Torsion Theory, Elementary Duality and their applications to the Ziegler Spectrum

MSc Pure Mathematics and Mathematical Logic

Alexis Arrowsmith

A dissertation submitted to The University of Manchester for the degree of Master of Science in the Faculty of Science and Engineering

Student ID: 11333345

Supervisor: Dr. Rose Wagstaffe

2025

Department of Mathematics, School of Natural Sciences, The University of Manchester

Contents

Abstract

This dissertation is a continuation of a prior Main Project, which can be found at [Arr25]. That project focused on developing the functorial language for modules and linking the functorial and model theoretic approaches. In this dissertation, we apply that foundation to study the Ziegler spectrum for a fixed ring. This is a topological space defined by Martin Ziegler and now arguably the central object of study in the model theory of modules. To do this, we draw on tools developed both on the model theoretic side and algebraic side of the study of modules and rely heavily on the categorical language of functor categories to synthesise them. From algebra, we draw upon torsion theories, which abstract the notion of a torsion group, whilst from model theory we draw upon elementary duality, which links the pp-formulae in the left and right languages of modules. We develop the theory of each separately and then bring them together to prove a duality of the left and right Ziegler spectra for a given ring.

Declaration

The work contained in this dissertation is, save where explicitly specified and referenced, the work of the author. It has not been submitted for for another degree or qualification at this or any other university.

Intellectual Property Statement

Acknowledgements

I would like to thank my supervisor, Dr. Rose Wagstaffe, for her help and encouragement with this dissertation. Her guidance through learning the topic and her feedback on structuring of the presentation have both been invaluable. I would also like to thank my friends and fellow mathematicians Dr. Rose Berry and Anton Farmar for listening to my endless pondering and for helping me with the fine details of Representation theory, which were far beyond me.

1 Introduction

The (right) Ziegler spectrum on a ring, introduced in [Zie84], is a topological space that captures much of the model theoretic information of the category of modules. Its points are the indecomposable pure-injective modules- indecomposable in the sense of having no proper direct summands and pure-injective capturing a notion of model theoretic saturation, that is, they realise many pp-types. The topology on the Ziegler spectrum is given by a specification of its closed sets as the intersection of the set of indecomposable pure-injective objects with the definable subcategories of **Mod-**R. The definable subcategories are those full subcategories which can be described as the vanishing locus of a set of finitely presented functors, which were introduced in the Main Project [Arr25].

From the Ziegler spectrum, we can obtain invariants such as its Cantor-Bendixson rank, which is an ordinal invariant determined by repeatedly removing isolated points. We can also use it to formulate some powerful connections between the left and right categories of modules of a ring. In particular, we shall present a result from Herzog (see [Her93]) that the left and right sets of complete theories are bijective in a tightly controlled way- that is, they are essentially the same.

To study the Zieler spectrum, we will make use of two particular tools, both introduced to the Model Theory of Modules by Prest: localisation and elementary duality. The former uses two notions of localisation in Abelian Categories, Dickinson's notion of torsion theories from [Dic63] (though our main reference is [Ste75]) and Serre subcategories as introduced in [Ser53]. In particular, we show a sequence of bijections between definable subcategories, torsion theories and Serre subcategories that we later combine with the notion of duality to prove results about the Ziegler spectrum. Elementary duality, introduced in [Pre88], initially gives us a way to take pp-formulas in the language of right-modules and obtain pp-formulas in the language of left-modules, hence giving us a lattice anti-isomorphism of left and right pp-formulas. Herzog in [Her93] extends this to a duality of functors. It is here that we then appply our torsion theoretic work to obtain an isomorphism of lattices of open sets of the left and right Ziegler spectra. We then present, without proof, Herzog's extension of this to the sets of complete theories.

For this dissertation, we presume knowledge of the contents of the Main Project ([Arr25]) and will refer back to it often. We hence inherit its assumptions of familiarity with ring and module theory. We assume here further some basic familiarity with the definitions of toplogical spaces and lattices.

In Section 2, we develop further background Abelian category theory necessary for the dissertation. The notion of a Grothendieck category is introduced and we state, without proof, a theorem of Crawley that locally finitely presentable Abelian categories, and hence all module categories under our consideration, are Grothendieck. We then move on to discusssing injective objects, which dual the projective objects introduced in [Arr25]. Many analogous results to those for projective objects are shown before moving on to some injective specific results, namely the structure theorem. The last topic we cover in Section 2 is the tensor product for modules over a small preadditive category. We first introduce the tensor product for Abelian groups and its universal property, before moving onto tensor products over general rings, where the tensor product for Abelian groups is used as part of a coend construction. We then prove that tensoring with a fixed module is right-exact and discuss how to compute the tensor product concretely in a locally finitely presentable category.

Section 3 is focused on the notion of purity. This is initially via pure-embeddings, which are monomorphisms that reflect the solutions of pp-formula, and later via pure-epimorphisms, which restrict to surjections on pp-definable groups, and pure exact sequences, which are directed colimits of split exact sequences. These are used to formulate pure-injective modules, which analogise injective modules but restricting attention from all monomorphisms to only pure-embeddings. We then develop the theory of pure-injectives analogously to that of injectives. In particular, this leads us to the tensor embedding, which is a full and faithful functor from $\mathbf{Mod}\text{-}R$ to $[R\text{-}\mathbf{mod}, \mathbf{Ab}]$ that sends each module M to its tensor functor $M \otimes -$. This embedding preserves finite limits and directed colimits and has the vital property of turning pure-exact sequences in $\mathbf{Mod}\text{-}R$ into exact sequences in $[R\text{-}\mathbf{mod}, \mathbf{Ab}]$ an vice versa. We use this embedding to make the analogy of pure-injectives to injectives rather literal, proving many similar results simply by moving to the functor category and applying the result for injectives there. We obtain, for instance, a structure theorem for pure-injectives.

The Ziegler spectrum itself is introduced in Section 4. We open with a discussion of definable subcategories of $\mathbf{Mod}\text{-}R$, with a particular focus on characterising them in terms of their categorical closure properties. We then define use them to define the Ziegler spectrum. Several detailed examples of Ziegler spectra are given for some standard rings and we define the Cantor-Bendixson rank of a point of the spectrum. The rest of Section 4 is dedicated to the development of torsion theory. Torsion theories are introduced in terms of radicals and then in terms of the defining closure properties of their torsion and torsionfree classes. We then discuss a means by which to localise or quotient a category at a torsion theory and see how this matches our usual understanding of quotienting. In the last subsection, we introduce both torsion theories of finite types and Serre subcategories and establish the bijections between definable subcategories, torsion theories of finite type and Serre subcategories.

The final section is focused on Elementary duality, beginning with the formulation of duality for formulae. To aid this formulation, we develop a calculus of matrices that correspond to pp-formulae and use this to understand the structure of the lattice of pp-formulae in terms of basic matrices. We use this to show that Elementary duality is indeed a lattice anti-isomorphism of left and right formulae. We then move on to a parallel duality, that of the functorial duality between functors on finitely presented left and right modules. This duality is exact and restricts to a duality of categories of finitely presented functors on each side. In order to understand the connection between Elementary duality and functorial duality, we study the interplay between duality of formulae and the tensor product. This leads us to Herzog's criterion, which allows us to fully describe the Abelian group structure of the tensor product in terms of pp-formulae and their duals. We use Herzog's criterion to express the pp-functor associated to a dual formula as a kernel of corresponding functorial duals. In the final subsection of the dissertation, we connect the torsion theory to the duality and use this to show the isomorphism of topologies between the left and right Ziegler spectra. We further state, without proof, the result of Herzog on bijections between left and right complete theories and explain its importance.

2 Abelian Category Theory

In this section we shall develop some more of the basic Abelian category theory required for the rest of the dissertation. Our primary source is [Pop73], though we have modernised much of the language. For the section on tensor products, we draw several definitions from [nLa25] and [Mac71].

2.1 Grothendieck Categories

In the Main Project, we identified module categories as archetypal Abelian categories. We also showed that module categories over skeletally small preadditive categories are locally finitely presentable. There are additional axioms that we can put on an Abelian category in order to obtain more structure, mostly notably in the existence and behaviour of injective and projective objects. Specifically, we are interested in Grothendieck cateories, which add exactness conditions onto the existence of directed colimits. This lets us prove, for instance, that every object embeds minimally in an injective object, called its injective hull. Every locally finitely presentable Abelian categories is Grothendieck, indeed module catgories are once again the archetypal Grothendieck category. It is nevertheless useful to identify the specific properties that make a category Grothendieck as it clarifies precisely what our proofs rely on.

We will now define the necessary structures for Grothendieck categories. Firstly, we have the AB3 and AB5 axioms, which control the shape of colimits in our category. Secondly, we have the existence of a generator, which, in a sense, gives us a single object to act as a lens to view the entire category. Hereon, let $\mathbf{C}$ be an Abelian category.

Definition 2.1. We say that $\mathbf{C}$ is an **AB3** category if it has all small coproducts.

Remark 2.2. Since Abelian categories have all coequalizers, this equivalently states that $\mathbf{C}$ has all small colimits.

Definition 2.3. We say that $\mathbf{C}$ is an **AB5** category if it is AB3 and, additionally, for every directed set I, the colimit functor $\operatorname{colim}\colon [I, \mathbf{C}] \to \mathbf{C}$ is exact.

Definition 2.4. An object U is called a **generator** if for every pair of objects X and Y and morphisms $f \neq g\colon X \to Y$, there is a morphism $h\colon U \to X$ such that $f \circ h \neq g \circ h$.

Remark 2.5. A generator is hence an object that is able to detect whether two morphisms in the category are equal.

If a category is AB3, to have a generator it suffices for there to be a small **generating family** of objects. This is a family of objects $\{U_\lambda\}_{\lambda\in\Lambda}$ such that for every pair of morphisms $f \neq g\colon X \to Y$, there is some λ_f and a morphism $h_f\colon U_{\lambda_f} \to X$ such that $f \circ h \neq g \circ h$. Since our category is AB3, we can form the direct sum $U = \bigoplus_{\lambda\in\Lambda} U_\lambda$. If ι_λ is the coprojection, it follows that there is some morphism $u_f\colon U \to X$ such that $u \circ \iota_{\lambda_f} = h_f$. Thus, U is a generating object for the category. It is frequently more convenient to work with a generating family rather than a single generator. For instance, in a module category, we have a generating family given by the class of represetable modules.

Definition 2.6. We say that $\mathbf{C}$ is a **Grothendieck category** if it is AB5 and has a generator.

The following theorem, the proof of which is far beyond our scope, also provides a whole class of examples.

Theorem 2.7 ([Cra94] Section 2.4 Discussion)**.** *Every locally finitely presentable Abelian category is Grothendieck.*

2.2 Injective Objects

Recall in the Main Project we define projective objects as a generalisation of free modules. Injective objects are the dual notion, swapping epimorphisms for monomorphisms and the covariant Hom functor for the contravariant Hom functor. Injective objects in Grothendieck categories behave very nicely and have a structure theorem that exhibits every injective object as a sum of indecomposable and superdecomposable parts. Later, we will see pure-injective objects, which are an extension of the class of injective objects formed by restricting from monomorphisms to only additive elementary embeddings. Pure-injectives are central to the model theoretic side of modules and many results about them can be converted into the corresponding result for injectives using what is called the tensor embedding. Here, we will define injective objects and prove the aforementioned structure results.

Definition 2.8. An object X is called **injective** if $\mathrm{Hom}(-,X)\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Ab}$ is exact.

We note that from this definition, it is immediate that X is a projective object of $\mathbf{C}^{\mathrm{op}}$, as indeed the functor $\mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(X,-)$ is the same as $\mathrm{Hom}_{\mathbf{C}}(-,X)$. As with projectives, there is also an alternate formulation of this in terms of factoring monomorphisms.

Proposition 2.9 ([Pop73] Chapter 3 Lemma 3.1)**.** *Let X be an object of $\mathbf{C}$. Then X is injective if and only if for every morphism $f\colon A \to X$ and monomorphism $m\colon A \to B$ there is a morphism $h\colon B \to X$ such that $f = h \circ m$. This is equivalent to the following diagram commuting.*

$$\begin{tikzcd} A \arrow[r, tail, "m"] \arrow[d, "f"'] & B \arrow[dl, dashed, "\exists h"] \\ X & \end{tikzcd}$$

Proof. Recall that Hom functors are always left-exact. Thus we only need to prove the implications for right-exactness.

For the forwards direction, given a monomorphism $m\colon A \to B$, we obtain a short exact sequence in $\mathbf{C}^{\mathrm{op}}$ from reversing the direction of m.

$$0 \longrightarrow \mathrm{coker}(m) \longrightarrow B \xrightarrow{m^{\mathrm{op}}} A \longrightarrow 0$$

Exactness of $\mathrm{Hom}(-,X)$ then says that the following sequence is exact.

$$0 \longrightarrow \mathrm{Hom}(\mathrm{coker}(m),X) \longrightarrow \mathrm{Hom}(B,X) \xrightarrow{-\circ m} \mathrm{Hom}(A,X) \longrightarrow 0$$

Hence, $j \in \mathrm{Hom}(A, X)$ is in the image of $- \circ m$ and thus there is some $h \in \mathrm{Hom}(B, X)$ with $j = h \circ m$.

For the reverse direction, we have right-exactness precisely when $\mathrm{Hom}(A, X)$ is the image of $- \circ m$. This is the condition that every $j \in \mathrm{Hom}(A, X)$ must factor as $j = h \circ m$, which is the condition formulated in the hypothesis. $\square$

Example 2.10. An Abelian group G is called **divisible** if for every $n \in \mathbb{Z} \setminus \{0\}$ and every $g \in G$, there is some $h \in G$ such that $g = nh$. That is, every element of G can be divided by every non-zero integer. Every injective Abelian group must be divisible, which can be seen from the following diagram.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, tail, "n\cdot -"] \arrow[d, "1 \mapsto g"'] & \mathbb{Z} \arrow[dl, dashed, "\exists"] \\ G & \end{tikzcd}$$

In fact, the divisible groups are exactly the injective Abelian groups [Fuc15, Chapter 4 Theorem 2.1].

The injective hull of an object X is an injective object containing X as a subobject and that is minimal satisfying this property. We equivalently define these in terms of an injective object containing X as what is called an essential subobject.

Definition 2.11. A monomorphism $f \colon X \to Y$ is called **essential** if for every $g \colon Y \to Z$, if $g \circ f$ is a monomorphism then g is a monomorphism.

Definition 2.12. An **injective hull** of an object X is an injective object $E(X)$ with an essential monomorphism $m \colon X \to E(X)$.

To see this as a minimality condition, let $m \colon X \to E(X)$ be the injective hull of X and let $n \colon X \to Y$ be another monomorphism, where Y is an injective object. By injectivity of Y, there exists some h with $n = h \circ m$. Then, $\ker(h \circ m) = \ker(n) = 0$ since n is a monomorphism. Thus, we see that $\ker(h \circ m) = \ker(m)$ and so by essentiality of m, h must be a monomorphism. We have hence exhibited $E(X)$ as a subobject of Y such that the monomorphism making X a subobject of Y factors through $E(X)$.

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, tail, "n"'] & E(X) \arrow[dl, dashed, tail, "\exists"] \\ Y & \end{tikzcd}$$

Injective hulls need not exist in arbitrary Abelian categories, but for module categories they always will. This is a property of Grothendieck categories in general.

Theorem 2.13 ([Pop73] Section 3, Theorem 10.10)**.** *Let* $\mathbf{C}$ *be a Grothendieck category. Then every object has an injective hull.*

Just as with projective objects, when an injective object appears at the beginning of a short exact sequence, we can obtain a presentation of the central object as a direct sum of the outer objects. The full definition of a split exact sequence was ommitted from the Main Project, so we include it here for completeness.

Definition 2.14. Given a short exact sequence $0 \longrightarrow A \xrightarrow{j} B \xrightarrow{k} C \longrightarrow 0$, we call it **split-exact** if there are morphisms $p\colon B \to A$ and $s\colon C \to B$ making (B, p, k, j, c) a biproduct of A and C.

Proposition 2.15. *If A is injective then the sequence splits.*

Proof. Dualise the exact sequence, thus making A a projective object at the end of the sequence in the opposite category $\mathbf{C}^{\mathrm{op}}$. Then, the sequence splits via the result for projective objects (see Proposition 3.37 of [Arr25]). We obtain B as a biproduct in $\mathbf{C}$ since biproducts are self-dual. □

We now turn to the structure theorem for injective objects. For this, we need to define both indecomposable and superdecomposable objects. The former are familiar from the theory of modules over commutative rings, whereas superdecomposable objects are more exotic.

Definition 2.16. An object A is called

- **indecomposable** if it is non-zero and there are no two subobjects A_1, A_2 of A such that $A = A_1 \oplus A_2$.
- **superdecomposable** if it is non-zero and, for every decomposition $A = A_1 \oplus A_2$, neither A_1 nor A_2 is indecomposable.

We state the theorem without proof, as it requires a significant detour beyond the topics of this dissertation.

Theorem 2.17 ([Pop73] Section 5.4 Exercise 5)**.** *Let* $\mathbf{C}$ *be a Grothendieck category and* A *be an injective object. Then there are injective indecomposables* $(A_\lambda)_{\lambda\in\Lambda}$*, unique up to isomorphism and reordering, and an injective superdecomposable* B*, unique up to isomorphism, such that* $A = E(\bigoplus_{\lambda\in\Lambda} A_\lambda) \oplus B$*.*

Example 2.18. By Theorem 3.1 of [Fuc15], the indecomposable divisble Abelian groups are precisely the Prüfer groups and their sums are also divisible and hence injective. The Prüfer groups for a prime p, written $\mathbb{Z}[p^\infty]$, is the group of all p^nth roots of unity as n varies from 0 to infinity. The superdecomposable divisible groups are direct sums of the rationals. Applying Theorem 2.17, we then see that any divisible Abelian group G can be written uniquely as

$$G \cong \bigoplus_{p \text{ prime}} (\bigoplus_{\kappa_p} \mathbb{Z}[p^\infty]) \oplus \bigoplus_{\kappa} \mathbb{Q}$$

for κ and κ_p cardinals.

Example 2.19. A (right) module is called **simple** if its only submodules are itself and 0. A module is called **semi-simple** if it is a direct sum of simple modules. A ring is called (right) semi-simple if it is semi-simple as a (right) module over itself. By Proposition 7.7 of [Ste75], all modules over a semi-simple ring are injective and so Theorem 2.17 gives us a structure theory for all modules over the ring.

For instance, a field k is simple over itself as a ring. Hence, all vector spaces are injective. Theorem 2.17 then recovers the familiar dimension theorem from linear algebra that every vector space is a sum of copies of k.

2.3 The Tensor Product

Let G, H and K be Abelian groups. Recall that a **bilinear map** from $G \times H$ to K is a function $\phi\colon G \times H \to K$ such that for every $g, g' \in G$ and $h, h' \in H$, we have that $\phi(g + g', h + h') = \phi(g, h) + \phi(g, h') + \phi(g', h) + \phi(g', h')$. Note that this is explicitly not a morphism in **Ab**. The tensor product (over $\mathbb{Z}$) of G and H has for data an Abelian group, written $G \otimes_{\mathbb{Z}} H$, combined with a bilinear map $- \otimes -\colon G \times H \to G \otimes_{\mathbb{Z}} H$ (note that we write $\otimes$ as an infix). We will drop the subscript $\mathbb{Z}$ when it is clear. These satisfy the universal property that for every bilinear map $\phi\colon G \times H \to K$, there is a unique morphism of Abelian groups $\overline{\phi}\colon G \otimes H \to K$ such that $\phi(g, h) = \overline{\phi}(g \otimes h)$. Thus, the tensor product is a universal construction turning bilinear maps out of $G \times H$ into group homomorphisms out of $G \otimes H$. This is often written as the following commuting diagram.

$$\begin{tikzcd} G \times H \arrow[r, "- \otimes -"] \arrow[dr, "\phi"'] & G \otimes H \arrow[d, dashed, "\exists ! \overline{\phi}"] \\ & K \end{tikzcd}$$

Given morphisms $\phi\colon G \to G'$ and $\psi\colon H \to H'$ of Abelian groups, we obtain a bilinear map $G \times H \to G' \otimes H'$ acting as $(g, h) \mapsto \phi(g) \otimes \psi(h)$. By the universal property of the tensor product, this factors uniquely through $G \otimes H$. We denote this factor as $\phi \otimes \psi\colon G \otimes H \to G' \otimes H'$. By stitching diagrams together, we can see that this preserves compositions. Thus, the tensor product is functorial $\mathbf{Ab} \times \mathbf{Ab} \to \mathbf{Ab}$.

$$\begin{tikzcd}
G \times H \arrow[rrr] \arrow[dr, "\phi\times\psi"] & & & G \otimes H \arrow[d, dashed, "\exists!"] \\
& G' \times H' \arrow[rr] \arrow[dr, "\theta\times\chi"] & & G' \otimes H' \arrow[dd, dashed, "\exists!"] \\
& & G'' \times H'' \arrow[dr] & \\
& & & G'' \otimes H''
\end{tikzcd}$$

Recall that an Abelian group is equivalent to a module over $\mathbb{Z}$, where the action is given by $n \cdot g = \sum_{i=1}^{n} g$. The tensor product satisfies the property that $(\sum_{i=1}^{n} g) \otimes h = \sum_{i=1}^{n}(g \otimes h) = g \otimes (\sum_{i=1}^{n} h)$. Thus, we have "moved" the action of $\mathbb{Z}$ over the tensor sign from g to h. We can take this idea and expand the notion of a tensor product over a general ring. We can even drop commutativity, instead taking the tensor product of a right module and a left module and moving the action on the right to the action on the left.

Let R be a skeletally small preadditive category, M a right-module and N a left-module. For each object A of R, we can form the tensor product over $\mathbb{Z}$ of MA and NA, as these are just Abelian groups as usual. We will use the these to define our functorial equivalent of bilinear maps.

Definition 2.20. Let $F\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ be a functor and e be an object of $\mathbf{D}$. An **extranatural transformation** $\alpha\colon F \to e$ consists of a family of maps $\{\alpha_A\colon F(A, A) \to e \mid A \in \mathbf{C}\}$ making, for all $f\colon A \to B$ in $\mathbf{C}$, the following coherence diagram commute.

$$\begin{tikzcd}
F(B, A) \arrow[r, "F(f, \mathrm{id}_A)"] \arrow[d, "F(\mathrm{id}_B, f)"'] & F(A, A) \arrow[d, "\alpha_A"] \\
F(B, B) \arrow[r, "\alpha_B"'] & e
\end{tikzcd}$$

The functors we shall consider are $M - \otimes N - \colon R^{\mathrm{op}} \times R \to \mathbf{Ab}$. That is, the functor computing the componentwise tensor product of Abelian groups. Given an Abelian group G, our analogue of a bilinear map will be an extranatural transformation $\alpha\colon M-\otimes N- \to G$. Note that the components of this transformation are morphisms $\alpha_A\colon MA\otimes NA \to G$ and hence correspond to genuine bilinear maps $MA \times NA \to G$. The extranaturality condition describes the movement of the action of the ring over the tensor sign.

Definition 2.21. The tensor product of a right-module M and left-module N has for data an Abelian group, written $M \otimes_R N$, along with an extranatural transformation $\otimes\colon M-\otimes N- \to M\otimes_R N$. This satisfies the universality condition that for every Abelian group G and extranatural transformation $\alpha\colon M-\otimes N- \to G$, there exists a unique $h\colon M\otimes N \to G$ such that $\alpha_A = h\circ\otimes_A$ for every A in R.

Remark 2.22. This is a special case of a construction called a **coend**. In general, a pair (G,α) of a group and an extranatural transformation is called a **cowedge**.

As with the tensor product over $\mathbb{Z}$, the tensor product of modules is functorial. To observe this, consider morphisms $f\colon M \to M'$ and $g\colon N \to N'$. Then, for each A, B in R, we have morphism f_A and g_B in **Ab** that have a tensor product $f_A \otimes g_B\colon MA\otimes NB \to M'A\otimes N'B$. This gives us a family of maps $f\otimes g = \{f_A \otimes g_B \mid A, B \in R\}$ which we claim gives a natural transformation $M-\otimes N- \to M'-\otimes N'-$. Indeed, let $r\colon A' \to A$ and $s\colon B \to B'$. Consider the naturality diagram below.

$$\begin{tikzcd}
MA\otimes NB \arrow[r, "Mr\otimes Ns"] \arrow[d, "f_A\otimes g_B"'] & MA'\otimes NB' \arrow[d, "f_{A'}\otimes g_{B'}"] \\
M'A\otimes N'B \arrow[r, "M'r\otimes N's"'] & M'A'\otimes N'B'
\end{tikzcd}$$

We know that $f_{A'}\circ Mr = M'r\circ f_A$ by naturality of f and similiarly for g and s. That the two paths are equal thus follows from the functoriality of the tensor product.

Given such a natural transformation $f\otimes g$ and the extranatural transformation $\otimes\colon M'-\otimes N'- \to M'\otimes_R N'$, we can compose the extranaturality diagram of $\otimes$ with the naturality diagram of $f\otimes g$ to obtain a new extranatural transformation obtain by pointwise composing $\otimes_A \circ (f_A \otimes g_A)$. That is, for each $r\colon A \to B$, the following diagram commutes.

$$\begin{tikzcd}
 & MB \otimes NA \arrow[r, "Mr\otimes \mathrm{id}_{NS}"] \arrow[d, "f_B\otimes g_A"'] \arrow[dl, equal] & MA \otimes NA \arrow[d, "f_A\otimes g_A"] \\
MB \otimes NA \arrow[r, "f_B\otimes g_A"] \arrow[d, "\mathrm{id}_{MB}\otimes Nr"'] & M'B \otimes N'A \arrow[r, "M'r\otimes \mathrm{id}_{N'A}"'] \arrow[d, "\mathrm{id}_{M'B}\otimes N'r"'] & M'A \otimes N'A \arrow[d, "\otimes_A"] \\
MB \otimes NB \arrow[r, "f_B\otimes g_B"'] & M'B \otimes N'B \arrow[r, "\otimes_B"'] & M' \otimes_R N'
\end{tikzcd}$$

Then, by the universality of the tensor product, we obtain a unique morphism $M \otimes_R N \to M' \otimes_R N'$ through which $\otimes \circ (f \otimes g)$ factors. We denote this as $f \otimes_R g$. When it is clear what we mean, we will abuse notation somewhat and drop the subscript. Similarly to the Abelian groups case, this process respects composition. We thus obtain a functor $- \otimes -\colon R\textbf{-Mod} \times \textbf{Mod-}R \to \textbf{Ab}$.

We now move on to proving several basic facts about the tensor product. We begin by computing the value of the tensor product given an arbitrary right module M and a representable left module $\text{よ}^{\mathrm{c}}A$. We will then show that the functor $M \otimes -\colon R\textbf{-Mod} \to \textbf{Ab}$ is both right-exact and preserves directed colimits. This immediately dualises for $- \otimes N$ for a fixed left module N. From this, we can compute the tensor product for arbitrary modules using local finite presentability.

Consider an extranatural transformation $\alpha\colon M - \otimes \operatorname{Hom}(A, -) \to G$. For each $r\colon A \to B$ in R, we have the following commuting diagram.

$$\begin{tikzcd}
MB \otimes \operatorname{Hom}(A, A) \arrow[r, "Mr\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes(r\circ -)"'] & MA \otimes \operatorname{Hom}(A, B) \arrow[d, "\alpha_A"] \\
MB \otimes \operatorname{Hom}(A, B) \arrow[r, "\alpha_B"'] & G
\end{tikzcd}$$

Now, suppose $b, b' \in MB$ are such that there is an $r\colon A \to B$ with $Mr(b) =$

$Mr(b')$. Then,

$$\begin{aligned}\alpha_B(b \otimes r) &= \alpha_A(Mr(b) \otimes \mathrm{id}) \\ &= \alpha_A(Mr(b') \otimes \mathrm{id}) \\ &= \alpha_B(b' \otimes r)\end{aligned}$$

That is, α is invariant under the action of the ring. We hence define the morphism $\overline{\alpha}\colon MA \to G$ by $\overline{\alpha}(a) = \alpha_B(b, r)$ for b and r such that $Mr(b) = a$. The previous discussion shows that this is well-defined. This provides a factoring for α through the evaluation map. Now, suppose we have another map $h\colon MA \to G$ such that $\alpha(b \otimes r) = h(Mr(b))$. For all $a \in MA$, we can compute $h(a)$ by taking $r = \mathrm{id}_A$ to get $h(a) = \alpha(a \otimes \mathrm{id}_A)$. Thus, h is uniquely specified. We sum up this discussion in the following proposition.

Proposition 2.23 ([Pop73] Chapter 3 Theorem 7.1)**.** *Let M be a right module and A an object of the ring R. Then, tensor product $M \otimes_R \text{よ}^c A$ is given by the Abelian group MA along with the extranatural transformation $\mathrm{ev}_B\colon MB \otimes \mathrm{Hom}(A, B) \to MA$ which acts by $\mathrm{ev}_B(b \otimes r) = Mr(b)$.*

To show that the tensor product, when fixing one side, is right exact and commutes with directed colimits, we start by proving that these hold for the tensor product of Abelian groups. It will be then be fairly simple to lift this property to the general tensor product.

Lemma 2.24 ([Pop73] Chapter 3 Theorem 7.1)**.** *The tensor product of Abelian groups is right exact and preserves directed colimits. Equivalently, the tensor product preserves all colimits.*

Proof. We start with right exactness. Fix a group G and consider the following short exact sequence of Abelian groups:

$$0 \longrightarrow H \xrightarrow{t} H' \xrightarrow{s} H'' \longrightarrow 0\ .$$

By functoriality of the tensor product, we know that $G \otimes s \circ G \otimes t = G \otimes (s \circ t) = 0$.

To show that $G \otimes s$ is the cokernel of $G \otimes t$, we need to show that $G \otimes s$ is universal composing with $G \otimes t$ to make 0.

Let $u\colon G \otimes H' \to K$ be a morphism such that $u \circ G \otimes t = 0$. By pre-composing with $\otimes\colon G \times H' \to G \otimes H'$, we get a bilinear map $u\colon G \times H' \to K$, given by $\overline{u}(g,h) = u(g \otimes h)$. Now, for each $g \in G$, we write $\overline{u}_g$ for the linear map given by $\overline{u}_g(h) = \overline{u}(g,h)$ for each $g \in G$ and $h \in H'$. Since $u(g \otimes t(h)) = 0$ for all g, we see that $\overline{u}_g \circ t = 0$. Thus, there is a unique v_g such that $\overline{u}_g = v_g \circ s$ and this extends to a bilinear map $v(g,h) = v_g(s(h))$. By the universal property of the tensor product, there is then a unique $\overline{v}\colon G \otimes H' \to K$ such that $\overline{v} \circ \otimes = v$. It follows that $u = \overline{v} \circ G \otimes s$ and this factorisation is unique.

We now prove preservation of directed colimits. Note that preserving cokernels implies preserving coequalizers, since the coequalizer of f and g is the same as the cokernel of $f - g$. Further, since all colimits decompose into a coequalizer of coproducts, it is sufficient to show that the tensor product preserves direct sums.

We consider a set $\{H_\lambda\}_{\lambda \in \Lambda}$ of groups and denote by H and ι_λ the coproduct and its coprojections respectively. Consider a family of morphisms $t_\lambda\colon G \otimes H_\lambda \to K$. For each λ, we obtain a bilinear map $\overline{t}_\lambda = t_\lambda \circ \otimes$. As before, we write $\overline{t}_{\lambda,g}(h) = \overline{t}_\lambda(g,h)$. Then, there is a unique $u_g\colon H \to K$ such that $\overline{t}_{\lambda,g} = u_g \circ \iota_\lambda$. We let u be the bilinear map given by $u(g,h) = u_g(h)$ and $\overline{u}$ the unique map $G \otimes H \to K$. It follows that $\overline{u} \circ G \otimes \iota_\lambda = G \otimes t_\lambda$, as required. □

Proposition 2.25 ([Pop73] Chapter 3 Theorem 7.1)**.** *Fix a right-module M. Then, the functor $M \otimes_R -$ preserves small colimits.*

Proof. Fix a diagram $J\colon X \to R\text{-}\mathbf{Mod}$ with colimit $(N, \eta_x\colon Jx \to N)_{x \in X}$. Consider a cocone $\varepsilon_x\colon M \otimes_R Jx \to K$ in $\mathbf{Ab}$. For each x, we obtain an extranatural transformation $\alpha_x = \{\alpha_{x,A}\colon MA \otimes Jx(A) \to K\}_{A \in R}$ given by $\alpha_{x,A} = \varepsilon_x \circ \otimes_{x,A}$.

Now, since colimits of functors are computed pointwise, we know that NA is the colimit of $(J-)(A)$ and using the previous lemma, we see that $MA \otimes NA$

is the colimit of $MA \otimes (J-)(A)$. We thus wish to show that $M \otimes_R N$ is the colimit of $M \otimes_R J-$. To see this, we note that $\alpha_{x,A}$ must factor uniquely through $MA \otimes \eta_{x,A}$ by our quick computation just now. We write $\alpha_{x,A} = h_A \circ (MA \otimes \eta_{x,A})$.

Consider the following diagram, which we wish to show commutes. The top squares is the naturality diagram for $MB \otimes \eta_x$ and the left square is the naturality diagram for $Mr \otimes \mathrm{id}$.

$$\begin{tikzcd}
 & MB \otimes Jx(A) \arrow[r] \arrow[d, "MB\otimes\eta_{x,A}"] & MB \otimes Jx(B) \arrow[d, "MB\otimes\eta_{x,B}"] \\
MB \otimes Jx(A) \arrow[ur, equal] \arrow[r, "MB\otimes\eta_{x,A}"'] \arrow[d] & MB \otimes NA \arrow[r, "\mathrm{id}\otimes Nr"'] \arrow[d, "Mr\otimes\mathrm{id}"] & MB \otimes NB \arrow[d, "h_B"] \\
MA \otimes Jx(A) \arrow[r] & MA \otimes NA \arrow[r, "h_A"'] & K
\end{tikzcd}$$

To show that the bottom right square commutes, we do the following steps.

$$\begin{aligned}
h_B \circ \mathrm{id} \otimes Nr \circ MB \otimes \eta_{x,A} &= h_b \circ MB \otimes \eta_{x,B} \circ MB \otimes Jx(r) && \text{(Naturality of } MB \otimes \eta_x\text{)} \\
&= h_A \circ MA \otimes \eta_{x,A} \circ Mr \otimes \mathrm{id} && \text{(Extranaturality of } \alpha_x\text{)} \\
&= h_A \circ Mr \otimes \mathrm{id} \circ MB \otimes \eta_{x,A}
\end{aligned}$$

We then use the fact that colimiting cocones are jointly epimorphic and the fact that we proved the above for arbitrary x to conclude that $h_B \circ \mathrm{id} \otimes Nr = h_A \circ Mr \otimes \mathrm{id}$. By the universal property of the tensor product, there is a unique $\overline{h}\colon M \otimes_R N \to K$ such that $h_A = \overline{h} \circ \otimes_A$ for all A. It follows that $\varepsilon_x = \overline{h} \circ M \otimes_R \eta_x$. This exhibits $(M \otimes_R N, M \otimes_R \eta_x)$ as a colimiting cocone for $M \otimes J-$. □

Remark 2.26. This dualises to $- \otimes_R N$ for N a left-module, for instance by viewing N as a right module over R^{op}.

3 Pure-Injectives

While purity was introduced before modern model theory had crystalised (see [Kap54] Chapter 20 for a brief history of purity), it is very elegantly expressed in model theoretic language. Namely, pure embeddings in the algebraist's sense would perhaps be called 'additively elementary embeddings' from the model theorist's perpsective: they are monomorphisms that preserve and reflect pp-definable subgroups.

Pure-injective modules are analogous to injective objects seen in Section 2.2 where we restrict the definition of injectivity from all monomorphisms to solely the pure embeddings. Pure-injective groups played a central role in Eklof and Fisher's work ([EF72]), in which they use pure-injectives to classify the complete theories of Abelian groups, systematising the earlier work of Smielew ([Szm55]). For us, pure-injectives will serve as the underlying elements of the Ziegler Spectrum, which is now the central tool in the Model Theory of Modules.

We will start by introducing purity and pure-injectives and proving several useful technical facts on how purity, exactness and finite presenability interact. We will then introduce the tensor embedding, which uses the tensor product introduced in Section 2.3 to convert pure-injective modules into injective functors and thus lets us reflect familiar results on injective objects to the pure-injective case.

3.1 Purity

Let R be a small preadditive category. We will work in **Mod-**R. Now, let $f\colon N \to M$ be any monomorphism of right-modules.

Definition 3.1. We say that f is a **pure embedding** if for every pp-formula $\phi(\overline{x})$ of context $(A_1, \ldots, A_n)$, we have that the following is a pullback diagram.

$$\begin{tikzcd} \phi(N) \arrow[r, "F_\phi f"] \arrow[d] & \phi(M) \arrow[d] \\ \bigoplus_{i=1}^n NA_i \arrow[r, "\bigoplus_{i=1}^n f_{A_i}"'] & \bigoplus_{i=1}^n MA_i \end{tikzcd}$$

Remark 3.2. For a single object ring R, this is equivalent to the condition $f(\phi(N)) = f(N) \cap \phi(M)$. For the many object case, if we identify $\phi(N)$ and $\phi(M)$ with their inclusions, this is the condition $F_\phi f(\phi(N)) = \bigoplus f_{A_i}(\bigoplus NA_i) \cap \phi(M)$.

We often phrase this as saying that ϕ does not 'grow' when passing from N to M via f. For instance, in the case where f is an inclusion of Abelian groups, while we may have the case that $\phi(M)$ is strictly larger than $\phi(N)$, ultimately no *new* elements of $\phi(M) \cap N$ are introduced by passing to M. An illustrative non-example here is the inclusion of groups $\mathbb{Z} \hookrightarrow \mathbb{Q}$ and the pp-formula $\phi(x) = \exists y.2y - x \doteq 0$, the formula expressing divisibility by 2. Then, $\phi(\mathbb{Z}) = 2\mathbb{Z}$, but in $\mathbb{Q}$, every integer is divisible by 2 and so $\phi(\mathbb{Q}) \cap \mathbb{Z} = \mathbb{Z} \neq \phi(\mathbb{Z})$.

A positive example is the first-coordinate embedding $\mathbb{Z} \hookrightarrow \mathbb{Z}^2$. Indeed, by Lemma 2.12 of [Pre09], we have that any embedding of a direct summand into a sum is a pure embedding. We find further that compositions of pure-embeddings are pure-embeddings, as are products and directed colimits. We spell out this last point more explicitly.

Lemma 3.3. *Let F and G be directed diagrams of shape Λ and let $\iota\colon F \to G$ be a natural transformation such that ι_λ is a pure-embedding for each λ. Then,* $\operatorname{colim} \iota$ *is a pure embedding.*

Proof. Let (N, η) and (M, ε) be the colimits of F and G respectively and write $f = \operatorname{colim} \iota$. Consider a pp-formula $\phi(\overline{x}) = \exists \overline{y}.\psi(\overline{x}, \overline{y})$ of context $(A_1, \dots, A_n)$ and pick some $\overline{a} \in N$ such that $f(\overline{a}) \in \phi(M)$. We must show that $\overline{a} \in \phi(N)$.

Pick some $\overline{b}$ of context $(B_1, \dots, B_m)$ such that $M \vDash \psi(f(\overline{a}), \overline{b})$. We write $C = \bigoplus_{k=1}^{t} よ^{\mathrm{c}} C_k$ for the codomain of the linear polynomials in ψ and $A = \bigoplus_{i=1}^{n} よ^{\mathrm{c}} A_i$ and $B = \bigoplus_{j=1}^{m} よ^{\mathrm{c}} B_j$ for the domain. Then ψ corresponds to a morphism $r\colon C \to A \oplus B$ and we have $(f(\overline{a})\, \overline{b}) \circ r = 0$.

We first try to find a preimage in the diagram G that satisfies ψ. Since $A \oplus B$ is finitely presented, $(f(\overline{a})\, \overline{b})$ factorises essentially uniquely through G as $\varepsilon_\mu \circ$

$(f(\overline{a})_\mu\, \overline{b}_\mu)$. Now, since $\varepsilon_\mu \circ (f(\overline{a})_\mu\, \overline{b}_\mu) \circ r = 0 = \varepsilon_\mu \circ 0$, it follows by essential uniqueness that there is some $\mu' \geq \mu$ such that $G(\mu \to \mu') \circ (f(\overline{a})_\mu\, \overline{b}_\mu) \circ r = G(\mu \to \mu') \circ 0 = 0$. We hence consider $(f(\overline{a})_{\mu'}\, \overline{b}_{\mu'}) = G(\mu \to \mu') \circ (f(\overline{a})_\mu\, \overline{b}_\mu)$ which is an element of $G\mu'$ satisfying $G\mu' \vDash \psi(f(\overline{a})_{\mu'}, \overline{b}_\mu^{'})$.

$$\begin{tikzcd}
 & M & \\
G\mu \arrow[ur, "\varepsilon_\mu"] \arrow[r] & G_{\mu'} \arrow[u, "\varepsilon_{\mu'}"'] & G_\mu \arrow[l] \arrow[ul, "\varepsilon_\mu"'] \\
A \oplus B \arrow[u, "(f(\overline{a})_\mu\, \overline{b}_\mu)"] \arrow[ur, dashed, "(f(\overline{a}_{\mu'})\, \overline{b}_{\mu'})"] & & C \arrow[u, "0"'] \arrow[ll, "r"]
\end{tikzcd}$$

Now, on the N side, we can factorise $\overline{a}$ essentially uniquely through F as $\eta_\lambda \circ \overline{a}_\lambda$. Then, by definition of f, $f \circ \overline{a} = f \circ \eta_\lambda \circ \overline{a}_\lambda = \varepsilon_\lambda \circ \iota_\lambda \circ \overline{a}_\lambda$. We also have that $\varepsilon_{\mu'} \circ f(\overline{a})_{\mu'} = f \circ \overline{a}$. By essential uniqueness, there is some $\nu \geq \lambda, \mu'$ such that $G(\lambda \to \mu) \circ \iota_\lambda \circ \overline{a}_\lambda = G(\mu' \to \nu) \circ f(\overline{a})_{\mu'}$.

Naturality of ι gets us that $G(\lambda \to \nu) \circ \iota_\lambda = \iota_\nu \circ F(\lambda \to \nu)$. We hence have the following diagram.

$$\begin{tikzcd}
N \arrow[r, "f"] & M \\
F\nu \arrow[u, "\eta_\nu"] \arrow[r, "\iota_\nu"] & G\nu \arrow[u, "\varepsilon_\nu"'] \\
A \arrow[u, "\overline{a}_\nu"] \arrow[ur, "f(\overline{a})_\nu"'] &
\end{tikzcd}$$

Now, $G\nu \vDash \phi(f(\overline{a})_\nu)$ and ι_ν is pure, so we have that $F\nu \vDash \phi(\overline{a}_\mu)$. That is, there is some $\overline{d}_\mu$ such that $(\overline{a}_\mu \circ \overline{d}_\mu) \circ r = 0$. Then, $\eta_\mu \circ (\overline{a}_\mu \circ \overline{d}_\mu) \circ r = 0$ and so $N \vDash \psi(\overline{a}, \eta_\mu \circ \overline{d}_\mu)$. Thus, $N \vDash \phi(\overline{a})$, as required. $\square$

We now define pure-injectives, essentially by taking the definition of injective and replacing all instances of "monomorphism" with "pure embedding".

Definition 3.4. An object M is called **pure-injective** if for every morphism $f\colon A \to M$ and every pure embedding $m\colon A \to B$, there is an $h\colon B \to M$ such that $f = h \circ m$. This is equivalent to the following diagram commuting for pure embeddings m.

$$\begin{tikzcd} A \arrow[r, tail, "m"] \arrow[d, "f"'] & B \arrow[dl, dashed, "\exists h"] \\ M & \end{tikzcd}$$

Over any ring, every injective module is pure-injective since pure embeddings are a restricting of the class of embeddings. An example of a non-injective but pure-injective Abelian group is the p-adic integers $\mathbb{Z}_p$ (by Theorem 5.2 of [Zie84] and following discussion). It is, in general, quite hard to identify the pure-injectives over a given ring and indeed determining the indecomposable pure-injectives (the points of the Ziegler spectrum) is an important step in bringing to bear model theoretic technology.

As with injectives, pure-injectives split certain short exact sequences, namely the pure-exact sequences which restrict from all monomorphisms to only pure embeddings.

Definition 3.5. Given a short exact sequence $0 \longrightarrow A \xrightarrow{j} B \longrightarrow C \longrightarrow 0$, we call it **pure-exact** if j is a pure embedding.

We define **pure epimorphisms** to be epimorphisms whose kernels are pure embeddings. A **pure-projective** object is then analogous to a projective object in the same was as pure-injectives are analogous to injective objects. We hence get the following splitting result by replacing all mention of epimorphism with pure-epimorphism in the proof of projectives splitting and similarly for monomorphisms.

Proposition 3.6. *Consider a pure exact sequence as follows.*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

If A is pure-injective or C is pure-projective then the sequence splits.

Pure exact sequences also interact nicely with respect to the locally finitely presentable structure of our module categories, decomposing into directed colimits of split exact sequences via the decomposition of modules into directed colimits of finitely presented modules. To prove this, we first need a few lemmas about pure epimorphisms and how they interact with finitely presented modules. We start with a free-standing characterisation of pure epimorphisms.

Proposition 3.7 ([Pre09] Proposition 2.1.14)**.** *Let $\pi\colon M \to N$ be a morphism. Then π is a pure epimorphism if and only if for each pp-formula ϕ, $F_\phi \pi$ is an epimorphism in* $\mathbf{Ab}$.

Proof. We start with the forwards direction. We consider a pp-formula $\phi(\overline{x}) = \exists \overline{y}.\psi(\overline{x}, \overline{y})$ with ψ quantifier free. Let the context of ψ be $(A_1, \dots, A_n, B_1, \dots, B_m)$ with domain $(C_1, \dots, C_t)$ for its linear polynomials. Setting $A = \bigoplus \text{よ}^{\mathrm{c}} A_i$, $B = \bigoplus \text{よ}^{\mathrm{c}} B_j$ and $C = \bigoplus C_k$, we represent ψ with a morphism $r\colon C \to A \oplus B$. Let $k\colon K \to M$ be the kernel of π, which is a pure embedding by definition of π as a pure epimorphism.

We now pick some $(\overline{c}\,\overline{d})$ in $\psi(N)$. Since π is an epimorphism, π_D is a surjection of Abelian groups for each D in R and so $\pi_{A\oplus B}$ is an epimorphism $MA \oplus MB \to NA \oplus NB$. We pick some $(\overline{e}\,\overline{b})$ preimages of $(\overline{c}\,\overline{d})$ under $\pi_{A\oplus B}$. Since $(\overline{c}\,\overline{d}) \circ r = 0$, we have that $\overline{a} = (\overline{e}\,\overline{b}) \circ r \in \mathrm{im}(k)$.

We define a new pp-formulae, $\chi(\overline{z}) = \exists \overline{x}, \overline{y}.(\overline{x}\,\overline{y}) \circ r = \overline{z}$. Thus, $M \vDash \chi(\overline{a})$. Since $\overline{a} \in \mathrm{im}(k)$, there is some $\overline{\alpha} \in KC$ such that $\overline{a} = k_C(\overline{\alpha})$. By purity of k, we have that $K \vDash \chi(\overline{\alpha})$ and we pick witnesses $(\overline{\alpha}', \overline{\alpha}'')$ for this. We set $\overline{a}' = k_C(\overline{\alpha}')$ and $\overline{a}'' = k_C(\overline{\alpha}'')$.

Now, $(\overline{e} - \overline{a}'\,\overline{b} - \overline{a}'') \circ r = (\overline{e}\,\overline{b}) \circ r - (\overline{a}'\,\overline{a}'') \circ r = \overline{a} - \overline{a} = 0$. Thus, $M \vDash \phi(\overline{e}\overline{a}')$. Further, $\pi(\overline{e} - \overline{a}) = \pi(\overline{e})$ since $\overline{a} \in \mathrm{im}(\ker(\pi))$. Thus, we have a preimage of $\overline{c}$ in $\phi(M)$, and hence surjectivity of $F_\phi \pi$.

For the backwards direction, we suppose that $F_\phi \pi$ is an epimorphism for each ϕ. In particular, for each A in R, we have the formulae $\phi(x) = x \doteq x$, which defines the whole of MA for each module M. Then, $F_\phi \pi = \pi_A$ is an epimorphism for each A and so π is an epimorphism.

Copying the set up of $\phi(\overline{x}) = \exists \overline{y}.\psi(\overline{x}, \overline{y})$ from the forwards direction, we pick some $\overline{a} \in K$ such that $M \vDash \phi(k_A(\overline{a}))$. Hence, there is some $\overline{b}$ with $M \vDash \psi(k_A(\overline{a}), \overline{b})$. We define the new pp-formulae $\chi(\overline{y}) = (\overline{0}\, \overline{y}) \circ r = 0$. Then, since $\overline{a} \in \ker(\pi)$, we have that $\pi \circ (k(\overline{a})\, \overline{b}) \circ r = (\overline{0}\, \pi(\overline{b})) \circ r = 0$ and so $N \vDash \chi(\overline{b})$.

Since $F_\chi \pi$ is epimorphic, there is some $\overline{b}^{\prime} \in M$ such that $M \vDash \chi(\overline{b}^{\prime})$ and $\pi(\overline{b}^{\prime}) = \pi(\overline{b})$. Then, $\pi(\overline{b} - \overline{b}^{\prime}) = 0$ and so $\overline{b} - \overline{b}^{\prime} \in \mathrm{im}(k)$. Pick some preimage $\overline{\beta}$ in K for $\overline{b} - \overline{b}^{\prime}$. Then, $k \circ (\overline{a}\, \overline{\beta}) \circ r = (k(\overline{a})\, \overline{b}) \circ r - (\overline{0} \circ \overline{b}^{\prime}) \circ r = 0$. Since k is monomorphic, we cancel on the left to find $(\overline{a}\, \overline{\beta}) \circ r = 0$ and so $K \vDash \psi(\overline{a}, \overline{\beta})$. Thus, $K \vDash \phi(\overline{a})$, showing that k is a pure embedding. $\square$

In the following corollary, we restrict our attention solely to the finitely presented modules. We recall here that, under the functor-pair duality from the Main Project (see Theorem 6.11 [Arr25]), pp-formulae correspond to pairs of finitely presented modules and distinguished elements (those being the free realisers of the formulae).

Corollary 3.8 ([Pre09] Corollary 2.1.16)**.** *Let $\pi \colon M \to N$ be a morphism of modules. Then π is a pure epimorphism if and only if for each finitely presented module C, every morphism $g \colon C \to N$ factors through π as $g = \pi \circ g'$.*

Proof. For the forwards direction, for an arbitrary finitely presented module C, let $\overline{c} \colon \bigoplus_{i=1}^{n} よ^{\mathrm{c}} C_i \to C$ be an epimorphism (which exists by Proposition 4.10 of [Arr25]). By Proposition 5.9 of the Main Project, there is some pp-formula ϕ such that $(C, \overline{c})$ is a free realiser of ϕ.

Now, $g(\overline{c}) \in \phi(N)$ and since π is a pure epimorphism, using Propsition 3.7, there is some $\overline{b} \in \phi(M)$ with $\pi(\overline{b}) = g(\overline{c})$. Since $(C, \overline{c})$ is a free realiser for ϕ, there is a unique morphism $g' \colon C \to M$ such that $g' \circ \overline{c} = \overline{b}$. Then, we have that $\pi \circ g' \circ \overline{c} = g \circ \overline{c}$ and since $\overline{c}$ is an epimorphism, we find that $\pi \circ g' = g$ as required.

For the backwards direction, we pick an arbitrary pp-formulae ϕ. This has a free realisation $(C, \overline{c})$. For each $\overline{a} \in \phi(N)$, there is a unique morphism $g \colon C \to N$

such that $\overline{a} = g \circ \overline{c}$. By assumption, g factors through π as $g = \pi \circ g'$. Set $\overline{b} = g' \circ \overline{c}$, which must be in $\phi(M)$ since $(C, \overline{c})$ is a free realiser. Then, we have that $\overline{a} = \pi(\overline{b})$, which is the preimage required for π to be a pure epimorphism as in Proposition 3.7. $\square$

The other important lemma we show is that, just as with normal epimorphisms, pure epimorphisms are preserved under pullback.

Proposition 3.9. *Let $\pi\colon M \to N$ be a pure epimorphism and let $f\colon Q \to N$ be any morphism. Then, in the following pullback diagram, p_π is a pure epimorphism.*

$$\begin{tikzcd} M \arrow[r, "\pi"] & N \\ P \arrow[u, "p_f"] \arrow[r, "p_\pi"'] \arrow[ur, phantom, "\urcorner", very near start] & Q \arrow[u, "f"'] \end{tikzcd}$$

Proof. Pick an arbitrary pp-formulae ϕ with free realisation $(C, \overline{c})$. Suppose we have $\overline{d} \in \phi(Q)$. To show p_π is a pure epimorphism, using Proposition 3.7, we want to show that $\overline{d}$ has a preimage in $\phi(P)$.

Now, since $(C, \overline{c})$ is a free realisation, there is a unique morphism $g\colon C \to Q$ such that $\overline{d} = g \circ \overline{c}$. Let $\overline{a}\colon A \to C$ be an epimorphism, where A is a finite sum of representables and let $s\colon B \to A$ be its kernel. We can define a new pp-formulae $\psi(\overline{x}) = \overline{x} \circ s \doteq 0$.

We have that $N \vDash \psi(f \circ g \circ \overline{a})$ and so, since π is a pure epimorphism, there is some $\overline{b}$ with $M \vDash \psi(\overline{b})$ and $\pi \circ \overline{b} = f \circ g \circ \overline{a}$. Since $\overline{a}$ is an epimorphism, it is the cokernel of s and so, since $\overline{b} \circ s = 0$, there is a unique morphism $h\colon C \to N$ with $\overline{b} = h \circ \overline{a}$ from the cokernel universal property. We then have that $\pi \circ h \circ \overline{a} = f \circ g \circ \overline{a}$ and since $\overline{a}$ is an epimorphism, we obtain $\pi \circ h = f \circ g$.

We have thus built a cone for the pullback diagram and hence can apply the universal property of the pullback as shown in the diagram below.

$$\begin{tikzcd}
 & M \arrow[r, "\pi"] & N \\
 & P \arrow[u, "p_f"'] \arrow[r, "p_\pi"] & Q \arrow[u, "f"'] \\
C \arrow[uur, "h"] \arrow[ur, dashed, "\exists! u"] \arrow[urr, "g"'] & &
\end{tikzcd}$$

It then follows that $u \circ \overline{c}$ is an element of $\phi(P)$ and satisfies $p_\pi \circ u \circ \overline{c} = g \circ \overline{c} = \overline{d}$. This is what was required to show that p_π is a pure epimorphism. $\square$

We now apply these facts to show how pure-exact sequences can be decomposed into directed colimits of split exact sequences.

Proposition 3.10 ([Pre09] Proposition 2.14)**.** *The pure-exact sequences in* ***Mod-****R are precisely the directed colimits of split exact sequences of finitely presented modules.*

Proof. We first show that directed colimits of split exact sequences are pure. This amounts to showing that split monomorphisms are pure, since directed colimits of pure monomorphisms are pure. We let $m\colon M \to N$ be a split monomorphism with an epimorphism $r\colon N \to M$ satisfying the splitting condition $r \circ m = \mathrm{id}_M$. Let $\phi(\overline{x})$ be a pp-condition.

Now, consider an element $\overline{a}$ of M such that $m(\overline{a}) \in \phi(N)$. By functoriality of pp-formulae, we know that $r(\phi(N)) \subset \phi(M)$ and so $r(m(\overline{a})) \in \phi(M)$. Since $r \circ m = \mathrm{id}$, we have hence shown that $\overline{a} \in \phi(M)$, as required.

We now move to showing that every pure exact sequence can be exhibited as a directed colimit of split exact sequences. We consider an arbitrary pure exact sequence as below.

$$0 \longrightarrow M \xrightarrow{m} N \xrightarrow{\pi} Q \longrightarrow 0$$

Now, since **Mod-**R is locally finitely presentable, there is some directed diagram $F\colon X \to \textbf{Mod-}R$ such that each Fx is finitely presented and $(Q, \eta) = \operatorname{colim} F$

for some co-cone η. We now define a new diagram $G\colon X \to \mathbf{Mod}\text{-}R$ by setting, for each x, Gx to be the pullback of η_x along π and defining the action on morphisms by functoriality of limits. We hence get a cocone $\varepsilon\colon G \to N$ and a natural transformation $\pi\colon N \to Q$ (abusing notation somewhat) as shown in the following diagrams.

$$
\begin{tikzcd}
N \arrow[r, "\pi"] & Q \\
Gx \arrow[u, "\varepsilon_x"] \arrow[r, "\pi_x"'] & Fx \arrow[u, "\eta_x"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & & Q \arrow[r] & Q \\
N \arrow[urr] \arrow[r] & N \arrow[urr] & Fx \arrow[u, dashed] \arrow[r, "Ff"] & Fy \arrow[u] \\
Gx \arrow[u] \arrow[r, "Gf"'] \arrow[urr, dashed, "\pi_x"] & Gy \arrow[u] \arrow[urr, "\pi_y"] & &
\end{tikzcd}
$$

We use the fact that pullbacks preserve kernels to get the following parallel exact sequences.

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "m"] \arrow[d, equal] & N \arrow[r, "\pi"] & Q \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r, "\ker(\pi_x)"'] & Gx \arrow[u, "\varepsilon_x"] \arrow[r, "\pi_x"'] & Fx \arrow[u, "\eta_x"] \arrow[r] & 0
\end{tikzcd}
$$

Since we have a constant diagram on the left, we know that the colimit there is A. That N is the colimit of G follows from Proposition 1.78 of [Arr25], which states that directed colimits commute with finite limits in locally finitely presentable categories. By Theorem 2.7, $\mathbf{Mod}\text{-}R$ is a Grothendieck category and hence directed colimits are exact, which verifies that the directed colimit of the system of exact sequences is our original exact sequence.

Now, by Proposition 3.9, we know that π_x is a pure epimorphism and hence each component sequence of our directed sequence is pure exact. We now apply Corollary 3.8 for $g = \mathrm{id}_{Fx}$, since Fx is finitely presented for each x. This shows that π_x must split as $\mathrm{id}_{Fx} = \pi_x \circ \sigma_x$ for some $\sigma_x\colon Fx \to Gx$. Following the argument from Proposition 3.37 of [Arr25], we see that the component sequences are split. This completes the proof. $\square$

3.2 The Tensor Embedding

Recall the tensor product from Section 2.3. For each right module M, we have a functor $M \otimes - \colon R\textbf{-mod} \to \mathbf{Ab}$ on finitely presented left modules. This process of sending each module to its corresponding tensor functor is in itself functorial. We call this functor the **tensor embedding** and denote it by ϵ.

The tensor embedding is defined on morphisms as follows. For a given morphism of right modules $f \colon M \to N$, ϵf is a natural transformation whose component at a left-module L is the map $f \otimes L$.

The name "embedding" is indicative- we prove in the following theorem that it is full and faithful. Further, it behaves nicely with respect to our local finitely presentability and, perhaps most importantly, translates pure-exactness in the module category into exactness in the functor category. We shall use this property to reflect proofs about exact sequences and injective objects into proofs about pure-exact sequences and pure-injective objects. We note that several proofs have been omitted as requiring more model theory than developed in this dissertation.

We start with a lemma which states that natural transformations between right-exact functors on finitely presented modules (in particular, tensor functors) are determined by their components on representables.

Lemma 3.11. *Let $F, F' \colon R\textbf{-mod} \to \mathbf{Ab}$ be right-exact functors and $\tau, \tau' \colon F \to F'$ be natural transformations such that, for each A in R, $\tau_{\text{よ}^c A} = \tau'_{\text{よ}^c A}$. Then $\tau = \tau'$.*

Proof. Let L be an arbitrary finitely presented left module with a presentation given by the following right exact sequence, for A and B finite sums of representables.

$$B \xrightarrow{f} A \xrightarrow{\pi} L \longrightarrow 0$$

By right-exactness of F and F', we get the following commuting diagram.

$$
\begin{tikzcd}
FB \arrow[r, "Ff"] \arrow[d, "\tau_B=\tau'_B"'] & FA \arrow[r, "F\pi"] \arrow[d, "\tau_A=\tau'_A"'] & FL \arrow[r] \arrow[d, shift right, "\tau_L"'] \arrow[d, shift left, "\tau'_L"] & 0 \\
F'B \arrow[r] & F'A \arrow[r] & FL \arrow[r] & 0
\end{tikzcd}
$$

Then, by functoriality of colimits (Proposition 1.48 of [Arr25]), we know that there is a unique morphism on the right making the diagram commute and hence we must have $\tau_L = \tau'_L$. Since we picked arbitrary L, this shows that $\tau = \tau'$. $\square$

We now make use of this lemma in the proof of full and faithfullness in the following theorem.

Theorem 3.12 ([Pre09] Theorem 12.1.3)**.** *The tensor embedding is full, faithful and commutes with directed colimits and finite limits.*

Further, each sequence $0 \to M \to N \to P \to 0$ *of right modules is pure-exact in* ***Mod**-R* *if and only if* $0 \to \epsilon M \to \epsilon N \to \epsilon P \to 0$ *is exact in* $[\boldsymbol{mod}\text{-}R, \mathbf{Ab}]$.

Proof. Let M and N be right modules and consider a natural transformation $\tau\colon \epsilon M \to \epsilon N$. For each A in R, we recall from Proposition 2.23 that $(\epsilon N)(\text{よ}^{c} A) = NA$ and so we obtain morphisms $\tau_{\text{よ}^{c} A}\colon MA \to NA$. This is the data of a natural transformation $\eta\colon M \to N$. We see it is natural as, for every morphism $f\colon A \to B$, $(\epsilon M)f = Mf$ and we have $\eta_A \circ Mf = \tau_{\text{よ}^{c} A} \circ M \otimes f = N \otimes f \circ \tau_{\text{よ}^{c} B} = Nf \circ \eta_B$.

We now apply Lemma 3.11 to the fact that $\tau_{\text{よ}^{c} A} = (\eta \otimes -)_{\text{よ}^{c} A}$ for each A in R to get $\tau = \epsilon\eta$. This shows that ϵ is full. We also obtain faithfullness, since η is fully determined by its image $\epsilon\eta$ and so if $\epsilon\eta = \epsilon\eta'$ then we quickly get $\eta = \eta'$.

For the fact about colimits and finite limits, we omit the proof here, referring back to [Pre09] Theorem 12.1.4 as it uses more theory than we have developed.

We delay but do not omit the proof of the last part of the theorem to Section 5.3 as the proof makes use of Elementary duality. $\square$

This theorem can be strengthened to show preservation of pure-exact sequences, which we shall state but not prove.

Theorem 3.13 ([Pre09] Theorem 12.1.6)**.** *Let* $0 \to M \to N \to P \to 0$ *be an exact sequence of right modules. This sequence is pure-exact if and only if* $0 \to \epsilon M \to \epsilon N \to \epsilon P \to 0$ *is pure-exact.*

The tensor embedding further restricts to a functor between the finitely presented modules and the finitely presented functors. We omit the proof for this as it requires further homological methods than covered in our scope.

Proposition 3.14 ([Aus66] Lemma 6.1)**.** *A right-module* M *is finitely presented if and only if* ϵM *is a finitely presented functor.*

We now introduce pure-essential morphisms and pure-injective hulls, which are wholly analogous to their injective counterparts in the pure setting.

Definition 3.15. A pure-embedding $f \colon X \to Y$ is called **pure-essential** if, for every morphism $g \colon Y \to Z$, if $g \circ f$ is a pure-embedding then so is g.

Definition 3.16. A **pure-injective hull** of an object X is a pure-injective object $H(X)$ and a pure-essential morphism $m \colon X \to H(X)$.

We can then use Theorem 3.12 to prove several basic facts about pure essential embeddings. These will allow us to prove the existence of pure-injective hulls.

Corollary 3.17. *A morphism* $f \colon X \to Y$ *is a pure-embedding if and only if* ϵf *is monomorphic.*

Proof. Consider the following right exact sequence.

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{\mathrm{coker}(f)} Z \longrightarrow 0$$

This is pure exact if and only if the image under ϵ is exact. This occurs only when ϵf is a monomorphism. $\square$

We will make use of the following important lemma from [Pre09] which we present without proof as it requires more theory than we cover in this dissertation.

Lemma 3.18 ([Pre09] Theorem 12.1.6 and Lemma 4.3.12)**.** *Every injective functor in* $[R\text{-}\boldsymbol{mod}, \mathbf{Ab}]$ *is isomorphic to a functor of the form* ϵN *for some right module* N*.*

The second corollary of Theorem 3.12 is analogous to Corollary 3.17 but for specifically essential embeddings.

Corollary 3.19. *A pure embedding* $m\colon X \to Y$ *is pure-essential if and only if* ϵm *is an essential embedding.*

Proof. Suppose ϵm is an essential embedding. Let $f\colon Y \to Z$ be a morphism such that $f \circ m$ is a pure embedding. Then, $\epsilon(f \circ m) = \epsilon f \circ \epsilon m$ is an embedding by Corollary 3.17 and so by essentiality of ϵm, we have that ϵf is a monommorphism. Then again by Corollary 3.17, f is a pure embedding. This shows that m is pure-essential.

Now, suppose that m is pure essential and let $\alpha\colon \epsilon Y \to F$ be a morphism such that $\alpha \circ \epsilon m$ is a monomorphism. We let $e\colon F \to E(F)$ be an injective hull for F. By Lemma 3.18 we know that $E(F)$ is in fact of the form ϵN for some module N. We set $\alpha' = e \circ \alpha$. By Theorem 3.12, ϵ is a full functor and so $\alpha' = \epsilon f$ for some $f\colon Y \to N$.

Since $\alpha' \circ \epsilon m$ is a monomorphism, we have that $f \circ m$ is a pure embedding and hence f is a pure embedding by pure-essentiality of m. We then move back to the functor category and see that $\epsilon f = \alpha'$ is a monomorphism. Since $\alpha' = e \circ \alpha$ and e is monic by definition, we obtain that α is monic. This is as was required to show that ϵm is essential. □

We can now prove the existence of pure-injective hulls as our first major application of Theorem 3.12.

Theorem 3.20. *Every object in* ***Mod**-R* *has a pure-injective hull that is unique up to isomorphsim.*

Proof. Let M be a right module. Then, we consider ϵM in the functor category and an injective hull $e \colon \epsilon M \to X$. Once again, we use Lemma 3.18 to write $X = \epsilon N$ for some module N. Then, by fullness of ϵ, $e = \varepsilon f$ for some $f \colon M \to N$. Then, by Corollary 3.19, f is a pure-essential embedding since ϵf is an essential embedding. □

Remark 3.21. We note that, as well as existence of pure-injective hulls, we have that the pure-injective hull of M is sent under the tensor embedding to the injective hull of ϵM and so we have a specific form for the pure-injective hull.

We can use this to identify the pure-injective modules precisely in terms of the injective functors.

Corollary 3.22. *A module M is pure-injective if and only if ϵM is injective.*

Proof. Note that the identity map is always a pure-essential (and hence essential) embedding. We then note that M is pure injective if and only if $\mathrm{id}_M \colon M \to M$ is a pure-injective hull for M. By Remark 3.21, this occurs only when $\mathrm{id}_{\epsilon M}$ is an injective hull for ϵM which is the case only when ϵM is injective. □

Our other major application of Therorem 3.12 is a pure-injective version of Theorem 2.17, the structure theorem for injective objects in a Grothendieck category.

Theorem 3.23 ([Pre09] Theorem 4.4.2)**.** *Let* $\mathbf{C}$ *be a module category and A be a pure-injective object. Then there are pure-injective indecomposables $(A_\lambda)_{\lambda \in \Lambda}$, unique up to isomorphism and reordering, and a pure-injective superdecomposable B such that $A = H(\bigoplus_{\lambda \in \Lambda} A_\lambda) \oplus B$.*

Proof. By Corollary 3.22, A is an injective functor and so we apply Theorem 2.17 to obtain a decomposition $A \cong E(\bigoplus_{\lambda \in \Lambda} A_\lambda) \oplus B$ for A_λ indecomposable injectives and B a superdecomposable injective. We then use Lemma 3.18 to find modules M_λ and N such that $A_\lambda \cong \epsilon M_\lambda$ and $B \cong \epsilon N$. Then, the M_λ and N are pure-injectives and indecomposability and superdecomposability are reflected by the fact that ϵ is a full additive embedding. The pure-injective hull $H(\bigoplus_{\lambda\in\Lambda} M_\lambda)$ is then mapped (up to isomorphism) to the injective hull $E(\bigoplus_{\lambda\in\Lambda} A_\lambda)$ by Remark 3.21.

From all this, and the fact that ϵ is full and faithful and so reflects isomorphisms, we see that $A \cong H(\bigoplus_{\lambda\in\Lambda} M_\lambda) \oplus N$. Uniqueness up to isomorphism and reordering simply follows by mapping any alternate decomposition under ϵ and implying uniqueness for the injective decomposition. $\square$

4 The Ziegler Spectrum

The Ziegler Spectrum is the central object in the Model Theory of Modules. It is a topological space defined on the set of indecomposable pure-injective modules for a ring, with a topology defined by the pp-pairs, equivalently the definable categories. It is thus a Morita-invariant of the ring (that is, an invariant up to equivalence of module categories) and simpler invariants can be derived from it- for instance a numerical invariant in the form of the Cantor-Bendixson rank of the spectrum. We will see that various properties of the ring are captured by and can be derived from the Ziegler spectrum and in later sections we will analyse it to derive broad-scale model-theoretic results.

We note that there are both a left and right Ziegler spectrum of a ring, depending on which module category we are looking at. For the time being, we will look at the right spectrum and all our results will transfer to the left spectrum via viewing left modules as right modules over the opposite ring. Later, we will investigate dualities between left and right modules and both spectra will be of importance.

4.1 Definable Categories

We start with defining Definable Subcategories, which are full subcategories of the category of modules of a ring determined by a class of modules on which a set of pp-pairs (equivalently, finitely presented functors) vanishes. We fix a small preadditive category R.

Definition 4.1. Given a set of pp-pairs $T = \{\phi_\lambda/\psi_\lambda\}_{\lambda\in\Lambda}$ over R, the category **defined by** T is the full subcategory of **Mod-**R given on objects by $\mathrm{Mod}(T) = \{M \in \mathbf{Mod}\text{-}R \mid \phi_\lambda(M) = \psi_\lambda(M) \text{ for all } \lambda \in \Lambda\}$.

Such a subcategory (resp. its class of objects) is called a **definable subcategory** (resp. **definable subclass**) of **Mod-**R.

These subcategories satisfy several closure properties which in fact entirely determine the class of definable subcategories, allowing us to omit mention of pp-pairs in the definition. We will prove that definable subcategories satisfy these closure properties and omit the reverse proof as it requires a diversion into some more model theoretic machinery that is beyond our scope.

Theorem 4.2 ([Pre09] Theorem 3.4.7)**.** *A given subclass $\mathcal{X}$ of **Mod-**R is definable if and only if it is closed under direct products, directed colimits and pure submodules.*

Proof. By definition of a pure submodule, if $N \hookrightarrow M$ is a pure embedding and ϕ/ψ is a pp-pair, then we have the following pullbacks.

$$\begin{tikzcd} \phi(N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \phi(M) \arrow[d] \\ \bigoplus NA \arrow[r] & \bigoplus MA \end{tikzcd} \qquad \begin{tikzcd} \psi(N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \psi(M) \arrow[d] \\ \bigoplus NA \arrow[r] & \bigoplus MA \end{tikzcd}$$

Now, if $\phi/\psi(M) = 0$, then $\phi(M) \cong \psi(N)$. Since pullbacks are determined up to isomorphism, it follows that the pullback of $\phi(M)$ along $\bigoplus NA \to \bigoplus MA$ is isomorphism to that of the pullback of $\psi(M)$. That is, $\phi(N) \cong \psi(N)$ and hence $\phi/\psi(N) = 0$.

To see closure under direct products, consider a pp-formula ϕ and modules $\{M_\lambda\}_{\lambda\in\Lambda}$ and setting $M = \prod_{\lambda\in\Lambda} M_\lambda$. Then, the addition and ring action on M is defined componentwise, since limits are computed componentwise in functor categories. Consider a tuple of tuples in M, that is some $(\overline{a}_\lambda)_{\lambda\in\Lambda}$ where each $\overline{a}_\lambda = (a_{\lambda 1}, \dots a_{\lambda n}) \in M_\lambda A_1 \oplus \cdots \oplus M_\lambda A_n$. It follows that, $(\overline{a}_\lambda)H = 0$ if and only if for each $\lambda \in \Lambda$, $\overline{a}_\lambda H = 0$. Similarly, existence can be determined componentwise. We thus see that $\phi(M) = \prod_{\lambda\in\Lambda} \phi(M_\lambda)$.

To apply this to closure, we simply observe that, given a pp-pair ϕ/ψ, if $\phi(M_\lambda) = \psi(M_\lambda)$ for each λ, then $\phi(M) = \prod \phi(M_\lambda) = \prod \psi(M_\lambda) = \psi(M)$.

For closure under directed colimits, we consider a directed system $(M_\lambda, f_{\lambda\mu} \colon M_\lambda \to M_\mu)$ and its colimit $(M, f_{\lambda\infty} \colon M_\lambda \to M)$. Now, suppose that for some pp-pair ϕ/ψ, we have that $\phi(M_\lambda) = \psi(M_\lambda)$ for each λ. We consider some tuple a of $\phi(M)$. By Proposition 5.8 of [Arr25], there is some free realiser (C, c) of ϕ and hence a morphism $g \colon C \to M$ such that $a = g(c)$.

Since C is finitely presented, by Proposition 1.68 of [Arr25], g must factor essentially uniquely through the system as $g = f_{\lambda\infty} \circ g'$. Now, $\phi(M_\lambda) = \psi(M_\lambda)$ and so $g'(a) \in \psi(M_\lambda)$. By functoriality of pp-formulae, we know that $f_{\lambda\infty}(\psi(M_\lambda)) \hookrightarrow \psi(M)$. Hence, $g(c) = f_{\lambda\infty}(g'(a)) \in \psi(M)$. This shows that $\phi(M) = \psi(M)$. $\square$

We use this theorem to prove a characterisation of finite unions of definable categories which will be necessary for defining the Ziegler spectrum.

Corollary 4.3 ([Pre09] 3.4.9)**.** *Let $\mathcal{X}_1, \dots, \mathcal{X}_n$ be definable subcategories of* ***Mod**-R. Them, smallest definable subcategory containing $\mathcal{X}_1 \cup \cdots \cup \mathcal{X}_2$ is the full subcategory consisting of exactly those modules that can be purely embedded in some sum $M_1 \oplus \cdots \oplus M_n$ where each M_i is in $\mathcal{X}_i$.*

Proof. We shall suppose that $\mathcal{X}$ is category of pure submodules of sums $M_1 \oplus \cdots \oplus M_n$. If $\mathcal{Y}$ is another definable subcategory containing $\mathcal{X}_1 \cup \cdots \cup \mathcal{X}_n$, then by Theorem 4.2 it is closed under pure submodules and direct products (and hence finite direct sums) and thus must contain $\mathcal{X}$. Thus, if $\mathcal{X}$ is definable, it is minimal. We now show that $\mathcal{X}$ satisfies the conditions of Theorem 4.2. We first note that closure under pure submodules follows from the definition of $\mathcal{X}$.

For products, we suppose that M_λ is a pure submodule of $M_{1\lambda} \oplus \cdots M_{n\lambda}$. Then, by the discussion before Lemma 3.3, we know that products of pure embeddings are pure and so $\prod_{\lambda\in\Lambda} M_\lambda$ is pure in $\prod_{\lambda\in\Lambda}(M_{1\lambda} \oplus \cdots M_{n\lambda})$. The finite sum is also a finite product and limits commute with limits, so we have that $\prod_{\lambda\in\Lambda}(M_{1\lambda} \oplus \cdots M_{n\lambda}) \cong \prod_{\lambda\in\Lambda} M_{1\lambda} \oplus \cdots \oplus \prod_{\lambda\in\Lambda} M_{n\lambda}$. Each of these summands is in their corresponding $\mathcal{X}_i$ by closure under products and hence we see that $\prod_{\lambda\in\Lambda} M_\lambda$ is pure in $\prod_{\lambda\in\Lambda} M_{1\lambda} \oplus \cdots \oplus \prod_{\lambda\in\Lambda} M_{n\lambda}$ which satisfies the condition for $\prod_{\lambda\in\Lambda} M_\lambda$ to be in $\mathcal{X}$.

Analogously for directed colimits, we recall from Lemma 3.3 that directed colimits of pure embeddings are pure. Now, consider a directed system $(M_\lambda, f_{\lambda\mu})_{\lambda,\mu\in\Lambda}$ such that for each λ, M_λ is pure in $M_{1\lambda} \oplus \cdots M_{n\lambda}$. For each i, $M_i = \bigoplus_{\lambda\in\Lambda} M_{i\lambda}$ is in $\mathcal{X}_i$ by closure under directed colimits and so we have that M_λ is pure in $M_1 \oplus \cdots \oplus M_n$ by composing with the coprojection. We now have a directed system as shown in the following diagram.

$$\begin{tikzcd}
M_\mu \arrow[rr] \arrow[dr] & & \bigoplus M_i \arrow[dr, equal] & \\
& M_\nu \arrow[rr] & & \bigoplus M_i \\
M_\lambda \arrow[rr] \arrow[ur] & & \bigoplus M_i \arrow[ur, equal] &
\end{tikzcd}$$

It then follows that $M = \operatorname{colim} M_\lambda$ is purely embedded in $M_1 \oplus \cdots \oplus M_n$ and hence in $\mathcal{X}$. $\square$

4.2 The Ziegler Spectrum

We now use definable subcategories to define a topology on the class of indecomposable pure-injective modules over a ring. We note that this is class is, up to isomorphism, just a set, and hence we do not have to worry about topological size issues.

Proposition 4.4 ([Pre09] Corollary 4.3.38)**.** *Up to isomorphism, there is only a set of indecomposable pure-injective modules.*

The (right) **Ziegler spectrum** Zg_R is a topological space associated to R whose underlying set is the set of isomorphism classes of indecomposable pure-injective right modules. For a given definable subcategory $\mathcal{X}$, we write $\mathcal{X} \cap \mathrm{pinj}_R$ for the class of pure-injectives in $\mathcal{X}$. The closed sets of Zg_R are then given by the indecomposable modules of $\mathcal{X} \cap \mathrm{pinj}_R$, which we write as $\mathcal{X} \cap \mathrm{Zg}_R$.

Proposition 4.5 ([Pre09] Theorem 5.11)**.** *These closed sets give a valid topology on Zg_R.*

Proof. We first note that the whole set is defined by the pp-pair ϕ/ϕ which is zero for all modules, while the empty set is defined by ϕ/ψ for $\phi(x) = x \doteq x$ and $\psi(x) = x \doteq 0$. This quotient is zero only on the zero module, which is not indecomposable.

For closure under intersections, let $\mathcal{X}_i = \{M \in \mathbf{Mod}\text{-}R \mid \phi_\lambda(M) = \psi_\lambda(M), \lambda \in \Lambda_i\}$ for $i = 1, \dots, n$. Then, we find that $\bigcap_{i=1}^n \mathcal{X}_i = \{M \in \mathbf{Mod}\text{-}R \mid \phi_\lambda(M) = \psi_\lambda(M), \lambda \in \bigcup_{i=1}^n \Lambda_i\}$ which is itself definable. Simply intersect with Zg_R to find that the closed sets on Zg_R are closed under intersections.

For closure under finite unions, we note that $\bigcup(\mathcal{X}_i \cap \mathrm{Zg}_R) \subseteq (\bigcup \mathcal{X}_i) \cap \mathrm{Zg}_R$ from elementary set calculus. Hence, we need only prove the reverse inclusion. Now, recall from Corollary 4.3 that M is in $\bigcup \mathcal{X}_i$ if and only if it is a pure submodule of some sum $M_1 \oplus \dots \oplus M_n$ with each M_i in $\mathcal{X}_i$. By Lemma 4.4.1 of [Pre09], which uses substantial ring theory beyond our scope, N is a direct summand of one of the M_i. Coprojections are pure, so this shows that N is a pure submodule of that M_i and so contained in $\mathcal{X}_i \cap \mathrm{Zg}_R$. This shows the reverse inclusion. $\square$

We shall now describe the Ziegler spectrum in some simple cases and discuss how this relates to the model theory of the rings. As a general rule, determining the Ziegler spectrum requires significant effort in analysing the fine structure of the ring, so we will restrict mostly to summaries.

For our first example, we consider the ring of integers, whose modules are the Abelian groups. Theorem 5.2 of [Zie84] gives us a list of the indecomposable pure-injective Abelian groups, which are as follows:

- The cyclic groups $\mathbb{Z}/p^n\mathbb{Z}$ for p prime.
- The Prüfer groups $\mathbb{Z}(p^\infty)$ which consist of all p^nth roots of unity for all $n \geq 0$.
- The p-adic integers $\mathbb{Z}_p$ which consist of infinite sequences $(x_1, x_2, ...)$ where each x_i is an integer satisfying $0 \leq x_i \leq p^i$ and the coherency relations $x_i \cong x_j (\text{mod} p^i)$ for all $i < j$.
- The rational numbers.

We describe the topology with an eye towards what is called **Cantor-Bendixson Analysis**, which analyses the isolated points of a topological space. For each point, we assign it a Cantor-Bendixson rank or **CB-rank**. For a given space X, the isolated points of X are assigned rank 0 and then we determine a new space X' consisting of X minus the isolated points and given the subset topology. We call this space the **derived set**. The isolated points of X' then have CB-rank of 1. We continue this process transfinitely, taking infinite intersections at the limit ordinals. If a point is never isolated in any derived set then it is given a CB-rank of ∞.

From Theorem 5.2.3 of [Pre09], we find that the CB-ranks of the indecomposable pure-injective Abelian groups are as follows:

- The cyclic groups have rank 0.
- The Prüfer groups and p-adic integers have rank 1.
- The rational numbers are the sole group of rank 2.

We say that a module is of **finite length** if its longest chain of submodules is finite. Theorem 5.2.3 of [Pre09] states that the closed points of $\mathrm{Zg}_{\mathbb{Z}}$ are precisely the finite length groups and the rational numbers. We now list the closed sets of $\mathrm{Zg}_{\mathbb{Z}}$, organised by amount of finite length points.

- The closed sets with no finite length points consist of any subset of $\mathrm{Zg}'_{\mathbb{Z}}$ along with the rationals.
- The closed sets with finitely many finite length points are as above along with their finite length points (which are closed points and so we can freely adjoin finitely many of them).
- The closed sets with infinitely many finite length points are as above along with the restriction that, for each prime p, if there are infinitely many cyclic groups of order p^n are in the set then $\mathbb{Z}_p, \mathbb{Z}(p^\infty)$ and the rationals are also in the set.

As a second example, we consider a field k. Basic linear algebra tells us that all modules over k are of the form k^λ for some cardinal λ and so there is a unique indecomposable module: k itself.

A morphism $t\colon k^\lambda \to k^\mu$ consists of determining where to send the basis of k^λ. Suppose $(e_k)_{k\in\lambda}$ is a basis of k^λ, then if t is monic, $(t(e_k))_{k\in\lambda}$ is a linearly independent set and so we can freely extend it to a basis of k^μ. Thus, for any morphism $f\colon k^\lambda \to k$, we can determine a morphism $h\colon k^\mu \to k$ by setting $h(t(e_k)) = f(e_k)$ and freely extending. Thus, k is an injective module and hence pure-injective.

Thus, we see that the Ziegler spectrum Zg_k consists of a single point.

For our third example, we consider the preadditive category that has two objects a and b such that as endomorphism rings we have $\mathrm{End}(a) \cong \mathrm{End}(b) \cong k$ and whose Hom-group $\mathrm{Hom}(a, b)$ we generate from a single morphism α under addition and composition with the endomorphisms on each end along with the coindition that $\lambda_b \circ r = r \circ \lambda_a$ for all $\lambda \in k$ and $r\colon a \to b$. This last condition enforces linearity of the maps between objects in our modules. We represent this as with the graph $K_2 = \; a \xrightarrow{\alpha} b$.

A module over this category will consist of two k-vector spaces, Ma and Mb, a linear map $M\alpha\colon Ma \to Mb$ and all the maps it generates. Assuming M is non-zero, we will work through decomposing it as a sum of indecomposables. Given a basis $(e_k)_{k\in\lambda}$ for Ma, $M\alpha$ is determined by $(M\alpha(e_k))_{k\in\lambda}$. We rewrite Mb as $\mathrm{im}(M\alpha) \oplus k^\mu$ for some cardinal μ and hence we can write M as a sum of the module $Ma \to \mathrm{im}(M\alpha)$ and $\bigoplus_{i=1}^{k} 0 \to k$. By rank-nullity, we can write $Ma \cong \ker(M\alpha)\oplus\mathrm{im}(M\alpha)$. This then lets us decompose $\mathrm{im}(M\alpha)$ as $\ker(M\alpha) \to 0 \oplus \mathrm{im}(M\alpha) \to \mathrm{im}(M\alpha)$. The former is isomorphic to a sum of copies of $k \to 0$ and the latter is isomorphic to a sum of copies of the identity $k \to k$.

We thus have decomposed every non-zero module into a sum of copies of three modules, each of which is indecomposable (since k is indecomposable as a k-vector space). These are:

- $S_1 = k \to 0$
- $S_2 = 0 \to k$
- $T = k \xrightarrow{\mathrm{id}} k$

We show now that S_1 and T are injective, while S_2 is not. Fix an arbitrary monomorphism of modules $m\colon M \to N$ and $f\colon M \to I$ a morphism into an indecomposable. We work by cases.

First, $I = S_1$. Consider the components m_a and f_a. Since k is an injective vector space, we find a morphism g_a such that $f_a = g_a \circ m_a$. We define a natural transformation $g\colon N \to I$ by g_a and $g_b = 0$. Since $Ib = 0$, we know that $f_b = 0$ and so we have $f_b = g_b \circ m_b$. Further, g is natural since $I\alpha = 0$ and so $I\alpha \circ g_a = 0 = 0 \circ N\alpha$.

$$\begin{tikzcd}
Ma \arrow[r, "m_a"] \arrow[d, "M\alpha"] & Na \arrow[r, "g_a"] \arrow[d, "N\alpha"] & k \arrow[d] \\
Mb \arrow[r, "m_b"'] & Nb \arrow[r] & 0
\end{tikzcd}$$

For $I = T$, we once again use injectivity of k, this time to find g_b such that $f_b = g_b \circ m_b$. We now set $g_a = g_b \circ N\alpha$. Since $I\alpha = \mathrm{id}_k$, we have that g is natural by definition. We then compute $g_a \circ m_a = g_b \circ N_\alpha \circ m_a = g_b \circ m_b \circ M_\alpha = f_b \circ M_\alpha = I\alpha \circ f_a = f_a$. Hence, we have the factorisation $f = g \circ m$ of module homomorphisms. This is the $n = 1$ case of Theorem 2.3 of [Par08].

$$\begin{tikzcd}
Ma \arrow[r, "m_a"] \arrow[d, "M\alpha"] & Na \arrow[r, "g_b \circ N_\alpha"] \arrow[d, "N\alpha"] & k \arrow[d, "\mathrm{id}"] \\
Mb \arrow[r, "m_b"'] & Nb \arrow[r, "g_b"'] & k
\end{tikzcd}$$

For $I = S_2$, we construct an explicit counterexample. Set $M = I = S_2$ and $N = T$ with the map m given by $m_a = 0$ and $m_b = \mathrm{id}_k$ and f the identity on S_2. We then have the following diagram, with g a potential factor for f through m.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & k \arrow[r] \arrow[d, "\mathrm{id}_k"] & 0 \arrow[d] \\
k \arrow[r, "\mathrm{id}_k"'] & k \arrow[r, "g_b"'] & k
\end{tikzcd}$$

To have $f = g \circ m$, we must satisfy both $g_b \circ \mathrm{id}_k = 0$ for naturality of g and $g_b \circ \mathrm{id}_k = f_b = \mathrm{id}_k$. This is impossible, hence no such g exists. Thus, S_2 is not injective.

It is the case, however, that S_2 is *pure*-injective. To see this, we pick some pure-monomorphism $m\colon M \to N$ and morphism $f\colon M \to S_2$. Now, consider the pp-formulae $\phi(x\colon b) = \exists(y\colon a).(\alpha(y) \doteq x)$. This defines the image of α as a pp-definable subgroup. In particular, we see that $m_b(\mathrm{im}(M\alpha)) = \mathrm{im}(N\alpha) \cap m_b(M)$ by purity of m.

Now, using injectivity of k as a vector space, we find a factorisation for f_b as $f_b = h \circ m_b$ for $h\colon Nb \to k$. Then, using the fact that every vector space is injective, we can decompose Nb as $Nb = m_b(\mathrm{im}(M\alpha)) \oplus V$ for some space V. We define a new morphism g_b such that g_b agrees with h on $m_b(\mathrm{im}(M\alpha))$ and is equal to 0 on V. Thus, g_b still satisfies $g_b \circ m_b = f_b$. We define $g_a = 0$ to obtain the data for a natural transformation g which is a factor of f through m. It remains to show that g is indeed natural, that is $g_b \circ N\alpha = 0$. This is equivalent to showing that the following diagram commutes.

$$\begin{tikzcd}
Ma \arrow[r, "m_a"] \arrow[d, "M\alpha"'] & Na \arrow[r] \arrow[d, "N\alpha"] & 0 \arrow[d] \\
Mb \arrow[r, "m_b"'] & m_b(\mathrm{im}(M\alpha)) \oplus V \arrow[r, "h\oplus 0"'] & k
\end{tikzcd}$$

To prove naturality, we pick some $n \in Na$. Then, we can write $N\alpha(n) = m_b(M\alpha(n')) + n''$ for some $n' \in Ma$ and $n'' \in V$. Then, using naturality of f, we compute that $g_b(N\alpha(n)) = g_b(m_b(M\alpha(n'))) + g_b(n'') = f_b(M\alpha(n')) + 0 = (0 \circ f_a)(n') = 0$. Since we picked arbitrary n, we obtain that $g_b \circ N\alpha = 0$ as required for naturality. We have hence obtainted the necessary factorisation to show that S_2 is pure-injective.

We conlude that the Ziegler spectrum has three points. The points S_1 and S_2 are closed, which we can see using the pp-formulae $\phi_i(x\colon i) = x \doteq x$ and $\psi_i(x\colon i) = x \doteq 0$ for $i = a, b$. Then, S_1 is the only indecomposable having $\phi_b/\psi_b = 0$ and S_2 is the only indecomposable having $\phi_a/\psi_a = 0$. Consequently, T is an isolated point.

4.3 Torsion Theory

A **torsion Abelian group** is an Abelian group each one of whose elements has finite order. Conversely, a **torsionfree** Abelian group has only elements of infinite order. Given an Abelian group G, we can take the subgroup of torsion elements $\tau(G)$. The quotient $G/\tau(G)$ is then torsionfree. The map τ is functorial, and we can think of the torsion groups as those on which τ acts as the identity and the torsionfree groups as those which are sent to 0 by τ. We note that τ is a subfunctor of the identity functor, by definition, and further it is left-exact.

Torsion theories are a generalisation of the notion of a torsion group, focusing on the structural properties of the classes of torsion and torsionfree objects in an Abelian category, rather than on annihilation by some action.

For this section, fix some Grothendieck category $\mathbf{C}$. We shall omit several proofs as they include long detours into homological methods that are beyond our scope.

Definition 4.6 ([Ste75] Chapter VI)**.** A **torsion radical** is a left-exact subfunctor of the identity functor on C, τ, such that for every object X we have that $\tau(X/\tau X) = 0$.

Thus, our archetypal example is the functor $\mathbf{Ab} \to \mathbf{Ab}$ taking a group to its torsion subgroup.

To each torsion radical τ, we can define two classes of objects. The **torsion class** of τ is $\mathcal{T} = \{C \in \mathbf{C} \mid \tau C \cong C\}$. The **torsionfree class** of τ is $\mathcal{F} = \{C \in \mathbf{C} \mid \tau C \cong 0\}$. The data $(\mathcal{T}, \mathcal{F}, \tau)$ is called a **(hereditary) torsion theory**. Note that each of these classes is closed under isomorphism. We will often conflate the class and its corresponding induced full subcategory.

We now state some basic properties of these classes, starting with how they relate to each other and the structure of $\mathbf{C}$. We fix some torsion theory $(\mathcal{T}, \mathcal{F}, \tau)$ and a natural monomorphism $\eta \colon \tau \to \mathrm{id}_{\mathbf{C}}$.

Proposition 4.7 ([Bor94] Proposition 1.12.3)**.** *Let $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Then,* $\mathrm{Hom}(T, F) = 0$.

Proof. Let $f \colon T \to F$ be a morphism. Then, we have the following commutative diagram from the fact that τ is a subfunctor of $\mathrm{id}_{\mathbf{C}}$.

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, equal] & F \\ T \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

It follows that $f = 0$. ☐

Proposition 4.8 ([Bor94] Definition 1.12.1)**.** *Let X be an object of* $\mathbf{C}$. *Then,* *there are $T \in \mathcal{T}$ and $F \in \mathcal{F}$ such that the following sequence is exact.*

$$0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$$

Proof. We set $T = \tau X$, with inclusion $\eta_X \colon \tau X \to X$. We then set $F = \operatorname{coker} \eta_X$. By definition of a radical, F must be torsionfree, so it suffices to show that T is torsion. First, we draw the following diagram by applying τ and using naturality of η.

$$\begin{tikzcd}
0 \arrow[r] & \tau X \arrow[r, "\eta_X"] & X \arrow[r, "f"] & F \arrow[r] & 0 \\
0 \arrow[r] & \tau^2 X \arrow[r, "\tau\eta_X"'] \arrow[u, "\eta_{\tau X}"] & \tau X \arrow[r, "\tau f"'] \arrow[u, "\eta_X"'] & \tau F \arrow[u, "\eta_F"'] &
\end{tikzcd}$$

It follows that $\eta_F \circ \tau f = f \circ \eta_X = 0$. Since η_F is a monomorphism, it follows that $\tau f = 0$. By exactness, $\tau\eta_X = \ker(\tau f) = \ker(0) = \mathrm{id}_{\tau X}$. Thus, $\tau^2 X = \tau X$ and so T is torsion. □

Remark 4.9. We can extract from this proof that τ is idempotent, that is, $\tau^2 = \tau$. We also showed that τX is torsion for all X. Indeed, τX is in fact the maximal torsion subobject of X. To see this, suppose $Y \hookrightarrow X$ is a torsion subobject. Then, applying τ we see that $\tau Y = Y \hookrightarrow \tau X$.

We move on to stating some closure properties of torsion and torsionfree classes. In particular, we see that torsion classes are closed under subobjects and torsionfree classes are closed under injective hulls - these are the hereditary part of hereditary torsion theories. Some authors choose to work with more general torsion theories not satisfying this closure property, but all relevant torsion theories for us will be hereditary and so we will continue to drop the word hereditary.

Proposition 4.10 ([Ste75] Chapter VI, Propositions 1.2 and 1.7)**.** *Let $(\mathcal{T}, \mathcal{F}, \tau)$ be a torsion theory. Then $\mathcal{T}$ is closed under quotients, extensions, direct sums and subobjects.*

Given a torsion radical τ on $\mathbf{C}$, we can define a radical τ^{-1} on $\mathbf{C}^{\mathrm{op}}$ by setting, for each object X, $\tau^{-1}X = X/\tau X$. We observe that if X in $\mathbf{C}$ is torsion, then $\tau X = X$ and so $\tau^{-1}X = 0$. Conversely, $\tau^{-1}X = 0$ must imply X is torsion, and so we see that $\mathcal{T}_\tau$ is the class of torsionfree objects of τ^{-1}. Similarly, $\mathcal{F}_\tau$ is the class of torsion objects of τ^{-1} and so we obtain a torsion theory $(\mathcal{F}, \mathcal{T}, \tau^{-1})$ on $\mathbf{C}^{\mathrm{op}}$. This gives us a duality property for torsion and torsionfree classes and we use that along with Proposition 4.10 to observe the following closure properties of torsionfree classes.

Proposition 4.11. *Let $(\mathcal{T}, \mathcal{F}, \tau)$ be a torsion theory. Then, $\mathcal{F}$ is closed under subobjects, products and injective hulls.*

By Proposition 2.1 of [Ste75] Chapter VI, these closure properties are in fact the defining properties of torsion theories. That is, given a class of objects $\mathcal{T}$ in $\mathbf{C}$, $\mathcal{T}$ is the torsion class of some torsion theory if and only if it is closed under quotients, subobjects, extensions and direct sums. By dualising, we obtain an analogous fact about torsionfree classes. Given a torsion class $\mathcal{T}$, we can recover the radical τ by setting τX to be the colimit of the diagram of all torsion subobjects of X with their inclusion maps. That this is indeed a radical follows from the discussion in [Pre09] Section 11.1.1 and before Proposition 1.4 in [Ste75] Chapter VI. We can recover the torsionfree class $\mathcal{F}$ corresponding to $\mathcal{T}$ by setting $\mathcal{F} = \{F \in \mathbf{C} \mid \mathrm{Hom}(T, F) = 0 \text{ \textbf{for all} } T \in \mathcal{T}\}$.

Our primary use for torsion theories will be **localisation**, a process by which we find some full subcategory of $\mathbf{C}$ that is simplified along τ. In the torsion group example, we would be looking at the quotients of Abelian groups by their largest torsion subgroup, which will necessarily be torsionfree. We can think of this process as a kind of quotienting for entire categories, and indeed the resulting category will often be referred to as the **quotient category**, $\mathbf{C}_\tau$.

The objects of the quotient category will all be torsionfree and **τ-injective**, a weakening of injectivity which we will now define. This weaknening is via restricting to our injectivity to **τ-dense** morphisms.

Definition 4.12. An embedding $\iota\colon C' \to C$ is called **τ-dense** if C/C' is torsionfree.

Definition 4.13. An object X is called **τ-injective** if for all morphisms $f\colon C' \to X$ and τ-dense embeddings $\iota\colon C' \to C$, there is a morphism $h\colon C \to X$ such that $f = h \circ \iota$. That is, the following diagram fills in.

$$\begin{tikzcd} C' \arrow[r, tail, "\iota"] \arrow[d, "f"'] & C \arrow[dl, dashed, "\exists h"] \\ X & \end{tikzcd}$$

To get some intuition for these τ-injective objects, we recall the torsion group example. Given a dense subgroup $G' \leq G$, their quotient G/G' is torsion. Hence, in the process of quotienting all groups by their largest torsion subgroup, G/G' goes to 0 and thus G and G' are mapped together. By their factorisation property, the τ-injective objects of $\mathbf{C}$ cannot 'see' the difference between G and G'.

We now define the **localisation** functor for τ, which we denote by Q_τ. Recall that we denote the injective hull of an object C by $E(C)$. Given some C in $\mathbf{C}$, we let $C_1 = C/\tau C$ and construct the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & C_1 \arrow[r, tail] & E(C_1) \arrow[r, two heads, "\pi"] & E(C_1)/C_1 \arrow[r] & 0 \\ & & P \arrow[u, tail] \arrow[r, two heads] \arrow[ur, phantom, "\lrcorner", very near start] & \tau(E(C_1)/C_1) \arrow[u, tail, "\eta_{E(C_1)/C_1}"'] & \end{tikzcd}$$

We denote the pullback P as $E_\tau(C_1)$, the **τ-injective hull** of C_1. We then set $Q_\tau(C) = E_\tau(C_1)$, also denoting this object as C_τ. The action on morphisms Q_τ is induced naturally by the functoriality of τ and of taking injective hulls, quotients and pullbacks.

The localisation functor relates to τ-injective objects via the following proposition.

Proposition 4.14. *Let $(\mathcal{T}, \mathcal{F}, \tau)$ be a torsion theory. If C is τ-torsionfree and τ-injective, then $C \cong Q_\tau C$.*

Proof. Since C is torsionfree, we have that $C = C/\tau C$. We then recall the diagram from defining localisation and use the fact that pullbacks preserve kernels ([Arr25] Proposition 3.20) to obtain the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, tail, "e"] & E(C) \arrow[r, two heads, "\pi"] & E(C)/C \arrow[r] & 0 \\
0 \arrow[r] & C \arrow[u, equal] \arrow[r, tail, "\iota"'] & Q_\tau C \arrow[u, tail, "p"'] \arrow[r, two heads] \arrow[ur, phantom, "\lrcorner"] & \tau(E(C)/C) \arrow[u, tail, "\eta_{E(C_1)/C_1}"'] \arrow[r] & 0
\end{tikzcd}$$

Now, the cokernel of ι is $\tau(E(C)/C)$ and so torsion by Proposition 4.8. Thus, ι is τ-dense and so by τ-injectivity, id_C must factor through ι as $\mathrm{id}_C = h \circ \iota$. Since it is right-invertible, h must be epimorphic. Further, since $p = e \circ h$ and p is monomorphic, we see that h must be monomorphic. Thus, h is an isomorphism. $\square$

That we can think of localisation as a kind of quotienting is cemented by the following theorem, in which we show that the localisation functor satisfies a universal property analogous to (but not the same as) the cokernel property that we usual consider for quotients.

Theorem 4.15 ([Pre09] Theorem 11.1.5)**.** *Let* $\mathbf{C}$ *be a Grothendieck category and* τ *a torsion theory. Then,*

- *the quotient category* $\mathbf{C}_\tau$ *is Grothendieck .*
- *the localisation functor* $Q_\tau \colon \mathbf{C} \to \mathbf{C}_\tau$ *is exact.*
- *for any Grothendieck catgeory* C' *and exact functor* $F \colon C \to C'$ *such that* F *commutes with directed colimits and* $\mathcal{T}_\tau \subseteq \ker(F)$*,* F *factors uniquely through* Q_τ*.*

4.4 Torsion Theories of Finite Type

We now turn to applying the torsion theory technology to studying the Ziegler spectrum. This will make use of the tensor embedding from Section 3.2 to embed definable subcategories of the module category as intersections of torsionfree classes in the functor category. We are specifically interested in a subclass of torsion theories called **torsion theories of finite type**.

Definition 4.16. A torsion radical $\tau \colon \mathbf{C} \to \mathbf{C}$ defines a torsion theory **of finite type** if it commutes with directed colimits.

We can characterise torsion theories of finite type via a closure property on their torsionfree classes, just as we could characterise torsion theories in general.

Proposition 4.17 ([Pre09] Proposition 11.1.12)**.** *Let $T = (\mathcal{T}, \mathcal{F}, \tau)$ be a torsion theory with inclusion $\eta \colon \tau \to \mathrm{id}$. Then T is of finite type if and only if $\mathcal{F}$ is closed under directed colimits.*

Proof. Let $(F_\lambda, f_{\lambda\mu})$ be a directed system of torsionfree objects with colimit (F, η). Then, $\tau F = \operatorname{colim} \tau F_\lambda$ and $\tau F_\lambda = 0$ for all λ. Hence, $\tau F = 0$ and so F is torsionfree.

For the backwards direction, let $(C_\lambda, c_{\lambda\mu})$ be an arbitrary directed system with colimit (C, η). By Proposition 4.8, we induce a system of exact sequences of the following form.

$$0 \longrightarrow \tau C_\lambda \longrightarrow C_\lambda \longrightarrow C/\tau C_\lambda \longrightarrow 0$$

Since $\mathbf{C}$ is Grothendieck, taking directed colimits is exact and so we find the following exact sequence.

$$0 \longrightarrow \operatorname{colim}(\tau C_\lambda) \xrightarrow{m} C \longrightarrow \operatorname{colim}(C/\tau C_\lambda) \xrightarrow{q} 0$$

By proposition 4.10, we know that torsion classes are closed under quotients and direct sums and hence under directed colimits. Thus, $\operatorname{colim}(\tau C_\lambda)$ is torsion and so from Remark 4.9, we have an embedding $\tau m \colon \operatorname{colim}(\tau C_\lambda) \hookrightarrow \tau C$ such that $m = \eta_C \circ \tau m$.

On the other hand, by assumption, $\operatorname{colim}(C/\tau C_\lambda)$ is torsionfree and so, by Proposition 4.7, $q \circ \eta_C = 0$. Hence, η_C must factor through the kernel of q as $\eta_C = m \circ h$ for $h \colon \tau C \to \operatorname{colim}(\tau C_\lambda)$. Since η_C is monic, we find that h is monic.

Combining these two factorisations, we get $m = \eta_C \circ \tau m = m \circ h \circ \tau$. Applying monicity we get $h \circ \tau = \mathrm{id}$ and so h is an epimorphism. Since $\mathbf{C}$ is balanced, h is an isomorphism and so we obtain $\tau C \cong \mathrm{colim}(\tau C_\lambda)$. This shows that τ commutes directed colimits. $\square$

Combining Proposition 4.17 and Proposition 4.11, we have that finite type torsionfree classes in Grothendieck categories are closed under directed colimits, products and subobjects (and hence pure subobjects). We can hence apply Theorem 4.2 to see that finite type torsionfree classes are definable. In the case of locally finitely presented categories, the converse is true, though we omit its proof.

Theorem 4.18 ([Pre09] Theorem 10.1.20)**.** *Let* $\mathbf{C}$ *be locally finitely presented Grothendieck and* $\mathcal{F}$ *a class of objects. Then* $\mathcal{F}$ *is definable if and only if it is the torsionfree class of some torsion theory of finite type.*

We shall use this theorem in order to sketch out a proof of the following characterisation of definable subcategories of $\mathbf{Mod}\text{-}R$ in terms of their images under the tensor embedding. As notation, we will denote by Abs the closure under isomorphisms in $[R\text{-}\mathbf{mod}, \mathbf{Ab}]$ of $\epsilon(\mathbf{Mod}\text{-}R)$. Here, Abs refers to the 'absolutely pure' objects of the functor category, but for our purposes it is sufficient to consider these as the tensor functors.

Theorem 4.19 ([Pre09] Proposition 12.3.2)**.** *Let* $\mathcal{X}$ *be a subclass of* $\boldsymbol{Mod}\text{-}R$*. Then,* $\mathcal{X}$ *is definable if and only if* $\epsilon\mathcal{X}$ *is of the form* $\mathcal{F} \cap \mathrm{Abs}$ *where* $\mathcal{F}$ *is the torsionfree class of some torsion theory of finite type.*

Sketch of Proof. Since ϵ is preserves finite limits and directed colimits, the image $\epsilon\mathcal{X}$ is closed under products and directed colimits. We use Theorem 12.1.6 and Proposition 2.3.2 of [Pre09] to see that pure subfunctors of tensor functors are themselves tensor functors. It then follows from Theorem 3.13 that if $M' \otimes -$ is a pure subfunctor of $M \otimes -$, then M' is a pure subobject of M and so $M' \in \mathcal{X}$ and $M' \otimes - \in \epsilon\mathcal{X}$ whenever $M \otimes - \in \epsilon\mathcal{X}$. Hence, by Theorem 4.2, $\epsilon\mathcal{X}$ is a definable subcategory of $[R\text{-}\mathbf{mod}, \mathbf{Ab}]$. We define the class $\mathcal{F}$ to be the closure of $\epsilon\mathcal{X}$ under subobjects. By Theorem 4.3.21 of [Pre09], $\mathcal{X}$ is closed under pure-injective hulls and hence by Corollary 3.22, $\epsilon\mathcal{X}$ is closed under injective hulls. Since, if $X \hookrightarrow Y$, we have that $E(X) \hookrightarrow E(Y)$, this shows that $\mathcal{F}$ is closed under injective hulls. Similarly for seeing that $\mathcal{F}$ is closed under products and direct sums, and hence a torsionfree class.

By Theorem 4.18, to see that $\mathcal{F}$ is a torsionfree class of finite type, it suffices to show that it is definable. Since we have already shown closure under finite limits and directed colimits, we only need closure under directed colimits. This proves rather difficult and so we refer back to the original proof in of Proposition 12.3.2 [Pre09] which makes use of an alternate characterisation of definable categories that requires the model-theoretic technology of reduced products rather than directed colimits. Once $\mathcal{F}$ being definable is established, we have finished the forwards direction of the theorem, seeing $\mathcal{X} = \mathcal{F} \cap \text{Abs}$.

For the backwards direction, for a given torsionfree class of finite type $\mathcal{F}$, we note that both $\mathcal{F}$ and Abs are closed under directed colimits, and hence so is their intersection. Similarly for both products and pure subobjects. If $\epsilon\mathcal{X} = \mathcal{F} \cap \text{Abs}$ for some class $\mathcal{X}$ in $\mathbf{Mod}\text{-}R$, we use the fact that ϵ reflects limits and colimits (Proposition 5.14) to see that $\mathcal{X}$ is closed under products and directed colimits. That $\mathcal{X}$ is closed under pure-submodules follows from combining Theorems 3.12 and 3.13 to see that pure-submodules must be reflected under ϵ. Then, we apply Theorem 4.2 to see that $\mathcal{X}$ is definable. $\square$

We have hence obtained a bijection between the definable subcategories of $\mathbf{Mod}\text{-}R$ and the torsion theories of finite type on $[R\text{-}\mathbf{mod}, \mathbf{Ab}]$. We will write $\tau_{D\mathcal{X}}$ for the torsion radical determined by $\mathcal{X}$. Note that we write $D\mathcal{X}$ rather than $\mathcal{X}$. This is related to notation from Section 5.2 and in essence is because we have moved from *right* modules to functors on *left* modules.

As well as being a bijection, this map is inclusion preserving. To see this, consider definable subclasses $\mathcal{X} \subseteq \mathcal{X}'$. Then, every subobject of an object in $\epsilon\mathcal{X}$

must also be a subobject of some object in $\epsilon\mathcal{X}'$, and so we obtain $F_{D\mathcal{X}} \subseteq F_{D\mathcal{X}'}$. This will be useful in Section 5.4 for relating left and right Ziegler spectra.

A final notion that we shall introduce in this section is that of a **Serre Subcategory**. These were introduced by Serre in [Ser53] and provide a distinct route to localisations of Abelian categories than that of torsion theories. In our specific case of locally finitely presentable Grothendieck categories, however, the two converge.

Definition 4.20. Let $\mathcal{S}$ be a full subcategory of an Abelian category $\mathbf{C}$. We say that $\mathcal{S}$ is **Serre** if, for every exact sequence $0 \to A \to B \to C \to 0$, B is in $\mathcal{S}$ if and only if both A and C are in $\mathcal{S}$.

We will consider in particular Serre subcategories of $[R\text{-}\mathbf{mod}, \mathbf{Ab}]^{\mathrm{fp}}$, recalling from Proposition 6.4 of [Arr25] that $[R\text{-}\mathbf{mod}, \mathbf{Ab}]^{\mathrm{fp}}$ is an Abelian subcategory of $[R\text{-}\mathbf{mod}, \mathbf{Ab}]$. Given a Serre subcategory $\mathcal{S}$ of $[R\text{-}\mathbf{mod}, \mathbf{Ab}]^{\mathrm{fp}}$, we denote by $\overrightarrow{\mathcal{S}}$ its closure under directed colimits in $[R\text{-}\mathbf{mod}, \mathbf{Ab}]$. We use this closure to state the following connection between Serre subcategories of $[R\text{-}\mathbf{mod}, \mathbf{Ab}]^{\mathrm{fp}}$ and torsion theories on $[R\text{-}\mathbf{mod}, \mathbf{Ab}]$.

Proposition 4.21 ([Pre09] Proposition 11.1.36)**.** *Let $\mathcal{S}$ be a full subcategory of $[R\text{-}\mathbf{mod}, \mathbf{Ab}]^{fp}$. If $\mathcal{S}$ is a Serre subcategory then $\overrightarrow{\mathcal{S}}$ is the torsion class of a torsion theory of finite type in $[R\text{-}\mathbf{mod}, \mathbf{Ab}]$. Further, $\mathcal{S} = [R\text{-}\mathbf{mod}, \mathbf{Ab}]^{fp} \cap \overrightarrow{\mathcal{S}}$.*

Conversely, if $\mathcal{T}$ is the torsion class of a torsion theory of finite type, then $[R\text{-}\mathbf{mod}, \mathbf{Ab}]^{fp} \cap \mathcal{T}$ is Serre in $[R\text{-}\mathbf{mod}, \mathbf{Ab}]^{fp}$ and $\mathcal{T} = \overrightarrow{[R\text{-}\mathbf{mod}, \mathbf{Ab}]^{fp} \cap \mathcal{T}}$.

Proof. Suppose $\mathcal{S}$ is a Serre subcategory. Then, we must show that $\overrightarrow{\mathcal{S}}$ satisfies the conditions of Proposition 4.10. This will show that it is a torsion class, after which we shall show finite type.

First, we show that $\overrightarrow{\mathcal{S}}$ is closed under subobjects. Let $(A_\lambda, a_{\lambda\mu})$ be a directed system of objects of $\mathcal{S}$ and let (C, g_λ) be its colimit. This is the general form for an object for an object of $\overrightarrow{\mathcal{S}}$. We then consider a subobject $m\colon C' \to C$. Since finite limits and directed colimits commute in module categories, we know that

C' is the directed colimit of the system given by pulling back each g_λ along m, which we shall write as $(g_\lambda^{-1}C', g^{-1}a_{\lambda\mu})$.

$$\begin{tikzcd} C' \arrow[r, "m"] & C \\ g_\lambda^{-1}C' \arrow[u] \arrow[r] & A_\lambda \arrow[u, "g_\lambda"'] \end{tikzcd}$$

Since pullbacks preserve monomorphisms, these $g_\lambda^{-1}C'$ are subobjects of the corresponding A_λ. If we consider the epimorphism onto each $g_\lambda^{-1}C'$ from Proposition 4.18 of [Arr25] and looking at its finite summands, we can write each $g_\lambda^{-1}C'$ as a directed colimit over its finitely generated subobjects. Each of these finitely generated subobjects is then a subobject of A_λ and, by local coherence of $[R\text{-}\mathbf{mod}, \mathbf{Ab}]$ (Corollary 6.3 of [Arr25]), they must be finitely presented. Since $\mathcal{S}$ is Serre, it follows that these subobjects are in $\mathcal{S}$ and so we can write C' as a directed colimit of objects in $\mathcal{S}$, showing that C' is in $\overrightarrow{\mathcal{S}}$.

For quotients, consider an epimorphism $f\colon C \to C''$. We get the following commuting diagram by considering the kernels of $f \circ g_\lambda$. Each h_λ exists by factoring $g_\lambda \circ \ker(f \circ g_\lambda)$ through $\ker(f)$ by the universal property of the kernel. We then take the cokernels of $\ker(f)$ and $\ker(f \circ g_\lambda)$ to obtain the arrow on the right.

$$\begin{tikzcd} 0 \arrow[r] & C' \arrow[r, "\ker(f)"] & C \arrow[r, "f"] & C'' \arrow[r] & 0 \\ 0 \arrow[r] & B_\lambda \arrow[u, "h_\lambda"] \arrow[r, "\ker(f\circ g_\lambda)"'] & A_\lambda \arrow[u, "g_\lambda"] \arrow[r] & A_\lambda/B_\lambda \arrow[u] \arrow[r] & 0 \end{tikzcd}$$

As before, we view B_λ as a colimit of its finitely generated subobjects $B_{\lambda\mu}$, each of which is a subobject of A_λ and hence finitely presented. Since Serre subcategories are closed under quotients, we have that $A_\lambda/B_{\lambda\mu}$ is in $\mathcal{S}$ for each λ and μ and hence when we note that $C'' = \operatorname{colim} A_\lambda/B_{\lambda\mu}$, we obtain that C'' is in $\overrightarrow{\mathcal{S}}$.

Next, we show that $\overrightarrow{\mathcal{S}}$ is closed under extensions. We consider an exact sequence of the form $0 \to C \to E \to D \to 0$ and write $D = \operatorname{colim}(B_\lambda)$ for $(B_\lambda, b_{\lambda\mu})$ a directed system of finitely presented objects in $\mathcal{S}$. For each λ, we pull back along $E \to D$ to obtain the following diagram and indeed a directed system $(P_\lambda, p_{\lambda\mu})$ by functoriality of limits.

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "m"] & E \arrow[r, "\pi"] & D \arrow[r] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[u, equal] & P_\lambda \arrow[r, "\pi_\lambda"'] \arrow[u, "\varepsilon_\lambda"] \arrow[ur, phantom, "\lrcorner"] & B_\lambda \arrow[r] \arrow[u, "\eta_\lambda"'] & 0
\end{tikzcd}$$

By exactness of directed colimits, we have that $E = \operatorname{colim} P_\lambda$ and so it suffices to show that P_λ is in $\overrightarrow{\mathcal{S}}$. We omit here certain homological details and refer back to the proof of Proposition 11.1.36 of [Pre09] in which, since B_λ is finitely presented and $[R\text{-}\mathbf{mod}, \mathbf{Ab}]$ is locally coherent, we find that taking extensions of B_λ commutes with directed colimits. That is, the extension of B_λ by C is the directed colimit of the extensions of B_λ by A_μ, where $C = \operatorname{colim} A_\mu$. Since $\mathcal{S}$ is closed under extensions, each extension of B_λ by A_μ is in $\mathcal{S}$ and hence P_λ is in $\overrightarrow{\mathcal{S}}$ as required

We have hence shown that $\overrightarrow{\mathcal{S}}$ is a torsion class, and so move on to showing it is of finite type. Let τ be the torsion radical corresponding to $\overrightarrow{\mathcal{S}}$ with inclusion $\eta\colon \tau \hookrightarrow \mathrm{id}$ and suppose $(C, c_\lambda) = \operatorname{colim} C_\lambda$. Since each τC_λ is a subobject of C_λ and hence of C, we obtain the following diagrams. On the left, we use naturality of η and the fact that τC_λ is torsion whilst on the right we take the directed colimit over the morphism $\tau C_\lambda \to \tau C$.

$$\begin{tikzcd}
\tau C_\lambda \arrow[r] & C \\
\tau C_\lambda \arrow[r, "\tau c_\lambda"'] \arrow[u, equal] & \tau C \arrow[u, "\eta_C"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{colim} \tau C_\lambda \arrow[r, "m"] & \tau C \\
\tau C_\lambda \arrow[r, "\tau c_\lambda"'] \arrow[u, "\alpha_\lambda"] & \tau C \arrow[u, equal]
\end{tikzcd}$$

Note that m is a monomorphism since each τc_λ is monic and the fact that directed colimits preserve pullbacks, recall that monomorphisms are determined by pullbacks. We now show that m is an isomorphism. We know that τC is torsion and hence we can write$(\tau C, g_\mu) = \operatorname{colim} A_\mu$ for A_μ in $\mathcal{S}$. For a fixed μ, we compose g_μ with η_C to obtain a morphism $\eta_C \circ g_\mu \colon A_\mu \to C$. Since A_μ is finitely presented, this must factor essentially uniquely through the diagram as $\eta_C \circ g_\mu = c_\nu \circ h_\mu$ for some ν. By naturality of η and the fact that A is torsion, we find the following diagram.

$$\begin{tikzcd}
A_\mu \arrow[r, "h"] & C_\nu \arrow[r, "c_\nu"] & C \\
A_\mu \arrow[u, equal] \arrow[r, "\tau h"'] & \tau C_\nu \arrow[u, "\eta_{C_\nu}"'] \arrow[r, "\tau c_\nu"'] & \tau C \arrow[u, "\eta_C"']
\end{tikzcd}$$

Hence, $\eta_C \circ g_\mu = \eta_C \circ \tau c_\nu \circ \tau h$. Since η_C is a monomorphism, we obtain $g_\mu = \tau c_{\nu_\mu} \circ \tau h$. Now, $\tau c_\nu = m \circ \alpha_\nu$ and so we have factored g_μ through m. Since we picked arbitrary μ and from the fact that colimiting cocones are jointly epimorphic, we obtain that m is epimorphic. Since Abelian categories are balanced and we started with m being a monomorphism, we obtain that m is an isomorphism.

Thus, $\operatorname{colim} \tau C_\lambda \cong \tau \operatorname{colim} C_\lambda$ and so τ preserved directed colimits, which by definition means it is of finite type.

For the claim that $\mathcal{S} = [R\text{-}\mathbf{mod}, \mathbf{Ab}]^{\mathrm{fp}} \cap \vec{\mathcal{S}}$, we note that $\mathcal{S} \subset [R\text{-}\mathbf{mod}, \mathbf{Ab}]^{\mathrm{fp}} \cap \vec{\mathcal{S}}$ is immediate and so focus on the other inclusion. Let $(C, c_\lambda) = \operatorname{colim} C_\lambda$ as before, assuming that each C_λ is in $\mathcal{S}$ and assume further that each C is finitely presented. Now, consider the identity morphism on C. Since C is finitely presented, it must factor essentially uniquely through the directed system of C_λ. Hence, there is some ν and $h \colon C \to C_\nu$ such that $\mathrm{id}_C = c_\nu \circ h$. Since id_C is an isomorphism, we have that h is a monomorphism and so C is a subobject of some C_ν and so, by definition of a Serre subcategory, we have that C is in $\mathcal{S}$, as required.

For the converse, we consider some exact sequence of objects in $[R\text{-}\mathbf{mod}, \mathbf{Ab}]$.

$$0 \longrightarrow C \longrightarrow C' \longrightarrow C'' \longrightarrow 0$$

We note that $\mathcal{T}$ is closed under subobjects, quotients and extensions as it is a torsion class, so to show that $[R\text{-}\mathbf{mod}, \mathbf{Ab}]^{\mathrm{fp}} \cap \mathcal{T}$ is Serre, we only need to show that C' is finitely presented if and only if C and C'' are finitely presented. This follows directly from Proposition 6.4 of [Arr25], since $[R\textbf{-mod}, \mathbf{Ab}]$ is locally coherent.

Finally, to show that $\mathcal{T} = \overrightarrow{[R\textbf{-mod}, \mathbf{Ab}]^{\mathrm{fp}} \cap \mathcal{T}}$, we let τ be the torsion radical for $\mathcal{T}$. By Proposition 4.10, all torsion classes are closed under quotients and direct sums and therefore directed colimits, thus we have $\overrightarrow{[R\textbf{-mod}, \mathbf{Ab}]^{\mathrm{fp}} \cap \mathcal{T}} \subset \mathcal{T}$.

For the other inclusion, we consider a torsion object C and write it as a directed colimit $C = \operatorname{colim} Di$ for $D \colon I \to [R\textbf{-mod}, \mathbf{Ab}]$ a directed diagram and assume further that each Di is finitely presented. Then, since τ preserves directed colimits and C is torsion, we find that $C = \tau C = \tau \operatorname{colim} Di = \operatorname{colim} \tau Di$. Each τDi can be written as a directed colimit of its finitely generated subobjects. These are subobjects of τDi and hence torsion and subobjects of Di and so finitely presented, since $[R\textbf{-mod}, \mathbf{Ab}]$ is locally coherent. Thus, we can write C as a directed colimit of finitely presented torsion objects. This shows that $\mathcal{T} \subset \overrightarrow{[R\textbf{-mod}, \mathbf{Ab}]^{\mathrm{fp}} \cap \mathcal{T}}$ and combining this with the other inclusion gives us equality as required. $\square$

This proposition gives us an inclusion-preserving bijection between the torsion theories of finite type on $[R\textbf{-mod}, \mathbf{Ab}]$ and the Serre subcategories on $[R\textbf{-mod}, \mathbf{Ab}]^{\mathrm{fp}}$. We shall discuss this further in Section 5.4 where we will make use of a duality between left and right finitely presented functors to extend the bijections we have found here.

5 Elementary Duality

Elementary Duality of formulae provides a means of moving between the left and right pp-formulae for a given ring. This is roughly done by converting all 'divisibility' formulae (i.e. $\exists y. yr \doteq x$) into 'annihilation' formulae ($rx \doteq 0$) and vice versa. This is extended in a coherent way to all pp-formulae, giving us a lattice anti-isomorphism of left and right pp-formulae. We shall parallel this duality for formulae with a duality for finitely presented functors, phrased in terms of the tensor product. We shall prove that these two dualities are intimately linked and then derive some important consequences from them.

5.1 Duality Of Formulae

Fix a ring R and a context $\overline{A} = (A_1, \dots, A_n)$. We denote the set of $T_{\mathbf{Mod}\text{-}R}$ equivalence classes of pp-formulae of context $\overline{A}$ in the language of right-modules by $\mathrm{pp}_R^{\overline{A}}$ and equivalence classes of pp-formulae in the language of left-modules by ${}_R\mathrm{pp}^{\overline{A}}$. We shall drop the superscript when the context is clear or not relevant.

These sets come with a natural ordering by implication, namely that $\phi(\overline{x}) \leq \psi(\overline{x})$ if and only if $T_{\mathbf{Mod}\text{-}R} \models \forall \overline{x}.\phi(\overline{x}) \to \psi(\overline{x})$ In fact, this ordering is a lattice. Recall that a lattice is a partially ordered set for which every pair of elements has a least upper bound, called a join, and a greatest lower bound, called a meet.

The meet of $\phi(\overline{x})$ and $\psi(\overline{x})$ is given by their conjunction, equivalently the intersection of the corresponding definable subgroups in each module. The join is given by the (indirect) sum of the definable subgroups. Syntactically, this sum is written $\chi(\overline{x}) = \exists \overline{z}, \overline{z}'.(\phi(\overline{z}) \wedge \psi(\overline{z}') \wedge \overline{x} \doteq \overline{z} + \overline{z}')$.

Proposition 5.1. *Conjunction and sum are the greatest lower bound and least upper bounds, respectively, on $(pp_R, \leq)$.[[Pop73] Chapter 3 Theorem 7.1]*

Elementary Duality is an operator that moves between these two lattices and gives us a lattice anti-isomorphism between them. Here, by anti-isomorphism we mean a bijecction which reverses the ordering. This is equivalent to a bijection that swaps meets and joins. This is a very strong connection between the left and right modules over a ring which are otherwise a priori disconnected.

To discuss Elementary Duality, we will need a few more syntactic tools for working with pp-formulae. We adopt a new convention for writing the ring actions, namely if $*_r$ is a function symbol in the language of right-modules over R, we will write xr for $*_r(x)$. This brings us in line with the usual notation for right modules. We now translate the linear polynomial form of our pp-formulae into a matrix form.

Remark 5.2. Let $\phi(\overline{x}) = \exists \overline{y}.(\bigwedge_{i=1}^{n} p_i(\overline{x}, \overline{y}) \doteq 0)$ where each p_i is of the form $p_i(\overline{x}, \overline{y}) = \sum_{j=1}^{m} x_j a_{ji} + \sum_{k=1}^{t} y_k b_{ki}$. We can define a matrix H.

$$H = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \\ a_{m1} & & & a_{mn} \\ b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & \ddots & & \\ \vdots & & & \\ b_{t1} & & & b_{tn} \end{bmatrix}$$

Now, we can rewrite ϕ as $\phi(\overline{x}, \overline{y}) = \exists \overline{y}.(\overline{x}\, \overline{y})H \doteq 0$, where $\overline{x}$ and $\overline{y}$ are viewed as elements of a row vector.

Note that these elements of these matrices are not all elements of one ring, but rather are the function symbols associated with elements of each sort of the contexts of $\overline{x}$ and $\overline{y}$. Matrix multiplication is hence interpreted as summing over the application of these function symbols.

Similarly to how we write quantifier free pp-formulae as morphisms between direct sums of representable modules, we can write such morphisms as matrices. Explicitly, suppose we have sums of representables $A = \bigoplus_{i=1}^{n} \text{よ} A_i$ and $B = \bigoplus_{j=1}^{m} \text{よ} B_j$. Let $\iota_i \colon \text{よ} A_i \to A$ be the coprojections for A and let $\pi_j \colon B \to \text{よ} B_j$ be the projections for B. Then, given a morphism $r \colon A \to B$, we can break it into $i \cdot j$ many components by $r_{ij} = \pi_j \circ r \circ \iota_i$. Thus, r is represented by the following matrix.

$$H_r = \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ \vdots & \ddots & \\ r_{n1} & & r_{nm} \end{bmatrix}$$

Tuples of elements of a module M can be viewed as row vectors and composition of a tuple $\overline{a} \colon B \to M$ with r is done by multiplying by H_r on the right, so that $\overline{a} \circ r$ is the tuple given by the row vector $\overline{a}H_r$. Similarly, composing morphisms $r\colon A \to B$ and $s\colon B \to C$ between representables is given by matrix multiplication, so that $s \circ r$ is represented by $H_s H_r$.

We use these matrix forms to formulate a condition for implication of pp-conditions in terms of how their matrices relate.

Proposition 5.3 ([Pre09] Lemma 1.1.13)**.** *Let $\phi(\overline{x}) = \exists\overline{y}.(\overline{x}\,\overline{y})H_\phi \doteq 0$ and $\psi(\overline{x}) = \exists\overline{z}.(\overline{x}\,\overline{y})H_\psi \doteq 0$. Then, $\psi \leq \phi$ if and only if there are matrices G', G'' and K such that $\begin{bmatrix} I & G' \\ 0 & G'' \end{bmatrix} H_\phi = H_\psi K$.*

Here, I denotes the identity matrix in the context of $\overline{x}$.

Proof. We write $H_\phi = \begin{bmatrix} A \\ -B \end{bmatrix}$ and $H_\psi = \begin{bmatrix} A' \\ -B' \end{bmatrix}$ where A and A' match the context of $\overline{x}$, B matches that of $\overline{y}$ and B' matches that of $\overline{z}$. Thus, we can rewrite ϕ and ψ as $\phi(\overline{x}) = \exists\overline{y}.(\overline{x}A - \overline{y}B \doteq 0)$ and $\psi(\overline{x}) = \exists\overline{z}.(\overline{x}A' - \overline{z}B') \doteq 0$. We can also write $\begin{bmatrix} I & G' \\ 0 & G'' \end{bmatrix} H_\phi = H_\psi K$ as the pair of equations $A - G'B = A'K$ and $-G''B = -B'K$.

We start with the backwards direction. Fix some arbitrary module M and tuple $\overline{a}$ such that $M \vDash \psi(\overline{a})$. Let $\overline{b}$ be such that $\overline{a}A' - \overline{b}B' = 0$. We prove that $M \vDash \phi(\overline{a})$ in several steps.

$$\begin{aligned}
& & \overline{a}A' - \overline{b}B' &= 0 & & \\
&\implies & \overline{a}A'K - \overline{b}B'K &= 0 & &\text{(Multiply on the right by } K\text{.)} \\
&\implies & \overline{a}(A - G'B) - \overline{b}G''B &= 0 & &\text{(Substitute from hypothesis.)} \\
&\implies & \overline{a}A - \overline{a}G'B - \overline{b}G''B &= 0 & & \\
&\implies & \overline{a}A - \overline{a}G'B - \overline{b}'B &= 0 & &\text{(Rewrite } \overline{b}G'' \text{ as } \overline{b}'\text{.)} \\
&\implies & \overline{a}A - (\overline{a}G' - \overline{b}')B &= 0 & & \\
&\implies & \overline{a}A - \overline{b}''B &= 0 & &\text{(Rewrite } \overline{a}G' - \overline{b}' \text{ as } \overline{b}''\text{.)}
\end{aligned}$$

The last line is precisely what is needed for $M \vDash \phi(\overline{a})$.

For the forwards direction, we let $\chi(\overline{x}, \overline{y}) = \bigwedge_{k=1}^{t} p_k(\overline{x}, \overline{y}) \doteq 0$ be the quantifier-free part of ψ and $\theta(\overline{x}, \overline{z}) = \bigwedge_{l=1}^{h} q_l(\overline{x}, \overline{z}) \doteq 0$ be the quantifier free part of ϕ. Let the context of $\overline{x}$ be $(W_1, \dots, W_n)$, the context of $\overline{y}$ be $(Y_1, \dots, Y_m)$, the context of $\overline{z}$ be $(Y_1', \dots, Y_g')$, the context of each p_k be Z_k and the context of each q_l be Z_l'. We write $W = \bigoplus_{i=1}^{n} \text{よ} W_i$ and analogously for Y and Z.

We can represent χ as a morphism $r_\chi \colon Z \to W \oplus Y$ and represent θ by $r_\theta \colon Z' \to W \oplus Y'$. We let $(M, (\overline{a}\,\overline{b}))$ be a free realiser for χ.

Now, by assumption, $M \vDash \psi(\overline{a})$ implies that $M \vDash \phi(\overline{a})$ and so there is some $\overline{d} \colon Y' \to M$ such that $(\overline{a}\,\overline{d}) \circ r_\theta = 0$. Since it is a finite sum of representables, $W \oplus Y'$ is projective. Then, because $(\overline{a}\,\overline{b}) \colon W \oplus Y \to M$ is epimorphic, $(\overline{a}\,\overline{d})$ must factor through it as $(\overline{a}\,\overline{d}) = (\overline{a}\,\overline{b}) \circ u$.

It follows that $(\overline{a}\,\overline{b}) \circ u \circ r_\theta = 0$. The kernel of $(\overline{a}\,\overline{b})$ is r_χ by definition of the free realiser. Thus, $u \circ r_\theta$ factorises through r_χ as $u \circ r_\theta = r_\chi \circ v$.

$$
\begin{tikzcd}
Z \arrow[r, "r_\chi"] & W \oplus Y \arrow[r, tail, "(\overline{a}\,\overline{b})"] & M \\
 & & W \oplus Y' \arrow[u, "(\overline{a}\,\overline{d})"'] \arrow[ul, dashed, "\exists u"] \\
 & & Z' \arrow[u, "r_\theta"'] \arrow[uull, dashed, "\exists! v"]
\end{tikzcd}
$$

What remains is to convert this equation into matrix form. Immediately by the description of the matrix form for a morphism between representables, we see that r_θ is represented by H_ϕ and r_χ is represented by H_ψ. Now, $u \colon W \oplus Y' \to W \oplus Y$ can be broken down into 4 blocks. Namely, if $\iota_w \colon W \to W \oplus Y'$ and $\iota_y \colon Y' \to W \oplus Y'$ are coprojections and $\pi_w \colon W \oplus Y \to W$ and $\pi_y \colon W \oplus Y \to Y$ are projections, we have that

$$
u = \begin{bmatrix} \pi_w \circ u \circ \iota_w & \pi_y \circ u \circ \iota_w \\ \pi_w \circ u \circ \iota_y & \pi_y \circ u \circ \iota_y \end{bmatrix}.
$$

We know that $u \circ (\overline{a}\,\overline{b}) = (\overline{a}\,\overline{d})$ and so u acts as the identity on the W component. Note that projectivity does not require a unique u, so we can simply choose an appropriate u that works. This gives us that $\pi_w \circ u \circ \iota_w$ is represented by I and $\pi_w \circ u \circ \iota_y$ is 0.

Finally, we can simply set G' to represent $\pi_y \circ u \circ \iota_w$, G'' to represent $\pi_y \circ u \circ \iota_y$ and K to represent v. Translating $u \circ r_\theta = r_\chi \circ v$ into matrix form then gives us the required equation.

$\square$

Remark 5.4. From the backwards direction of the proof, we used three different kinds of implication:

$$\exists \overline{y}. \begin{bmatrix} \overline{x} & \overline{y} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \doteq 0 \to \exists \overline{y}. \begin{bmatrix} \overline{x} & \overline{y} \end{bmatrix} \begin{bmatrix} AK \\ BK \end{bmatrix} \doteq 0 \tag{1}$$

$$\exists \overline{y}. \begin{bmatrix} \overline{x} & \overline{y} \end{bmatrix} \begin{bmatrix} A \\ KB \end{bmatrix} \doteq 0 \to \exists \overline{y}. \begin{bmatrix} \overline{x} & \overline{y} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \doteq 0 \tag{2}$$

$$\exists \overline{y}. \begin{bmatrix} \overline{x} & \overline{y} \end{bmatrix} \begin{bmatrix} A+GB \\ B \end{bmatrix} \doteq 0 \to \exists \overline{y}. \begin{bmatrix} \overline{x} & \overline{y} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \doteq 0 \tag{3}$$

From the forwards direction, we know that every implication $\psi \leq \phi$ can be written as a matrix equation $GH_\phi = H_\psi K$. We can then apply the backwards direction to break down the implication $\psi \leq \phi$ into one of these three **basic implications**. This gives a simplified structure for the ordering on the lattice of pp-formulae which we will use for proving that Elementary Duality holds.

We note that each of these implications also has a dual form for left pp-formulae:

$$\exists \overline{y}. \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} \doteq 0 \to \exists \overline{y}. \begin{bmatrix} KA & KB \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} \doteq 0 \tag{4}$$

$$\exists \overline{y}. \begin{bmatrix} A & BK \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} \doteq 0 \to \exists \overline{y}. \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} \doteq 0 \tag{5}$$

$$\exists \overline{y}. \begin{bmatrix} A+BG & B \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} \doteq 0 \to \exists \overline{y}. \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} \doteq 0 \tag{6}$$

We now define the Elementary Duality operator. The idea here is that we have two basic kinds of pp-formulae: annihilators and divisors. An annihilator formula is of the form $\phi(\overline{x}) = xr \doteq 0$. A divisor is of the form $\exists y.(x \doteq yr)$. We then have more general annihilator formulae of the form $\overline{x}H \doteq 0$ and more general divisors of the form $\exists \overline{y}.\overline{x} = \overline{y}H$. Our duality will send annihilator formulae on right modules to divisior formulae on left modules and similarly will send divisors to annihilators.

Fully, the duality operator, which we write as D for both directions, sends the formula $\phi(\overline{x}) = \exists \overline{y}. \begin{bmatrix} \overline{x} & \overline{y} \end{bmatrix} \begin{bmatrix} A \\ -B \end{bmatrix} \doteq 0$ to $D\phi(\overline{x}) = \exists \overline{z}. \begin{bmatrix} I & A \\ 0 & -B \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{z} \end{bmatrix} \doteq 0$. Note that all our conventions for writing formulae as matrices carry wholesale to the left-module case, all that changes is that we now act on the left and hence we treat our tuples as column vectors of elements. By setting $B = 0$, we get the annihilator formula $\phi(\overline{x}) = \overline{x}A \doteq 0$ and this dualises to $D\phi(\overline{x}) = \exists \overline{z}.(\overline{x} \doteq -A\,\overline{z})$. This is equivalent over $T_{\textbf{Mod-}R}$ to $\exists \overline{z}.(\overline{x} = A\,\overline{z})$, so we will usually write this latter form for $D\phi$. By setting $A = I$, we get the divisor formula $\phi(\overline{x}) = \exists \overline{y}.(\overline{x} \doteq \overline{y}B)$. This dualises to $D\phi(\overline{x}) = \exists \overline{z}.(\overline{x} \doteq -\overline{z} \wedge -B\,\overline{z} \doteq 0)$. Similarly to the annihilators, we will replace $\overline{z}$ by $-\overline{z}$ to get the formula $D\phi(\overline{x}) = \exists \overline{z}.(\overline{x} \doteq \overline{z} \wedge B\,\overline{z} \doteq 0)$.

For going from left to right formulae, we define D as taking $\phi(\overline{x}) = \exists \overline{y}. \begin{bmatrix} A & -B \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} \doteq 0$ to $D\phi(\overline{x}) = \exists \overline{z}. \begin{bmatrix} \overline{x} & \overline{z} \end{bmatrix} \begin{bmatrix} I & 0 \\ A & -B \end{bmatrix} \doteq 0$.

Proposition 5.5 ([Pre09] Proposition 1.3.1)**.** *For each context $\overline{A}$, the operator D is a lattice anti-isomorphism between the left and right lattices of equivalence classes of pp-formulae of context $\overline{A}$.*

Proof. We first show that D is order-reversing from right to left lattices. For this, it suffices to show order-reversal for the basic implications from Remark 5.4 as these generate the whole ordering. Thus, we split into cases.

For case 1, we have

$$\phi(\overline{x}) = \exists \overline{y}.\begin{bmatrix}\overline{x} & \overline{y}\end{bmatrix}\begin{bmatrix}A \\ B\end{bmatrix} \doteq 0 \leq \psi(\overline{x}) = \exists \overline{y}.\begin{bmatrix}\overline{x} & \overline{y}\end{bmatrix}\begin{bmatrix}AK \\ BK\end{bmatrix} \doteq 0.$$

These dualise to

$$D\phi(\overline{x}) = \exists \overline{z}.\begin{bmatrix}I & A \\ 0 & B\end{bmatrix}\begin{bmatrix}\overline{x} \\ \overline{z}\end{bmatrix} \doteq 0 \text{ and } D\psi(\overline{x}) = \exists \overline{z}.\begin{bmatrix}I & AK \\ 0 & BK\end{bmatrix}\begin{bmatrix}\overline{x} \\ \overline{z}\end{bmatrix} \doteq 0.$$

We obtain $D\psi \leq D\phi$ by rewriting $\overline{z}' = K\overline{z}$.

For case 2, set

$$\phi(\overline{x}) = \exists \overline{y}.\begin{bmatrix}\overline{x} & \overline{y}\end{bmatrix}\begin{bmatrix}A \\ KB\end{bmatrix} \doteq 0 \text{ and } \psi(\overline{x}) = \exists \overline{y}.\begin{bmatrix}\overline{x} & \overline{y}\end{bmatrix}\begin{bmatrix}A \\ B\end{bmatrix} \doteq 0.$$

Then, these dualise as

$$D\phi(\overline{x}) = \exists \overline{z}.\begin{bmatrix}I & A \\ 0 & KB\end{bmatrix}\begin{bmatrix}\overline{x} \\ \overline{z}\end{bmatrix} \doteq 0 \text{ and } D\psi(\overline{x}) = \exists \overline{z}.\begin{bmatrix}I & A \\ 0 & B\end{bmatrix}\begin{bmatrix}\overline{x} \\ \overline{z}\end{bmatrix} \doteq 0.$$

Then $D\psi \leq D\phi$ holds since $M \models \forall \overline{z}. B\,\overline{z} \doteq 0 \to KB\,\overline{z} \doteq 0$.

For case 3, set

$$\phi(\overline{x}) = \exists \overline{y}.\begin{bmatrix}\overline{x} & \overline{y}\end{bmatrix}\begin{bmatrix}A + GB \\ B\end{bmatrix} \doteq 0 \text{ and } \psi(\overline{x}) = \exists \overline{y}.\begin{bmatrix}\overline{x} & \overline{y}\end{bmatrix}\begin{bmatrix}A \\ B\end{bmatrix} \doteq 0.$$

Their duals are

$$D\phi(\overline{x}) = \exists \overline{z}.\begin{bmatrix}I & A + GB \\ 0 & B\end{bmatrix}\begin{bmatrix}\overline{x} \\ \overline{z}\end{bmatrix} \doteq 0 \text{ and } D\psi(\overline{x}) = \exists \overline{z}.\begin{bmatrix}I & A \\ 0 & B\end{bmatrix}\begin{bmatrix}\overline{x} \\ \overline{z}\end{bmatrix} \doteq 0.$$

Expanding the equations to a system, we get that $M \models D\psi(\overline{a})$ when $\overline{a} + A\,\overline{b} = 0$ and $B\,\overline{b} = 0$ for some realiser $\overline{b}$. Then, for $D\phi$, we want that $\overline{a} + (A + GB)\,\overline{c} = 0$ for some $\overline{c}$. But, if $B\,\overline{b} = 0$, we see that $(A + GB)\,\overline{b} = A\,\overline{b} + GB\,\overline{b} = A\,\overline{b}$. Thus, we obtain that $\overline{b}$ realises $M \models D\phi(\overline{a})$. Since we picked arbitrary $\overline{a}$, this shows that $D\psi \leq D\phi$.

Thus, we have shown order-reversal. The above cases can be dualises to show that D is also order-reversing going from left to right. Thus, what remains is to show that D^2 is the identity on each of the right and left lattices. We will show it for right lattices and then the proof will easily dualise to left lattices.

Let $\phi(\overline{x}) = \exists \overline{y}. \begin{bmatrix} \overline{x} & \overline{y} \end{bmatrix} \begin{bmatrix} A \\ -B \end{bmatrix} \doteq 0$. Then $D\phi = \exists \overline{z}. \begin{bmatrix} I & A \\ 0 & -B \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{z} \end{bmatrix}$ and so $D^2\phi = \exists \overline{w}. \begin{bmatrix} \overline{x} & \overline{w} \end{bmatrix} \begin{bmatrix} I & 0 \\ I & A \\ 0 & -B \end{bmatrix} \doteq 0$. We write $\overline{w} = \begin{bmatrix} \overline{w}' & \overline{w}'' \end{bmatrix}$ where $\overline{w}'$ matches up with A and $\overline{w}''$ matches up with B, so that $\overline{w} \begin{bmatrix} A \\ -B \end{bmatrix} = \begin{bmatrix} \overline{w}'A & -\overline{w}''B \end{bmatrix}$.

Hence, we write $D^2\phi = \exists \overline{w}', \overline{w}''. \begin{bmatrix} \overline{x} & \overline{w}' & \overline{w}'' \end{bmatrix} \begin{bmatrix} I & 0 \\ I & A \\ 0 & -B \end{bmatrix} \doteq 0.$

This gives us the following system of equations.

$$\overline{x} + \overline{w}' = 0 \tag{7}$$

$$\overline{w}'A - \overline{w}''B = 0 \tag{8}$$

We can hence eliminate $\overline{w}'$ to obtain $D^2\phi(\overline{x}) \equiv \exists \overline{w}''. \begin{bmatrix} \overline{x} & \overline{w}'' \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \doteq 0$. By replacing $\overline{w}''$ with $-\overline{w}''$ we then recover ϕ. Thus, $\phi \equiv D^2\phi$ and, since the elements of our lattices are equivalence classes of formulae, we have that $D^2 = \mathrm{id}_{\mathrm{pp}_R}$. The dual argument shows the same for the left lattice. This completes the proof that D is a bijection and hence an anti-isomorphism. □

5.2 Duality Of Functors

Recall the definition of the tensor product from Section 2.3. We will now use the tensor product to define a duality between finitely presented functors on finitely presented left modules and finitely presented functors on finitely presented right modules. This duality is not exactly the same as the elementary Duality for formulae, but in the next section we will show a close relationship between them.

Let $F\colon R\text{-}\mathbf{mod} \to \mathbf{Ab}$ be a finitely presented functor on finitely presented right modules. We define the **dual functor** $dF\colon \mathbf{mod}\text{-}R \to Ab$ by $dF(A) = \mathrm{Hom}(F, A\otimes -)$ for every A in $\mathbf{mod}\text{-}R$. Similarly, for a morphism $g\colon A \to B$ in $\mathbf{mod}\text{-}R$, we define $dF(g) = \mathrm{Hom}(F, g\otimes -)$.

Given a natural transformation $f\colon F \to G$ of finitely presented functors, for each M in $\mathbf{mod}\text{-}R$, the component of df at M acts by $\tau \mapsto \tau \circ f$ for each $\tau \in \mathrm{Hom}(G, M\otimes -)$.

As with the duality for formulae, we use d also for the functor going from functors on left modules to functors on right modules. Given a functor $F\colon \mathbf{mod}\text{-}R \to \mathbf{Ab}$ and a finitely presented right module A, we define $dF(A) = \mathrm{Hom}(F, -\otimes A)$. The actions of dF and d on morphisms are defined analogously.

The proof of the following proposition uses homological techniques which are beyond our scope and hence is omitted.

Proposition 5.6 ([Pre09] Lemma 10.2.3)**.** *The functor $d\colon [R\text{-}\boldsymbol{mod}, \mathbf{Ab}]^{fp} \to [\boldsymbol{mod}\text{-}R, \mathbf{Ab}]^{op}$ is exact.*

We now show that the application of d always gives us a finitely presented functor. Thus, d is a well-defined functor $[R\text{-}\mathbf{mod}, \mathbf{Ab}]^{\mathrm{fp}} \to [\mathbf{mod}\text{-}R, \mathbf{Ab}]^{\mathrm{op}}$.

Proposition 5.7 ([Pre09] Proposition 10.3.3)**.** *Let $F \in [R\text{-}\boldsymbol{mod}, \mathbf{Ab}]^{fp}$. Then $dF \in [\boldsymbol{mod}\text{-}R, \mathbf{Ab}]$ is finitely presented.*

Proof. Suppose F has a presentation as a cokernel of a morphism between representables given by the exact sequence

$$(L, -) \xrightarrow{\mathrm{Hom}(f,-)} (K, -) \longrightarrow F \longrightarrow 0$$

where K and L are finitely presented left R modules. The duality for representables is computed by $d(\mathrm{Hom}(K, -))A = \mathrm{Hom}(\mathrm{Hom}(K, -), A\otimes -)$. By Yoneda, we then obtain that $d(\mathrm{Hom}(K, -))A$ is naturally isomorphic to $A\otimes K$ and hence $d(\mathrm{Hom}(K, -))$ is isomorphic to $-\otimes K$. Thus, when we apply the duality to the above presentation (which by, Proposition 5.6, is exact) we obtain the following.

$$0 \longrightarrow dF \longrightarrow - \otimes K \xrightarrow{-\otimes f} - \otimes L$$

The functors $- \otimes K$ and $- \otimes L$ are finitely presented by Proposition 3.14. The image I of $- \otimes f$ is finitely generated as the epimorphic image of a finitely presented functor. Further, I is a subobject of $- \otimes L$ and hence, by local coherence of $[\textbf{mod-}R, \mathbf{Ab}]$, is finitely presented. We obtain the following exact sequence.

$$0 \longrightarrow dF \longrightarrow - \otimes K \longrightarrow I \longrightarrow 0$$

Since $- \otimes K$ and I are both finitely presented, we must have that dF is finitely generated. Then, once again by local coherence, we obtain that dF is finitely presented as a finitely generated subobject of $- \otimes K$. □

We can then extend this to the full duality. As with Elementary Duality, we abuse some notation and denote also by d the corresponding functor $d\colon ([\textbf{mod-}R, \mathbf{Ab}]^{\mathrm{fp}})^{\mathrm{op}} \to [R\textbf{-mod}, \mathbf{Ab}]^{\mathrm{fp}}$.

Theorem 5.8 ([Pre09] Theorem 10.3.4)**.** *The functor d^2 is naturally equivalent to the identity on $[R\textbf{-mod}, \mathbf{Ab}]^{fp}$, making each d an equivalence of categories.*

Proof. We start by computing the action of $d\colon ([\textbf{mod-}R, \mathbf{Ab}]^{\mathrm{fp}})^{\mathrm{op}} \to [R\textbf{-mod}, \mathbf{Ab}]^{\mathrm{fp}}$ on $- \otimes K$ for a finitely presented left module K. For another finitely presented left module L, we know that $[d(- \otimes K)]L = \mathrm{Hom}(- \otimes K, - \otimes L)$. By Theorem 3.12, sending a module K to $- \otimes K$ is full and faithful and so $\mathrm{Hom}(- \otimes K, - \otimes L) \cong \mathrm{Hom}(K, L)$. This isomorphism being natural in K follows directly from full and faithfulness. It follows that $d(- \otimes K) \cong \mathrm{Hom}(K, -)$.

We now take a presentation $\mathrm{Hom}(L, -) \to \mathrm{Hom}(K, -) \to F \to 0$ of an arbitrary finitely presented functor F. Applying d^2, we obtain, up to isomorphism, $\mathrm{Hom}(L, -) \to \mathrm{Hom}(K, -) \to d^2F \to 0$. Then, cokernels are unique up to isomorphism, so we see that $d^2F \cong F$.

What remains is to show that this isomorphism is natural in F. Thus, take a morphism $f\colon F \to G$. We take a presentation $\mathrm{Hom}(L', -) \to \mathrm{Hom}(K', -) \to G \to 0$ of G. We are in the following situation.

$$\begin{tikzcd}
\mathrm{Hom}(L,-) \arrow[r, "k"] & \mathrm{Hom}(K,-) \arrow[r, "m"] & F \arrow[r] \arrow[d, "f"] & 0 \\
\mathrm{Hom}(L',-) \arrow[r, "l"'] & \mathrm{Hom}(K',-) \arrow[r, "n"'] & G \arrow[r] & 0
\end{tikzcd}$$

First, we note that $f \circ m$ factors uniquely through n by the cokernel property of G. Thus, we have $f \circ m = n \circ h$. We will then use the image-coimage factorisation of k and l to obtain the following diagram.

$$\begin{tikzcd}
\mathrm{Hom}(L,-) \arrow[r, "\mathrm{coim}(k)"] & \mathrm{im}(k) \arrow[r, "\mathrm{im}(k)"] & \mathrm{Hom}(K,-) \arrow[r, "m"] \arrow[d, "h"] & F \arrow[r] \arrow[d, "f"] & 0 \\
\mathrm{Hom}(L',-) \arrow[r, "\mathrm{coim}(l)"'] & \mathrm{im}(l) \arrow[r, "\mathrm{im}(l)"'] & \mathrm{Hom}(K',-) \arrow[r, "n"'] & G \arrow[r] & 0
\end{tikzcd}$$

Exactness of our original diagram means that $\mathrm{im}(k)$ is the kernel of m and $\mathrm{im}(l)$ is the kernel of n. Thus, $h \circ \mathrm{im}(k)$ factors uniquely through $\mathrm{im}(l)$ as $h \circ \mathrm{im}(k) = \mathrm{im}(l) \circ s$ by the kernel property of $\mathrm{im}(l)$. Finally, we will use the fact that $\mathrm{Hom}(L, -)$ is projective ands $\mathrm{coim}(l)$ is an epimorphism to factor $s \circ \mathrm{coim}(k)$ as $s \circ \mathrm{coim}(k) = \mathrm{coim}(l) \circ t$.

$$\begin{tikzcd}
\mathrm{Hom}(L,-) \arrow[r, "\mathrm{coim}(k)"] \arrow[d, "t"] & \mathrm{im}(k) \arrow[r, "\mathrm{im}(k)"] \arrow[d, "s"] & \mathrm{Hom}(K,-) \arrow[r, "m"] \arrow[d, "h"] & F \arrow[r] \arrow[d, "f"] & 0 \\
\mathrm{Hom}(L',-) \arrow[r, "\mathrm{coim}(l)"'] & \mathrm{im}(l) \arrow[r, "\mathrm{im}(l)"'] & \mathrm{Hom}(K',-) \arrow[r, "n"'] & G \arrow[r] & 0
\end{tikzcd}$$

We then use right-exactness of d^2 to obtain the following diagram.

$$\begin{tikzcd}
\mathrm{Hom}(L,-) \arrow[r] \arrow[d, "d^2 t"] & \mathrm{Hom}(K,-) \arrow[r, "d^2 m"] \arrow[d, "d^2 h"] & d^2 F \arrow[r] \arrow[d, "d^2 f"] & 0 \\
\mathrm{Hom}(L',-) \arrow[r] & \mathrm{Hom}(K',-) \arrow[r, "d^2 n"'] & d^2 G \arrow[r] & 0
\end{tikzcd}$$

By stitching this diagram with the diagrams showing $d^2 F \cong F$ and $d^2 G \cong G$ (forming a rather unwiedly cuboid that we omit), we obtain the naturality diagram for $d^2 \cong \mathrm{id}$ as functors. $\square$

5.3 Duality and Tensors

Before we can prove the relation between the Functorial Duality and the Elementary Duality, we need to take a detour into the tensor product, introduced in Section 2.3. We will prove the important theorem known as Herzog's criterion which characterises the structure of tensor products in terms of the Elementary Duality. From this, we will be able to prove the connection between dualities discussed before. This will end up being rather natural, noting that the Functorial Duality acts as a duality between Hom and Tensor functors on representables.

We start by noting how the tensor product acts with regard to the actions on our modules. Consider a right-module M and a left-module N, a morphism $r\colon B \to A$ of the ring and elements $a \in MA$ and $b \in NB$. We recall the extranatural transformation diagram for $\otimes$.

$$\begin{tikzcd}
MA \otimes NB \arrow[r, "Mr\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes Nr"'] & MB \otimes NB \arrow[d, "\otimes_B"] \\
MA \otimes NA \arrow[r, "\otimes_A"'] & M \otimes_R N
\end{tikzcd}$$

From this we derive $Mr(a) \otimes b = a \otimes Nr(b)$. Writing this the usual way for module actions, this is $ar \otimes b = a \otimes rb$. We now wish to extend this to tuples. We hence consider the tuples of sorts $(A_1, \dots, A_m)$ and $(B_1, \dots, B_n)$ and morphisms $r_{ij}\colon B_i \to A_j$ which are represented by a matrix H. We also wish to consider the sums $S_A = \bigoplus_{j=1}^{m} MA_i \otimes NA_i$ and $S_B = \bigoplus_{i=1}^{n} MB_i \otimes NB_i$. By the universal property of the coproduct, we get a map combining all of the $\otimes_{B_i}$ into a map $S_B \to M \otimes_R N$. For some notation, if $\overline{c}$ and $\overline{d}$ are tuples in M and N respectively of context $\overline{B}$, we will write $\overline{c} \otimes \overline{d} = \sum_{i=1}^{n} c_i \otimes d_i$, which is the image of the aforementioned map.

Writing MH and NH for the matrices composed of Mr_{ij} and Nr_{ij} respectively, we find the following diagram, recalling that the tensor product commutes with direct sums in order to simplify.

$$\begin{tikzcd}
(\bigoplus MA_j) \otimes (\bigoplus NB_i) \arrow[r, "MH\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes NH"'] & \bigoplus_i (MB_i \otimes NB_i) \arrow[d, "\otimes_{S_B}"] \\
\bigoplus_j (MA_j \otimes NA_j) \arrow[r, "\otimes_{S_A}"'] & M \otimes_R N
\end{tikzcd}$$

Hence, we obtain the matrix equation $\overline{a}H \otimes \overline{b} = \overline{a} \otimes H\,\overline{b}$. We note that this should really be $\overline{a}H \otimes \overline{b} = \overline{a} \otimes H^T\,\overline{b}$ to match our convention that acting on right modules should be right multiplication on row vectors and left multiplication on column vectors for left modules.

We then move onto the following lemma which characterises the tensor product structure in terms of matrix equations.

Lemma 5.9 ([Pre09] Proposition 1.3.5)**.** *Let $\overline{a}$ be a tuple from a right-module M and $\overline{b}$ a tuple from a left-module N, each of the context $(A_1, \dots, A_n)$. Then $\overline{a} \otimes \overline{b} = 0$ if and only if there are tuples $\overline{c}$ in M and $\overline{d}$ in N and matrices G and H such that the following equations hold:*

$$\begin{bmatrix} \overline{a} & 0 \end{bmatrix} = \overline{c} \begin{bmatrix} G & H \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} G & H \end{bmatrix} \begin{bmatrix} \overline{b} \\ \overline{d} \end{bmatrix} = 0 \tag{10}$$

Proof. For the backwards direction, we directly compute:

$$\begin{aligned}
\overline{a} \otimes \overline{b} &= \overline{c}G \otimes \overline{b} \\
&= \overline{c} \otimes G\overline{b} \\
&= \overline{c} \otimes -H\overline{d} \\
&= \overline{c}H \otimes -\overline{d} \\
&= 0 \otimes -\overline{d} \\
&= 0
\end{aligned}$$

For the forwards direction, we extend $\overline{b}$ to a generating tuple $(\overline{b}\,\overline{b}')$. That is, extend $\overline{A}$ to $(A_1, \dots, A_n, A_{n+1}, \dots)$ and such that $(\overline{b}\,\overline{b}') \colon Y = \bigoplus よ^{c} A_i \to N$ is an epimorphism. This is by the method of [Arr25, Proposition 4.18]. We complete this to an exact sequence

$$0 \longrightarrow K \xrightarrow{j} Y \xrightarrow{(\overline{b}\,\overline{b}')} N \longrightarrow 0$$

and then apply the tensor functor $M \otimes_R -$

$$M \otimes K \xrightarrow{\mathrm{id}_M \otimes j} M \otimes Y \xrightarrow{\mathrm{id}_M \otimes (\overline{b}\,\overline{b}')} M \otimes N \longrightarrow 0$$

We denote by e_i the element id_{A_i} of $(\text{よ}^{\mathrm{c}} A_i)A_i$. We will write $\overline{e}$ for $(e_1, \ldots, e_n)$ and $\overline{e}'$ for $(e_{n+1}, \ldots)$. It then follows that $[\mathrm{id}_M \otimes (\overline{b}\,\overline{b}')](\overline{a} \otimes \overline{e}) = \overline{a} \otimes \overline{b} = 0$, by hypothesis. Thus, $\overline{a} \otimes \overline{e}$ is in the kernel of $\mathrm{id}_M \otimes (\overline{b}\,\overline{b}')$ and so by exactness it is in the image of $\mathrm{id}_M \otimes j$.

Thus, we set $\overline{a} \otimes \overline{e} = \overline{c} \otimes j(\overline{k})$ for some $\overline{k}$ in K. By Yoneda, we know that $j(\overline{k})$ corresponds to a morphism $Y \to Y$ and we can write this as a matrix J. By setting $Y_1 = \bigoplus_{i=1}^{n} \text{よ}^{\mathrm{c}} A_i$ and similarly for Y_2 so that $Y = Y_1 \oplus Y_2$, we can decompose J into a block matrix $\begin{bmatrix} G & H \end{bmatrix}$ by viewing $j(\overline{k})$ as a morphism $Y_1 \oplus Y_2 \to Y$. This matrix satisfies the property of $J(\overline{e}\,\overline{e}') = j(\overline{k})$ since $(\overline{e}\,\overline{e}')$ Yoneda corresponds to the identity map on Y.

Now, in Proposition 2.23 we computed that $M \otimes \text{よ}^{\mathrm{c}} A = MA$ and we established in 2.25 that $M \otimes -$ commutes with direct sums. Hence, $M \otimes Y \cong \bigoplus MA_i$ which we write as MY. There is hence a canonical isomorphism mapping $\overline{x} \otimes (\overline{e}\,\overline{e}')$ to $\overline{x}$ for all $\overline{x} \in MY$. We now compute

$$\begin{aligned}
(\overline{a}\,\overline{0}) \otimes (\overline{e}\,\overline{e}') &= \overline{c} \otimes j(\overline{k}) \\
&= \overline{c} \otimes J(\overline{e}, \overline{e}') \\
&= \overline{c}J \otimes (\overline{e}, \overline{e}')
\end{aligned}$$

By the above isomorphism, we then obtain $(\overline{a}\,\overline{0}) = \overline{c}J$, as required.

Now, we didn't assume finiteness for our extended $\overline{A}$, however, $j(\overline{k})$ has only finitely many non-zero indices and so we can reduce everything down to only the non-zero indices of $j(\overline{k})$ and $\overline{a}$. We then set $\overline{d} = (\overline{b}\,\overline{b}^{'})(\overline{0}, \overline{e}')$. We then find that $J \begin{bmatrix} \overline{b} \\ \overline{d} \end{bmatrix} = (\overline{b}\,\overline{b}^{'})((j(\overline{k})\,\overline{e}')) = 0$, since $(\overline{b}\,\overline{b}^{'})$ is a module homomorphism and so commutes with the action of J and because j is its kernel. This is the second required equation. □

We can then move directly into Herzog's Criterion, which is in effect a nicer version of Lemma 5.9 using model-theoretic language.

Theorem 5.10 ([Pre09] Theorem 1.3.7)**.** *Let M be a right-module and N a left-module and let $\overline{a} \in M$ an $\overline{b} \in N$ be tuples of sort $(A_1, \dots, A_n)$. Then $\overline{a} \otimes_R \overline{b} = 0$ if and only if there is some pp-formula $\phi(\overline{x})$ of sort $(A_1, \dots, A_n)$ such that $M \vDash \phi(\overline{a})$ and $N \vDash D\phi(\overline{b})$.*

Proof. By Lemma 5.9, $\overline{a} \otimes \overline{b} = 0$ if and only if there are $\overline{c}, \overline{d}, G$ and H such that $(\overline{a}\,\overline{0}) = \overline{c} \begin{bmatrix} G & H \end{bmatrix}$ and $\begin{bmatrix} G & H \end{bmatrix} \begin{bmatrix} \overline{b} \\ \overline{d} \end{bmatrix} = 0$.

Thus, we set

$$\phi(\overline{x}) = \exists\, \overline{y}.\, \begin{bmatrix} \overline{x} & \overline{y} \end{bmatrix} \begin{bmatrix} I & 0 \\ G & H \end{bmatrix} \doteq 0.$$

This has dual formula

$$D\phi(\overline{x}) = \exists\, \overline{z}.\, \begin{bmatrix} G & H \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{z} \end{bmatrix} \doteq 0.$$

It then follows that $M \vDash \phi(\overline{a})$ if and only if there exists some $\overline{c}$ with

$$(\overline{a}\,\overline{0}) = \overline{c} \begin{bmatrix} G & H \end{bmatrix}.$$

Similarly, $M \vDash D\phi(\overline{b})$ if and only if there is some $\overline{d}$ with

$$\begin{bmatrix} G & H \end{bmatrix} \begin{bmatrix} \overline{b} \\ \overline{d} \end{bmatrix} = 0.$$

Thus, the two conditions from the theorem statement are equivalent more or less by definition. □

We can now use Herzog's criterion to relate the two kinds of duality we've encountered together. We first prove a corollary of Herzog's criterion that relates the tensor product structure to the free realisations of formulae that were central to the Main Project.

Corollary 5.11. *Let $(C, \overline{c})$ be a free realisation for $\phi(\overline{x})$. Then, if $\bar{l} \in L$ is a tuple of the same sort as $\overline{c}$ for L a left module, we have that $\overline{c} \otimes \bar{l} = 0$ if and only if $L \vDash D\phi(\bar{l})$.*

Proof. Note that $C \vDash \phi(\overline{c})$ by definition of a free realisation. Thus, if $L \vDash D\phi(\bar{l})$, we simply apply Herzog's criterion.

Now, suppose $\overline{c} \otimes \bar{l} = 0$ and hence, by the criterion, there is some $\psi(\overline{x})$ with $C \vDash \psi(\overline{c})$ and $L \vDash D\psi(\bar{l})$. Then, since $\overline{c}$ is a free realiser of ϕ, it follows that $T_{\mathbf{Mod}\text{-}R} \vDash \forall \overline{x}.\phi(\overline{x}) \to \psi(\overline{x})$. Then, by Elementary Duality, we have $D\psi \leq D\phi$ and hence $L \vDash D\phi(\bar{l})$. $\square$

Given a formula ϕ, this corollary allows us to exhibit the Elementary dual DF_ϕ as a kernel of canonical morphisms out of the functorial dual dF_ϕ.

Proposition 5.12. *Let $\phi(\overline{x})$ be a formula of sort $(A_1, \dots, A_n)$ and denote $Y = \bigoplus_{i=1}^{n} \text{よ} A_i$. Let $(C, \overline{c})$ be a free realisation, hence exhibiting a monomorphism* $\mathrm{Hom}(\overline{c}, -)\colon \mathrm{Hom}(C, -) \to \mathrm{Hom}(Y, -)$. *Then, DF_ϕ is the kernel of* $d\,\mathrm{Hom}(\overline{c}, -)$.

Proof. Recall from [Arr25, Proposition 6.8] that F_ϕ is the image of $\mathrm{Hom}(\overline{c}, -)$, giving us the following diagram.

$$\begin{tikzcd}
\mathrm{Hom}(C,-) \arrow[rr, "\mathrm{Hom}(\overline{c},-)"] \arrow[dr, two heads] & & \mathrm{Hom}(Y,-) \\
& F_\phi \arrow[ur, tail] &
\end{tikzcd}$$

Then, we apply functorial duality to obtain the following.

$$
\begin{tikzcd}
C \otimes - & & Y \otimes - \arrow[ll, "\overline{c} \otimes -"'] \arrow[dl, two heads] \\
& dF_\phi \arrow[ul, tail] &
\end{tikzcd}
$$

By Corollary 5.11, we observe that for each left module L, $\ker(\overline{c} \otimes L) = D\phi(L)$. Hence, since limits are computed pointwise, $\ker(\overline{c} \otimes -) = D\phi$. Then, the proposition follows from $\ker(\overline{c} \otimes -) = \ker(\mathrm{im}(\overline{c} \otimes -))$. □

As a second application of Herzog's criterion, we complete the proof of Theorem 3.12. We make use of the following lemma characterising pure embeddings in terms of the tensor product.

Lemma 5.13 ([Pre09] Proposition 2.1.28)**.** *Let $f\colon M \to N$ be a monomorphism in **Mod**-R. Then f is pure if and only if for each left module L, the map $f \otimes_R \mathrm{id}_L$ is monic.*

Proof. We start wih the forward direction. Pick some arbitrary L. Since $f \otimes_R \mathrm{id}_L$ is a morphism of Abelian groups, it is sufficient to show that it is injective. Thus, we pick $\overline{a} \in M$ and $\overline{l} \in L$ such that $f\overline{a} \otimes_R \overline{l} = 0$. By Herzog's criterion (Theorem 5.10), there is hence some $\phi(\overline{x}) = \exists \overline{y}.\psi(\overline{x})$ such that $N \models \phi(f(\overline{a}))$ and $L \models D\phi(\overline{l})$. By purity of f, we have that $M \models \phi(\overline{a})$. But then $\overline{a} \otimes_R \overline{l}$ satisfies Herzog's criterion and so must be zero. Hence, f has zero kernel and so is monic.

For the reverse direction, it is sufficient to assume the condition holds for all finitely presented L. We pick some $\overline{a}$ and suppose that $f(\overline{a}) \in \phi(N)$ for some pp-formula ϕ. Let $(C, \overline{c})$ be a free realiser for $D\phi$. Then, $f(\overline{a}) \otimes \overline{c}$ by Herzog's criterion, noting that C is finitely presented and $C \models \phi(\overline{c})$ by definition. Since $f \otimes \mathrm{id}_C$ is monic, we hence find that $\overline{a} \otimes \overline{c} = 0$ and so $M \models \phi(\overline{a})$ by Corollary 5.11. This shows that f is pure. □

We shall also use the following general categorical fact about full and faithful functors, stating that they **reflect** all limits and colimits. We explain what this means formally in the proposition statement.

Proposition 5.14 ([Mit65] Theorem II.7.1)**.** *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be arbitrary categories and* $F\colon \mathbf{C} \to \mathbf{D}$ *be a full and faithful functor. Consider a diagram* $J\colon X \to \mathbf{C}$ *and suppose that* (L, η) *is a cocone for* J*. If* $(FL, F\eta)$ *is a colimiting cocone for* $F \circ J$ *then* (L, η) *is a colimiting cocone for* J*.*

Remark 5.15. The equivalent for limits follows by duality. Note that this is distinct from *preserving* colimits.

Proof. Let (M, ε) be a second cocone for J in $\mathbf{C}$. Then, since $(FL, F\eta)$ is a colimiting cocone, there is a unique morphism $h\colon FL \to FM$ such that $F\varepsilon = h \circ F\eta$. By fullness of F, there is then some morphism $h'\colon L \to M$ such that $\varepsilon = h' \circ \eta$. This is unique by faithfulness of F. Thus, (L, η) is a colimiting cocone. $\square$

We now restate the latter part of Theorem 3.12 and prove it.

Theorem 3.12. *Consider the following sequence in* $\boldsymbol{Mod}$-R*.*

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

This sequence is pure-exact if and only if its image under ϵ *is exact in* $[R\text{-}\boldsymbol{mod}, \mathbf{Ab}]$*.*

$$0 \longrightarrow \epsilon M \longrightarrow \epsilon N \longrightarrow \epsilon P \longrightarrow 0$$

Proof. For the forwards direction, we recall from Proposition 2.25 that tensoring with a fixed module is right exact, and so, for each finitely presented left module L, we have that the following sequence is exact.

$$M \otimes_R L \longrightarrow N \otimes_R L \longrightarrow P \otimes_R L \longrightarrow 0$$

The leftmost arrow is then monic by Lemma 5.13. Since kernels and cokernels of functors are computed pointwise and since this holds for arbitrary L, it follows that the exact sequence of tensor functors is exact.

For the backwards direction, we start by applying Proposition 5.14 to see that the sequence in **Mod-**R is exact. This is because exactness is determined solely by limits and colimits and because ϵ is full and faithful by the first part of Theorem 3.12.

For purity, we once again use pointwise computation of limits to obtain that the following sequence is exact for each L in R**-mod**.

$$0 \longrightarrow M \otimes_R L \longrightarrow N \otimes_R L \longrightarrow P \otimes_R L \longrightarrow 0$$

This shows that the map $M \to N$ is a pure monomorphism and hence that the sequence is pure-exact as required. $\square$

5.4 Duality Of Spectra

We now turn our attention to applying the results on duality to the Ziegler spectrum introduced in Section 4. We shall use the torsion theoretic technology from that section along with the functorial duality to obtain a bijection between the closed sets of the left and right Ziegler spectra. This will extend to an isomorphism of locales (that is, the spaces have isomorphic lattices of closed sets). When this is a true homeomorphism of topological spaces is still an open question, see the discussion at the beginning of Section 5.4 of [Pre09]. We will finish by stating some results of Herzog in which he uses these dualities to show some very strong model-theoretic dualities on the levels of theories.

Suppose we have a Serre subcategory $\mathcal{S}$ of $[R\textbf{-mod}, \textbf{Ab}]^{\text{fp}}$. We define the **dual Serre subcategory** $d\mathcal{S}$ to be the subcategory of $[\textbf{mod-}R, \textbf{Ab}]^{\text{fp}}$ given by $d\mathcal{S} = \{dF \mid F \in \mathcal{S}\}$. The next theorem shows that this is indeed a Serre subcategory and establishes a bijection of Serre subcategories of the left and right categories of finitely presented functors.

Theorem 5.16 ([Pre09] Theorem 11.2.9)**.** *Let S be a Serre subcategory of $[R\textbf{-mod}, \textbf{Ab}]^{fp}$. Then, the category $d\mathcal{S}$ is a Serre subcategory of $[\textbf{mod-}R, \textbf{Ab}]^{fp}$*

and $d(d\mathcal{S}) = \mathcal{S}$*.*

Proof. Let $0 \to F \to G \to H \to 0$ be an exact sequence in $[\textbf{mod-}R, \textbf{Ab}]^{\text{fp}}$. By Propositions 5.6 and 5.7, we have that $0 \to dF \to dG \to dH \to 0$ is an exact sequence in $[R\textbf{-mod}, \textbf{Ab}]^{\text{fp}}$.

Suppose G is in $d\mathcal{S}$. Then, dG is in $\mathcal{S}$ and so dF, dH are also in $\mathcal{S}$ since $\mathcal{S}$ is Serre. By Theorem 5.8, we know that $d^2 \cong \text{id}$ and so $F \cong d(dF)$ and $H \cong H \cong d(dH)$. Thus, by definition of $d\mathcal{S}$ we find that F, H are in $d\mathcal{S}$.

Similarly, if F, H are in $d\mathcal{S}$, we find that dF, dH are in $\mathcal{S}$ and so dG is in $\mathcal{S}$. Then, applying $d^2 \cong \text{id}$ again, we find that $G \in d\mathcal{S}$. Thus, we have shown that $d\mathcal{S}$ is Serre.

That $d(dS) = S$ is also from $d^2 \cong \text{id}$. □

From this theorem, we obtain an inclusion-preserving bijection between the Serre subcategories of $[R\textbf{-mod}, \textbf{Ab}]^{\text{fp}}$ and $[\textbf{mod-}R, \textbf{Ab}]^{\text{fp}}$. We can combine this with our two previous bijections, one between definable categories of $\textbf{Mod-}R$ and torsion theories of finite type on $[R\textbf{-mod}, \textbf{Ab}]^{\text{fp}}$ (Theorem 4.19 and the other between said torsion theories and Serre subcategories of $[R\textbf{-mod}, \textbf{Ab}]^{\text{fp}}$ (Proposition 4.21). Putting these all together, we get the following corollary.

Corollary 5.17 ([Pre09] Theorem 12.4.1)**.** *There is a bijection between definable subcategories of* $\textbf{\textit{Mod-}}R$ *and definable subcategories of* $R\textbf{\textit{-Mod}}$*.*

In fact, we can derive something even stronger. Recall from the discussion after Proposition 4.11 that, given a torsion class $\mathcal{T}$, its corresponding torsionfree class can be determined by $\mathcal{F} = \{F \mid \text{Hom}(T, F) = 0 \text{ for all } T \in \mathcal{T}\}$. Suppose we have torsion classes $\mathcal{T} \subset \mathcal{T}'$. It follows that if an object F satisfies $\text{Hom}(T, F) = 0$ for all $T \in \mathcal{T}'$, then it satisfies it for all $T \in \mathcal{T}$. Thus, we find that the corresponding torsionfree classes have the opposite inclusion, $\mathcal{F}' \subset \mathcal{F}$.

We then have an inclusion-preserving bijection from definable subcategories to torsion classes of finite type, an inclusion-reversing bijection from torsion classes to torsionfree classes, an inclusion-prerserving bijection from torsionfree classes to Serre subcategories and an inclusion-preserving bijection from Serre subcategories to dual Serre subcategories. Going both ways, we reverse inclusions precisely twice and so gain an overall inclusion-preserving bijection between the left and right definable subcategories.

We now recall a basic fact from lattice theory.

Proposition 5.18 ([PD02] Lemma 2.27)**.** *Let $f\colon X \to Y$ be an order-preserving bijection of ordered sets. Then, f preserves all existing joins and meets.*

From this we conclude that our inclusion-preserving bijection of closed sets is in fact a lattice isomorphism of closed sets.

Theorem 5.19 ([Pre09] Theorem 5.4.1)**.** *The left and right Ziegler spectra of a given small preadditive category have isomorphic lattices of closed sets.*

This duality is obtained directly from working with formulas in [Her93]. Herzog then extends these results to complete theories of modules obtain the following bijection.

Theorem 5.20 ([Her93] Theorem 6.6)**.** *There is a bijection, arising from Elementary duality, between the sets of complete theories of left and right modules.*

This is a very strong and unexpected result. In general, non-commutative rings can have very different behaviour in their left and right modules. For instance, consider the ring of matrices of the form $\left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right]$ where $a \in \mathbb{Q}$ and b, c are constructible numbers. This ring is right Noetherian, but not left Noetherian and similarly right but not left Artinian (see [rsc23] Ring R_{20}). As another example, it is still an open question whether decomposition of right modules into sums of indecomposable finitely generated modules implies an equivalent decomposition for left modules (this is known as the pure semi-simplicity conjecture, see [Ša21] for more details). On the level of elementary equivalence, however, this theorem shows that the equvialence classes on the right are essentially the same as those on the left.

References

[Arr25] Alexis Arrowsmith. "An Equivalence of Functorial and Model-Theoretic perspectives on Modules". In: *MSc Pure Mathematics Main Project* (2025) (cit. on pp. 1, 4, 5, 12, 26, 29, 31, 37, 49, 53, 54, 57, 71, 73).

[Aus66] Maurice Auslander. "Coherent Functors". In: *Proceedings of the Conference on Categorical Algebra*. Ed. by S. Eilenberg et al. Berlin, Heidelberg: Springer Berlin Heidelberg, 1966, pp. 189–231. ISBN: 978-3-642-99902-4 (cit. on p. 32).

[Bor94] Francis Borceux. *Handbook of Categorical Algebra*. Vol. 2. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1994 (cit. on p. 45).

[Cra94] William Crawley-Boevey. "Locally finitely presented additive categories". In: *Communications in Algebra* 22 (1994), pp. 1641–1674 (cit. on p. 9).

[Dic63] Spencer Ernest Dickinson. "Torsion Theories For Abelian Categories". PhD thesis. New Mexico State University, 1963 (cit. on p. 4).

[EF72] Paul C. Eklof and Edward R. Fischer. "The elementary theory of abelian groups". In: *Annals of Mathematical Logic* 4.2 (1972), pp. 115–171. ISSN: 0003-4843. DOI: `https://doi.org/10.1016/0003-4843(72)90013-7` (cit. on p. 21).

[Fuc15] László Fuchs. *Abelian Groups*. Springer Monographs in Mathematics. Springer Cham, 2015 (cit. on pp. 11, 13).

[Her93] Ivo Herzog. "Elementary Duality of Modules". In: *Transactions of the American Mathematical Society* 340.1 (1993) (cit. on pp. 4, 78).

[Kap54] Irving Kaplansky. *Infinite Abelian Groups*. University of Michigan Publications in Mathematics. University of Michigan Press, 1954 (cit. on p. 21).

[Mac71] Saunders MacLane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics, Vol. 5. New York: Springer-Verlag, 1971, pp. ix+262 (cit. on p. 8).

[Mit65] Barry Mitchell. *Theory of Categories*. Pure and Applied Mathematics 17. Academic Press, 1965. ISBN: 9780080873299 (cit. on p. 75).

[nLa25] nLab authors. *tensor product of functors*. `https://ncatlab.org/nlab/show/tensor+product+of+functors`. Sept. 2025 (cit. on p. 8).

[Par08] Sangwon Park. "Injective and Projective Properties of Representations of Quivers with n Edges". In: *Korean Journal of Mathematics* (2008) (cit. on p. 43).

[PD02] Hilary Priestley and Brian Davey. *Introduction to Lattices and Order*. 2. Cambridge University Press, 2002 (cit. on p. 78).

[Pop73] Nicolae Popescu. *Abelian categories with applications to rings and modules*. London Mathematical Society Monologues 3. Academic Press, 1973 (cit. on pp. 8, 10, 12, 13, 18, 19, 58).

[Pre09] Mike Prest. *Purity, Spectra and Localisation*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2009 (cit. on pp. 22, 25, 26, 28, 31–34, 36, 37, 39–41, 47, 49–53, 55, 60, 63, 66, 67, 70, 72, 74, 76–78).

[Pre88] Mike Prest. *Model Theory and Modules*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1988 (cit. on p. 4).

[rsc23] rschwiebert. *rschwiebert/dart_data: Release of version 23.03*. Version v23.03. Mar. 2023. DOI: `10.5281/zenodo.7745128`. URL: `https://doi.org/10.5281/zenodo.7745128` (cit. on p. 78).

[Ser53] Jean-Pierre Serre. "Groupes D'Homotopie Et Classes De Groupes Abeliens". In: *Annals of Mathematics, Second Series* 58 (1953), pp. 258–294 (cit. on pp. 4, 53).

[Ste75] Bo Stenström. *Rings of Quotients*. Grundlehren der mathematischen Wissenschaften. 1975 (cit. on pp. 4, 13, 45–47).

[Szm55] Wanda Szmielew. "Elementary properties of Abelian groups". eng. In: *Fundamenta Mathematicae* 41.2 (1955), pp. 203–271 (cit. on p. 21).

[Zie84] Martin Ziegler. "Model theory of modules". In: *Annals of Pure and Applied Logic* 26.2 (1984). SPECIAL ISSUE, pp. 149–213. ISSN: 0168-

0072. DOI: https://doi.org/10.1016/0168-0072(84)90014-9 (cit. on pp. 4, 24, 40).

[Ša21] Jan Šaroch. *Pure semisimplicity conjecture and Artin problem for dimension sequences*. 2021. arXiv: 1909.13864 [math.RA]. URL: https://arxiv.org/abs/1909.13864 (cit. on p. 78).