Computations and lower bounds for scl and the relative Gromov seminorm

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Statement of originality

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the preface and specified in the text. It is not substantially the same as any work that has already been submitted, or, is being concurrently submitted, for any degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the preface and specified in the text. It does not exceed the prescribed word limit for the relevant Degree Committee.

Abstract

The main purpose of this thesis is to give computations and estimates of stable commutator length, an invariant of groups that measures homological complexity and is connected to several notions of negative curvature in geometric group theory.

A first aspect of this is to give exact computations, motivated by questions of rationality. A particular case of interest is that of surface groups. We introduce the relative Gromov seminorm, a new invariant that serves as an intermediate step in computations of scl. We show that several results about scl in free groups generalise to the relative Gromov seminorm in surface groups. We explain how bounded cohomology provides a dual to the relative Gromov seminorm, and apply those ideas to obtain new computations of scl.

Another aspect of this thesis is to obtain lower bounds for stable commutator length in the presence of negative curvature. We do this by developing a new geometric method, where surfaces estimating scl are equipped with a combinatorial geometric structure called an angle structure, and for which there is a notion of curvature and a version of the Gauß–Bonnet formula leading to estimates of the Euler characteristic. We show in particular that our method yields a new proof of a theorem of Heuer on a sharp spectral gap for right-angled Artin groups.

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CONTENTS

INTRODUCTION

Let us start by taking a step back in time and returning briefly to the prehistory of geometric group theory. The subject is usually considered to have been born from Misha Gromov's essay on hyperbolic groups [45], which continues to have a strong influence on today's research. Before that, a key idea that led to Gromov's work was that of small-cancellation theory, which aims to understand the word problem by analysing the combinatorics of group presentations. Small-cancellation theory was created by abstracting Max Dehn's solution to the word problem in surface groups [31], and we will discuss some of the ideas behind Dehn's work for a moment.

Isoperimetry. Dehn's algorithm for the word problem in surface groups was based on the idea of *isoperimetry*. The question, in algebraic terms, is as follows: given a group presentation $G = \langle S \mid R \rangle$, and a word w over the finite generating set S which is known to represent the trivial element in G, what is the size of a shortest expression of w as a product of conjugates of relators in R, and how does this size increase with the word length of w?

The application to the word problem is that, if one knows a bound on the minimal size of an expression of a w as a product of conjugates of relators given the word length of w, then one can decide whether or not w is trivial — and hence solve the word problem in G — by only testing such expressions of size at most the given bound, of which there are a finite number since the generating set S is finite.

Let us rephrase the isoperimetry problem topologically. The group G can be realised as the fundamental group of a presentation 2-complex X of $\langle S \mid R \rangle$. A word w

over S representing the trivial element in G corresponds to a based loop $\gamma : S^1 \to X$ in the 1-skeleton of X, which is contractible in X. An expression of w as a product of conjugates of relators corresponds to a *Van Kampen diagram*, which gives rise in particular to a map $f : D^2 \to X$ from a disc whose restriction to the boundary $\partial D^2 = S^1$ is γ. Hence, an imprecise topological reformulation of the isoperimetry problem would be as follows: given a contractible loop $\gamma : S^1 \to X$ in a space X, what is the minimal 'complexity' of maps $f : D^2 \to X$ such that $f_{|\partial D^2} = \gamma$?

Today, the word and isoperimetry problems are very well-understood in the context of hyperbolic groups. Now let us note that those problems are fundamentally concerned with *homotopy*. Considering the wealth of ideas that their investigation brought to geometric group theory, we ask: what about their *homological* counterpart?

The homological isoperimetry problem. Understanding the homological analogue of the isoperimetry problem is the main concern of this thesis.

Let us explain what this means more precisely. Observe that a loop γ is homotopically trivial — i.e. contractible — if and only if it bounds a disc; analogously, γ is homologically trivial if and only if it bounds an oriented compact surface.

This leads to what we call the *homological isoperimetry problem*: given a homologically trivial loop $\gamma : S^1 \to X$ in a space X, what is the minimal complexity of an oriented compact surface bounding γ? For us, the complexity of an oriented compact surface Σ will be measured by its (negative) Euler characteristic, or more precisely by its *reduced Euler characteristic* $\chi^-(\Sigma)$ — i.e. the Euler characteristic of Σ after discarding all disc- and sphere-components. At this point, we also observe that, in the homological case, there is no reason to restrict one's attention to a single loop in X — one can instead consider any compact 1-manifold mapping to X, or in other words, any map $\gamma : \coprod S^1 \to X$ from a disjoint union of circles. After stabilising under finite covers, this naturally leads to the definition of *stable commutator length* — usually abbreviated scl — which will be our main object of interest:

Definition. The *stable commutator length* of $\gamma : \coprod S^1 \to X$ is defined by

$$\mathrm{scl}_X(\gamma) := \inf_{f,\Sigma} \frac{-\chi^-(\Sigma)}{2n(\Sigma)},$$

where the infimum is over all maps $f : \Sigma \to X$ from oriented compact surfaces whose restriction to $\partial\Sigma$ represents a power $n(\Sigma)$ of γ. Such maps are called *admissible surfaces*, and we agree that $\mathrm{scl}_X(\gamma) = \infty$ if there is no admissible surface — i.e. if γ is not homologically trivial.

In addition, an admissible surface attaining the above infimum is called *extremal*.

A precise definition of what it means for $f_{|\partial\Sigma}$ to represent a power of γ will be given in §I.1.d.

The above definition of scl is given for families of loops in a topological space X, but this also induces a function on conjugacy classes of 1-chains (which will be defined in §I.1.b) on the group $G \cong \pi_1 X$. Throughout this introduction, we will freely talk about scl on groups or on topological spaces, and precise definitions will be given in Chapter I.

We should point out that Gersten [41] introduced another homological analogue of isoperimetry — which he calls *weak isoperimetry*. The main differences between Gersten's notion and scl is that Gersten works in the universal cover and in a cellular setting, whereas we work in the base space and in a singular setting.

The role of scl in geometric group theory. Like the word and isoperimetry problems, stable commutator length has now been revealed to have many connections with various topics in geometric group theory and geometric topology. Among many examples, scl was successfully used by Calegari [14] to give partial answers to a famous question of Gromov on closed surface subgroups of hyperbolic groups. It was shown by Bavard [4] that scl is dual to quasimorphisms, which are related to bounded cohomology and also play a prominent role in symplectic topology. Stable commutator length was utilised by Heuer and Löh [49] to manufacture examples of manifolds of specified simplicial volume. Calegari [15] also exhibited connections with the dynamics of circle actions via the rotation quasimorphism.

Most prominent of all is the connection between stable commutator length and hyperbolic-like behaviour in groups. On the one hand, amenable groups are known to have vanishing stable commutator length, while on the opposite end of the spectrum, many groups with some notion of negative curvature are known to exhibit large values of scl. This includes for instance free groups [32], Gromov-hyperbolic groups [19], mapping class groups [5], certain graphs of groups and notably 3-manifold groups [26], as well as right-angled Artin groups [48]. The general philosophy is that the stable commutator length of a group element seems to measure the negative curvature of that element.

Computing scl and the homological Dehn problem. Hence, stable commutator length is an invariant of groups that is defined as some kind of homological complexity and appears to be related to group-theoretical negative curvature, as well as many other topics. The next question is: how to compute scl? In general, this seems to be a very difficult problem. A breakthrough was however made by Calegari [17], who proved that in free groups, not only is scl computable, but there is in fact an algorithm which computes an extremal surface given a map $\gamma : \coprod S^1 \to X$, where X is a bouquet of circles. In particular, scl has rational values in free groups, and this is why Calegari's Theorem is often called the *Rationality Theorem*. Calegari's work has led to many generalisations [18, 22, 24, 27, 65, 68], usually in various kinds of graphs of groups.

Considering the impact that Dehn's understanding of homotopic isoperimetry in surface groups has had on modern geometric group theory, the next goal seems to be to understand scl — that is, homological isoperimetry — in closed surface groups. It is somewhat surprising that fifteen years after Calegari's work, and more than a century after Dehn's algorithm, the problem of computing scl in surface groups — which we call the *homological Dehn problem* — remains elusive.

Isometric embeddings. A lot of the work that we present in this thesis is aiming to improve understanding of stable commutator length in surface groups in order to make progress towards the homological Dehn problem. A first aspect of this is the construction of *isometric embeddings*, i.e. maps of spaces $\iota : X \to Y$ that are π_1-injective and such that for any map $\gamma : \coprod S^1 \to X$ that is homologically trivial in X, there should be an equality $\mathrm{scl}_X(\gamma) = \mathrm{scl}_Y(\iota\gamma)$.

Specifically, we would like to construct isometric embeddings between oriented compact surfaces. One motivation for this problem is that it could lead to computations of scl for non-filling maps: if a map $\gamma : \coprod S^1 \to S$ has image contained in a proper subsurface T of S, then we hope to be able to understand $\mathrm{scl}_S(\gamma)$ by computing scl in T, which is usually easier because T will be a surface with boundary, and will therefore have free fundamental group.

In order for an embedding $\iota : T \hookrightarrow S$ to be isometric for scl, it is natural to ask in addition that ι should preserve homological triviality, in the sense that $\gamma : \coprod S^1 \to T$ should be homologically trivial if and only if $\iota\gamma$ is homologically trivial. This amounts to saying that $\mathrm{scl}_T(\gamma) < \infty$ if and only if $\mathrm{scl}_S(\iota\gamma) < \infty$, and is also equivalent to imposing that ι induces an injective map $H_1(T;\mathbb{Q}) \to H_1(S;\mathbb{Q})$. Under this condition, we can indeed prove that ι is an isometric embedding, at least in the non-closed case:

Theorem IV.3.2 (scl-isometric embedding of surfaces)**.** *Let S be an oriented, compact, connected surface with non-empty boundary, and let $T \subseteq S$ be a subsurface*[1] *such that the inclusion-induced map $H_1(T) \to H_1(S)$ is injective. Then the embedding*

$$\iota : T \hookrightarrow S$$

is isometric for scl.

Unfortunately, Theorem IV.3.2 is a result about surfaces with boundary, and hence really about scl in free groups. In fact, it becomes false if S is a closed surface:

Example. Consider the inclusion of surfaces $T \hookrightarrow S$ of Figure 1. Then the induced

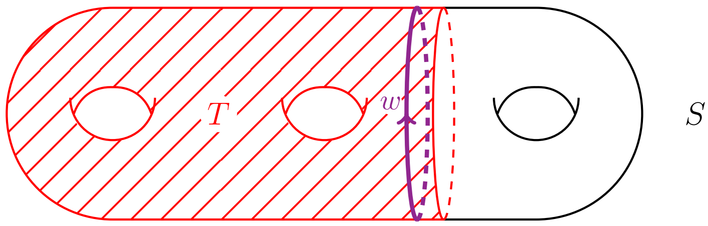


Figure 1: An inclusion of surfaces that is H_1-injective but not isometric for scl.

map $H_1(T) \to H_1(S)$ is injective. However, let $\gamma : S^1 \to T$ be the boundary loop of T. It is a general fact that an oriented compact surface is extremal for its boundary (see Corollary V.1.13), and therefore

$$\mathrm{scl}_T(\gamma) = \frac{-\chi^-(T)}{2} = \frac{3}{2}.$$

On the other hand, there is a surface Σ of genus 1 with one boundary component bounding γ in S, so that

$$\mathrm{scl}_S(\gamma) \leqslant \frac{-\chi^-(\Sigma)}{2} = \frac{1}{2}.$$

Therefore, the inclusion $T \hookrightarrow S$ does not preserve scl.

[1] We do not require the subsurface T of S to have its boundary contained in ∂S.

However, this is not the end of the story. Given a map $\gamma : \coprod S^1 \to X$, there is a relative singular homology $H_2(X, \gamma)$, such that any admissible surface $f : \Sigma \to X$ for γ represents a class $f_*[\Sigma] \in H_2(X, \gamma)$ — see §I.2.a. In the above example, the two admissible surfaces $T \to S$ and $\Sigma \to S$ represent classes in $H_2(S, \gamma)$ that differ by the fundamental class $[S] \in H_2(S) \hookrightarrow H_2(S, \gamma)$. Note that this is a phenomenon that cannot occur if S has nonempty boundary, because then $H_2(S) = 0$ and all admissible surfaces bounding γ projectively represent the same class in $H_2(S, \gamma)$. One might therefore ask whether we still get an isometric embedding when a relative homology class is fixed.

The relative Gromov seminorm. It is this question that leads us to consider the *relative Gromov seminorm*, that is to say the ℓ^1-seminorm $\|\cdot\|_1$ on $H_2(S, \gamma)$ with respect to the usual basis of singular simplices — see §I.2 for more details. Given a class $\alpha \in H_2(S, \gamma)$, its seminorm $\|\alpha\|_1$ is also equal to the infimum of the complexities of all admissible surfaces for γ which represent α in relative homology — see Proposition I.2.11. Hence, the question of whether an embedding is isometric for scl with fixed relative homology can be made more precise by asking whether it is isometric for $\|\cdot\|_1$.

The relative Gromov seminorm is related to scl in the following way:

Proposition I.2.12 (Gromov seminorm and scl)**.** *Let X be a topological space and $\gamma : \coprod S^1 \to X$. Then*

$$\mathrm{scl}_X(\gamma) = \frac{1}{4} \inf \left\{ \|\alpha\|_1 \mid \alpha \in H_2(X, \gamma; \mathbb{Q}),\ \partial\alpha = \left[\coprod S^1\right] \right\},$$

where $\partial : H_2(X, \gamma; \mathbb{Q}) \to H_1\left(\coprod S^1; \mathbb{Q}\right)$ is the boundary map in the exact sequence of the pair (X, γ) (see Proposition I.2.2(i)*).*

One of the points of this thesis is to promote the study of the relative Gromov seminorm on $H_2(X, \gamma)$ as an intermediate step in the computation of $\mathrm{scl}_X(\gamma)$ when X has nontrivial second homology. This approach separates the problem of computing scl into two steps: one can try to understand the relative Gromov seminorm first, and then investigate the infimum over $H_2(X, \gamma)$.

In particular, working with the relative Gromov seminorm leads to the following generalisation of Theorem IV.3.2 to closed surfaces:

Theorem IV.3.4 (ℓ^1-isometric embedding of surfaces)**.** *Let S be an oriented, compact, connected surface, let $T \subseteq S$ be a π_1-injective subsurface. For any $\gamma : \coprod S^1 \to T$, if*

$\iota : T \hookrightarrow S$ is the inclusion, then the induced map

$$\iota_* : H_2(T, \gamma) \to H_2(S, \iota\gamma)$$

is isometric for $\|\cdot\|_1$.

Theorem IV.3.4 does in fact imply a stronger version of Theorem IV.3.2 where we only assume that the embedding $\iota : T \hookrightarrow S$ is π_1-injective, rather than H_1-injective — see Corollary IV.3.5.

Observe that, if S is a closed surface and a relative class $\alpha \in H_2(S, \gamma)$ actually lives in the image of $H_2(T, \gamma)$ for some proper, π_1-injective subsurface $T \subseteq S$, then Theorem IV.3.4 says that one can compute $\|\alpha\|_1$ indifferently in S or in T, and Calegari's Rationality Theorem implies that $\|\cdot\|_1$ is rational in T since $\partial T \neq \varnothing$; it follows that $\|\alpha\|_1 \in \mathbb{Q}$. Note that it is also known by different arguments that $\mathrm{scl}_S(\gamma)$ is rational when γ is non-filling [35].

The core of the work in proving Theorems IV.3.2 and IV.3.4 is to reduce admissible surfaces to certain standard forms for the purpose of computing stable commutator length or the relative Gromov seminorm. This is explained in Chapter II, and we believe that those standard forms can be used for further work on scl in surface groups. The remaining part of the proofs of Theorems IV.3.2 and IV.3.4 is given in Chapter IV.

Bavard duality. Our next goal is to better understand the relative Gromov seminorm and to what extent it can be used to carry our understanding of scl on free groups to the world of closed surfaces. The discovery by Bavard [4] that scl is dual to quasimorphisms was a pioneering result that opened the way to many developments around stable commutator length, and our next step is to find an analogue for the relative Gromov seminorm. Such an analogue exists, and is given by bounded cohomology:

Theorem III.2.4 (Bavard duality for the relative Gromov seminorm)**.** *Let X be a topological space and $\gamma : \coprod S^1 \to X$. Given a real class $\alpha \in H_2(X, \gamma; \mathbb{R})$, the relative Gromov seminorm of α is given by*

$$\|\alpha\|_1 = \sup \left\{ \frac{\langle u, \alpha \rangle}{\|u\|_\infty} \;\middle|\; u \in H_b^2(X; \mathbb{R}) \smallsetminus \{0\} \right\}.$$

The background on bounded cohomology leading to Theorem III.2.4 is given in Chapter III.

It should be said that Theorem III.2.4 is obtained by combining several deep results in bounded cohomology due to other authors and was certainly known to some experts. Our contribution here is to point out the importance of the relative Gromov seminorm and its connection with stable commutator length on the one hand, and with bounded cohomology on the other hand.

As a first application, Theorem III.2.4 can be used to translate a result of Bucher et al. [11] in bounded cohomology into the fact that, in graphs of groups with amenable edge groups, vertex groups are isometrically embedded for the relative Gromov seminorm — see Theorem IV.4.2. This recovers Theorem IV.3.4 above — though the proof takes a completely different route — and can also lead to computations of scl in certain graphs of groups — see in particular §VI.5.

Another aspect of Theorem III.2.4 that we explore is an algebraic interpretation via an analogue of the Hopf formula. Recall that the Hopf formula computes the second homology of a group given by a presentation; we show that this can be adapted to compute $H_2\left(X, \gamma\right)$ when X is a $K\left(G, 1\right)$ space, $\gamma : S^1 \to X$ is a loop, and the group G is given by a presentation — see Theorem III.3.2. We then explain in Theorem III.3.4 how this gives a purely algebraic interpretation of our analogue of Bavard duality for the relative Gromov seminorm (Theorem III.2.4).

The bounded Euler class. With this version of Bavard duality in hand, we can attempt to compute the relative Gromov seminorm from a dual perspective, by estimating it with bounded cohomology. For stable commutator length, the Hahn–Banach Theorem implies that every map $\gamma : \coprod S^1 \to X$ admits a quasimorphism that is *extremal*, in the sense that it achieves the supremum in Bavard duality. For the same reason, every $\alpha \in H_2\left(X, \gamma\right)$ admits an extremal class $u \in H_b^2(X)$ — i.e. a class attaining the supremum in Theorem III.2.4. However finding extremal quasimorphisms or classes explicity is in general difficult.

In the context of an oriented, compact, connected, hyperbolic surface S with non-empty boundary, Calegari [15] showed that, for a map $\gamma : \coprod S^1 \to S$ which is bounded by an *immersed* admissible surface $f : \Sigma \looparrowright S$, the *rotation quasimorphism* — which arises from the dynamics of the action of $\pi_1 S$ on $\partial\mathbb{H}^2 = S^1$ — is extremal, and so is f — see §V.1.f.

In the same vein as Theorem IV.3.4, we generalise Calegari's Theorem to a result about the relative Gromov seminorm in closed surfaces. In this setting, the bounded Euler class acts as the analogue of the rotation quasimorphism:

Theorem V.1.10 (Extremality of the bounded Euler class)**.** *Let $\gamma : \coprod S^1 \to S$ be a family of geodesic loops in a compact hyperbolic surface S. Let $\alpha \in H_2(S, \gamma; \mathbb{Q})$ be projectively represented by a positive immersion $f : (\Sigma, \partial\Sigma) \looparrowright (S, \gamma)$. Then*

$$\|\alpha\|_1 = \frac{-2\chi^-(\Sigma)}{n(\Sigma)} = -2\left\langle \mathrm{eu}_b^{\mathbb{R}}(S), \alpha \right\rangle,$$

where $\mathrm{eu}_b^{\mathbb{R}}(S)$ *is the* real bounded Euler class *of S.*

In other words, f is an extremal surface and $-\mathrm{eu}_b^{\mathbb{R}}(S)$ is an extremal class for α.

Theorem V.1.10 was hinted at by Calegari [15, Remark 3.18]. Our notion of relative Gromov seminorm, together with the analogue of Bavard duality provided by Theorem III.2.4, allow us to turn this remark into a precise statement. This is explained in Chapter V, together with the necessary background on the bounded Euler class and the rotation quasimorphism.

In particular, Theorem V.1.10 implies rationality of the relative Gromov seminorm for certain relative classes in closed surfaces, though it seems unlikely that one could tackle all of $H_2(S, \gamma)$ with similar techniques.

We also discuss in §V.2 how the rotation quasimorphism and the bounded Euler class behave under the isometric embeddings of surfaces given by Theorems IV.3.2 and IV.3.4 above.

A perspective on the homological Dehn problem. Let us summarise our contribution to the homological Dehn problem and a possible roadmap for solving it.

The main idea, perhaps, emerging from this thesis is the importance of the relative Gromov seminorm as a tool to understand stable commutator length. We have seen several examples in which we can prove some results about the relative Gromov seminorm and use them to deduce new computations of scl — this is the case in particular of Corollaries IV.3.5 and IV.4.3 about isometric embeddings of surfaces and graphs of groups, respectively. Going via the relative Gromov seminorm can be a fruitful route in those cases, as our version of Bavard duality (Theorem III.2.4) allows one to use existing knowledge in bounded cohomology — this is how we obtain isometric embeddings for graphs of groups (Corollary IV.4.3) for instance, and it is not clear how one could reprove this by thinking only about scl and quasimorphisms.

In those cases however, we are able to deduce computations of scl from information about the relative Gromov seminorm because we are working in groups where the second homology vanishes. One important task lying ahead is therefore to understand how to compute $\mathrm{scl}_X(\gamma)$ as an infimum over $H_2(X, \gamma)$ (as given by Proposition I.2.12), when $\gamma : \coprod S^1 \to X$ is a map in a space X with $H_2(X) \neq 0$.

The results of this thesis do in fact point at the idea that having non-zero second homology might be the only obstruction to obtaining rationality in closed surface groups. Indeed, we have seen at least two examples of theorems on scl in free groups that generalise very naturally to results about the relative Gromov seminorm in closed surface groups, namely our scl-isometric embedding theorem for non-closed surfaces (Theorem IV.3.2) and Calegari's theorem on the extremality of the rotation quasimorphism (Corollary V.1.13). A natural next step is to understand if the proof of Calegari's Rationality Theorem can be adapted to obtain rationality of the relative Gromov seminorm in closed surfaces, and we are hoping that some of our work on standard forms for admissible surfaces (see Chapter II) can be used towards this goal.

The strategy for solving the homological Dehn problem that this thesis suggests is the following: one should understand how to extend rationality from scl in free groups to the relative Gromov seminorm in closed surface groups, and one should find methods to infimise $\|\cdot\|_1$ over all relative homology classes. These two points together would lead to computations, and possibly rationality, of scl in closed surface groups.

Spectral gaps and negative curvature. Another part of this thesis is concerned with the connection between stable commutator length and negative curvature. This connection goes via the notion of *spectral gap*: a group G has a spectral gap for scl if there is a constant $\varepsilon > 0$ (depending on G only) such that every loop $\gamma : S^1 \to X$ (where X is a $K(G, 1)$) has either $\mathrm{scl}_X(\gamma) = 0$ or $\mathrm{scl}_X(\gamma) \geqslant \varepsilon$. We have already mentioned several examples of groups which are known to have spectral gaps [5, 19, 23, 26, 27, 32, 34, 48]. In most cases, the existence of a spectral gap is proved using one of two major techniques: either by constructing explicit quasimorphisms — which are often variations of Brooks' counting quasimorphisms [9] — and deducing lower bounds on scl via Bavard duality, or via Chen's method of *linear programming duality* [23–26].

In the final chapter of this thesis, we promote a geometric method for proving spectral gaps. Our method is inspired by Duncan and Howie's $\frac{1}{2}$-spectral gap in free groups

[32], and can also be seen as a geometric reformulation of Chen's linear programming duality. Rather than using Bavard duality, we prove lower bounds on scl via its topological definition as an infimum over admissible surfaces. This means that, instead of constructing *one* quasimorphism, we must estimate the Euler characteristic of *every* admissible surface. The strategy to do so is to equip admissible surfaces with a combinatorial geometric structure called an *angle structure* and for which there is a version of the Gauß–Bonnet formula relating the Euler characteristic to some notion of curvature.

All the existing linear programming duality arguments can be translated into this language — as an example, we reprove Chen's spectral gap in free products [23] in §VI.2.

In order to prove that right-angled Artin groups have a spectral gap, Heuer [48] introduced letter-quasimorphisms, which are free-group-valued analogues of quasimorphisms — see §VI.3 for a precise definition. Heuer proved in particular that letter-quasimorphisms lead to large values of scl, and we are able to reprove this result using our geometric method:

Theorem VI.4.1 (Angle structures from letter-quasimorphims)**.** *Let X be a connected 2-complex and let $g \in \pi_1 X \smallsetminus \{1\}$, represented by an immersed loop $\gamma : S^1 \to X$. Assume that there is a letter-quasimorphism $\Phi : \pi_1 X \to \mathcal{A}$ with $\Phi(g) \neq 1$ and $\Phi\left(g^n\right) = \Phi(g)^n$ for all $n \in \mathbb{Z}$.*

Then given a monotone, incompressible, disc- and sphere-free scl-admissible surface $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$, there is an angle structure on Σ whose total curvature satisfies

$$\kappa\left(\Sigma\right) \leqslant -2\pi \cdot n(\Sigma).$$

In particular, $\frac{-\chi^-(\Sigma)}{2n(\Sigma)} \geqslant \frac{1}{2}$, and $\mathrm{scl}_X(\gamma) \geqslant \frac{1}{2}$.

Theorem VI.4.1 says that every letter-quasimorphism gives a lower bound for the scl of each element on which it has a 'sufficiently non-trivial' value; this can be seen as an analogue of Bavard duality [4] for letter-quasimorphisms. Another perspective is that Theorem VI.4.1 is a coarse version of Duncan and Howie's $\frac{1}{2}$-spectral gap in free groups [32], since their result can be restated as saying that whenever a group element admits a morphism to a free group with non-trivial image, then this element has stable commutator length at least equal to $\frac{1}{2}$. Theorem VI.4.1 says that the same is true when morphisms are replaced with letter-quasimorphisms, and our proof reflects this fact and uses a similar method to Duncan and Howie's. We should also note that our theorem is logically equivalent to Heuer's (see Corollary VI.4.2), but the proofs are completely different. The general method and the proof of Theorem VI.4.1 are explained in Chapter VI.

This opens up two future research directions on spectral gaps. Firstly, Theorem VI.4.1 shows that letter-quasimorphisms satisfy a Bavard duality statement analogous to classical quasimorphisms, and can be used to yield sharp lower bounds for scl (see §VI.2.a). However, the only existing constructions of letter-quasimorphisms to date are the ones due to Heuer [48] for right-angled Artin groups, and more generally for graphs of groups with a certain orderability condition. This calls for new constructions of letter-quasimorphisms.

Moreover, we expect that our geometric method in terms of angle structures can be used both to improve existing lower bounds in groups which are already known to have spectral gaps for scl, and to uncover new cases where some notion of group-theoretic negative curvature gives rise to a spectral gap. By using this method in groups where there is additional geometric structure, it is our hope that we can eventually reach a better understanding of the connection between stable commutator length and curvature.

Outline of the thesis. Chapter I starts by introducing scl and the relative Gromov seminorm. Both objects are given several equivalent definitions which are shown to coincide, and we also explain the connection between them.

Chapter II discusses various topological operations that one can perform on admissible surfaces used to compute scl or $\|\cdot\|_1$. Some of these operations are general, while some are specific to the case of surfaces and are used later in the thesis to obtain isometric embeddings of surfaces. Most of this chapter is taken from [58].

Chapter III introduces the necessary background on quasimorphisms, Bavard duality, and bounded cohomology. We then explain how existing results in bounded cohomology can be used to deduce our version of Bavard duality for the relative Gromov seminorm. Finally, we prove a relative Hopf formula and reinterpret our Bavard duality statement in this light. Most of this chapter is taken from [57].

Chapter IV starts with a general discussion on isometric embeddings for scl and the relative Gromov seminorm, and then goes on to prove specific isometric embedding results for surfaces and graphs of groups. Part of this chapter is taken from [58].

Chapter V introduces the bounded Euler class, and explains how it can be used to generalise Calegari's theorem on immersed admissible surfaces and the rotation quasimorphism [15]. A second part of this chapter discusses the behaviour of the bounded Euler class and of the rotation quasimorphism under the isometric embeddings of surfaces of Chapter IV. The first part of this chapter is taken from [57] while the second part contains some generalisations of results proved in [58].

Finally, Chapter VI introduces our geometric method for proving spectral gaps for scl via angle structures. We apply this method to Chen's theorem on the spectral gap of free products [23], then introduce letter-quasimorphisms and show that our method applies in their presence, recovering Heuer's theorem on the spectral gap of right-angled Artin groups [48]. We conclude this chapter by using our isometric embedding results for graphs of groups to construct groups with controlled spectral gaps. Most of this chapter is taken from [59].

CHAPTER I

SCL AND THE RELATIVE GROMOV SEMINORM

In this chapter, we introduce the main protagonists of this thesis. The most important one will be stable commutator length, of which we give an algebraic definition for single elements, and a more general topological definition for chains. We also introduce the homology of a space relative to a chain, and the associated ℓ^1-seminorm, that we call the relative Gromov seminorm. Those two objects and the interplay between them will be at the centre of the thesis. Most of the material of this chapter is taken from [58, §2] and [57, §2]. A much more detailed treatment of scl can be found in Calegari's book [16].

I.1 Stable commutator length

I.1.a Stable commutator length of elements

Let G be a group. A *commutator* in G is an expression of the form $[a,b] = aba^{-1}b^{-1}$, for some $a, b \in G$. The *commutator length* $\mathrm{cl}_G(w)$ of an element $w \in G$ is its word length with respect to the set of all commutators:

$$\mathrm{cl}_G(w) := \inf\{k \in \mathbb{N}_{\geqslant 0} \mid \exists a_1, b_1, \ldots, a_k, b_k \in G,\ w = [a_1,b_1]\cdots[a_k,b_k]\} \in \mathbb{N}_{\geqslant 0} \cup \{\infty\},$$

where we agree that $\inf\varnothing = \infty$. Note that $\mathrm{cl}_G(w) < \infty$ if and only if $w \in [G,G]$, or equivalently, if and only if the image of w in $H_1(G;\mathbb{Z})$ vanishes.

Definition I.1.1. Given $w \in G$, the *stable commutator length* $\mathrm{scl}_G(w)$ is defined by

$$\mathrm{scl}_G(w) := \lim_{n\to\infty} \frac{\mathrm{cl}_G(w^n)}{n} \in [0,+\infty].$$

The subadditivity of the sequence $(\mathrm{cl}_G(w^n))_{n\geqslant 1}$ guarantees that the above limit exists and is equal to $\inf_{n\geqslant 1}\frac{\mathrm{cl}_G(w^n)}{n}$. We have $\mathrm{scl}_G(w) < \infty$ if and only if the image of w in $H_1(G;\mathbb{Q})$ vanishes.

There are very few known exact computations of commutator length, but we give one as an example:

Example I.1.2 (Culler [29, Example 2.6])**.** Let F be a free group and let a, b be two basis elements of F. Then, for all $n \geqslant 1$,

$$\mathrm{cl}_F([a,b]^n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Therefore, $\mathrm{scl}_F([a,b]) = \frac{1}{2}$. We will discuss this equality again from a topological perspective in Example I.1.21.

Throughout this thesis, we will not only be concerned with the stable commutator length of single elements, but also with that of chains. The stable commutator length of chains can be given an algebraic definition [16, §2.6.1], but we will instead introduce it from a topological point of view.

I.1.b Conjugacy classes of 1-chains

Before giving the topological definition of scl, we introduce conjugacy classes of chains, which will be our algebraic framework for this definition.

Suppose that $G \cong \pi_1 X$, where X is a path-connected topological space. There is a canonical correspondence between free homotopy classes of loops $S^1 \to X$ and conjugacy classes of elements of $\pi_1 X$. In this thesis, we are interested in free homotopy classes of finite families of loops $\coprod S^1 \to X$, which can be encoded by certain classes of 1-chains on $\pi_1 X$, as we explain below.

Fix a coefficient ring $R = \mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$. We denote by $C_n(G; R)$ the group of n-chains on G with coefficients in R:

$$C_n(G; R) := \bigoplus_{g_1, \dots, g_n \in G} R(g_1, \dots, g_n).$$

We define boundary maps $d_n : C_n(G; R) \to C_{n-1}(G; R)$ by

$$\begin{aligned} d_n(w_1, \dots, w_n) := (w_2, \dots, w_n) - (w_1 w_2, w_3, \dots, w_n) + (w_1, w_2 w_3, \dots, w_n) \\ - \dots + (-1)^{n-1}(w_1, \dots, w_{n-2}, w_{n-1} w_n) + (-1)^n (w_1, \dots, w_{n-1}). \end{aligned}$$

Hence, $C_*(G; R)$ is a chain complex, whose homology is the group homology $H_*(G; R)$. See [70, §6.5] for more details: our $C_*(G; R)$ is the tensor product of the $\mathbb{Z}G$-module R (with trivial G-action) with the bar resolution of G.

We will also use the following notations:

- $Z_n(G; R) := \operatorname{Ker}(d_n) \subseteq C_n(G; R)$ is the group of n-*cycles*.
- $B_n(G; R) := \operatorname{Im}(d_{n+1}) \subseteq Z_n(G; R)$ is the group of n-*boundaries*.

We now focus on 1-chains, i.e. elements of $C_1(G; R) = \bigoplus_{g \in G} Rg$. The *support* of a 1-chain $c = \sum_g \lambda_g g$ (with $\lambda_g \in R$ for $g \in G$) is the finite set $\operatorname{Supp} c := \{g \in G \mid \lambda_g \neq 0\}$.

We consider the sub-R-module $K(G; R)$ of $C_1(G; R)$ spanned by elements of the form $(w - w^t)$ for $w, t \in G$, where we write $w^t = t^{-1} w t$.

Remark I.1.3. $K(G; R) \subseteq B_1(G; R) \subseteq Z_1(G; R) = C_1(G; R)$.

Proof. The equality $Z_1(G; R) = C_1(G; R)$ follows from the fact that $d_1 = 0$, and the inclusion $B_1(G; R) \subseteq Z_1(G; R)$ is because $C_*(G; R)$ is a chain complex. It remains to show that $K(G; R) \subseteq B_1(G; R)$. This follows from the following computation, for $w, t \in G$:

$$\begin{aligned} d_2\left((t^{-1}, wt) + (w, t) - (t^{-1}, t) - (1, 1)\right) &= \left(wt - w^t + t^{-1}\right) + (t - wt + w) \\ &\quad - \left(t - 1 + t^{-1}\right) - (1 - 1 + 1) \\ &= w - w^t. \end{aligned}$$

□

Definition I.1.4. The R-module of *conjugacy classes of chains* on G with coefficients in R is the quotient

$$C_1^{\mathrm{conj}}(G;R) := C_1(G;R)/K(G;R).$$

We will also denote by $B_1^{\mathrm{conj}}(G;R)$ the image of $B_1(G;R)$ in $C_1^{\mathrm{conj}}(G;R)$ — elements of $B_1^{\mathrm{conj}}(G;R)$ are called *conjugacy classes of boundaries*[1].

We denote by $\pi : C_1(G;R) \twoheadrightarrow C_1^{\mathrm{conj}}(G;R)$ the projection. Given a 1-chain $c \in C_1(G;R)$, we write $[c] := \pi(c) \in C_1^{\mathrm{conj}}(G;R)$.

Note that there is a new chain complex

$$\cdots \xrightarrow{d_{n+1}} C_n(G;R) \xrightarrow{d_n} \cdots \xrightarrow{d_3} C_2(G;R) \xrightarrow{d_2^c := \pi\circ d_2} C_1^{\mathrm{conj}}(G;R) \xrightarrow{d_1=0} C_0(G;R),$$

which we denote by $C_*^{\mathrm{conj}}(G;R)$. In fact, this chain complex can be used to compute the first homology of G:

Remark I.1.5. The projection $C_1(G;R) \xrightarrow{\pi} C_1^{\mathrm{conj}}(G;R)$ induces an isomorphism

$$\pi_* : H_1(G;R) = C_1(G;R)/B_1(G;R) \xrightarrow{\cong} C_1^{\mathrm{conj}}(G;R)/B_1^{\mathrm{conj}}(G;R).$$

Proof. There is a commutative diagram with exact rows:

$$\begin{tikzcd}
C_2(G;R) \arrow[r, "d_2"] \arrow[d, equal] & C_1(G;R) \arrow[r] \arrow[d, two heads, "\pi"] & H_1(G;R) \arrow[r] \arrow[d, two heads, "\pi_*"] & 0 \\
C_2(G;R) \arrow[r, "d_2^c"] & C_1^{\mathrm{conj}}(G;R) \arrow[r] & H_1\left(C_*^{\mathrm{conj}}(G;R)\right) \arrow[r] & 0
\end{tikzcd}$$

Given $\alpha \in \operatorname{Ker}\pi_* \subseteq H_1(G;R)$, pick $a \in C_1(G;R)$ mapping to α under $C_1(G;R) \to H_1(G;R)$. Then $\pi(a) \in \operatorname{Im}(d_2^c)$, i.e. there is $b \in C_2(G;R)$ such that $a - d_2 b \in \operatorname{Ker}\pi = K(G;R)$. But $K(G;R) \subseteq B_1(G;R) = \operatorname{Im} d_2$ (see Remark I.1.3), so $a \in \operatorname{Im} d_2$ and $\alpha = 0$. Therefore, $\pi_* : H_1(G;R) \to H_1\left(C_*^{\mathrm{conj}}(G;R)\right) = C_1^{\mathrm{conj}}(G;R)/B_1^{\mathrm{conj}}(G;R)$ is an isomorphism. $\square$

I.1.c Standard form for conjugacy classes of chains

We now give for each conjugacy class of chains in $C_1^{\mathrm{conj}}(G;R)$ a standard representative in $C_1(G;R)$ having a natural topological counterpart.

This standard representative will be unique up to reordering and conjugacy, and we will use the following lemma to prove uniqueness:

[1] Compare with [16, Definition 2.78], where Calegari introduces a quotient of $B_1^{\mathrm{conj}}(G;R)$.

Lemma I.1.6. *Let $\kappa \in K(G;R)$. Suppose that there is no pair of distinct conjugate elements in* $\operatorname{Supp}\kappa$*. Then* $\kappa = 0$*.*

Proof. By definition, κ can be written as a linear combination

$$\kappa = \sum_{i=1}^{r} \lambda_i \left(w_i - w_i^{t_i}\right), \tag{$*$}$$

with $\lambda_1, \dots, \lambda_r \in R \smallsetminus \{0\}$, $w_1, \dots, w_r \in G$, $t_1, \dots, t_r \in G$, and $w_i \neq w_i^{t_i}$ for all i. We choose a decomposition ($*$) such that r is minimal. Assume for contradiction that $\kappa \neq 0$. In particular, $r \geqslant 1$ and $\operatorname{Supp}\kappa \neq \varnothing$. After reordering, we may assume that at least one of w_r and $w_r^{t_r}$ lies in $\operatorname{Supp}\kappa$. However, there is no pair of distinct conjugate elements in $\operatorname{Supp}\kappa$. Without loss of generality, we can therefore assume that $w_r \in \operatorname{Supp}\kappa$ and $w_r^{t_r} \notin \operatorname{Supp}\kappa$. Now define

$$\begin{aligned} I_1 &\coloneqq \left\{ i < r \;\middle|\; w_i = w_r^{t_r} \right\}, \\ I_2 &\coloneqq \left\{ i < r \;\middle|\; w_i^{t_i} = w_r^{t_r} \right\}. \end{aligned}$$

The sets I_1 and I_2 are disjoint since $w_i \neq w_i^{t_i}$ for all i. We also set $I_0 \coloneqq \{1, \dots, r-1\} \smallsetminus (I_1 \amalg I_2)$, so that I_0, I_1, I_2 form a partition of $\{1, \dots, r-1\}$. Since the coefficient of $w_r^{t_r}$ in κ vanishes, we have

$$\lambda_r = \sum_{i \in I_1} \lambda_i - \sum_{i \in I_2} \lambda_i.$$

Therefore, setting $p_i = \left(w_i - w_i^{t_i}\right)$ for each $i \in \{1, \dots, r\}$, we can rewrite

$$\kappa = \sum_{i \in I_0} \lambda_i p_i + \sum_{i \in I_1} \lambda_i \left(p_i + p_r\right) + \sum_{i \in I_2} \lambda_i \left(p_i - p_r\right). \tag{$\dagger$}$$

Now note that:

- For $i \in I_1$, $p_i + p_r = w_r - w_i^{t_i} = w_r - w_r^{t_r t_i}$.
- For $i \in I_2$, $p_i - p_r = w_i - w_r = w_i - w_i^{t_i t_r^{-1}}$.

Therefore, ($\dagger$) is a decomposition of κ of the form ($*$) with at most $|I_0| + |I_1| + |I_2| = r - 1$ terms. This contradicts the minimality of r, so $\kappa = 0$. □

We can now obtain our standard form:

Lemma I.1.7 (Standard form for conjugacy classes of chains)**.** *Let* $[c] \in C_1^{\mathrm{conj}}(G;R)$*.*

(i) *There is a* 1*-chain*

$$c_0 = \sum_{i=1}^{d} \lambda_i w_i \in C_1(G; R),$$

such that $[c_0] = [c]$ *in* $C_1^{\mathrm{conj}}(G; R)$*, where* $d \in \mathbb{N}_{\geqslant 0}$*,* $\lambda_1, \ldots, \lambda_d \in R \smallsetminus \{0\}$*, and* $w_1, \ldots, w_d \in G$ *are pairwise non-conjugate.*

(ii) *Assume that* $c_0' = \sum_{i=1}^{d'} \lambda_i' w_i' \in C_1(G; R)$ *also satisfies* $[c_0] = [c_0']$*, where* $d' \in \mathbb{N}_{\geqslant 0}$*,* $\lambda_1', \ldots, \lambda_{d'}' \in R \smallsetminus \{0\}$*, and* $w_1', \ldots, w_d'$ *are pairwise non-conjugate. Then* $d = d'$*, and there is a permutation* $\sigma \in \mathfrak{S}_d$*, and elements* $t_i \in G$*, such that* $w_{\sigma(i)}' = w_i^{t_i}$ *and* $\lambda_{\sigma(i)}' = \lambda_i$ *for all* i*.*

Proof. (i) Write $c = \sum_{i=1}^{d} \lambda_i w_i \in C_1(G; R)$, with $\lambda_1, \ldots, \lambda_d \in R \smallsetminus \{0\}$ and $w_1, \ldots, w_d \in G$. Assume moreover that d is minimal among all representatives c of the class $[c]$. If there are $j \neq k$ such that $w_j = w_k^t$ for some $t \in G$, then

$$c \equiv (\lambda_j + \lambda_k) w_j + \sum_{\substack{1 \leqslant i \leqslant d \\ i \neq j,k}} \lambda_i w_i \mod K(G; R),$$

which contradicts the minimality of d. Therefore, no two of the w_i's are conjugate as wanted.

(ii) We argue by induction on $d + d'$. If $d + d' = 0$, then $c_0 = 0 = c_0'$. Assume that $d + d' \geqslant 1$ and consider $\kappa := c_0 - c_0' \in K(G; R)$. If $\kappa = 0$, then $c_0 = c_0'$ and there is nothing to prove; otherwise, Lemma I.1.6 implies the existence of a pair of distinct conjugate elements in $\operatorname{Supp} \kappa$. But $\operatorname{Supp} \kappa \subseteq \{w_i\}_{1 \leqslant i \leqslant d} \cup \{w_i'\}_{1 \leqslant i \leqslant d'}$. By assumption on the w_i's and w_i''s, this implies that one of the w_i's is conjugate to one of the w_i''s. After relabelling, we can assume that w_d is conjugate to $w_{d'}'$. We then consider

$$c_1 := \left(\lambda_d - \lambda_{d'}'\right) w_d + \sum_{i<d} \lambda_i w_i \quad \text{and} \quad c_1' := \sum_{i<d'} \lambda_i' w_i'.$$

Note that $c_1 \equiv c_1' \mod K(G; R)$, so the induction hypothesis applies to c_1 and c_1'. If $\lambda_d \neq \lambda_{d'}'$, then we deduce that w_d is conjugate to some w_i' with $i < d'$, and therefore $w_{d'}'$ is conjugate to w_i', which is a contradiction. Therefore, $\lambda_d = \lambda_{d'}'$, and the result follows from the induction hypothesis applied to c_1 and c_1'. □

Remark I.1.8. (i) A group element can be seen as an element of $C_1(G; \mathbb{Z})$, and conjugacy classes of chains generalise conjugacy classes of group elements, in the sense that, for $w \in G$, Lemma I.1.7 implies that $\pi^{-1}([w]) \cap G$ is exactly the conjugacy class of w in G.

(ii) The equivalence relation given by $K(G; R)$ on 1-chains should be thought of as the algebraic analogue of (free) homotopy. This is parallel to the equivalence relation given by $B_1(G; R)$, which is the algebraic analogue of homology. Hence, Remark I.1.3 is an algebraic formulation of the fact that homotopic maps also represent the same class in homology.

We now see G as the fundamental group of a path-connected space X. Pick an integral conjugacy class $[c] \in C_1^{\mathrm{conj}}(G; \mathbb{Z})$, and let $c_0 = \sum_i \lambda_i w_i \in C_1(G; \mathbb{Z})$ be a standard representative of $[c]$ given by Lemma I.1.7. For each i, pick a loop $\gamma_i : S^1 \to X$ whose free homotopy class in X corresponds to the conjugacy class of $w_i^{\lambda_i}$ in G, and consider the map

$$\gamma := \coprod_i \gamma_i : \coprod_i S^1 \to X.$$

By Lemma I.1.7(ii), the free homotopy class $[\gamma]$ of γ only depends on the class $[c]$. If Ξ is the set of free homotopy classes of finite unordered families of pairwise non-homotopic oriented loops $\coprod S^1 \to X$, then this defines a map

$$C_1^{\mathrm{conj}}(G; \mathbb{Z}) \to \Xi$$

given by $[c] \mapsto [\gamma]$. We will say that the map γ *represents* the conjugacy class $[c]$.

Conversely, consider the free homotopy class of a map $\gamma : \coprod S^1 \to X$, with components $\{\gamma_i : S^1 \to X\}_i$. For each i, pick an element $w_i \in G$ whose conjugacy class corresponds to the free homotopy class of γ_i. Sending the free homotopy class of γ to $[\sum_i w_i] \in C_1^{\mathrm{conj}}(G; \mathbb{Z})$ defines a right inverse to the map $C_1^{\mathrm{conj}}(G; \mathbb{Z}) \to \Xi$ constructed above.

Hence, the map $C_1^{\mathrm{conj}}(G; \mathbb{Z}) \to \Xi$ is surjective, but note that it is not injective in general: given $w \in G \smallsetminus \{1\}$ and $\lambda \in \mathbb{Z} \smallsetminus \{1\}$, the conjugacy classes $[\lambda w]$ and $[w^\lambda]$ are distinct but are represented by the same (homotopy class of) loop $\gamma : S^1 \to X$.

Remark I.1.9. The algebraic definition of stable commutator length (scl) in terms of products of commutators as a function $G \to [0, \infty]$ can be shown to extend to $C_1^{\mathrm{conj}}(G; \mathbb{Z})$, and then by linearity to $C_1^{\mathrm{conj}}(G; \mathbb{R})$ — see [16, §2.6]. For us, scl will be defined by its topological interpretation, and this definition will naturally be given for classes in $C_1^{\mathrm{conj}}(G; \mathbb{Z})$.

I.1.d Stable commutator length topologically

We now introduce the topological interpretation of scl, which will be our working definition throughout this thesis.

Fix a path-connected topological space X with $\pi_1 X \cong G$, and consider a finite unordered family of oriented loops $\gamma : \coprod S^1 \to X$.

An *admissible surface*[2] for γ in X is the data of an oriented compact (possibly disconnected) surface Σ, and of maps $f : \Sigma \to X$ and $\partial f : \partial\Sigma \to \coprod S^1$ making the following diagram commute:

$$\begin{tikzcd} \partial\Sigma \arrow[r, "\iota"] \arrow[d, "\partial f"'] & \Sigma \arrow[d, "f"] \\ \coprod S^1 \arrow[r, "\gamma"] & X \end{tikzcd} \tag{‡}$$

where $\iota : \partial\Sigma \hookrightarrow \Sigma$ is the inclusion. Such an admissible surface will be denoted by $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$.

Remark I.1.10. If $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$ is an admissible surface, then Σ is oriented by definition. This induces an orientation on $\partial\Sigma$, which defines a fundamental class $[\partial\Sigma] \in H_1(\partial\Sigma; \mathbb{Z})$. Moreover, $\gamma : \coprod S^1 \to X$ is seen as a family of oriented loops, so there is also an orientation on $\coprod S^1$ corresponding to a fundamental class $\left[\coprod S^1\right] \in H_1\left(\coprod S^1; \mathbb{Z}\right)$.

The complexity of a compact connected surface Σ is measured by its *reduced Euler characteristic* $\chi^-(\Sigma) = \min\{0, \chi(\Sigma)\}$. If Σ is disconnected, we set $\chi^-(\Sigma) = \sum_K \chi^-(K)$, where the sum ranges over all connected components K of Σ.

Definition I.1.11. Let X be a topological space and $\gamma : \coprod S^1 \to X$. We define

$$\mathrm{scl}_X(\gamma) := \inf_{f,\Sigma} \frac{-\chi^-(\Sigma)}{2n(\Sigma)},$$

where the infimum is over all admissible surfaces $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$ such that $\partial f_* [\partial\Sigma] = n(\Sigma) \left[\coprod S^1\right]$ in $H_1\left(\coprod S^1; \mathbb{Z}\right)$ for some $n(\Sigma) \in \mathbb{N}_{\geqslant 1}$.

Such an admissible surface will be called an scl-*admissible surface* for γ.

We first give a simple but important property of scl:

[2]Note that this differs from the usual definition of admissible surfaces [16, §2.6.1] in that we do not impose any condition on $\partial f_* [\partial\Sigma]$ at this point.

Lemma I.1.12 (Monotonicity of scl)**.** *Let $h : X \to Y$ be a continuous map between topological spaces, and let $\gamma : \coprod S^1 \to X$. Then*

$$\mathrm{scl}_Y (h \circ \gamma) \leqslant \mathrm{scl}_X(\gamma).$$

Proof. If $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$ is an scl-admissible surface for γ, then $h \circ f$ is an scl-admissible surface for $h \circ \gamma$, with the same degree and complexity as f. □

I.1.e Equivalence of the algebraic and topological definitions

We now prove that, for single elements, the algebraic and topological definitions of scl are equivalent:

Proposition I.1.13 (Equivalence of the algebraic and topological definitions of scl)**.** *Let X be a path-connected topological space with $\pi_1 X \cong G$. Given a loop $\gamma : S^1 \to X$ representing the conjugacy class of an element $w \in G$, there is an equality*

$$\mathrm{scl}_X(\gamma) = \mathrm{scl}_G(w).$$

Proof. Fix a basepoint $o \in X$, such that γ represents $w \in G \cong \pi_1 (X, o)$.

($\leqslant$) To prove that $\mathrm{scl}_X (\gamma) \leqslant \mathrm{scl}_G(w)$, fix an integer $n \in \mathbb{N}_{\geqslant 1}$ and consider an expression of w^n as a product of k commutators:

$$w^n = [a_1, b_1] \cdots [a_k, b_k], \tag{§}$$

for some $a_1, b_1, \ldots, a_k, b_k \in G$. For each $i \in \{1, \ldots, k\}$, let α_i be a based loop at o representing a_i, and let β_i be a based loop at o representing b_i.

Let $\Sigma_{k,1}$ be an oriented genus-k surface with one boundary component. The surface $\Sigma_{k,1}$ has a cell structure with one 0-cell $\bullet$, $(2k + 1)$ 1-cells with labels $u_1, v_1, \ldots, u_k, v_k, d$, and one 2-cell glued along the word $d^{-1} [u_1, v_1] \cdots [u_k, v_k]$ — see Figure I.1.

One can define a map $f^{(1)} : \Sigma_{k,1}^{(1)} \to X$ from the 1-skeleton of $\Sigma_{k,1}$ by sending $\bullet$ to o, each u_i to the loop α_i, each v_i to β_i, and d to γ^n. The equality (§) means that this map extends over the 2-cell to a map $f : \Sigma_{k,1} \to X$. Since f sends $\partial\Sigma_{k,1} = d$ to γ^n, there is an induced degree-n map $\partial f : \partial\Sigma_{k,1} \to S^1$ yielding a commutative diagram as in (‡). Therefore, Definition I.1.11 gives an upper bound

$$\mathrm{scl}_X (\gamma) \leqslant \frac{-\chi^- (\Sigma_{k,1})}{2n} = \frac{2k-1}{2n}.$$

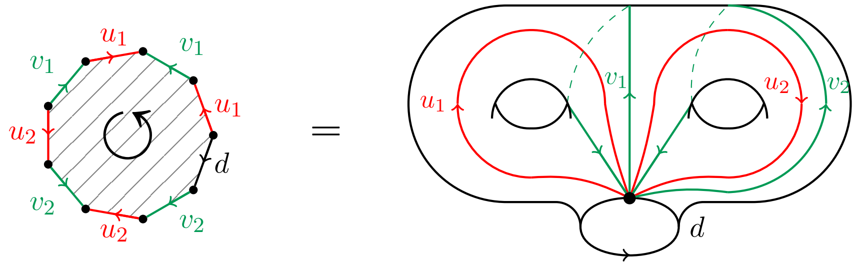


Figure I.1: The cell structure on $\Sigma_{k,1}$ (with $k = 2$).

By taking the infimum over k, this yields $\mathrm{scl}_X(\gamma) \leqslant \frac{1}{n}\,\mathrm{cl}_G(w^n) - \frac{1}{2n}$, which implies the desired inequality $\mathrm{scl}_X(\gamma) \leqslant \mathrm{scl}_G(w)$ after passing to the limit as $n \to \infty$.

($\geqslant$) For the reverse inequality, consider an scl-admissible surface $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$. If Σ has only one boundary component, then all the closed components of Σ can be discarded (as this will decrease $-\chi^-(\Sigma)$ without changing $n(\Sigma)$), so that Σ is a genus-k surface $\Sigma_{k,1}$ with one boundary component for some $k \in \mathbb{N}_{\geqslant 1}$. Hence, considering the standard cell structure on $\Sigma_{k,1}$ as above (see in particular Figure I.1) gives an expression of $w^{n(\Sigma)}$ as a product of k commutators, proving that

$$\mathrm{scl}_G(w) \leqslant \frac{\mathrm{cl}_G\left(w^{n(\Sigma)}\right)}{n(\Sigma)} \leqslant \frac{k}{n(\Sigma)} \leqslant \frac{-\chi^-(\Sigma)}{2n(\Sigma)} + \frac{1}{2n(\Sigma)}.$$

Since $n(\Sigma)$ can be made arbitrarily large by replacing Σ with a finite-degree covering (which does not change $\frac{-\chi^-(\Sigma)}{2n(\Sigma)}$), this proves that $\mathrm{scl}_G(w)$ is at most equal to the infimum of $\frac{-\chi^-(\Sigma)}{2n(\Sigma)}$ over all scl-admissible surfaces $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$, where Σ has only one boundary component.

To reduce a general scl-admissible surface $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$ to one with only one boundary component, fix a small $\varepsilon > 0$. We'll show that we can replace Σ with a surface with exactly one boundary component while increasing $\frac{-\chi^-(\Sigma)}{2n(\Sigma)}$ by no more than ε. We follow an argument of Calegari [16, Proposition 2.10 and Lemma 1.12].

Assume that Σ has $p \geqslant 2$ boundary components. We pick an integer $N \in \mathbb{N}_{\geqslant 1}$ large enough so that $\frac{1}{N} \leqslant \frac{2\varepsilon}{p}$, and with N coprime to $p-1$. The homology $H_1(\partial\Sigma)$ has a $\mathbb{Z}$-basis $(\delta_1, \ldots, \delta_p)$, where the δ_i's are the classes corresponding to the p boundary components of Σ (with orientation induced by that of Σ). Moreover, the kernel of the orientation-induced map $\iota_* : H_1(\partial\Sigma) \to H_1(\Sigma)$ is one-dimensional, generated by $\sum_i \delta_i$. Therefore, the family $(\iota_*\delta_1, \ldots, \iota_*\delta_{p-1})$ can be completed to a basis of $H_1(\Sigma)$, and there exists a surjective morphism $\alpha : H_1(\Sigma) \to \mathbb{Z}/N$ satisfying $\alpha(\iota_* \delta_i) = 1$ for each $i \in \{1, \ldots, p-1\}$. Now consider the degree-N covering $\Sigma_0 \to \Sigma$ corresponding to the kernel of the morphism $\pi_1 \Sigma \to H_1(\Sigma) \xrightarrow{\alpha} \mathbb{Z}/N$. Since N was chosen coprime to $p-1$, each $\alpha(\iota_* \delta_i)$ is primitive for $i \in \{1, \ldots, p\}$, so Σ_0 is a surface with p boundary components.

Composing f with the covering map $\Sigma_0 \to \Sigma$ yields an admissible surface $f_0 : (\Sigma_0, \partial \Sigma_0) \to (X, \gamma)$. We can now connect two of the boundary components of Σ_0 by attaching a 1-handle and extending f over the 1-handle by a trivial map to the basepoint o of X — see Figure I.2. This yields a new admissible surface

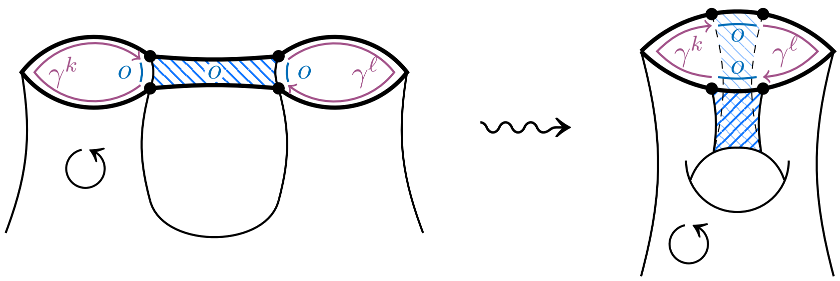


Figure I.2: Connecting two boundary components by attaching a 1-handle: the 1-handle is depicted with a blue striped pattern and the new boundary is the bold curve.

$f_0' : (\Sigma_0', \partial \Sigma_0') \to (X, \gamma)$ with $n(\Sigma_0') = n(\Sigma_0) = N \cdot n(\Sigma)$, and

$$-\chi^-(\Sigma_0') \leqslant -\chi^-(\Sigma_0) + 1 = -N\chi^-(\Sigma) + 1.$$

Therefore,

$$\frac{-\chi^-(\Sigma_0')}{2n(\Sigma_0')} \leqslant \frac{-\chi^-(\Sigma)}{2n(\Sigma)} + \frac{\varepsilon}{p}.$$

By iterating at most $(p-1)$ times, we construct from Σ an admissible surface Σ' with only one boundary component, with

$$\frac{-\chi^-(\Sigma')}{2n(\Sigma')} \leqslant \frac{-\chi^-(\Sigma)}{2n(\Sigma)} + \varepsilon.$$

Since ε was arbitrary, this proves that $\mathrm{scl}_X(\gamma)$ is equal to the infimum of $\frac{-\chi^-(\Sigma)}{2n(\Sigma)}$ over all admissible surfaces $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$, where Σ has only one boundary component. This concludes the proof that $\mathrm{scl}_G(w) \leqslant \mathrm{scl}_X(\gamma)$. $\square$

An scl-admissible surface is called *extremal* if it realises the infimum in Definition I.1.11.

Remark I.1.14. The algebraic definition of scl can be restated as

$$\mathrm{scl}_G(w) = \inf\left\{ \frac{k}{n} \,\middle|\, k \in \mathbb{N}_{\geqslant 0},\ n \in \mathbb{N}_{\geqslant 1},\ \exists a_1, b_1, \ldots, a_k, b_k \in G,\ w^n = [a_1, b_1] \cdots [a_k, b_k] \right\}. \tag{¶}$$

It follows from the proof of Proposition I.1.13 that for every pair (k, n) as in the above infimum, there is an inequality

$$\mathrm{scl}_G(w) \leqslant \frac{k}{n} - \frac{1}{2n}.$$

Hence, there cannot be an extremal pair (k, n) realising the infimum (¶).

I.1.f Group invariance

Proposition I.1.13 implies that if X and Y are two path-connected spaces, and $h : X \to Y$ is a continuous map such that $h_* : \pi_1 X \to \pi_1 Y$ is an isomorphism, then for any loop $\gamma : S^1 \to X$,

$$\mathrm{scl}_Y(h \circ \gamma) = \mathrm{scl}_X(\gamma).$$

Indeed, $\mathrm{scl}_X(\gamma)$ only depends on the conjugacy class represented by γ in $\pi_1 X$. This turns out to be true for families of loops as well, and we give a proof in the case where X and Y are CW-complexes[3]:

Proposition I.1.15 (scl is a group invariant)**.** *Let X and Y be path-connected CW-complexes. Let $h : X \to Y$ be a continuous map such that $h_* : \pi_1 X \to \pi_1 Y$ is an isomorphism. Then for any map $\gamma : \coprod S^1 \to X$,*

$$\mathrm{scl}_Y(h \circ \gamma) = \mathrm{scl}_X(\gamma).$$

Proof. Note first that the definition of $\mathrm{scl}_X(\gamma)$ is in terms of admissible surfaces, which are continuous maps from compact surfaces to X. Since compact surfaces can be given the structure of 2-dimensional CW-complexes, it follows from the Cellular Approximation Theorem [47, Theorem 4.8] that any admissible surface can be homotoped to a cellular map. In particular, any admissible surface can be homotoped to one with image contained in the 2-skeleton $X^{(2)}$. It follows that

$$\mathrm{scl}_X(\gamma) = \mathrm{scl}_{X^{(2)}}(\gamma),$$

[3] A proof in the general case can be found in Calegari's book [16, §2.6]. His approach is to extend Definition I.1.1 to an algebraic definition of scl for conjugacy classes of chains, and show that this coincides with the topological interpretation for any space X with $\pi_1 X \cong G$.

and the same is true in Y. Therefore, we can assume that X and Y are 2-dimensional.

Definition I.1.11 is invariant under homotopy. Therefore, we can contract spanning trees in the 1-skeletons of X and Y and assume that they have one 0-cell each; in other words, X and Y are now presentation complexes.

Now one can construct a cellular map $g : Y \to X$ such that $g_* = (h_*)^{-1} : \pi_1 Y \to \pi_1 X$. In particular, $g \circ h \circ \gamma$ is homotopic to γ since $g_* \circ h_* = \mathrm{id}_{\pi_1 X}$, so they have the same scl. Hence, Lemma I.1.12 gives

$$\mathrm{scl}_X(\gamma) = \mathrm{scl}_X(g \circ h \circ \gamma) \leqslant \mathrm{scl}_Y(h \circ \gamma) \leqslant \mathrm{scl}_X(\gamma). \qquad \square$$

I.1.g Stable commutator length of conjugacy classes of chains

In essence, Proposition I.1.15 says that, for a path-connected CW-complex X, scl_X is an invariant of $\pi_1 X$. Hence, it becomes natural to redefine scl as a function of conjugacy classes of chains in a given group:

Definition I.1.16. Let G be a group and $[c] \in C_1^{\mathrm{conj}}(G; \mathbb{Z})$. Let X be a $K(G,1)$ space, and let $\gamma : \coprod S^1 \to X$ be a map whose free homotopy class corresponds to $[c]$. Define

$$\mathrm{scl}_G([c]) := \mathrm{scl}_X(\gamma).$$

It is clear from Definition I.1.11 that $\mathrm{scl}_X(\gamma)$ remains unchanged if X is replaced with a space X' homotopy equivalent to X and γ is replaced with a map γ' freely homotopic to γ. Since the homotopy types of X and γ are uniquely determined by G and $[c]$ [47, Theorem 1B.8], this shows that the above notion of scl for conjugacy classes of chains is well-defined.

In fact, the group invariance of scl tells us that it is not necessary in Definition I.1.16 to assume that X is a $K(G,1)$:

Corollary I.1.17. *Let X be a path-connected CW-complex with $\pi_1 X \cong G$, and let $[c] \in C_1^{\mathrm{conj}}(G; \mathbb{Z})$ corresponding to the free homotopy class of a map $\gamma : \coprod S^1 \to X$. Then*

$$\mathrm{scl}_G([c]) = \mathrm{scl}_X(\gamma).$$

Proof. Pick a $K(G,1)$ space Y; then there exists a continuous map $h : X \to Y$ inducing an isomorphism on fundamental groups [47, Proposition 1B.9]. Now the free homotopy class of $h \circ \gamma$ in Y corresponds to $[c]$, so $\mathrm{scl}_G([c]) = \mathrm{scl}_Y(h \circ \gamma)$ by Definition I.1.16. But $\mathrm{scl}_Y(h \circ \gamma) = \mathrm{scl}_X(\gamma)$ by the group invariance of scl (Proposition I.1.15). $\square$

Remark I.1.18. Proposition I.1.13 proves that, if w is an element of a group G, then

$$\mathrm{scl}_G([w]) = \mathrm{scl}_G(w).$$

We give two basic properties of scl:

Lemma I.1.19 (Monotonicity of scl for conjugacy classes of chains)**.** *Let $\varphi : G \to H$ be a group homomorphism, and let $[c] \in C_1^{\mathrm{conj}}(G;\mathbb{Z})$. Then*

$$\mathrm{scl}_H([\varphi(c)]) \leqslant \mathrm{scl}_G([c]).$$

Proof. This is a consequence of Lemma I.1.12 and the definition of scl on conjugacy classes of chains, since any group homomorphism $\varphi : G \to H$ is induced by a continuous map $h : K(G,1) \to K(H,1)$ [47, Proposition 1B.9]. □

Lemma I.1.20 (Homogeneity of scl)**.** *Let $[c] \in C_1^{\mathrm{conj}}(G;\mathbb{Z})$. Given $n \in \mathbb{N}_{\geqslant 0}$, we have*

$$\mathrm{scl}_G(n[c]) = n \cdot \mathrm{scl}_G([c]).$$

Proof. Fix a $K(G,1)$ space X and a map $\gamma : \coprod S^1 \to X$ representing $[c]$. Then $\gamma \circ \theta$ represents $n[c]$, where $\theta : \coprod S^1 \to \coprod S^1$ is a degree-n map on each component of $\coprod S^1$.

Given an scl-admissible surface $f : (\Sigma, \partial\Sigma) \to (X, \gamma \circ \theta)$, there is an scl-admissible surface $f' : (\Sigma', \partial\Sigma') \to (X, \gamma)$, where $\Sigma' := \Sigma$, $f' := f$, and $\partial f' := \theta \circ \partial f$. This satisfies $n(\Sigma') = n \cdot n(\Sigma)$, so that

$$\mathrm{scl}_X(\gamma) \leqslant \frac{-\chi^-(\Sigma')}{2n(\Sigma')} = \frac{1}{n} \cdot \frac{-\chi^-(\Sigma)}{2n(\Sigma)}.$$

By taking the infimum over Σ, one gets

$$\mathrm{scl}_G([c]) = \mathrm{scl}_X(\gamma) \leqslant \frac{1}{n} \cdot \mathrm{scl}_X(\gamma \circ \theta) = \frac{1}{n} \cdot \mathrm{scl}_G(n[c]).$$

For the reverse inequality, start with an scl-admissible surface $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$. Let $p : \Sigma_0 \to \Sigma$ be a degree-n covering with the same number of boundary components as Σ. In particular, there is a homeomorphism $\partial h : \partial\Sigma_0 \to \partial\Sigma$. Moreover, by observing the maps $H_1(\partial\Sigma_0) \to H_1\left(\coprod S^1\right)$ induced by $\partial f \circ p_{|\partial\Sigma_0}$ and by $\partial f \circ \partial h$, we see that

$$\partial f \circ p_{|\partial\Sigma_0} = \theta' \circ \partial f \circ \partial h,$$

where $\theta' : \coprod S^1 \to \coprod S^1$ is a degree-n map on each component — in particular, θ' is homotopic to θ (since the homotopy class of a map $\coprod S^1 \to \coprod S^1$ is determined by the induced map on homology). Now set $\Sigma_0' := \Sigma_0$, $f_0' := f \circ p$, and $\partial f_0' = \partial f \circ \partial h$. This defines an scl-admissible surface $f_0' : (\Sigma_0', \partial\Sigma_0') \to (X, \gamma \circ \theta')$, with $n(\Sigma_0') = n(\Sigma)$ and $\chi^-(\Sigma_0') = n \cdot \chi^-(\Sigma)$. It follows after infimising over Σ that

$$\mathrm{scl}_G(n[c]) = \mathrm{scl}_X(\gamma \circ \theta') \leqslant n \cdot \mathrm{scl}_X(\gamma) = n \cdot \mathrm{scl}_G([c]). \qquad \square$$

We finish with a simple example showing how one can use the topological interpretation to give estimates of scl:

Example I.1.21 (Stable commutator length of a commutator)**.** Let a, b be two elements in a group G and consider $w = [a, b] \in G$. Given a path-connected CW-complex X with $\pi_1 X \cong G$, and a loop γ representing the conjugacy class of w, there is an scl-admissible surface $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$, where Σ is a genus-1 surface with one boundary component, with equator mapping to (a loop in X representing) a and meridian mapping to b, as in Figure I.3.

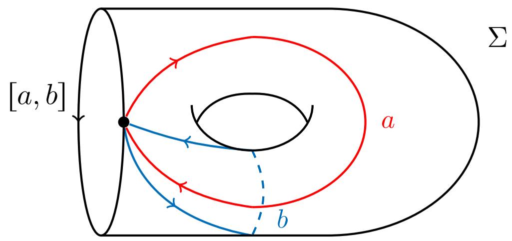


Figure I.3: scl-admissible surface for a commutator.

This gives the estimate

$$\mathrm{scl}_G([a, b]) \leqslant \frac{-\chi^-(\Sigma)}{2} = \frac{1}{2}.$$

If G is a free group and $[a, b] \neq 1$, then it turns out that $\mathrm{scl}_G([a, b]) = \frac{1}{2}$, though this is not obvious at this point. The inequality $\mathrm{scl}_G([a, b]) \geqslant \frac{1}{2}$ will follow from two different theorems that we will discuss later in this thesis: the first one is Corollary V.1.13, which gives a topological condition under which admissible surfaces are extremal when X is a hyperbolic surface, and the second one is the Duncan–Howie Theorem (Theorem VI.2.1), which is a general lower bound on the scl of non-trivial elements in free groups.

I.2 The relative Gromov seminorm

I.2.a Homology of a space relative to a chain

The second invariant that we will consider in this thesis is the relative Gromov seminorm. It will be defined on a certain relative homology group that we now introduce.

Let $\gamma : \coprod S^1 \to X$ be a finite (unordered) family of oriented loops. We denote by X_γ the *mapping cylinder* of γ:

$$X_\gamma := \left(X \amalg \left(\coprod S^1 \times [0,1]\right)\right) / \sim,$$

where $\sim$ is the equivalence relation generated by $(u,0) \sim \gamma(u)$ for $u \in \coprod S^1$. There is an embedding $\coprod S^1 \hookrightarrow X_\gamma$ via $u \mapsto (u,1)$, and we will identify $\coprod S^1$ with its image under this embedding.

Definition I.2.1. The homology of the pair (X,γ) over the coefficient ring $R = \mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$ is defined as the singular homology of the pair $\left(X_\gamma, \coprod S^1\right)$:

$$H_*(X,\gamma;R) := H_*\left(X_\gamma, \coprod S^1; R\right).$$

We remark that the homotopy type of the pair $\left(X_\gamma, \coprod S^1\right)$ — and therefore the homology $H_*(X,\gamma)$ — only depends on the free homotopy class of γ.

It is useful to write down the long exact sequence of the pair (X,γ):

Proposition I.2.2 (Exact sequences in relative homology)**.** *Let X be a topological space and $\gamma : \coprod S^1 \to X$.*

(i) *There is an exact sequence*

$$0 \to H_2(X;R) \to H_2(X,\gamma;R) \xrightarrow{\partial} H_1\left(\coprod S^1; R\right) \xrightarrow{\gamma_*} H_1(X;R).$$

(ii) *If $Y \subseteq X$ and $\gamma\left(\coprod S^1\right) \subseteq Y$, then there is a long exact sequence*

$$\cdots \to H_n(Y,\gamma;R) \to H_n(X,\gamma;R) \to H_n(X,Y;R) \xrightarrow{\partial} H_{n-1}(Y,\gamma;R) \to \cdots.$$

Proof. These are simply the long exact sequences of pairs and triples in homology [47, p.118], together with the fact that X_γ deformation retracts to X [47, p.2]. □

We now compute $H_*(X,\gamma)$ in a few special cases:

Example I.2.3. (i) If γ is the empty family of loops, then

$$H_*(X,\gamma;R) \cong H_*(X;R).$$

(ii) If γ is an embedding $\coprod S^1 \hookrightarrow X$, then the pair $\left(X_\gamma, \coprod S^1\right)$ deformation retracts onto $\left(X, \gamma\left(\coprod S^1\right)\right)$, inducing an isomorphism

$$H_*(X,\gamma;R) \cong H_*\left(X, \gamma\left(\coprod S^1\right); R\right).$$

In general, there is still a morphism $H_*(X,\gamma) \to H_*\left(X,\gamma\left(\coprod S^1\right)\right)$ given by collapsing the mapping cylinder, but this might not be an isomorphism, as shown for instance by item (iii) below.

(iii) If $\gamma : S^1 \to X$ is a contractible loop, then the quotient X_γ/S^1 is homotopy equivalent to $X \vee S^2$, and collapsing the pair gives

$$H_*(X,\gamma;R) \cong H_*(X;R) \oplus H_*\left(S^2;R\right).$$

(iv) Suppose that $\gamma : S^1 \to X$ is rationally homologically trivial, in the sense that $\gamma_* : H_1\left(S^1;\mathbb{Q}\right) \to H_1(X;\mathbb{Q})$ vanishes (see Proposition I.2.13 below). Then the map $\gamma_* : H_1\left(S^1;R\right) \to H_1(X;R)$ in the exact sequence of Proposition I.2.2(i) has kernel $q\left[S^1\right]$ for some $q \in R$, so the image of the boundary map ∂ is isomorphic to R, which gives a split short exact sequence, and an isomorphism

$$H_2(X,\gamma;R) \cong H_2(X;R) \oplus R.$$

Note that there is a natural isomorphism

$$H_*(X,\gamma;\mathbb{R}) \cong H_*(X,\gamma;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R},$$

allowing us to view $H_*(X,\gamma;\mathbb{Q})$ as a subset of $H_*(X,\gamma;\mathbb{R})$. We will say that $\alpha \in H_*(X,\gamma;\mathbb{Z})$ is an *integral class*, while $\alpha \in H_*(X,\gamma;\mathbb{Q})$ is *rational* and $\alpha \in H_*(X,\gamma;\mathbb{R})$ is *real*.

Definition I.2.4. Let G be a group, and let X be a $K(G,1)$ space. Given a conjugacy class of chains $[c] \in C_1^{\mathrm{conj}}(G;\mathbb{Z})$, we set

$$H_*(G,[c];R) := H_*(X,\gamma;R),$$

where $\gamma : \coprod S^1 \to X$ is a map whose free homotopy class corresponds to $[c]$ as explained in §I.1.b.

Note that the homotopy type of X is uniquely defined by G, and the free homotopy class of γ is determined by $[c]$, so the group $H_*(G,[c];R)$ only depends on G and the class $[c]$.

The case where $[c]=0$ corresponds to γ being the empty family of loops, and so $H_*(G,0;R)\cong H_*(G;R)$ by Example I.2.3(i).

I.2.b Rational points in real vector spaces

The difference between real and rational classes in $H_2(X,\gamma)$ will play a role in the sequel, and we make a brief digression to introduce some general terminology related to this.

Definition I.2.5. Let V be a $\mathbb{R}$-vector space. A *rational structure* on V is the choice of an equivalence class of bases of V, where two bases are considered equivalent if each vector of one basis has rational coordinates in the second basis. Any basis in the equivalence class is called a *rational basis*.

Given a rational structure on V, a *rational point* is a vector of V that has rational coordinates in a rational basis. The set $V_{\mathbb{Q}}$ of rational points of V is naturally a $\mathbb{Q}$-vector space, and satisfies $V=V_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{R}$. In fact, a rational structure on V can be defined equivalently as the choice of a $\mathbb{Q}$-subspace $V_{\mathbb{Q}}$ of V such that $V=V_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{R}$.

Example I.2.6. The space $\mathbb{R}^n$ has a rational structure given by the equivalence class of the standard basis, and its set of rational points is $\mathbb{Q}^n$.

A *rational subspace* W of V is a $\mathbb{R}$-subspace spanned by rational points. It naturally inherits a rational structure from V.

If V and W are $\mathbb{R}$-vector spaces equipped with rational structures, a *rational linear map* $f:V\to W$ is a linear map such that the image of each vector in a rational basis of V has rational coordinates in a rational basis of W. This implies that the kernel and the image of f are rational subspaces of V and W respectively.

Let $C_*^{\mathbb{Q}}$ be a chain complex over $\mathbb{Q}$ and let $C_*^{\mathbb{R}}=C_*^{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{R}$. Hence, each vector space $C_n^{\mathbb{R}}$ has a rational structure whose set of rational points is $C_n^{\mathbb{Q}}$. The boundary map $d_n:C_n^{\mathbb{R}}\to C_{n-1}^{\mathbb{R}}$ is rational, and the space $Z_n^{\mathbb{R}}=\operatorname{Ker} d_n$ of n-cycles is a rational subspace. In particular, the set of rational points of $Z_n^{\mathbb{R}}$ is the space $Z_n^{\mathbb{Q}}$ of n-cycles for $C_*^{\mathbb{Q}}$. Moreover, the inclusion $C_*^{\mathbb{Q}}\hookrightarrow C_*^{\mathbb{R}}$ induces an isomorphism

$$H_n\left(C_*^{\mathbb{R}}\right)\cong H_n\left(C_*^{\mathbb{Q}}\right)\otimes_{\mathbb{Q}}\mathbb{R},$$

giving $H_n\left(C_*^{\mathbb{R}}\right)$ a rational structure whose set of rational points is $H_n\left(C_*^{\mathbb{Q}}\right)$.

The following lemma says that any real cycle representing a rational homology class can be approximated by a rational cycle:

Lemma I.2.7 (Rational approximation in homology)**.** *Let $C_*^{\mathbb{Q}}$ be a chain complex over $\mathbb{Q}$ and let $C_*^{\mathbb{R}} = C_*^{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$. Let $\|\cdot\|$ be a norm on $C_*^{\mathbb{R}}$. Consider a real n-cycle $a \in Z_n^{\mathbb{R}}$ whose homology class $[a]$ is rational:*

$$[a] \in H_n\left(C_*^{\mathbb{Q}}\right) \hookrightarrow H_n\left(C_*^{\mathbb{R}}\right).$$

Then for any $\varepsilon > 0$, there exists a rational n-cycle $a' \in Z_n^{\mathbb{Q}}$ such that

- $[a] = [a']$ *in* $H_n\left(C_*^{\mathbb{R}}\right)$*, and*
- $\|a - a'\| \leqslant \varepsilon$*.*

Proof. We follow an argument of Calegari [16, Remark 1.5]; see also for instance [64, Lemma 2.9].

Pick a rational basis $(e_i)_{i\in I}$ of $C_n^{\mathbb{R}}$, and note that we can write $a = \sum_i \lambda_i e_i$, where the family $(\lambda_i)_{i\in I}$ of real numbers has finite support J. Now $\bigoplus_{j\in J} \mathbb{R}e_j$ is a finite-dimensional rational subspace of $C_n^{\mathbb{R}}$ containing a; by working in this subspace, we may assume that $C_n^{\mathbb{R}}$ has finite dimension.

Observe that the natural projection map

$$p : Z_n^{\mathbb{R}} \to H_n\left(C_*^{\mathbb{R}}\right)$$

is rational. Hence, since $[a]$ is a rational point of $H_n\left(C_*^{\mathbb{R}}\right)$, the affine subspace $p^{-1}([a])$ is rational in $Z_n^{\mathbb{R}}$, so its rational points are contained in $Z_n^{\mathbb{Q}}$. Since $C_n^{\mathbb{R}}$ has finite dimension, so does $Z_n^{\mathbb{R}}$, and rational points are dense. Moreover, the real n-cycle a lies in $p^{-1}([a])$, so there is $a' \in p^{-1}([a])$ rational arbitrarily close to a for $\|\cdot\|$. This rational n-cycle a' lies in $Z_n^{\mathbb{Q}}$ and is homologous to a as wanted. □

I.2.c The Gromov seminorm as an ℓ^1-seminorm

We now give a first definition of the Gromov seminorm.

Given $\gamma : \coprod S^1 \to X$, recall that $H_*\left(X_\gamma; \mathbb{R}\right)$ is the homology of the singular chain complex $C_*^{\mathrm{sg}}\left(X_\gamma; \mathbb{R}\right)$. Each $\mathbb{R}$-vector space $C_n^{\mathrm{sg}}\left(X_\gamma; \mathbb{R}\right)$ can be equipped with the ℓ^1-norm defined by

$$\left\|\sum_\sigma \lambda_\sigma \sigma\right\|_1 := \sum_\sigma |\lambda_\sigma|,$$

with $\lambda_\sigma \in \mathbb{R}$ for each singular n-simplex $\sigma : \Delta^n \to X_\gamma$. The quotient

$$C_n^{\mathrm{sg}}\left(X_\gamma, \coprod S^1; \mathbb{R}\right) := C_n^{\mathrm{sg}}\left(X_\gamma; \mathbb{R}\right) / C_n^{\mathrm{sg}}\left(\coprod S^1; \mathbb{R}\right)$$

inherits a quotient seminorm that we also denote by $\|\cdot\|_1$, and that is defined by

$$\|a\|_1 := \inf_{\underline{a} \in a} \|\underline{a}\|_1 ,$$

where the infimum is over all absolute n-chains $\underline{a} \in C_n^{\mathrm{sg}}(X_\gamma; \mathbb{R})$ representing $a \in C_n^{\mathrm{sg}}\left(X_\gamma, \coprod S^1; \mathbb{R}\right)$. Now $\|\cdot\|_1$ restricts to a seminorm on the subspace $Z_n^{\mathrm{sg}}\left(X_\gamma, \coprod S^1; \mathbb{R}\right)$ of relative n-cycles, which descends to a seminorm, still denoted by $\|\cdot\|_1$, on homology:

Definition I.2.8. Let X be a topological space and $\gamma : \coprod S^1 \to X$. The *relative Gromov seminorm* on $H_n(X, \gamma; \mathbb{R})$ is defined by

$$\|\alpha\|_1 := \inf \left\{ \|a\|_1 \;\middle|\; a \in Z_n^{\mathrm{sg}}\left(X_\gamma, \coprod S^1; \mathbb{R}\right),\ [a] = \alpha \right\}.$$

Remark I.2.9. Given a group G and an integral conjugacy class $[c] \in C_1^{\mathrm{conj}}(G; \mathbb{Z})$, the relative homology $H_2(G, [c]; \mathbb{R})$ is by definition $H_2(X, \gamma; \mathbb{R})$, where X is a $K(G,1)$ space, and $\gamma : \coprod S^1 \to X$ represents $[c]$ — see Definition I.2.4. If X' is another choice of $K(G,1)$ and $\gamma' : \coprod S^1 \to X'$ is another map representing $[c]$, then there is a homotopy equivalence $h : X \xrightarrow{\simeq} X'$ sending γ to $h\gamma$, and a free homotopy between $h\gamma$ and γ'. Hence, there are induced homotopy equivalences of pairs

$$\left(X_\gamma, \coprod S^1\right) \simeq \left(X'_{h\gamma}, \coprod S^1\right) \simeq \left(X'_{\gamma'}, \coprod S^1\right).$$

Since $\|\cdot\|_1$ is preserved by homotopy equivalences[4], the above homotopy equivalences induce isometric isomorphisms

$$H_2(X, \gamma; \mathbb{R}) \cong H_2(X', h\gamma; \mathbb{R}) \cong H_2(X', \gamma'; \mathbb{R}).$$

Hence, one can extend the definition of the Gromov seminorm to $H_2(G, [c]; \mathbb{R})$.

The above definitions still make sense if $\mathbb{R}$ is replaced with $\mathbb{Q}$ everywhere. Given $\alpha \in H_n(X, \gamma; \mathbb{Q}) \hookrightarrow H_n(X, \gamma; \mathbb{R})$, it is natural to ask whether the Gromov seminorm of α as a rational class coincides with its Gromov seminorm as a real class. The following lemma gives an affirmative answer:

[4] This follows from Gromov's Mapping Theorem [44, §3.1] (see also [36, Corollary 5.11]), together with the duality principle between ℓ^1-homology and bounded cohomology [36, Corollary 6.2].

Lemma I.2.10 (Equality of the rational and real Gromov seminorms)**.** *Let X be a topological space and $\gamma : \coprod S^1 \to X$. Given a rational class $\alpha \in H_n(X, \gamma; \mathbb{Q})$, the Gromov seminorm of α (seen as a real class) can be computed over rational cycles:*

$$\|\alpha\|_1 = \inf\left\{\|a\|_1 \,\middle|\, a \in Z_n^{\mathrm{sg}}\left(X_\gamma, \coprod S^1; \mathbb{Q}\right),\ [a] = \alpha\right\}$$

Proof. This follows from Lemma I.2.7. □

In other words, Lemma I.2.10 says that the inclusion

$$H_n(X, \gamma; \mathbb{Q}) \hookrightarrow H_n(X, \gamma; \mathbb{R})$$

is an isometric embedding if $H_n(X, \gamma; \mathbb{Q})$ and $H_n(X, \gamma; \mathbb{R})$ are equipped with the rational and real Gromov seminorms respectively.

I.2.d Topological interpretation of the Gromov seminorm

Analogously to the topological definition of stable commutator length (see Definition I.1.11), we now give a topological interpretation of the Gromov seminorm for rational classes in H_2. This extends the topological interpretation of the absolute Gromov seminorm on H_2 [16, §1.2.5].

Let $\gamma : \coprod S^1 \to X$ be a family of loops in a space X. Consider an admissible surface $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$. Let Σ_ι be the mapping cylinder of the inclusion map $\iota : \partial\Sigma \hookrightarrow \Sigma$:

$$\Sigma_\iota := (\Sigma \amalg (\partial\Sigma \times [0,1])) / \sim,$$

where $\sim$ is the equivalence relation generated by $(u, 0) \sim \iota(u)$ for $u \in \partial\Sigma$. Hence, there is a natural map of pairs

$$f_\# : (\Sigma_\iota, \partial\Sigma \times \{1\}) \to \left(X_\gamma, \coprod S^1\right)$$

defined by f and ∂f — see §I.2.a for the definition of the pair $\left(X_\gamma, \coprod S^1\right)$. Since the pair $(\Sigma_\iota, \partial\Sigma \times \{1\})$ deformation retracts to $(\Sigma, \partial\Sigma)$, the map $f_\#$ induces a morphism

$$f_* : H_*(\Sigma, \partial\Sigma) \to H_*(X, \gamma).$$

In particular, f represents a class $f_*[\Sigma] \in H_2(X, \gamma)$, where $[\Sigma] \in H_2(\Sigma, \partial\Sigma)$ is the (integral, rational, or real) fundamental class of Σ.

Proposition I.2.11 (Topological interpretation of the Gromov seminorm)**.** *Let X be a topological space and $\gamma : \coprod S^1 \to X$. If $\alpha \in H_2(X, \gamma; \mathbb{Q})$ is a rational class, then there is an equality*

$$\|\alpha\|_1 = \inf_{f,\Sigma} \frac{-2\chi^-(\Sigma)}{n(\Sigma)},$$

where the infimum is taken over all admissible surfaces $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$ such that $f_*[\Sigma] = n(\Sigma)\alpha$ for some $n(\Sigma) \in \mathbb{N}_{\geqslant 1}$.

*Such an admissible surface will be called an ℓ^1-*admissible surface *for α.*

Proof. First consider an admissible surface $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$ with $f_*[\Sigma] = n(\Sigma)\alpha$. Since $\|\cdot\|_1$ is *functorial* in the sense of Gromov [44] (see also [28]), we can estimate

$$\|\alpha\|_1 = \frac{\|f_*[\Sigma]\|_1}{n(\Sigma)} \leqslant \frac{\|[\Sigma]\|_1}{n(\Sigma)}.$$

But the ℓ^1-seminorm of $[\Sigma]$ is known as the *simplicial volume* of Σ, and it is equal to $-2\chi^-(\Sigma)$ [44, p.9] (see also [36, Corollary 7.5] and [1]). This proves the inequality ($\leqslant$) of the proposition.

For the reverse inequality, we follow the same line of reasoning as in Calegari's proof that scl is not greater than the Gersten boundary norm [16, Lemma 2.69], which is based on an argument of Bavard [4, Proposition 3.2]. Consider a rational relative 2-cycle $a \in Z_2\left(X_\gamma, \coprod S^1; \mathbb{Q}\right)$ representing α, and let $a_0 \in C_2(X_\gamma; \mathbb{Q})$ be a 2-chain mapping to a. By Lemma I.2.10, the infimum of $\|a_0\|_1$ over such a_0 is equal to $\|\alpha\|_1$.

Since a_0 is rational, there exists $q \in \mathbb{N}_{\geqslant 1}$ such that qa_0 is integral; we can write $qa_0 = \sum_j \varepsilon_j \sigma_j$, with $\varepsilon_j \in \{\pm 1\}$ and $\sigma_j : \Delta^2 \to X_\gamma$ a singular 2-simplex. We can assume that no singular 2-simplex appears twice with opposite signs in the above expression, so that

$$\|qa_0\|_1 = \sum_j |\varepsilon_j|.$$

The fact that a is a relative 2-cycle means that da_0 has support contained in $\coprod S^1$. Therefore, we can construct a partial pairing on the edges of the simplices σ_j such that paired edges have the same map to X_γ, and non-paired edges all map to $\coprod S^1$. We then construct a 2-dimensional simplicial complex Σ by taking a collection $\left\{\Delta_j^2\right\}_j$ of 2-simplices and gluing them along this pairing. The complex Σ thus constructed is a simplicial 2-complex such that each edge is incident to either one or two faces; moreover, there is no vertex identification, so the links of vertices remain connected at the end of the gluing process: they are homeomorphic to circles or arcs. Hence, Σ

is a (possibly disconnected) surface with boundary (see Lemma IV.2.1 below for more details), and the singular simplices σ_j define a map $f : \Sigma \to X_\gamma$ by $f_{|\Delta_j^2} = \sigma_j$, with $f(\partial\Sigma) \subseteq \coprod S_1$. After homotoping $f(\Sigma)$ into the image of X in X_γ, and $f(\partial\Sigma)$ into $\gamma\left(\coprod S^1\right)$, this induces an admissible surface $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$, and $f_*[\Sigma] = q\alpha$ in $H_2(X, \gamma; \mathbb{R})$. As above, $-2\chi^-(\Sigma)$ is the simplicial volume $\|[\Sigma]\|_1$ of Σ (see [44, p.9] or [36, Corollary 7.5]); on the other hand, our triangulation of Σ by the simplices Δ_j^2 gives an upper bound on the simplicial volume:

$$\|a_0\|_1 = \frac{\|qa_0\|_1}{q} = \frac{1}{q}\sum_j |\varepsilon_j| \geqslant \frac{\|[\Sigma]\|_1}{q} = \frac{-2\chi^-(\Sigma)}{q}.$$

By taking the infimum over a_0 representing α, we obtain the inequality ($\geqslant$). □

I.2.e Connection with scl

The topological interpretation of the relative Gromov seminorm relates it to scl:

Proposition I.2.12 (Gromov seminorm and scl)**.** *Let X be a topological space and $\gamma : \coprod S^1 \to X$. Then*

$$\mathrm{scl}_X(\gamma) = \frac{1}{4}\inf\left\{\|\alpha\|_1 \mid \alpha \in H_2(X, \gamma; \mathbb{Q}),\ \partial\alpha = \left[\coprod S^1\right]\right\},$$

*where $\partial : H_2(X, \gamma; \mathbb{Q}) \to H_1\left(\coprod S^1; \mathbb{Q}\right)$ is the boundary map in the exact sequence of the pair (X, γ) (see Proposition I.2.2(*i*)).*

Proof. This follows from Proposition I.2.11 and Definition I.1.11. □

This homological point of view yields in particular a simple characterisation of finiteness of scl:

Proposition I.2.13 (Finiteness of scl)**.** *Let X be a path-connected CW-complex with $\pi_1 X \cong G$, let $[c] \in C_1^{\mathrm{conj}}(G; \mathbb{Z})$ be represented by a map $\gamma : \coprod S^1 \to X$. Then the following are equivalent:*

(i) $\mathrm{scl}_G([c]) < \infty$.

(ii) *There is an integer $n \in \mathbb{N}_{\geqslant 1}$ such that $n[c] \in B_1^{\mathrm{conj}}(G; \mathbb{Z})$.*

(iii) *The image of $[c]$ in $C_1^{\mathrm{conj}}(G; \mathbb{Q})$ lies in $B_1^{\mathrm{conj}}(G; \mathbb{Q})$.*

(iv) *The image of $[c]$ in $H_1(G; \mathbb{Q})$ is trivial.*

(v) *The map* $\gamma_* : H_1\left(\coprod S^1; \mathbb{Q}\right) \to H_1(X; \mathbb{Q})$ *satisfies* $\gamma_*\left[\coprod S^1\right] = 0$.

If those conditions are satisfied, we say that γ *(or* $[c]$*) is* rationally homologically trivial.

Proof. (ii) $\Leftrightarrow$ (iii) This follows from the fact that for any $c \in C_1(G; \mathbb{Z}) \cap B_1(G; \mathbb{Q})$, there is an integer $n \in \mathbb{N}_{\geqslant 1}$ such that $nc \in B_1(G; \mathbb{Z})$.

(iii) $\Leftrightarrow$ (iv) This follows from Remark I.1.5.

(iv) $\Leftrightarrow$ (v) This follows from the fact that the image of $[c]$ in $H_1(G; \mathbb{Q})$ corresponds to $\gamma_*\left[\coprod S^1\right]$ under the isomorphism $H_1(G; \mathbb{Q}) \cong H_1(X; \mathbb{Q})$.

(v) $\Leftrightarrow$ (i) By the exact sequence of Proposition I.2.2(i), we have

$$\gamma_*\left[\coprod S^1\right] = 0 \Longleftrightarrow \left[\coprod S^1\right] \in \operatorname{Im}\left(H_2(X, \gamma; \mathbb{Q}) \xrightarrow{\partial} H_1\left(\coprod S^1; \mathbb{Q}\right)\right).$$

But by Proposition I.2.12, this is equivalent to $\mathrm{scl}_G([c]) = \mathrm{scl}_X(\gamma)$ being finite. $\square$

Proposition I.2.12 suggests that computations of scl could be tackled in two successive steps: first fix a relative class $\alpha \in H_2(X, \gamma)$ with $\partial\alpha = \left[\coprod S^1\right]$ and estimate $\|\alpha\|_1$, and then find the infimum over all such classes α. Note that, if $H_2(X) = 0$ (which happens for instance if G is free and X is a $K(G, 1)$), then the exact sequence of Proposition I.2.2(i) tells us that $\partial : H_2(X, \gamma) \to H_1\left(\coprod S^1\right)$ is injective. If in addition γ is rationally homologically trivial, then the map $\gamma_* : H_1\left(\coprod S^1; \mathbb{Q}\right) \to H_1(X; \mathbb{Q})$ in the exact sequence of Proposition I.2.2(i) satisfies $\gamma_*\left[\coprod S^1\right] = 0$, and there is a unique $\alpha \in H_2(X, \gamma)$ such that $\partial\alpha = \left[\coprod S^1\right]$. In this case, $\mathrm{scl}_X(\gamma) = \frac{1}{4}\|\alpha\|_1$ by Proposition I.2.12, and scl and $\|\cdot\|_1$ measure the same thing.

However, our point is that in some cases, computations of scl can be made difficult by the presence of nonzero classes in $H_2(X)$, and in those cases, one may hope to obtain information on the relative Gromov seminorm as a stepping stone towards scl.

CHAPTER II

REDUCTION OF ADMISSIBLE SURFACES

In order to give estimates or exact computations of scl or the Gromov seminorm using their topological interpretation, it is often useful to reduce admissible surfaces to having certain topological properties. The purpose of this chapter is to show how admissible surfaces can be modified — while keeping some control over their complexity — to give them those properties. The first part of the chapter is devoted to properties that one can obtain in the general case; this material is well-known and most of it can also be found in Calegari's book [16]. The goal of the second part is to obtain sharper properties, leading us to standard forms for admissible surfaces, in the context of surface groups; these are original results that first appeared in [58, §4] and that we will use in Chapter IV to prove that certain embeddings of surface groups are isometric for scl or the relative Gromov seminorm.

II.1 General properties

II.1.a Incompressibility, simplicity, etc.

Let X be a path-connected topological space and let $\gamma : \coprod S^1 \to X$. We begin with three basic properties of admissible surfaces:

Definition II.1.1. An admissible surface $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$ is said to be

- *Disc- and sphere-free* if no connected component of Σ is a disc or a sphere,
- *Incompressible* if every simple closed curve in Σ with nullhomotopic image in X is itself nullhomotopic in Σ,
- *Simple* if there are no two boundary components of Σ whose image under f represent powers of the same conjugacy class in $\pi_1 X$.

Given an scl- or ℓ^1-admissible surface $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$, we can readily discard any disc- or sphere-component of Σ as this will not change $-\chi^-(\Sigma)$.

Assume now that Σ has a non-contractible simple closed curve β with nullhomotopic image. If β is homotopic to a boundary component of Σ, then we can glue a disc onto that boundary component, thereby reducing $-\chi^-(\Sigma)$. Otherwise, we can cut Σ along β and glue two discs onto the resulting boundary components mapping with opposite orientations to the same disc in X; this will also decrease $-\chi^-(\Sigma)$. As none of the above operations change $f_*[\Sigma]$, they can be iterated until one obtains a disc- and sphere-free and incompressible admissible surface.

Obtaining a simple admissible surface is slightly more subtle: if Σ has two boundary components whose image under f represent powers of the same conjugacy class in $\pi_1 X$, then one should replace Σ with a cover $\hat{\Sigma}$ of large degree N and with the same number of boundary components as Σ, and then attach a 1-handle connecting two boundary components of $\hat{\Sigma}$, as in Figure I.2; this will increase the stabilised complexity $\frac{-\chi^-(\Sigma)}{n(\Sigma)}$ by only $\frac{1}{N}$. Since N can be chosen arbitrarily large, this allows one to make Σ simple while only increasing the complexity by a controlled small amount. Note also that after this operation, Σ remains an scl- or ℓ^1-admissible surface for the same class: $f_*[\Sigma]$ was only multiplied by N (as the new 1-handle is mapped to a point). This is the same process that was explained in the proof of Proposition I.1.13, and it is an example of a type of argument that we call *asymptotic promotion* after Chen [24] —

we have performed a topological modification on our surface in an asymptotic way to control its cost. There is however a drawback: while we can use simple admissible surfaces to compute scl or $\|\cdot\|_1$, we cannot assume that extremal surfaces are simple, because of the small but positive cost of the operation.

By first making our admissible surface simple, and then performing the two previous operations to make it disc- and sphere-free and incompressible, we can combine all three properties. This proves the following:

Lemma II.1.2 (Simple, incompressible, disc- and sphere-free admissible surfaces)**.** *Let $\alpha \in H_2(X,\gamma;\mathbb{Q})$, and consider an admissible surface $f : (\Sigma, \partial\Sigma) \to (X,\gamma)$ with $f_*[\Sigma] = n(\Sigma)\alpha$.*

(i) *There exists an incompressible and disc- and sphere-free admissible surface $f' : (\Sigma', \partial\Sigma') \to (X,\gamma)$ with $f'_*[\Sigma'] = n(\Sigma')\alpha$, and such that*
$$\frac{-\chi^-(\Sigma')}{n(\Sigma')} \leqslant \frac{-\chi^-(\Sigma)}{n(\Sigma)}.$$

(ii) *For every $\varepsilon > 0$, there is a simple, incompressible, disc- and sphere-free admissible surface $f' : (\Sigma', \partial\Sigma') \to (X,\gamma)$ with $f'_*[\Sigma'] = n(\Sigma')\alpha$, and such that*
$$\frac{-\chi^-(\Sigma')}{n(\Sigma')} \leqslant \frac{-\chi^-(\Sigma)}{n(\Sigma)} + \varepsilon.$$ □

II.1.b Monotonicity

Let $f : (\Sigma, \partial\Sigma) \to (X,\gamma)$ be an admissible surface. Recall from §I.1.d that, as part of the data of f, there is a continuous map $\partial f : \partial\Sigma \to \coprod S^1$. If ∂_j is a boundary component of Σ, then ∂f sends ∂_j to a component $S^1_{i_j}$ of $\coprod S^1$.

Definition II.1.3. An admissible surface $f : (\Sigma, \partial\Sigma) \to (X,\gamma)$ is *monotone* if the sign of the degree of the restriction
$$\partial f_{|\partial_j} : \partial_j \to S^1_{i_j}$$
only depends on the component $S^1_{i_j}$ of $\coprod S^1$. In other words, two boundary components of Σ mapping to the same component of $\coprod S^1$ do so with the same orientation[1].

[1] This definition is in general different from the usual definition of a monotone admissible surface (see [16, Definition 2.12]), but the two definitions coincide if f is an scl-admissible surface, or more generally if the coordinates of $\partial\alpha$ in the basis $\left([S^1_i]\right)_i$ of $H_1\left(\coprod S^1\right)$ all have the same sign. Indeed, recall from §I.1.d that each circle in $\coprod S^1$ comes with an orientation, and that α might not necessarily map to all circles with positive orientation under ∂. In particular, if $\partial\alpha$ has two components of opposite orientations, then there is no monotone ℓ^1-admissible surface for α in the usual sense.

Monotonicity is a weaker property than simplicity. We have seen that admissible surfaces could be made simple via an asymptotic promotion argument; we will now show that asymptotic promotion can be bypassed to obtain monotonicity. Hence extremal surfaces, if they exist, can be assumed to be monotone, even if they might not be simple.

The reduction to monotone admissible surfaces is standard in the context of scl [16, Proposition 2.13]. Our definition of monotonicity allows us to adapt this to the relative Gromov seminorm[2]:

Lemma II.1.4 (Monotone admissible surfaces)**.** *Let* $\alpha \in H_2(X, \gamma; \mathbb{Q})$*, and consider an admissible surface* $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$ *with* $f_*[\Sigma] = n(\Sigma)\alpha$*. Then there is an incompressible, monotone, disc- and sphere-free admissible surface* $f' : (\Sigma', \partial\Sigma') \to (X, \gamma)$ *with* $f'_*[\Sigma'] = n(\Sigma')\alpha$*, and such that*

$$\frac{-\chi^-(\Sigma')}{n(\Sigma')} \leqslant \frac{-\chi^-(\Sigma)}{n(\Sigma)}.$$

Proof. Start by using Lemma II.1.2 to ensure that f is disc- and sphere-free and incompressible. For each component ∂_j of $\partial\Sigma$, denote by $n_j \in \mathbb{Z}$ the (signed) degree of the restriction $f_{|\partial_j} : \partial_j \to S^1_{i_j}$. The integer n_j must be non-zero by incompressibility.

We first assume that Σ is connected, and follow Calegari's argument [16, Proposition 2.13] in this case. Since Σ is disc- and sphere-free, it has non-positive Euler characteristic. If $\chi(\Sigma) = 0$, then Σ is a torus or an annulus, in which case it is automatically monotone. We can therefore assume that $\chi(\Sigma) < 0$, and also that $\partial\Sigma \neq \varnothing$. After passing to a finite cover, we can further assume that Σ has positive genus. Thus, Σ admits a degree-2 cover in which every boundary component of Σ has exactly two preimages. After replacing Σ with this cover, the boundary components of Σ can be paired so that paired components have the same degree n_j. We now denote by $\left\{\partial_j^\pm\right\}_{j\in J}$ the set of boundary components of Σ, so that ∂_j^+ is paired with ∂_j^-, and both have degree n_j.

Fix a basepoint ω of Σ, and for each $\partial_j^\pm$, a based loop $\delta_j^\pm \in \pi_1(\Sigma, \omega)$ in the free homotopy class of $\partial_j^\pm$. Let $\Delta = \left\{\delta_j^\pm\right\}_{j\in J}$ and pick an element $\delta_0 \in \Delta$. Let $N = \operatorname{lcm}\{n_j\}_{j\in J}$ and define

$$\underline{\theta} : \delta_j^\pm \in \Delta \longmapsto (\pm n_j \mod N) \in \mathbb{Z}/N.$$

[2]This would be false with the usual definition — see Footnote 1.

Note that $\sum_{\delta\in\Delta}\underline{\theta}(\delta) = 0$. Now the group $\pi_1\Sigma$ is free and its subset $\Delta \setminus \{\delta_0\}$ can be completed to a free basis B. Pick any map $\theta : B \to \mathbb{Z}/N$ such that

$$\forall\delta \in \Delta \setminus \{\delta_0\}\,,\ \theta(\delta) = \underline{\theta}(\delta).$$

This extends to a group homomorphism $\theta : \pi_1\Sigma \to \mathbb{Z}/N$. Moreover, we have seen that $\sum_{\delta\in\Delta}\underline{\theta}\,(\delta) = 0$; since the image of $\sum_{\delta\in\Delta}\delta$ in $(\pi_1\Sigma)_{\mathrm{ab}}$ is zero, it follows that $\theta\,(\delta_0) = \underline{\theta}\,(\delta_0)$. In other words, we have shown the existence of a group homomorphism

$$\theta : \pi_1\Sigma \to \mathbb{Z}/N$$

with $\theta\left(\delta_j^{\pm}\right) = (\pm n_j \mod N)$ for all $j \in J$. Replace Σ with its finite cover corresponding to $\operatorname{Ker}\theta$. Now every component of $\partial\Sigma$ maps to a component of $\coprod S^1$ with degree $\pm N$. Boundary components mapping to the same component of $\coprod S^1$ with opposite degrees can be glued to one another — which does not change $-\chi^-(\Sigma)$ — until we obtain a monotone admissible surface, with the additional property that each boundary component has degree $\pm N$. After passing to a finite cover if necessary, we can further replace N with any positive multiple.

Now assume that Σ has two or more connected components. After applying the previous process to each component of Σ, and after possibly taking further finite covers to ensure that the homology classes represented by all individual components have been multiplied by the same integer N, we can assume that the restriction of Σ to each component is monotone, and that each component of $\partial\Sigma$ has degree $\pm N$ for some fixed integer $N \in \mathbb{N}_{\geqslant 1}$. If Σ is still not monotone, then it must have two boundary components ∂_1 and ∂_2 contained in distinct connected components, and mapping to the same component of $\coprod S^1$ with opposite degrees. In this case, we can glue ∂_1 to ∂_2. The resulting admissible surface might now have a non-monotone connected component, but then we can apply the previous process again to ensure that the restriction to each component is monotone. Since each iteration of this process decreases the number of connected components of Σ by 1, we can repeat until Σ is monotone. We can then apply Lemma II.1.2 again to make sure that our admissible surface is still incompressible and disc- and sphere-free — this will not impact monotonicity. □

II.1.c Transversality

We now assume that X is a 2-dimensional cell complex, with $\gamma : \coprod S^1 \to X$. In this setting, we will use the notion of transversality from [13, §VII.2] to obtain a nice decomposition of admissible surfaces. The same notion was used by Brady, Clay and Forester in their proof of the Rationality Theorem [7].

Consider an admissible surface $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$ with $f_*[\Sigma] = n(\Sigma)\alpha$. We can first use Lemma II.1.2 to make f incompressible. We then apply the Transversality Theorem from [13, §VII.2] and modify γ, f and ∂f by homotopies to make $\gamma : \coprod S^1 \to X$ and $f : \Sigma \to X$ transverse:

Lemma II.1.5 (Transversality)**.** *Let X be a connected 2-dimensional CW-complex, $\gamma : \coprod S^1 \to X$. Then any incompressible admissible surface $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$ may be homotoped to $f' : (\Sigma, \partial\Sigma) \to (X, \gamma')$ (with γ' homotopic to γ) that is* transverse*, in the sense that Σ (with the map f') has a decomposition into the following pieces:*

- Vertex discs*, i.e. discs mapping to vertices of X,*
- 1-handles*, i.e. trivial I-bundles over edges of X, and*
- Cellular discs*, i.e. discs mapping homeomorphically onto 2-cells of X.*

Proof. This follows from Buoncristiano, Rourke and Sanderson's Transversality Theorem [13, §VII.2]. □

Figure II.1 represents a part of a transverse incompressible admissible surface.

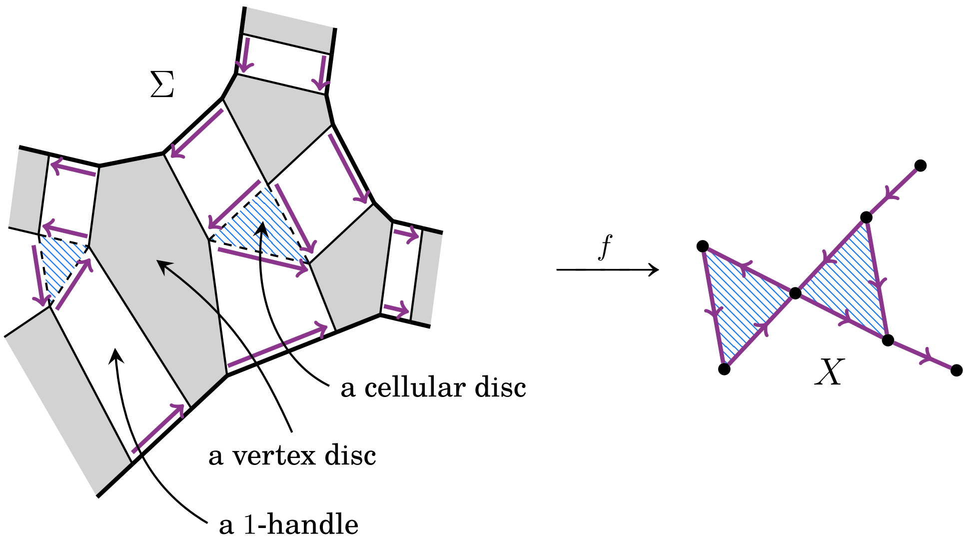


Figure II.1: Part of a transverse incompressible admissible surface.

II.2 Properties specific to the surface group case

We henceforth assume that the 2-complex X is a cellulated, oriented, compact, connected surface, and we'll denote it by S. We will see that in this case, we can obtain much finer topological properties for admissible surfaces.

II.2.a Connectedness of links

Let $\gamma : \coprod S^1 \to S$ be a transverse family of loops and consider an scl- or ℓ^1-admissible surface $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ with $f_*[\Sigma] = n(\Sigma)\alpha$ in $H_2(S, \gamma)$. We may assume that f is transverse, incompressible, monotone, and disc- and sphere-free, as explained in §II.1.

We now want to use the fact that S is a surface to ensure that Σ is 'thick enough', in the sense that its vertex discs have connected links.

More precisely, consider the 2-complex $\bar{\Sigma}$ obtained from Σ by collapsing all vertex discs to vertices and all 1-handles to edges — hence, f induces a combinatorial map $\bar{f} : \bar{\Sigma} \to S$, but $\bar{\Sigma}$ may not be a surface.

Definition II.2.1. We say that a transverse incompressible admissible surface $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ has *connected links* if the 2-complex $\bar{\Sigma}$ has connected vertex links.

If f does not have connected links, let D be a vertex disc in Σ whose corresponding vertex in $\bar{\Sigma}$ has disconnected link. Let v be the image of D under f; so v is a vertex of S. There are two cases: either v lies in the interior of S, or on the boundary.

Assume first that v lies in the interior of S; an example is depicted in Figure II.2 (all the 2-cells on the picture are triangles for simplicity, but the general case is similar). The main point is that there is a 2-cell in S between any two consecutive edges around v; in particular, there is a sequence of 2-cells $\sigma_1, \dots, \sigma_\ell$ lying between two successive edges whose preimages are 1-handles with one end each on $\partial\Sigma$, as in Figure II.2. Now we perform a homotopy that moves the image of $\partial\Sigma$ across the 2-cells $\sigma_1, \dots, \sigma_\ell$: see Figure II.3. The new map $f : \Sigma \to S$ defines an admissible surface for a map $\gamma' : \coprod S^1 \to S$ homotopic to γ. Note that f (and γ) have been modified by a homotopy (but $\partial f : \partial\Sigma \to S^1$ can remain unchanged), so we still have $f_*[\Sigma] = n(\Sigma)\alpha$ in $H_2(S, \gamma') \cong H_2(S, \gamma)$.

This operation decreases the number of connected components in the link of D (or more precisely, of its image in $\bar{\Sigma}$). Note that new vertex discs may have been created

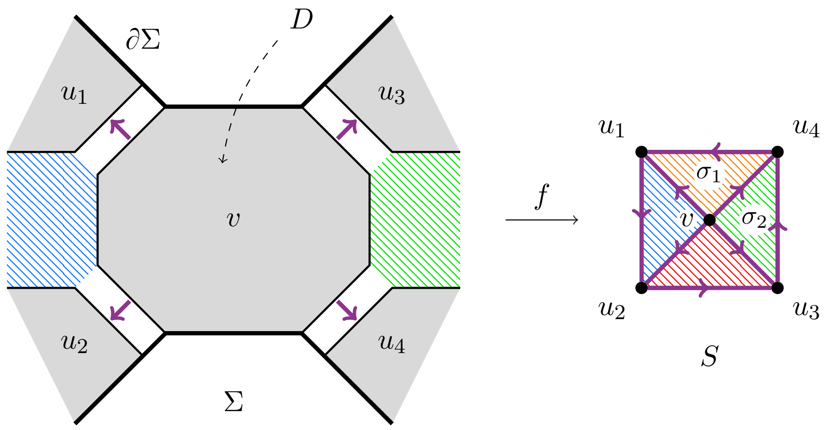


Figure II.2: A vertex disc with disconnected link mapping to a vertex in the interior of S.

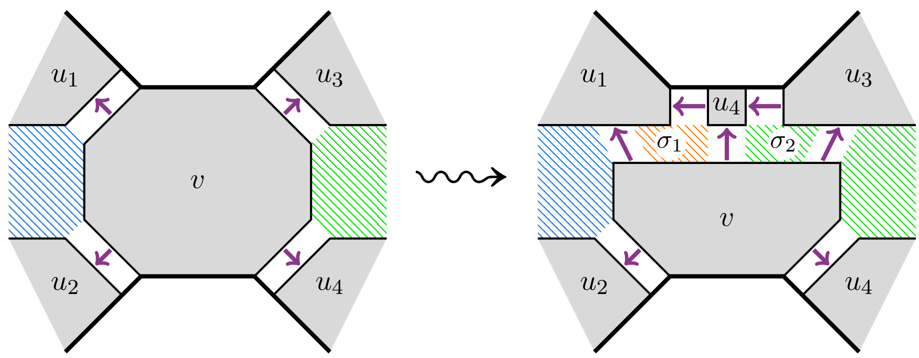


Figure II.3: Making links of vertex discs connected (interior case).

(such as the one mapping to u_4 in Figure II.3), but they all have connected link. As for preexisting vertex discs of Σ, the operation doesn't impact the number of connected components of their links. Hence, if $\{D_i\}_i$ is the set of vertex discs of Σ mapping to the interior of S, and k_i denotes the number of connected components of the link of D_i, then we have made the quantity $\sum_i (k_i - 1)$ decrease strictly. We may therefore iterate to ensure that all vertex discs mapping to the interior of S have connected link.

We deal with the case where v lies on the boundary in the following way. We thicken S by gluing a cellulated annulus to each of its boundary components. This modifies the cellular structure of S but not its homeomorphism type (in other words, S has been replaced with another cell complex S' with more cells, but with S' homeomorphic to S), and this preserves all the properties of the map f — in particular, f is transverse for the new cellulation of S. Now all of the vertex discs with disconnected link map to the interior of S. Therefore, we can apply the operation described above to make all their links connected. This may create some new vertex discs mapping to the boundary, but they will have connected link. Hence, after both operations, all vertex discs have connected link.

We therefore obtain the following:

Lemma II.2.2 (Admissible surfaces with connected links)**.** *Fix $\alpha \in H_2(S, \gamma; \mathbb{Q})$, and let $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ be a transverse incompressible admissible surface with $f_*[\Sigma] = n(\Sigma)\alpha$. Then, after possibly changing the cellular structure on S, and changing γ by a homotopy, the map f may be homotoped to a transverse incompressible admissible surface with connected links.* □

Note that since our admissible surface was modified by a homotopy, the properties of being incompressible, monotone, and disc- and sphere-free are preserved.

II.2.b Folding

The key properties of admissible surfaces that we will need in the surface group case are related to orientation. Indeed, both S and the admissible surface Σ are oriented. Since f is transverse and cellular discs map homeomorphically into S, they can be of two types: either they preserve the orientation or they reverse it. Having cellular discs of opposite orientations is undesirable, and we are now going to modify Σ to avoid this situation as much as possible.

We assume that $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ is transverse, incompressible, monotone, disc- and sphere-free, and with connected links.

Suppose first that there is a connected component of Σ containing cellular discs of two different types — i.e. one is orientation-preserving with respect to f and the other is orientation-reversing. Since $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ has connected links, any two cellular discs in Σ mapping to cells in S with a common vertex on their boundaries must be connected by a path of cellular discs and 1-handles in Σ. It follows that any two cellular discs in the same connected component of Σ are connected by a path of cellular discs and 1-handles.

Therefore, Σ must contain two cellular discs of opposite types that are adjacent via a 1-handle. Since S is a surface, those two cellular discs must map to the same 2-cell of S, and we are in the situation of Figure II.4 (pictures are given for the case of a 2-cell of degree 3, but the general case is similar) — in other words, f *folds* those two cellular discs onto one another.

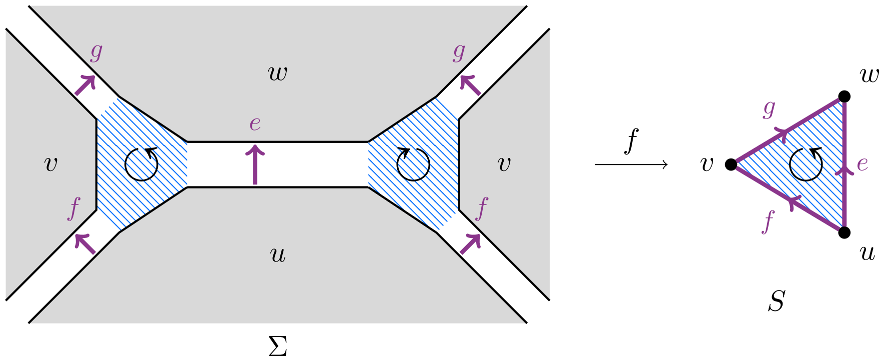


Figure II.4: Adjacent cellular discs of opposite orientations.

In this case, we can delete the two adjacent cellular discs as illustrated in Figure II.5. This operation does not change the homotopy type of Σ nor the boundary map

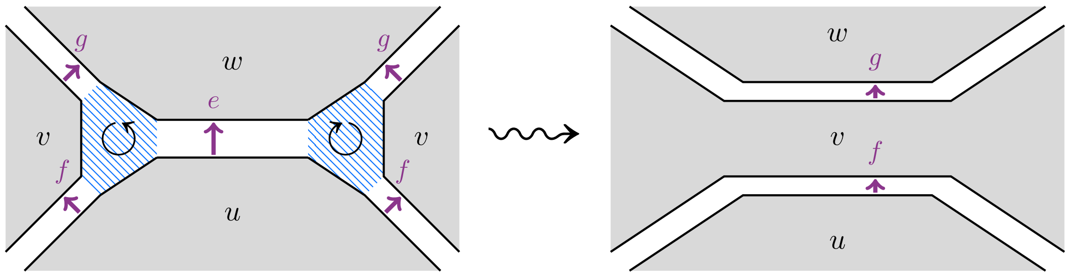


Figure II.5: Eliminating cellular discs of opposite orientations.

$\partial f : \partial\Sigma \to \coprod S^1$. It amounts to changing f by a homotopy across the blue cell in Figure II.5, so it preserves the class $f_*[\Sigma] = n(\Sigma)\alpha$, as well as incompressibility, monotonicity of f, and the property of being disc- and sphere-free; finally, the resulting admissible surface is still transverse.

Moreover, it makes the number of cellular discs of Σ decrease strictly. Hence, after repeating a finite number of times, there is no connected component of Σ containing two adjacent cellular discs of opposite orientations as in Figure II.4. Therefore, each connected component of Σ contains cellular discs of only one type: either orientation-preserving or orientation-reversing.

In other words, we have reduced to the case where Σ has the following property:

Definition II.2.3. Let S be a cellulated, oriented, compact, connected surface. A transverse incompressible admissible surface $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ is said to be *non-folded* if each connected component of Σ contains only cellular discs that are either all orientation-preserving or all orientation-reversing.

However, the operation just described could have created some vertex discs with disconnected link in Σ (as removing cellular discs amounts to deleting edges in the links of vertex discs).

To fix this, we would like to successively apply the operation described above and the one of §II.2.a. This relies on the following crucial observation: in the process described in §II.2.a, we can always choose the orientation of the cellular discs that we add. Indeed, the added cellular discs correspond to a path in the link of v in S, and this link is a circle, so there are two possible paths, one corresponding to adding cellular discs of positive orientation to Σ, and the other corresponding to adding cellular discs of negative orientation. For example, Figure II.6 shows two possible choices that make the link of the vertex disc of Figure II.2 connected. In particular, when applying the operation of §II.2.a, we can assume that we are only adding cellular discs of *positive* orientation.

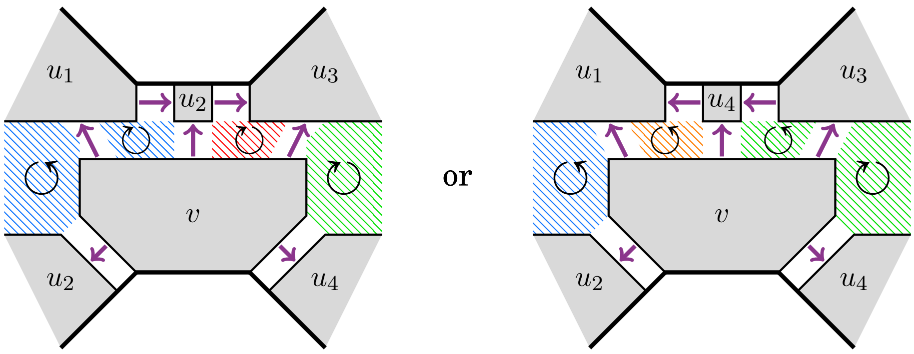


Figure II.6: Two possible choices for making the link of the vertex disc of Figure II.2 connected.

We can now apply the following procedure: we alternately apply the operation of §II.2.a to make links connected — and with the choice of only adding cellular discs of positive orientation — and then the operation described above to remove folding. Each iteration of the latter removes one cellular disc of each orientation, and each iteration of the former does not increase the number of cellular discs of negative orientation. Hence, the total number of discs of negative orientation decreases strictly at each pair of iterations, ensuring that the process terminates in an admissible surface that is both non-folded and with connected links.

This proves the following:

Lemma II.2.4 (Non-folded admissible surfaces)**.** *Fix $\alpha \in H_2(S, \gamma; \mathbb{Q})$. Given a transverse incompressible admissible surface $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ with $f_*[\Sigma] = n(\Sigma)\alpha$, there is a map γ' homotopic to γ and a non-folded admissible surface $f' : (\Sigma', \partial\Sigma') \to (S, \gamma')$ with connected links, with $f'_*[\Sigma'] = n(\Sigma')\alpha$, and such that*

$$\frac{-\chi^-(\Sigma')}{n(\Sigma')} \leqslant \frac{-\chi^-(\Sigma)}{n(\Sigma)}.$$

Moreover, f' can be assumed to be monotone and disc- and sphere-free. □

II.2.c Asymptotic promotion to orientation-perfect surfaces

In order to obtain isometric embedding results for surface groups in §IV.3, we will need admissible surfaces to satisfy the following orientation property:

Definition II.2.5. Let S be a cellulated, oriented, compact, connected surface. A transverse incompressible admissible surface $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ is *orientation-perfect* if there are no two cellular discs in Σ that map to the same 2-cell of S with opposite orientations.

In order to make an admissible surface orientation-perfect, we will need to use an asymptotic promotion argument, similar to the idea used in §II.1.b for monotonicity.

We start with a transverse, incompressible, monotone, disc- and sphere-free, non-folded admissible surface with connected links $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ with $f_*[\Sigma] = n(\Sigma)\alpha$, and we assume that f is not orientation-perfect. Fix a small $\varepsilon > 0$, and pick a large $N \in \mathbb{N}_{\geqslant 1}$ such that $\frac{1}{N} \leqslant \varepsilon$. Let $\Sigma_0 \to \Sigma$ be a degree-N covering under which the preimage of every connected component of Σ is connected. The composite map $\Sigma_0 \to \Sigma \to S$ is also a transverse, incompressible, monotone, disc- and sphere-free, non-folded admissible surface with connected links, with $\chi^-(\Sigma_0) = N\chi^-(\Sigma)$ and $n(\Sigma_0) = Nn(\Sigma)$. Since f is non-folded but not orientation-perfect, there are two cellular discs in distinct components of Σ_0 that map to the same 2-cell of S with opposite orientations. We remove those two discs and glue the resulting boundary components to one another in a way that is compatible with the map f. There is an admissible surface $f' : (\Sigma_0', \partial\Sigma_0') \to (S, \gamma)$ resulting from this operation, which is still transverse, incompressible, monotone, disc- and sphere-free; it satisfies $f'_*[\Sigma_0'] = Nn(\Sigma)\alpha$, and

$$-\chi^-\left(\Sigma_0'\right) = -\chi^-(\Sigma_0) + 2 = -N\chi^-(\Sigma) + 2.$$

Therefore

$$\frac{-\chi^-(\Sigma_0')}{n(\Sigma_0')} \leqslant \frac{-\chi^-(\Sigma)}{n(\Sigma)} + 2\varepsilon.$$

We can then perform the process of §II.2.b again to ensure that Σ_0' is non-folded and with connected links.

After the complete operation, the number of connected components of Σ has decreased by one, while the quantity $\frac{-\chi^-(\Sigma)}{n(\Sigma)}$ has not increased more than a controlled arbitrarily small amount. Since Σ has a finite number of connected components, we may iterate until we obtain an orientation-perfect surface. We obtain the following:

Lemma II.2.6 (Orientation-perfect admissible surfaces). *Fix $\alpha \in H_2(S, \gamma; \mathbb{Q})$. Given an $\varepsilon > 0$ and a transverse incompressible admissible surface $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ with $f_*[\Sigma] = n(\Sigma)\alpha$, there is a map γ' homotopic to γ and an orientation-perfect admissible surface $f' : (\Sigma', \partial\Sigma') \to (S, \gamma')$, with $f'_*[\Sigma'] = n(\Sigma')\alpha$, and such that*

$$\frac{-\chi^-(\Sigma')}{n(\Sigma')} \leqslant \frac{-\chi^-(\Sigma)}{n(\Sigma)} + \varepsilon.$$

Moreover, f' can be assumed to be monotone, disc- and sphere-free, non-folded, and with connected links. □

Remark II.2.7. If γ is a single loop $S^1 \to S$, and if $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ is an scl-admissible surface for γ (rather than an ℓ^1-admissible surface), then we can bypass the asymptotic promotion argument and in fact replace Σ with a connected admissible surface. Indeed, consider the connected components $\{\Sigma_i\}_i$ of Σ, and observe that the restriction of f to each Σ_i is an scl-admissible surface for γ (but note that distinct components may represent distinct classes in $H_2(S, \gamma; \mathbb{Q})$). We have

$$\frac{-\chi^-(\Sigma)}{n(\Sigma)} = \frac{\sum_i(-\chi^-(\Sigma_i))}{\sum_i n(\Sigma_i)} \geqslant \min_i \frac{-\chi^-(\Sigma_i)}{n(\Sigma_i)}.$$

Hence there is a component Σ_i of Σ for which $\frac{-\chi^-(\Sigma_i)}{n(\Sigma_i)} \leqslant \frac{-\chi^-(\Sigma)}{n(\Sigma)}$, and we may replace Σ with Σ_i. Now Σ is connected, so making it non-folded is enough to guarantee that it is orientation-perfect.

II.2.d Standard form in the surface group case

We have shown the following:

Proposition II.2.8 (Standard form). *Let S be an oriented, compact, connected surface, let $\gamma : \coprod S^1 \to S$, and $\alpha \in H_2(S, \gamma; \mathbb{Q})$. Then*

(i) *The relative Gromov seminorm of* α *can be computed via*

$$\|\alpha\|_1 = \inf_{f,\Sigma} \frac{-2\chi^-(\Sigma)}{n(\Sigma)},$$

where the infimum is taken over all admissible surfaces $f : (\Sigma, \partial\Sigma) \to (S, \gamma')$ *that are transverse, incompressible, monotone, disc- and sphere-free, orientation-perfect, non-folded, with connected links for some cellulation of* S *and some map* γ' *homotopic to* γ.

Such an admissible surface is said to be in perfect standard form.

(ii) *If there exists an* ℓ^1*-extremal surface for* α *(i.e. realising the infimum in Proposition I.2.11), then there exists one which is transverse, incompressible, monotone, disc- and sphere-free, non-folded, with connected links for some cellulation of* S.

Such an admissible surface is said to be in standard form.

Proof. This follows from the lemmas of §II.1 and §II.2. □

It follows from Proposition I.2.12 that the obvious analogue of Proposition II.2.8 holds for scl: the stable commutator length of γ can be computed with surfaces in perfect standard form, and if there exists an extremal surface, then there exists one in standard form.

Moreover, if γ consists of a single loop $S^1 \to S$, and if there exists an scl-extremal surface for γ, then there exists one in perfect standard form (see Remark II.2.7).

CHAPTER III

BAVARD DUALITY

Bavard [4] proved that the dual space of the scl-seminorm on $C_1^{\mathrm{conj}}(G;\mathbb{R})$ can be interpreted in terms of quasimorphisms. This can be thought of as a kind of ℓ^1–ℓ^∞ duality, and has had a wide range of applications in giving lower bounds for scl [5, 19, 27, 34, 48]. We start with some background on classical Bavard duality and bounded cohomology, and then we explain how a result analogous to Bavard's Theorem can be obtained for the relative Gromov seminorm. We then give a purely algebraic interpretation of our duality theorem for the relative Gromov seminorm using a relative version of the Hopf formula. The content of this chapter is taken from [57, §3-4].

III.1 Bavard duality for scl

A *quasimorphism* on a group G is a map $\phi : G \to \mathbb{R}$ such that

$$\sup_{g,h\in G} |\phi(gh) - \phi(g) - \phi(h)| < \infty.$$

The above supremum is then called the *defect* of ϕ and denoted by $D(\phi)$. We say in addition that ϕ is *homogeneous* if $\phi(g^n) = n\phi(g)$ for all $g \in G$ and $n \in \mathbb{Z}$.

We denote by $Q(G)$ the $\mathbb{R}$-vector space of homogeneous quasimorphisms on G. The defect defines a seminorm $D : Q(G) \to [0,\infty)$, which vanishes exactly on the subspace $\mathrm{Hom}(G,\mathbb{R}) \hookrightarrow Q(G)$ consisting of homomorphisms to $\mathbb{R}$. In particular, the defect descends to a genuine norm on the quotient $Q(G)/\mathrm{Hom}(G,\mathbb{R})$.

If $\phi : G \to \mathbb{R}$ is a homogeneous quasimorphism, then ϕ extends to a $\mathbb{Z}$-linear map $C_1(G;\mathbb{Z}) \to \mathbb{R}$. The extension satisfies

$$\left|\phi\left(w - w^t\right)\right| = \frac{1}{n}\left|\phi\left(w^n - t^{-1}w^n t\right)\right| \leqslant \frac{2D(\phi)}{n} \xrightarrow[n\to\infty]{} 0$$

for all $w, t \in G$. It follows that ϕ vanishes on the sub-$\mathbb{Z}$-module $K(G;\mathbb{Z})$ of $C_1(G;\mathbb{Z})$ spanned by elements of the form $\left(w - w^t\right)$, as in §I.1.b. Therefore, ϕ descends to a $\mathbb{Z}$-linear map $C_1^{\mathrm{conj}}(G;\mathbb{Z}) \to \mathbb{R}$, which then extends to a $\mathbb{R}$-linear map $C_1^{\mathrm{conj}}(G;\mathbb{R}) \to \mathbb{R}$.

Classical Bavard duality says that the (semi)normed vector space $(Q(G), D)$[1] is dual to $\left(C_1^{\mathrm{conj}}(G;\mathbb{R}), \mathrm{scl}\right)$:

Theorem III.1.1 (Bavard [4]). *Let G be a group and $[c] \in C_1^{\mathrm{conj}}(G;\mathbb{R})$. Then there is an equality*

$$\mathrm{scl}_G([c]) = \sup\left\{ \frac{\phi([c])}{2D(\phi)} \;\middle|\; \phi \in Q(G) \smallsetminus \mathrm{Hom}(G;\mathbb{R}) \right\}.$$

III.2 ℓ^1-ℓ^∞ duality in homology

III.2.a Bounded cohomology of topological spaces

Our analogue of Bavard duality for the relative Gromov seminorm will be based on bounded cohomology, of which we recall the definition here. We refer the reader to Frigerio's book [36] for a much more detailed treatment.

Let X be a topological space. Recall that the singular cohomology of X (with real coefficients) is the cohomology of the singular cochain complex $C^*_{\mathrm{sg}}(X;\mathbb{R})$ given by

$$C^n_{\mathrm{sg}}(X;\mathbb{R}) := \mathrm{Hom}_{\mathbb{R}}\left(C_n^{\mathrm{sg}}(X;\mathbb{R}), \mathbb{R}\right).$$

[1] Or $Q(G)/\mathrm{Hom}(G,\mathbb{R})$, where D defines a honest norm.

Since the n-th chain group $C_n^{\mathrm{sg}}(X;\mathbb{R})$ is the free $\mathbb{R}$-module on the set $\mathcal{S}_n$ of singular n-simplices $\sigma : \Delta^n \to X$, the n-th cochain group $C^n_{\mathrm{sg}}(X;\mathbb{R})$ can equivalently be defined as the set of all maps $\mathcal{S}_n \to \mathbb{R}$, with coboundary maps $d^n : C^{n-1}_{\mathrm{sg}}(X;\mathbb{R}) \to C^n_{\mathrm{sg}}(X;\mathbb{R})$ given by

$$d^n\psi(\sigma) := \sum_{i=0}^{n}(-1)^i\psi\left(\sigma_{|F_i}\right),$$

where $\sigma : \Delta^n \to X$ is a singular simplex, and F_i denotes the i-th face of the standard simplex Δ^n for $i \in \{0,\ldots,n\}$ (for more details, see for instance [47]).

The ℓ^∞*-norm* of a cochain $\psi \in C^n_{\mathrm{sg}}(X;\mathbb{R}) = \mathbb{R}^{\mathcal{S}_n}$ is

$$\|\psi\|_\infty := \sup_{\sigma\in\mathcal{S}_n}|\psi(\sigma)| \in [0,+\infty].$$

The *bounded cochain complex* of X with coefficients in $\mathbb{R}$ is the sub-cochain complex $C^*_b(X;\mathbb{R})$ of $C^*_{\mathrm{sg}}(X;\mathbb{R})$ consisting of all *bounded* maps $\mathcal{S}_n \to \mathbb{R}$:

$$C^n_b(X;\mathbb{R}) := \left\{\psi \in C^n_{\mathrm{sg}}(X;\mathbb{R}) \;\middle|\; \|\psi\|_\infty < \infty\right\} \hookrightarrow C^n_{\mathrm{sg}}(X;\mathbb{R}).$$

Note that $d^n\left(C^{n-1}_b(X;\mathbb{R})\right) \subseteq C^n_b(X;\mathbb{R})$ (because each $d^n : C^{n-1}_{\mathrm{sg}}(X;\mathbb{R}) \to C^n_{\mathrm{sg}}(X;\mathbb{R})$ is a bounded linear operator for $\|\cdot\|_\infty$), so $C^*_b(X;\mathbb{R})$ is indeed a sub-cochain complex of $C^*_{\mathrm{sg}}(X;\mathbb{R})$. The *bounded cohomology* of X is the cohomology of this cochain complex:

$$H^*_b(X;\mathbb{R}) := H^*\left(C^*_b(X;\mathbb{R})\right).$$

Throughout this thesis, bounded cohomology should always be understood to be with real coefficients, even though we might sometimes drop $\mathbb{R}$ from the notation.

The ℓ^∞-norm descends to a seminorm — still denoted $\|\cdot\|_\infty$ — on $H^*_b(X;\mathbb{R})$.

It turns out that $\|\cdot\|_\infty$ defines a genuine norm in degree 2:

Theorem III.2.1 (Matsumoto–Morita–Ivanov [52, 60])**.** *Let X be a topological space. Then $\|u\|_\infty > 0$ for every $u \in H^2_b(X;\mathbb{R}) \smallsetminus \{0\}$.*

Duality between the ℓ^∞-norm on bounded cohomology and the ℓ^1-norm on singular homology plays a central role in this chapter. It comes from the natural pairing

$$\langle -,-\rangle : C^*_b(X;\mathbb{R}) \times C^{\mathrm{sg}}_*(X;\mathbb{R}) \to R$$

which is the restriction of the duality pairing $C^*_{\mathrm{sg}}(X;\mathbb{R}) \times C^{\mathrm{sg}}_*(X;\mathbb{R}) \to R$ given by $\langle\psi,c\rangle := \psi(c)$ for $\psi \in C^n_{\mathrm{sg}}(X;\mathbb{R})$ and $c \in C^{\mathrm{sg}}_n(X;\mathbb{R})$. This descends to a pairing

$$\langle -,-\rangle : H^*_b(X;\mathbb{R}) \times H_*(X;\mathbb{R}) \to \mathbb{R},$$

which is called the *Kronecker product*. This pairing leads to a duality statement between bounded cohomology and singular homology:

Proposition III.2.2 (ℓ^1–ℓ^∞ duality in bounded cohomology [36, Lemma 6.1])**.** *Let X be a topological space and $\alpha \in H_n(X;\mathbb{R})$. Then the ℓ^1-seminorm of α satisfies*

$$\|\alpha\|_1 = \sup\left\{ \frac{\langle u, \alpha\rangle}{\|u\|_\infty} \;\middle|\; u \in H_b^n(X;\mathbb{R}),\ \|u\|_\infty > 0 \right\}.$$

III.2.b Bounded cohomology of groups

It follows from Gromov's Mapping Theorem [44, §3.1] (see also [36, Corollary 5.11]) that, for any continuous map $f : X \to Y$ between path-connected spaces inducing an isomorphism on fundamental groups, the induced map $f^* : H_b^*(Y;\mathbb{R}) \to H_b^*(X;\mathbb{R})$ is an isometric isomorphism (*isometric* means that it preserves the ℓ^∞-seminorm).

Hence, given a group G, one can define the *bounded cohomology* of G to be the bounded cohomology of any path-connected space X with $\pi_1 X \cong G$:

$$H_b^*(G;\mathbb{R}) := H_b^*(X;\mathbb{R}).$$

Such a space X always exists — for instance, one can take X to be a (potentially infinite) presentation complex of G. Since there are isometric isomorphisms $H_b^*(X;\mathbb{R}) \cong H_b^*(X';\mathbb{R})$ for any two choices of X, X' as above, there is a well-defined ℓ^∞-seminorm on $H_b^*(G;\mathbb{R})$.

But the bounded cohomology of a group can be given a more algebraic interpretation as follows. The *bar cochain complex* of G with real coefficients is defined by

$$C^n(G;\mathbb{R}) := \mathbb{R}^{G^n},$$

where $\mathbb{R}^{G^n}$ is the space of all maps $G^n \to \mathbb{R}$. This defines a cochain complex, with coboundary maps $d^n : C^{n-1}(G;\mathbb{R}) \to C^n(G;\mathbb{R})$ given by

$$\begin{aligned} d^n\psi(g_1,\dots,g_n) := {}& \psi(g_2,\dots,g_n) - \psi(g_1g_2, g_3,\dots,g_n) + \psi(g_1, g_2g_3,\dots,g_n) \\ & - \cdots + (-1)^{n-1}\psi(g_1,\dots,g_{n-2}, g_{n-1}g_n) + (-1)^n\psi(g_1,\dots,g_{n-1}). \end{aligned}$$

This is the dual of the chain complex $C_*(G;\mathbb{R})$ introduced in §I.1.b.

Given a cochain $\psi \in C^n(G;\mathbb{R})$, its ℓ^∞-norm is

$$\|\psi\|_\infty := \sup_{(g_1,\dots,g_n)\in G^n} |\psi(g_1,\dots,g_n)| \in [0,+\infty].$$

The *bounded cochain complex* of G is

$$C_b^*(G;\mathbb{R}) := \{\psi \in C^n(G;\mathbb{R}) \mid \|\psi\|_\infty < \infty\}.$$

As before, boundedness is preserved by the coboundary maps d^n, so the latter define a structure of cochain complex on $C_b^*(G;\mathbb{R})$. It turns out that the bounded cohomology of G can be defined as the cohomology of $C_b^*(G;\mathbb{R})$, and we now explain — in the aspherical case — how to write an explicit isomorphism between this cohomology and the bounded cohomology of a space X with fundamental group G as defined in §III.2.a.

Let X be a $K(G,1)$ space with fixed basepoint ω. Each element g of G can be represented by a loop $\gamma_g : S^1 \to X$ based at ω, which can also be described as a map $\sigma_g : \Delta^1 \to X$, where Δ^1 is a 1-simplex (i.e. a segment), and σ_g maps both endpoints of Δ^1 to ω. For all $g_1, g_2 \in G$, the concatenation $\sigma_{g_1} \cdot \sigma_{g_2}$ is homotopic (with fixed endpoints) to $\sigma_{g_1 g_2}$, and one can construct a map $\sigma_{g_1,g_2} : \Delta^2 \to X$ (where Δ^2 is a 2-simplex) such that the restrictions of σ_{g_1,g_2} to its three faces are σ_{g_2}, $\sigma_{g_1 g_2}^{-1}$, and σ_{g_1} (where σ^{-1} is the singular simplex σ with reversed orientation), as in Figure III.1. Since X is aspher-

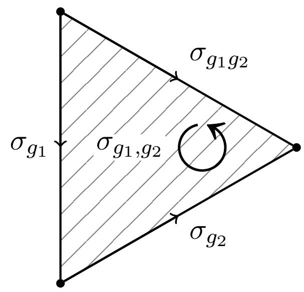


Figure III.1: Construction of the chain map $h_* : C_*(G;\mathbb{R}) \to C_*^{\mathrm{sg}}(X;\mathbb{R})$.

ical, we can iterate this construction and choose, for each n-tuple $(g_1,\ldots,g_n) \in G^n$, a singular simplex $\sigma_{g_1,\ldots,g_n}$ whose restriction to its i-th face is $\sigma^{\varepsilon_i}_{g_1,\ldots,g_i g_{i+1},\ldots,g_n}$ (respectively $\sigma^{\varepsilon_0}_{g_2,\ldots,g_n}$ for $i = 0$ and $\sigma^{\varepsilon_n}_{g_1,\ldots,g_{n-1}}$ for $i = n$), with $\varepsilon_i = (-1)^i$. In this expression, given a singular simplex $\sigma : \Delta^n \to X$, we set $\sigma^{-1} = \sigma \circ \tau_n$, where τ_n is a fixed choice of orientation-reversing homeomorphism $\Delta^n \to \Delta^n$ (for example a reflection) — hence, σ^{-1} is the singular simplex σ with reversed orientation.

The map $h : (g_1,\ldots,g_n) \mapsto \sigma_{g_1,\ldots,g_n}$ induces a chain homotopy equivalence $h_* : C_*(G;\mathbb{R}) \xrightarrow{\sim} C_*^{\mathrm{sg}}(X;\mathbb{R})$ (see [10, §I.4]) and therefore a cochain homotopy equivalence

$$h^* : C^*_{\mathrm{sg}}(X;\mathbb{R}) \xrightarrow{\sim} C^*(G;\mathbb{R}),$$

which induces an isomorphism

$$H^*\left(C^*\left(G;\mathbb{R}\right)\right) \cong H^*\left(X;\mathbb{R}\right).$$

The image under h^* of the bounded cochain complex $C_b^*\left(X;\mathbb{R}\right)$ is $C_b^*\left(G;\mathbb{R}\right)$. In fact, h also induces a cochain homotopy equivalence $h^* : C_b^*\left(X;\mathbb{R}\right) \xrightarrow{\sim} C_b^*\left(G;\mathbb{R}\right)$, and an isometric isomorphism [36, Theorem 5.5]

$$H^*\left(C_b^*\left(G;\mathbb{R}\right)\right) \cong H_b^*\left(X;\mathbb{R}\right).$$

We will denote this cohomology by $H_b^*\left(G;\mathbb{R}\right)$ and interpret it using both points of view.

Remark III.2.3. There is a connection between quasimorphisms and bounded cohomology: a quasimorphism $\phi : G \to \mathbb{R}$ can be seen as an element of $C^1\left(G;\mathbb{R}\right)$, and its coboundary $d^2\phi$ is given by

$$d^2\phi\left(g,h\right) = \phi(g) - \phi(gh) + \phi(h).$$

Hence, the quasimorphism condition means precisely that $d^2\phi$ is a *bounded* cochain, and in fact a bounded cocycle. Therefore, it defines a class $\left[d^2\phi\right] \in H_b^2\left(G;\mathbb{R}\right)$. This gives a morphism $\left[d^2-\right] : Q(G) \to H_b^2\left(G;\mathbb{R}\right)$ whose kernel is the subspace $\operatorname{Hom}\left(G,\mathbb{R}\right)$ of $Q(G)$ consisting of homomorphisms to $\mathbb{R}$. In fact, this extends to an exact sequence [16, Theorem 2.50]

$$0 \to \operatorname{Hom}\left(G,\mathbb{R}\right) \to Q(G) \xrightarrow{\left[d^2-\right]} H_b^2\left(G;\mathbb{R}\right) \to H^2\left(G;\mathbb{R}\right),$$

where $H_b^2\left(G;\mathbb{R}\right) \to H^2\left(G;\mathbb{R}\right)$ is the map induced by the inclusion of cochain complexes $C_b^*\left(G;\mathbb{R}\right) \hookrightarrow C^*\left(G;\mathbb{R}\right)$.

III.2.c Bavard duality for the relative Gromov seminorm

Our aim is now to use bounded cohomology in order to obtain a statement analogous to Bavard duality (Theorem III.1.1) for the relative Gromov seminorm on $H_2\left(X,\gamma\right)$, where $\gamma : \coprod S^1 \to X$ is a finite family of loops in a path-connected topological space X.

There is a notion of *relative bounded cohomology*, giving rise to a long exact sequence as in the case of singular cohomology — we refer the reader to Frigerio's book [36, §5.7] for a definition. The duality principle [36, Lemma 6.1] then implies that $H_b^2\left(X,\gamma\right) := H_b^2\left(X_\gamma, \coprod S^1\right)$ (with the ℓ^∞-seminorm) is the dual of $H_2\left(X,\gamma\right)$ (with the

ℓ^1-seminorm). But since $\pi_1 S^1 = \mathbb{Z}$ is amenable (and our family of loops is finite), the bounded cohomology $H_b^*\left(\coprod S^1\right)$ vanishes [36, Theorem 3.6] and the long exact sequence of $\left(X_\gamma, \coprod S^1\right)$ shows that the inclusion induces an isomorphism

$$H_b^2\left(X, \gamma; \mathbb{R}\right) \cong H_b^2\left(X; \mathbb{R}\right).$$

This isomorphism, with the Kronecker product $H_b^2\left(X, \gamma\right) \times H_2\left(X, \gamma\right) \to \mathbb{R}$ (which is the relative analogue of the absolute Kronecker product introduced in §III.2.a) defines a pairing

$$\langle \cdot, \cdot \rangle : H_b^2\left(X; \mathbb{R}\right) \times H_2\left(X, \gamma; \mathbb{R}\right) \to \mathbb{R}.$$

It turns out that the ℓ^∞-seminorm (which is a norm in degree 2 by Theorem III.2.1) is dual to the ℓ^1-seminorm under this pairing:

Theorem III.2.4 (Bavard duality for the relative Gromov seminorm)**.** *Let X be a topological space and $\gamma : \coprod S^1 \to X$. Given a real class $\alpha \in H_2(X, \gamma; \mathbb{R})$, the relative Gromov seminorm of α is given by*

$$\|\alpha\|_1 = \sup \left\{ \frac{\langle u, \alpha \rangle}{\|u\|_\infty} \;\middle|\; u \in H_b^2\left(X; \mathbb{R}\right) \smallsetminus \{0\} \right\}.$$

Proof. Duality between ℓ^1-homology and bounded cohomology [36, Lemma 6.1] (applied to the cochain complex $C_b^*\left(X, \gamma; \mathbb{R}\right)$ defining $H_b^*\left(X, \gamma; \mathbb{R}\right)$ equipped with the ℓ^∞-norm, which is the dual of the singular chain complex $C_*\left(X, \gamma; \mathbb{R}\right)$ with the ℓ^1-norm) implies that

$$\|\alpha\|_1 = \sup \left\{ \frac{\langle u, \alpha \rangle}{\|u\|_\infty} \;\middle|\; u \in H_b^2\left(X, \gamma; \mathbb{R}\right),\ \|u\|_\infty \neq 0 \right\}.$$

Moreover, a result proved independently by Bucher et al. [11, Theorem 1.2] and by Kim and Kuessner [53, Theorem 1.2] implies that the isomorphism $H_b^2\left(X, \gamma\right) \cong H_b^2\left(X\right)$ coming from the long exact sequence of the pair $\left(X, \gamma\right)$ as explained above is isometric for the ℓ^∞-norm. Together with the fact that $\|u\|_\infty = 0$ only if $u = 0$ in $H_b^2(X)$ (Theorem III.2.1), this implies the result. ☐

We'll say that a class $u \in H_b^2\left(X; \mathbb{R}\right)$ is *extremal* for $\alpha \in H_2\left(X, \gamma; \mathbb{R}\right)$ if it realises the supremum in Theorem III.2.4. Note that extremal classes exist for all $\alpha \in H_2\left(X, \gamma; \mathbb{R}\right)$ by the Hahn–Banach Theorem (see [36, Lemma 6.1]).

III.3 An algebraic interpretation à la Hopf

We now prove a relative version of the Hopf formula, and explain how this can be used to provide a purely algebraic interpretation of Theorem III.2.4. We focus on the special case of the homology of a group relative to the conjugacy class of an element (rather than that of a chain). An analogous Hopf formula could be given in the general case, but the notation would become cumbersome.

III.3.a A relative Hopf formula

Recall that the classical Hopf formula computes $H_2(G)$ when G is a group given by a presentation (see [10, Theorem II.5.3]):

Theorem III.3.1 (Hopf formula [50])**.** *Let F be a free group, $R \trianglelefteq F$, and $G = F/R$. Then there is an isomorphism*

$$H_2(G;\mathbb{Z}) \cong (R \cap [F,F])/[F,R].$$

The isomorphism of Theorem III.3.1 will be made explicit by our proof of Theorem III.3.2 below — see Remark III.3.3(iii).

With the same setup as in Theorem III.3.1, our goal is to compute $H_2(G,[w];\mathbb{Z})$, where $[w] \in C_1^{\mathrm{conj}}(G;\mathbb{Z})$ is an integral conjugacy class represented by an element $w \in G$ (see Remark I.1.8(i) for the relation between conjugacy classes of chains and of elements). This is provided by the following theorem; our proof is topological and inspired by [16, §1.1.6] and [63].

Theorem III.3.2 (Relative Hopf formula)**.** *Let F be a free group, $R \trianglelefteq F$, and $G = F/R$. Let w be an infinite-order element of G, and let $\bar{w} \in F$ be a preimage of w under $F \xrightarrow{p} F/R$. Then there is an isomorphism*

$$H_2(G,[w];\mathbb{Z}) \cong (\langle \bar{w} \rangle R \cap [F,F])/[F,R].$$

In the above equality, $\langle \bar{w} \rangle R$ is the subset of the group F consisting of elements of the form $\bar{w}^n r$, for some $n \in \mathbb{Z}$ and $r \in R$.

Proof. Let X be a $K(G,1)$ with a fixed basepoint x_0 and let $\gamma : S^1 \to X$ be a loop based at x_0 representing w. Then $H_2(G,[w]) = H_2(X,\gamma)$ (see Definition I.2.4), and we construct a morphism

$$\Phi : \langle \bar{w} \rangle R \cap [F,F] \to H_2(X,\gamma;\mathbb{Z})$$

as follows. Let $\bar{g} \in \langle \bar{w} \rangle R \cap [F, F]$. Since $\bar{g} \in [F, F]$, one can write

$$\bar{g} = \left[\bar{a}_1, \bar{b}_1\right] \cdots \left[\bar{a}_k, \bar{b}_k\right],$$

with $\bar{a}_i, \bar{b}_i \in F$. Set $a_i = p\left(\bar{a}_i\right) \in G$, $b_i = p\left(\bar{b}_i\right) \in G$ and $g = p\left(\bar{g}\right) \in G$. The assumption that $\bar{g} \in \langle \bar{w} \rangle R$ in F means that $g \in \langle w \rangle$ in G, so one can write $g = w^n$ for some $n \in \mathbb{Z}$. Moreover, since w has infinite order, the integer n is uniquely determined by $\bar{g}$. Let $\Sigma_{k,1}$ be an oriented genus-k surface with one boundary component. The surface $\Sigma_{k,1}$ has a cell structure with one 0-cell $\bullet$, $(2k+1)$ 1-cells with labels $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k, \delta$, and one 2-cell glued along the word $\delta^{-1}\left[\alpha_1, \beta_1\right] \cdots \left[\alpha_k, \beta_k\right]$ — see Figure III.2.

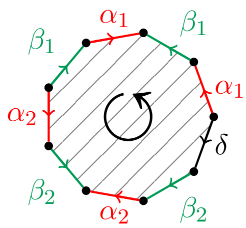


Figure III.2: The cell structure on $\Sigma_{k,1}$ (with $k = 2$).

First pick a degree-n map $\partial f : \partial\Sigma_{k,1} \to S^1$. Then define a map $f^{(1)} : \Sigma_{k,1}^{(1)} \to X$ on the 1-skeleton of $\Sigma_{k,1}$ by sending $\bullet$ to the basepoint x_0 of X, each 1-cell α_i to a loop representing a_i in $\pi_1\left(X, x_0\right) \cong G$, each β_i to a loop representing b_i, and define $f^{(1)}$ on δ by $f^{(1)}_{|\delta} = \gamma \circ \partial f$; in particular, δ is mapped to a loop representing $g = w^n$. Since $g = \left[a_1, b_1\right] \cdots \left[a_k, b_k\right]$ in $G \cong \pi_1\left(X, x_0\right)$, the map $f^{(1)} : \Sigma_{k,1}^{(1)} \to X$ can be extended over the 2-cell of $\Sigma_{k,1}$ to $f : \Sigma_{k,1} \to X$. Now the data of f and ∂f define an admissible surface

$$f : \left(\Sigma_{k,1}, \partial\Sigma_{k,1}\right) \to (X, \gamma).$$

Note moreover that the homotopy types of f and ∂f are uniquely determined by the choice of an expression $\left[\bar{a}_1, \bar{b}_1\right] \cdots \left[\bar{a}_k, \bar{b}_k\right]$; this would fail if w had torsion because in that case, the integer n is not unique — see Remark III.3.3(ii) below.

Now we define $\Phi(\bar{g})$ by

$$\Phi(\bar{g}) := f_*\left[\Sigma_{k,1}\right] \in H_2\left(X, \gamma; \mathbb{Z}\right),$$

where $\left[\Sigma_{k,1}\right] \in H_2\left(\Sigma_{k,1}, \partial\Sigma_{k,1}; \mathbb{Z}\right)$ is the integral fundamental class of $\Sigma_{k,1}$ (note that $\Sigma_{k,1}$ was chosen with an orientation).

The construction of $\Phi\left(\bar{g}\right)$ explained above depends *a priori* on the choice of an expression $\bar{g} = \left[\bar{a}_1, \bar{b}_1\right] \cdots \left[\bar{a}_k, \bar{b}_k\right]$, which might not be unique. For now, we see Φ as a map defined on the monoid Θ of all formal expressions $[\bar{a}_1, \bar{b}_1] \cdots [\bar{a}_k, \bar{b}_k]$ whose image in F lie in $\langle \bar{w} \rangle R$, and we'll show that this induces a well-defined map on $\langle \bar{w} \rangle R \cap [F, F]$.

Claim. The map $\Phi : \Theta \to H_2(X, \gamma; \mathbb{Z})$ is a monoid homomorphism.

Proof of the claim. Consider two formal expressions $\theta = [\bar{a}_1, \bar{b}_1] \cdots [\bar{a}_k, \bar{b}_k]$ and $\theta' = [\bar{a}'_1, \bar{b}'_1] \cdots [\bar{a}'_\ell, \bar{b}'_\ell]$ in Θ. As explained above, this gives rise to admissible surfaces $f : (\Sigma_{k,1}, \partial\Sigma_{k,1}) \to (X, \gamma)$ and $f' : (\Sigma_{\ell,1}, \partial\Sigma_{\ell,1}) \to (X, \gamma)$, and we have $\Phi(\theta) = f_*[\Sigma_{k,1}]$ and $\Phi(\theta') = f'_*[\Sigma_{\ell,1}]$. Consider the wedge sum

$$\Sigma_\vee := \Sigma_{k,1} \vee \Sigma_{\ell,1}.$$

The maps f and f' naturally induce $f_\vee : \Sigma_\vee \to X$, and the fundamental classes of $\Sigma_{k,1}$ and $\Sigma_{\ell,1}$ sum to a class $[\Sigma_\vee] \in H_2(\Sigma_\vee, \partial\Sigma_\vee; \mathbb{Z})$, where we define $\partial\Sigma_\vee = \partial\Sigma_{k,1} \vee \partial\Sigma_{\ell,1} \subseteq \Sigma_\vee$. Hence,

$$\Phi(\theta_1) + \Phi(\theta_2) = (f_\vee)_*[\Sigma_\vee].$$

Now there is a homotopy equivalence $(\Sigma_\vee, \partial\Sigma_\vee) \simeq (\Sigma_{k+\ell,1}, \partial\Sigma_{k+\ell,1})$, as illustrated in Figure III.3. This yields an admissible surface $(\Sigma_{k+\ell,1}, \partial\Sigma_{k+\ell,1}) \to (X, \gamma)$ representing

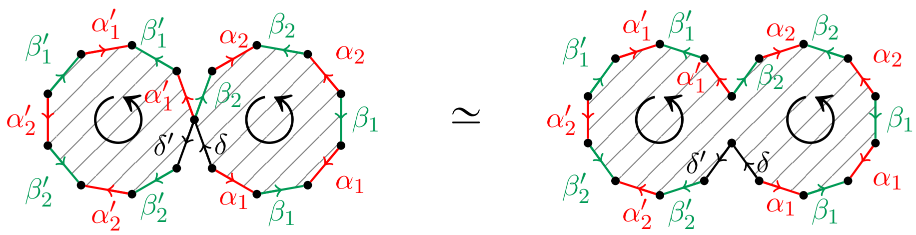


Figure III.3: The homotopy equivalence $\Sigma_{k,1} \vee \Sigma_{\ell,1} \simeq \Sigma_{k+\ell,1}$ (here $k = \ell = 2$).

the class $\Phi(\theta) + \Phi(\theta')$ in $H_2(X, \gamma; \mathbb{Z})$. But note that this admissible surface is exactly the one obtained when the above construction is applied to $\theta\theta'$. This proves that $\Phi(\theta) + \Phi(\theta') = \Phi(\theta\theta')$, so Φ is a monoid homomorphism. $\square$

Using the claim, we now prove that Φ induces a well-defined map on $\langle \bar{w} \rangle R \cap [F, F]$. Consider two formal expressions $\theta, \theta' \in \Theta$ defining the same element of $\langle \bar{w} \rangle R \cap [F, F]$. Write $\theta = [\bar{a}_1, \bar{b}_1] \cdots [\bar{a}_k, \bar{b}_k]$, and consider its formal inverse $\theta^{-1} = [\bar{b}_k, \bar{a}_k] \cdots [\bar{b}_1, \bar{a}_1] \in \Theta$ (which, despite our choice of notation, is not an inverse of θ in the monoid Θ!). Then the formal expression $\theta^{-1}\theta'$ represents the trivial element of F. This means that the above construction for the formal expression $\theta^{-1}\theta'$ can actually be performed when the $K(G, 1)$ space X is replaced with a $K(F, 1)$ space X_F. In other words, the admissible surface $f : (\Sigma_{m,1}, \partial\Sigma_{m,1}) \to (X, \gamma)$ associated to $\theta^{-1}\theta'$ factors through the map $X_F \to X$ induced by $F \to G$. Moreover, the image of $\partial\Sigma_{m,1}$ is nullhomotopic in X_F, from which it follows that

$$f_* [\Sigma_{m,1}] \in H_2 (X_F; \mathbb{Z}) \hookrightarrow H_2 (X_F, \bar{\gamma}; \mathbb{Z}),$$

where $\bar{\gamma} : S^1 \to X_F$ is a representative of $\bar{w} \in F$. But $H_2 (X_F; \mathbb{Z}) = H_2 (F; \mathbb{Z}) = 0$ since F is a free group, so $[\Sigma_{m,1}]$ maps to zero in $H_2 (X_F, \bar{\gamma}; \mathbb{Z})$, and hence also in $H_2 (X, \gamma; \mathbb{Z})$. Therefore, it follows from the claim that

$$0 = \Phi \left(\theta^{-1}\theta'\right) = \Phi \left(\theta^{-1}\right) + \Phi \left(\theta'\right),$$

and it is clear from the construction that $\Phi \left(\theta^{-1}\right) = -\Phi (\theta)$, so $\Phi (\theta) = \Phi (\theta')$ as wanted. This proves that Φ induces a well-defined map

$$\Phi : \langle \bar{w} \rangle R \cap [F, F] \to H_2 (X, \gamma; \mathbb{Z}),$$

which is a group homomorphism by the claim.

The homomorphism Φ is surjective since every element of $H_2 (X, \gamma; \mathbb{Z})$ can be represented by a map from an orientable compact connected surface with one boundary component — this follows from Lemma II.1.2(ii).

It remains to show the following:

Claim. $\operatorname{Ker} \Phi = [F, R]$.

Proof of the claim. To prove that $[F, R] \subseteq \operatorname{Ker} \Phi$, it suffices to show that for every $\bar{g} \in F$ and $\bar{r} \in R$, we have $[\bar{g}, \bar{r}] \in \operatorname{Ker} \Phi$. But $\Phi ([\bar{g}, \bar{r}]) = f_* [\Sigma_{1,1}]$, where $\Sigma_{1,1}$ is a torus with one boundary component, with equator mapping to $\bar{g}$ and meridian mapping to $\bar{r}$. Since the image of $\bar{r}$ in G is trivial, we may cut $\Sigma_{1,1}$ along the meridian and fill in the two resulting discs, obtaining a map $f_1 : \left(D^2, \partial D^2\right) \to (X, \gamma)$. We can glue f_1 to itself with reversed orientation along ∂D^2 to obtain $f_2 : S^2 \to X$. But X is assumed to be a $K(G, 1)$ so it is aspherical, and f_2 is nullhomotopic. Therefore, the map $f : \Sigma_{1,1} \to X$ is nullhomotopic. Note also that the word $[\bar{g}, \bar{r}]$ lies in $[F, R] \subseteq R$, so it represents the trivial element of G; hence, with the notations of the above construction, we have $n = 0$, and the map $\partial f : \partial\Sigma_{1,1} \to S^1$ is also nullhomotopic. It follows that $f : (\Sigma_{1,1}, \partial\Sigma_{1,1}) \to (X, \gamma)$ is nullhomotopic as a map of pairs, and $f_* [\Sigma_{1,1}] = 0$. This proves that $\Phi ([\bar{g}, \bar{r}]) = 0$, so $[F, R] \subseteq \operatorname{Ker} \Phi$.

Conversely, let $\bar{g} \in \operatorname{Ker} \Phi$. Let $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$ be an admissible surface associated to an expression of $\bar{g}$ as a product of commutators by the above construction, with $\Sigma = \Sigma_{k,1}$. The assumption that $\bar{g} \in \operatorname{Ker}\Phi$ means that $f_*[\Sigma] = 0$, so the map $f_* : H_2(\Sigma, \partial\Sigma; \mathbb{Z}) \to H_2(X, \gamma; \mathbb{Z})$ is zero. Long exact sequences of pairs give a commutative diagram with exact rows (with omitted $\mathbb{Z}$-coefficients):

$$\begin{tikzcd}
 & 0 \arrow[r] & H_2(\Sigma, \partial\Sigma) \arrow[r, "\partial"] \arrow[d, "0"] & H_1(\partial\Sigma) \arrow[r] \arrow[d, "f_*"] & H_1(\Sigma) \arrow[r] \arrow[d, "f_*"] & \cdots \\
0 \arrow[r] & H_2(X) \arrow[r] & H_2(X, \gamma) \arrow[r, "\partial"] & H_1(S_1) \arrow[r] & H_1(X) \arrow[r] & \cdots
\end{tikzcd}$$

If $f_* : H_1(\partial\Sigma) \to H_1(S^1)$ were nonzero, then since $H_1(\partial\Sigma) \cong H_1(S^1) \cong \mathbb{Z}$, the map $f_* : H_1(\partial\Sigma) \to H_1(S^1)$ would in fact be injective. But $f_* \circ \partial = 0$, so the map $\partial : H_2(\Sigma, \partial\Sigma) \to H_1(\partial\Sigma)$ would be zero, implying by exactness that $H_1(\partial\Sigma) = 0$ since $H_1(\partial\Sigma) \to H_1(\Sigma)$ is zero. This is a contradiction, and therefore the map

$$f_* : H_1(\partial\Sigma) \to H_1(S^1)$$

is zero. Therefore, the restriction of f to $\partial\Sigma$ is nullhomotopic, which implies in particular that the image of $\bar{g}$ in G is trivial, i.e. $\bar{g} \in R \cap [F, F]$. Therefore, we are reduced to the setting of the classical Hopf formula (Theorem III.3.1), i.e. $\bar{g} \in R \cap [F, F]$ and $\Phi(\bar{g}) = 1$ in $H_2(X; \mathbb{Z})$. Since Φ coincides with the morphism giving the classical Hopf formula (see for instance [63]), it follows that $\bar{g} \in [F, R]$. □

We have constructed a surjective group homomorphism

$$\Phi : \langle \bar{w} \rangle R \cap [F, F] \to H_2(X, \gamma; \mathbb{Z})$$

with $\operatorname{Ker}\Phi = [F, R]$, so Φ induces the desired isomorphism. □

Remark III.3.3. (i) In the proof of Theorem III.3.2, the assumption that the space X is a $K(G, 1)$ is essential. This is why we state the theorem in terms of the relative homology of groups, rather than topological spaces.

(ii) Theorem III.3.2 becomes false if w has finite order q in G. Indeed, the (usual) Hopf formula (Theorem III.3.1) says that the absolute homology $H_2(G; \mathbb{Z})$ is isomorphic to $(R \cap [F, F]) / [F, R]$, which has finite index in the right-hand side $(\langle \bar{w} \rangle R \cap [F, F]) / [F, R]$ of Theorem III.3.2 when w has finite order. But we know from Example I.2.3(iv) that

$$H_2(G, [w]; \mathbb{Z}) \cong H_2(G; \mathbb{Z}) \oplus \mathbb{Z},$$

so $H_2(G;\mathbb{Z})$ must have infinite index in $H_2(G,[w];\mathbb{Z})$!

Let us explain briefly where the missing homology classes are. Pick $\gamma : S^1 \to X$ a loop at $x_0 \in X$ representing w. Then the homology class corresponding to the integer $n \in \mathbb{Z}$ in the $\mathbb{Z}$-summand of $H_2(G,[w];\mathbb{Z})$ is represented by an admissible surface $f : (D^2, \partial D^2) \to (X, \gamma)$, where D^2 is the disc, $\partial f : \partial D^2 \to S^1$ is a map of degree nq, and $f : D^2 \to X$ is an extension of $\gamma \circ \partial f$ to the disc (which exists since γ^{nq} is nullhomotopic).

Note that the underlying maps $f : D^2 \to X$ of the admissible surfaces representing these 'missing classes' are nullhomotopic since the disc is contractible. This underlines the importance of defining an admissible surface as the data of both maps f *and* ∂f.

(iii) One can recover the classical Hopf formula (Theorem III.3.1) from our proof by observing that our isomorphism

$$(\langle \bar{w} \rangle R \cap [F,F]) / [F,R] \xrightarrow{\cong} H_2(G,[w];\mathbb{Z})$$

sends $(R \cap [F,F]) / [F,R]$ to $H_2(G;\mathbb{Z}) \hookrightarrow H_2(G,[w];\mathbb{Z})$. In other words, our construction maps an element $\bar{g} \in \langle \bar{w} \rangle R \cap [F,F]$ to an absolute homology class if and only if $\bar{g}$ has trivial image in G.

III.3.b Bavard duality through the lens of the Hopf formula

We next explain how to obtain an algebraic restatement of Theorem III.2.4 using the relative Hopf formula (Theorem III.3.2).

Recall from §III.2.b the definition of the bounded cochain complex $C_b^*(G;\mathbb{R})$. We denote by $Z_b^2(G;\mathbb{R})$ the subspace of $C_b^2(G;\mathbb{R})$ consisting of *bounded* 2*-cocycles* on G, i.e. bounded maps $\psi : G^2 \to \mathbb{R}$ such that

$$\psi(g_2, g_3) - \psi(g_1 g_2, g_3) + \psi(g_1, g_2 g_3) - \psi(g_1, g_2) = 0$$

for all $g_1, g_2, g_3 \in G$.

Theorem III.3.4 (Bavard duality via the Hopf formula)**.** *Let F be a free group, $R \trianglelefteq F$, and $G = F/R$. Let $w \in G$ and let $\bar{w} \in F$ be a preimage of w under $F \xrightarrow{p} G$.*

Let $\alpha \in H_2(G,[w];\mathbb{Z})$ and let

$$[\bar{a}_1, \bar{b}_1] \cdots [\bar{a}_k, \bar{b}_k] \in \langle \bar{w} \rangle R \cap [F,F],$$

be a representative of $\Psi(\alpha)$*, where* $\Psi : H_2(G,[w];\mathbb{Z}) \xrightarrow{\cong} (\langle \bar{w}\rangle R \cap [F,F])/[F,R]$ *is the isomorphism of Theorem III.3.2. Set* $a_i = p(\bar{a}_i) \in G$ *and* $b_i = p(\bar{b}_i) \in G$*.*

Then

$$\begin{aligned}\|\iota\alpha\|_1 = \sup\Bigg\{ \frac{1}{\|\psi\|_\infty} \Big(\psi(a_1,b_1) + \psi\left(a_1b_1,a_1^{-1}\right) + \psi\left(a_1b_1a_1^{-1},b_1^{-1}\right)\\ + \psi([a_1,b_1],a_2) + \psi([a_1,b_1]a_2,b_2) + \psi\left([a_1,b_1]a_2b_2,a_2^{-1}\right) + \cdots\\ + \psi\left([a_1,b_1]\cdots[a_{k-1},b_{k-1}]a_kb_ka_k^{-1},b_k^{-1}\right)\Big) \,\Bigg|\, \psi \in Z_b^2(G;\mathbb{R}) \smallsetminus \{0\}\Bigg\},\end{aligned}$$

where $\iota : H_2(G,[w];\mathbb{Z}) \to H_2(G,[w];\mathbb{R})$ *is the change-of-coefficients map.*

Proof. Let X be a $K(G,1)$ and let $\gamma : S^1 \to X$ represent w. Recall that the isomorphism $\Psi : H_2(G,[w];\mathbb{Z}) \xrightarrow{\cong} \langle \bar{w}\rangle R \cap [F,F]/[F,R]$ was constructed in the proof of Theorem III.3.2 by starting with a product of k commutators in $\langle \bar{w}\rangle R \cap [F,F]$, labelling the edges in a cellular decomposition of the compact surface $\Sigma_{k,1}$ with those commutators, mapping $\Sigma_{k,1}$ to X and considering the image of the fundamental class $[\Sigma_{k,1}]$ in $H_2(X,\gamma) = H_2(G,[w])$. We will now be a bit more specific about the choice of the map $\Sigma_{k,1} \to X$. As X is a $K(G,1)$, we start by picking singular simplices $\sigma_{g_1,\dots,g_n} : \Delta^n \to X$ for each n-uple $(g_1,\dots,g_n) \in G^n$ as in §III.2.b (see in particular Figure III.1), so that the map $(g_1,\dots,g_n) \mapsto \sigma_{g_1,\dots,g_n}$ induces a chain homotopy equivalence $C_*(G;\mathbb{R}) \xrightarrow{\sim} C_*^{\mathrm{sg}}(X;\mathbb{R})$. Take a one-vertex triangulation of $\Sigma_{k,1}$ as in Figure III.4. We can construct the map $f : \Sigma_{k,1} \to X$ explicitly by sending each triangle of

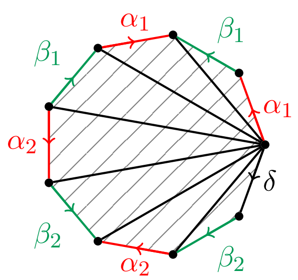


Figure III.4: One-vertex triangulation of $\Sigma_{k,1}$.

$\Sigma_{k,1}$ to the correct singular 2-simplex among the σ_{g_1,g_2}'s. We obtain in particular that

$$\begin{aligned}\alpha = f_*[\Sigma_{k,1}] = \Big[\sigma_{a_1,b_1} + \sigma_{a_1b_1,a_1^{-1}} + \sigma_{a_1b_1a_1^{-1},b_1^{-1}} + \sigma_{[a_1,b_1],a_2}\\ + \sigma_{[a_1,b_1]a_2,b_2} + \cdots + \sigma_{[a_1,b_1]\cdots[a_{k-1},b_{k-1}]a_kb_ka_k^{-1},b_k^{-1}}\Big] \in H_2(X,\gamma;\mathbb{Z}). \qquad (*)\end{aligned}$$

Now Bavard duality for $H_2(X,\gamma)$ (Theorem III.2.4) gives

$$\|\iota\alpha\|_1 = \sup\left\{ \frac{\langle u,\alpha\rangle}{\|u\|_\infty} \,\middle|\, u \in H_b^2(X;\mathbb{R}) \smallsetminus \{0\} \right\}.$$

Pick some $u \in H_b^2(X;\mathbb{R}) \cong H_b^2(G;\mathbb{R})$ and let $\psi \in Z_b^2(G;\mathbb{R})$ be a 2-cocycle such that $u = [\psi]$. By definition, we have $\langle \psi, (g_1, g_2)\rangle = \psi(g_1, g_2)$ for $g_1, g_2 \in G$. Since X is a $K(G,1)$, the isomorphism $H_b^2(X;\mathbb{R}) \cong H_b^2(G;\mathbb{R})$ is induced by the chain homotopy equivalence $(g_1, \ldots, g_n) \mapsto \sigma_{g_1,\ldots,g_n}$ constructed in §III.2.b, which implies together with $(*)$ that the Kronecker product $\langle u, \alpha\rangle$ is given by

$$\begin{aligned}\langle u,\alpha\rangle = {}& \psi(a_1,b_1) + \psi\left(a_1b_1, a_1^{-1}\right) + \psi\left(a_1b_1a_1^{-1}, b_1^{-1}\right) + \psi\left([a_1,b_1], a_2\right) \\ & + \psi\left([a_1,b_1]a_2, b_2\right) + \cdots + \psi\left([a_1,b_1]\cdots[a_{k-1},b_{k-1}]\,a_kb_ka_k^{-1}, b_k^{-1}\right).\end{aligned}$$

The result follows, remembering that $\|u\|_\infty = \inf\{\|\psi\|_\infty \mid [\psi] = u\}$. □

Remark III.3.5. Given a quasimorphism $\phi \in Q(G)$, the coboundary $d^2\phi$ is a bounded 2-cocycle as explained in Remark III.2.3. For $\alpha \in H_2(G,[w];\mathbb{Z})$ with $\partial\alpha = n\left[S^1\right]$, one can use the formula of Theorem III.3.4 to obtain

$$\left\langle \left[d^2\phi\right], \alpha \right\rangle = n \cdot \phi(w).$$

Using the lower bound on $\|\cdot\|_1$ given by Theorem III.3.4 together with the connection between scl and $\|\cdot\|_1$ (Proposition I.2.12), it follows that

$$\mathrm{scl}([w]) \geqslant \frac{1}{2} \sup_{\phi\in Q(G)} \frac{\phi(w)}{2\left\|d^2\phi\right\|_\infty}.$$

On the other hand, classical Bavard duality (Theorem III.1.1) says that

$$\mathrm{scl}([w]) = \sup_{\phi\in Q(G)} \frac{\phi(w)}{2\left\|d^2\phi\right\|_\infty}.$$

Feeding quasimorphisms into Theorem III.3.4 has yielded a non-optimal lower bound on scl. The reason for this is the difference between a cocycle $\psi \in Z_b^2(G;\mathbb{R})$ and its class $[\psi] \in H_b^2(G;\mathbb{R})$: given $\phi \in Q(G)$, there are inequalities [16, Lemma 2.58]

$$\frac{1}{2}\left\|d^2\phi\right\|_\infty \leqslant \left\|\left[d^2\phi\right]\right\|_\infty \leqslant \left\|d^2\phi\right\|_\infty,$$

and $\left\|\left[d^2\phi\right]\right\|_\infty$ might not be realised by the coboundary of a quasimorphism.

CHAPTER IV

ISOMETRIC EMBEDDINGS

One of the main concerns of this thesis, and of this chapter and the next one in particular, is to compute exact values of stable commutator length and the relative Gromov seminorm. Calegari's Rationality Theorem [17] fully answers this question in the case of free groups. This has been generalised by Chen [24] to certain graphs of groups, encompassing previous results of various authors [18, 22, 27, 65, 68]. However, computations of scl in closed surface groups remain elusive. This chapter aims to improve understanding of scl in closed surface groups by embedding them into free groups in an scl- or ℓ^1-isometric manner. We start with a general discussion of scl-preserving maps, then we introduce some topological notions for 2-complexes that we use in the next part of the chapter to prove that certain inclusions of surfaces are scl- or ℓ^1-isometric. We finish with an application of our Bavard duality theorem for the relative Gromov seminorm to isometric embeddings of vertex groups in graphs of groups. Most of the material of this chapter is based on [58].

IV.1 Isometries for scl and the relative Gromov seminorm

IV.1.a scl-isometric embeddings

Recall from Lemma I.1.19 that every group homomorphism is scl-nonincreasing. In this chapter, we would like to understand when a group homomorphism preserves scl. There are slight variations as to what different authors mean by scl-isometric maps, and we try to clarify the terminology below:

Definition IV.1.1. Let $\varphi : G \to H$ be a group homomorphism.

- We say that φ is scl-*preserving* if for every $[c] \in B_1^{\mathrm{conj}}(G;\mathbb{Z})$, the following equality holds:
$$\mathrm{scl}_G([c]) = \mathrm{scl}_H([\varphi(c)])\,. \tag{$*$}$$
- We say that φ is an scl-*isometric embedding* if it is injective and scl-preserving.
- We say that φ is a *strong* scl-*isometric embedding* if it is injective and ($*$) holds for every $[c] \in C_1^{\mathrm{conj}}(G;\mathbb{Z})$.

It is clear that a strong scl-isometric embedding is also an isometric embedding since $B_1^{\mathrm{conj}}(G;\mathbb{Z}) \subseteq C_1^{\mathrm{conj}}(G;\mathbb{Z})$. The following clarifies the relation between isometries and strong isometries:

Proposition IV.1.2. *Let $\varphi : G \to H$ be an* scl*-isometric embedding. Then φ is a strong* scl*-isometric embedding if and only if the induced map*
$$\varphi_* : H_1(G;\mathbb{Q}) \to H_1(H;\mathbb{Q})$$
is injective.

Proof. Observe first that, as soon as φ is an scl-isometric embedding, every $[c] \in C_1^{\mathrm{conj}}(G;\mathbb{Z})$ for which there is some $n \in \mathbb{N}_{\geqslant 1}$ with $n[c] \in B_1^{\mathrm{conj}}(G;\mathbb{Z})$ satisfies
$$\mathrm{scl}_H([\varphi(c)]) = \frac{1}{n}\,\mathrm{scl}_H([\varphi(nc)]) = \frac{1}{n}\,\mathrm{scl}_G([nc]) = \mathrm{scl}_G([c])\,.$$
(This uses the homogeneity of scl — see Lemma I.1.20.) Moreover, it follows from the definition of group homology that φ_* is injective if and only if every $c \in Z_1(G;\mathbb{Q}) \smallsetminus B_1(G;\mathbb{Q})$ satisfies $\varphi(c) \in Z_1(H;\mathbb{Q}) \smallsetminus B_1(H;\mathbb{Q})$. Since $Z_1(G;\mathbb{Q}) = C_1(G;\mathbb{Q})$, this is equivalent to every homogeneous chain $[c] \in C_1^{\mathrm{conj}}(G;\mathbb{Q}) \smallsetminus B_1^{\mathrm{conj}}(G;\mathbb{Q})$ having image under φ in $C_1^{\mathrm{conj}}(H;\mathbb{Q}) \smallsetminus B_1^{\mathrm{conj}}(H;\mathbb{Q})$.

Now Proposition I.2.13 says that $[c] \in C_1^{\mathrm{conj}}(G;\mathbb{Q}) \smallsetminus B_1^{\mathrm{conj}}(G;\mathbb{Q})$ if and only if $\mathrm{scl}_G([c]) = \infty$; the result follows. □

IV.1.b Overview of previous results

We gather here a few known results about isometries of stable commutator length, focusing on free groups:

Proposition IV.1.3. *Any map $\varphi : G \to H$ admitting a left inverse (i.e. a map $\psi : H \to G$ such that $\psi \circ \varphi = \mathrm{id}_G$) is a strong* scl*-isometric embedding.*

Proof. This follows from monotonicity — see Lemma I.1.19. □

Theorem IV.1.4. (i) (Calegari [18, Corollary 3.16]) *Let F_m and F_n be free groups with respective free bases $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_n)$, with $m \leqslant n$, and let $\varphi : F_m \to F_n$ be given by*

$$\varphi : a_i \mapsto b_i^{k_i},$$

with $k_i \in \mathbb{Z} \smallsetminus \{0\}$. Then φ is a strong scl*-isometric embedding.*

(ii) (Chen [24, Theorem 3.8]) *Let $\mathbb{X}, \mathbb{Y}$ be graphs of groups such that* scl *vanishes on all vertex groups. Assume that we are given an edge-injective morphism $h : X \to Y$ between the underlying graphs, monomorphisms $h_v : X_v \hookrightarrow Y_{h(v)}$ between the vertex groups, and isomorphisms $h_e : X_e \xrightarrow{\cong} Y_{h(e)}$ between the edge groups, that commute with the inclusions of the edge groups in the vertex groups, and such that each map h_v induces a morphism that is injective in homology on the sum of the images of the incident edge groups. Then there is an induced map $\underline{h} : \pi_1\mathbb{X} \to \pi_1\mathbb{Y}$, and $\underline{h}$ is an* scl*-isometric embedding.*

(iii) (Calegari–Walker [20, Theorem 3.16]) *Let F_m and F_n be free groups of respective ranks m and n. Then there is a constant $C > 1$ such that a random homomorphism $\varphi : F_m \to F_n$ of length k is* scl*-preserving with probability $1 - O\left(C^{-k}\right)$.*

IV.1.c ℓ^1-isometric embeddings

We now introduce the analogue property for the relative Gromov seminorm:

Definition IV.1.5. A group homomorphism $\varphi : G \to H$ is an *ℓ^1-isometric embedding* if φ is injective, and for each homogeneous chain $[c] \in C_1^{\mathrm{conj}}(G;\mathbb{Z})$, the induced map

$$\varphi_* : H_2(G, [c]; \mathbb{R}) \to H_2(H, [\varphi(c)]; \mathbb{R})$$

preserves $\|\cdot\|_1$.

To prove that a morphism φ is an scl-isometric embedding, we will usually show that any scl-admissible surface in a certain standard form in the target can be homotoped to one that factors through φ. With this technique, it is easier to prove ℓ^1-isometry, because we can then work with the extra assumption that admissible surfaces represent a relative homology class in the image of the morphism φ. Hence, an ℓ^1-isometric embedding can be thought of as a map that preserves scl only up to fixing a relative homology class. The following clarifies the connection between scl- and ℓ^1-isometric embeddings:

Proposition IV.1.6. *If $\varphi : G \to H$ is an ℓ^1-isometric embedding such that $\varphi_* : H_2(G;\mathbb{Q}) \to H_2(H;\mathbb{Q})$ is surjective, then φ is* scl*-isometric.*

Proof. Pick a $K(G,1)$ space X, a $K(H,1)$ space Y, and a continuous map $h : X \to Y$ such that $h_* = \varphi$. Given a map $\gamma : \coprod S^1 \to X$ representing a homogeneous boundary $[c] \in B_1^{\mathrm{conj}}(G;\mathbb{Z})$, the map $\gamma_* : H_1\left(\coprod S^1\right) \to H_1(X)$ vanishes, and Proposition I.2.2(i) gives a commutative diagram with exact rows (with omitted $\mathbb{Q}$-coefficients):

$$\begin{tikzcd}
0 \arrow[r] & H_2(X) \arrow[r] \arrow[d, "h_*"] & H_2(X,\gamma) \arrow[r, "\partial"] \arrow[d, "h_*"] & H_1\left(\coprod S^1\right) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_2(Y) \arrow[r] & H_2(Y,h\gamma) \arrow[r, "\partial"] & H_1\left(\coprod S_1\right) \arrow[r] & 0
\end{tikzcd}$$

Now the map $H_2(X) \xrightarrow{h_*} H_2(Y)$ is surjective by assumption, so the Five Lemma implies that $H_2(X,\gamma) \xrightarrow{h_*} H_2(Y,h\gamma)$ is surjective. Hence, given $\beta \in H_2(Y,h\circ\gamma;\mathbb{Q})$ with $\partial\beta = \left[\coprod S^1\right]$, there exists $\alpha \in H_2(X,\gamma;\mathbb{Q})$ with $h_*\alpha = \beta$. Since the diagram commutes, we have $\partial\alpha = \partial\beta = \left[\coprod S^1\right]$. Therefore, Proposition I.2.12 gives the estimate

$$\mathrm{scl}_G([c]) = \mathrm{scl}_X(\gamma) \leqslant \frac{1}{4}\|\alpha\|_1 = \frac{1}{4}\|h_*\alpha\|_1 = \frac{1}{4}\|\beta\|_1 .$$

Taking the infimum over β gives $\mathrm{scl}_G([c]) \leqslant \mathrm{scl}_H([\varphi(c)])$. The reverse inequality follows from monotonicity of scl (Lemma I.1.19), so φ is an isometric embedding. $\square$

IV.2 Links and orientability in 2-complexes

This section is a digression into some topological properties of 2-dimensional cell complexes that will be useful in the sequel. In particular, we will need a notion of orientability for 2-complexes that are not surfaces. The right setting to make this work will be 2-complexes with small links. The goal of this section is to introduce those notions.

IV.2.a 2-complexes

We first specify the category of topological spaces we will be working with throughout this chapter.

Following the terminology of [37, Chapter 2], we say that a continuous map $f : X \to Y$ between CW-complexes is *cellular* if, for each $n \in \mathbb{N}_{\geqslant 0}$, $f\left(X^{(n)}\right) \subseteq Y^{(n)}$, where $X^{(n)}$ and $Y^{(n)}$ denote the n-skeleta of X and Y respectively. We say that f is *combinatorial* if it maps each open cell of X homeomorphically onto an open cell of Y.

A 2*-complex* is a 2-dimensional CW-complex X such that the attaching map $S^1_\sigma \to X^{(1)}$ of each 2-cell σ of X is combinatorial for a suitable subdivision of the circle S^1_σ. An edge in this subdivision of S^1_σ is called a *side* of σ, and the *degree* of σ is its number of sides. This follows Gersten's terminology [40].

We will assume that all 2-complexes are locally finite, but we allow non-compact 2-complexes.

Each vertex v in a 2-complex X has a *link* — denoted by $\mathrm{Lk}_X(v)$ — which is the graph with vertices corresponding to half-edges e of X at v, and with an edge between e_1 and e_2 in $\mathrm{Lk}_X(v)$ corresponding to each face of X whose boundary traverses e_1^{-1} and e_2 successively.

A *cellulated surface* is a 2-complex that is also a topological surface, possibly with boundary.

IV.2.b A surface criterion for 2-complexes

To decide whether or not a given 2-complex is a surface, it suffices to examine the topology of links of vertices; this is the content of the following lemma:

Lemma IV.2.1 (Surface criterion)**.** *A* 2*-complex X is a cellulated surface if and only if the link of every vertex in X is a circle or a nondegenerate arc (i.e. an arc that is not reduced to a point). In this case, a vertex v of X lies on the boundary if and only if its link is homeomorphic to an arc.*

Proof. The direct implication ($\Rightarrow$) is clear from the definition of the link $\mathrm{Lk}_X(v)$ as the boundary of a regular neighbourhood of v. For ($\Leftarrow$), the key point is that every vertex v has a neighbourhood homeomorphic to a cone over $\mathrm{Lk}_X(v)$. If $\mathrm{Lk}_X(v)$ is a circle, then a cone over $\mathrm{Lk}_X(v)$ is homeomorphic to a (2-dimensional) disc; if $\mathrm{Lk}_X(v)$ is a nondegenerate arc, then a cone over $\mathrm{Lk}_X(v)$ is homeomorphic to a half-disc. It is also clear that points in the interior of 2-cells have neighbourhoods homeomorphic to $\mathbb{R}^2$. It remains to consider points in the interior of edges. Given an edge e, let v be one of its endpoints; then $\mathrm{Lk}_X(v)$ has a vertex $\hat{e}$ corresponding to e, and by assumption, $\hat{e}$ has one or two neighbours in $\mathrm{Lk}_X(v)$. Since our 2-complexes are assumed to be combinatorial, this means that e is incident to one or two 2-cells of X; in both cases, points in the interior of e have neighbourhoods homeomorphic to $\mathbb{R}^2$ or $\mathbb{R} \times \mathbb{R}_{\geqslant 0}$. $\square$

This motivates the following:

Definition IV.2.2. A 2-complex X has *small links* if one of the following equivalent conditions holds:

(i) The link of every vertex of X is homeomorphic to a circle or a union of arcs (that might be degenerate — i.e. reduced to points).

(ii) Every edge e of X is incident to at most two 2-cells, counted with multiplicity (i.e. a 2-cell is counted as many times as it has sides that are glued to e).

The main motivation for Definition IV.2.2 will be Lemma IV.2.7 which says that the property of having small links descends to subcomplexes. This will boil down to the fact that a subgraph of a circle or a union of arcs is again a circle or a union of arcs, which is why there is an apparent asymmetry in the definition (we do not need to consider unions of circles, because they never arise from taking subgraphs of arcs and circles).

In other words, Lemma IV.2.1 says that a 2-complex X is a surface if and only if it has nondegenerate small connected links.

IV.2.c Orientability of 2-complexes

We will need a notion of orientation for 2-complexes. To define it, we will work with *locally finite homology*, denoted by H_*^{lf}. For a CW-complex X, this is defined as the homology of the chain complex $C_*^{\mathrm{lf,cell}}(X)$, where $C_n^{\mathrm{lf,cell}}(X)$ consists of infinite formal sums of oriented n-cells of X with locally finite support. Note that, if X is compact, then $H_*^{\mathrm{lf}}(X) = H_*(X)$. See [39, Chapter 11] or [54, §5.1.1] for more details on locally finite homology.

Definition IV.2.3. Given a 2-complex X, we define the *boundary* ∂X of X to be the 1-dimensional subcomplex consisting of all the edges of X (and their endpoints) that are incident to a single 2-cell, and along only one side of this 2-cell. In other words, these are the edges e for which each point in the interior of e has a neighbourhood in X that is homeomorphic to a half-disc.

Note that, in a 2-complex, $C_3^{\mathrm{lf,cell}}(X) = 0$, so $H_2^{\mathrm{lf}}(X) = Z_2^{\mathrm{lf,cell}}(X)$. In particular, it makes sense to speak of the *support* of a 2-class: this is just the support of the corresponding 2-cycle.

Definition IV.2.4. Let X be a 2-complex and let A be an abelian group. We say that X is *A-orientable* if there is a class $\beta \in H_2^{\mathrm{lf}}(X, \partial X; A)$ whose support contains every 2-cell of X.

Note that, if $X = S$ is a cellulated surface, then our definition of boundary coincides with the usual one, and $\mathbb{Z}$-orientability of S is equivalent to orientability of S in the usual sense (see for example [67, Corollary 6.7] in the closed case). Our definition of orientation applies to any 2-complex, but in the context of surfaces, it is less intrinsic and flexible than the usual one because it requires one to fix a cellular structure first.

We will use orientability via the following lemma:

Lemma IV.2.5. *Let X be an A-orientable 2-complex. Consider a subcomplex Y of X such that $\partial X \subseteq Y \subseteq X$ and $H_2^{\mathrm{lf}}(X, Y; A) = 0$. Then Y contains every 2-cell of X.*

Proof. The long exact sequence of the triple $(X, Y, \partial X)$ shows that the inclusion induces a surjective morphism

$$H_2^{\mathrm{lf}}(Y, \partial X; A) \twoheadrightarrow H_2^{\mathrm{lf}}(X, \partial X; A).$$

Since X is A-orientable relative to ∂X, there is a class $\beta \in H_2^{\mathrm{lf}}(X, \partial X; A)$ with support containing every 2-cell of X. Let $\beta_0 \in H_2^{\mathrm{lf}}(Y, \partial X; A)$ be a preimage of β. Then the support of β_0 is contained in Y and must contain every 2-cell of X. □

Corollary IV.2.6. *Let S be an orientable closed surface and let $T \subseteq S$ be a subsurface such that $H_2(S, T) = 0$. Then $S = T$.* □

IV.2.d Subcomplex stability

For our purpose, it will be necessary to check that certain properties of 2-complexes are inherited by subcomplexes. We start with the following easy observation:

Lemma IV.2.7 (Subcomplex stability of small links)**.** *If X is a 2-complex with small links, then any subcomplex X_0 of X also has small links.*

Proof. For each vertex $v \in X_0$, there is an embedding $\mathrm{Lk}_{X_0}(v) \hookrightarrow \mathrm{Lk}_X(v)$, and any subgraph of a circle or a union of arcs is again a circle or a union of arcs. □

We also need to check that orientability, as well as vanishing of relative homology, descend to subcomplexes:

Lemma IV.2.8 (Subcomplex stability of orientability)**.** *Let A be an abelian group and let X be an A-orientable 2-complex with small links. Then any subcomplex $X_0 \subseteq X$ is A-orientable.*

Proof. Orientability of X means that there is a relative cellular 2-cycle $p = \sum_\sigma \lambda_\sigma \sigma \in Z_2^{\mathrm{lf,cell}}(X, \partial X; A)$ (with $\lambda_\sigma \in A$ for each 2-cell σ of X) whose support contains all 2-cells of X. Set

$$p_0 = \sum_{\sigma \subseteq X_0} \lambda_\sigma \sigma.$$

Since $dp \in C_1^{\mathrm{cell}}(\partial X; A)$, the support of dp_0 consists of 1-cells of X that lie in ∂X (and are therefore incident to exactly one 2-cell in X, and at most one in X_0) or are incident to at least one 2-cell in $X \setminus X_0$ (since X has small links, they are incident to at most one other 2-cell, which can lie either in X or in X_0). In both cases, they are incident to at most one 2-cell of X_0; moreover, they are incident to at least one 2-cell of X_0 as they lie in the support of dp_0. Therefore, the support of dp_0 is contained in ∂X_0, showing that p_0 is a relative 2-cycle in $Z_2^{\mathrm{lf,cell}}(X_0, \partial X_0; A)$ whose support contains all the 2-cells of X_0. Hence, X_0 is A-orientable. □

Lemma IV.2.9 (Injectivity of relative homology)**.** *Let X be a 2-complex, and let $Y, X_0 \subseteq X$ be two subcomplexes. Set $Y_0 = Y \cap X_0$. Then for any abelian group A, the inclusion-induced map*

$$H_2(X_0, Y_0; A) \to H_2(X, Y; A)$$

is injective.

Proof. We follow an argument of Howie [51, Lemma 3.2]. Applying excision to the triple $(X_0 \cup Y, Y, Y \smallsetminus Y_0)$ shows that the inclusion induces an isomorphism

$$H_2\left(X_0, Y_0; A\right) \cong H_2\left(X_0 \cup Y, Y; A\right).$$

But since X is a 2-complex, $H_3\left(X, X_0 \cup Y; A\right) = 0$, so the long exact sequence of the triple $(X, X_0 \cup Y, Y)$ shows that the inclusion induces an embedding

$$H_2\left(X_0 \cup Y, Y; A\right) \hookrightarrow H_2\left(X, Y; A\right).$$

This proves that the inclusion-induced map $H_2\left(X_0, Y_0; A\right) \to H_2\left(X, Y; A\right)$ is injective. □

IV.3 Isometric embeddings of surfaces

We now have all the tools we need to prove our isometric embedding theorems for surfaces. We consider S an oriented, compact, connected surface, and $T \subseteq S$ a compact (connected) embedded subsurface (with boundary not necessarily contained in ∂S), and that is π_1-*injective*, in the sense that the induced morphism

$$\iota : \pi_1 T \hookrightarrow \pi_1 S$$

is injective. We would like to understand when ι is a (strong) scl- or ℓ^1-isometric embedding.

IV.3.a Main theorem

Our main technical result is the following, which says that, with our standard form from Propositon II.2.8 and with appropriate homology vanishing conditions, an admissible surface in S for a family of loops in T is in fact entirely contained in T.

Theorem IV.3.1. *Let S be a cellulated, oriented, compact, connected surface, let $T \subseteq S$ be a π_1-injective subcomplex, let $\gamma : \coprod S^1 \to T$, and let $\alpha \in H_2(S, \gamma; \mathbb{Q})$. Consider an ℓ^1-admissible surface $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ in S with $f_*[\Sigma] = n(\Sigma)\alpha$ for some $n(\Sigma) \in \mathbb{N}_{\geqslant 1}$.*

Let $R = \mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$ and assume that one of the following holds:

(i) *f is in standard form and $H_2(S, T; R) = 0$, or*

(ii) *f is in perfect standard form and $f_*[\Sigma] = 0$ in $H_2(S, T; R)$.*

Then $f(\Sigma) \subseteq T$.

Proof. Consider $S_0 = \operatorname{Im} f \subseteq S$, and let $T_0 = S_0 \cap T$. Note that S_0 is a subcomplex of S since f is transverse, and T_0 is a subcomplex of S_0 since T is a subcomplex of S. The map f induces $f_0 : \Sigma \to S_0$. Our subcomplex stability lemmas imply that

- S_0 is an R-orientable 2-complex with small links as it inherits these properties from S (see Lemmas IV.2.7 and IV.2.8);
- With assumption (i) (i.e. $H_2(S, T; R) = 0$), we also have $H_2(S_0, T_0; R) = 0$ by Lemma IV.2.9;
- With assumption (ii) (i.e. $f_*[\Sigma] = 0$ in $H_2(S, T; R)$), we also have $f_{0*}[\Sigma] = 0$ in $H_2(S_0, T_0; R)$ by Lemma IV.2.9.

Claim. We have an inclusion $\partial S_0 \subseteq f_0(\partial\Sigma)$ (where ∂S_0 should be understood in the sense of Definition IV.2.3).

Proof of the claim. Let e be an edge of ∂S_0. By definition, $e \subseteq S_0 = \operatorname{Im} f$; by transversality of f, there is a 1-handle H in Σ that maps to e. Now each end of the 1-handle H can be either incident to a cellular disc or to $\partial\Sigma$. If each end is incident to a cellular disc, then those two cellular discs must map to the same 2-cell σ of S_0, because e is incident to only one 2-cell as it lies on ∂S_0. In this case, the two cellular discs map to σ with opposite orientations (as in Figure II.4), which contradicts the non-folding property — which f has since it is in standard form. Therefore, at least one end of H must be incident to $\partial\Sigma$. This implies that $e \subseteq f_0(\partial\Sigma)$. □

By assumption, $f_0(\partial\Sigma) \subseteq T_0$. Hence we have

$$\partial S_0 \subseteq f_0(\partial\Sigma) \subseteq T_0 \subseteq S_0.$$

With assumption (i), we have $H_2(S_0, T_0; R) = 0$, so it follows immediately from Lemma IV.2.5 that every 2-cell of S_0 is contained in T_0.

With assumption (ii), we have $f_{0*}[\Sigma] = 0$ in $H_2(S_0, T_0; R)$. Let σ be a 2-cell in $S_0 = \operatorname{Im} f$. Recall that f_0 is in perfect standard form, so it is orientation-perfect. This means that all cellular discs of Σ mapping to σ do so with the same orientation. Hence, the image $f_{0*}[\Sigma]$ in $H_2(S_0, T_0; R)$ has a term in σ with nonzero coefficient. But $f_{0*}[\Sigma] = 0$ in $H_2(S_0, T_0; R)$, so we must have $\sigma \subseteq T_0$. This shows that every 2-cell in S_0 is contained in T_0.

Claim. Every 0- or 1-cell of S_0 is incident to a 2-cell of S_0.

Proof of the claim. Note first that there is no isolated 0-cell in S_0 since f_0 is incompressible. Now assume for contradiction that there is a 1-cell $e \subseteq S_0$ without any incident 2-cell. Let H be a 1-handle of Σ mapping to e. Then both ends of H lie on $\partial\Sigma$. Hence any vertex disc in Σ incident to H meets $\partial\Sigma$ on both sides of H. But links of vertex discs are connected since f is in standard form, so neither of the vertex discs incident to H is incident to any other 1-handle — see Figure IV.1. Hence, Σ has a

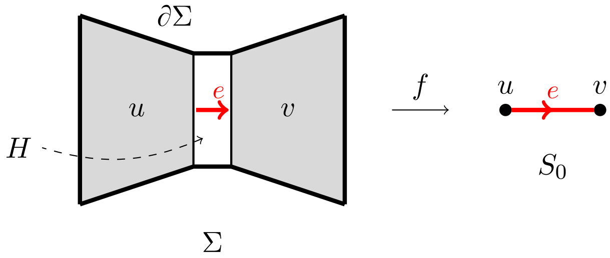


Figure IV.1: A 1-handle mapping to an edge with no incident 2-cell.

disc component consisting of H and its two incident vertex discs (note that these two discs are distinct by connectedness of their link). This is impossible since admissible surfaces in standard form are assumed to be disc-free. □

Since every 2-cell of S_0 is contained in T_0, it follows from the last claim that $S_0 = T_0$, and therefore $\operatorname{Im} f = S_0 \subseteq T$ as wanted. □

IV.3.b Isometric embedding results

We now discuss applications of Theorem IV.3.1 to isometric embeddings for stable commutator length and the relative Gromov seminorm.

If S is a surface, we say that a subsurface $T \subseteq S$ is H_1*-injective* if the induced map $H_1(T;\mathbb{Z}) \to H_1(S;\mathbb{Z})$ is injective.

Theorem IV.3.2 (scl-isometric embedding of surfaces)**.** *Let S be an oriented, compact, connected surface with non-empty boundary, and let $T \subseteq S$ be an H_1-injective subsurface. Then the inclusion-induced map*

$$\iota : \pi_1 T \hookrightarrow \pi_1 S$$

is a strong scl*-isometric embedding.*

Proof. Note first that $\pi_1 T$ and $\pi_1 S$ are free groups as $\partial S \neq \varnothing$. Since $\iota_* : H_1(\pi_1 T) \to H_1(\pi_1 S)$ is injective, we have $\operatorname{rk}(\pi_1 T) = \operatorname{rk}(H_1(\pi_1 T)) = \operatorname{rk}(\operatorname{Im} \iota_*) = \operatorname{rk}(\operatorname{Im} \iota)$, and it follows from the Hopf property for free groups that $\iota : \pi_1 T \to \pi_1 S$ is injective.

Moreover, by the Universal Coefficient Theorem, the inclusion induces an injective map $H_1(T; \mathbb{Q}) \to H_1(S; \mathbb{Q})$, or equivalently, the map

$$\iota_* : H_1(\pi_1 T; \mathbb{Q}) \to H_1(\pi_1 S; \mathbb{Q})$$

is injective.

It remains to show that ι preserves the scl of homogeneous boundaries (strong isometry will follow from Proposition IV.1.2).

Consider a homogeneous boundary $[c] \in B_1^{\mathrm{conj}}(\pi_1 T; \mathbb{Z})$, represented by a map $\gamma : \coprod S^1 \to T$. By Proposition II.2.8, $\mathrm{scl}_{\pi_1 S}([\iota c]) = \mathrm{scl}_S(\iota\gamma)$ can be computed as an infimum over all admissible surfaces $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ in standard form. But since S has non-empty boundary, $H_2(S) = 0$, so injectivity of $H_1(T) \to H_1(S)$ implies that $H_2(S, T) = 0$ by the long exact sequence of (S, T). Therefore, by Theorem IV.3.1, every admissible surface in standard form in S can be homotoped to an admissible surface in T. It follows that

$$\mathrm{scl}_{\pi_1 T}([c]) = \mathrm{scl}_T(\gamma) \leqslant \mathrm{scl}_S(\iota\gamma) = \mathrm{scl}_{\pi_1 S}([\iota c]),$$

and the reverse inequality always holds by monotonicity of scl (see Lemma I.1.19). □

Remark IV.3.3. In Theorem IV.3.2, observe that H_1-injectivity of T could be replaced with the equivalent assumption that $H_2(S, T) = 0$. With that assumption, the theorem would also hold when S is closed since in that case, $H_2(S, T) = 0$ implies that $S = T$ by Corollary IV.2.6. However, this would only add a trivial statement and therefore the generality of the theorem would not be increased.

Our second isometric embedding theorem is the following, which also applies — and gives a non-trivial result — in the closed case:

Theorem IV.3.4 (ℓ^1-isometric embedding of surfaces)**.** *Let S be an oriented, compact, connected surface, let $T \subseteq S$ be a π_1-injective subsurface. Then the inclusion-induced map*

$$\iota : \pi_1 T \hookrightarrow \pi_1 S$$

is an ℓ^1-isometric embedding.

Proof. Fix a family of loops $\gamma : \coprod S^1 \to T$, and let $\alpha \in H_2(T, \gamma; \mathbb{Q})$. Then Proposition II.2.8 says that $\|\iota_* \alpha\|_1$ can be computed as an infimum over all admissible surfaces $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ in perfect standard form. Let f be such an admissible surface, with $f_*[\Sigma] = n(\Sigma)\iota_*\alpha$ in $H_2(S, \gamma; \mathbb{Q})$. By the long exact sequence of the triple (S, T, γ) (see Proposition I.2.2(ii)), $\iota_*\alpha$ maps to zero in $H_2(S, T; \mathbb{Q})$, and so the image of $f_*[\Sigma]$ in $H_2(S, T; \mathbb{Q})$ is also zero. Therefore, Theorem IV.3.1 applies and f can be homotoped to an admissible surface in T. This proves that $\|\alpha\|_1 \leqslant \|\iota_*\alpha\|_1$, and the reverse inequality always holds since $\|\cdot\|_1$ is monotone with respect to continuous maps. □

In fact, applying Theorem IV.3.4 to the context of surfaces with non-empty boundary yields a stronger version of Theorem IV.3.2:

Corollary IV.3.5. *Let S be an oriented, compact, connected surface with non-empty boundary and let $T \subseteq S$ be a π_1-injective subsurface. Then the inclusion-induced map*

$$\iota : \pi_1 T \hookrightarrow \pi_1 S$$

is an scl*-isometric embedding.*

Proof. The morphism ι is ℓ^1-isometric by Theorem IV.3.4, and $H_2(S) = 0$ since $\partial S \neq \varnothing$, so Proposition IV.1.6 implies that ι is scl-isometric. □

IV.3.c Extremal surfaces

We have seen that certain inclusions of surfaces induce scl-isometric embeddings. This section is concerned with understanding when those scl-isometric embeddings preserve extremal surfaces, i.e. scl-admissible surfaces that realise the infimum in Definition I.1.11. More precisely, given an inclusion $\iota : T \hookrightarrow S$ inducing an scl-isometric embedding, and given a family of loops $\gamma : \coprod S^1 \to T$, we are interested in understanding whether one can find an extremal surface $f : (\Sigma, \partial\Sigma) \to (T, \gamma)$ for γ in T such that $\iota \circ f$ is extremal for $\iota \circ \gamma$ in S.

Proposition II.2.8 says that, for the purpose of finding extremal surfaces, we can assume that admissible surfaces are in standard form — but not necessarily in perfect standard form. In the context of Theorem IV.3.2, this is sufficient: H_1-injectivity implies that $H_2(S, T) = 0$ since $H_2(S) = 0$, so Theorem IV.3.1 with assumption (i) says that any admissible surface $(\Sigma, \partial\Sigma) \to (S, \gamma)$ in standard form is in fact contained in

T. Note also that $\pi_1 S$ and $\pi_1 T$ are free groups, so extremal surfaces exist for all homologically trivial γ by Calegari's Rationality Theorem [17]. This gives the following:

Corollary IV.3.6. *Let S be an oriented, compact, connected surface with non-empty boundary, and let $T \subseteq S$ be a H_1-injective subsurface. Consider a rationally homologically trivial map $\gamma : \coprod S^1 \to T$.*

Then there exists an extremal scl*-admissible surface $f : (\Sigma, \partial\Sigma) \to (T, \gamma)$ such that $\iota \circ f : (\Sigma, \partial\Sigma) \to (S, \iota\gamma)$ is also extremal for* scl. □

Note however that, even if extremal surfaces were known to exist for the relative Gromov seminorm, we would not obtain an analogue of Corollary IV.3.6 in that setting. Indeed, to prove Theorem IV.3.4, we needed to apply Theorem IV.3.1 with assumption (ii) and work with admissible surfaces in perfect standard form. But an asymptotic promotion argument was necessary to obtain the perfect standard form (see §II.2.c), and this does not preserve extremal surfaces.

In the case where S has non-empty boundary and the subsurface $T \subseteq S$ is only π_1-injective rather than H_1-injective, then Corollary IV.3.5 says that $\pi_1 T \hookrightarrow \pi_1 S$ is still an isometric embedding for scl. In general, as Corollary IV.3.5 relies on Theorem IV.3.4, this isometric embedding might not preserve extremal surfaces, but in the special case of a single loop $\gamma : S^1 \to T$, Remark II.2.7 says that the asymptotic promotion argument can be bypassed, and therefore extremal surfaces can be assumed to be in perfect standard form. We obtain:

Corollary IV.3.7. *Let S be an oriented, compact, connected surface with non-empty boundary, and let $T \subseteq S$ be a π_1-injective subsurface. Consider a rationally homologically trivial loop $\gamma : S^1 \to T$.*

Then there exists an extremal scl*-admissible surface $f : (\Sigma, \partial\Sigma) \to (T, \gamma)$ such that $\iota \circ f : (\Sigma, \partial\Sigma) \to (S, \iota\gamma)$ is also extremal for* scl. □

IV.4 Vertex group embeddings in graphs of groups

We finish this section with one more ℓ^1-isometric embedding result, in the context of graphs of groups. One of the fundamental facts in the theory of graphs of groups is that a vertex group embeds into the fundamental group of the graph of groups. It is natural to try to upgrade this to an scl-isometric embedding, but that unfortunately does not work in general, even if edge groups are amenable (and hence have vanishing stable commutator length [16, Proposition 2.65]):

Example IV.4.1. Let S be a closed genus-3 surface, and let β be a non-separating simple closed curve in S. Consider the HNN-splitting $\pi_1 S = \pi_1 T *_{\mathbb{Z}}$ obtained by cutting S along β, where T is a genus-2 surface with two boundary components, and the HNN-extension identifies the two boundary components of T — see Figure IV.2. Then the

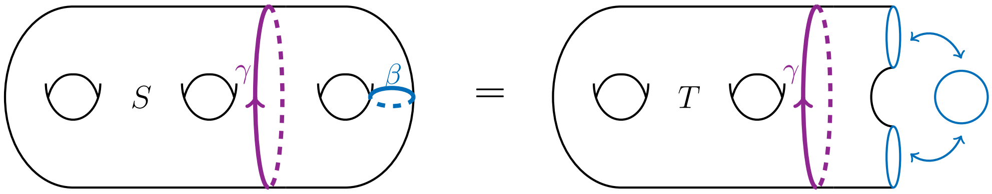


Figure IV.2: HNN-splitting of a closed genus-3 surface.

embedding $\pi_1 T \hookrightarrow \pi_1 S$ is *not* scl-isometric. To see this, consider the loop γ represented in the picture. Note that T is a surface with non-empty boundary, and γ bounds an immersed (and in fact embedded) genus-2 surface with one boundary component in T, so Corollary V.1.13 implies that $\mathrm{scl}_T(\gamma) = \frac{3}{2}$. In S however, γ bounds a genus-1 surface with one boundary component, so $\mathrm{scl}_S(\gamma) \leqslant \frac{1}{2}$. This shows that the morphism

$$\pi_1 T \hookrightarrow \pi_1 T *_{\mathbb{Z}}$$

is not an scl-isometric embedding.

Example IV.4.1 shows that vertex groups can fail to embed scl-isometrically into graphs of groups. However, using our Bavard duality statement for the relative Gromov seminorm (Theorem III.2.4), we are able to translate an isometric embedding result in bounded cohomology of Bucher et al. [11] into the fact that vertex groups embed ℓ^1-isometrically if edge groups are amenable:

Theorem IV.4.2 (ℓ^1-isometric embedding of vertex groups in graphs of groups)**.** *Let $\mathbb{X}$ be a graph of groups whose underlying graph X is finite, with countable vertex groups $\{X_v\}_{v \in V(X)}$, and amenable edge groups $\{X_e\}_{e \in E(X)}$. Then the embedding*

$$X_v \hookrightarrow \pi_1 \mathbb{X}$$

of each vertex group into the fundamental group is ℓ^1-isometric.

Proof. For each vertex v of X, denote by $i_v : X_v \hookrightarrow \pi_1\mathbb{X}$ the embedding of the vertex group into the fundamental group. By [11, Theorem 1.1], there is an isometric embedding

$$\Theta : \bigoplus_v H_b^2(X_v;\mathbb{R}) \hookrightarrow H_b^2(\pi_1\mathbb{X};\mathbb{R}),$$

which is a right inverse to

$$\bigoplus_v i_v^* : H_b^2(\pi_1\mathbb{X};\mathbb{R}) \twoheadrightarrow \bigoplus_v H_b^2(X_v;\mathbb{R}).$$

Now let $[c] \in C_1^{\mathrm{conj}}(X_v;\mathbb{Z})$ and $\alpha \in H_2(X_v,[c];\mathbb{R})$. Bavard duality for $\|\cdot\|_1$ (Theorem III.2.4) implies that

$$\|\alpha\|_1 = \sup\left\{ \frac{\langle u,\alpha\rangle}{\|u\|_\infty} \;\middle|\; u \in H_b^2(X_v;\mathbb{R}) \right\}.$$

Since $i_v^*\Theta u = u$ for all $u \in H_b^2(X_v)$, and since Θ preserves $\|\cdot\|_\infty$, it follows that

$$\begin{aligned} \|\alpha\|_1 &= \sup\left\{ \frac{\langle i_v^*\Theta u,\alpha\rangle}{\|\Theta u\|_\infty} \;\middle|\; u \in H_b^2(X_v) \right\} = \sup\left\{ \frac{\langle \Theta u, i_{v*}\alpha\rangle}{\|\Theta u\|_\infty} \;\middle|\; u \in H_b^2(X_v) \right\} \\ &\leqslant \sup\left\{ \frac{\langle u', i_{v*}\alpha\rangle}{\|u'\|_\infty} \;\middle|\; u' \in H_b^2(\pi_1\mathbb{X}) \right\} = \|i_{v*}\alpha\|_1 . \end{aligned}$$

This proves that $\|\alpha\|_1 \leqslant \|i_{v*}\alpha\|_1$, and the reverse inequality follows from the fact that group homomorphisms are $\|\cdot\|_1$-non-increasing. □

Corollary IV.4.3 (scl-isometric embedding of vertex groups in graphs of groups)**.** *Let $\mathbb{X}$ be a graph of groups whose underlying graph X is finite, with countable vertex groups $\{X_v\}_{v\in V(X)}$, and amenable edge groups $\{X_e\}_{e\in E(X)}$. Assume that $H_2(\pi_1\mathbb{X};\mathbb{Q}) = 0$. Then the embedding*

$$X_v \hookrightarrow \pi_1\mathbb{X}$$

of each vertex group into the fundamental group is scl*-isometric.*

Proof. This follows from Theorem IV.4.2 and Proposition IV.1.6. □

Note that Theorem IV.4.2 recovers our ℓ^1-isometric embedding theorem for surfaces (Theorem IV.3.4): indeed, given an oriented, compact, connected surface S and a π_1-injective subsurface T, the fundamental group $\pi_1 S$ splits as a graph of groups, with vertex groups given by $\pi_1 T$ and the fundamental groups of all connected components of $S \setminus T$, and with edge groups isomorphic to $\mathbb{Z}$ — and corresponding to cutting S along simple closed curves. However, this approach has the disadvantage of relying on deep results in bounded cohomology from [11], whereas the proof of Theorem IV.3.4 given in §IV.3 is self-contained.

CHAPTER V

EXTREMALITY OF THE BOUNDED EULER CLASS

Calegari [15] exhibited a connection between the rotation quasimorphism, area, and stable commutator length for loops bounding immersed surfaces in compact hyperbolic surfaces with non-empty boundary. The goal of this chapter is to explain how this generalises to a statement about the relative Gromov seminorm for all compact (possibly closed) hyperbolic surfaces. This uses our analogue of Bavard duality for the relative Gromov seminorm via bounded cohomology, and the bounded Euler class as the generalisation of the rotation quasimorphism to this setting. The first section explains our generalisation of Calegari's Theorem to possibly closed hyperbolic surfaces; this material is from [57, §5]. The second section then goes on to discuss the behaviour of the bounded Euler class under the isometric embeddings of surfaces of Chapter IV; the content of this section is a generalisation of that of [58, §6].

V.1 Immersed surfaces and bounded Euler class

V.1.a Equivariant (bounded) cohomology

To define the bounded Euler class, we will use the language of equivariant cohomology.

Given a set X with an action of a group G, a *degree-n homogeneous G-cochain* (with real coefficients) is a map $\psi : X^{n+1} \to \mathbb{R}$ which is invariant under the diagonal action of G on X^{n+1}, in the sense that

$$\psi(x_0, \dots, x_n) = \psi(gx_0, \dots, gx_n)$$

for all $x_0, \dots, x_n \in X$ and $g \in G$. We denote by $C^n(G \curvearrowright X; \mathbb{R})$ the $\mathbb{R}$-vector space of such cochains; they form a cochain complex $C^*(G \curvearrowright X; \mathbb{R})$ with coboundary maps $d^n : C^{n-1}(G \curvearrowright X; \mathbb{R}) \to C^n(G \curvearrowright X; \mathbb{R})$ given by

$$d^n\psi(x_0, \dots, x_n) := \sum_{i=0}^{n} (-1)^i \psi(x_0, \dots, \hat{x}_i, \dots, x_n),$$

where the hat denotes omission. The cohomology of $C^*(G \curvearrowright X; \mathbb{R})$ is denoted by $H^*(G \curvearrowright X; \mathbb{R})$.

Remark V.1.1. (i) If G acts on itself by (left) multiplication, then there is an isomorphism

$$H^*(G \curvearrowright G; \mathbb{R}) \cong H^*(G; \mathbb{R}),$$

which is induced by the map $\theta : C^*(G; \mathbb{R}) \to C^*(G \curvearrowright G; \mathbb{R})$ given by

$$(\theta\psi)(g_0, \dots, g_n) := \psi\left(g_0^{-1}g_1, g_1^{-1}g_2, \dots, g_{n-1}^{-1}g_n\right)$$

for $\psi \in C^n(G; \mathbb{R})$ (see §III.2.b for our definition of $C^*(G; \mathbb{R})$).

(ii) Given a choice of basepoint x in a G-set X, there is a morphism

$$H^*(G \curvearrowright X; \mathbb{R}) \to H^*(G \curvearrowright G; \mathbb{R})$$

induced by the map $\varpi_x : C^*(G \curvearrowright X; \mathbb{R}) \to C^*(G \curvearrowright G; \mathbb{R})$ given by

$$(\varpi_x\psi)(g_0, \dots, g_n) = \psi(g_0x, \dots, g_nx)$$

for $\psi \in C^n(G \curvearrowright X; \mathbb{R})$. In fact, on the level of cohomology, this morphism is independent of the choice of x. Combining this with (i), we obtain a morphism

$$\eta : H^*(G \curvearrowright X; \mathbb{R}) \to H^*(G; \mathbb{R}).$$

Similar to the definition of bounded cohomology for spaces and groups (see §III.2.a and §III.2.b), there is a bounded version of equivariant cohomology: the complex $C_b^*(G \curvearrowright X;\mathbb{R})$ of bounded homogeneous G-cochains is defined to be the subcomplex of $C^*(G \curvearrowright X;\mathbb{R})$ consisting of *bounded* G-equivariant maps $X^{n+1} \to \mathbb{R}$. The corresponding cohomology is denoted by $H_b^*(G \curvearrowright X;\mathbb{R})$, and there is a morphism

$$H_b^*(G \curvearrowright X;\mathbb{R}) \to H_b^*(G;\mathbb{R})$$

as in Remark V.1.1(ii). We refer the reader to [12, §3.1] for more details on equivariant cohomology.

V.1.b Bounded Euler class of a circle action

A choice of hyperbolic structure on a connected surface S defines an action of $\pi_1 S$ on the hyperbolic plane $\mathbb{H}^2$. This induces an action on the boundary of $\mathbb{H}^2$, which is homeomorphic to the circle S^1. In general, the dynamics of an action of a group G on the circle is encoded by the bounded Euler class, which is a class in $H_b^2(G)$ that was introduced by Ghys [42] as a generalisation of Poincaré's rotation number [61, 62].

The bounded Euler class has several equivalent definitions [12], and for our purpose, it will be helpful to define it from the point of view of the orientation cocycle.

Consider the action of the group $\mathrm{Homeo}^+(S^1)$ of orientation-preserving homeomorphisms of the circle on S^1. The *orientation cocycle* $\mathrm{Or} \in C_b^2(\mathrm{Homeo}^+(S^1) \curvearrowright S^1;\mathbb{R})$ is given by

$$\mathrm{Or}(x,y,z) = \begin{cases} +1 & \text{if the triple } (x,y,z) \in (S^1)^3 \text{ is positively oriented} \\ -1 & \text{if the triple } (x,y,z) \in (S^1)^3 \text{ is negatively oriented} \\ 0 & \text{if the triple } (x,y,z) \in (S^1)^3 \text{ is degenerate} \end{cases} \cdot$$

This turns out to be a cocycle, so it defines $[\mathrm{Or}] \in H_b^2(\mathrm{Homeo}^+(S^1) \curvearrowright S^1;\mathbb{R})$.

Definition V.1.2. The *universal real bounded Euler class for circle actions* is

$$\mathrm{eu}_b^{\mathbb{R}} = -\frac{1}{2}\eta\,[\mathrm{Or}] \in H_b^2(\mathrm{Homeo}^+(S^1);\mathbb{R}),$$

where $\eta : H_b^2(\mathrm{Homeo}^+(S^1) \curvearrowright S^1) \to H_b^2(\mathrm{Homeo}^+(S^1))$ is the morphism described in Remark V.1.1(ii).

Given an action $\rho : G \to \mathrm{Homeo}^+\left(S^1\right)$ of a group on the circle, the (real) *bounded Euler class* of the action is

$$\mathrm{eu}_b^{\mathbb{R}}(\rho) = \rho^* \, \mathrm{eu}_b^{\mathbb{R}} \in H_b^2\left(G; \mathbb{R}\right).$$

This measures how far ρ is from being a rotation action on S^1 [36, Corollary 10.27]. By definition, $\left\|\mathrm{eu}_b^{\mathbb{R}}(\rho)\right\|_\infty \leqslant \left\|\mathrm{eu}_b^{\mathbb{R}}\right\|_\infty \leqslant \frac{1}{2}\left\|\mathrm{Or}\right\|_\infty = \frac{1}{2}$. See [12, 43] for more details on the bounded Euler class.

V.1.c Area of a relative 2-class

In [15], Calegari defines a notion of area for a (rationally) homologically trivial map $\gamma : \coprod S^1 \to S$ in an oriented, connected, hyperbolic surface S with non-empty boundary. In his definition, it is crucial that S has non-empty boundary because then $H_2(S) = 0$, so the map $\partial : H_2\left(S, \gamma\right) \to H_1\left(\coprod S^1\right)$ of Proposition I.2.2(i) is injective and there is a unique class $\alpha \in \partial^{-1}\left(\left[\coprod S^1\right]\right)$. We now explain how to generalise Calegari's notion of area to the case where S is closed by defining the area of a class in $H_2(S, \gamma)$.

Let S be an oriented, compact, hyperbolic surface with (possibly empty) geodesic boundary. Let $\gamma : \coprod S^1 \to S$ be a finite family of geodesic loops in S, and let $\alpha \in H_2\left(S, \gamma; \mathbb{R}\right)$. By definition, $H_2(S, \gamma) = H_2\left(S_\gamma, \coprod S^1\right)$. The mapping cylinder S_γ has no geometric structure allowing us to measure areas, but there is a map of pairs $\left(S_\gamma, \coprod S^1\right) \to \left(S, \gamma\left(\coprod S^1\right)\right)$ defined by collapsing the cylinder (see Example I.2.3(ii)). This induces a morphism

$$H_2\left(S, \gamma; \mathbb{R}\right) \to H_2\left(S, \gamma\left(\coprod S^1\right); \mathbb{R}\right),$$

and we'll measure the area of α in the image.

Note that γ consists of finitely many geodesic loops in a compact hyperbolic surface, so it has finitely many multiple points — indeed, if one looks at a fundamental domain D in the universal cover, each loop has finitely many lifts meeting D, and any two geodesics in the universal cover intersect in at most one point.

We pick a (finite) cell structure on S such that

- The 0-skeleton of S contains all multiple points of γ (of which there are finitely many by the above argument),
- The 1-skeleton of S contains $\gamma\left(\coprod S^1\right)$, and

- Each 2-cell is positively oriented (for the orientation inherited by S).

One can choose a cellular relative 2-cycle c representing the image of α in the homology $H_2\left(S, \gamma\left(\coprod S^1\right); \mathbb{R}\right)$, and c is in fact unique because both $C_3^{\text{cell}}(S)$ and $C_2^{\text{cell}}\left(\gamma\left(\coprod S^1\right)\right)$ vanish.

Definition V.1.3. Let $\gamma : \coprod S^1 \to S$ be a family of geodesic loops in an oriented, compact, hyperbolic surface S. Given $\alpha \in H_2(S, \gamma; \mathbb{R})$, the *area* of α is defined by

$$\operatorname{area}(\alpha) = \sum_{\sigma} \lambda_\sigma \operatorname{area}(\sigma),$$

where $\sum_\sigma \lambda_\sigma \sigma \in Z_2^{\text{cell}}\left(S, \gamma\left(\coprod S^1\right); \mathbb{R}\right)$ (with $\lambda_\sigma \in \mathbb{R}$ for each 2-cell σ) is the unique cellular relative 2-cycle representing the image of α in $H_2\left(S, \gamma\left(\coprod S^1\right); \mathbb{R}\right)$.

Remark V.1.4. Let $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ be an admissible surface. Assume that Σ is equipped with a hyperbolic structure with respect to which the map $f : \Sigma \to S$ is a local isometric embedding. Then there is an equality

$$\operatorname{area}(f_*[\Sigma]) = \operatorname{area}(\Sigma),$$

where $f_*[\Sigma]$ is seen as a class in $H_2(S, \gamma; \mathbb{R})$.

V.1.d Pleated surfaces

In order to obtain good estimates on the Gromov seminorm for a hyperbolic surface S, it will be helpful to measure it with special admissible surfaces, called pleated surfaces. The heuristics behind pleated surfaces is the following: if Σ is an orientable compact connected surface, then its simplicial volume is given by $\|[\Sigma]\|_1 = -2\chi^-(\Sigma)$; however, there is no triangulation of Σ realising this equality. Instead, the simplicial volume is realised by an *ideal triangulation*. The idea is therefore to endow admissible surfaces Σ with ideal triangulations that are compatible with the hyperbolic structure on S.

Pleated surfaces, which were introduced by Thurston [66, §8.8], will achieve this.

A *geodesic lamination* Λ in a hyperbolic surface Σ is a closed subset of Σ which decomposes as a disjoint union of complete embedded geodesics. Each such geodesic is called a *leaf* of Λ.

Definition V.1.5. Let M be a hyperbolic manifold. A *pleated surface* in M is a map $f : \Sigma \to M$, where Σ is a finite-area hyperbolic surface, such that

(i) f sends each arc in Σ to an arc of the same length in M,

(ii) There is a geodesic lamination $\Lambda \subseteq \Sigma$ such that f sends each leaf of Λ to a geodesic of M, and f is totally geodesic (i.e. sends every geodesic to a geodesic) on $\Sigma \smallsetminus \Lambda$, and

(iii) If Σ is non-compact, then f sends each small neighbourhood of each cusp of Σ to a small neighbourhood of a cusp of M.

The geodesic lamination Λ is called a *pleating locus* for f.

For a more detailed introduction to pleated surfaces in hyperbolic manifolds, we refer the reader to [6, 21, 38].

We now show, following Calegari [16, §3.1.3], how to obtain pleated admissible surfaces. The fundamental tool for this is Thurston's *spinning construction*:

Lemma V.1.6 (Thurston [66, §8.10]). *Let P be a pair of pants (i.e. a compact hyperbolic surface of genus 0 with three boundary components) and let M be a compact, hyperbolic surface or a closed, hyperbolic manifold. Given a map $f : P \to M$, either*

(i) *The image of $\pi_1 P$ under f_* is a cyclic subgroup of $\pi_1 M$, or*

(ii) *The map f can be homotoped to a pleated surface.*

Proof. Consider a lift $\tilde{f} : \tilde{P} \to \tilde{M}$ of f to universal covers. Note that $\tilde{M}$ is a convex subset of the hyperbolic n-space $\mathbb{H}^n$, and $\tilde{P}$ is a convex subset of $\mathbb{H}^2$. Pick a geodesic triangle Δ in P with one vertex on each boundary component. This lifts to a geodesic triangle $\tilde{\Delta}$ in a fundamental domain of $\tilde{P} \subseteq \mathbb{H}^2$.

Now the spinning construction consists in simultaneously dragging vertices of $\tilde{\Delta}$ along the lifts of ∂P to $\mathbb{H}^2$, and moving them to the boundary $\partial \mathbb{H}^2$. See Figure V.1. Note that the process simultaneously modifies $\tilde{\Delta}$ as well as all other lifts of Δ to $\tilde{P}$; in particular, Figure V.1 depicts only one fundamental domain but there are many others. This construction is called *spinning* because, in P, the triangle Δ has been spun around the boundary components of P. In this way, one obtains a geodesic lamination Λ on P with three leaves, whose complement consists of two open ideal triangles.

There are two cases:

(i) If $f(\Lambda)$ is degenerate (i.e. the images of the three leaves of Λ have the same axis in $\tilde{M}$), then $f_* (\pi_1 P)$ generates a cyclic subgroup of $\pi_1 M$.

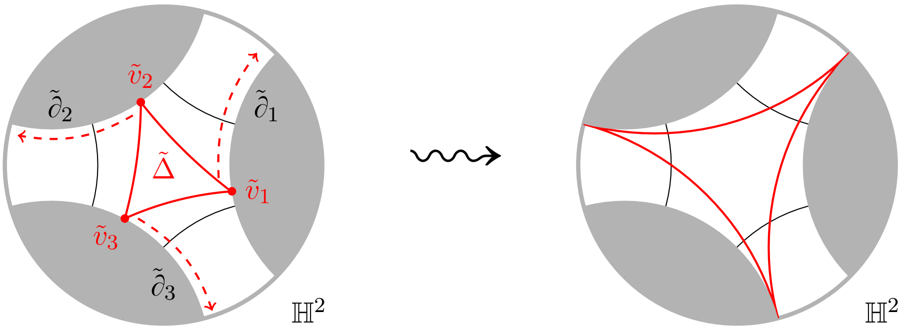


Figure V.1: Thurston's spinning construction.

(ii) Otherwise, construct a map $f' : P \to M$ homotopic to f as follows. For each boundary component ∂_i of P, we define $f'(\partial_i)$ to be the unique closed geodesic in the homotopy class of $f(\partial_i)$. Each leaf λ_i of Λ is mapped under f to a quasi-geodesic in M, which can be straightened to a geodesic γ_i. Set $f'(\lambda_i) = \gamma_i$. Finally, each component of $P \setminus \Lambda$ is an open ideal triangle, and since its image is nondegenerate in M, there is a unique totally geodesic extension of f' to this triangle. $\square$

Using Thurston's spinning construction, we can obtain pleated admissible surfaces. This is an adaptation of a lemma of Calegari [16, Lemma 3.7]:

Lemma V.1.7 (Pleated admissible surfaces)**.** *Let M be a compact, hyperbolic surface or a closed, hyperbolic manifold. Let $\gamma : \coprod S^1 \to M$ be a family of geodesic loops in M, no two components of which have the same image in M. Then for every rational class $\alpha \in H_2(M, \gamma; \mathbb{Q})$ and for every $\varepsilon > 0$, there is a pleated admissible surface $f : (\Sigma, \partial\Sigma) \to (M, \gamma)$ such that $f_*[\Sigma] = n(\Sigma)\alpha$ for some $n(\Sigma) \in \mathbb{N}_{\geqslant 1}$, and*

$$\|\alpha\|_1 \leqslant \frac{-2\chi^-(\Sigma)}{n(\Sigma)} \leqslant \|\alpha\|_1 + \varepsilon. \qquad (*)$$

Proof. By Lemma II.1.2, there is a simple, incompressible, admissible surface $f : (\Sigma, \partial\Sigma) \to (M, \gamma)$ satisfying $(*)$, with $f_*[\Sigma] = n(\Sigma)\alpha$ for some $n(\Sigma) \in \mathbb{N}_{\geqslant 1}$. Now take a pants decomposition $\{P_i\}_i$ of Σ, as in Figure V.2. The idea is to apply the spinning construction (Lemma V.1.6) to each P_i. We can perform the construction separately on each connected component of Σ; to simplify notations, we therefore assume that Σ is connected. There are three types of components in the pants decomposition:

(i) Pairs of pants that are part of a twice-punctured torus (e.g. $P_2, \ldots, P_5$ in Figure V.2),

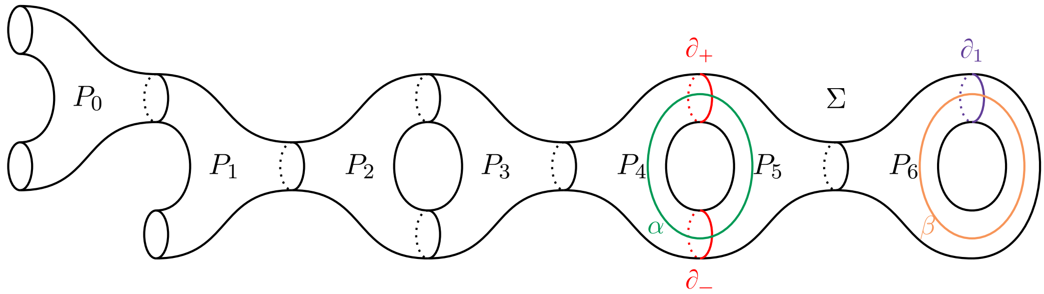


Figure V.2: Pants decomposition of Σ.

(ii) Pairs of pants that are glued to themselves to form a once-punctured torus (e.g. P_6 in Figure V.2),

(iii) Pairs of pants that are not of type (i) or (ii) (e.g. P_0, P_1 in Figure V.2).

Fix a pants component P_i. We want to apply Lemma V.1.6 to the restriction $f_{|P_i} : P_i \to M$; we need to ensure that $f_*(\pi_1 P_i)$ is non-cyclic. We distinguish three cases, based on the type of P_i.

We first show that, if P_i of type (iii), then $f_*(\pi_1 P_i)$ cannot be cyclic. Recall from §I.1.b that γ represents $[c] = \left[\sum_j w_j\right] \in C_1^{\mathrm{conj}}(\pi_1 M; \mathbb{Z})$, where no two of the w_j's are conjugate or generate a cyclic subgroup of $\pi_1 M$ (by assumption on γ). Since f is an admissible surface, each boundary component of Σ maps to a power of some w_j, and simplicity implies that no two boundary components of Σ map to powers of the same w_j. We can assume that the components $\{P_i\}_i$ of type (iii) are ordered as $\{P_0, \ldots, P_k\}$, in such a way that P_0 has two boundary components on $\partial\Sigma$, and each P_i is glued to P_{i-1} along one boundary component and has one boundary component on $\partial\Sigma$ (this is consistent with the notations of Figure V.2, where $k = 1$). With these notations, we can order the w_j's in such a way that $f_*(\pi_1 P_0) = \langle w_0, w_1 \rangle$, and each P_i has one boundary component glued to P_{i-1} and whose image represents an element of $\langle w_0, \ldots, w_i \rangle$, and one boundary component lying on $\partial\Sigma$, and whose image represents a power of w_{i+1}. In particular, it follows that $f_*(\pi_1 P_i)$ is not cyclic for any P_i of type (iii).

Now assume that P_i is of type (i). Two of the boundary components ∂_+ and ∂_- of P_i are meridians in a twice-punctured torus ($\partial_\pm$ are depicted in Figure V.2 for $P_i = P_4$). Let α be the equator of this twice-punctured torus and let $\delta_\alpha : \Sigma \to \Sigma$ denote the Dehn twist along α. If $f_*(\pi_1 P_i) = \langle f_*(\partial_+), f_*(\partial_-) \rangle$ is cyclic, then replace $\partial_\pm$ with $\delta_\alpha \partial_\pm$; this amounts to defining a new pants decomposition of Σ. For this pants decomposition, $\langle f_* (\partial_+), f_* (\partial_-) \rangle$ is not cyclic because $f_*\alpha$ and $f_*\partial_\pm$ do not commute by incompressibility (otherwise $[\alpha, \partial_\pm]$ would be represented by a simple closed curve in Σ with nullhomotopic image in M). It might be that, after this modification, the adjacent pair of pants P_j in the same twice-punctured torus as P_i has cyclic image in $\pi_1 M$. In this case, one applies the Dehn twist δ_α a second time.

Assume finally that P_i is of type (ii). Then P_i is glued to itself to form a once-punctured torus. Denote by ∂_1 one of the two boundary components of P_i that is glued to form a meridian in the once-punctured torus, and by β the equator (∂_1 and β are depicted in Figure V.2 for $P_i = P_6$). Then $f_* (\pi_1 P_i) = \left\langle f_* (\partial_1), f_* (\partial_1)^{f_*(\beta)} \right\rangle$, where the exponent denotes conjugation. If $f_* (\pi_1 P_i)$ is cyclic, then there are $w \in \pi_1 M$ and $k, \ell \in \mathbb{Z}$ such that

$$f_* (\partial_1) = w^k = f_*(\beta) w^\ell f_*(\beta)^{-1}.$$

But $\pi_1 M$ is Gromov-hyperbolic, and therefore it is known to be a *CSA group*, in the sense that all its maximal abelian subgroups are malnormal — see [30, Example 10]. Hence $\langle w \rangle$ is malnormal (after possibly replacing w with a generator of a maximal abelian subgroup containing it), and $f_* (\partial_1) \in \langle w \rangle \cap \langle w \rangle^{f_*(\beta)} \smallsetminus \{1\}$, so $f_*(\beta) \in \langle w \rangle$. In particular, $f_* [\partial_1, \beta] = 1$, which contradicts incompressibility. This proves that $f_* (\pi_1 P_i)$ cannot be cyclic.

Therefore, after performing the above modifications, we have a pants decomposition of Σ for which $f_* (\pi_1 P_i)$ is never a cyclic subgroup of $\pi_1 M$. By Lemma V.1.6, the restriction of f to each P_i can be homotoped to a pleated map. Moreover, these homotopies can be performed simultaneously as the image of each boundary component of a pair of pants is homotoped to the unique geodesic in its homotopy class. Hence, we obtain a pleated map homotopic to f, which is still an admissible surface and satisfies ($*$). ☐

Remark V.1.8. In fact, we will not need the estimate ($*$) on the Gromov seminorm in Lemma V.1.7: it will be enough for us to know that every rational class is represented by a pleated admissible surface.

V.1.e Bounded Euler class and area

A hyperbolic structure on a surface S induces an action of $\pi_1 S$ on the boundary of the hyperbolic plane, which is a circle. Hence, we get a circle action $\rho : \pi_1 S \to$

$\mathrm{Homeo}^+\left(S^1\right)$, defining a bounded Euler class $\mathrm{eu}_b^{\mathbb{R}}(\rho) \in H_b^2\left(\pi_1 S;\mathbb{R}\right)$ as explained in §V.1.b. We will call it the *bounded Euler class of* S and denote it by $\mathrm{eu}_b^{\mathbb{R}}(S)$. It can also be seen as an element of $H_b^2(S;\mathbb{R})$ — see §III.2.b.

The following is implicit in Calegari's book [16, Lemma 4.68]:

Lemma V.1.9 (Bounded Euler class and area)**.** *Let* $\gamma : \coprod S^1 \to S$ *be a family of geodesic loops in a compact hyperbolic surface* S*. Let* $\alpha \in H_2\left(S,\gamma;\mathbb{Q}\right)$ *be a rational class. Then*

$$\mathrm{area}(\alpha) = -2\pi \left\langle \mathrm{eu}_b^{\mathbb{R}}(S), \alpha \right\rangle.$$

Proof. Lemma V.1.7 yields a pleated admissible surface $f : (\Sigma, \partial\Sigma) \to (S,\gamma)$ with $f_*[\Sigma] = n(\Sigma)\alpha$ for some $n(\Sigma) \in \mathbb{N}_{\geqslant 1}$. Hence,

$$\left\langle \mathrm{eu}_b^{\mathbb{R}}(S), \alpha \right\rangle = \frac{1}{n(\Sigma)} \left\langle \mathrm{eu}_b^{\mathbb{R}}(S), f_*[\Sigma] \right\rangle.$$

Recall from §III.2.c that the above Kronecker product is under the (isometric) identification $H_b^2\left(S,\gamma;\mathbb{R}\right) \cong H_b^2\left(S;\mathbb{R}\right)$.

The bounded Euler class $\mathrm{eu}_b^{\mathbb{R}}(S)$ was defined as the image of $-\frac{1}{2}\,[\mathrm{Or}]$ under

$$H_b^2\left(\mathrm{Homeo}^+\left(S^1\right) \curvearrowright S^1\right) \xrightarrow{\eta} H_b^2\left(\mathrm{Homeo}^+\left(S^1\right)\right) \xrightarrow{\rho^*} H_b^2\left(\pi_1 S\right)$$

(where η is the map constructed in Remark V.1.1, and $\rho : \pi_1 S \to \mathrm{Homeo}^+\left(S^1\right)$ corresponds to the action $\pi_1 S \curvearrowright S^1 = \partial\mathbb{H}^2$ given by the hyperbolic structure on S). Tracking down the definitions, we see that $\mathrm{eu}_b^{\mathbb{R}}(S) = [e_{\pi_1 S}] \in H_b^2\left(\pi_1 S;\mathbb{R}\right)$, where $e_{\pi_1 S} \in C_b^2\left(\pi_1 S;\mathbb{R}\right)$ is the bounded 2-cocycle given by

$$e_{\pi_1 S} : (g_1, g_2) \in \pi_1 S \times \pi_1 S \longmapsto -\frac{1}{2}\,\mathrm{Or}\left(x, g_1 x, g_2 x\right) \in \mathbb{R}$$

for an arbitrary choice of $x \in S^1$, where the action $\pi_1 S \curvearrowright S^1 = \partial\mathbb{H}^2$ is given by the hyperbolic structure on S.

On the other hand, the pleated structure of f defines a finite ideal triangulation $\{\sigma_i\}_{i\in I}$ of Σ: each σ_i is an ideal triangle in Σ, the σ_i's have pairwise disjoint interior, and $\Sigma = \bigcup_{i\in I} \sigma_i$. For each $i \in I$, the ideal triangle σ_i is mapped under f to a (compact) geodesic triangle $f\left(\sigma_i\right)$ in S. Hence, $c = \sum_{i\in I} f\left(\sigma_i\right)$ is a relative 2-cycle in $Z_2\left(S, \mathrm{Im}\,\gamma;\mathbb{R}\right)$ representing $f_*[\Sigma]$ in $H_2\left(S, \mathrm{Im}\,\gamma;\mathbb{R}\right)$. The relative 2-cycle c can also be seen as an absolute chain in $C_2\left(S;\mathbb{R}\right)$, which corresponds to a certain 2-chain

$$c_{\pi_1 S} = \sum_{i\in I} \left(g_i, h_i\right) \in C_2\left(\pi_1 S;\mathbb{R}\right),$$

where each $(g_i, h_i) \in \pi_1 S \times \pi_1 S$ corresponds to the singular simplex $f(\sigma_i)$. Evaluating $\mathrm{eu}_b^{\mathbb{R}}(S)$ on $f_*[\Sigma]$ corresponds to evaluating $e_{\pi_1 S}$ on $c_{\pi_1 S}$:

$$\left\langle \mathrm{eu}_b^{\mathbb{R}}(S), f_*[\Sigma] \right\rangle = \langle e_{\pi_1 S}, c_{\pi_1 S} \rangle = -\frac{1}{2} \sum_{i \in I} \mathrm{Or}\,(x, g_i x, h_i x)\,,$$

for an arbitrary $x \in S^1 = \partial \mathbb{H}^2$.

Now each σ_i is an ideal triangle in Σ, so it has (signed) area $\pm\pi$ by the Gauß–Bonnet Theorem. Since f is a pleated map, the (compact) triangle $f(\sigma_i)$ also has area $\pm\pi$, and its orientation is the same as that of the triple $(x, g_i x, h_i x)$ (for a fixed $x \in S^1$). Therefore, there is an equality

$$\mathrm{area}\,(f\,(\sigma_i)) = \pi\,\mathrm{Or}\,(x, g_i x, h_i x)\,.$$

It follows that

$$\begin{aligned}
\mathrm{area}(\alpha) &= \frac{1}{n(\Sigma)}\,\mathrm{area}\,(f_*[\Sigma]) = \frac{1}{n(\Sigma)} \sum_{i \in I} \mathrm{area}\,(f\,(\sigma_i)) \\
&= \frac{\pi}{n(\Sigma)} \sum_{i \in I} \mathrm{Or}\,(x, g_i x, h_i x) = -\frac{2\pi}{n(\Sigma)} \left\langle \mathrm{eu}_{\mathbb{R}}^{b}(S), f_*[\Sigma] \right\rangle \\
&= -2\pi \left\langle \mathrm{eu}_b^{\mathbb{R}}(S), \alpha \right\rangle.
\end{aligned}$$

□

A class $\alpha \in H_2\,(S, \gamma; \mathbb{R})$ is said to be *projectively represented by a positive immersion* if there is an admissible surface $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ with $f_*[\Sigma] = n(\Sigma)\alpha$ for some $n(\Sigma) \in \mathbb{N}_{\geqslant 1}$, and such that f is an orientation-preserving immersion.

The following is now a straightforward generalisation of a result of Calegari [16, Lemma 4.62]:

Theorem V.1.10 (Extremality of the bounded Euler class). *Let $\gamma : \coprod S^1 \to S$ be a family of geodesic loops in a compact hyperbolic surface S. Let $\alpha \in H_2(S, \gamma; \mathbb{R})$ be projectively represented by a positive immersion $f : (\Sigma, \partial\Sigma) \looparrowright (S, \gamma)$. Then*

$$\|\alpha\|_1 = \frac{-2\chi^-(\Sigma)}{n(\Sigma)} = -2 \left\langle \mathrm{eu}_b^{\mathbb{R}}(S), \alpha \right\rangle.$$

In other words, f is an extremal surface and $-\,\mathrm{eu}_b^{\mathbb{R}}(S)$ is an extremal class for α.

In particular, $\|\alpha\|_1 \in \mathbb{Q}$.

Proof. Note that Σ inherits a hyperbolic structure from S for which f is a local isometric embedding, and $\mathrm{area}(\Sigma) = n(\Sigma)\,\mathrm{area}(\alpha)$ (see Remark V.1.4). By the Gauß–Bonnet Theorem,

$$-2\pi\chi^-(\Sigma) = -2\pi\chi(\Sigma) = \mathrm{area}(\Sigma) = n(\Sigma)\,\mathrm{area}(\alpha).$$

Therefore (using the topological interpretation of $\|\cdot\|_1$ — see Proposition I.2.11),

$$\|\alpha\|_1 \leqslant \frac{-2\chi^-(\Sigma)}{n(\Sigma)} = \frac{1}{\pi}\operatorname{area}(\alpha) = -2\left\langle \mathrm{eu}_b^{\mathbb{R}}(S), \alpha \right\rangle,$$

where the last equality follows from Lemma V.1.9. We have $\left\|\mathrm{eu}_b^{\mathbb{R}}(S)\right\|_\infty \leqslant \frac{1}{2}$, so Bavard duality for $\|\cdot\|_1$ (Theorem III.2.4) gives

$$-2\left\langle \mathrm{eu}_b^{\mathbb{R}}(S), \alpha \right\rangle \leqslant \frac{\left\langle -\mathrm{eu}_b^{\mathbb{R}}(S), \alpha \right\rangle}{\left\|\mathrm{eu}_b^{\mathbb{R}}(S)\right\|_\infty} \leqslant \|\alpha\|_1 . \qquad \square$$

V.1.f Rotation quasimorphism and scl

We finish this section by explaining how Theorem V.1.10 specialises to Calegari's statement [15] on the rotation quasimorphism in the case of compact hyperbolic surfaces with non-empty boundary.

Ghys' bounded Euler class [42] was introduced as a generalisation of Poincaré's rotation number [61, 62] containing information about the dynamics of an entire group (rather than a single element) acting on the circle. The rotation number gives rise to a quasimorphism in compact hyperbolic surfaces with non-empty boundary. For the purpose of deducing a statement on scl from Theorem V.1.10, we follow the anachronistic approach of defining the rotation quasimorphism in terms of the bounded Euler class.

Let S be an oriented compact hyperbolic surface with non-empty boundary. Recall from Remark III.2.3 that there is an exact sequence

$$0 \to \operatorname{Hom}(\pi_1 S; \mathbb{R}) \to Q(\pi_1 S) \xrightarrow{[d^2-]} H_b^2(S;\mathbb{R}) \to H^2(S;\mathbb{R}).$$

Since $\partial S \neq \varnothing$, the degree-2 cohomology group $H^2(S;\mathbb{R})$ vanishes and the map $\left[d^2-\right] : Q(\pi_1 S) \to H_b^2(S;\mathbb{R})$ is therefore surjective. In particular, the bounded Euler class $\mathrm{eu}_b^{\mathbb{R}}(S) \in H_b^2(S;\mathbb{R})$ admits preimages in $Q(\pi_1 S)$.

Definition V.1.11. Given an oriented, compact, connected, hyperbolic surface S with non-empty boundary, a *rotation quasimorphism* on S is an element $r \in Q(\pi_1 S)$ satisfying

$$-\left[d^2 r\right] = \mathrm{eu}_b^{\mathbb{R}}(S) \in H_b^2(S;\mathbb{R}).$$

Different rotation quasimorphisms on S differ by homomorphisms to $\mathbb{R}$. In particular, they all agree on the commutator subgroup $[\pi_1 S, \pi_1 S]$, and more generally on any rationally homologically trivial $[c] \in C_1^{\mathrm{conj}}(\pi_1 S; \mathbb{Z})$. We will denote by $\mathrm{rot}_S([c])$ the common value of all rotation quasimorphisms on such a homogeneous chain $[c]$.

We will sometimes make the abuse of saying that *the* rotation quasimorphism rot_S is extremal for a given rationally homologically trivial conjugacy class $[c]$ — this should be understood as meaning that *every* rotation quasimorphism is extremal for $[c]$.

We will make frequent use of the following characterisation:

Lemma V.1.12 (Computation of the rotation quasimorphism)**.** *Let S be an oriented, compact, connected, hyperbolic surface with non-empty boundary. Consider a rationally homologically trivial map $\gamma : \coprod S^1 \to S$ representing $[c] \in C_1^{\mathrm{conj}}(\pi_1 S; \mathbb{Z})$. Then*

$$\mathrm{rot}_S([c]) = -\left\langle \mathrm{eu}_b^{\mathbb{R}}(S), \alpha \right\rangle,$$

where α is the unique element of $H_2(S, \gamma; \mathbb{R})$ such that $\partial\alpha = \left[\coprod S^1\right]$ for the exact sequence of Proposition I.2.2(i)*.*

Proof. Pick a rotation quasimorphism $r \in Q(\pi_1 S)$. The image of c in $C_1(\pi_1 S; \mathbb{R})$ is the boundary of a 2-chain $a \in C_2(\pi_1 S; \mathbb{R})$ corresponding to a singular chain $a_{\mathrm{sg}} \in C_2^{\mathrm{sg}}(S; \mathbb{R})$ which represents $\alpha \in H_2(S, \gamma; \mathbb{R})$. Therefore

$$\mathrm{rot}_S([c]) = r(c) = \langle r, d_2 a \rangle = \langle d^2 r, a \rangle = \langle [d^2 r], \alpha \rangle = -\left\langle \mathrm{eu}_b^{\mathbb{R}}(S), \alpha \right\rangle. \qquad \square$$

We refer the reader to [43, §5.1], [36, Chapter 10], [12], and [16, §2.3.3] for more on rotation numbers and their connection with the bounded Euler class.

We can now specialise Theorem V.1.10 to a statement about scl in the case of surfaces with non-empty boundary.

We say that a family of loops $\gamma : \coprod S^1 \to S$ *projectively bounds a positive immersion* if there is an scl-admissible surface $f : (\Sigma, \partial\Sigma) \to (S, \gamma)$ which is an orientation-preserving immersion (here, *immersion* is understood in the sense of differential topology [69, p.21]).

Corollary V.1.13 (Calegari [15])**.** *Let $\gamma : \coprod S^1 \to S$ be a family of geodesic loops in an oriented, compact, connected, hyperbolic surface S with non-empty boundary, and let $[c] \in C_1^{\mathrm{conj}}(\pi_1 S; \mathbb{Z})$ be the conjugacy class represented by γ. If γ projectively bounds a positive immersion $f : (\Sigma, \partial\Sigma) \looparrowright (S, \gamma)$, then*

$$\mathrm{scl}_{\pi_1 S}([c]) = \frac{-\chi^-(\Sigma)}{2n(\Sigma)} = \frac{\mathrm{rot}_S([c])}{2}.$$

In other words, f is an extremal surface and rot_S *is an extremal quasimorphism for* $[c]$.

In particular, $\mathrm{scl}_{\pi_1 S}([c]) \in \mathbb{Q}$.

Remark V.1.14. (i) The assumption that γ projectively bounds a positive immersion in Corollary V.1.13 implies that $\mathrm{scl}_S(\gamma) < \infty$, and therefore the quantity $\mathrm{rot}_S([c])$ is well-defined as explained above.

(ii) Pick a rotation quasimorphism $r \in Q(\pi_1 S)$. Combining Corollary V.1.13 with Bavard duality for scl (Theorem III.1.1) yields $\frac{r([c])}{2} = \mathrm{scl}_{\pi_1 S}([c]) \geqslant \frac{r([c])}{2D(r)}$. It follows that $D(r) \geqslant 1$. On the other hand, we have

$$D(r) \leqslant 2 \left\| \mathrm{eu}_b^{\mathbb{R}}(S) \right\|_\infty \leqslant 1$$

by [16, Lemma 2.58]. This shows that every rotation quasimorphism on S has defect 1 and is indeed extremal in the setting of Corollary V.1.13.

Proof of Corollary V.1.13. Let $\alpha = \frac{1}{n(\Sigma)} f_* [\Sigma] \in H_2(S, \gamma; \mathbb{Q})$. Since $H_2(S) = 0$, we have $\mathrm{scl}_{\pi_1 S}([c]) = \frac{1}{4}\|\alpha\|_1$ by Proposition I.2.12. Therefore, Theorem V.1.10 and Lemma V.1.12 imply

$$\mathrm{scl}_{\pi_1 S}([c]) = \frac{-\chi^-(\Sigma)}{2n(\Sigma)} = -\frac{1}{2}\left\langle \mathrm{eu}_b^{\mathbb{R}}(S), \alpha \right\rangle = \frac{\mathrm{rot}_S([c])}{2}. \qquad \square$$

Remark V.1.15. The converse of Corollary V.1.13 is also true: if $\mathrm{scl}_{\pi_1 S}([c]) = \frac{\mathrm{rot}_S([c])}{2}$, then γ projectively bounds a positive immersion [16, Lemma 4.62]. However, this uses the existence of extremal surfaces for $\mathrm{scl}_{\pi_1 S}([c])$ (see [16, Remark 4.65]), which is not known if S is closed (even for the relative Gromov seminorm), so this might not generalise to a converse of Theorem V.1.10.

V.2 Isometric embeddings and the bounded Euler class

As opposed to extremal surfaces, extremal quasimorphisms achieving the supremum in Bavard duality (Theorem III.1.1) exist for all conjugacy classes of boundaries [16, Proposition 2.88], and the same is true of extremal classes achieving the supremum in Theorem III.2.4 by the Hahn–Banach Theorem. However, finding an explicit extremal quasimorphism or bounded cohomology class is usually a hard problem. Theorem V.1.10 and Corollary V.1.13 tackle this question and say that, under a topological assumption, the rotation quasimorphism or the bounded Euler class is extremal.

In Chapter IV, we have seen that certain inclusions of surfaces are isometric for scl or $\|\cdot\|_1$. We are now concerned with understanding how the rotation quasimorphism and the bounded Euler class behave under those isometric embeddings.

V.2.a Pullback of the bounded Euler class

We start with the following observation. Note that, as we now work in a geometric setting, we need to assume that subsurfaces are convex. This is a stronger property than π_1-injectivity (since every conjugacy class in the fundamental group admits a geodesic representative).

Proposition V.2.1 (Pullback of the bounded Euler class to a subsurface)**.** *Let S be an oriented, compact, connected, hyperbolic surface, and let $T \overset{\iota}{\hookrightarrow} S$ be a convex subsurface. Fix a family of geodesic loops $\gamma : \coprod S^1 \to T$ representing $[c] \in C_1^{\mathrm{conj}}(\pi_1 T; \mathbb{Z})$.*

(i) *For all $\alpha \in H_2(T, \gamma; \mathbb{Q})$, we have $\langle \mathrm{eu}_b^{\mathbb{R}}(T), \alpha \rangle = \langle \mathrm{eu}_b^{\mathbb{R}}(S), \iota_* \alpha \rangle$.*

In other words, $\iota^* \mathrm{eu}_b^{\mathbb{R}}(S) = \mathrm{eu}_b^{\mathbb{R}}(T)$.

(ii) *Assume that $\partial S \neq \varnothing$, and that $[c]$ is rationally homologically trivial in T. Then $\mathrm{rot}_T([c]) = \mathrm{rot}_S([\iota_* c])$.*

In other words, $\iota^* \mathrm{rot}_S = \mathrm{rot}_T$.

Proof. (i) Pick a cell structure on S as in §V.1.c and such that T is a subcomplex of S. Let $a = \sum_\sigma \lambda_\sigma \sigma \in Z_2^{\mathrm{cell}}(T, \gamma(\coprod S^1); \mathbb{Q})$ be the unique cellular relative 2-cycle representing the image of α in $H_2(T, \gamma(\coprod S^1); \mathbb{Q})$. Note that the image $\iota_* a \in Z_2^{\mathrm{cell}}(S, \gamma(\coprod S^1); \mathbb{Q})$ represents the image of $\iota_* \alpha$ in $H_2(S, \gamma(\coprod S^1); \mathbb{Q})$, and Definition V.1.3 says that

$$\mathrm{area}(\alpha) = \sum_\sigma \lambda_\sigma \,\mathrm{area}(\sigma) = \mathrm{area}(\iota_* \alpha).$$

Now it follows from Lemma V.1.9 that

$$\left\langle \mathrm{eu}_b^{\mathbb{R}}(T), \alpha \right\rangle = -\frac{1}{2\pi} \mathrm{area}(\alpha) = -\frac{1}{2\pi} \mathrm{area}(\iota_* \alpha) = \left\langle \mathrm{eu}_b^{\mathbb{R}}(S), \iota_* \alpha \right\rangle.$$

(ii) Since $H_2(T) = H_2(S) = 0$ and γ is rationally homologically trivial in T (and therefore in S), there is a unique class $\alpha \in H_2(T, \gamma; \mathbb{Q})$ such that $\partial \alpha = [\coprod S^1]$. We also have $\partial(\iota_* \alpha) = [\coprod S^1]$. Hence, Lemma V.1.12 gives

$$\mathrm{rot}_T([c]) = -\left\langle \mathrm{eu}_b^{\mathbb{R}}(T), \alpha \right\rangle = -\left\langle \mathrm{eu}_b^{\mathbb{R}}(S), \iota_* \alpha \right\rangle = \mathrm{rot}_S([\iota_* c]). \qquad \square$$

V.2.b Positive immersions and subsurfaces

Theorem V.1.10 says that, if $\alpha \in H_2(T,\gamma;\mathbb{Q})$ is projectively represented by a positive immersion, then $-\operatorname{eu}_b^{\mathbb{R}}(T)$ is an extremal class for α, and thus

$$\|\alpha\|_1 = -2\left\langle \operatorname{eu}_b^{\mathbb{R}}(T), \alpha \right\rangle = -2\left\langle \operatorname{eu}_b^{\mathbb{R}}(S), \alpha \right\rangle \leqslant \|\iota_*\alpha\|_1 \leqslant \|\alpha\|_1 .$$

Hence, in this situation, $-\operatorname{eu}_b^{\mathbb{R}}(S)$ is also an extremal class for $\iota_*\alpha$. Therefore, we have an extremal class for $\iota_*\alpha$ which pulls back to an extremal class for α. This should be seen as a dual analogue to the situation of Corollaries IV.3.6 and IV.3.7, which gave the existence of extremal surfaces in T whose pushforward in S is also extremal.

It is natural at this point to ask whether it is equivalent for α and $\iota_*\alpha$ to be projectively represented by a positive immersion. The following proposition answers this question affirmatively:

Proposition V.2.2. *Let S be an oriented, compact, connected surface, and let $T \overset{\iota}{\hookrightarrow} S$ be a subsurface. Consider $\gamma : \coprod S^1 \to T$ and $\alpha \in H_2(T,\gamma;\mathbb{Q})$. Then the following are equivalent:*

(i) *$\alpha \in H_2(T,\gamma;\mathbb{Q})$ is projectively represented by a positive immersion (in T).*

(ii) *$\iota_*\alpha \in H_2(S,\gamma;\mathbb{Q})$ is projectively represented by a positive immersion (in S).*

Proof. It is clear that (i) $\Rightarrow$ (ii) since $T \subseteq S$, so it remains to prove that (ii) $\Rightarrow$ (i).

Assume that (ii) holds: there is an ℓ^1-admissible surface $f : (\Sigma, \partial\Sigma) \to (S,\gamma)$ for $\iota_*\alpha$ that is immersed and orientation-preserving.

The map $f : \Sigma \to S$ can be perturbed by a diffeotopy to make it *self-transverse* [69, Proposition 4.6.6] in the sense of differential topology [69, p.105]. Moreover, being an orientation-preserving immersion is an open condition, so we can assume that $f : \Sigma \to S$ is a self-transverse orientation-preserving immersion (see again [69, Proposition 4.6.6]).

Now since S is compact and f is a self-transverse immersion, we can pick a cellulation of S fine enough so that the preimage of each 2-cell σ of S is a disjoint union of subsets of Σ mapping homeomorphically under f onto σ. Pulling back this cellulation defines a cellulation of Σ for which the map $f : \Sigma \to S$ is cellular. After possibly further subdividing the cellulation, we can also assume that T is homotopic to a subcomplex of S; after relabelling, we identify T with this subcomplex. Likewise, we can assume that γ has image contained in the 1-skeleton $T^{(1)}$.

The surface Σ admits a cellular fundamental cycle whose image is a cellular 2-chain $b \in C_2^{\mathrm{cell}}(S)$ with $db \in C_1^{\mathrm{cell}}(T)$ (because $f(\partial\Sigma) \subseteq \gamma\left(\coprod S^1\right) \subseteq T$). Since f is orientation-preserving, no two 2-cells of Σ map to the same cell in S with opposite orientations — in other words, there is no cancellation. It follows that each 2-cell σ of Σ has image appearing in the support of b. But in cellular homology, b represents a class in $H_2(T, \operatorname{Im}\gamma; \mathbb{Q})$, so the image of σ must lie in T. Therefore, $f(\Sigma) \subseteq T$ and α is projectively represented by a positive immersed surface in T. $\square$

In the case where S has non-empty boundary, Calegari [16, Lemma 4.62] proved a converse to Corollary V.1.13: γ projectively bounds a positive immersion as soon as there is an equality $\mathrm{scl}_{\pi_1 S}([c]) = \frac{1}{2}\mathrm{rot}_S([c])$ (see Remark V.1.15). This gives the following:

Corollary V.2.3. *Let S be an oriented, compact, connected, hyperbolic surface with non-empty boundary, let $T \overset{\iota}{\hookrightarrow} S$ be a convex, connected subsurface, and consider $\gamma : \coprod S^1 \to T$ a rationally homologically trivial family of geodesic loops representing $[c] \in C_1^{\mathrm{conj}}(\pi_1 T; \mathbb{Z})$. Then the following are equivalent:*

(i) *γ projectively bounds a positive immersion (in T).*

(ii) *$\iota\gamma$ projectively bounds a positive immersion (in S).*

(iii) $\mathrm{scl}_{\pi_1 T}([c]) = \frac{1}{2}\mathrm{rot}_T([c])$.

(iv) $\mathrm{scl}_{\pi_1 S}([\iota_* c]) = \frac{1}{2}\mathrm{rot}_S([\iota_* c])$.

In particular, if those conditions hold, then rot_S is an extremal quasimorphism for $[\iota_* c]$ which pulls back under ι to an extremal quasimorphism for $[c]$.

Proof. (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv) follow from Calegari's Theorem [16, Lemma 4.62], while (i) $\Leftrightarrow$ (ii) is a consequence of Proposition V.2.2. $\square$

Remark V.2.4. We could also have used our isometric embedding theorem (Theorem IV.3.2) and Proposition V.2.1(ii) on the pullback of the rotation quasimorphism to prove that (iii) $\Leftrightarrow$ (iv) in Corollary V.2.3.

However, the previous proof has the advantage of being independent of Theorem IV.3.2. In fact, this gives an alternative proof that the embedding $T \hookrightarrow S$ preserves the stable commutator length of every $\gamma : \coprod S^1 \to T$ which projectively bounds a positive immersed surface in S.

CHAPTER VI

SPECTRAL GAPS VIA ANGLE STRUCTURES

The aim of this final chapter is to exhibit connections between notions of negative curvature from geometric group theory and large values of stable commutator length. In the past, such connections have often been obtained by constructing quasimorphisms and using Bavard duality — see for example [5, 19, 26, 27, 48]. Instead, our aim is to show that group-theoretic negative curvature can give rise to combinatorial notions of negative curvature on scl-admissible surfaces, which can then be used to estimate the Euler characteristic using an analogue of the Gauß–Bonnet formula. This chapter starts by introducing angle structures, which will be our carrier of combinatorial negative curvature; then we show that our method can be used to obtain sharp lower bounds for scl in free products — recovering a result of Chen [22] — and then much more generally in groups with enough letter-quasimorphisms — recovering a result of Heuer [48]. Most of the material of this chapter is taken from [59].

VI.1 Angle structures

Wise [71] introduced angle structures to study the interplay of geometric and topological properties of 2-complexes: the idea is to assign a numerical angle to each corner of the 2-complex of interest. We will work with the notion of 2-complex of §IV.2.a.

VI.1.a Definition and Gauß–Bonnet Formula

Let X be a 2-complex. We call the 0-cells of X *vertices*, its 1-cells *edges*, and its 2-cells *faces*. A *corner* of a face f at a vertex v is an edge in the link $\mathrm{Lk}_X(v)$ of v corresponding to the face f. We denote by $\mathcal{C}(f)$ the set of corners of the face f and by $\mathcal{C}(v)$ the set of corners at the vertex v.

Definition VI.1.1 (Wise [71])**.** An *angle structure* on X is the assignment of a real number $\angle c$ to each corner c in X. We then say that X is an *angled* 2*-complex*.

Note that any real angle value is allowed — in particular, there is no restriction that angles be positive.

Associated to an angle structure, there is a notion of curvature:

- Let f be a face of X. The *curvature* of f is
$$\kappa(f) := 2\pi + \sum_{c \in \mathcal{C}(f)} (\angle c - \pi)\,.$$
- Let v be a vertex of X. The *curvature* of v is
$$\kappa(v) := 2\pi - \pi \cdot \chi\left(\mathrm{Lk}_X(v)\right) - \sum_{c \in \mathcal{C}(v)} \angle c.$$

An angle structure on X is said to be *non-positively curved* if $\kappa(f) \leqslant 0$ for every face f of X and $\kappa(v) \leqslant 0$ for every vertex v of X.

The *total curvature* of an angled 2-complex X is the quantity
$$\kappa(X) := \sum_{f \in F(X)} \kappa(f) + \sum_{v \in V(X)} \kappa(v),$$
where $F(X)$ and $V(X)$ are the face set and vertex set of X, respectively.

Angle structures arise naturally — a prominent example is that of $\mathrm{CAT}(0)$ cell complexes [8, Chapter II.3]. The natural angle structure on a $\mathrm{CAT}(0)$ cell complex is non-positively curved.

The relevance of angle structures comes from the *Combinatorial Gauß–Bonnet Formula*, which constrains the topology of a 2-complex depending on its total curvature.

Proposition VI.1.2 (Combinatorial Gauß–Bonnet)**.** *Let X be an angled 2-complex. Then*

$$2\pi \cdot \chi(X) = \kappa(X).$$

Proof. By definition, the total curvature $\kappa(X)$ is equal to

$$\sum_{f\in F(X)} \left(2\pi + \sum_{c\in\mathcal{C}(f)} (\angle c - \pi)\right) + \sum_{v\in V(X)} \left(2\pi - \pi \cdot \chi\left(\mathrm{Lk}_X(v)\right) - \sum_{c\in\mathcal{C}(v)} \angle c\right).$$

Note that every corner appears as a corner of a unique face at a unique vertex, so that

$$\sum_{f\in F(X)} \sum_{c\in\mathcal{C}(f)} \angle c = \sum_{v\in V(X)} \sum_{c\in\mathcal{C}(v)} \angle c. \tag{$*$}$$

Hence,

$$\kappa(X) = 2\pi\left(\#F(X) + \#V(X)\right) - \pi \sum_{f\in F(X)} \#\mathcal{C}(f) - \pi \sum_{v\in V(X)} \chi\left(\mathrm{Lk}_X(v)\right).$$

Now the link of a vertex v is a graph, with vertices corresponding to half-edges of X incident to v, and edges corresponding to corners of X at v. It follows that

$$\sum_{v\in V(X)} \chi\left(\mathrm{Lk}_X(v)\right) = 2\#E(X) - \sum_{v\in V(X)} \#\mathcal{C}(v).$$

Similar to ($*$), we have $\sum_f \#\mathcal{C}(f) = \sum_v \#\mathcal{C}(v)$. We finally obtain

$$\kappa(X) = 2\pi\left(\#F(X) - \#E(X) + \#V(X)\right) = 2\pi \cdot \chi(X).$$ □

VI.1.b Interior curvature of surfaces

We have defined angle structures on 2-complexes, which are constructed by gluing a collection of discs to a graph. Each disc then has a notion of curvature as explained above. The objects of which we will want to estimate the curvature in the sequel will have a more natural decomposition into compact subsurfaces, rather than discs; this is our motivation for introducing the following notions.

We consider a compact surface Λ with non-empty boundary, with a cellulation whose vertex set $V(\Lambda)$ is contained in $\partial\Lambda$. Suppose that Λ is equipped with an angle structure.

- The *total angle* of a vertex v of Λ is

$$\angle_{\mathrm{tot}}^{\Lambda}(v) := \sum_{c\in\mathcal{C}(v)} \angle c.$$

The reason why we include Λ in the notation is that we will typically be considering a decomposition of a 2-complex into subsurfaces, so that a vertex can be seen as belonging to one subsurface or another.

- The *interior curvature* of Λ is

$$\kappa_{\text{int}}(\Lambda) := 2\pi \cdot \chi(\Lambda) + \sum_{v \in V(\Lambda)} \left(\angle_{\text{tot}}^{\Lambda}(v) - \pi \right).$$

Note in particular that if Λ is a disc, then $\chi(\Lambda) = 1$ and $\kappa_{\text{int}}(\Lambda)$ is the curvature of Λ seen as a face.

We'll use the following elementary observation:

Lemma VI.1.3. *Let Λ be a compact surface equipped with an angle structure supported on a cellulation of Λ with $V(\Lambda) \subseteq \partial\Lambda$. Then*

$$\kappa_{\text{int}}(\Lambda) = \sum_{f \in F(\Lambda)} \kappa(f),$$

where $F(\Lambda)$ is the face set of Λ.

Proof. Each vertex $v \in V(\Lambda)$ lies on $\partial\Lambda$, so its link is homeomorphic to a line segment, and we have

$$\kappa(v) = 2\pi - \pi \cdot \chi\left(\mathrm{Lk}_{\Lambda}(v)\right) - \sum_{c \in \mathcal{C}(v)} \angle c = \pi - \angle_{\text{tot}}^{\Lambda}(v).$$

Hence, the Gauß–Bonnet Formula (Proposition VI.1.2) gives

$$\sum_{f \in F(\Lambda)} \kappa(f) = 2\pi \cdot \chi(\Lambda) + \sum_{v \in V(\Lambda)} \left(\angle_{\text{tot}}^{\Lambda}(v) - \pi \right) = \kappa_{\text{int}}(\Lambda). \qquad \square$$

This says that, if 2-cells — which are normally discs — are replaced with more general compact surfaces with non-empty boundary, then the interior curvature is the right analogue of the curvature of a face.

Corollary VI.1.4. *Let X be an angled 2-complex. Suppose that there is a finite collection $(\Lambda_i)_{i \in I}$ of compact surfaces with non-empty boundary cellularly embedded in X, such that*

- *Each face of X is contained in a unique Λ_i, and*

- *Each vertex of X is contained in $\partial\Lambda_i$ for some (possibly several) $i \in I$.*

Then the total curvature of X can be computed via

$$\kappa(X) = \sum_{i \in I} \kappa_{\text{int}}\left(\Lambda_i\right) + \sum_{v \in V(X)} \kappa(v). \qquad \square$$

VI.2 Spectral gap in free products

VI.2.a Historical overview

One of the first major lower bound results for stable commutator length is the following:

Theorem VI.2.1 (Duncan–Howie [32]). *Let F be a free group. Then any $w \in F \smallsetminus \{1\}$ satisfies*

$$\mathrm{scl}_F(w) \geqslant \frac{1}{2}.$$

A group G in which $\mathrm{scl}_G(w) \geqslant \frac{1}{2}$ for every $w \in G \smallsetminus \{1\}$ is said to have a *strong spectral gap*. More generally, a group G is said to have a *spectral gap* at some $\varepsilon > 0$ if $\mathrm{scl}_G(w) = 0$ or $\mathrm{scl}_G(w) \geqslant \varepsilon$ for every $w \in G$. The relevance of strong spectral gaps is twofold: firstly, $\frac{1}{2}$ is the largest possible spectral gap that one can hope to find in a group with non-trivial scl:

Remark VI.2.2. If a group G has a spectral gap at $\varepsilon > \frac{1}{2}$, then scl vanishes on $[G, G]$.

Proof. Assume that G has a spectral gap at $\varepsilon > \frac{1}{2}$. Since the scl of a commutator is always at most $\frac{1}{2}$ (Example I.1.21), we must have $\mathrm{scl}_G([a, b]) = 0$ for all $a, b \in G$. Now it is a general fact that, if scl vanishes on all commutators, then it vanishes on the commutator subgroup. Indeed, pick a homogeneous quasimorphism $\phi \in Q(G)$. Bavard duality (Theorem III.1.1) implies that $\phi([a, b]) = 0$ for all $a, b \in G$. But a lemma of Bavard [4, Lemma 3.6] (see also [16, Lemma 2.24]) gives

$$D(\phi) = \sup_{a,b \in G} |\phi([a, b])| = 0,$$

so ϕ is a homomorphism. This proves that every homogeneous quasimorphism is a homomorphism, and hence vanishes on $[G, G]$; it follows from Bavard duality that scl vanishes on $[G, G]$. $\square$

Secondly, a strong spectral gap also says that the only element with vanishing scl is trivial; consequently, the property of having a strong spectral gap descends to subgroups by monotonicity of scl (Lemma I.1.19), and can therefore carry information on the subgroup structure.

Two notable generalisations of the Duncan–Howie Theorem were given by Chen [23] for free products, and by Heuer [48] for right-angled Artin groups. However, those results use three different approaches — Duncan and Howie (in modern language) show that admissible surfaces can be given an angle structure with an upper bound on the curvature, Chen's *LP duality method* [23–26] is based on ideas from rationality theorems and linear programming, while Heuer constructs quasimorphisms and uses them to give lower bounds for scl via Bavard duality.

Our contribution is to give a geometric method that applies to all three theorems. The idea is similar to Duncan and Howie's in that the method is to construct a negatively curved angle structure on admissible surfaces, but we show that it applies with the more general assumption of admitting a *letter-quasimorphism* — a notion introduced by Heuer [48] and that we define in §VI.3 — which recovers Heuer's Theorem for right-angled Artin groups [48]. It should be noted that our method is fundamentally a reformulation of Chen's LP duality method [23–26], in the sense that proofs can easily be translated between Chen's language and ours, but we believe that the use of angle structures has the advantage of making the connection with negative curvature apparent and has potential for future applications.

VI.2.b Geometric proof of the spectral gap in free products

We first show how our angle structure method can be used to recover the following theorem on spectral gaps in free products:

Theorem VI.2.3 (Chen [23]). *Let $G = A * B$ be a free product. Let $w = a_1 b_1 \cdots a_\ell b_\ell \in [G, G]$, with $a_i \in A \smallsetminus \{1\}$ of order $p_i \in \mathbb{N}_{\geqslant 1} \cup \{\infty\}$ and $b_i \in B \smallsetminus \{1\}$ of order $q_i \in \mathbb{N}_{\geqslant 1} \cup \{\infty\}$. Then*

$$\mathrm{scl}_G(w) \geqslant \frac{1}{2} - \frac{1}{N},$$

where $N := \min\{p_1, q_1, \ldots, p_\ell, q_\ell\}$.

Before proving Theorem VI.2.3, observe that it yields lower bounds for scl in free products with an arbitrary (possibly infinite) number of factors:

Corollary VI.2.4 (Chen [23]). *Let $G = \mathop{\Large *}_{\lambda \in \Lambda} G_\lambda$ be a free product of torsion-free groups. Consider an element $w \in [G, G]$ that is not conjugate into any of the G_λ's. Then*

$$\mathrm{scl}_G(w) \geqslant \frac{1}{2}.$$

Proof. Note that w can be written using letters from a finite subset M of the index set Λ, and the embedding

$$\mathop{\ast}_{\mu \in M} G_\mu \hookrightarrow \mathop{\ast}_{\lambda \in \Lambda} G_\lambda$$

is isometric (since it admits a left-inverse — see Proposition IV.1.3), so we can assume that the free product has only finitely many factors.

We argue by induction on the number n of factors. The base case $n = 2$ follows from Theorem VI.2.3. If $n > 2$, write $G = H * G_{\lambda_n}$, with $H := G_{\lambda_1} * \cdots * G_{\lambda_{n-1}}$. If w is conjugate into H, then the induction hypothesis implies that

$$\mathrm{scl}_G(w) = \mathrm{scl}_H(w) \geqslant \frac{1}{2},$$

since H is isometrically embedded into G. If w is not conjugate into H, then the result follows from applying Theorem VI.2.3 to the free product $G = H * G_{\lambda_n}$. □

We now prove Theorem VI.2.3 using angle structures; this is a geometric reformulation of Chen's proof [23]:

Proof of Theorem VI.2.3. Pick a $K(A, 1)$ space X_A and a $K(B, 1)$ space X_B. Hence, the wedge $X := X_A \vee X_B$ is a $K(G, 1)$. We denote by p the wedge point of X (which is also assumed to be the common basepoint of X, X_A, and X_B), and we consider $\gamma : S^1 \to X$ an immersed based loop at p representing w.

Now consider an scl-admissible surface $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$. Our goal is to prove that

$$-\chi^-(\Sigma) \geqslant \left(1 - \frac{2}{N}\right) n(\Sigma),$$

as this will imply the desired lower bound for scl after passing to the infimum.

Using Lemmas II.1.2 and II.1.5, we may assume that f is transverse, incompressible, and disc- and sphere-free. Each vertex disc of Σ maps to p; we collapse each of them to a point and denote by Σ_0 the resulting 2-complex. We call Σ_0 the *collapsed admissible surface*. There is a homotopy equivalence $\Sigma \to \Sigma_0$ through which the map f factors, defining $f_0 : \Sigma_0 \to X$. Consider the subcomplexes $\Sigma_0^A := f_0^{-1}(X_A)$ and $\Sigma_0^B := f_0^{-1}(X_B)$.

Define a new cellular structure on Σ_0 as follows. The preimage $f_0^{-1}(p)$ is a finite union of points in Σ_0; place a vertex at each point in this set. Next, the image of $\partial\Sigma$ in Σ_0 is a union of arcs between points in $f^{-1}(p)$; add one edge for each such arc. Finally, each of Σ_0^A and Σ_0^B decomposes as a union of surfaces with boundary contained in the 1-skeleton $\Sigma_0^{(1)}$ and meeting at vertices of Σ_0 only; call those surfaces A- and B-*surfaces* respectively. For each A- or B-surface, pick a cellulation containing the cells just added, and with vertex set contained in the boundary, and add the corresponding cells to Σ_0. This defines a cellulation of Σ_0. Importantly, Σ_0 has a decomposition into subsurfaces mapping either to X_A or to X_B, with boundary composed of edges mapping to loops representing some a_i or b_i, with $i \in \{1, \ldots, \ell\}$; we will think of a_i or b_i as the label of the corresponding edge of Σ_0 — see Figure VI.1.

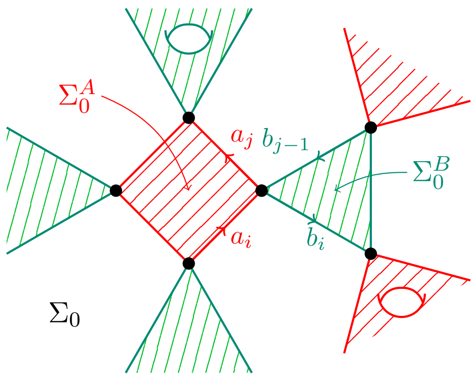


Figure VI.1: Collapsed admissible surface.

We now endow Σ_0 with an angle structure in order to estimate its Euler characteristic via the Gauß–Bonnet formula. This angle structure will be constructed by defining the total angle of each vertex inside each A- or B-surface — if one wanted to extend this to a completely defined angle structure on the cellulation of Σ_0, it would suffice to pick any choice of angles having the right angle sums — note that such a choice is possible because it amounts to solving a system of affine equations where each variable — corresponding to an angle — appears in exactly one equation — corresponding to the total angle of the vertex inside the A- or B-surface. But by Corollary VI.1.4, the choice of a cellulation of each A- or B-surface and of a precise angle at each corner are unimportant: it suffices to know the total angles in order to compute the total curvature of Σ_0.

Pick a vertex v of Σ_0, and suppose that it lies within an A-surface Λ_A. The vertex v lies between two edges labelled a_i and a_j, which are ordered by the orientation of Λ_A inherited from Σ. We then set

$$\angle_{\mathrm{tot}}^{\Lambda_A}(v) := \theta_{i,j} := \begin{cases} +\left(1 - \frac{2}{N}\right)\pi & \text{if } i \geqslant j \\ -\left(1 - \frac{2}{N}\right)\pi & \text{if } i < j \end{cases}.$$

Likewise, if v is contained in a B-surface Λ_B, then it lies between two sides labeled b_i and b_j, and we set $\angle_{\mathrm{tot}}^{\Lambda_B}(v) := \theta_{i,j}$. See Figure VI.2.

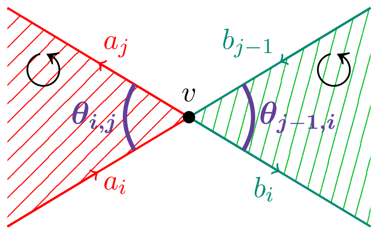


Figure VI.2: Angle structure on Σ_0.

We now estimate the curvature of vertices and A- and B-surfaces in order to apply the Gauß–Bonnet formula.

Claim. Each A- or B-surface Σ_0 has non-positive interior curvature.

Proof of the claim. As the situation is symmetric, it suffices to prove the claim for an A-surface Λ_A. Recall from §VI.1.b that the interior curvature of Λ_A is defined by

$$\kappa_{\mathrm{int}}(\Lambda_A) = 2\pi \cdot \chi(\Lambda_A) + \sum_{v \in V(\Lambda_A)} \left(\angle_{\mathrm{tot}}^{\Lambda_A}(v) - \pi\right).$$

Since each total angle is at most π, it is clear that $\kappa_{\mathrm{int}}(\Lambda_A) \leqslant 0$ as soon as $\chi(\Lambda_A) \leqslant 0$. But Λ_A is a surface with non-empty boundary (as Σ was sphere-free), so the only case where $\chi(\Lambda_A) > 0$ is if Λ_A is a disc, and thus $\chi(\Lambda_A) = 1$.

We can now assume that Λ_A is a disc, with boundary edges labelled by $a_{i_1}, \dots, a_{i_k}$, in clockwise order. The total angles of the corners of Λ_A are $\theta_{i_1,i_2}, \dots, \theta_{i_{k-1},i_k}, \theta_{i_k,i_1}$, and we have

$$\kappa_{\mathrm{int}}(\Lambda_A) = 2\pi + \sum_{j=1}^{k} \left(\theta_{i_j,i_{j+1}} - \pi\right),$$

with cyclic notations: $i_{k+1} = i_1$. There are two cases:

- $i_1 = \dots = i_k$. In this case, we have $\theta_{i_j,i_{j+1}} = \pi - \frac{2\pi}{N}$ for all j, and $\kappa_{\mathrm{int}}(\Lambda_A) = 2\pi\left(1 - \frac{k}{N}\right)$. But the restriction of f_0 to the disc Λ_A defines a homotopy from a loop labeled $a_{i_1}^k$ to a point, so that $a_{i_1}^k = 1$ in A, and therefore $k \geqslant N$ by definition of N. Hence, $\kappa_{\mathrm{int}}(\Lambda_A) \leqslant 0$.

- The i_j's are not all equal. Then there must be some index $u \in \{1, \dots, k\}$ such that $i_u \geqslant i_{u+1}$, and some index v such that $i_v < i_{v+1}$ (with cyclic notations). Since no total angle in Λ_A is greater than π, it follows that

$$\kappa_{\mathrm{int}}(\Lambda_A) \leqslant 2\pi + \left(\theta_{i_u,i_{u+1}} - \pi\right) + \left(\theta_{i_v,i_{v+1}} - \pi\right) = 0. \qquad \square$$

By the above claim, the Gauß–Bonnet formula (Proposition VI.1.2), together with Corollary VI.1.4, implies an upper bound on the Euler characteristic of Σ_0 (which is equal to that of Σ):

$$2\pi\chi(\Sigma) = 2\pi\chi(\Sigma_0) \leqslant \sum_{v\in V(\Sigma_0)} \kappa(v). \tag{†}$$

It remains to estimate the curvature of vertices of Σ_0.

Claim. There is an equality $\sum_v \kappa(v) = -2\pi\left(1-\frac{2}{N}\right)n(\Sigma)$.

Proof. Because Σ is an scl-admissible surface, successive edges on $\partial\Sigma$ must be labelled by successive letters in $w = a_1 b_1 \cdots a_\ell b_\ell$. During the collapsing process that we performed to obtain Σ_0, it might be that some vertices were identified. However, if this happens, then the link of the resulting vertex is the disjoint union of the links of all the vertices that were identified, so these identifications have no impact on the value of $\sum_v \kappa(v)$. We can therefore assume that each vertex of Σ_0 is of the form depicted in Figure VI.2 — which is what one obtains if there is no vertex identification — for some values of i and j in $\{1,\dots,\ell\}$, where we work with cyclic notations and take the convention that $b_0 = b_\ell$. The curvature of such a vertex v is

$$\kappa(v) = 2\pi - \pi\cdot\underbrace{\chi\left(\mathrm{Lk}_{\Sigma_0}(v)\right)}_{=2} - \theta_{i,j} - \theta_{j-1,i} = -\theta_{i,j} - \theta_{j-1,i}.$$

- If $j \neq 1$, then $i \geqslant j$ if and only if $j < i-1$, and $\theta_{i,j} = -\theta_{j,i-1}$, so $\kappa(v) = 0$.
- If $j = 1$, then
$$\kappa(v) = -\theta_{i,1} - \theta_{\ell,i} = -2\pi\left(1-\frac{2}{N}\right).$$

Note that exactly $\frac{1}{\ell}$ of the vertices have $j = 1$. Since the total number of vertices is $\ell\cdot n(\Sigma)$ (working, again, in the scenario where no vertex identification occurs during the collapse), we obtain

$$\sum_{v\in V(\Sigma_0)} \kappa(v) = \frac{\ell\cdot n(\Sigma)}{\ell}(-2\pi)\left(1-\frac{2}{N}\right) = -2\pi\left(1-\frac{2}{N}\right)n(\Sigma). \qquad \square$$

It now follows from (†) that

$$-\chi^-(\Sigma) \geqslant -\chi(\Sigma) = -\chi(\Sigma_0) \geqslant \left(1-\frac{2}{N}\right)n(\Sigma),$$

as wanted. $\square$

VI.3 Letter-quasimorphisms

Letter-quasimorphisms were introduced by Heuer [48] as a means of obtaining spectral gaps for scl in certain amalgams and right-angled Artin groups.

We denote by $\mathcal{A}$ the set of *alternating words* in the free group $F_2 := F(a,b)$:

$$\mathcal{A} := \left\{ b_0 a_1 b_1 \cdots a_\ell b_\ell a_{\ell+1} \in F_2 \,\middle|\, a_1, \dots, a_\ell \in \left\{a^{\pm 1}\right\}, \right.$$
$$\left. b_1, \dots, b_\ell \in \left\{b^{\pm 1}\right\},\ b_0 \in \left\{1, b^{\pm 1}\right\},\ a_{\ell+1} \in \left\{1, a^{\pm 1}\right\} \right\}.$$

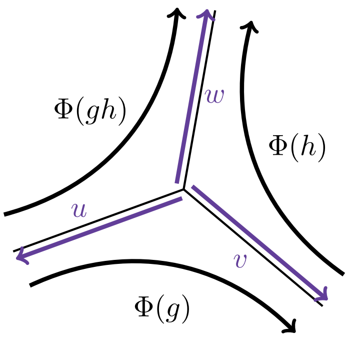
Definition VI.3.1 (Heuer [48]). A *letter-quasimorphism* on a group G is a map $\Phi : G \to \mathcal{A}$ subject to the following conditions:

(i) $\Phi\left(g^{-1}\right) = \Phi\left(g\right)^{-1}$ for every $g \in G$, and

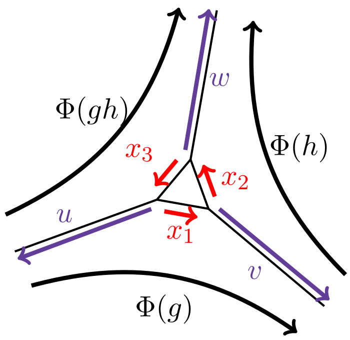
(ii) For every $g, h \in G$, either

  (a) $\Phi\left(gh\right) = \Phi\left(g\right)\Phi\left(h\right)$ — in this case, paths labelled by $\Phi(g)$, $\Phi(h)$, and $\Phi(gh)$ form tripods in the Cayley graph of $F(a,b)$, as shown in Figure VI.3a — or

  (b) $\Phi(g) = u^{-1}x_1 v$, $\Phi(h) = v^{-1}x_2 w$, and $\Phi\left(gh\right)^{-1} = w^{-1}x_3 u$ as reduced expressions, for some alternating words $u, v, w \in \mathcal{A}$, and letters x_1, x_2, x_3 such that either $x_1, x_2, x_3, x_1x_2x_3 \in \left\{a^{\pm 1}\right\}$, or $x_1, x_2, x_3, x_1x_2x_3 \in \left\{b^{\pm 1}\right\}$ — this situation is depicted in Figure VI.3b, but it should be noted that, contrary to VI.3a, this picture does not live in the Cayley graph of $F(a,b)$ but is merely a diagrammatic representation of the above equalities.



(a) Degenerate case.



(b) Non-degenerate case.

Figure VI.3: Letter-quasimorphism condition.

Remark VI.3.2. Let $\Phi : G \to \mathcal{A}$ be a letter-quasimorphism and let $g, h \in G$ be such that $\Phi(gh) = \Phi(g)\Phi(h)$. Write $\Phi(g) = u^{-1}v$, $\Phi(h) = v^{-1}w$, and $\Phi(gh)^{-1} = w^{-1}u$ as reduced words, as in Figure VI.3a. Then in fact one of u, v, and w must be trivial. Indeed, otherwise, since $\Phi(g)$ is a reduced alternating word, if for example the first letter of u is a or a^{-1}, then the first letter of v is b or b^{-1}. Similarly, $\Phi(h)$ is an alternating word, so the first letter of w must be a or a^{-1}. But now both u and w start with a or a^{-1}, which contradicts $\Phi(gh)$ being reduced and alternating.

We will need groups to have enough letter-quasimorphisms in the following sense:

Definition VI.3.3. We will say that a group G is *quasi-residually free* if for every $g \in G$ with $\mathrm{scl}_G(g) < \infty$, there is a letter-quasimorphism $\Phi : G \to \mathcal{A}$ such that $\Phi(g) \neq 1$, and $\Phi(g^n) = \Phi(g)^n$ for all $n \in \mathbb{Z}$.

Example VI.3.4 (Heuer [48]). The following groups are quasi-residually free:

(i) Residually free groups [48, Example 4.2],

(ii) Right-angled Artin groups [48, Theorem 7.2].

Heuer [48] showed that quasi-residually free groups have a strong spectral gap for scl. To do this, he explains how one can use a letter-quasimorphism $G \to \mathcal{A}$ to construct a classical, real-valued quasimorphism $G \to \mathbb{R}$; he then uses Bavard Duality to obtain lower bounds on scl.

We take a different route and show that a letter-quasimorphism gives rise to an angle structure on admissible surfaces, which can in turn be used to estimate the Euler characteristic. This approach is closer in spirit to Duncan and Howie's strong spectral gap in free groups [32], and is inspired by the above proof of Theorem VI.2.3.

VI.4 From letter-quasimorphisms to angle structures

We now reach the core of this chapter. The goal is to prove the following:

Theorem VI.4.1 (Angle structures from letter-quasimorphims)**.** *Let X be a connected 2-complex and let $g \in \pi_1 X \smallsetminus \{1\}$, represented by an immersed loop $\gamma : S^1 \to X$. Assume that there is a letter-quasimorphism $\Phi : \pi_1 X \to \mathcal{A}$ with $\Phi(g) \neq 1$ and $\Phi(g^n) = \Phi(g)^n$ for all $n \in \mathbb{Z}$.*

Then given a monotone, incompressible, disc- and sphere-free scl*-admissible surface $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$, there is an angle structure on Σ whose total curvature satisfies*

$$\kappa(\Sigma) \leqslant -2\pi \cdot n(\Sigma).$$

Before entering the proof of Theorem VI.4.1, we explain its implications for scl:

Corollary VI.4.2 (Heuer [48, Theorem 4.7]). *Let G be a group and $g \in G \smallsetminus \{1\}$. Assume that there is a letter-quasimorphism $\Phi : G \to \mathcal{A}$ with $\Phi(g) \neq 1$ and $\Phi(g^n) = \Phi(g)^n$ for all $n \in \mathbb{Z}$.*

Then $\mathrm{scl}_G(g) \geqslant \frac{1}{2}$.

Proof. Fix X a connected 2-complex with $\pi_1 X \cong G$ (for example, X can be a presentation 2-complex of G). Lemma II.1.4 says that scl can be computed as the infimum of $\frac{-\chi^-(\Sigma)}{2n(\Sigma)}$ over all monotone, incompressible, disc- and sphere-free admissible surfaces. For any such admissible surface Σ, Theorem VI.4.1 gives an angle structure on Σ with a bound on the total curvature. Now the Gauß–Bonnet Formula (Proposition VI.1.2) translates the bound on $\kappa(\Sigma)$ into a bound on $\chi(\Sigma)$, and one obtains

$$-\chi^-(\Sigma) \geqslant -\chi(\Sigma) = -\frac{1}{2\pi}\kappa(\Sigma) \geqslant n(\Sigma).$$

After passing to the infimum, this implies that $\mathrm{scl}_G(g) \geqslant \frac{1}{2}$. ☐

Corollary VI.4.3 (Heuer [48]). *Every quasi-residually free group has a strong spectral gap for* scl. ☐

The rest of this section is devoted to proving the main theorem.

Proof of Theorem VI.4.1. Note that $\Phi(g)$ is an alternating word in $F_2 = F(a, b)$. Furthermore, the assumption that $\Phi(g^n) = \Phi(g)^n$ for all n, and the fact that each $\Phi(g^n)$ is an alternating word implies that $\Phi(g)$ is cyclically reduced, and cannot start and end in the same letter. Let us assume to fix notations that it starts with a or a^{-1} and ends with b or b^{-1}, so that

$$\Phi(g) = a_1 b_1 \cdots a_\ell b_\ell,$$

with $a_i \in \{a^{\pm 1}\}$ and $b_i \in \{b^{\pm 1}\}$.

Start with $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$ a monotone, incompressible, disc- and sphere-free admissible surface with $f_*[\partial\Sigma] = n(\Sigma)[S^1]$ for some $n(\Sigma) \in \mathbb{N}_{\geqslant 1}$.

After contracting a spanning tree in the 1-skeleton, one may assume that X has only one vertex. One may also subdivide the faces of X to ensure that they are all triangles. After applying Proposition II.1.5, we can further assume that f is transverse.

VI.4.a Subdividing 1-handles via the letter-quasimorphism

Each 1-handle $\mathcal{H}$ in Σ is a trivial I-bundle over an edge e of X, which is a loop and represents an element $g_{\mathcal{H}} \in \pi_1 X$. We consider the image of $g_{\mathcal{H}}$ under our letter-quasimorphism $\Phi : \pi_1 X \to \mathcal{A}$; this is an alternating word in F_2, and we write for example

$$\Phi(g_{\mathcal{H}}) = \alpha_1 \beta_1 \cdots \alpha_k \beta_k, \tag{‡}$$

with $\alpha_i \in \{a^{\pm 1}\}$ and $\beta_i \in \{b^{\pm 1}\}$. Note that $\Phi(g_{\mathcal{H}})$ might have odd length and might not be cyclically reduced, but the construction remains the same. The element $\Phi(g_{\mathcal{H}}) \in F_2$ is represented by an immersed loop $\gamma_{\Phi(g_{\mathcal{H}})} : S^1 \looparrowright B_2$, where $B_2 := S^1_a \vee S^1_b$ is a bouquet of two oriented circles labelled by a and b.

The 1-handle $\mathcal{H}$ being a trivial I-bundle over e means that it is homeomorphic to the product $I \times e$ (where $I := [0, 1]$), and the map $\mathcal{H} \to e$ is given by projection onto the second coordinate. Now pick an orientation-preserving homeomorphism $e \cong S^1$ and consider the composition

$$\mathcal{H} = I \times e \xrightarrow{\text{proj}_2} e \cong S^1 \xrightarrow{\gamma_{\Phi(g_{\mathcal{H}})}} B_2 = S^1_a \vee S^1_b.$$

This map is transverse, and defines a decomposition of $\mathcal{H}$ into $2k$ 1-handles (if $\Phi(g_{\mathcal{H}})$ has even length as in (‡)) mapping successively to $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$. To distinguish those 1-handles from the original ones, we call them *stripes*. Stripes can be divided into two types:

- a-*stripes*, with image contained in S^1_a, and
- b-*stripes*, with image contained in S^1_b.

We'll depict those two types of stripes with two different colours (red and green). The map from each stripe to S^1_a or S^1_b will be encoded in pictures by arrows parallel to the base edge of the I-bundle, indicating the positive direction of S^1_a or S^1_b — see Figure VI.4.

VI.4.b Extension to cellular discs

The above construction defines a new cellular structure on the 1-handles of Σ, and we now want to do something similar to cellular discs. At the end of the construction, Σ will be decomposed into its preexisting vertex discs, a-regions (extending a-stripes), and b-regions (extending b-stripes).

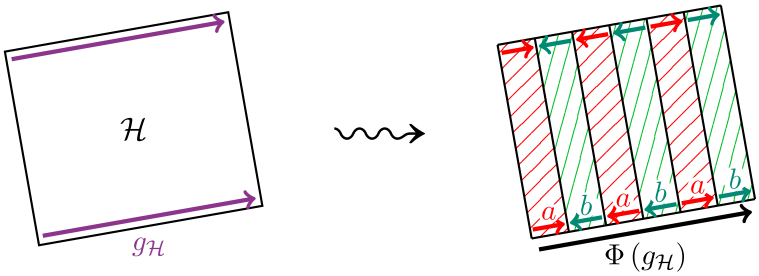


Figure VI.4: Decomposition of a 1-handle into stripes via the letter-quasimorphism. Here, $\Phi(g_{\mathcal{H}}) = ab^{-1}a^{-1}b^{-1}ab$.

Each cellular disc $\mathcal{D}$ maps homeomorphically to a (triangular) 2-cell of X, and has three sides mapping under f to edges representing elements $g_1, g_2, g_3 \in \pi_1 X$. The presence of the 2-cell means that $g_3 = (g_1 g_2)^{-1}$. Now the three 1-handles incident to $\mathcal{D}$ have been subdivided as explained above into parallel stripes labelled by $a^{\pm 1}$ and $b^{\pm 1}$, and the concatenation of those labels along each of the three edges of the triangle are given by $\Phi(g_1)$, $\Phi(g_2)$, and $\Phi(g_1 g_2)^{-1}$. By the letter-quasimorphism condition, they satisfy one of the patterns of Figure VI.3: we can write

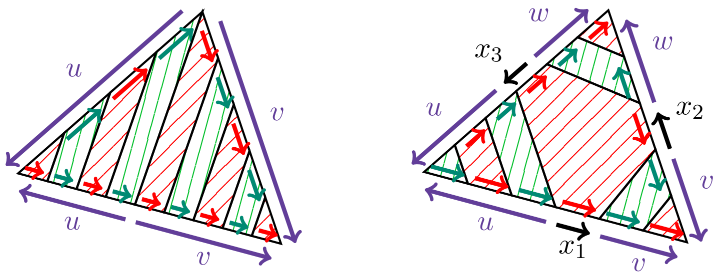
$$\Phi(g_1) = u^{-1} x_1 v, \quad \Phi(g_2) = v^{-1} x_2 w, \quad \Phi(g_1 g_2)^{-1} = w^{-1} x_3 u,$$

with either $x_1 = x_2 = x_3 = 1$, or $x_1, x_2, x_3, x_1x_2x_3 \in \{a^{\pm 1}\}$, or $x_1, x_2, x_3, x_1x_2x_3 \in \{b^{\pm 1}\}$. Moreover, Remark VI.3.2 says that, in the degenerate case (when $x_1 = x_2 = x_3 = 1$), we have $u = 1$ or $v = 1$ or $w = 1$ — we will assume that $w = 1$ to fix notations.

As illustrated by Figure VI.5, the boundary of $\mathcal{D}$ has two consecutive sections labelled respectively by u and u^{-1}, and these sections lie on different sides of $\mathcal{D}$. We can connect the vertices of these two sections and extend the a- and b-stripes across part of $\mathcal{D}$. We then apply the same process to v and w. The new stripes that we have constructed inside $\mathcal{D}$ are called a- or b-stripes depending on whether they extend a- or b-stripes. Note that some of the new stripes, at the vertices of the cellular disc, are triangular (one of the fibres of the I-bundle is collapsed to a point), but we still call them stripes.

In the degenerate case (Figure VI.5a), the new stripes now fill $\mathcal{D}$. In the non-degenerate case (Figure VI.5b), we are left with one unfilled hexagon inside $\mathcal{D}$; this hexagon has three sides lying on $\partial\mathcal{D}$ and mapping either all to S^1_a or all to S^1_b. We declare the hexagon to be of type a or b accordingly.

This construction yields a new cellular structure on Σ, with a decomposition into



(a) Degenerate case.

(b) Non-degenerate case.

Figure VI.5: Extension of stripes across cellular discs.

vertex discs (which remain unchanged), a- and b-stripes, and a- and b-hexagons — see Figure VI.6. We call this data a *stripe pattern*.

It is important to note that the new cellular structure does not correspond to any transverse map to a bouquet of two circles: such a map can be defined on a- and b-stripes as explained above, but it cannot be extended to hexagons since their boundary maps to a non-trivial element of F_2 (either $a^{\pm 1}$ or $b^{\pm 1}$).

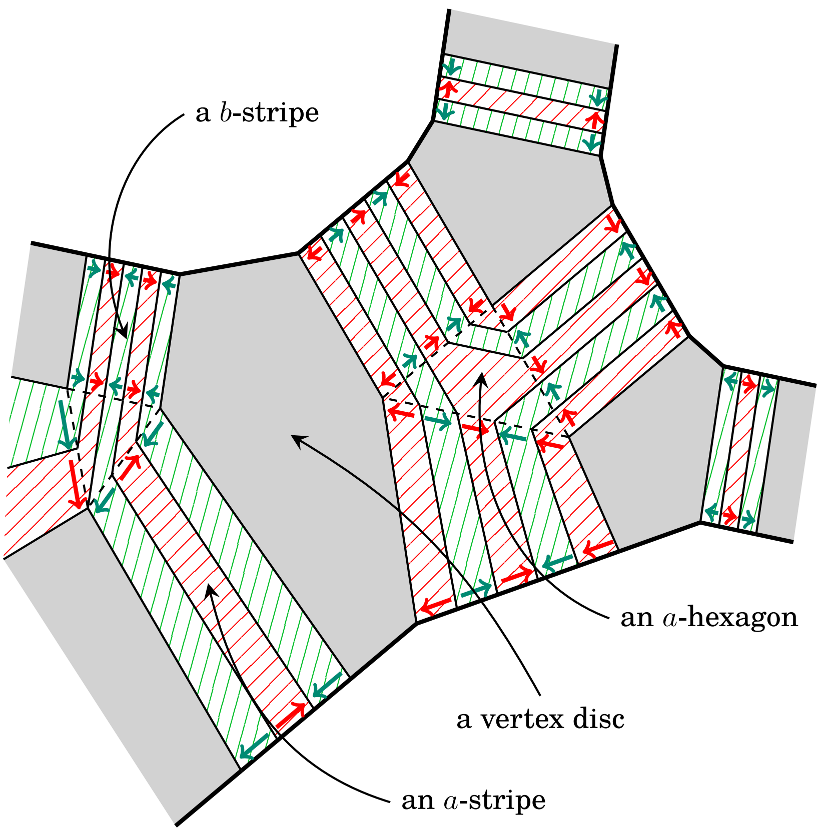


Figure VI.6: Part of a stripe pattern.

VI.4.c Boundary labelling

We now turn our attention to the boundary of Σ. The (oriented) boundary edges of Σ are either part of a vertex disc, or they bound a stripe and are then labelled by some letter in $\{a^{\pm 1}, b^{\pm 1}\}$. Moreover, the concatenation of the oriented labels of the boundary edges along a 1-handle $\mathcal{H}$ of Σ mapping via f to $g_{\mathcal{H}}$ is the (reduced, but possibly not cyclically reduced) word $\Phi(g_{\mathcal{H}})$ (see Figure VI.4). Now pick a boundary component ∂_j of Σ. The image of ∂_j under f represents a word $g_1 \cdots g_\ell = g^k$ for some $k \in \mathbb{N}_{\geqslant 1}$, where each g_i is an element in $\pi_1 X$ represented by a single edge loop in X, and g is the element of $\pi_1 X$ with respect to which $f : \Sigma \to X$ is an admissible surface. It follows that the labelling of ∂_j obtained by concatenating the labels of the a- and b-stripes along it is a (not necessarily reduced) word representing $\Phi(g_1) \cdots \Phi(g_\ell)$.

We want to modify Σ so that the labelling of ∂_j becomes $\Phi(g_1 \cdots g_\ell) = \Phi(g)^k$. We will use the letter-quasimorphism condition to do so. Consider two successive sections of ∂_j that are labelled by $\Phi(g_i)$ and $\Phi(g_{i+1})$. Definition VI.3.1 says that either $\Phi(g_i)\Phi(g_{i+1}) = \Phi(g_i g_{i+1})$ (in the degenerate case), or there are alternating words $u, v, w \in F_2$, and letters x_1, x_2, x_3 in $\{a^{\pm 1}\}$ or $\{b^{\pm 1}\}$ such that

$$\Phi(g_i) = u^{-1} x_1 v, \quad \Phi(g_{i+1}) = v^{-1} x_2 w, \quad \Phi(g_i g_{i+1})^{-1} = w^{-1} x_3 u$$

(as reduced words). We can use this data to glue some new stripes and (in the non-degenerate case) one a- or b-hexagon to ∂_j as in Figure VI.7, modifying Σ by a homeomorphism so that the portion of ∂_j that was labelled by $\Phi(g_i)\Phi(g_{i+1})$ is now labelled by $\Phi(g_i g_{i+1})$. Note that this operation relies on the fact that the labelling of each 1-handle on $\partial\Sigma$ is a reduced word, and this property still hold after gluing the new pieces.

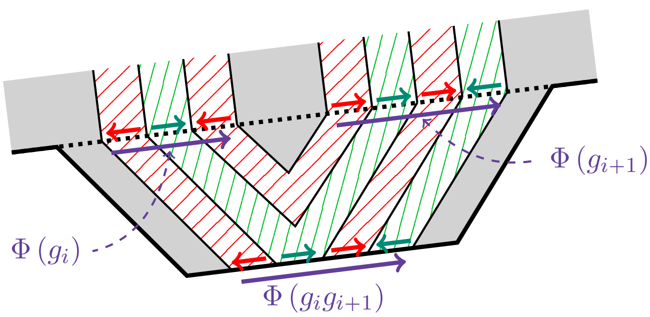


Figure VI.7: Correcting the boundary labelling. (The former boundary is depicted as a dotted line and the new one as a full line.)

By repeating this process a finite number of times (as most as many times as the number of 1-handles of Σ), we obtain a surface homeomorphic to Σ with a new stripe pattern. Each boundary component is now labelled by a word representing some positive power of $\Phi(g)$ — and which is therefore cyclically reduced. Notice for instance in Figure VI.7 that this construction has eliminated a pair of cancelling edges.

Since boundary components are labelled by powers of $\Phi(g) = a_1 b_1 \cdots a_\ell b_\ell$, with $a_i \in \{a^{\pm 1}\}$ and $b_i \in \{b^{\pm 1}\}$, each boundary edge bounding an a-stripe corresponds to some a_i and each boundary edge bounding a b-stripe corresponds to some b_i. We will remember the index i as part of the stripe pattern. Hence, reading the successive indices of the edges along a boundary component of Σ yields a cyclic permutation of a positive iterate of the sequence $(1, 1, \dots, \ell, \ell)$.

Example VI.4.4. Suppose that X is a bouquet of two oriented circles labelled by a and b, so that $\pi_1 X \cong F(a, b)$. Let $g = [a, b] = aba^{-1}b^{-1} \in \pi_1 X$. There is a letter-quasimorphism $\Phi : \pi_1 X \to \mathcal{A}$ given by

$$a^{m_1} b^{n_1} \cdots a^{m_k} b^{n_k} \longmapsto a^{\operatorname{sign}(m_1)} b^{\operatorname{sign}(n_1)} \cdots a^{\operatorname{sign}(m_k)} b^{\operatorname{sign}(n_k)},$$

with $m_i, n_i \in \mathbb{Z}$, all non-zero except possibly for m_1 and n_k (see [48, Example 4.2]).

Applying the above construction to the admissible surface $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$ which is a once-punctured torus with generators mapping to X in the standard way yields the stripe pattern of Figure VI.8. The only boundary component of Σ is labelled by $(1, 1, 2, 2)$.

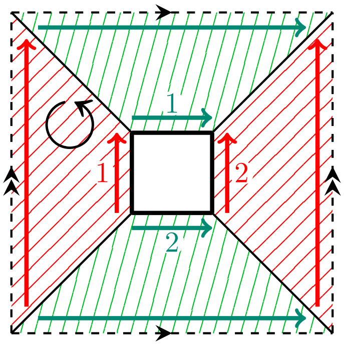


Figure VI.8: Stripe pattern on an admissible surface for $g = [a, b] \in F(a, b)$ (black arrows indicate edge identifications).

VI.4.d Unzipping Σ

With its current stripe pattern, Σ is subdivided into:

- The union Σ^D of its closed vertex discs,
- The union Σ^a of its closed a-stripes and a-hexagons, and
- The union Σ^b of its closed b-stripes and b-hexagons.

Consider the boundary of those different regions:

$$\Gamma := \left(\Sigma^a \cap \Sigma^b\right) \cup \left(\Sigma^a \cap \Sigma^D\right) \cup \left(\Sigma^b \cap \Sigma^D\right).$$

By observing the local structure of the stripe pattern, one can see that Γ is an embedded graph in Σ.

We want to perform one last modification on the stripe pattern to remove all singular points of Γ (i.e. to ensure that Γ is a 1-dimensional submanifold of Σ). Note that the only points where Γ might not be locally homeomorphic to a line are the apices of degenerate cellular discs (see Figure VI.5a). We can *unzip* each degenerate cellular disc as indicated in Figure VI.9, extending an existing vertex disc between the a- and b-stripes meeting at the apex of the triangle.

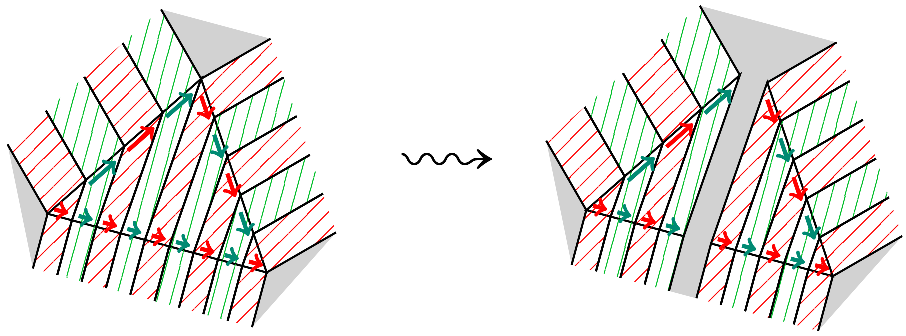


Figure VI.9: Unzipping a degenerate cellular disc.

After unzipping one cellular disc, one needs to keep unzipping along the same leaf of Γ. One can iterate until there is no more degenerate cellular disc. After this process, Γ is locally homeomorphic to a line. We call *unzipping path* each of the paths in the original surface along which we have unzipped the cellular discs. Note that the unzipping process might have connected several vertex discs to one another; however, if any region resulting from these identifications contained a non-contractible simple closed curve β, then β could be homotoped to an *unzipping loop* — i.e. an unzipping path which is a simple closed curve — in the original surface; but then β must be contractible by the following claim:

Claim. Any unzipping loop has contractible image under $f : \Sigma \to X$ and is therefore contractible by incompressibility.

Proof. Consider a cellular disc that is unzipped by the above process (this cellular disc is of the degenerate type — see Figure VI.5a). Such a cellular disc had its boundary edges mapping to loops representing elements g_1, g_2, and $(g_1 g_2)^{-1}$ of $\pi_1 X$ under $f : \Sigma \to X$ (see §VI.4.b). The restriction of any unzipping path to this cellular disc is the singular leaf (i.e. the middle edge in Figure VI.5a); this is homotopic in Σ to the path following one of the boundary edges and half of the next edge, which maps to a loop representing $g_1 g_1^{-1} = 1$ in $\pi_1 X$.

Now any unzipping loop alternates between vertex discs in the original surface (which map under f to the vertex of X), and the singular leaves of cellular discs of the degenerate type (which have contractible image under f as explained above). This proves the claim. □

Therefore, at the end of the unzipping process, the new vertex discs are simply connected and are therefore topological discs.

Moreover, Γ is compact as its intersection with each vertex disc, 1-handle, or cellular disc of Σ is compact, and Σ has only finitely many such pieces. Therefore, Γ is a compact 1-dimensional submanifold of Σ. We then say that the stripe pattern is an *unzipped stripe pattern*.

VI.4.e The angle structure

Before constructing the angle structure on Σ, we describe the cellulation that will support it.

An a*-region* of Σ is a connected component of Σ^a, while a b*-region* is a connected component of Σ^b. A *region* is either an a-region, a b-region, or a vertex disc. A *transition arc* is a connected component of the 1-dimensional submanifold Γ discussed above, while a *boundary arc* is an edge of $\partial\Sigma$ that bounds a stripe. Recall that each boundary arc has an index $i \in \{1, \ldots, \ell\}$ such that the arc is labelled by the letter $a_i \in \{a^{\pm 1}\}$ (or $b_i \in \{b^{\pm 1}\}$) of the word $\Phi(g)$. Note that a transition arc might be a loop, but a boundary arc cannot because $\Phi(g)$ must be a cyclically reduced alternating word starting and ending in distinct letters since $\Phi(g)^n$ is alternating for all n.

Observe that each a- or b-region is a subsurface of Σ, and each of its boundary components alternates between transition and boundary arcs. We put one vertex at each endpoint of each transition or boundary arc — those vertices are called *arc endpoints* — and an additional vertex in the interior of each transition arc — those are called *transition vertices*. In the exceptional case of a transition arc that is a loop, there is one arc endpoint at the unique endpoint, and one transition vertex in the interior of the arc. Then we add boundary arcs and half-transition arcs as edges. In order to complete this to a cellulation of Σ, we need to add more edges (until the edge set cuts Σ into discs), but in light of Corollary VI.1.4, the way we do this is irrelevant as we will estimate the total curvature of Σ by counting the interior curvature of each region, and the interior curvature does not depend on how a subsurface is subdivided into discs. Likewise, it suffices to define the total angle of each vertex inside each region, without specifying how the total angle is split between the faces of Σ. Note that, for any choice of cellulation of Σ, we can pick angles for all corners in the cellulation with specified total angles in the regions of Σ — as in the proof of Theorem VI.2.3, such a choice amounts to solving a system of affine equations where each variable — corresponding to an angle — appears in exactly one equation — corresponding to the total angle of the associated vertex in the relevant region.

The angle structure is now defined as follows.

Each arc endpoint v is contained in two regions (on both sides of the corresponding transition arc). The total angle of v in each of these regions is defined to be

- A right angle ($\pi/2$) if $v \in \partial\Sigma$, or
- A flat angle (π) if $v \notin \partial\Sigma$.

Consider a transition vertex v. It lies in two regions, at most one of which is a vertex disc. Consider an a- or b-region Λ on the boundary of which v lies. Recall that the boundary of Λ alternates between transition and boundary arcs; v lies on a transition arc, which is preceded by a boundary arc with index $i \in \{1, \dots, \ell\}$, and succeeded by another boundary arc with index $j \in \{1, \dots, \ell\}$ — here, the order is induced by the orientation that Λ inherits from Σ. We define the total angle of v in the a- or b-region Λ to be

$$\angle_{\mathrm{tot}}^{\Lambda}(v) := \theta_{i,j} := \begin{cases} 2\pi & \text{if } i \geqslant j \\ 0 & \text{if } i < j \end{cases}. \tag{§}$$

In the exceptional case where v is the endpoint of a transition arc which is a loop, there is no preceding or succeeding boundary arc, and we set $\angle^{\Lambda}_{\text{tot}}(v) := \pi$.

Figure VI.10 shows an annular a-region and the definition of the angle structure in that region.

If the other region in which v lies is also an a- or b-region, then its total angle in that region is defined in the same way. Otherwise, v lies in a vertex disc Δ, and its total angle in Δ is defined to be

$$\angle^{\Delta}_{\text{tot}}(v) := 2\pi - \angle^{\Lambda}_{\text{tot}}(v). \qquad (¶)$$

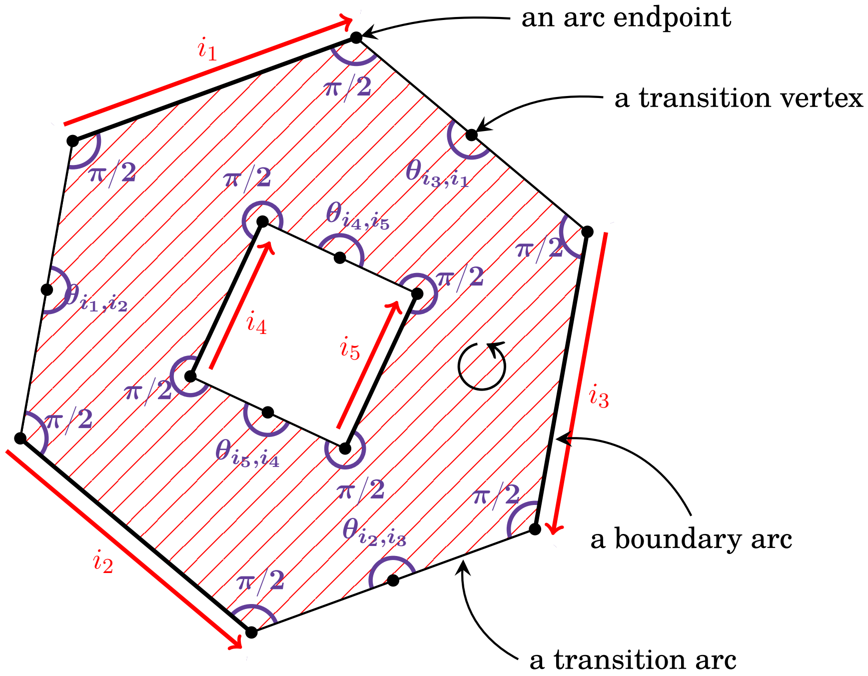


Figure VI.10: The angle structure in an a-region. Boundary arcs are in bold, with labels indicated by parallel arrows. Angles are indicated in purple.

VI.4.f Boundary orientation of a- and b-regions

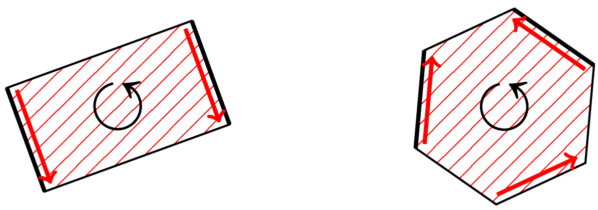
In order to estimate the interior curvature of a- and b-regions, we need a better understanding of their boundary labelling. Each boundary arc of Σ is labelled by a letter in $\{a^{\pm 1}, b^{\pm 1}\}$. Another way to see this labelling is as an orientation of the boundary arc, as indicated by bold arrows in Figures VI.5, VI.6, VI.7, VI.8, VI.9, VI.10 — the label of the boundary arc is a^{+1} (or b^{+1}) if its orientation matches that of Σ, and a^{-1}

(or b^{-1}) otherwise. We say that two boundary arcs have *opposite orientations* if one of them is labelled by a^{+1} (or b^{+1}) while the other one is labelled by a^{-1} (or b^{-1}) — in other words, one is oriented consistently with Σ, and the other one isn't.

Lemma VI.4.5. *Let Λ be an a- or b-region of Σ which is topologically a disc and whose boundary contains at least two distinct boundary arcs. Then $\partial\Lambda$ contains two boundary arcs with opposite orientations.*

Proof. The region Λ is built out of stripes and hexagons. Each stripe has four boundary edges, alternating between sections of transition arcs, and edges that have an orientation given by whether they map positively to S^1_a or S^1_b. Those edges are called *directed edges*. The orientations of directed edges are indicated by bold arrows in Figures VI.5, VI.6, VI.7, VI.8, VI.9, VI.10. Likewise, each hexagon has six boundary edges, alternating between sections of transitions arcs, and directed edges. Stripes and hexagons are glued along directed edges, in such a way that orientations of directed edges which are glued together agree.

Now the key observation is that each stripe and each hexagon contains two directed edges with opposite orientations (i.e. one of them matches the orientation of the stripe or hexagon, while the other does not). In stripes (Figure VI.11a), this is because parallel directed edges have parallel orientations, so one of them matches the orientation of the stripe, and the other does not. In hexagons (Figure VI.11b), one must come back to the construction of §VI.4.b to see that the labels of the boundary edges correspond to letters x_1, x_2, x_3 as in the definition of letter-quasimorphisms (Definition VI.3.1), and so in particular $x_1x_2x_3 \in \{a^{\pm 1}, b^{\pm 1}\}$. This means that each hexagon has two directed edges with opposite orientations.



(a) Stripe. (b) Hexagon.

Figure VI.11: Directed edges with opposite orientations (highlighted in bold) in stripes and hexagons.

Using this observation, and the fact that stripes and hexagons are glued so that directed edges have matching orientations, we can construct two boundary arcs of Λ with opposite orientations as follows. Start at a boundary arc of Λ; this arc is a directed edge e_1 of a stripe or a hexagon, which we denote by Δ_1. We have observed that $\partial\Delta_1$ has a directed edge e_2 with orientation opposite to e_1; pick a point p_1 in the interior of e_1 and a point p_2 in the interior of e_2 and connect them via an arc in the interior of Δ_1. Now e_2 borders another stripe or hexagon, which we denote by Δ_2; extend the previous arc to a point p_3 in the interior of a directed edge e_3 of Δ_3 with orientation opposite to e_2. Iterate this process; it will follow from the following claim that we cannot visit the same stripe or hexagon twice:

Claim. There is no non-trivial sequence of successively adjacent stripes and hexagons in Λ (without backtracking) starting and finishing on the same stripe or hexagon.

Proof of the claim. Suppose for contradiction that there is a sequence of stripes and hexagons as above in Λ. Draw a loop β in Λ connecting the midpoints of the successive connecting edges of this sequence. We will show that β is non-contractible (in Λ), contradicting the assumption that Λ is topologically a disc. Suppose otherwise: hence, there is a homotopy $\beta_\bullet : [0,1]^2 \to \Lambda$ from $\beta = \beta_0$ to a constant loop β_1. Note that the restriction of β to each stripe or hexagon enters and exits using two opposite edges; since those are separated by edges contained in $\partial\Lambda$, this must also be true of each β_t (for $t \in [0,1]$). Therefore, the number of stripes and hexagons visited by β_t does not depend on t. But this number is at least 2 for β_0, while it is exactly 1 for β_1; this is a contradiction. □

Moreover, Λ is compact, so we must eventually reach a boundary arc of Λ. The construction is illustrated in Figure VI.12. Hence, we obtain an arc η visiting an injective sequence of stripes and hexagons, with endpoints on distinct boundary arcs of Λ, with the property that all directed edges of stripes or hexagons crossed by η have the same transverse orientation with respect to η. In particular, the boundary arcs at the extremities of η must have opposite orientations (with respect to the orientation of Λ). □

VI.4.g Estimating the curvature

We will estimate the total curvature of Σ using its decomposition into a-regions, b-regions, and vertex discs. Corollary VI.1.4 says that it suffices to estimate the curva-

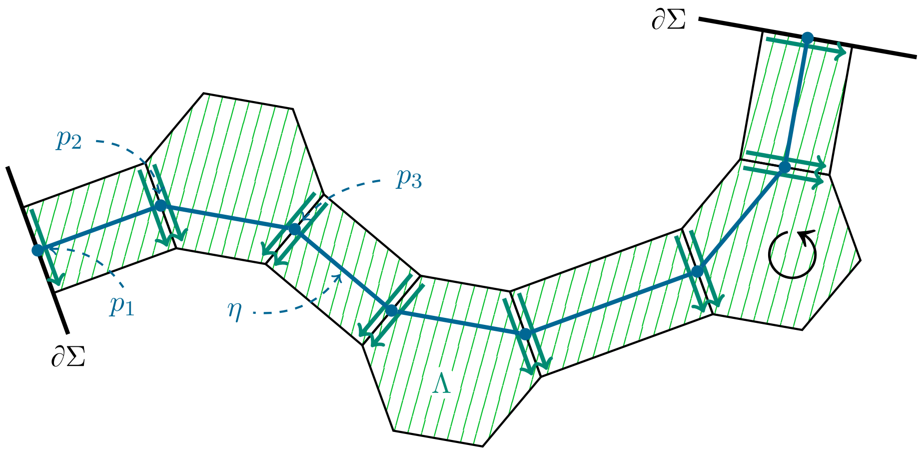


Figure VI.12: Construction of a path connecting two boundary arcs with opposite orientations in an a- or b-region.

ture of each vertex and the interior curvature of each region.

Claim. Each arc endpoint of Σ has zero curvature.

Proof of the claim. Let v be an arc endpoint.

- Suppose that v lies at the endpoint of a transition arc and a boundary arc. In particular, $v \in \partial\Sigma$, and v is contained in two regions. The angle of v in each region is $\pi/2$, and $\chi(\mathrm{Lk}_\Sigma(v)) = 1$, so that $\kappa(v) = 0$.
- If v is only the endpoint of a transition arc (which has to be a loop again), then $v \notin \partial\Sigma$, v is contained in two regions, and its angle is π in each. Since $\chi(\mathrm{Lk}_\Sigma(v)) = 0$ in this case, we get $\kappa(v) = 0$ again. □

Claim. Each a- or b-region of Σ has non-positive interior curvature.

Proof of the claim. Let Λ be an a- or b-region of Σ. Recall from §VI.1.b that the interior curvature of Λ is

$$\kappa_{\mathrm{int}}(\Lambda) = 2\pi \cdot \chi(\Lambda) + \sum_{v \in V(\Lambda)} \left(\angle^{\Lambda}_{\mathrm{tot}}(v) - \pi\right).$$

We first show that the sum $S := \sum_v \left(\angle^{\Lambda}_{\mathrm{tot}}(v) - \pi\right)$ is non-positive. To do so, observe that each vertex of Λ lies on a unique transition arc. Therefore, for the purpose of computing the sum S, we can partition $V(\Lambda)$ into subsets corresponding to transition arcs. A transition arc which is a loop contains two vertices of Λ, each of which has total angle π in Λ, so that its contribution to S is zero. Each transition arc which is not a loop contains three vertices of Λ: two arc endpoints with total angle $\pi/2$ in Λ each, and one transition vertex with total angle 0 or 2π in Λ — see Figure VI.10. Hence, the total contribution of a transition arc to S is at most $\left(\frac{\pi}{2}-\pi\right)+\left(\frac{\pi}{2}-\pi\right)+(2\pi-\pi)=0$. It follows that $S \leqslant 0$. This proves that $\kappa_{\mathrm{int}}(\Lambda) \leqslant 0$ as soon as $\chi(\Lambda) \leqslant 0$.

It remains to analyse the case where $\chi(\Lambda) > 0$. Since Λ has non-empty boundary, the assumption that $\chi(\Lambda) > 0$ implies that Λ is a disc, and $\chi(\Lambda) = 1$. Looking back at the previous argument, the total contribution to the sum S of a transition arc preceded by a boundary arc with index $i \in \{1, \ldots, \ell\}$ and succeeded by a boundary arc with index $j \in \{1, \ldots, \ell\}$ is exactly $(\theta_{i,j} - 2\pi)$. Now let $i_1, \ldots, i_k$ be the successive indices of the boundary arcs of Λ, ordered according to the orientation of Λ. Then we have

$$\kappa_{\mathrm{int}}(\Lambda) = 2\pi + \sum_{j=1}^{k} \left(\theta_{i_j, i_{j+1}} - 2\pi\right),$$

with the convention that $i_{k+1} = i_1$. If the indices $i_1, \ldots, i_k$ are not all equal, then there is some j_0 such that $i_{j_0} < i_{j_0+1}$, and so $\left(\theta_{i_{j_0}, i_{j_0+1}} - 2\pi\right) = -2\pi$. Since $\theta_{i_j, i_{j+1}} \leqslant 2\pi$ for all j by definition, it then follows that

$$\kappa_{\mathrm{int}}(\Lambda) \leqslant 2\pi + \left(\theta_{i_{j_0}, i_{j_0+1}} - 2\pi\right) = 0.$$

Therefore, the only case where $\kappa_{\mathrm{int}}(\Lambda)$ can be positive is if Λ is topologically a disc and all its boundary arcs have the same index i. In particular, all the boundary arcs are labelled by the same letter (either a, a^{-1}, b, or b^{-1}), which contradicts Lemma VI.4.5. □

Claim. The combined contribution of vertex discs and transition vertices to the total curvature of Σ is at most $-2\pi \cdot n(\Sigma)$.

Proof of the claim. Let us start with transition vertices. Each transition vertex v lies on a transition arc α, bounding two distinct regions of Σ. If α bounds a vertex disc Δ and an a- or b-region Λ, then by (¶), we have

$$\kappa(v) = 2\pi - \angle^{\Delta}_{\mathrm{tot}}(v) - \angle^{\Lambda}_{\mathrm{tot}}(v) = 0.$$

Otherwise, α bounds an a-region Λ_a and a b-region Λ_b. Let i and j be the indices of the boundary arcs preceding and succeeding α in $\partial\Lambda_a$. Hence, the indices of the boundary arcs preceding and succeeding α in $\partial\Lambda_b$ must be $j-1$ and i (with indices taken modulo ℓ), as in Figure VI.13.

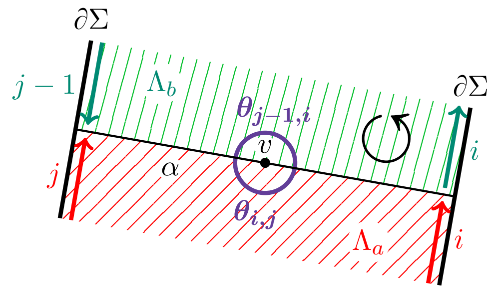


Figure VI.13: Estimating the curvature of transition vertices.

It follows that

$$\kappa(v) = 2\pi - \angle_{\text{tot}}^{\Lambda_a}(v) - \angle_{\text{tot}}^{\Lambda_b}(v) = 2\pi - \theta_{i,j} - \theta_{j-1,i}.$$

If $j \neq 1$, then we have $i \geqslant j$ if and only if $j - 1 < i$, so that $\theta_{i,j} = 2\pi$ if and only if $\theta_{j-1,i} = 0$ (see (§)), and therefore $\kappa(v) = 0$.

On the other hand, if $j = 1$, then $j - 1 = \ell$, and we have $\theta_{i,j} = \theta_{j-1,i} = 2\pi$ and $\kappa(v) = -2\pi$.

We now turn our attention to vertex discs. By definition, each vertex disc Δ is a topological disc, with boundary alternating between transition arcs and edges of $\partial\Sigma$ (which are not boundary arcs since they do not bound a stripe). The successive transition arcs of $\partial\Delta$ alternatively bound an a-region $\Lambda_a^{(1)}$, then a b-region $\Lambda_b^{(1)}$, then another a-region $\Lambda_a^{(2)}$, etc. Here, we are ordering the transition arcs in clockwise order — i.e. in the order opposite to the orientation of $\partial\Delta$. Hence there is an even number of transition arcs, say $2r$, so that the regions adjacent to Δ are (in clockwise order) $\Lambda_a^{(1)}, \Lambda_b^{(1)}, \dots, \Lambda_a^{(r)}, \Lambda_b^{(r)}$. See Figure VI.14, where $r = 2$. In $\Lambda_a^{(k)}$, let i_k and j_k be the indices of the boundary arcs preceding and succeeding the transition arc bounding Δ. Then in $\Lambda_b^{(k)}$, the indices of the boundary arcs preceding and succeeding the transition arc bounding Δ must be $j_k - 1$ and i_{k+1} (with the convention $i_{r+1} = i_1$). Again, we refer the reader to Figure VI.14 for an illustration of these notations.

Now Δ has one transition vertex and two arc endpoints on each transition arc, and each arc endpoint has a total angle of $\pi/2$ in Δ. Therefore, the interior curvature of Δ is

$$\kappa_{\text{int}}(\Delta) = 2\pi - \sum_{k=1}^{r} \left(\theta_{i_k,j_k} + \theta_{j_k-1,i_{k+1}}\right). \qquad (\|)$$

The assumption that Σ is disc-free implies that $r \geqslant 1$. From (‖), observe first that $\kappa_{\text{int}}(\Delta) \leqslant 0$. Indeed, if $\kappa_{\text{int}}(\Delta)$ were positive, then each term in the sum would have to

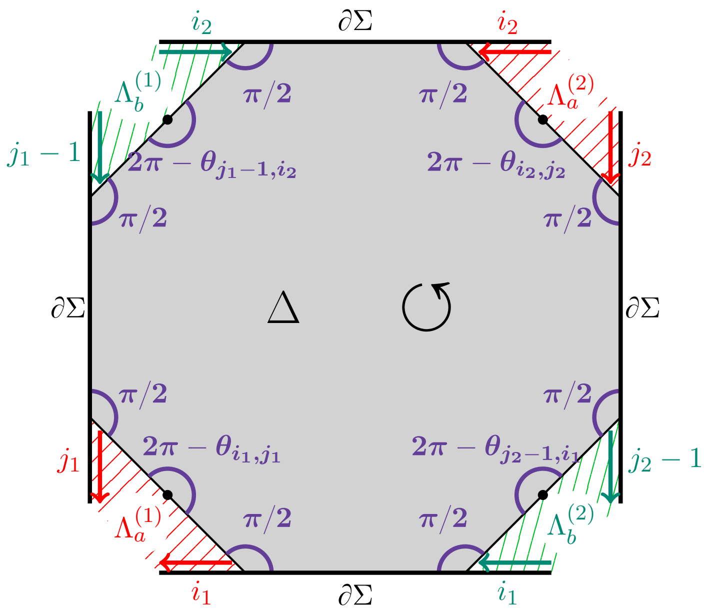


Figure VI.14: An octagonal vertex disc.

equal zero, so that we would have

$$i_k < j_k \quad \text{and} \quad j_k - 1 < i_{k+1}$$

for all k, implying that $i_1 < j_1 \leqslant i_2 < j_2 \leqslant \cdots \leqslant i_r < j_r \leqslant i_1$, which is impossible.

Further, note that each $k \in \{1, \dots, r\}$ such that $j_k = 1$ adds an extra contribution of -2π to the right-hand side of ($\|$).

Wrapping everything up, the above computations show that transition vertices and vertex discs have non-positive (interior) curvature; moreover, every a-boundary arc α with index 1 will contribute -2π to the curvature

- Of a transition vertex if α is preceded by a b-boundary arc along $\partial\Sigma$, as in Figure VI.13 with $j = 1$, or
- Of a vertex disc if α is preceded by an edge that is part of a vertex disc, as in Figure VI.14 with for instance $j_1 = 1$.

Since the indices along each component of $\partial\Sigma$ read $1, 1, \dots, \ell, \ell$, alternating between a- and b-boundary arcs, with a total of $n(\Sigma)$ repetitions of this pattern across all boundary components of Σ, there are in total $n(\Sigma)$ a-boundary arcs with index 1, and their total contribution to the curvature of transition vertices and vertex discs is at most $-2\pi \cdot n(\Sigma)$ as wanted. $\square$

Using the decomposition of Σ into vertex discs and a- and b-regions to compute its total curvature via Corollary VI.1.4 now yields

$$\kappa(\Sigma) = \sum_{\substack{v \text{ arc} \\ \text{endpoint}}} \overbrace{\kappa(v)}^{=0} + \overbrace{\sum_{\substack{v \text{ transition} \\ \text{vertex}}} \kappa(v) + \sum_{\substack{\Delta \text{ vertex} \\ \text{disc}}} \kappa_{\text{int}}(\Delta)}^{\leqslant -2\pi \cdot n(\Sigma)} + \sum_{\substack{\Lambda\ a\text{- or} \\ b\text{-region}}} \overbrace{\kappa_{\text{int}}(\Lambda)}^{\leqslant 0}$$
$$\leqslant -2\pi \cdot n(\Sigma). \qquad \square$$

VI.5 Groups with controlled spectral gaps

Heuer's Theorem (Corollary VI.4.2) says that right-angled Artin groups have a spectral gap at exactly $\frac{1}{2}$, which is the largest possible spectral gap a group can have (see Remark VI.2.2). The aim of this final section is to use the previous results of this thesis to construct groups in which we can control the optimal spectral gap.

We start with a very concrete example:

Example VI.5.1 (Dyck's surface)**.** Let $\varDelta$ be the non-orientable surface given by the side-pairing of Figure VI.15, so that $\pi_1\varDelta = \langle a, b, c \mid [a, b] = c^2 \rangle$. Dyck's Theorem [33]

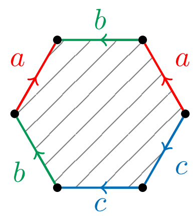


Figure VI.15: Side-pairing for Dyck's surface $\varDelta$.

asserts that $\varDelta \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$.

Then for all $g \in \pi_1\varDelta \smallsetminus \{1\}$, there is an inequality

$$\mathrm{scl}_{\pi_1\varDelta}(g) \geqslant \frac{1}{4}.$$

Moreover, this bound is sharp: $\mathrm{scl}_{\pi_1\varDelta}(c) = \frac{1}{4}$.

Proof. The group $\pi_1\varDelta$ splits as an HNN-extension $\pi_1\varDelta = G*_{\mathbb{Z}}$, where

$$G := \langle a_1, a_2, c \mid a_1 a_2 = c^2 \rangle,$$

and the HNN-extension is given by the isomorphism $\langle a_1 \rangle \cong \langle a_2 \rangle$ sending a_1 to a_2.

Note that G is a free group, and both $\{a_1, c\}$ and $\{a_2, c\}$ are free bases of G. It follows that $\langle a_1 \rangle$ and $\langle a_2 \rangle$ are *left relatively convex* in G, meaning that the G-sets $G/\langle a_1 \rangle$ and $G/\langle a_2 \rangle$ admit G-invariant orders. This will allow us to apply results of Chen and Heuer [26] on spectral gaps in graphs of groups.

For $g \in \pi_1 \Delta \smallsetminus \{1\}$, there are two cases:

- If g is hyperbolic for the HNN-splitting $G *_{\mathbb{Z}}$, then since $\langle a_1 \rangle$ and $\langle a_2 \rangle$ are left relatively convex in G, it follows from [26, Theorem 5.19] that $\mathrm{scl}_{\pi_1 \Delta}(g) \geqslant \frac{1}{2}$.

- If g is elliptic, then g is conjugate to a non-trivial element g_0 of G. Since G is a free group, the Duncan–Howie Theorem (Theorem VI.2.1) implies that $\mathrm{scl}_G(g_0) \geqslant \frac{1}{2}$. Now Corollary IV.4.3 implies that $G \hookrightarrow \Delta$ is an scl-isometric embedding; this means that $\mathrm{scl}_{\pi_1 \Delta}(g) = \mathrm{scl}_G(g_0) \geqslant \frac{1}{2}$ whenever $\mathrm{scl}_G(g_0) < \infty$.

Therefore, if $g \in \pi_1 \Delta$ satisfies $\mathrm{scl}_{\pi_1 \Delta}(g) < \frac{1}{2}$, then g must be conjugate to $g_0 \in G$ with $\mathrm{scl}_G(g_0) = \infty$. In particular, the image of g_0 in $H_1(G; \mathbb{Q})$ must be a non-trivial element of $\mathrm{Ker}(H_1(G; \mathbb{Q}) \to H_1(\pi_1 \Delta; \mathbb{Q}))$. But we have

$$H_1(G; \mathbb{Q}) = \mathbb{Q}a_1 \oplus \mathbb{Q}c \xrightarrow[c \mapsto 0]{a_1 \mapsto a} \mathbb{Q}a \oplus \mathbb{Q}b = H_1(\pi_1 \Delta; \mathbb{Q}),$$

so $\mathrm{Ker}(H_1(G; \mathbb{Q}) \to H_1(\pi_1 \Delta; \mathbb{Q})) = \mathbb{Q}c$. Hence, any element g of $\pi_1 \Delta$ with $\mathrm{scl}_{\pi_1 \Delta}(g) < \frac{1}{2}$ must be conjugate into $\langle c \rangle$.

Now $\mathrm{scl}_{\pi_1 \Delta}(c) = \frac{1}{2} \mathrm{scl}_{\pi_1 \Delta}([a, b])$ since $c^2 = [a, b]$. We have $\mathrm{scl}_{\pi_1 \Delta}([a, b]) \leqslant \frac{1}{2}$ (by Example I.1.21) and $\mathrm{scl}_{\pi_1 \Delta}([a, b]) \geqslant \frac{1}{2}$ (since $[a, b]$ is hyperbolic in $G *_{\mathbb{Z}}$), so $\mathrm{scl}_{\pi_1 \Delta}(c) = \frac{1}{4}$ as wanted. □

In particular, the Duncan–Howie Theorem (Theorem VI.2.1) implies that $\pi_1 \Delta$ is not residually free (this also follows from a result of Lyndon [55, 56]). Moreover, it follows from Heuer's Theorem (Corollary VI.4.2) that $\pi_1 \Delta$ is not a subgroup of any right-angled Artin group, and thus not special[1] in the sense of Haglund and Wise [46]. Note however that $\pi_1 \Delta$ is virtually special since its orientation double cover is an orientable closed surface of genus 2.

In fact, except for $\mathbb{RP}^2$ and $\mathbb{RP}^2 \# \mathbb{RP}^2$ — which have virtually abelian fundamental groups and therefore vanishing scl — the fundamental groups of all other non-orientable closed surfaces have a strong spectral gap since they are residually free

[1] To be more precise, $\pi_1 \Delta$ is not *A-special* in the terminology of [46].

by a result of Benjamin Baumslag [2, p.414]. The fundamental groups of orientable closed surfaces are all residually free [3], so they satisfy a strong spectral gap. Therefore, the fundamental group of Dyck's surface is the only surface group containing elements with scl in $\left(0, \frac{1}{2}\right)$.

We can easily generalise Example VI.5.1 and construct groups with arbitrarily small spectral gap:

Proposition VI.5.2. *For* $n \in \mathbb{N}_{\geqslant 1}$*, consider the group*

$$\Gamma_n := \langle a, b, c \mid [a, b] = c^n \rangle .$$

Then $\forall g \in \Gamma_n \setminus \{1\}$, $\mathrm{scl}_{\Gamma_n}(g) \geqslant \frac{1}{2n}$*. Moreover, this bound is sharp:* $\mathrm{scl}_{\Gamma_n}(c) = \frac{1}{2n}$.

Proof. As for Example VI.5.1, we write $\Gamma_n = G_n * \mathbb{Z}$, with $G_n := \langle a_1, a_2, c \mid a_1 a_2 = c^n \rangle$, and show that any element g of Γ_n with $\mathrm{scl}_{\Gamma_n}(g) < \frac{1}{2}$ must be conjugate into $\langle c \rangle$. Since $\mathrm{scl}_{\Gamma_n}(c) = \frac{1}{2n}$, the result follows. □

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