Integration of the Walrasian Paradigm into the Statistical Paradigm

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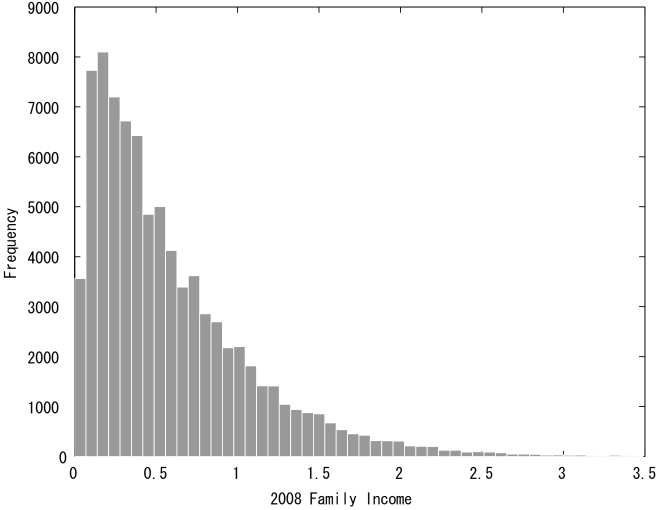
June 23, 2010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2008 Family Income

Contribution

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2. Prove existence of equilibrium.
3. Prove that Walrasian equilibrium is a special case of statistical equilibrium. Hence

$$\text{Walrasian equilibrium theory} \subsetneq \text{statistical equilibrium theory},$$
$$\text{statistical equilibrium} = \text{“general general equilibrium”}.$$

Axioms

Walrasian Equilibrium

1. well-functioning market,
2. agent optimization,
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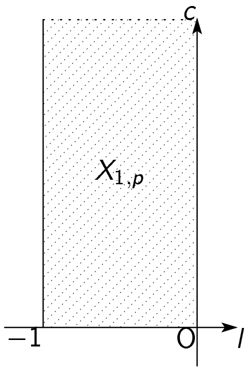
Statistical Equilibrium

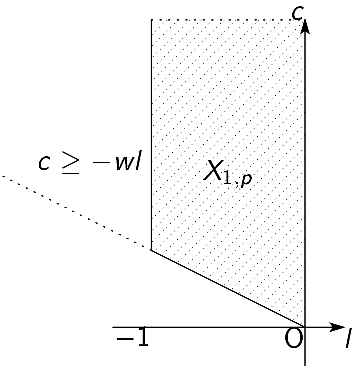
1. agent satisfaction,
2. entropy maximization,
3. market clearing,
4. rational expectations (informational consistency).

Households' offer set

No minimum wage:

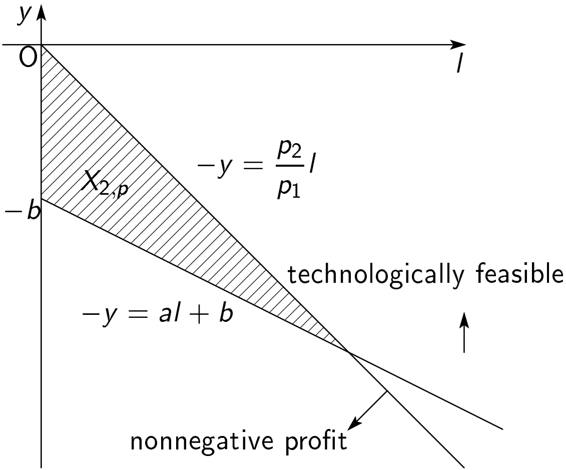


Minimum wage w:

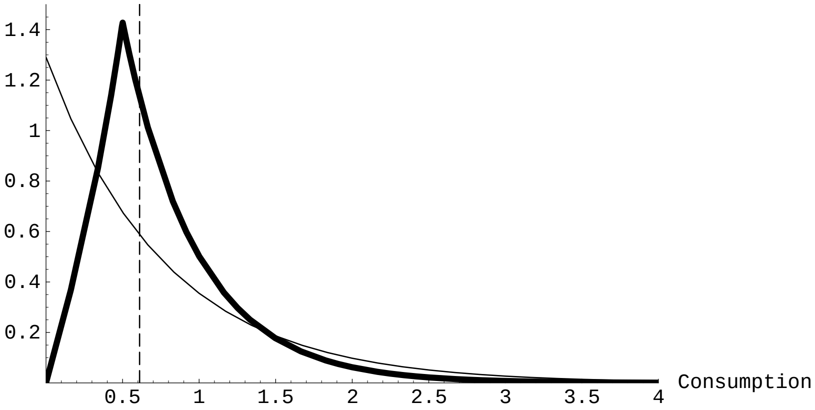


Firms' offer set

Firms have technology $y \leq al + b$; produce iff nonnegative profit.



Equilibrium income distribution



Histogram

Model

- $i \in \mathcal{I} = \{1, 2, \ldots, I\}$: agent types.
- $w_i > 0$, $w_1 + w_2 + \cdots + w_I = 1$: each type's proportion. Continuum of agents in each type.

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  - Represents type i's prior on transactions given p.
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- $x \in X_{i,p}$: transaction. $x_c > 0$: demand; $x_c < 0$: supply.
- $(X_{i,p}, \mu_{i,p})$: type i's offer space.

Statistical Economy

Definition (Statistical Economy)

A statistical economy is the object

$$\mathcal{E} = \left\{ \mathcal{I}, \{w_i\}_{i\in\mathcal{I}}, \{\mu_{i,p}\}_{i\in\mathcal{I}, p\in\Delta^{C-1}} \right\}.$$

$$\text{Economy} = \{\text{types}, \text{population}, \text{beliefs}$$
$$\text{(given the scarcity of each good)}\}.$$

Entropy, average transaction

Let f_i be a density on $X_{i,p}$, and $f = (f_1, \ldots, f_I)$.
The entropy of f is

$$H_p(f) := -\sum_{i=1}^{I} w_i \int f_i \log f_i d\mu_{i,p}.$$

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The average transaction of f is

$$\bar{x}_p(f) := \sum_{i=1}^{I} w_i \int x f_i(x) \mu_{i,p}(dx).$$

Equilibrium

Two cases to consider: genuine & degenerate.

Definition (Genuine Statistical Equilibrium)

Densities $f = (f_i)$, scarcity parameter $p \in \Delta^{C-1}$, and vector $\pi \in \mathbb{R}^C_+$ are called a genuine statistical equilibrium if

1. $f = (f_i)$ maximizes entropy subject to the feasibility constraint: f solves

$$\max H_p(f) \text{ subject to } \bar{x}_p(f) \leq 0,$$

2. π is the Lagrange multiplier of above,
3. π and p are collinear.

π (Lagrange multiplier, shadow price) is called *entropy price* (Foley 1994, Toda 2009).

Equilibrium

Definition (Degenerate Statistical Equilibrium)

A scarcity parameter $p \in \Delta^{C-1}$ and points $x_i \in \operatorname{co} X_{i,p}$ are called a degenerate statistical equilibrium if

1. $\sum_{i=1}^{I} w_i x_i \leq 0$,
2. For all i, $p'x \geq 0$ for all $x \in X_{i,p}$.

- Need to consider the degenerate case to allow distributions to concentrate on some points (density like Dirac delta function).
- In this case all transactions must have nonnegative "value", otherwise can increase entropy by spreading density.

Equilibrium

Definition (Statistical Equilibrium)

A genuine or a degenerate statistical equilibrium is simply called a statistical equilibrium.

- Foley (1994) first defined when offer sets $X_{i,p}$ are independent of p.
- Toda (2009) defined (genuine) statistical equilibrium when offer sets $X_{i,p}$ depend on p.
- Generalize to incorporate the degenerate case (point mass).

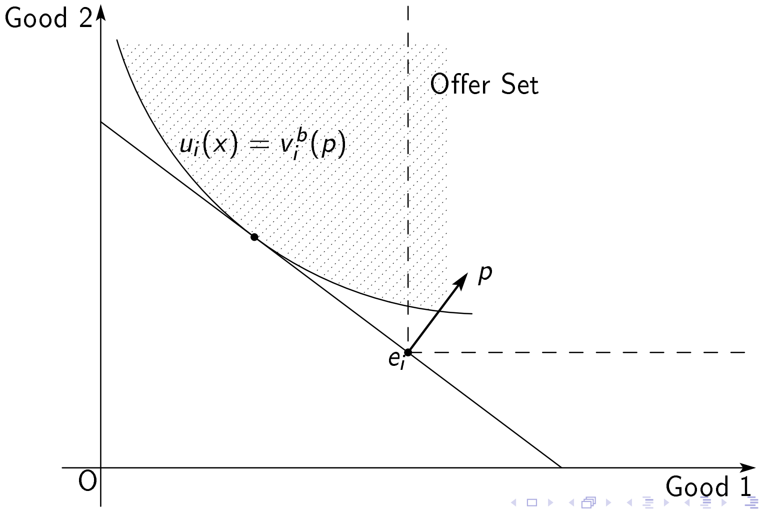
Existence

Theorem

Under some reasonable assumptions ▸ detail*, a statistical equilibrium exists.*

Corollary

A Walrasian equilibrium is a statistical equilibrium. detail

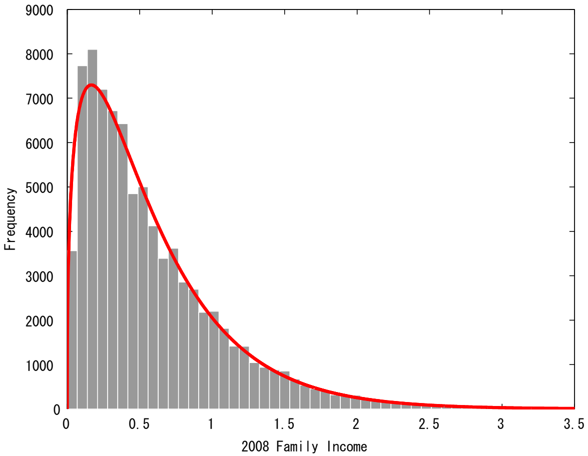


Walrasian eq $\subsetneq$ statistical eq

The corollary implies:

- Walrasian equilibrium is a special case of statistical equilibrium.
- Statistical economics can explain whatever Walrasian economics can, but not vice versa (e.g., income distribution).

Gamma density $f(x) = Cx^{\alpha}e^{-\pi x}$ maximizes entropy for the improper prior $\mu(dx) = x^{\alpha}dx$.



Conclusion

- Build a general equilibrium theory
(called *statistical equilibrium theory*)
that incorporates agents' ex post heterogeneity.
- Prove existence of equilibrium.
- Prove that Walrasian equilibrium is a special case of statistical equilibrium.
- Statistical equilibrium theory can explain "stylized facts".
- Micro-foundation of macroeconomics?

Reference

Duncan K. Foley.
A statistical equilibrium theory of markets.
Journal of Economic Theory, 62:321–345, 1994.

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Existence of a statistical equilibrium for an economy with endogenous offer sets.
Economic Theory, forthcoming.

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Duality relationships for entropy-like minimization problems.
SIAM Journal of Control and Optimization, 29:325–338, 1991.

Typical Set

$X_1, \dots, X_n$: discrete, i.i.d., with probability $p(x)$. By LLN,

$$-\frac{1}{n}\log p(X_1, \dots, X_n) = -\frac{1}{n}\sum_{i=1}^{n} \log p(X_i) \to -\operatorname{E}[\log p(X)]$$
$$= -\sum_x p(x)\log p(x) = H(\mathbf{p}).$$

Hence define

$$A_\epsilon^{(n)} := \left\{ (x_1, \dots, x_n) : \left| H(\mathbf{p}) + \frac{1}{n}\log p(x_1, \dots, x_n) \right| < \epsilon \right\}$$

to be the typical set.
Then $\# A_\epsilon^{(n)} \approx e^{nH}$.

Example

Jaynes 1982

Tossed an unfair die many times and average number was 4.5. What is the probability of each face?

Answer: Maximize $H(\mathbf{p}) = -\sum_{k=1}^{6} p_k \log p_k$ subject to

$$\sum_{k=1}^{6} p_k = 1,$$

$$\sum_{k=1}^{6} k p_k = 4.5.$$

Assumptions

A statistical equilibrium exists under reasonable assumptions.

A1

For all agent types i and scarcity parameter p, the measure $\mu_{i,p}$ is finite.

- Note that $\mu_{i,p}$ is a prior, so $\mu_{i,p}(\mathbb{R}^C) = 1$.
- Later allow improper priors, so $\mu_{i,p}$ can be infinite measures (e.g., Lebesgue).

Assumptions

A2 (boundedness from below)

$X_{i,p}$ is uniformly bounded below, i.e., there exists a vector a such that for all i, p and $x \in X_{i,p}$, we have $x \geq a$.

- Free disposal, but only up to the amount a.
- Not unrealistic since there is only a finite amount of everything in the world.

Assumptions

A3 (realistic agents)

For all agent types i and scarcity parameter p, we have

$$\inf\left\{p'x : x \in X_{i,p}\right\} \leq 0.$$

- Offer set $X_{i,p} = \operatorname{supp} \mu_{i,p}$ are those transactions that type i agents expect to engage with positive probability.
- $p'x \leq 0$ for some transactions implies that agents are realistic: agents put some probability on trades within their "budget" (evaluated at scarcity parameter p).

Assumptions

A4 (continuity of measure)

The mapping $p \mapsto \mu_{i,p}$ is weakly continuous, *i.e.*, for every sequence $\{p_n\}$ such that $p_n \to p$ and bounded measurable function f, we have

$$\lim_{n\to\infty} \int f d\mu_{i,p_n} = \int f d\mu_{i,p}.$$

A5 (continuity of offer set)

The correspondence $p \mapsto \prod_{i\in\mathcal{I}} \operatorname{cl}\operatorname{co} X_{i,p}$ is closed at those points such that $\sum_{i=1}^{I} w_i \inf\{p'x : x \in X_{i,p}\} = 0$, *i.e.*, $p_n \to p$, $x_i^n \in X_{i,p_n}$, and $x_i^n \to x_i^\infty$ implies $x_i^\infty \in X_{i,p}$ for all $i \in \mathcal{I}$ whenever $\sum_{i=1}^{I} w_i \inf\{p'x : x \in X_{i,p}\} = 0$.

Outline of the proof

- Define the log-partition function

$$Q_p(\xi) = \sum_{i=1}^{I} w_i \log\left(\int e^{-\xi' x} d\mu_{i,p}\right).$$

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- Let $b \to \infty$ and get full (genuine or degenerate) equilibrium.

Walrasian eq $\subsetneq$ statistical eq

Corollary

Let $\mathcal{E} = \{\mathcal{I}, \{u_i\}, \{e_i\}\}$ *be an endowment economy such that*

- $u_i : \mathbb{R}_+^C \to \mathbb{R}$ *is a continuous, locally non-satiated utility function of type* i *agents (with population* $w_i > 0$*),*
- *the endowments satisfy* $e_i \gg 0$ *for all* i*.*

Then,

1. *there exists a statistical economy* $\mathcal{E}'$ *such that all Walrasian equilibria of* $\mathcal{E}$ *are statistical equilibria of* $\mathcal{E}'$*,*
2. *the existence of Walrasian equilibria can be shown by using statistical equilibrium theory.*

Outline of the proof

- Take $b > 0$ large enough ($\sum_i e_i \ll b\mathbf{1}$).
  Let the constrained indirect utility be

$$v_i^b(p) := \max\left\{u_i(x) : p'x \leq p'e_i, x \in [0,b]^C\right\}.$$

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- All assumptions of existence theorem satisfied.
  Can show all statistical equilibria are degenerate.
- By construction statistical equilibria are also Walrasian.

Computation of Equilibria

In general, similar to Newton-Raphson method.

- Take initial p_0, π_0.
- Iterate over

$$\begin{aligned} \pi_{k+1} &= \pi_k - [D_\xi^2 Q_{p_k}(\pi_k)]^{-1} D_\xi Q_{p_k}(\pi_k), \\ p_{k+1} &= \pi_{k+1} / \|\pi_{k+1}\|_1 , \end{aligned}$$

where

$$Q_p(\xi) = \sum_{i=1}^{I} w_i \log \left(\int e^{-\xi' x} \mu_{i,p}(dx) \right).$$

Computation of Equilibria

If offer sets are of the form $X_{i,p} = x_{i,p} + \mathbb{R}_+^C$, then reduces to solving

$$\forall c,\, T = -p_c \sum_{i=1}^{I} w_i x_{ic,p},$$

$$\sum_{c=1}^{C} p_c = 1.$$

These are $C+1$ equations in $C+1$ $(p_1, \ldots, p_C, T)$ unknowns.
T: economic temperature; $\pi = \frac{1}{T} p$: entropy price.

Solving MEP

Duality Theorem (Borwein & Lewis 1991, 1992)

Let (X, μ) be a measure space, $a : X \to \mathbb{R}^n$ continuous, and $f \in L^1(X, \mu)$ a density. The dual problem of the MEP

$$(P): \quad \max_f \left[-\int f \log f \right] \text{ s.t. } \int af \leq 0$$

is

$$(D): \quad \min_{\xi \geq 0} \log \left(\int e^{-\xi' a} d\mu \right)$$

and the MEP has a unique solution (under mild conditions)

$$f(x) = e^{-\xi' a(x)} / \int e^{-\xi' a} d\mu.$$