An Introduction to Abelian Categories

`MATH61000, Mini-project`

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1 Introduction

Abelian categories are a tool first introduced by Alexander Grothendieck in a now famous 1957 paper for the Tōhoku Mathematical Journal. The goal of this paper was to unify the study of Homological Algebra of modules with that of sheaf cohomology of a space. As a result, Abelian categories are now the setting in which modern Homological Algebra is developed.

In Homological Algebra, we wish to find algebraic invariants of other mathematical objects, most notably topological spaces. These algebraic invariants are often simpler to compute and study than the spaces themselves and are used to show the non-existence of equivalences between spaces.

The particular invariants we are interested in are chain complexes. These consist of a two-sided sequence $(C_i)_{-\infty}^{+\infty}$ of modules and homomorphisms $d_i \colon C_{i+1} \to C_i$ satisfying the condition $d_i \circ d_{i+1} = 0$ for all i. The morphisms are called the differential maps. The equational condition on differentials ensures that $\operatorname{im}(d_{i+1}) \subseteq \ker(d_i)$. We can then define the homology modules of the chain as $H_i = \ker(d_i)/\operatorname{im}(d_{i+1})$.

The homology modules of a chain reflect important properties of the underlying space. For example, in the theory of simplicial homology, the modules C_i consist of formal sums of oriented i-simplices. The homology modules of the chain then count how many "higher dimensional holes" the space has.

The axioms of an Abelian category provide the means to define chain complexes and homology objects in categories other than the category of modules over a ring. In particular, we will have zero maps, kernels, images and quotients. The category of modules over a commuative ring R, which we will denote by R-**Mod**, is a natural example and many of our definitions will be explained by referencing back to the modules case. The category of sheaves valued in Abelian groups on a topological space also form an Abelian category. This the Abelian category in which the sheaf cohomology mentioned earlier is formulated.

In this project, we will explore two definitions of an Abelian category and prove their equivalence. Doing so, we elucidate the necessary properties of an Abelian category and highlight the key tools used in proving results about them.

Throughout the project, we will assume knowledge of both basic commutative algebra and basic category theory. In particular, we will make reference to common limits and colimits. For any new limit notions, we introduce we will make efforts to translate them into module language for clarity. Occasionally, a less illuminating proof has been omitteed for brevity. These can be found in the sources mentioned in individual sections.

The ideas of Homological Algbra appearing in the Introduction have been taken from [Wei94]. The Tōhoku paper was originally published in French, however there is a high quality English translation in [GBB08].

2 Categorical Preliminaries

In this section we introduce limit and colimit notions dependent on our category having what is called a “zero object”, analogous to the zero module in R-**Mod**. This allows us to build the kernels and cokernels which are central to Abelian categories. Our basic reference for these is [Bor94] and most of these definitions and proofs can be found in Volume 2, Chapter 1.

Definition 2.1. A zero object is an object $\mathbf{0}$ that is both an initial and a terminal object. For every pair of objects X and Y, the existence of a zero object induces a zero morphism $0\colon X \to Y$ given by the composition of the unique map $X \to \mathbf{0}$ and the unique map $\mathbf{0} \to Y$.

Example 2.2. The zero module, denoted as $\mathbf{0}$, consisting of a singleton set $\{0\}$ and all trivial module operations, is a zero object in R-**Mod**. For any module M, there is a unique morphism $\mathbf{0} \to M$ sending the single element to the additive identity of M. Similarly, there is a unique morphism $M \to \mathbf{0}$ sending all elements of M to the single element of $\mathbf{0}$. Hence $\mathbf{0}$ is both initial and terminal.

We will now work in a locally small category $\boldsymbol{C}$ which has a zero object.

Definition 2.3. For a morphism f, its kernel $\ker(f)$ is the equalizer $\mathrm{eq}(f, 0)$ and its cokernel is the coequalizer $\mathrm{coeq}(f, 0)$.

Example 2.4. Let $T\colon M \to N$ be an R-module homomorphism. Then there is a kernel of T, $\ker(T)$ which is a submodule of M. In R-**Mod**, the inclusion map $i\colon \ker(T) \to M$ is then a kernel of T. Similarly, $T(M)$ is a submodule of N and we can form the quotient $q\colon N \to N/T(M)$. The map q is a cokernel of T.

We now develop a few basic properties of kernels and cokernels. Note that the notions are dual, and hence proving a result for one proves the dual result for the other. We notice, in particular, that as equalizers and coequalizers, kernels and cokernels are monic and epic, respectively.

Proposition 2.5. *Let* $0\colon A \to A$ *be the zero map on* A. *Then* $\mathrm{coker}(0) = \mathrm{id}_A$.

Proof. Clearly $\mathrm{id}_A \circ 0 = 0$. Further, for any map $f\colon A \to B$, f factors through id_A as $f = f \circ \mathrm{id}_A$. Uniqueness of this factorisation is also simple, since $g \circ \mathrm{id}_A = g$ and hence if $g \circ \mathrm{id}_A = f$ we get that $g = f$. $\square$

Proposition 2.6. *Let* $f\colon X \to Y$ *be monic. Then* f *has a kernel and* $\ker(f) = 0$.

Proof. Let $g\colon Z \to Y$ be such that $f \circ g = 0$. Notice also that $f \circ 0 = 0$ by definition. But f is monic, so $g = 0$. Hence 0 is the unique morphism such that $f \circ 0 = 0$ and so 0 is the kernel of f by definition of the kernel. $\square$

From Section 1.3 of [Gro57] we draw the notions of image and coimage, which are essential for formulating Grothendieck's original definition of an Abelian category.

Definition 2.7. For a morphism f we call $\ker(\operatorname{coker}(f))$ the image of f, $\operatorname{im}(f)$. We call $\operatorname{coker}(\ker(f))$ the coimage of f, $\operatorname{coim}(f)$.

Proposition 2.8. *Let $f\colon X \to Y$ have both an image $\operatorname{im}(f)\colon V \to Y$ and coimage $\operatorname{coim}(f)\colon X \to U$. Then there is a unique morphism $\bar{f}\colon V \to U$ such that $f = \operatorname{im}(f) \circ \bar{f} \circ \operatorname{coim}(f)$.*

Proof. We are in the following situation:

$$\begin{tikzcd}
K \arrow[r, "\ker(f)"] & X \arrow[r, "f"] \arrow[d, "\operatorname{coim}(f)=\operatorname{coker}(\ker(f))"'] & Y \arrow[r, "\operatorname{coker}(f)"] & C \\
& U \arrow[r, dashed, "\bar{f}"'] & V \arrow[u, "\operatorname{im}(f)=\ker(\operatorname{coker}(f))"'] &
\end{tikzcd}$$

Now observe that $f \circ \ker(f) = 0$ by definition of the kernel, and hence f coequalizes $\ker(f)$ and 0. Thus f must factor uniquely through $\operatorname{coim}(f)$ as $f = h\circ\operatorname{coim}(f)$. Now, $\operatorname{coker}(f)\circ h\circ\operatorname{coim}(f) = \operatorname{coker}(f)\circ f = 0$ and $0 = 0\circ\operatorname{coim}(f)$. $\operatorname{coim}(f)$ is an epimorphism, so we obtain that $\operatorname{coker}(f) \circ h = 0$. Therefore h factors through $\operatorname{im}(f)$ as $h = \operatorname{im}(f) \circ \bar{f}$. We have thus obtained $\bar{f}$ as required, and its uniqueness follows from the uniqueness of the factorisations of f and h above (in particular, any other morphism g making this diagram commute would induce such factorisations). □

Example 2.9. Fix a morphism $T\colon M \to N$ in R-**Mod**. The kernel of T is the inclusion map $i\colon \ker(T) \to M$ and hence $\operatorname{coim}(T)$ is the quotient $M/\operatorname{im}(i) = M/\ker(T)$, since $\ker(T)$ is a submodule of M. The cokernel of T is the map $M \to N/T(M)$, and its kernel is hence the inclusion map $j\colon T(M) \to N$, as this is exactly the module of elements of N sent to zero under this quotient. The unique map $\bar{T}\colon M/\ker(T) \to T(M)$ should hopefully look familiar, as it is the map constructed when proving the first isomorphism theorem for modules.

The next set of definitions and propositions formalise categories in which we can add morphisms together, giving an Abelian group structure on the hom-collections for pairs of objects. This gives a kind of algebraic substrate to the category and we can use results from commutative algebra to prove things about the structure of the category.

Definition 2.10. We say a category $\boldsymbol{C}$ is preadditive if for every triple of objects X, Y, Z, the following two conditions holds:

1. There is an Abelian group structure on $\operatorname{Hom}(X, Y)$
2. The composition map $\circ\colon \operatorname{Hom}(X,Y) \times \operatorname{Hom}(Y,Z) \to \operatorname{Hom}(X,Z)$ is a bilinear map.

Example 2.11. For two homomorphisms $T, R\colon M \to N$, we can add them by pointwise addition in N so that $(T + R)(m) = T(m) + R(M)$. The hom-set

$\mathrm{Hom}(M, N)$ hence inherits an Abelian group structure from N and indeed the structure of an R-module. To see bilinearity of composition, observe that

$$T(R(m+n)) = T(R(m) + R(n)) = T(R(m)) + T(R(n)) = TR(m) + TR(n)$$

simply by applying pointwise additivity of T and R. This gives us linearity in the right hand side of composition, but the left hand side is a near identical argument.

The following generalises the notion of direct sums for modules.

Definition 2.12 ([Mac71])**.** Let A, B be objects in $\boldsymbol{C}$. Consider an object X and morphisms $p_A \colon X \to A$ and $p_B \colon X \to B$ and morphisms $s_A \colon A \to X$ and $s_B \colon B \to X$. We say that (X, p_A, p_B, s_A, s_B) is a biproduct for A and B if (X, p_A, p_B) is a product for A and B, (X, s_a, s_B) is their coproduct and the following coherence equations holds:

$$\begin{aligned} p_A \circ s_A &= \mathrm{id}_A, \quad p_A \circ s_B = 0 \\ p_B \circ s_B &= \mathrm{id}_B, \quad p_B \circ s_A = 0 \end{aligned}$$

This matches the situation in R-**Mod**, where $M \oplus N$, having underlying set $N \times M$, is both a product and coproduct for M and N. In fact, this is the case for products in every preadditive category:

Proposition 2.13. *Let $\boldsymbol{C}$ be a preadditive and objects A, B in $\boldsymbol{C}$. Suppose A and B have a product $(A \times B, p_A, p_B)$. Then there are s_A, s_b such that $A \times B$ is a biproduct for A and B.*

Our final notion for the section is that of an additive category. Proving that certain categories are additive will be central to Section 3.

Definition 2.14. We say $\boldsymbol{C}$ is additive if it is preadditive and additional has a zero object and all finite products (and hence all finite biproducts).

3 Abelian Categories

3.1 Definitions

In this section, we introduce the two notions of an Abelian category. One of these is the definition given by Grothendieck in his famed Tōhoku paper in which he lays down a modernisation of homological algebra in terms of Abelian and Derived categories. The second notion is a more modern formulation which at first blush appears significantly weaker.

The rest of the section will be taken up by the proof that this weaker formulation is in fact equivalent to the original formulation. The second definition does not require the structure of Abelian groups to define. The additive structure of the category is instead defined from categorical first principles. This provides a philosophical argument that the notion of Abelian category is fundamentally categorical, rather than algebraic.

Definition 3.1 ([Gro57])**.** We say that $\boldsymbol{C}$ is a Tōhoku Category if:

1. $\boldsymbol{C}$ is additive
2. $\boldsymbol{C}$ has all kernels and cokernels
3. For a morphism f, the unique map $\bar{f}\colon \operatorname{coim}(f) \to \operatorname{im}(f)$ is an isomorphism.

Please note that "Tōhoku Category" is not standard terminology. This is simply the definition of an Abelian category from the Tōhoku paper.

Remark 3.2. When we set $\boldsymbol{C} = R$-**Mod**, the final axiom here is exactly the statement of the first isomorphism theorem. In this case, the coimage of f is the quotient of the domain by the the kernel.

Definition 3.3 ([Bor94])**.** We say that $\boldsymbol{C}$ is an Abelian category if:

1. $\boldsymbol{C}$ has a zero object
2. $\boldsymbol{C}$ has all finite products and coproducts
3. $\boldsymbol{C}$ has all kernels and cokernels
4. Every monomorphism in $\boldsymbol{C}$ is a kernel
5. Every epimorphism in $\boldsymbol{C}$ is a cokernel

Remark 3.4 (Duality Principle)*.* Observe that for Abelian $\boldsymbol{C}$, $\boldsymbol{C}^{\mathrm{op}}$ is also Abelian since dualising swaps the limits and colimits that already existed in $\boldsymbol{C}$. Hence, when we prove a result for Abelian categories, we get the dual result for free. This is another advantage of the modern definition over the Tōhoku definition.

We will first work towards proving the isomorphism axiom for Abelian categories. Along the way, we will build up some basic theory of Abelian categories. Some of these constructions will be seen to apply also in the Tōhoku category case and we will hence be able to show that every Tōhoku category is Abelian.

We will then turn to constructing an additive structure on the hom-collections of our Abelian categories. These two results will then be sufficient to show that every Abelian category is Tōhoku. This will complete the equivalence of the two definitions.

3.2 Limits and Colimits

Our first step is to show that Abelian categories have sufficient limits and colimits which will we use to construct necessary maps down the line. We will first show that pairs of monomorphisms have pullbacks. We combine these with existence of products to obtain all equalizers.We will use these limits to show that monomorphisms in an Abelian category have well behaved kernels which will set us up for the proof of the isomorphism theorem.

Fix, from hereon, an Abelian category $\boldsymbol{C}$.

Proposition 3.5. *Let $a\colon A \to C$ and $b\colon B \to C$ be monomorphisms with the same codomain. Then their pullback exists.*

Proof. As a and b are monomorphic, they are kernels and so there are maps $f\colon C \to F$ and $g\colon C \to G$ such that $a = \ker(f)$ and $b = \ker(g)$. By the universal property for products, we can then form the map $(f,g)\colon C \to F \times G$ and consider its kernel k. Some simple compositions give us that $f \circ k = p_F \circ (f,g) \circ k = p_F \circ 0 = 0$ and similarly for $g \circ k$ and hence k factors through the kernels of f and g.

$$\begin{tikzcd}
K \arrow[r, "b'"] \arrow[d, "a'"'] \arrow[dr, "k"] & B \arrow[d, "b"] & \\
A \arrow[r, "a"'] & C \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, "{(f,g)}"] & F \\
 & G & F \times G \arrow[l, "p_G"] \arrow[u, "p_F"']
\end{tikzcd}$$

(K, a', b') forms our candidate for a pullback of a and b. It now suffices to show that it is universal. Let (W, u, v) be an object and morphisms making $a \circ u = b \circ v$. Now $p_F \circ (f,g) \circ a \circ u = f \circ a \circ u = 0$ and $p_G \circ (f,g) \circ a \circ u = g \circ b \circ v = 0$. Hence $(f,g) \circ a \circ u = 0$ and so $a \circ u$ factorises uniquely through $\ker((f,g)) = k$ as $a \circ u = k \circ \bar{k}$. Then $a \circ u = a \circ a' \circ \bar{k}$ and $b \circ v = b \circ b' \circ \bar{k}$. Thus $\bar{k}$ is the map required for (W, u, v) to factor through (K, a', b') and its uniqueness follows from the uniqueness of the kernel factorisation. Hence (K, a', b') is indeed the pullback of a and b. □

Proposition 3.6. *$\boldsymbol{C}$ has all equalizers.*

Proof. Recall that if $f \circ g$ is a monomorphism, then g is a monomorphism since $g \circ u = g \circ v$ implies $f \circ g \circ u = f \circ g \circ v$ implies $u = v$.

Now, given morphisms $f, g \colon A \to B$, we can construct monomorphisms from them by looking at the product $A \times B$ and taking the morphisms (id_A, f) and (id_A, g). Since $p_A \circ (\mathrm{id}_A, f) = \mathrm{id}_A$ is a monomorphism, it follows that (id_A, f) is monic and similarly for (id_A, g). We can now take the pullback of these morphisms, (P, u, v).

Then, $p_A \circ (\mathrm{id}_A, f) \circ u = \mathrm{id}_A \circ u = u$ and similarly $p_A \circ (\mathrm{id}_A, g) \circ v = v$. But $p_A \circ (\mathrm{id}_A, f) \circ u = p_A \circ (\mathrm{id}_A, g) \circ v$ and so $u = v$. It then follows that $f \circ u = p_B \circ (\mathrm{id}_A, f) \circ u = p_B \circ (\mathrm{id}_A, g) \circ u = g \circ u$. Then universality of (P, u) as the equalizer for f and g follows from universality of (P, u, u) as the pullback. $\square$

Having established the existence of equalizers, we now move on to using them to prove some niceness properties of our morphisms that will be used to prove the isomorphism theorem.

Proposition 3.7. *Let $f \colon A \to B$ be a morphism. Then f is monomorphic if and only if* $\ker(f) = 0$.

Proof. The forwards direction is given is Proposition 2.6.

For the other direction, let $u, v \colon C \to A$ be such that $f \circ u = f \circ v$. By Proposition 3.6 and a quick dualisation, $\boldsymbol{C}$ has all coequalizers, so let $q \colon A \to E$ be the coequalizer of u and v. Coequalizers are epimorphic and hence $q = \mathrm{coker}(w)$ for some $w \colon W \to A$ since $\boldsymbol{C}$ is Abelian. Since $f \circ u = f \circ v$, f factors through the coequalizer q as $f = m \circ q$. Then we can deduce that $f \circ w = m \circ q \circ w = m \circ 0 = 0$ and so w must factor through the kernel of f. Our morphisms now look like this:

$$\begin{tikzcd}
 & W \arrow[r, "n"] \arrow[dr, "w"'] & K \arrow[d, "\ker(f)"] & \\
C \arrow[rr, shift left, "u"] \arrow[rr, shift right, "v"'] & & A \arrow[r, "f"] \arrow[dr, "q"'] & B \\
 & & & E \arrow[u, "m"']
\end{tikzcd}$$

We now apply the assumption that $\ker(f) = 0$. It follows that $w = 0$. By Proposition 2.5 we see that q is the identity map on A and so in particular is monomorphic. Hence from $q \circ u = q \circ v$ we obtain $u = v$, which is what we required to show that f is monomorphic. $\square$

Remark 3.8. We have in particular that f is monic if and only if $f \circ x = 0$ implies $x = 0$. The right to left implication is immediate from $f \circ \ker(f) = 0$ and the other implication follows since $f \circ x = 0$ implies x factors through $\ker(f) = 0$.

Proposition 3.9. *A morphism in $\boldsymbol{C}$ is an isomorphism if and only if it is both a monomorphism and an epimorphism. That is, $\boldsymbol{C}$ is balanced.*

We are now ready to prove the isomorphism theorem for Abelian categories.

Proposition 3.10. *For a morphism f, the unique map $\bar{f}\colon \operatorname{coim}(f) \to \operatorname{im}(f)$ is an isomorphism.*

Proof. Recall that we have the following maps:

$$\begin{tikzcd}
K \arrow[r, "\ker(f)"] & X \arrow[r, "f"] \arrow[d, "\operatorname{coim}(f)"'] & Y \arrow[r, "\operatorname{coker}(f)"] & C \\
 & U \arrow[r, "\bar{f}"'] & V \arrow[u, "\operatorname{im}(f)"'] &
\end{tikzcd}$$

We let $h = \operatorname{im}(f) \circ \bar{f}$, which is also the unique map such that $f = h \circ \operatorname{coim}(f)$ by the cokernel universal property on $\ker(f)$. Similarly, we obtain $g = \bar{f} \circ \operatorname{coim}(f)$ and $f = \operatorname{im}(f) \circ g$. To show $\bar{f}$ is an isomorphism, we will show that h is monic and g is epic, from which we observe $\bar{f}$ as monic and epic.

To show h is monic, we will use Proposition 3.7. Thus, let $x = \ker(h)$ and $q = \operatorname{coker}(x) = \operatorname{coim}(h)$. Then h factors through its coimage as $h = r \circ q$. We will show that q is monic, and hence $q \circ x = 0$ from q as the cokernel of x implies $x = 0$.

As q and $\operatorname{coim}(f)$ are both epis, their composition is also an epi and hence is the cokernel of some map i. This satisfies $f \circ i = r \circ q \circ \operatorname{coim}(f) \circ i = r \circ 0 = 0$. Hence i factors through $\ker(f)$ as $i = \ker(f) \circ l$. Now, $\operatorname{coim}(f) \circ i = \operatorname{coim}(f) \circ \ker(f) \circ l = 0$ and hence $\operatorname{coim}(f)$ factors through $\operatorname{coker}(i) = q \circ \operatorname{coim}(f)$. This gives us the equation $\mathrm{id} \circ \operatorname{coim}(f) = s \circ q \circ \operatorname{coim}(f)$ and since $\operatorname{coim}(f)$ is epic, we obtain $\mathrm{id} = s \circ q$ and hence q is monic.

$$\begin{tikzcd}
 & I \arrow[dl, "l"'] \arrow[d, "i"] & \\
K \arrow[r, "\ker(f)"'] & X \arrow[r, "f"] \arrow[d, "\operatorname{coim}(f)"'] & Y \\
X \arrow[r, "x"'] & U \arrow[r, "q"'] \arrow[ur, "h"'] & Q \arrow[u, "r"']
\end{tikzcd}$$

To get that g is an epi, we dualise:

$$\begin{tikzcd}
K & X \arrow[l, "\operatorname{coker}(f^{\mathrm{op}})"'] & Y \arrow[l, "f^{\mathrm{op}}"'] \arrow[d, "\operatorname{coim}(f^{\mathrm{op}})"] & C \arrow[l, "\ker(f^{\mathrm{op}})"'] \\
 & U \arrow[u, "\operatorname{im}(f^{\mathrm{op}})"] & V \arrow[l, "(\bar{f})^{\mathrm{op}}"] &
\end{tikzcd}$$

By uniqueness of $\overline{f^{\mathrm{op}}}$, it follows that $(\bar{f})^{\mathrm{op}} = \overline{f^{\mathrm{op}}}$. Then, we can construct $\bar{h}\colon V \to X$ as $h = \operatorname{im}(f^{\mathrm{op}}) \circ \overline{f^{\mathrm{op}}}$. By uniqueness of g, it follows that $g = (\bar{h})^{\mathrm{op}}$. Then $\bar{h}$ is monic by the earlier part of the proof and so g is epic. $\square$

Remark 3.11. We can obtain from Proposition 3.10 that U and V are isomorphic, and so U is both a coimage and image for f. Then $\bar{f} = \mathrm{id}_U$ and we obtain a factorisation of $f = \mathrm{im}(f) \circ \mathrm{coim}(f)$. This is called the "image factorisation" of f.

Having seen the image factorisation, we are now well positioned to prove one direction of our equivalence.

Proposition 3.12. *Every Tōhoku category is an Abelian category.*

Proof. Axioms 1 and 2 of a Usual Abelian category are capturing by additivity of a Tōhoku category while axiom 3 is simply also an axiom of a Tōhoku category. The only non-trivial part is to show that every monomorphism is a kernel and every epimorphism is a cokernel.

Let f be monic. We can factorise f as $f = \mathrm{coim}(f) \circ \mathrm{im}(f)$. $\mathrm{coim}(f) = \mathrm{coker}(\ker(f))$. Recall that, as f is monic, $\ker(f) = 0$ and hence $\mathrm{coker}(\ker(f)) = \mathrm{id}$. Hence $f = \mathrm{im}(f) = \ker(\mathrm{coker}(f))$. So f is a kernel, and in fact is the kernel of its own cokernel.

We can dualise to obtain the result for epimorphisms. This shows that all the Abelian category axioms hold in a Tōhoku category. □

3.3 Additivity of Abelian Categories

What remains is to show additivity of an Abelian category. We shall define the group structure on hom-collections using the following morphism.

Proposition 3.13. *Let A be an object in $\boldsymbol{C}$ and $\Delta \colon A \to A \times A$ be its diagonal map. Let $q = \mathrm{coker}(\Delta) \colon A \times A \to Q$ and $r = q \circ (\mathrm{id}_A, 0)$. Then r is an isomorphism.*

Remark 3.14. Suppose we have morphisms $f, g \colon B \to A$. Then, by the universal property of products, we obtain $(f, g) \colon B \to A \times A$. However, we can now compose with $r^{-1} \circ q$ to obtain $r^{-1} \circ q \circ (f, g) \colon B \to A$. Hence, we have a way of combining morphisms.

We introduce some notation for this binary operation on morphisms. The notation has been chosen to be indicative of additivity and indeed $r^{-1} \circ q \circ (f, g)$ will be the subtraction of our group.

Definition 3.15. Define $\sigma_A = r^{-1} \circ q$ and $f - g = \sigma_A \circ (f, g)$. We write $(-b)$ for $(0 - b)$ and $f + g$ for $f - (-g)$.

The next step is a technical lemma that provides a partial commutativity for our operation. We first state a diagrammatic form and then apply it several times to obtain a more useful form in terms of subtraction. It will turn out that, along with existence of a right identity and right inverse for substraction, this will be sufficient for subtraction to define an Abelian group.

Lemma 3.16. *Let $f\colon B \to A$ be a morphism. We denote by $f \times f$ the morphism $(f \circ p_1, f \circ p_2)\colon B \times B \to A \times A$. Then $f \circ \sigma_B = \sigma_A \circ (f \times f)$.*

Lemma 3.17. *Let $a, b, c, d\colon C \to A$ be morphisms. Then $(a - b) - (c - d) = (a - c) - (b - d)$.*

Proof. First, we must show that we can meaningfully extend our subtraction up to products. Specifically, we wish to show that $(a, b) - (c, d) = (a - c, b - d)$. Notice that this is half way to the right hand side of our claimed equation.

We will use Lemma 3.16, setting $B = A \times A$ and $f = p_1\colon A \times A \to A$. We consider the map $((a, b), (c, d))\colon C \to (A \times A) \times (A \times A)$. Then $p_1 \circ \sigma_{A\times A} \circ ((a, b), (c, d)) = p_1 \circ ((a, b) - (c, d))$. On the other side, $\sigma_A \circ (p_1 \times p_1) = \sigma_A \circ (a, c) = a - c$.

We now repeat this with $f = p_2$ to obtain the components of $\sigma_{A\times A} \circ ((a, b), (c, d))$ as $a - c$ and $b - d$, so $(a, b) - (c, d) = (a - c, b - d)$.

To obtain our desired equation, observe that we wish to apply σ_A both on the right and to both arguments on the left. To do this coherently, we will once again use Lemma 3.16, keeping $B = A \times A$ but this time setting $f = \sigma_A$.

Applying σ_A to both terms on the right hand side looks like $\sigma_A \circ \sigma_{A\times A} \circ ((a, b), (c, d))$. Above, we showed that $\sigma_{A\times A} \circ ((a, b), (c, d)) = (a - c, b - d)$, so we get $\sigma_A \circ \sigma_{A\times A} \circ ((a, b), (c, d)) = (a - c) - (b - d)$.

The lemma then tells us that $\sigma_A \circ \sigma_{A\times A} = \sigma_A \circ (\sigma_A \times \sigma_A)$. This side is simpler: $(\sigma_A \times \sigma_A) \circ ((a, b), (c, d)) = (a - b, c - d)$ so the left hand side is $(a - b) - (c - d)$.

The equality from the lemma hence gives us $(a - b) - (c - d) = (a - c) - (b - d)$ as required. $\square$

Proposition 3.18. ***C*** *is a pre-additive category.*

Proof. We fix objects A and B and establish that $+$ gives a group structure on $\mathrm{Hom}(B, A)$. We will use a, b and c to refer to arbitrary morphisms in this hom-set.

We quickly establish existence of right inverses via observing that $q \circ \Delta = 0$ as q is a the cokernel and hence that $a - a = \sigma_A \circ \Delta \circ a = 0$.

Then for existence of a right identity, recall that $\sigma_A = r^{-1} \circ q$, while $r = q \circ (\mathrm{id}_A, 0)$. Hence $a - 0 = \sigma_A \circ (a, 0) = r^{-1} \circ q \circ (\mathrm{id}_A, 0) \circ a = r^{-1} \circ r \circ a = a$.

We next show that taking inverses is involutive. That is, $-(-b) = b$:

$$\begin{aligned}
-(-b) &= 0 - (0 - b) \\
&= (b - b) - (0 - b) \text{ (by existence of inverses)} \\
&= (b - 0) - (b - b) \text{ (this is by Lemma 3.17)} \\
&= b - 0 \\
&= b
\end{aligned}$$

From these, we see that $b + (-a) = b - (-(-a)) = b - a$. We can now happily move between the additive and subtractive notation.

Next, we show commutativity of addition. Most of the steps will be applications of Lemma 3.17.

$$\begin{aligned}
a + b &= (0 - (0 - a)) - (0 - b) \\
&= (0 - 0) - ((0 - a) - b) \\
&= (0 - 0) - ((0 - a) - (b - 0)) \\
&= (0 - 0) - ((0 - b) - (a - 0)) \\
&= (0 - (0 - b)) - (0 - a) \\
&= b + (-(-a)) \\
&= b + a
\end{aligned}$$

A consequence of commutativity is that our right inverses and identity are two-sided. Hence all that remains is to show associativity. Again, we make heavy use of Lemma 3.17.

$$\begin{aligned}
(a + b) + d &= (a - (0 - b)) - (0 - d) \\
&= (a - 0) - ((0 - b) - d) \\
&= a - ((0 - b) - (0 - (0 - d))) \text{ (since } d = -(-d)) \\
&= a - ((0 - 0) - (b - (0 - d))) \\
&= a - (0 - (b + d)) \\
&= a + (b + d)
\end{aligned}$$

Thus we have shown that $\mathrm{Hom}(B, A)$ is an abelian group. To finish the proof of pre-additivity, we need to show that composition is bilinear. So, let $x, y \colon C \to B$. We wish to show that $(a-b)\circ x = (a\circ x)-(b\circ x)$ and that $a\circ(x-y) = (a\circ x)-(a\circ y)$.

Now, to see the first, we merely need the associativity of composition: $(a+b)\circ x = \sigma_A \circ (a, b) \circ x = \sigma_A \circ (a \circ x, b \circ x) = (a \circ x) - (b \circ x)$.

For the second, we will need to use Lemma 3.16 to move σ to the right place. In particular, we let $f = a$ and observe that $a \circ \sigma_B \circ (x, y) = \sigma_A \circ (a \times a) \circ (x, y) = (a \circ x) - (a \circ y)$. □

By definition of an Abelian category, we know that $\boldsymbol{C}$ has a zero object and finite products. Hence, $\boldsymbol{C}$ being preadditive is enough to show that $\boldsymbol{C}$ is additive. By definition, $\boldsymbol{C}$ also has all kernels and cokernels. We have proven that the isomorphism theorem holds in $\boldsymbol{C}$. Thus, all the axioms of a Tōhoku category hold for $\boldsymbol{C}$. Hence, every Abelian category is Tōhoku. We also proved earlier in this section that every Tōhoku category is Abelian. Hence, from this discussion we get the following theorem.

Theorem 3.19. *A category is Abelian if and only if it is Tōhoku.*

References

[Bor94] Francis Borceux. *Handbook of Categorical Algebra*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1994 (cit. on pp. 3, 6).

[GBB08] Alexander Grothendieck, Marcia Barr, and Michael Barr. *Some aspects of homological algebra*. Distributed online. 2008. URL: `https://www.math.mcgill.ca/barr/papers/gk.pdf` (cit. on p. 2).

[Gro57] Alexander Grothendieck. "Sur quelques points d'algèbre homologique, I". In: *Tohoku Mathematical Journal* 9.2 (1957), pp. 119 –221. DOI: `10.2748/tmj/1178244839`. URL: `https://doi.org/10.2748/tmj/1178244839` (cit. on pp. 3, 6).

[Mac71] Saunders MacLane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics, Vol. 5. New York: Springer-Verlag, 1971, pp. ix+262 (cit. on p. 5).

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994 (cit. on p. 2).