Note on discrete logistic distribution

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Consider modeling the age distribution of counties that are infected. The total mass of counties is normalized to 1. At $t = 0$, a certain fraction ϕ of counties are infected. Infected counties interact with uninfected counties and infect them at some rate $\rho > 0$. Letting $P(t)$ be the fraction of counties that are infected at time t, assume the differential equation

$$P' = \rho P(1 - P).$$

The solution is

$$P(t) = \frac{\phi}{\phi + (1 - \phi)\mathrm{e}^{-\rho t}}.$$

Now think of $t = 0$ as the present. Then the fraction of counties that have age at least $t \geq 0$ is

$$P(-t) = \frac{\phi}{\phi + (1 - \phi)\mathrm{e}^{\rho t}}.$$

Therefore the cumulative distribution function of age is

$$F(t) = 1 - P(-t) = \frac{(1 - \phi)\mathrm{e}^{\rho t}}{\phi + (1 - \phi)\mathrm{e}^{\rho t}} = \frac{1 - \phi}{1 - \phi + \phi\mathrm{e}^{-\rho t}},$$

which is a truncated logistic distribution. Setting $1 - p = \mathrm{e}^{-\rho}$, we obtain

$$F(t) = \frac{1 - \phi}{1 - \phi + \phi(1 - p)^t}.$$

Therefore a plausible model for the age distribution in discrete-time is the probability mass function

$$\Pr(T = t) = F(t) - F(t - 1) = \frac{1 - \phi}{1 - \phi + \phi(1 - p)^t} - \frac{1 - \phi}{1 - \phi + \phi(1 - p)^{t-1}}$$

for $t = 1, 2, \dots$ and $\Pr(T = 0) = 1 - \phi$. This distribution is the truncated version of the discrete logistic distribution introduced by [1].

Let us show that when the age distribution is discrete logistic with parameter (ϕ, p), the Pareto exponent formula is still

$$(1 - p)M(\zeta) = 1.$$

To show this, let $\kappa = \frac{\phi}{1-\phi}$ and $q = 1 - p$. Then for $t \geq 1$, we have

$$\begin{aligned} p_t &\coloneqq \Pr(T = t \mid T > 0) = \frac{1}{\phi}\left(\frac{1}{1 + \kappa q^t} - \frac{1}{1 + \kappa q^{t-1}}\right) \\ &= \frac{\kappa(1 - q)q^{t-1}}{\phi(1 + \kappa q^t)(1 + \kappa q^{t-1})}. \end{aligned}$$

Noting that $\kappa, q > 0$, we obtain

$$\begin{aligned}0 < \frac{\kappa}{\phi}(1-q)q^{t-1} - p_t &= \frac{\kappa(1-q)q^{t-1}(\kappa q^{t-1} + \kappa q^t + \kappa^2 q^{2t-1})}{\phi(1+\kappa q^t)(1+\kappa q^{t-1})} \\ &= \frac{\kappa^2(1-q)q^{2(t-1)}(1+q+\kappa q^t)}{\phi(1+\kappa q^t)(1+\kappa q^{t-1})} \le Cq^{2(t-1)},\end{aligned}$$

where

$$C = \frac{\kappa^2}{\phi}(1-q)(1+q+\kappa q).$$

If $M = M_X(z) = \mathrm{E}[\mathrm{e}^{zX}]$ is the MGF of log growth rate within a period and $Y = \sum_{t=1}^T X_t$ is the observed log growth rate in the entire sample period, then the MGF is

$$M_Y(z) = \sum_{t=1}^\infty p_t M^t.$$

The principal part is

$$\tilde{M}_Y(z) = \sum_{t=1}^\infty \frac{\kappa}{\phi}(1-q)q^{t-1}M^t = \frac{\kappa(1-q)M}{\phi(1-qM)}.$$

Taking the difference, we obtain

$$\begin{aligned}\left|M_Y(z) - \tilde{M}_Y(z)\right| &\le \sum_{t=1}^\infty \left|p_t - \frac{\kappa}{\phi}(1-q)q^{t-1}\right| M^t \\ &\le \sum_{t=1}^\infty Cq^{2(t-1)}M^t = \frac{CM}{1-q^2M}.\end{aligned}$$

Let $\zeta > 0$ be the unique number such that $qM(\zeta) = 1$. Then $q^2M(\zeta) = q < 1$, so $1 - q^2M > 0$ for $0 \le z \le \zeta$. Therefore

$$M_Y(z) = \tilde{M}_Y(z) + \epsilon(z) = \frac{\kappa(1-q)M_X(z)}{\phi(1-qM_X(z))} + \epsilon(z),$$

where $\epsilon(z)$ is analytic on the strip $0 \le \Re z \le \zeta$. Since $z = \zeta$ is a simple pole of $\tilde{M}_Y$, it is also a simple pole of M_Y.

References

[1] Subrata Chakraborty and Dhrubajyoti Chakravarty. A new discrete probability distribution with integer support on $(-\infty, \infty)$. *Communications in Statistics—Theory and Methods*, 45(2):492–505, 2016.