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HW2

1a.: Derive the steps of the ADMM algorithm in Alg 1.

Step 1: $x^{(k+1)} \leftarrow (A^TA + \rho I)^{-1}(A^Ty + \rho(z^{(k)} - u^{(k)}))$

Note that $x^{(k+1)} = \underset{x}{\operatorname{argmin}}\, l_\rho(x,v) = \underset{x}{\operatorname{argmin}} \{ \frac{1}{2}\|Ax-y\|_2^2 + \alpha\|z\|_1 + \frac{\rho}{2}\|x-z+u\|_2^2 \}$

$= \underset{x}{\operatorname{argmin}} \{ \frac{1}{2}\|Ax-y\|_2^2 + \frac{\rho}{2}\|x-z+u\|_2^2 \}.$

Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \in \mathbb{R}^m$, $z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^n$, $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$, $\rho \in \mathbb{R}_{>0}$.

The partial derivative of $l_\rho(x,v)$ w.r.t. the c^{th} coordinate x_c of x is

$$\frac{\delta l}{\delta x_c} = \frac{\delta}{\delta x_c}\{ \frac{1}{2}\|Ax-y\|_2^2 + \frac{\rho}{2}\|x-z+u\|_2^2 \}$$

$$= \frac{\delta}{\delta x_c}\{ \frac{1}{2}[(Ax)^T(Ax) - 2y^TAx + y^Ty + \rho(x-z+u)^T(x-z+u)] \}$$

$$= \frac{\delta}{\delta x_c}\{ \frac{1}{2}[\sum_{j=1}^m (\sum_{i=1}^n a_{ji}x_i)^2 - 2\sum_{j=1}^m (y_j \sum_{i=1}^n a_{ji}x_i) + y^Ty + \rho \sum_{i=1}^n (x_i - z_i + u_i)^2] \}$$

$$= \frac{1}{2} 2\sum_{j=1}^m \left(a_{jc}\left(\sum_{i=1}^n a_{ji}x_i - y_j\right)\right) + \rho(x_c - z_c + u_c).$$

Therefore,

$\nabla l := A^TAx - A^Ty + \rho(x-z+u)$. is the gradient of $l_\rho(x,v)$.

Set ∇l equal to zero and solve for x:

$$0 = A^TAx - A^Ty + \rho(x-z+u)$$

$$A^Ty + \rho(z-u) = (A^TA + \rho I)x$$

$$x = (A^TA + \rho I)^{-1}(A^Ty + \rho(z-u)).$$

Note: $A^TA + \rho I$ invertible for any matrix A

Step 2: $z^{(k+1)} \leftarrow S_{\alpha/\rho}(x^{(k+1)} + u^{(k)})$

Note that $z^{(k+1)} = \underset{z}{\operatorname{argmin}}\, l_\rho(x^{(k+1)}, w^{(k)}) = \underset{z}{\operatorname{argmin}}\{ \alpha\|z\|_1 + \frac{\rho}{2}\|x^{(k+1)} - z + u^{(k)}\|_2^2 \}$

$= \underset{z}{\operatorname{argmin}}\{ \alpha\sum_{i=1}^n |z_i| + \frac{\rho}{2}\sum_{i=1}^n (x_i^{(k+1)} - z_i + u_i^{(k)})^2 \}.$

$$\frac{\delta l}{\delta z_c} = \frac{\delta}{\delta z_c}\{ \alpha\sum_{i=1}^n |z_i| + \frac{\rho}{2}\sum_{i=1}^n (x_i^{(k+1)} - z_i + u_i^{(k)})^2 \}$$

$$= \alpha\frac{\delta}{\delta z_c}|z_c| - \rho(x_c^{(k+1)} - z_c + u_c^{(k)})$$

NOTE: subdifferential of $|z_c| \in \begin{cases} \{-1\}, & z_c < 0, \\ [-1,1], & z_c = 0, \\ \{1\}, & z_c > 0. \end{cases}$

Find $z_c^{(k+1)}$ by setting $\frac{\delta l}{\delta z_c}$ equal to zero; consider all possible values of z

Case: $z_c < 0$:

$$0 = -\alpha - \rho(x_c^{(k+1)} - z_c^{(k+1)} + u_c^{(k)})$$

$$z_c^{(k+1)} = x_c^{(k+1)} + u_c^{(k)} + \frac{\alpha}{\rho}$$

<u>Case $z_i > 0$:</u> $\quad 0 = \alpha - \rho\left(x_i^{(k+1)} - z_i^{(k+1)} + u_i^{(k)}\right)$

$$z_i^{(k+1)} = x_i^{(k+1)} + u_i^{(k)} - \frac{\alpha}{\rho}$$

<u>Case $z_i = 0$</u> $\quad 0 = \alpha[-1,1] - \rho\left(x_i^{(k+1)} - z_i^{(k+1)} + u_i^{(k)}\right)$

$$z_i^{(k+1)} \in x_i^{(k+1)} + u_i^{(k)} - \frac{\alpha}{\rho}[-1,1]$$

Combine the derived coordinatewise optimality conditions to obtain $z^{(k+1)}$:

$$z^{(k+1)} = \begin{cases} x^{(k+1)} + u^{(k)} + \alpha/\rho, & x^{(k+1)} + u^{(k)} < -\alpha/\rho, \\ x^{(k+1)} + u^{(k)} - \alpha/\rho, & x^{(k+1)} + u^{(k)} > \alpha/\rho, \\ 0, & -\frac{\alpha}{\rho} \le x^{(k+1)} + \mu^{(k)} \le \frac{\alpha}{\rho}. \end{cases}$$

Equivalently, $z^{(k+1)} = S_{\frac{\alpha}{\rho}}\left(x^{(k+1)} + u^{(k)}\right)$.

<u>**STEP 3**</u>: $\quad u^{(k+1)} \leftarrow u^{(k)} + x^{(k+1)} - z^{(k+1)}$

Given $u^{(k)}$ and steps 1 and 2, we obtain $u^{(k+1)}$.