^2-acyclic and*

$$\rho_{CW}^{(2)}\left(X \times Y; \mathrm{N}(G \times H)\right) = \chi\left(H \backslash Y\right) \cdot \rho_{CW}^{(2)}\left(X; \mathrm{N}G\right).$$

(iv) (Restriction) *Let H be a finite-index subgroup of G, and let $\operatorname{Res}_H^G X$ be the free H-CW-complex obtained from X by restricting the G-action to an H-action. Then X is L^2-acyclic if and only if $\operatorname{Res}_H^G X$ is, and in this case,*

$$\rho^{(2)}_{CW}(X; \mathrm{N}G) = [G : H] \cdot \rho^{(2)}_{CW}\left(\operatorname{Res}_H^G X; \mathrm{N}H\right).$$

(v) (Induction) *Let H be a subgroup of G, and let X be a free H-CW-complex of type F_∞. Then the free G-CW-complex $G \times_H X$ (with underlying set $G/H \times X$) of type F_∞ is L^2-acyclic if and only if X is L^2-acyclic, and in this case,*

$$\rho^{(2)}_{CW}(G \times_H X; \mathrm{N}G) = \rho^{(2)}_{CW}(X; \mathrm{N}H).$$

Next, we specialise to the case where G is the fundamental group of a CW-complex X acting on the universal cover $\tilde{X}$.

Definition 7.7. Let X be a finite, connected CW-complex. Then the universal cover $\tilde{X}$ is a free $\pi_1 X$-CW-complex of type F_∞, and we set

$$\rho^{(2)}_{\widetilde{CW}}(X) \coloneqq \rho^{(2)}_{CW}\left(\tilde{X}; \mathrm{N}(\pi_1 X)\right).$$

Proposition 7.8. *Throughout, X, Y, X_0, X_1, X_2 are finite, connected CW-complexes.*

(i) (Homotopy invariance) *Let $f : X \to Y$ be a homotopy equivalence. Assume that $\tilde{X}$ and $\tilde{Y}$ are L^2-acyclic and that $\pi_1 X = \pi_1 Y$ has vanishing Whitehead group (see* [18, §3.1.2]*). Then*

$$\rho^{(2)}_{\widetilde{CW}}(X) = \rho^{(2)}_{\widetilde{CW}}(Y).$$

(ii) (Sum formula) *Consider the following pushout square of π_1-injective maps:*

$$\begin{tikzcd} X_0 \arrow[r, "j_1"] \arrow[d, "j_2"] & X_1 \arrow[d, "i_1"] \\ X_2 \arrow[r, "i_2"] & X \end{tikzcd}$$

If $\tilde{X}_0, \tilde{X}_1, \tilde{X}_2$ are L^2-acyclic, then so is $\tilde{X}$, and

$$\rho^{(2)}_{\widetilde{CW}}(X) = \rho^{(2)}_{\widetilde{CW}}(X_1) + \rho^{(2)}_{\widetilde{CW}}(X_2) - \rho^{(2)}_{\widetilde{CW}}(X_0).$$

(iii) (Product formula) *Assume that $\tilde{X}$ is L^2-acyclic. Then $\widetilde{X \times Y}$ is L^2-acyclic, and*

$$\rho^{(2)}_{\widetilde{CW}}(X \times Y) = \chi(Y) \cdot \rho^{(2)}_{\widetilde{CW}}(X).$$

(iv) (Multiplicativity) *If $X \to Y$ is a d-sheeted covering, then X is L^2-acyclic if and only if Y is, and in this case*

$$\rho^{(2)}_{\widetilde{CW}}(X) = d \cdot \rho^{(2)}_{\widetilde{CW}}(Y).$$

7.3 Mapping tori

Definition 7.9. Given a cellular map $f : X \to Y$, we define

- The *mapping cylinder* $C_f := ((X \times I) \amalg Y) / ((x, 1) \sim f(x))$, and
- If $X = Y$, the *mapping torus* $T_f := (X \times I) / ((x, 1) \sim (f(x), 0))$.

Now consider a group G, which we assume to be of type F_∞, and of $(\det \geq 1)$ class. Let X be a $K(G, 1)$ space. An automorphism $\varphi \in \mathrm{Aut}(G)$ induces a cellular map $f : X \to X$, unique up to homotopy, such that $f_* = \varphi$ on $\pi_1 X = G$.

Obervation 7.10. $\pi_1 T_f = G \rtimes_\varphi \mathbb{Z}$.

Fact 7.11. *If G is of $(\det \geq 1)$ class, then so is $G \rtimes_\varphi \mathbb{Z}$.*

We now introduce the L^2-torsion of the automorphism φ.

Definition 7.12. Given $\varphi \in \mathrm{Aut}(G)$ inducing a cellular map $f : X \to X$ on a $K(G, 1)$ space X, its *L^2-torsion* is defined by

$$\rho^{(2)}_{\mathrm{aut}}(\varphi) := \rho^{(2)}_{\widetilde{CW}}(T_f) .$$

The previous properties of cellular L^2-torsion (Proposition 7.8) translate to the following properties for the L^2-torsion of an automorphism.

Proposition 7.13. (i) *If $G = G_1 *_{G_0} G_2$ and $\varphi = \varphi_1 *_{\varphi_0} \varphi_2$, then*

$$\rho^{(2)}_{\mathrm{aut}}(\varphi) = \rho^{(2)}_{\mathrm{aut}}(\varphi_1) + \rho^{(2)}_{\mathrm{aut}}(\varphi_2) - \rho^{(2)}_{\mathrm{aut}}(\varphi_0) .$$

(ii) *Given isomorphisms $\phi : G \to H$ and $\psi : H \to G$, we have*

$$\rho^{(2)}_{\mathrm{aut}}(\phi \circ \psi) = \rho^{(2)}_{\mathrm{aut}}(\psi \circ \phi) .$$

In particular, $\rho^{(2)}_{\mathrm{aut}}$ is conjugacy-invariant and descends to $\mathrm{Out}(G)$.

(iii) *For $n \in \mathbb{N}_{\geq 0}$, we have*

$$\rho^{(2)}_{\mathrm{aut}}(\varphi^n) = n \cdot \rho^{(2)}_{\mathrm{aut}}(\varphi) .$$

We conclude with a few facts on L^2-torsion.

Fact 7.14. (i) *There is a constant C_n such that, if X is a hyperbolic manifold of dimension $2n + 1$, then*

$$\rho^{(2)}_{\widetilde{CW}}(X) = (-1) C_n \operatorname{vol}(X).$$

(ii) *There is an algorithm to compute the cellular L^2-torsion from the matrices of the differentials.*

(iii) *If Σ is an oriented closed connected surface and $f \in \mathrm{MCG}(\Sigma)$, then f is pseudo-Anosov if and only if $\rho^{(2)}_{\mathrm{aut}}(f_*) = 0$.*

References

[1] Miklós Abért and Nikolay Nikolov, *Rank gradient, cost of groups and the rank versus Heegaard genus problem*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 5, 1657–1677. MR2966663

[2] Mladen Bestvina and Mark Feighn, *A hyperbolic* $\mathrm{Out}(F_n)$*-complex*, Groups Geom. Dyn. **4** (2010), no. 1, 31–58. MR2566300

[3] Mladen Bestvina and Koji Fujiwara, *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol. **6** (2002), 69–89. MR1914565

[4] Brian H. Bowditch, *Tight geodesics in the curve complex*, Invent. Math. **171** (2008), no. 2, 281–300. MR2367021

[5] Indira Chatterji, Sam Hughes, and Peter Kropholler, *The first* ℓ^2*-Betti number and groups acting on trees*, Proc. Edinb. Math. Soc. (2) **64** (2021), no. 4, 916–923. MR4349416

[6] F. Dahmani, V. Guirardel, and D. Osin, *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces*, Mem. Amer. Math. Soc. **245** (2017), no. 1156, v+152. MR3589159

[7] M. W. Davis and I. J. Leary, *The* l^2*-cohomology of Artin groups*, J. London Math. Soc. (2) **68** (2003), no. 2, 493–510. MR1994696

[8] Jacob Feldman and Calvin C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. I*, Trans. Amer. Math. Soc. **234** (1977), no. 2, 289–324. MR578656

[9] R. Frigerio, M. B. Pozzetti, and A. Sisto, *Extending higher-dimensional quasi-cocycles*, J. Topol. **8** (2015), no. 4, 1123–1155. MR3431671

[10] Damien Gaboriau, *Sur la (co-)homologie* L^2 *des actions préservant une mesure*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), no. 5, 365–370. MR1751672

[11] ———, *Invariants* l^2 *de relations d'équivalence et de groupes*, Publ. Math. Inst. Hautes Études Sci. **95** (2002), 93–150. MR1953191

[12] Damien Gaboriau and Camille Noûs, *On the top-dimensional* ℓ^2*-Betti numbers*, Ann. Fac. Sci. Toulouse Math. (6) **30** (2021), no. 5, 1121–1137. MR4401387

[13] Michael Hull and Denis Osin, *Transitivity degrees of countable groups and acylindrical hyperbolicity*, Israel J. Math. **216** (2016), no. 1, 307–353. MR3556970

[14] Holger Kammeyer, *Introduction to* ℓ^2*-invariants*, Lecture Notes in Mathematics, vol. 2247, Springer, Cham, 2019. MR3971279

[15] ———, ℓ^2*-Betti numbers*, Bounded cohomology and simplicial volume, 2023, pp. 57–62. MR4496345

[16] Yoshikata Kida, *The mapping class group from the viewpoint of measure equivalence theory*, Mem. Amer. Math. Soc. **196** (2008), no. 916, viii+190. MR2458794

[17] Wolfgang Lück, L^2*-Betti numbers of mapping tori and groups*, Topology **33** (1994), no. 2, 203–214. MR1273782

[18] Wolfgang Lück, L^2*-invariants: theory and applications to geometry and* K*-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 44, Springer-Verlag, Berlin, 2002. MR1926649

[19] Clara Löh, *Ergodic theoretic methods in group homology—a minicourse on* L^2*-Betti numbers in group theory*, SpringerBriefs in Mathematics, Springer, Cham, [2020] ©2020. MR4177798

[20] Ashot Minasyan and Denis Osin, *Acylindrical hyperbolicity of groups acting on trees*, Math. Ann. **362** (2015), no. 3-4, 1055–1105. MR3368093

[21] Donald S. Ornstein and Benjamin Weiss, *Ergodic theory of amenable group actions. I. The Rohlin lemma*, Bull. Amer. Math. Soc. (N.S.) **2** (1980), no. 1, 161–164. MR551753

[22] D. Osin, *On acylindrical hyperbolicity of groups with positive first ℓ^2-Betti number*, Bull. Lond. Math. Soc. **47** (2015), no. 5, 725–730. MR3403956

[23] ———, *Acylindrically hyperbolic groups*, Trans. Amer. Math. Soc. **368** (2016), no. 2, 851–888. MR3430352

[24] Jesse Peterson and Andreas Thom, *Group cocycles and the ring of affiliated operators*, Invent. Math. **185** (2011), no. 3, 561–592. MR2827095