An Equivalence of Functorial and Model-Theoretic perspectives on Modules

`MATH61000, Main Project`

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Introduction

The Model Theory of Modules arises as a generalisation of work of Szmielew ([Szm55]) and later Eklof & Fisher ([EF72]) on the Model Theory of Abelian Groups, in which they provided a classification of Abelian groups up to elementary equivalence and proved that the Theory of Abelian groups is decidable. An important ingredient in this classification is the notion of pure-injective group, equivalently algebraically compact groups, which satisfy a condition on existence of solutions to linear equations: an infinite system of linear equations over the group has a solution if and only if every finite subsystem has a solution.

Pure-injectivity lifts directly to modules over a general ring, simply by moving from coefficients in $\mathbb{Z}$ to an arbitrary ring R. Model-theoretically, finite systems of linear equations can be captured as the quantifier free positive primitive formulae in the language of modules over R. A result of Baur in 1976 showed that, up to certain well-controlled sentences, all the meaningful formulae are boolean combinations of positive primitive (pp-) formulae. The Model Theory proceeds by treating these as our central logical objects.

In his landmark 1984 paper on the Model Theory of Modules ([Zie84]), Ziegler introduced what is now called the Ziegler spectrum, a topological space associated with a category of modules over a ring which captures much of their model-theoretic content. This space has for an underlying set the isomorphism classes of pure-injective modules over the ring and closed sets given by the vanishing of sets of 'additive imaginaries', properly called pp-pairs. A pp-pair consists of an inclusion of one pp-formulae inside another, providing a quotient of corresponding definable groups in each module. By vanishing, we mean that this quotient is zero in every module in the closed set.

The Ziegler spectrum is a central tool in the Model Theory of Modules and a large number of results have been derived from studying it topologically. In particular, the Cantor-Bendixson rank of the spectrum provides a useful, if difficult to compute, Morita-invariant for rings. Further, decidability for the category of modules over a recursively axiomatised ring can be obtained by providing an effective description of the topological space.

In the early 90s, due to work of Herzog, attention turned to an alternative formulation of the Model Theory of Modules which used methods from Abelian Category Theory. In this setting, rings are understood as preadditive categories (meaning categories whose Hom-sets have an Abelian group structure) and modules are understood as Additive functors from preadditive categories into **Ab**, the Category of Abelian groups. Under this interpretation, the Ziegler spectrum is understood in terms of localisations of module categories relative to certain nice torsion theories - an Abelian Category Theoretic tool that arises as a generalisation of the class of torsion groups in **Ab**.

There is a canonical category of pp-pairs over a fixed ring R, written $\mathbb{L}_R^{\mathrm{eq}+}$, whose objects are logical equivalence classes of pairs and whose morphisms are equivalence classes of definable functions. The foundation of the functorial interpretation of the Model Theory of Modules is best seen as an equivalence between $\mathbb{L}_R^{\mathrm{eq}+}$ and the category $[\mathbf{mod}\text{-}R, \mathbf{Ab}]^{\mathrm{fp}}$ of finitely presented functors on the category of finitely presented right modules of R.

In this project, we provide an exposition of this equivalence. Along the way, we set the scene for the modern approach for the Model Theory of Modules and formulate many tools that will be necessary for the study of Definable Categories and the Ziegler spectrum, which will make up the content of the dissertation.

We assume little prior knowledge of the reader for this project, only that they have a comfortable knowledge of group, ring and module that might be covered in a first undergraduate algebra course. We develop the formal theory of categories, models and modules. This makes up the first half of the project, which are made up of Sections 1 through 3.

In Section 1, we cover the requisite Ordinary Category Theory required. We say Ordinary in contrast to the Additive Category Theory encountered in Section 3, which is significantly more Algebraic in nature. We first cover the basic notions of Category Theory and the primary structural tools for studing categories, namely limits and colimits. We then progress to the study of functor categories and presheaves, including the foundational Yoneda Lemma which will find vital usage throughout all of the project. At the end of Section 1, we also develop the basics of the theory of locally presentable categories and collect some essential facts about finite presentability and directed colimits.

In Section 2, we cover the requisite logic for the project. In contrast to most introductions to Model Theory, we work entirely multi-sortedly as this is necessary for understanding modules over preadditive categories in their full generality. We introduce multi-sorted languages, structures and theories, furnished with several examples, including the language and axioms for the usual notion of module first encountered in algebra.

In Section 3, we round out the preliminaries with the necessary tools from Abelian Category Theory. This includes definitions of preadditive, additive and Abelian categories as well as the central co(limits) that we make use of, namely, zero objects, kernels and cokernels. Several facts about the behaviour of co(kernels) in Abelian categories are established and we prove the existence of biproducts. This section also introduces the Homological notions of exact sequences, exact functors and projective modules, all of which find usage as technical tools for proofs.

For the second half of the project, made up of Sections 4 to 6, we move onto the study of categories of modules proper. We first introduce our general notion of modules in their own right and collect many useful facts about their categories. This makes up Section 4. We then introduce the relevant positive-primitive logic on its own in Section 5 and synthesise the two in Section 6, ultimately reaching the equivalence of $\mathbb{L}_R^{\mathrm{eq}+}$ and $[\mathbf{mod}\text{-}R, \mathbf{Ab}]^{\mathrm{fp}}$ discussed earlier.

The primary content of Section 4 is a characterisation of the finitely presented modules over a preadditive category. We have two notions of finitely presented module, the first from locally finitely presentable categories and the second from standard algebra. This first definition is discussed initially in Section 1 and is in terms of the preservation of directed colimits. The second notion is more familiar, in which finitely presented modules are those finitely generated modules that have only finitely many relations on their generators. Our characterisation shows that these two notions are equivalent. In order to do this, we develop several tools for talking about modules, including the idea of generalised elements. We also prove that the category of modules is locally finitely presentable. An application of this characterisation is to show that, up to isomorphism, there is only a set of finitely presented modules over a small ring.

For Section 5, we introduce positive-primitive formulae and their corresponding definable subgroups in a fixed module. We prove a normal form theorem for pp-formulae and present, without proof, the pp-elimination of quantifiers result of Baur which underlies their prevalence throughout the subject. We then develop the fundamental tool of free realisations, which associate with each pp-formula a class of finitely presented modules, each with a distinguished tuple. We see that this association in fact goes both ways. We then move on to defining the category $\mathbb{L}_R^{\mathrm{eq}+}$ and associate to every pp-pair its own additive functor on the category of modules.

Finally, in Section 6, we bring together the work from Sections 4 and 5 to prove the equivalence. We prove several structural results for categories of modules in the case where the underlying ring is itself a small category of modules.

This builds to a proof that the category $[R\text{-}\mathbf{mod}, \mathbf{Ab}]$ is locally coherent, and hence that $[R\text{-}\mathbf{mod}, \mathbf{Ab}]^{\mathrm{fp}}$ is Abelian. We then show that each of the functors associated to pp-pairs in Section 5 is in fact finitely presented and conversely associate to each finitely presented functor a pp-pair. The equivalence is then a simple matter of showing that this correspondence appropriately preserves morphisms. We finish the project by discussing how to view modules over R as modules over $\mathbb{L}_R^{\mathrm{eq}+}$, which is an Abelian category by the equivalence. In particular, we see that the modules over R are precisely the exact functors on $\mathbb{L}_R^{\mathrm{eq}+}$.

The historical content of this introduction is drawn from an article of Herzog [Her17] and the introduction to [Pre09].

1 Category Theory

We recall here many of the basic definitions and results of Category Theory which will be essential to the rest of the project. We draw primarily from [Lei16]. Where results have been cited as Exercises, we have given original proofs. In the last section, we move onto the theory of Accessible categories and draw from [AR94].

1.1 Basic Definitions

We start at the very beginning.

Definition 1.1. A **category C** consists of:

- A collection of objects $\mathrm{Ob}(\mathbf{C})$.
- For each ordered pair of objects (X, Y) in $\mathrm{Ob}(\mathbf{C})$, there is a collection of arrows (also called morphisms) $\mathrm{Hom}(X, Y)$. We write $f\colon X \to Y$ for an arrow in $\mathrm{Hom}(X, Y)$. X is called the domain of f and Y is the codomain, which we will write as $\mathrm{dom}(f) = X$ and $\mathrm{cod}(f) = Y$.
- For each ordered triple of objects (X, Y, Z), there is a composition function $\circ_{X,Y,Z}\colon \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$. Composition is written as an infix and the subscript is ommitted. When our meaning is clear, we will simply juxtapose morphisms for composition. That is, gf to mean $g \circ f$.
- For each object X there is a morphism id_X in $\mathrm{Hom}(X, X)$ such that for all Y and all f in $\mathrm{Hom}(X, Y)$ and all g in $\mathrm{Hom}(Y, X)$, we have that $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$.

Remark 1.2. A category is called **locally small** if for every pair of objects, their collection of morphisms is in fact a set (and not a proper class). A category is called **small** if it is locally small and, in addition, the collection of objects is a set. A category that is not small is called **large**. Most categories we work with will be large but locally small.

Example 1.3. There is a category **Set** whose objects are sets and whose morphisms are functions. Composition is function composition and identity morphisms are simply identity functions.

The definition of a category really is dependent on what the morphisms of the category are- there are often multiple possible ”natural” kinds of morphism for a given collection of objects that make it into a category. The following example, which has the same objects as **Set** but different morphisms makes this clear. It is also an example of a category whose morphisms are *not* functions.

Example 1.4. The category **Rel** has sets as objects, but for each pair of sets X and Y, the collection of morphisms $\mathrm{Hom}(X, Y)$ is given by the set of all relations between X and Y. Given sets X, Y, Z and relations $R \subset X \times Y$ and $S \subset Y \times Z$, we define $S \circ R \subset X \times Z$ as $\{(x, z) \in X \times Z \mid \text{there is a } y \in Y \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}$. The identity morphism is the diagonal relation $\{(x, y) \in X \times X \mid x = y\}$.

Example 1.5. There is a category **Grp** whose objects are groups and whose morphisms are group homomorphisms. Proving that this is a category is equivalent to proving that the function composition of group homomorphisms is still a group homomorphism, which is an early theorem in any course on group theory.

Example 1.6. We can restrict our attention to solely the Abelian groups to obtain the category **Ab**. The morphisms are still group homorphisms, as any group homorphism between Abelian groups "preserves commutativity".

An important example of a category whose morphisms are not functions is the category induced by a partially ordered set. Often we will identify a partially ordered set with its corresponding category.

Example 1.7. Let $(X, \leq)$ be a partially ordered set. We define a category, which we will abuse notation and also refer to as X, by taking the collection of objects to be the set of elements of X. Then, for any pair of objects x and y, if $x \leq y$ we set $\mathrm{Hom}(x, y) = \{\emptyset\}$ (or indeed, any arbitrary choice of one element set). If $x \nleq y$ then we set $\mathrm{Hom}(x, y) = \emptyset$. Composition is then defined uniquely, since there is a unique function from every set into $\{\emptyset\}$ and a unique function to every set from $\emptyset$.

Example 1.8. Let $\mathbf{C}$ be a category. We define a new category $\mathbf{C}^{\mathrm{op}}$ called the **opposite category** as follows: the collection of objects of $\mathbf{C}^{\mathrm{op}}$ is the same as that of $\mathbf{C}$. For each pair of objects X, Y, the collection of morphisms $\mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(X, Y)$ in $\mathbf{C}^{\mathrm{op}}$ is equal to $\mathrm{Hom}_{\mathbf{C}}(Y, X)$- that is, we reverse the order of all morphisms. For $f \in \mathrm{Hom}_{\mathbf{C}}(Y, X)$ we will often writen $f^{\mathrm{op}} \in \mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(X, Y)$ to make it clear which category we are working with. Composition is then defined as $f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$.

As with all mathematical objects, we want to have structure-preserving maps between categories. For categories, these are called functors, and the structure they preserve is that of morphism composition.

Definition 1.9. A **functor** $F\colon \mathbf{C} \to \mathbf{D}$ for $\mathbf{C}$ and $\mathbf{D}$ categories, consists of

- A map $F\colon \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$.
- For each pair of objects X, Y in $\mathrm{Ob}(\mathbf{C})$, a map $F\colon \mathrm{Hom}(X, Y) \to \mathrm{Hom}(FX, FY)$.

Notice that we abuse notation and write F for all the components of the functor, but this will always be clear from usage. We will usually suppress brackets for functor application. Additionally, the functor must satisfy two conditions:

- For each X in $\mathrm{Ob}(\mathbf{C})$, we have that $F(\mathrm{id}_X) = \mathrm{id}_{FX}$.
- For morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, we have that $F(g \circ f) = Fg \circ Ff$.

Remark 1.10. Given two functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{E}$, we can compose them to form a functor $GF\colon \mathbf{C} \to \mathbf{E}$ such that for each object A in $\mathbf{C}$, $GF(A) = G(F(A))$ and for each morphism $f\colon A \to B$, $GF(f) = G(F(f))$.

Example 1.11. Given a locally small category $\mathbf{C}$, for each pair of objects X and Y, $\mathrm{Hom}(X, Y)$ is a set. If we fix an object X we can obtain the **covariant Hom-functor** at X, written $\mathrm{Hom}(X, -)\colon \mathbf{C} \to \mathbf{Set}$. For each object Y, $\mathrm{Hom}(X, -)$ sends Y to $\mathrm{Hom}(X, Y)$. For each morphism $f\colon Y \to Z$, $\mathrm{Hom}(X, -)$ sends f to the morphism $\mathrm{Hom}(X, f)\colon \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$ which acts as composition with f. This means that for each $g \in \mathrm{Hom}(X, Y)$, $\mathrm{Hom}(X, f)(g) = f \circ g \in \mathrm{Hom}(X, Z)$.

As with categories, there is a notion of a structure-preserving map between functors. We will often speak of "natural" constructions, which usually indicates that there is a natural transformation at work.

Definition 1.12. Given functors $F, G\colon \mathbf{C} \to \mathbf{D}$, a **natural transformation** $\eta\colon F \to G$ consists of, for each object A in $\mathbf{C}$, a morphism $\eta_A\colon FA \to GA$. Additionally, for each morphism $f\colon A \to B$ in $\mathbf{C}$, η must satisfy the condition $Gf \circ \eta_A = \eta_B \circ Ff$. This is often written as the following diagram, called the **naturality diagram** of the transformation, commuting.

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\eta_A"'] & FB \arrow[d, "\eta_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}$$

Remark 1.13. Given functors $F, G, H\colon \mathbf{C} \to \mathbf{D}$ and natural transformations $\eta\colon F \to G$ and $\varepsilon\colon G \to H$, the **vertical composition** of η and ε is given by

$(\varepsilon \circ \eta)_A = \varepsilon_A \circ \eta_A$. This is natural by pasting the naturality diagrams of ε and η.

The term vertical composition of course implies the existence of a horizontal composition, which is distinct and not relevant to us here. We nonetheless specify verticality as this is standard terminology.

We can use natural transformations to define categories of functors, which find a great deal of use all throughout this project.

Definition 1.14. Given categories $\mathbf{C}$ and $\mathbf{D}$, we can define the **functor category** $[C, D]$ as the category whose objects are functors $\mathbf{C} \to \mathbf{D}$ and whose morphisms are natural transformations composed vertically.

In addition to the three layers of categories, functors and natural transformations, there are various special classes of morphisms within a given category that come up very often. The first are monomorphisms, which are a categorical generalisation of injective homomorphisms in algebra.

Definition 1.15. A morphism $f\colon X \to Y$ is called a **monomorphism** if for every pair of morphisms $g, h\colon W \to X$ with $f \circ g = f \circ h$, we have that $g = h$. Thus, monomorphisms are **left-cancellative** morphisms. We will often use "monic" as an adjectival form and "mono" as a shortening of monomorphism.

There is then also a categorical generalisation of surjective homomorphims, called epimorphisms.

Definition 1.16. Dually, f is a called an **epimorphism** if for every pair of morphisms $g, h\colon Y \to Z$ such that $g \circ f = h \circ f$, we have that $g = h$. Epimorphisms are hence **right-cancellative**. Their adjectival form is "epic" and their shortening is "epi".

We emphasise the difference between these concepts. In the case of monomorphisms, we are looking at pairs of morphisms going into the domain of f, whereas for epimorphisms we are looking at pairs coming out of the target of f.

The categorical notion of an isomorphism is in fact just the usual notion, that of a map with both a left and right inverse.

Definition 1.17. A morphism $f\colon A \to B$ in $\mathbf{C}$ is called an **isomorphism** if there is a morphism $g\colon B \to A$ such that $gf = \mathrm{id}_A$ and $fg = \mathrm{id}_B$.

Remark 1.18. If there is an isomorphism between X and Y, we write $X \cong Y$.

In categories such as **Set** and **Ab**, the isomorphisms are exactly the injective and surjective homomorphisms. This is not always the case, for instance in the category of topological spaces, **Top**, a bijective continuous map is not necessarily a homeomorphism. Categories in which a morphism that is both monic and epic is an isomorphism are called **balanced** categories. The converse, however, does hold in every category.

Proposition 1.19. *Suppose $f\colon X \to Y$ is an isomorphism. Then it is monic and epic.*

Proof. Let $g, h\colon W \to X$ be morphisms such that $f \circ g = f \circ h$. Let f^{-1} a two-sided inverse for f. Then $f^{-1} \circ f \circ g = f^{-1} \circ f \circ h$ implying that $g = h$. Thus, f is monic. The epic case is analogous. □

In the particular case of natural transformations, isomorphisms in a functor category are called **natural isomorphisms**. A useful fact is that a natural transformation being an isomorphism can be determined componentwise.

Proposition 1.20 ([Lei16] Lemma 1.3.11)**.** *Let $F, G\colon \mathbf{C} \to \mathbf{D}$ be functors and $\alpha\colon F \to G$ be a natural transformation. Then, α is an isomorphism if and only if α_X is an isomorphism for every X in $\mathbf{C}$.*

Proof. First, suppose α is an isomorphism and hence has a two-sided inverse α^{-1}. Then, by definition of composition, for all X in $\mathbf{C}$, $\alpha_X^{-1} \circ \alpha_X = (\mathrm{id}_F)_X = \mathrm{id}_{FX}$ and $\alpha_X \circ \alpha_X^{-1} = \mathrm{id}_{FX}$. Thus, for every X, α_X has a two-sided inverse and so is an isomorphism.

For the other direction, suppose that α_X is an isomorphism for every X, with each inverse being some β_X. We will show that the morphisms β_X are components of a natural transformation. In particular, let $f\colon X \to Y$ be a morphism in $\mathbf{C}$. Then, the upper square and outer rectangle of the following diagram commute:

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] & FY \arrow[d, "\alpha_Y"] \\
GX \arrow[r, "Gf"] \arrow[d, "\beta_X"'] & GY \arrow[d, "\beta_Y"] \\
FX \arrow[r, "Ff"'] & FY
\end{tikzcd}$$

Then, $\beta_Y \circ Gf \circ \alpha_X = \beta_Y \circ \alpha_Y \circ Ff = Ff \circ \beta_X \circ \alpha_X$. Now, α_X is an isomorphism and hence epimorphic, so we obtain $\beta_Y \circ Gf = Ff \circ \beta_X$, which is the commutativity of the lower square. This is the naturality diagram of β, so β is natural. It follows that β is a two-sided inverse for α and hence α is an isomorphism. □

We obtain an analogous partial result for monomorphisms and epimorphisms, namely that if a natural transformation is monic componentwise, then it is monic. The converse requires an amount of structure in $\mathbf{D}$ that is not necessarily a given.

Proposition 1.21 ([Lei16] Exercise 6.2.20)**.** *Let $\alpha\colon F \to G$ be a natural transformation such that for every X in $\mathbf{C}$, α_X is a monomorphism (resp. epimorphism). Then α is a monomorphism (resp. epimorphism).*

Proof. Let $\eta, \varepsilon\colon E \to F$ be such that $\alpha \circ \eta = \alpha \circ \varepsilon$. Then, for each X, we have that $\alpha_X \circ \eta_X = \alpha_X \circ \varepsilon_X$. Since α_X is monic, we obtain $\eta_X = \varepsilon_X$. This gives us $\eta = \varepsilon$ since natural transformations are determined by their components.

The epimorphism case is completely analogous. □

While it is possible to define isomorphims of categories as being a functor with a two-sided inverse, it turns out that this is far too strong a condition. Most properties of categories that we are interested in are in fact preserved under a condition called equivalence, which we can think as isomorphism "up to" natural isomorphism.

Definition 1.22. We say that two categories $\mathbf{C}$ and $\mathbf{D}$ are **equivalent** if there are functors $F\colon \mathbf{C} \to \mathbf{D}$, $G\colon \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $\eta\colon FG \to \mathrm{id}_{\mathbf{D}}$ and $\varepsilon\colon \mathrm{id}_{\mathbf{C}} \to GF$.

Remark 1.23*.* We write equivalence as $\mathbf{C} \cong \mathbf{D}$ since we will almost never need isomorphisms of categories.

Equivalences can also be characterised in terms of a single functor, somewhat analagous to the case for isomorphisms in balanced categories. The properties we need from a functor are the following.

Definition 1.24. Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor. Then we say that F is:

- **full** if for every pair of objects A, B in $\mathbf{C}$ the map $F\colon \mathrm{Hom}(A, B) \to \mathrm{Hom}(FA, FB)$ is surjective.
- **faithful** if for every pair of objects A, B in $\mathbf{C}$ the map $F\colon \mathrm{Hom}(A, B) \to \mathrm{Hom}(FA, FB)$ is injective.

- **essentially surjective** if for every object X in $\mathbf{D}$, there is some object A in $\mathbf{C}$ such that $X \cong FA$.

We say that a functor $F\colon \mathbf{C} \to \mathbf{D}$ is **part of an equivalence** if there is a functor $G\colon \mathbf{D} \to \mathbf{C}$ such that $FG \cong \mathrm{id}_{\mathbf{D}}$ and $GF \cong \mathrm{id}_{\mathbf{C}}$. We now prove the characterisation of equivalences.

Proposition 1.25 ([Lei16] Exercise 1.3.32)**.** *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor. Then F is part of an equivalence if and only if it is full, faithful and essentially surjective.*

Proof. For the forwards direction, let G be the other part of the equivalence with natural isomorphisms $\eta\colon \mathrm{id}_{\mathbf{C}} \to GF$ and $\varepsilon\colon FG \to \mathrm{id}_{\mathbf{D}}$.

To see that F is essentially surjective, fix X in $\mathbf{D}$. Then, ε_X^{-1} is an isomorphism $X \to FGX$.

To see that F is faithful, let $f, g\colon A \to B$ in $\mathbf{C}$ such that $Ff = Fg$. Consider the naturality diagram for η:

$$\begin{tikzcd} GFA \arrow[r, "GFf"] & GFB \\ A \arrow[u, "\eta_A"] \arrow[r, "f"'] & B \arrow[u, "\eta_B"'] \end{tikzcd}$$

Since $Ff = Fg$, we see that $GFf = GFg$. We can use naturality of η to write $f = \eta_B^{-1} \circ GFf \circ \eta_A$. Similarly for g. It follows that $f = \eta_B^{-1} \circ GFf \circ \eta_A = \eta_B^{-1} GFg \circ \eta_A = g$. Hence, F acts injectively on $\mathrm{Hom}(A, B)$ and so is faithful. We note that this argument works equally well for G, so we see that G is also faithful.

To show that F is full, let $u\colon FA \to FB$ be a morphism in $\mathbf{D}$. Now, for every $f\colon A \to B$, we know that $GFf = \eta_B \circ f \circ \eta_A^{-1}$. Consider the morphism $v = \eta_B^{-1} \circ Gu \circ \eta_A$. Then, $GFv = \eta_B \circ \eta_B^{-1} \circ Gu \circ \eta_A \circ \eta_A^{-1} = Gu$. But G is faithful, so $GFv = Gu$ implies that $Fv = u$. Hence, F is full.

Now, for the reverse direction, we must define a functor G to be the other part of the equivalence. First, we define G for objects. In particular, we know that F is essentially surjective and hence, for every X in $\mathbf{D}$, there is some A in $\mathbf{C}$ with $X \cong FA$. We set $GX = A$. For each A, we will in particular set $GFA = A$. Note that this is not canonical and there could be many possible choices for G.

Now, for each X, let $\varepsilon_X\colon FGX \to X$ be the isomorphism from before. Note that this does not yet lift to a natural transformation, indeed we have not made G

a functor. We will define G in order to make this family natural. In particular, for $u\colon X \to Y$, there is a morphism $u' = \varepsilon_Y^{-1} \circ u \circ \varepsilon_X$ acting $FGX \to FGY$. Since F is full, there must be some morphism v with $Fv = u'$. We set $Gu = v$. It follows that $FGu = u' = \varepsilon_Y^{-1} \circ u \circ \varepsilon_X$ which is exactly naturality of ε.

To see that G is functorial, let $u\colon X \to Y$ and $t\colon Y \to Z$. Then, $F(G(t \circ u)) = \varepsilon_Z^{-1} \circ t \circ u \circ \varepsilon_X = \varepsilon_Z^{-1} \circ t \circ \varepsilon_Y \circ \varepsilon_Y^{-1} \circ u \circ \varepsilon_X = FGt \circ FGu$. Since F is faithful, it follows that $G(t \circ u) = Gu \circ Gt$, which is functoriality of G.

By definition, we have picked $GFA = A$ and hence $GF = \mathrm{id}_{\mathbf{C}}$. We thus have natural isomorphisms $GF \cong \mathrm{id}_{\mathbf{C}}$ and $FG \cong \mathrm{id}_{\mathbf{D}}$ which shows that F is part of an equivalence. $\square$

In addition to equivalences, we are also interested in anti-equivalences, also called dualities. We say that two categories $\mathbf{C}$ and $\mathbf{D}$ are **anti-equivalent** or **dual** if $\mathbf{C}^{\mathrm{op}} \cong \mathbf{D}$. This is equivalent to $\mathbf{C} \cong \mathbf{D}^{\mathrm{op}}$.

1.2 Limits

Limits and colimits are methods of producing “minimal” and “maximal” objects satisfying certain conditions in a category. Many common ways of producing new structures across mathematics arise as limits, for instance direct products and sums in algebra or unions and intersections in set theory.

Definition 1.26. Given a category $\mathbf{C}$ that we are taking the limit in, a **diagram** is a category I and a functor $J\colon I \to \mathbf{C}$. I is called the **shape** of the diagram.

Remark 1.27. There is nothing special about this functor, we call it a diagram only to make it clear that we are using it to take a limit.

We make use of the following canonical functor in the definition of limits.

Definition 1.28. For every object A in $\mathbf{C}$ and every catgeory I, the **constant functor** is the functor $\Delta_A\colon I \to \mathbf{C}$ that sends all objects to A and all morphisms to the identity morphism on A.

Remark 1.29. Note that for two objects A, B and a morphism $f\colon A \to B$, we obtain a canonical natural transformation $\Delta f\colon \Delta_A \to \Delta_B$ whose component is f for every object in I. Similarly, for a natural transformation $\eta\colon \Delta_A \to \Delta_B$ and morphisms $u\colon i \to j$ in I, naturality gives us the following diagram.

$$\begin{tikzcd}
\Delta_A i \arrow[r, "\mathrm{id}_A"] \arrow[d, "\eta_i"'] & \Delta_A j \arrow[d, "\eta_j"] \\
\Delta_B i \arrow[r, "\mathrm{id}_B"'] & \Delta_B j
\end{tikzcd}$$

We conclude from this that $\eta_j = \mathrm{id}_B \circ \eta_i \circ \mathrm{id}_A^{-1} = \eta_i$. Thus, there is a morphism $f\colon A \to B$ with $\eta_i = f$ for every i. We hence see that there is a bijection between morphisms $A \to B$ and natural transformations $\Delta_A \to \Delta_B$.

As a result, will often abuse notation and write f for Δf since these uniquely determine each other and so there is no ambiguity.

We can now define cones over diagrams. These consist of objects and corresponding morphisms that make the diagram commute. A limit is given by a special kind of cone called a limiting cone.

Definition 1.30. Given a diagram $J\colon I \to \mathbf{C}$, a **cone** over this diagram consists of:

- An object A.
- A natural transformation $a\colon \Delta_A \to J$.

We will write cones as $(A, a_i\colon A \to Ji)_{i\in I}$, suppressing the indexing and function specification when it is clear.

Definition 1.31. A **limiting cone** is a cone (L, η) such that for every cone (A, a), there is a unique morphism $h\colon A \to L$ such that for each object i in the shape I, we have that $a_i = \eta_i \circ h$.

This unique morphism is sometimes called the **universal morphism** of the limit, and we will say that it exists via the **universal property** of the limit.

Example 1.32. A **discrete category** is a category whose only morphisms are identity morphisms. Consider a category $\mathbf{C}$ and set I to be a discrete category with two objects, which we will call 1 and 2. Then a functor $J\colon I \to \mathbf{C}$ picks out two (possibly equal) objects of $\mathbf{C}$, say A and B. A cone over J is then an object C and maps $f\colon C \to A$ and $g\colon C \to A$. The only naturality diagrams these have to satisfy are for the identity map, and since $f = \mathrm{id}_A f$, these are trivially satisfied.

A limiting cone for this diagram will be some object $A\times B$ and maps $\pi_A\colon A\times B \to A$ and $\pi_B\colon A\times B \to B$ such that there is a unqiue morphism $h\colon C \to A\times B$

with $f = \pi_A h$ and $g = \pi_B h$. We call such a cone a **product** of A and B. This is often written as the following diagram commuting.

$$\begin{tikzcd}
 & C \arrow[d, dashed, "\exists! h"'] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\
 & A \times B \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\
A & & B
\end{tikzcd}$$

Example 1.33. We let I be a category with three objects and a morphism from each of the first two objects into the third. We might draw this as $(\bullet \to \bullet \leftarrow \bullet)$. A diagram J then picks out three objects A, B, C in $\mathbf{C}$ and two maps $f\colon A \to C$ and $g\colon B \to C$. A cone over this diagram will consist of an object D and maps $d_A\colon D \to A$, $d_B\colon D \to B$ and $d_C\colon D \to C$ such that $d_C = f d_A$ and $d_C = g d_B$. Hence, $f d_A = g d_C$ and we will therefore usually suppress d_C as it is obtainable from the other morphisms. A limiting cone over this diagram is called a **pullback** and is written $A \times_C B$. The commuting diagram for this limit hence looks as follows:

$$\begin{tikzcd}
D \arrow[dr, dashed, "\exists! h"'] \arrow[drr, bend left, "d_A"] \arrow[ddr, bend right, "d_b"'] & & \\
 & A \times_C B \arrow[r, "p_A"] \arrow[d, "p_b"'] & A \arrow[d, "f"] \\
 & B \arrow[r, "g"'] & C
\end{tikzcd}$$

Example 1.34. Let I be the category with two parallel arrows: $(\bullet \rightrightarrows \bullet)$. This category is commonly called the **walking parallel pair**. A limit over a diagram of shape I is called an **equalizer**. Its commuting diagram, with limit (E, e) looks as follows.

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
C \arrow[u, dashed, "\exists! h"] \arrow[ur, "q"'] & &
\end{tikzcd}$$

Example 1.35. An important but irregular example of a limit is that of a **terminal object** in a category. An object 1 is called terminal if for every object X, there is a unique morphism $X \to 1$. We can exhibit this as a limit by considering the empty category, that is the category with an empty set of objects and empty set of morphisms. There is a unique functor from the empty category into any other category $\mathbf{C}$, namely the empty functor.

A cone over the empty functor consists of an object A of $\mathbf{C}$, a constant functor (which in fact must be the empty functor) and a natural transformation (which indeed must be the empty transformation). Of these, the functor and transformation are both trivial, and so the real data of a cone is an object in $\mathbf{C}$. Then, a limit cone is an object 1 such that for every other object A, there is a unique morphism $A \to 1$. We do not need to check any commuting condition on the natural transformations of the cones, since the collection of maps to make commute is empty. In this way, we can see that the terminal object is a limit over the empty diagram.

Limits are among the most important kinds of structure that a category can have. Because of this, we are very interested in how limits interact with functors.

Definition 1.36. Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor and $D\colon I \to \mathbf{C}$ a diagram with limit (L, η). We say that F **preserves** $\lim D$ if $\lim F \circ D$ exists and $(FL, F\eta)$ is a limit for $F \circ D$.

Remark 1.37. We say F **preserves limits** if F preserves $\lim D$ for every diagram D.

Remark 1.38. Regardless of whether F preserves D, $(FL, F\eta)$ will always be a cone for $F \circ D$.

The notion of a limit has a dual, that of colimits. For each type of limit described above, there is a corresponding colimit which mirrors its properties. We will write out explicit definitions and examples below, but due to this duality we shall leave out much of the detail since the cases are very similar.

Definition 1.39. A **cocone** over a diagram $J\colon I \to \mathbf{C}$ consists of

- An object A.
- A natural transformation $a\colon J \to \Delta_A$.

Definition 1.40. A **colimit cocone** is a cocone (L, η) such that for every cocone (A, a), there is a unique morphism $h\colon L \to A$ such that for each object i in the shape I, we have that $a_i = h \circ \eta_i$.

Notice that all we have changed from the limit definition is reversing the directions of the natural transformations and unique morphisms. This reflects a general fact about colimits and limits: for a given diagram, if it has a limit (resp.

colimit) (L, η) in $\mathbf{C}$, then (L, η^{op}) is a colimit (resp. limit) for the diagram in $\mathbf{C}^{\mathrm{op}}$.

Example 1.41. For each of the named limits given above, there is a corresponding named colimit.

- A colimit over a discrete diagram is called a **coproduct** and is written $A \sqcup B$.
- A colimit over $\bullet \to \bullet \leftarrow \bullet$ is called a **pushout**.
- A colimit over $\bullet \rightrightarrows \bullet$ is called a **coequalizer**.
- The dual notion to a terminal object is an initial object $\emptyset$, an object such that for each other object X, there is a unique morphism $\emptyset \to X$.

In general, our diagrams can be arbitrarily complicated, which can make limits hard to work with. Fortunately, in the case where our shape is a small category (where we also call the limit small), there is a nice characterisation in categories which have sufficient already existing structure. We shall prove this here, starting with some not so deep definitions that let us speak clearer.

Definition 1.42. We say that $\mathbf{C}$ is **complete** if there is a limit for every small diagram ("$\mathbf{C}$ has all small limits"). Similarly, $\mathbf{C}$ is **cocomplete** if there is a colimit for very small diagram.

Definition 1.43. A **small product** is a limit over a diagram from a small discrete category. For such a diagram $J \colon I \to \mathbf{C}$ we write the product as $\prod_{i \in I} Ji$.

We can now state the theorem.

Theorem 1.44 ([Lei16] Proposition 5.1.26)**.** *Suppose* $\mathbf{C}$ *has all small products and equalizers. Then* $\mathbf{C}$ *is complete.*

Remark 1.45. By dualising, we see that if $\mathbf{C}$ has all small coproducts and coequalizers then it is cocomplete.

Proof. Let $J \colon I \to \mathbf{C}$ be a small diagram. We form two products.

First, we can forget all non-identity arrows in I to make a discrete category $\overline{I}$ and a functor $\overline{J} \colon I \to \mathbf{C}$ that acts the same as J on objects. Then there is a product $\prod_{i \in \overline{I}} \overline{J}i$ which we will write as $\prod_{i \in I} Ji$ forgetting the bars, since this is clear.

Second, we can form a discrete category $\mathrm{Arr}(I)$ whose objects are the arrows in I. There is then a functor $\mathrm{cod}(J)\colon \mathrm{Arr}(I) \to \mathbf{C}$ which, for each $u\colon i \to j$ in I, acts as $\mathrm{cod}(J)(u) = Jj$. We write the limit over this as $\prod_{u\in I} J\mathrm{cod}(u)$. Note that this second product will have many copies of Ji for each i - one for each arrow into i. In particular, there will be a copy for the identity morphism on each i.

Now, for each u in Arr, we can write a morphism $\prod_{i\in I} Ji \to \prod_{u\in\mathrm{Arr}(I)} J\mathrm{cod}(u)$ by composing the projection morphism $\pi_{\mathrm{dom}(u)}\colon \prod_{i\in I} Ji \to J\mathrm{dom}(u)$ with the morphism $Ju\colon J\mathrm{dom}(u) \to J\mathrm{cod}(u)$. This projection morphism exists by definition of the product. Our composition is a morphism $f_u\colon \prod_{i\in I} Ji \to J\mathrm{cod}(u)$. There is such a morphism for each u. This is a cone for the functor $\mathrm{cod}(J)$ and therefore by the universal property of the product, there is a unique morphism $f\colon \prod_{\imath\in I} Ji \to \prod_{u\in\mathrm{Arr}(I)} J\mathrm{cod}(u)$ such that $\pi_{\mathrm{cod}(u)} \circ f = f_u$. Hence, the following diagram commutes for each $v \in \mathrm{Arr}(I)$.

$$\begin{tikzcd}
\prod_{i\in I} Ji \arrow[r, "f"] \arrow[d, "\pi_{\mathrm{dom}(v)}"'] & \prod_{u\in\mathrm{Arr}(I)} J\mathrm{cod}(u) \arrow[d, "\pi_v"] \\
J\mathrm{dom}(v) \arrow[r, "Jv"'] & J\mathrm{cod}(v)
\end{tikzcd}$$

We build another arrow $\prod_{i\in I} Ji \to \prod_{u\in\mathrm{Arr}(I)} J\mathrm{cod}(u)$ by considering the projection arrows $g_u = \pi_{\mathrm{cod}(u)}\colon \prod_{i\in I} Ji \to J\mathrm{cod}(u)$. This is once again a cone, and so we get a morphism $g\colon \prod_{i\in I} Ji \to \prod_{u\in\mathrm{Arr}(I)} J\mathrm{cod}(u)$ and the following triangle commutes.

$$\begin{tikzcd}
\prod_{i\in I} Ji \arrow[r, "g"] \arrow[d, "\pi_{\mathrm{cod}(v)}"'] & \prod_{u\in\mathrm{Arr}(I)} J\mathrm{cod}(u) \arrow[dl, "\pi_v"] \\
J\mathrm{cod}(v) &
\end{tikzcd}$$

We are left in the situation of having two parallel morphisms for which we can take an equalizer.

$$\begin{tikzcd}
E \arrow[r, "e"] & \prod_{i\in I} Ji \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \prod_{u\in\mathrm{Arr}(I)} J\mathrm{cod}(u)
\end{tikzcd}$$

We claim that E along with $\eta_i = \pi_i \circ e$ is a limit cone for J.

To see that E is a cone for J, let $v\colon i \to j$ be a morphism in I. Then $Jv \circ \eta_i = Jv \circ \pi_i \circ e$. Since e is an equalizer of f and g, we use that $Jv \circ \pi_i = f_v$ and $f_v \circ e = g_v \circ e$ to find that $Jv \circ \eta_i = g_v \circ e$. But $g_v = \pi_j$ so we conclude that $Jv \circ \eta_i = \eta_j$. Thus, η really is a natural transformation.

To see that this cone is limiting, suppose (D, d) is another cone. The idea is to show that D induces a cone of the equalizer, whose limit we already know. Since there is a morphism $d_i \colon D \to Ji$ for all i, it follows from the universal property that there is a morphism $p \colon D \to \prod_{i \in I} Ji$ such that $d_i = \pi_i \circ p$ for all i. That d is a natural transformation from the constant functor at D into J is precisely the statement that $Jv \circ d_i = d_j$ for all $v \colon i \to j$ in I. Therefore $Jv \circ \pi_i \circ p = \pi_j \circ p$ or, rewriting, $f_v \circ p = g_v \circ p$ for all v. Hence, p equalizes the pair of arrows f and g. The universal property of the equalizer says that there is a unique morphism $h \colon D \to E$ such that $p = e \circ h$.

We then move back to thinking about the original cone. By definition, $d_i = \pi_i \circ p$. Therefore, for all i, $d_i = \pi_i \circ e \circ h = \eta_i \circ h$. So h is a morphism $D \to E$ making their cones commute. The uniqueness of this morphism is shown by reversing the process for existence: if there is some morphism $h' \colon D \to E$ making the cones commute, then this h' makes the cones over the equalizer also commute. h is unique in this, so we must have that $h' = h$.

We conclude that (E, η) is a limit for J and since J was an arbitrary small diagram, all small limits must exist. □

Remark 1.46. This proof gives a way of decomposing limits into products and equalizers as long as the latter exist. For a fixed limit J, we don't need to have all products and equalizers to decompose J, we only need that $\prod_{i \in I} Ji$ and $\prod_{u \in \mathrm{Arr}(u)} J\mathrm{cod}(u)$ exist. Hence, we can use the decomposition in slightly weaker settings than having all small products and equalizers.

Example 1.47. In **Set**, products are given by the usual Cartesian products and equalizers of $f, g \colon X \to Y$ are subsets of X (that is, the largest subset of X on which f and g agree). Hence, every limit is a subset of a product.

Similarly, coproducts are disjoint unions and coequalizers are quotients of Y by the equivalence closure of the relation given by $y \sim z$ if there is an $x \in X$ such that $f(x) = y$ and $g(x) = z$. Thus, colimits in **Set** are quotients of disjoint unions.

1.3 Limits in Functor Categories

Categories of functors form a particularly nice class of functors due inheriting large amounts of structure from the target category. Thus, if we have an unpleasant category $\mathbf{C}$, which does not have many limits or colimits, then we can look at the category of functors from $\mathbf{C}$ into a nice category, e.g. **Set**, and then study the category via these functors. This is analogous to studying groups via their representations (indeed the category of representations of a group is given by functors from the one object category for the group into $k - \mathbf{Vect}$). It turns out that there is often a nice, partially structure preserving embedding of categories into functor categories built on them. This is the famous Yoneda Lemma.

The main structural result we want to prove is that $[\mathbf{C}, \mathbf{D}]$ has all the limits and colimits that exist in $\mathbf{D}$. These will in fact be computed "pointwise", meaning that we find the limit of functors by evaluating at each object and taking the limit in $\mathbf{D}$. To prove this, we first need the following result, which we refer to as the functoriality of limits.

Proposition 1.48 ([Lei16] Exercise 5.3.8)**.** *Let $\mathbf{C}$ a category and $F, G \colon I \to \mathbf{C}$ diagrams with limits $(L, \eta) = \lim F$ and $(M, \varepsilon) = \lim G$ in $\mathbf{C}$. Suppose there is a natural transformation $\alpha \colon F \to G$. Then there is a unique morphism $h \colon L \to M$ such that $\varepsilon \circ h = \alpha \circ \eta$. Then there is a functor $\lim \colon [I, \mathbf{C}] \to \mathbf{C}$ such that $\lim \alpha = h$ for each such natural transformation.*

Proof. Consider the maps $\alpha_i \circ \eta_i$ Then for $u \colon i \to j$ in I, we have that $Gu \circ \alpha_i \circ \eta_i = \alpha_j \circ Fu \circ \eta_i = \alpha_j \circ \eta_j$. Hence $(L, \alpha \circ \eta)$ is a cone for G. Then, by the universality of the limit, there is a unique morphism $h \colon L \to M$ such that $\alpha_i \circ \eta_i = \varepsilon_i \circ h$ for each i.

$$\begin{tikzcd}
L \arrow[d, "\eta_j"'] & L \arrow[l, "\mathrm{id}_L"'] \arrow[d, "\eta_i"] \\
Fj \arrow[d, "\alpha_j"'] & Fi \arrow[l, "Fu"'] \arrow[d, "\alpha_i"] \\
Gj & Gi \arrow[l, "Gu"]
\end{tikzcd}$$

Now, suppose we have H with $(N, \delta) = \lim H$ and morphisms $\alpha \colon F \to G$ and $\beta \colon G \to H$. Then, there are morphisms $h_\alpha \colon L \to M$, $h_\beta \colon M \to N$ and $h_{\beta \circ \alpha} \colon L \to N$. We need to show that $h_{\beta \circ \alpha} = h_\beta \circ h_\alpha$.

So, $\delta_i \circ h_\beta \circ h_\alpha = \beta_i \circ \varepsilon_i \circ h_\alpha = \beta_i \circ \alpha_i \circ \eta_i = (\beta \circ \alpha)_i \circ \eta_i$. Then, uniqueness of the morphism satisfying this property shows that $h_{\beta \circ \alpha} = h_\beta \circ h_\alpha$. Hence, $\lim(\beta \circ \alpha) = \lim \beta \circ \lim \alpha$ which gives us functoriality. $\square$

Remark 1.49. The opposite category of $[I, \mathbf{C}]$ is $[I^{\mathrm{op}}, \mathbf{C}^{\mathrm{op}}]$. The limit functor $[I^{\mathrm{op}}, \mathbf{C}^{\mathrm{op}}] \to \mathbf{C}^{\mathrm{op}}$ dualises to a colimit functor $[I, \mathbf{C}] \to \mathbf{C}$.

The following proposition has the tagline "limits in functor categories are computed pointwise".

Proposition 1.50 ([Lei16] Theorem 6.2.5)**.** *Let $F\colon I \to [\mathbf{C}, \mathbf{D}]$ be a diagram of functors where $\mathbf{D}$ has all limits of shape I. Consider, for each $X \in \mathbf{C}$, the functor $G_X\colon I \to \mathbf{D}$ given by $Gi = (Fi)X$ and for each $u\colon i \to j$ in I, $G_X u = (Fu)_X$. Then,* $(\lim F)X = \lim G_X$*.*

Each morphism $f\colon X \to Y$ in $\mathbf{C}$ lifts to a unique natural transformation $\phi\colon G_X \to G_Y$. Then, $(\lim F)f = \lim \phi$*, according to the functoriality of limits in Proposition 1.48.*

Proof. We start by considering the morphism $f\colon X \to Y$. We define ϕ by $\phi_i = (Fi)f$. Now, for $u\colon i \to j$ in I, we have that $G_X u = (Fu)_X$ and $G_Y u = (Fu)_Y$. So then $\phi_j \circ G_X u = (Fi)f \circ (Fu)_X$. Fu is a natural transformation $F_i \to F_j$ with the following naturality diagram.

$$\begin{tikzcd} (Fi)X \arrow[r, "(Fi)f"] \arrow[d, "Fu_X"'] & (Fi)Y \arrow[d, "Fu_Y"] \\ (Fj)X \arrow[r, "(Fj)f"'] & (Fj)Y \end{tikzcd}$$

Hence we obtain $(Fi)f \circ (Fu)_X = (Fu)_Y \circ (Fj)f$ which is exactly $G_Y u \circ \phi_j$. Hence the naturality diagram for ϕ commutes:

$$\begin{tikzcd} G_X i \arrow[r, "G_X u"] \arrow[d, "\phi_i"'] & G_X j \arrow[d, "\phi_j"] \\ G_Y i \arrow[r, "G_Y u"'] & G_Y j \end{tikzcd}$$

We now define a functor $G\colon \mathbf{C} \to \mathbf{D}$ such that $GX = \lim G_X$ and $Gf = \lim \phi$. We shall show that $G = \lim F$. Now, for each X, there is a morphism $GX \to (Fi)X$ given by the limit cone for G_X. Call this $(\eta_{i,X})_{i \in I}$. We claim that $(\eta_i\colon G \to Fi)_{i \in I}$ is a limiting cone for G.

To see that η forms a cone, let $u\colon i \to j$ be a morphism of I. Then, for each X, we have that $(Fu)_X \circ \eta_{i,X} = \eta_{j,X}$ by definition of $(\eta_{i,X})_{i \in I}$ being a cone. Therefore $Fu \circ \eta_i = \eta_j$, since composition of natural transformations is determined componentwise, and hence η is a cone.

To check that η is limiting, let (H, ε) be another cone. Then, for each X, there is a unique morphism $h_X \colon HX \to GX$ since $(GX, \eta_{i,X})_{i \in I}$ is limiting. Now, for $f \colon X \to Y$, we see that $(HX, \eta_{i,Y} \circ \phi \circ h_X)$ forms a cone for G_Y. Thus there is a unique morphism $k \colon HX \to GY$ such that $\eta_{i,Y} \circ k = \eta_{i,Y} \circ \phi \circ h_X$. Clearly, $k = \phi \circ h_X$. We can also see that $(HX, \eta_{i,Y} \circ h_Y \circ Hf)$ and so $\eta_{i,Y} \circ h_Y \circ Hf$ must factor through k. Hence, we conclude that $\phi \circ h_X = H_y \circ Hf$. This shows that there is a natural transformation $h \colon H \to G$ so that $\varepsilon = \eta \circ h$. This is unique due to the uniqueness of each h_X and natural transformations being determined by components.

We conclude that $G = \lim F$ and G is defined as required by the proposition. $\square$

We can use these two propositions to prove an important computational result with the slogan "limits commute with limits". This result encapsulates many seemingly obvious basic facts as well as allowing us to commute more complicated limits later. For instance, given sets A, B and C, we can prove directly that $(A \times B) \times C \cong A \times (B \times C)$ and that these are in fact both limiting cones for $A \times B \times C$ (as viewed as a limit with shape the three object discrete category). The two products are said to commute with each other, since we can do them in either order. The whole result subsumes this and all other combinations of products, and indeed lets us combine products with other limits such as equalizers so that we can see that the product of equalizers of two pairs of morphisms is isomorphic to the equalizer of the products.

Before we can properly formulate and prove this result, we first must make some observations. We want to consider limits of shape I and shape J at the same time in a category $\mathbf{C}$ having all limits of these shapes. To do this, we look at functors $F \colon I \times J \to \mathbf{C}$. Here, $I \times J$ is the category whose objects are pairs of objects (X_I, X_J), one from I and one from J and a morphism $f \colon (X_I, X_J) \to (Y_I, Y_J)$ is given as a pair of morphisms $f = (f_I, f_J)$ where f_I is a morphism $X_I \to Y_I$ and similarly for f_j. Composition is computed pairwise.

Continuing the example of products above, if we let I and J both be the two-object discrete category with objects a and b, then the objects of $I \times J$ are $(a, a), (a, b), (b, a)$ and (b, b) and all morphisms are of the form $(\mathrm{id}, \mathrm{id})$. A functor $F \colon I \times J \to \mathbf{C}$ then picks out objects X_1, X_2, X_3 and X_4.

Now, given such a functor, we can **curry** it by thinking of it as a family of one-argument functors paramaterised by the other argument. Specifically, for each $i \in I$, there is a functor $F(i, -) \colon J \to \mathbf{C}$ and for each $j \in J$ there is a functor $F(-, j) \colon I \to \mathbf{C}$. Each of these is a diagram in its own right, and we can hence take limits over them. Further, this paramaterisation is in fact functorial: there is a functor $F_I \colon I \to [J, \mathbf{C}]$ such that $F_I(i) = F(i, -)$ and similarly there is an $F_J \colon J \to [I, \mathbf{C}]$. This functoriality is not necessarily immediately obvious, so we will demonstrate it here in full. We need to demonstrate, for each $u \colon i \to i'$ in I that there is a natural transformation $F_I u \colon F(i, -) \to F(i', -)$ and this mapping preserves composition.

Constructing $F_I u$ is analogous to the proof of 1.50 above. For each $f \colon j \to j'$ in J, we have a morphism $F(i, f) = F(\mathrm{id}_i, f) \colon F(i, j) \to F(i, j')$. The component of $F_I u$ at j is given by $F(u, \mathrm{id}_j)$. Then $F(\mathrm{id}_j, f) \circ F(u, \mathrm{id}_i) = F(u, f) = F(u, \mathrm{id}_j) \circ F(\mathrm{id}_i, f)$ simply by functoriality of F. This shows that $F_I u$ is natural. Composition is then also clear from functoriality and behaviour of the identity, since $F(v \circ u, \mathrm{id}_j) = F(v, \mathrm{id}_j) \circ F(u, \mathrm{id}_j)$.

Since $\mathbf{C}$ has limits of shape I, so does $[J, \mathbf{C}]$ by Proposition 1.50, so we can happily take the limit over F_I in $[J, \mathbf{C}]$. Similarly for F_J. Then, $\lim F_I$ is a functor $J \to \mathbf{C}$ and so we can take its limit too, to get $\lim(\lim F_I)$ in $\mathbf{C}$. Equally, we could have taken $\lim(\lim F_J)$.

Returning to our example, let us say that $X_1 = A, X_2 = B, X_3 = 1$ and $X_4 = C$, where 1 is any choice of one element set. Recall that $X \times 1 \cong 1 \times X \cong X$ for all sets X. Then, our functor $F(a, -)$ acts as $a \mapsto A$ and $b \mapsto B$ while $F(b, -)$ acts as $a \mapsto 1$ and $b \mapsto C$. We now take the limit of the functor F_I, which will be a functor G acting as $G(a) = A \times B$ and $G(b) = C$. Hence, the limit of G is $(A \times B) \times C$. We can compute this the other way round, with functors $F(-, a)$ acting as $a \mapsto A$ and $b \mapsto 1$ and $F(-, b)$ acting as $a \mapsto B$ and $b \mapsto C$. Then we take the limit of F_J, which is the functor H such that $Ha = A$ and $Hb = B \times C$. Then $\lim H = A \times (B \times C)$. The commutativity of limits then tells us $\lim G \cong \lim H$, that is $(A \times B) \times C \cong A \times (B \times C)$, which matches what we computed directly.

We will now prove this result properly.

Proposition 1.51 ([Lei16] Proposition 6.2.8)**.** *Let* $\mathbf{C}$ *a category with all limits of shapes* I *and* J*. Then* $\mathbf{C}$ *has all limits of shape* $I \times J$*. Further, if* $F \colon I \times J \to \mathbf{C}$ *is a diagram with curried functors* $F_I \colon I \to [J, \mathbf{C}]$ *and* $F_J \colon J \to [I, \mathbf{C}]$ *then the limit of* F *is computed as* $\lim_{I \times J} F \cong \lim_J(\lim_I F_I) \cong \lim_I(\lim_J F_J)$*.*

Proof. Conceptually, the proof will be fairly simply: we take a limit cone for $\lim_I F_I$ and show that it lifts to a limit cone for F. We then apply symmetry to find the same for F_J and then uniqueness of the limit of F gives us the full isomorphism.

Let $(G, \eta_i \colon G \to F_I i)_{i \in I}$ be a limit cone for F_I and $(L, \varepsilon_j \colon L \to Gj)_{j \in J}$ be a limit cone for G. Now, for each (i, j), consider the morphism $\eta_{i,j} \circ \varepsilon_j \colon L \to F(i, j)$. Let $(u, v) \colon (i, j) \to (i', j')$ be a morphism in $I \times J$. Now, naturality of η is expressed in the following diagrams. The first is naturality of η as a transformation between functors $I \to [J, \mathbf{C}]$ and the second is naturality of each η_i as a transformation between functors $J \to \mathbf{C}$.

$$\begin{tikzcd} F(i,-) \arrow[rr, "F_I u"] & & F(i',-) \\ & G \arrow[ul, "\eta_i"] \arrow[ur, "\eta_{i'}"'] & \end{tikzcd} \qquad \begin{tikzcd} F(i,j) \arrow[r, "F(\mathrm{id}_i, v)"] & F(i,j') \\ Gj \arrow[u, "\eta_{i,j}"] \arrow[r, "Gv"'] & Gj' \arrow[u, "\eta_{i,j'}"'] \end{tikzcd}$$

Naturality of ε is this triangle:

$$\begin{tikzcd} Gj \arrow[rr, "Gv"] & & Gj' \\ & L \arrow[ul, "\varepsilon_j"] \arrow[ur, "\varepsilon_{j'}"'] & \end{tikzcd}$$

We decompose $(u, v) = (\mathrm{id}_{i'}, v) \circ (u, \mathrm{id}_j)$. Then,

$$\begin{aligned} F(u, v) &= F(\mathrm{id}_{i'}, v) \circ F(u, \mathrm{id}_j) \circ \eta_{i,j} \circ \varepsilon_j \\ &= F(\mathrm{id}_{i'}, v) \circ \eta_{i',j} \circ \varepsilon_j \\ &= \eta_{i',j'} \circ Gv \circ \varepsilon_j \\ &= \eta_{i',j'} \circ \varepsilon_{j'} \end{aligned}$$

This gives us naturality of $\eta_{i,j} \circ \varepsilon_j$ and hence $(L, \eta_{i,j} \circ \varepsilon_j)_{i,j \in I \times J}$ forms a cone for F. We need to verify that this is a limiting cone.

Let $(M, \mu_{i,j} \colon M \to F(i, j))$ be another cone. Then, we can view μ_i as a natural transformation $M \to F(i, -)$ with components $\mu_{i,j}$. Since G is limiting, there must be some unique $h \colon M \to G$ such that $\mu_i = \eta_i \circ h$ for all i. Now, above we were thinking of M as the constant functor, but now we can view M as an object in $\mathbf{C}$ and then for each j, we get a component $h_j \colon M \to G$ of the natural transformation h. This exhibits M as a cone of G and hence there is some unique k such that $h_j = \varepsilon_j \circ k$ for all j. Hence, we can write $\mu_{i,j} = \eta_{i,j} \circ \varepsilon_j \circ k$. So $\eta \circ \varepsilon$ is indeed a limiting cone and $\lim_{I \times J} F = \lim_J (\lim_I F_I) = L$.

The exact same proof follows for F_J in place of F_I. Indeed, the only difference is the ordering of I and J, but $I \times J$ and $J \times I$ are canonically isomorphic by twisting. With this observation, we see that $lim_{I\times J}F = \lim_I(\lim_J F_J)$. Now, limits are unique up to unique isomorphism, and hence it follows that $\lim_I(\lim_J F_J) \cong \lim_J(\lim_I F_I))$ and that this isomorphism is witnessed uniquely. $\square$

A very useful application of these facts is a partial converse to Proposition 1.21. Specifically, in a functor category $[\mathbf{C}, \mathbf{D}]$, if D has all pullbacks then we have that a monic natural transformation is componentwise monic. To show this, we first prove a presentation of monomorphisms in terms of pullbacks.

Lemma 1.52 ([Lei16] Exercise 5.1.41)**.** *Let* $\mathbf{D}$ *be a category with all pullbacks and* $f\colon X \to Y$ *be a morphism. Then* f *is monomorphic if and only if the following is a pullback diagram.*

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Proof. Suppose f is monomorphic. It is clear that the diagram commutes and hence is a cone for the pullback. Suppose (P, p_1, p_2) is another cone, meaning that $f \circ p_1 = f \circ p_2$. Since f is monic, we have that $p_1 = p_2$. Then, p_1 is a morphism $P \to X$ making the two cones commute. If $h\colon P \to X$ is another morphism making the cones commute, we have that $\mathrm{id}_X \circ h = \mathrm{id}_X \circ p_1$, hence p_1 is unique. This exhibits the original cone as a pullback.

Now, suppose the diagram is a pullback. Let $u, v\colon Z \to X$ be such that $f \circ u = f \circ v$. Then (Z, u, v) is another cone for the pullback and hence there is a unique morphism h such that $\mathrm{id}_X \circ h = u$ and $\mathrm{id}_X \circ h = v$. Thus, $u = h = v$, and so f is monic. $\square$

Proposition 1.53 ([Lei16] Exercise 6.2.20)**.** $\mathbf{C}, \mathbf{D}$ *be categories such that* $\mathbf{D}$ *has all pullbacks and let* $\eta\colon F \to G$ *be a monomorphic natural transformation. Then, for all* X *in* $\mathbf{C}$*,* η_X *is monomorphic.*

Proof. As $\mathbf{D}$ has all pullbacks, so does $[\mathbf{C}, \mathbf{D}]$. Hence, we can draw the pullback diagram exhibiting η as a monomorphism:

$$\begin{tikzcd} F \arrow[r, "\mathrm{id}_F"] \arrow[d, "\mathrm{id}_F"'] \arrow[dr, phantom, "\lrcorner" very near start] & F \arrow[d, "\eta"] \\ F \arrow[r, "\eta"'] & G \end{tikzcd}$$

Then, since limits are computed componentwise, we have that for each X, the componentwise diagram is a pullback:

$$\begin{tikzcd} FX \arrow[r, "\mathrm{id}_{FX}"] \arrow[d, "\mathrm{id}_{FX}"'] \arrow[dr, phantom, "\lrcorner" very near start] & FX \arrow[d, "\eta_X"] \\ FX \arrow[r, "\eta_X"'] & GX \end{tikzcd}$$

Hence, η_X is a monomorphism for each X. $\square$

Remark 1.54. If $\mathbf{D}$ has all pushouts then we obtain an analogous result for epimorphisms by dualising.

We include here also the fact that monomorphisms are "stable" under pullback. If we have morphisms $f\colon X \to Z$ and $g\colon Y \to Z$ with pullback (P, p_X, p_Y), we say that p_X is the pullback of g *along* f. Stability under pullbacks means that if g is a monomorphism, so is p_X.

Proposition 1.55 ([Lei16] Exercise 5.1.42)**.** *Let $f\colon X \to Z$ and $g\colon Y \to Z$ morphisms with g monomorphic. Let (P, p_X, p_Y) be a pullback for f and g. Then, p_X is monomorphic.*

Proof. Let $u, v\colon W \to P$ be morphisms such that $p_X \circ u = p_X \circ v$. It follows that $f \circ p_X \circ u = f \circ p_X \circ v$ and so $g \circ p_Y \circ u = g \circ p_Y \circ v$. Since g is monomorphic, we obtain that $p_Y \circ u = p_Y \circ v$. Hence, $(P, p_X \circ u, p_Y \circ u)$ and $(P, \circ p_X \circ v, p_Y \circ v)$ are in fact the same cone for the pullback diagram. It follows that there is a unique $h\colon W \to P$ such that $p_X \circ u = p_X \circ v = p_X \circ h$. Thus, $u = h = v$. This shows that p_X is monomorphic. $\square$

1.4 Presheaves and Yoneda

The category **Set** is particularly nice, having all small limits and colimits. In fact, in the sense of topos theory is "prototypically" nice, more details of which can be found in [MLM92]. Additionally, most of the categories we are really interested in are locally small and hence there is a canonical way of connecting sets to the category, the Hom-sets. For this reason, we are often interested in functor categories into **Set** and some very important tools arise around such categories.

First, there is a fair bit of notation we must introduce. We let $\mathbf{C}$ be a locally small category. For each object A of $\mathbf{C}$ we write $\text{よ}A$ for the Hom functor $\mathrm{Hom}(-, A)\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ and $\text{よ}^{\mathrm{c}}A$ for $\mathrm{Hom}(A, -)\colon \mathbf{C} \to \mathbf{Set}$. よ is the hiragana pronounced "yo" and is short for "Yoneda". We write $[\mathbf{C}, \mathbf{Set}]$ for the category of functors $\mathbf{C} \to \mathbf{Set}$ with natural transformations as morphisms. We will write $\mathrm{Nat}(-, -)$ for the Hom functor on $[\mathbf{C}, \mathbf{Set}]$ and $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ in order to distinguish from the Hom functor on $\mathbf{C}$. By the evaluation functor, we mean the functor $\mathrm{ev}_{\mathbf{C}}\colon \mathbf{C} \times [\mathbf{C}, \mathbf{Set}] \to \mathbf{Set}$ which acts as $(C, F) \mapsto FC$.

The functors $\mathrm{Hom}(-, A)$ and $\mathrm{Hom}(A, -)$ are often called representable functors. For slightly arcane reasons, functors $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ are usually called presheaves.

We can now prove what is arguably, despite often being called a lemma, one of the most foundational theorems in Category Theory, the Yoneda Lemma.

Theorem 1.56 ([Lei16] Theorem 4.2.1)**.** *There is a natural isomorphism*

$$\mathrm{ev}_{\mathbf{C}^{op}}(X, F) \to \mathrm{Nat}(\text{よ}X, F)$$

Remark 1.57*.* This is often written in the form of a specific object and functor: for all objects C in $\mathbf{C}^{\mathrm{op}}$ and functors $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, the Yoneda Lemma tells us that $\mathrm{Nat}(\text{よ}C, F) \cong FC$ and that this equivalence is natural in C and F.

Proof. Fix a functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ and an object A in $\mathbf{C}^{\mathrm{op}}$. We will define $h_{A,F}\colon \mathrm{ev}_{\mathbf{C}}(A, F) \to \mathrm{Nat}(\text{よ}A, F)$ as follows. Let $x \in FA$. We start by defining the component of $\tau = h_{A,F}(x)$ at A. In particular, set $\tau_A(\mathrm{id}_A) = x$. Now, the diagram that we *want* to commute, for all B in $\mathbf{C}$ and $f\colon B \to A$, is:

$$\begin{tikzcd}
\mathrm{Hom}(A, A) \arrow[r, "- \circ f"] \arrow[d, "\tau_A"'] & \mathrm{Hom}(B, A) \arrow[d, "\tau_B"] \\
FA \arrow[r, "Ff"'] & FB
\end{tikzcd}$$

We will define $\tau_B(f)$ in order to make this commute. Namely, we have that id_A is in $\mathrm{Hom}(A, A)$, so we set $\tau_B \circ (- \circ f) = Ff \circ \tau_A$. We have $(\tau_B \circ (- \circ f))(\mathrm{id}_A) = \tau_B(f)$, so we have hence defined $\tau_B(f) = (Ff \circ \tau_A)(\mathrm{id}_A) = Ff(x)$.

We define an inverse for $h_{A,F}$ as $k_{A,F}(\tau) = \tau_A(\mathrm{id}_A)$. The above shows that a natural transformation $\text{よ}(A) \to F$ is determined by the image of id_A under the map. Hence, it follows that $h_{A,F}(k_{A,F}(\tau)) = \tau$. Similarly, $h_{A,F}(x)$ is defined by $[h_{A,F}(x)]_A(\mathrm{id}_A) = x$, hence $k_{A,F}(h_{A,F}(x)) = x$. Thus $h_{A,F}$ is a bijection.

It remains to show that h is natural. Let $f\colon B \to A$ be a morphism in $\mathbf{C}$, $F, G\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ and $\eta\colon F \to G$ be a natural transformation. The map $\mathrm{ev}(f, \eta)$ acts by taking $x \in FA$ to $\eta_B(Ff(x)) \in GB$. The map $\mathrm{Nat}(よ f, \eta)$ acts by taking $\tau\colon よ A \to F$ to $\eta\tau(f \circ -)\colon よ B \to G$.

Now, for $x \in FA$, let $\tau = h_{A,F}(x)$ and $\nu = h_{B,G}(\eta_B(Ff(x)))$. We wish to show that $\eta\tau(f \circ -) = \nu$ as natural transformations $よ B \to G$. As mentioned earlier, such natural transformations are determined by how they act on id_B, so we evaluate both sides at id_B. $\nu(\mathrm{id}_B) = h_{B,G}(\eta_B(Ff(x)))(\mathrm{id}_B) = \eta_B(Ff(x))$ by definition. Then, $\eta_B\tau_B(f \circ \mathrm{id}_B) = \eta_B\tau_B(f) = \eta_B Ff(x)$ by naturality of τ. Both sides evaluate equally at id_B, so in fact they must define the same natural transformation. This completes naturality of h.

Finally, since h is natural and every component is an isomorphism, we conclude that h is an isomorphism. $\square$

There is a version of the lemma for covariant functors which follows very quickly from observing two facts. First, the covariant representable functor over $\mathbf{C}$, $\mathrm{Hom}_{\mathbf{C}}(A, -)$, corresponds exactly to the contravariant functor over $\mathbf{C}^{\mathrm{op}}$, $\mathrm{Hom}_{\mathbf{C}^{\mathrm{op}}}(-, A)$. Second, taking opposite categories is an involution, meaning that $(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}} \cong \mathbf{C}$. It then suffices to substitute $\mathbf{C}^{\mathrm{op}}$ into the Yoneda lemma to obtain the covariant form.

Corollary 1.58. *There is a natural isomorphism*

$$\mathrm{ev}_{\mathbf{C}}(-,-) \to \mathrm{Nat}(よ^{c}(-), -)$$

Perhaps the most important use of the Yoneda Lemma is to prove that taking objects in $\mathbf{C}$ to their corresponding representable functors in $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ is in fact an embedding of categories. This allows us to view every locally small category as a full subcategory of a functor category. This is called the Yoneda Embedding.

Corollary 1.59 ([Lei16] Corollary 4.3.7)**.** $よ\colon \mathbf{C} \to [\mathbf{C}^{op}, \mathbf{Set}]$ *is a full and faithful functor.*

Proof. We must first define how よ acts on morphisms. Suppose $f\colon A \to B$ is a morphism in $\mathbf{C}$. Then, we define $よ f$ as the natural transformation that, for each X in $\mathbf{C}$, has component $(\text{よ} f)_X \colon \text{よ} A \to \text{よ} B$ which acts by composition f. That is, for $u \in \mathrm{Hom}(X, A)$, $(\text{よ} f)_X$ is the function acting as $u \mapsto f \circ u$.

To see that this is natural, let $p \colon X \to Y$ in $\mathbf{C}$. Recall that the functor $\text{よ} A$ sends p to precomposition with p. It is clear, then, that for $u \colon Y \to A$, $[(\text{よ} A)p \circ (\text{よ} f)_X](u) = f \circ u \circ p = [(\text{よ}_f)_Y \circ (\text{よ} B)p](u)$ by associativity of composition.

Preservation of composition also follows from associativity of composition. That is, if $g \colon B \to C$ is another morphism, we see that $\text{よ}(g \circ f) = \text{よ} g \circ \text{よ} f$ since for all $u \colon X \to A$ we observe that $g \circ (f \circ u) = (g \circ f) \circ u$.

That よ is full and faithful then follows from the Yoneda lemma. Specifically we know that there is an isomorphism $h \colon (\text{よ} B)A \to \mathrm{Nat}(\text{よ} A, \text{よ} B)$. So, let $f \colon A \to B$ in C. Then, the proof of Yoneda tells us that the natural transformation has components given by, for all objects object X and morphisms $u \colon X \to A$, $[h(f)_X](u) = [(\text{よ} B)u](f) = f \circ u$. This is exactly the action of $(\text{よ} f)_X$. Hence, we see that $h(f) = \text{よ} f$. Hence, for every $g \colon \text{よ} A \to \text{よ} B$, we have that $g = \text{よ} h^{-1}(g)$, so よ is full. Similarly, if $f, g \colon A \to B$ are morphisms and $\text{よ} f = \text{よ} g$, we see that $h(f) = h(g)$ and so $f = g$. Thus, よ is faithful. $\square$

Additionally, the Yoneda embedding preserves limits, which we will use several times in our study of modules. To prove this, we first need to prove that the functor $\text{よ}^{\mathrm{c}} A$ preserves limits for every object A.

Lemma 1.60 ([Lei16] Proposition 6.22)**.** *Let A be an object of $\mathbf{C}$. Then $\mathrm{Hom}(A, -)$ preserves limits.*

Proof. Let $D \colon I \to \mathbf{C}$ be a diagram with limit (L, η) in $\mathbf{C}$. We invoke the proof of Theorem 1.44 to get a characterisation of $\lim \mathrm{Hom}(A, D-)$ as the subset X of $\prod \mathrm{Hom}(A, Di)$ such that for every $(x_i)_{i \in I} \in X$ and $u \colon i \to j$ in I, $x_j = \mathrm{Hom}(A, Du)(x_i) = Du \circ x_i$. The cone structure are the projections mapping $\pi_i \colon X \to \mathrm{Hom}(A, Di)$ mapping $\pi_j((x_i)) = x_j$.

We now consider the map $\varphi \colon \mathrm{Hom}(A, L) \to X$ which sends $f \colon A \to L$ to $(\eta_i \circ f)_{i \in I}$. This satisfies $\eta_j \circ f = Du \circ \eta_i \circ f$ since L is a cone for D. We will show that φ is a bijection.

Let $(x_i)_{i \in I} \in X$ and observe that this gives a cone on A precisely by the equations $x_j = Du \circ x_i$. Thus, there is a unique $h \colon A \to L$ such that $x_i = \eta_i \circ h$ for all i. Hence, $(x_i) = \varphi(h)$. This provides surjectivity of φ and uniqueness of h gives injectivity.

This isomorphism exhibits a cone on $\mathrm{Hom}(A, L)$ as $\pi_i \circ \varphi$. For each $f\colon A \to L$, we have $(\pi_i \circ \varphi)(f) = \eta_i \circ f = \mathrm{Hom}(A, \eta_i)(f)$. Hence, $(\mathrm{Hom}(A, L), \mathrm{Hom}(A, \eta))$ is a limit for $\mathrm{Hom}(A, D-)$ and so $\mathrm{Hom}(A, -)$ preserves $\lim D$.

Since we picked D arbitrarily, we see that $\mathrm{Hom}(A, D-)$ preserves all limits. $\square$

Proposition 1.61 ([Lei16] Corollary 6.2.12)**.** よ *preserves limits.*

Proof. Let $D\colon I \to \mathbf{C}$ be a diagram in $\mathbf{C}$ with limit (L, η). Then, consider the diagram よ $\circ D\colon I \to [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ and call its limit (F, α). Since limits in functor categories are computed pointwise by Proposition 1.50, we have that for every A in $\mathbf{C}$, $FA = \lim \mathrm{Hom}(A, Di)$. By Lemma 1.60, we compute $FA = \mathrm{Hom}(A, \lim Di) = \mathrm{Hom}(A, L)$. Thus, on objects F acts the same as よL.

For objects A, B in $\mathbf{C}$ and morphism $f\colon A \to B$ in I, we get a natural transformation $\mathrm{Hom}(f, D-)\colon \mathrm{Hom}(B, D-) \to \mathrm{Hom}(A, D-)$ that acts by precomposition with f. Combining Proposition 1.50 and 1.48, we know that $Ff = \lim \mathrm{Hom}(f, -)$ and that this is the unique morphism $h\colon \mathrm{Hom}(B, L) \to \mathrm{Hom}(A, L)$ such that $\mathrm{Hom}(A, \eta) \circ h = \mathrm{Hom}(f, D-) \circ \mathrm{Hom}(B, \eta)$. Now, for $u\colon B \to Di$, we know that $(\mathrm{Hom}(f, Di) \circ \mathrm{Hom}(B, \eta))(u) = \eta_i \circ u \circ f$. Hence $h = \mathrm{Hom}(f, L)$ and this is exactly (よ$L)f$. Thus, $Ff = ($よ$L)f$ and so $F =$ よL.

By Corollary 1.59, よ is full and faithful and so each $\alpha_i\colon$ よ$L \to$ よDi must Yoneda correspond to a morphism $a_i\colon L \to Di$. Further, since $\alpha_j = \alpha_i \circ$ よDu for every morphism $u\colon i \to j$ in I, we must get that $a_j = a_i \circ Du$ and so a is a cone for L. Thus, there is a unique morphism $h\colon L \to L$ such that $a = \eta \circ h$.

It follows that よ$h\colon$ よ$L \to$ よL is such that $\alpha =$ よ$\eta \circ$ よh. As (F, α) is a limit, there must be some よ$k\colon$ よ$L \to$ よL such that よ$\eta = \alpha \circ$ よk. There is a unique endomorphism $t\colon L \to L$ such that $\eta = t \circ \eta$ and both id_L and $h \circ k$ satisfy this. Thus, $h \circ k = \mathrm{id}_L$. This implies よ$h \circ$ よ$k = \mathrm{id}_{よL}$. Similarly, there is a unique よs such that $\alpha = \alpha \circ$ よs and so よ$s = \mathrm{id}_{よL} =$ よ$k \circ$ よh. We conclude that よh is an isomorphism and hence (よL, よη) is a limit for よ $\circ D$. $\square$

1.5 Accessibility

Here we develop the notions of locally presentable and accessible categories. These are categories which are generated from subsets of small objects via direct limits. It will turn out that almost all the categories in which we are interested are in fact accessible and indeed many of the methods can be generalised to the finitely accessible case. We draw primarily from [AR94].

Accessible categories are dependent on cardinal numbers in their definition and in general are quite sensitive to the set theory we work in. In particular, we will need the notion of a regular cardinal.

Definition 1.62. Let λ be a cardinal.

- We call a set Y **λ-small** if Y is of cardinality less than λ.
- We say that λ is **regular** if for every set X of cardinality λ and for every λ-small set Y of λ-small subsets of X, the union $\bigcup Y$ is λ-small.

In essence, the definition is saying that a regular cardinal cannot be built from a small collection of smaller pieces. The simplest example of this is the first infinite cardinal $\aleph_0$, the cardinality of the integers. If we have some family of finite subsets of integers, for their union to equal the whole integers, we must have infinitely many finite subsets. More details can be found in [Jec03].

From hereon, every cardinal will be regular. With this out of the way, we can define the important tools of accessible categories- directed systems and their corresponding colimits. In these definitions, let λ be a regular cardinal.

Definition 1.63. Let $(X, \leq)$ be a partially ordered set and $\mathbf{C}$ a category.

- We call X a λ-directed set if every λ-small subset has an upper bound.
- A λ-directed colimit in $\mathbf{C}$ is a colimit over a diagram whose shape is a λ-directed set, viewed as a category.

Remark 1.64. If $(X, \leq)$ is a λ-directed set and $J\colon X \to \mathbf{C}$ is a functor, we sometimes call the collection $(Dx, f_{xy}\colon Dx \to Dy)_{x,y\in X}$ a **directed system** and speak about the colimit being over the system.

Example 1.65. Consider a sequence of sets $(A_i)_{i\in\mathbb{N}}$ such that for each i we have that $A_i \subset A_{i+1}$. We can view this as a diagram in **Set** over the natural partial order on the natural numbers $(\mathbb{N}, \leq)$ by sending n to A_n and $n \leq m$ to the inclusion mapping $A_n \hookrightarrow A_m$. $(\mathbb{N}, \leq)$ is an $\aleph_0$-directed set, since for any finite subset of the natural numbers there is an upper bound- namely their maximum. Taking the colimit over this diagram, we obtain $\bigcup_{n\in\mathbb{N}} A_n$. For this reason, we call such a colimit a **directed union**. We can also extend this notion to transfinite chains by taking diagrams over larger ordinals with their natural well-ordering.

Example 1.66. Let X be a set and consider the collection of finite subsets of X, $\mathcal{P}^{<\aleph_0}(X)$. This is an $\aleph_0$-directed set ordered under subset inclusion, since the union of any finite family of finite sets is once again a finite set. Indeed, for any regular cardinal λ which is smaller than the cardinality of X, if we consider the collection of λ-small subsets of X, $\mathcal{P}^{<\lambda}(X)$, then this is a λ-directed set ordered by inclusion. The closure of this set under unions of λ-small families is precisely the definition of regularity of λ.

Example 1.67. Suppose $\mathbf{C}$ has all finite coproducts and directed colimits. Then, given some index J and collection of objects X_j, we can form the (possibly infinite) coproduct $\bigoplus_{j\in J} X_j$ as a directed colimit. We take the shape of our colimit to be $I = \mathcal{P}^{<\aleph_0}(J)$ and for each finite $J_0 \subset J$, set $DJ_0 = \bigoplus_{j\in J_0} X_j$. Then $\bigoplus_{j\in J} X_j = \operatorname{colim} D$.

We use directed colimits in order to define our notion of smallness:

Definition 1.68. Let K be an object in $\mathbf{C}$. We say that K is **λ-presented** if $\operatorname{Hom}(K, -)$ preserves all λ-directed colimits. As a special case, $\aleph_0$-presented objects are called finitely presented.

We can use the presentation of colimits in **Set** to provide an alternate characterisation of preserving directed colimits that is often more useful for proving that objects are λ-presentable.

Proposition 1.69 ([AR94] Remark after Definition 1.1)**.** *Let K be an object in $\mathbf{C}$. Then K is λ-presented if and only if for all λ-directed colimits $(L, \eta) = \operatorname{colim}(D\colon I \to \mathbf{C})$ and morphisms $f\colon K \to L$, there is some $i \in I$ and $h\colon K \to Di$ such that $f = \eta_i \circ h$ and this factorisation is essentially unique. This means that if $f = \eta_i \circ h = \eta_j \circ h'$ then there is some $k \geq i, j$ with $D(i \to k)h = D(j \to k)h'$.*

Remark 1.70. The catchphrase for this would be that the morphism f **factors through** the diagram D.

Proof. Fix a λ-directed diagram $D\colon I \to \mathbf{C}$ with a colimit (L, η) in $\mathbf{C}$. From the proof of Theorem 1.44 we know that colimits are presented as coequalizers of parallel morphisms between coproducts. Let $(\overline{L}, \varepsilon)$ be the colimit of $\operatorname{Hom}(K, D-)$. Then, $\overline{L}$ is presented in terms of parallel morphisms $u, v\colon \bigsqcup_{i\to j\in I} \operatorname{Hom}(K, Di) \to \bigsqcup_{i\in I} \operatorname{Hom}(K, Di)$ such that for the $i \to j$ component and $g \in \operatorname{Hom}(K, Di)$, $u(g) = g$ and $v(g) = [D(i \to j)](g)$. We let $q\colon \bigsqcup_{i\in I} \operatorname{Hom}(K, Di) \to \overline{L}$ be the coequalizer of u and v. This is necessarily surjective.

$$\begin{tikzcd}
\coprod_{i\to j\in I}\operatorname{Hom}(K,Di) \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & \coprod_{i\in I}\operatorname{Hom}(K,Di) \arrow[rr, "q"] & & \overline{L} \arrow[d, "h"] \\
& \operatorname{Hom}(K,Dn) \arrow[u] \arrow[rr, "\operatorname{Hom}(K,\eta_n)"'] & & \operatorname{Hom}(K,L)
\end{tikzcd}$$

We first prove the forwards direction. K being λ-presented means that $\operatorname{Hom}(K,L) = \overline{L}$ and $\eta = \varepsilon$. Now, let f be an element of $\operatorname{Hom}(K,L)$. We know q is surjective, so we pick some $\overline{f} \in \coprod_{i\in I}\operatorname{Hom}(K,Di)$ such that $f = q(\overline{f})$ and by the definition of the disjoint union, there must be some $n \in I$ such that $\overline{f} \in \operatorname{Hom}(K,Dn)$. There is an inclusion $\iota\colon \operatorname{Hom}(K,Dn) \to \coprod_{i\in I}\operatorname{Hom}(K,Di)$ and the presentation of colimits tells us that $q \circ \iota = \operatorname{Hom}(K,\eta_n)$. This morphism acts as composition on the Hom sets, so $\operatorname{Hom}(K,\eta_n)(\overline{f}) = f$ says that $\eta_n \circ \overline{f} = f$.

To see that this factorisation is essentially unique, suppose $f = \eta_n \circ \overline{f} = \eta_m \circ f^*$. Now, q is the coequalizer of u and v, meaning that, since $q(\overline{f}) = q(f^*)$, there is some finite chain $\overline{f} = f_0, \dots, f_k = f^*$ with $f_x \in \operatorname{Hom}(K, Di_x)$ such that for each x, either $f_{x+1} = [D(i_x \to i_{x+1})](f_x)$ or $f_x = [D(i_{x+1} \to i_x)](f_{x+1})$. This is directly from definition of a coequalizer in **Set**. Let a be the least upper bound of $i_0, \dots, i_k$. We show that $[D(n \to a)](\overline{f}) = [D(m \to a)](f^*)$ by going up the chain. If $i_x \leq i_{x+1}$ then we know that $D(i_x \to a) = D(i_x \to i_{x+1}) \circ D(i_{x+1} \to a)$ simply from how composition in categories works. It follows that $[D(i_x \to a)](f_x) = [D(i_{x+1} \to a)]([D(i \to i_{x+1}](f_x)) = [D(i_{x+1} \to a)](f_{x+1})$. Simply reverse the arrows to see this for $i_x \geq i_{x+1}$. Hence, $[D(i_x \to a)](f_x)$ is the same for all x, including $x = 0$ and $x = k$, which shows the needed equality.

For the backwards direction, we oberve that $\operatorname{Hom}(K,L)$ is a cocone and hence there is a unique function $h\colon \overline{L} \to \operatorname{Hom}(K,L)$ such that $\operatorname{Hom}(K,\eta_i) = h \circ \varepsilon_i$ for each i. We will show that h is a bijection, which is sufficient to exhibit $\operatorname{Hom}(K,L)$ as a colimit.

We suppose that every $f\colon K \to L$ factors essentially uniquely as $f = \eta_i \circ \overline{f}$ for some i as above. Surjectivity of h follows from writing $f = \eta_i \circ \overline{f} = \operatorname{Hom}(K,\eta_i)(\overline{f}) = h(\varepsilon_i(\overline{f}))$. Now, suppose there are $a, b \in \overline{L}$ with $h(a) = h(b)$. Let $a = \varepsilon_n(\overline{a})$ and $b = \varepsilon_m(\overline{b})$ for some $\overline{a} \in \operatorname{Hom}(K,Dn)$, $\overline{b} \in \operatorname{Hom}(K,Dm)$ and $n, m \in I$. Hence, $h(a) = (h \circ \varepsilon_n)(\overline{a}) = \operatorname{Hom}(K,\eta_n)(\overline{a})$. Thus, $h(a) = \eta_n \circ \overline{a} = \eta_m \circ \overline{b}$. By essentially unique factorisation, there is some $s \geq n, m$ and $c \in \operatorname{Hom}(K, Ds)$ with $[D(n \to s)](\overline{a}) = c = [D(m \to s)](\overline{b})$. That is, $(v \circ \iota)(\overline{a}) = c = (v \circ \iota)(\overline{b})$, where ι is the inclusion into $\coprod_{i\to j\in I} Hom(K,Di)$ and choosing the appropriate components for v. We know q coequalizes u, v

and therefore $(q \circ \iota)(\overline{a}) = q(c) = (q \circ \iota)(\overline{b})$. We know that $\varepsilon = q \circ \iota$ and so $a = \varepsilon(\overline{a}) = \varepsilon(\overline{b}) = b$, completing the proof of injectivity. $\square$

The notion of presentability matches the natural notion of size in many commonly encountered categories.

Example 1.71. In **Set**, a set X is λ-presented precisely when it is λ-small.

Example 1.72. In **Grp**, G being λ-presented in the above sense precisely corresponds to the usual notion of presentability from algebra. That is, G is λ-generated if it has a λ-small set of generators and G is λ-presented if it is the quotient of a λ-generated free group by a λ-generated normal subgroup.

Restricting to **Ab**, we find that the finitely presented Abelian groups are exactly the finitely generated Abelian groups, as a consequence of the structure theorem.

Definition 1.73. We say that a category **C** is **λ-accessible** if:

- **C** is closed under λ-directed colimits.
- There is a set B of λ-presented objects such that every object is the λ-directed colimit of objects in B.

A category is **accessible** if it is λ-accessible for some λ.

Remark 1.74. If $\lambda = \aleph_0$, we say that **C** is **finitely accessible**.

As a special case of accessible categories, we introduce locally presentable categories. Though the axioms for a locally presentable category do not seem to add much to those of accessible categories, they in fact induce a much more rigid structure. Every accessible category we see in this project will in fact be locally finitely presentable and we will make use of that structure in non-trivial ways.

Definition 1.75. We say that a category **C** is **locally λ-presentable** if it is λ-accessible and cocomplete.

Remark 1.76. Again, if $\lambda = \aleph_0$, we say that **C** is **locally finitely presentable**.

It is beyond the scope of this project to explore the details of the structure of locally finitely presentable categories but we shall state a couple useful facts. The general gist of these properties is that locally finitely presentable categories can be embedded as particularly nice subcategories of functor categories into **Set** and hence inherit many nice properties. The interested reader is advised to consult Chapter 1 of [AR94] for proofs.

Proposition 1.77 ([AR94] Remark 1.56)**.** *Every locally presentable category is complete.*

The following proposition has the slogan "directed colimits commute with finite limits".

Proposition 1.78 ([AR94] Proposition 1.59)**.** *Let* $\mathbf{C}$ *be locally finitely presentable and* $F\colon I \times J \to \mathbf{C}$ *be a diagram where* I *is directed and* J *is finite. Recall the notation for the curried functors* F_I *and* F_J *from Proposition 1.51.*

Then, there is an isomorphism $\mathrm{colim}_I(\lim_J F_J) \cong \lim_J(\mathrm{colim}_I F_I)$.

2 Model Theory

In this section, we will introduce the requisite mathematical logic required for the project. We focus solely on the Model Theoretic, that is, semantic side of the subject and hence introduce no proof systems. In contrast to most presentations, we will work directly in multi-sorted first order logic, as this is the setting that most of the logic of the project takes place. A general source for the topic can be found in Chapter 1 of [Mar02].

A **language** is a set of well defined words, called **formulae** that describe some class of mathematical structure. The formulae consist of logical symbols, which have the same meaning in every setting, and non-logical symbols, which represent sets, constants, functions and relations that we want to exist on our structures but can be interpreted in different ways.

The logical symbols are:

And - $\wedge$	Implies - $\rightarrow$	There exists - $\exists$
Or - $\vee$	True - $\top$	For all - $\forall$
Not - $\neg$	False - $\bot$	Equality - $\doteq$

The collection of non-logical symbols of a language is called its **signature**. The signature and language can be obtained from each other, so we will usually identify them and often talk about the signature *as* the language.

The non-logical symbols come in four kinds. The first are sort symbols, usually denoted by a capital latin latter. These describe the different sets that should exist in a structure that the language describes.

The second kind are constant symbols. These determine elements that must always exist in a structure. Each constant symbol has an **arity**, which is a sort from the language. When we interpret the constant as a real element in a structure, it must be contained in the set corresponding to its arity.

Next are function symbols. As the name implies, these will be interpreted as functions on the structure. Each function symbol has an arity of the form $(A_1, \dots, A_n; B, m)$ for $A_1, \dots, A_n, B$ (potentially repeating) sorts. When $m = 1$ we will simply write $(A_1, \dots, A_n; B)$. When we interpret the symbol, it must be a function $A_1 \times \cdots \times A_n \to B^m$ for the sets interpreting the A_i and B.

Last are relation symbols. These determine relations that exist on the structure, that is, specified subsets. A relation symbol has an arity $(A_1, A_2, \dots, A_n)$ of (potentially repeating) sorts and the interpreted relation must be a subset of $A_1 \times \cdots \times A_n$.

A signature is written as a (potentially infinite) tuple $(\mathcal{A};\mathcal{C};\mathcal{F};\mathcal{R})$ where $\mathcal{A}$ is the list of sorts, $\mathcal{C}$ the list of constant symbols, $\mathcal{F}$ the list of function symbols and $\mathcal{R}$ the list of relation symbols.

Given a language L with signature $\sigma = (\mathcal{A};\mathcal{C};\mathcal{F};\mathcal{R})$ an **L-structure** is a tuple $\mathcal{M} = (\mathcal{A}^{\mathcal{M}};\mathcal{C}^{\mathcal{M}};\mathcal{F}^{\mathcal{M}};\mathcal{R}^{\mathcal{M}})$. For each sort A in $\mathcal{A}$, there is an actual set $A^{\mathcal{M}}$ in $\mathcal{A}^{\mathcal{M}}$. Similarly, for each constant symbol $c \in \mathcal{C}$ with arity A, there is a constant $c^{\mathcal{M}} \in A^{\mathcal{M}}$ and we put $c^{\mathcal{M}}$ in $\mathcal{C}^{\mathcal{M}}$. For each function symbol $f \in \mathcal{F}$ of arity $(A_1, \dots, A_n; B, m)$, there is a function $f^{\mathcal{M}} \colon A_1^{\mathcal{M}} \times \dots \times A_n^{\mathcal{M}} \to (B^{\mathcal{M}})^m$ in $\mathcal{F}^{\mathcal{M}}$. For each relation symbol $R \in \mathcal{R}$ with arity $(A_1, \dots, A_n)$, there is a relation $R^{\mathcal{M}} \subset A_1^{\mathcal{M}} \times \dots \times A_n^{\mathcal{M}}$ in $\mathcal{R}^{\mathcal{M}}$.

There are several “canonical” languages for common mathematical structures, some of which we will go over now.

Example 2.1. The language of groups, L_{grp} has signature $(G; e; m, i)$. There is a unique sort G with a constant symbol e. The function symbol m has arity $(G, G; G)$ and i has arity $(G; G)$. There are no relation symbols.

An L_{grp}-structure is hence a set $G^{\mathcal{M}}$ with a constant $e^{\mathcal{M}} \in G^{\mathcal{M}}$ and functions $m^{\mathcal{M}} \colon G^{\mathcal{M}} \times G^{\mathcal{M}} \to G^{\mathcal{M}}$ and $i \colon G^{\mathcal{M}} \to G^{\mathcal{M}}$. For instance, every group $(H; 1; \cdot, (-)^{-1})$ defines an L_{grp} structure. However, not every L_{grp}-structure is a group. For instance, there is an L_{grp}-structure given by $(\mathbb{N}; 0; m^{\mathbb{N}}, i^{\mathbb{N}})$ where $m^{\mathbb{N}}$ and $i^{\mathbb{N}}$ are both constant functions valued at 1, which is clearly not a group.

We emphasise that the behaviour of the functions and relations is not determined by the language. The behaviour will be determined later on, when we introduce axioms and theories.

Example 2.2. The language of rings L_{ring} has signature $(R; 0, 1; +, -, \cdot)$. The function symbols $+$ and $\cdot$ both have arity $(R, R; R)$. When we only have a single sort like this, we will usually reduce the arity to a single positive integer, since this fully determines the arity. Hence, $+$ and $\cdot$ have an arity of 2, since they have domain R^2. Then, $-$ has arity 1.

Example 2.3. Recall that a left-module over a ring $(R; 0, 1; +, -, \cdot)$ is an Abelian group $(M; 0; +, -)$ with a left ring action of R on M, often called **scalar multiplication**. That is, there is a function $* \colon R \times M \to M$ such that:

1. For all $r \in R$, $*(r, 0) = 0$.
2. For all $m \in M$, $*(1, m) = m$
3. For all $m \in M$, $*(0, m) = 0$.

4. For all $r, s \in R$ and $m \in M$, $*(r, *(s, m)) = *(r \cdot s, m)$
5. For all $r, s \in R$ and $m \in M$, $*(r + s, m) = *(r, m) + *(s, m)$
6. For all $r \in R$ and $m, n \in M$, $*(r, m + n) = *(r, m) + *(r, n)$

Since it is usually clear which are ring elements and which are module elements, we write the action $*(r, m) = rm$.

Initially, it might seem that we want to use a two-sorted langage to describe modules, with one sort for the ring and one for the module, with the action being a function symbol of arity $(R, M; M)$. It will turn out, however, that there are weaknesses to our logic's descriptive power, and that such a language (and corresponding axioms) could not capture just the left-modules over R but instead would capture all left-modules over a class of rings containing R.

The language of left-R-modules is hence defined in an alternative, more maximalist way. We let $L_{R\text{-}\mathbf{Mod}}$ be the langage with signature $(M; 0; +, -, (*_r)_{r \in R})$. As with groups, $+$ has arity 2 and $-$ has arity 1. There is then a function symbol $*_r$ of arity 1 for *every* element r of R. This is intended to be interpreted as the function $*(r, -)\colon M \to M$.

So far we have only seen one-sorted languages. We shall enconunter some very non-trivially multi-sorted languages later in the project, after more theory of modules has been developed. The following are two examples of genuinely multi-sorted languages.

Example 2.4. A quiver is a directed multigraph. That is, a quiver Q consists of a set of vertices V and a set of edges E where each edge $e \in E$ has a source vertex $s(e) \in V$ and a target vertex $t(e) \in V$.

The language of quivers L_{quiv} has signature $(V, E; s, t)$ where s and t both have arity $(E; V)$.

Example 2.5. Given a quiver Q and a ring R, an R-representation of Q consists of an R-module M_v for every vertex v and for every edge e, a homomorphism $T_e\colon M_{s(e)} \to M_{t(e)}$.

The language of R-representations of Q has signature
$((M_v)_{v \in V}; (0_v)_{v \in V}; (+_v)_{v \in V}, (-_v)_{v \in V}, (*_{r,v})_{r \in R, v \in V}, (T_e)_{e \in E})$. Each $+_v$ has arity $(M_v, M_v; M_v)$, each $-_v$ and $*_{r,v}$ has arity $(M_v; M_v)$ and each T_e has arity $(M_{s(e)}; M_{t(e)})$. We can view this as a V-indexed disjoint union of many copies of the language of R-modules, plus the homomorphisms (T_e).

The full language - that is, set of formulae - generated by a signature is defined inductively. In addition to the logical and non-logical symbols we have already seen, we also draw from an infinite supply of variables. These can be any symbol not already included among the symbols defined, but is usually a lower case Latin letter, often x, y, z. Each variable has an associated sort, called its **context**. We write $x\colon A$ to say that x is a variable of context A. If we have a tuple of variables $(x_1, \dots, x_n)$ we will shorten this to $\overline{x}$.

We now define the **terms** of the language. These are not formulae, which are intended to say things about the language, but rather stand-ins for elements. They consist of variables, constants and applications of function symbols to other terms. When interpreted in a structure, each term becomes a specific element.

Definition 2.6. We define the terms inductively. The basic terms, T_0 are:

1. Any constant $c \in \mathcal{C}$. The context of c is its arity.
2. Any variable x.

We define T_{n+1} as the union of T_n with applications of functions symbols to terms in T_n. Specifically, given a function symbol $f \in \mathcal{F}$ of arity $(A_1, \dots, A_n; B, m)$ and terms $t_1, \dots, t_n$ such that the context of t_i is A_i, we there is a compound term $f(t_1, \dots, t_n)$ in T_{n+1} which has context m many copies of B, $(B, \dots, B)$.

We then define $\mathrm{Term}_L = \bigcup_{i=0}^{\infty} T_i$.

As mentioned before, we want to interpret our terms in the language. Consider L-structure $\mathcal{M}$ and a term $t(x_1, \dots, x_n)$ containing variables x_i each of arity A_i respectively. Given a choice of elements $(a_i \in A_i^{\mathcal{M}})$, an **interpretation** $t^{\mathcal{M}}(a_1, \dots, a_n)$ of t in M is defined inductively:

1. If t is a constant c, then $t^{\mathcal{M}}(\overline{a}) = c^{\mathcal{M}}$.
2. If t is the variable x_k, then $t^{\mathcal{M}}(\overline{a}) = a_k$.
3. If t is a compound term $f(t_1(\overline{x}), \dots, t_m(\overline{x}))$, then $t^{\mathcal{M}}(\overline{a}) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(\overline{a}), \dots, t_m^{\mathcal{M}}(\overline{a}))$.

We now define the formulae inductively.

Definition 2.7. The **atomic formulae**, L_0 are given by:

1. If t_1, t_2 are terms with the same context, then $t_1 \doteq t_2$ is an atomic formula.
2. If R is a relation symbol of arity $(A_1, \dots, A_n)$ and $t_1, \dots, t_n$ are terms of context $A_1, \dots, A_n$ respectively, then $R(t_1, \dots, t_n)$ is an atomic formula.

3. The always-true formula $\top$ is atomic.

4. The always-false formula $\bot$ is atomic.

Given L_n we can then define $(L_n)^c$ by adding in the connectives. That is, if $\phi(\overline{x})$ and $\psi(\overline{y})$ are formulae in L_n with (not necessarily distinct) variables $\overline{x}$ and $\overline{y}$ respectively, then:

1. The conjunction $\phi(\overline{x}) \wedge \psi(\overline{y})$ is in $(L_n)^c$.

2. The disjunction $\phi(\overline{x}) \vee \psi(\overline{y})$ is in $(L_n)^c$.

3. The negation $\neg\phi(\overline{x})$ is in $(L_n)^c$.

4. The implication $\phi(\overline{x}) \to \psi(\overline{x})$ is in $(L_n)^c$.

We also define $(L_n)^q$ by adding in the quantifiers. Let $\phi(x, \overline{y})$ be a formula where x has context A. Then,

1. The existential quantification $\exists(x\colon A).\phi(x, \overline{y})$ is a formula in $(L_n)^q$.

2. The universal quantification $\forall(x\colon A).\phi(x, \overline{y})$ is a formula in $(L_n)^q$.

If a variable appears after a quantifier it is called **bound**. Else, it is called **free**. A **sentence** is a formula that has no free variables.

We then let $L_{n+1} = L_n \cup (L_n)^c \cup (L_n)^q$. The language is then $L = \bigcup_{i=0}^{\infty} L_i$.

Now that we have formulae, we can start using the language to say things about our structures. We then use this to write axioms- things that we demand be true of a structure - and consider the collection of structures that satisfy them.

To do this, we need to define the interpretation of a formula in a structure. This becomes a little philosophically abstract. We need to assume some notion of mathematical objects and mathematical truths outside our logic. We have already ambiently been doing this, with our discussion of the sets of a structure, but now that we start talking about truth "in a structure" it can become a little muddy. The way we will address this is to treat the Model Theory like any other branch of mathematics, with the notion of truth being whatever the reader's preference is, whether that be platonist, formalist or something else. We then have a "model-theoretic truth", given by interpretations, which is an *entirely* mathematical construct.

Definition 2.8. Let L be a language of signature σ and let $\mathcal{M}$ be an L-structure. We define inductively the interpretation of formula. This is written in terms of

entailment. For a formula $\phi(\overline{x})$ and a tuple $\overline{a}$ in $\mathcal{M}$, we read $\mathcal{M} \vDash \phi(\overline{a})$ as "$\mathcal{M}$ entails $\phi(\overline{a})$". If $\mathcal{M}$ does not entail $\phi(\overline{a})$, we write $\mathcal{M} \nvDash \phi(\overline{a})$.

For the atomic formulae, we interpret as follows:

1. Let $t_1(\overline{x})$ and $t_2(\overline{y})$ be terms and $\overline{a}$ and $\overline{b}$ tuples in $\mathcal{M}$ matching the context of $\overline{x}$ and $\overline{y}$. We say that $\mathcal{M} \vDash t_1(\overline{a}) \doteq t_2(\overline{b})$ if and only if $t_1^{\mathcal{M}}(\overline{a}) = t_2^{\mathcal{M}}(\overline{b})$.
2. Let R a relation of arity $(A_1, \dots, A_n)$, $t_1(\overline{x}_1), \dots t_n(\overline{x}_n)$ terms of contexts A_1 to A_n respectively and $\overline{a}_1, \dots, \overline{a}_n$ be tuples in $\mathcal{M}$ matching the contexts of $\overline{x}_1$ to $\overline{x}_n$ respectively. Then $M \vDash R(t_1(\overline{a}_n), \dots, t_n(\overline{a}_n))$ if and only if $(t^{\mathcal{M}}(\overline{a}_1), \dots, t_n^{\mathcal{M}}(\overline{a}_n)) \in R^{\mathcal{M}}$.
3. It is always the case that $\mathcal{M} \vDash \top$.
4. It is always the case that $\mathcal{M} \nvDash \bot$

Now, let $\phi(\overline{x})$ and $\psi(\overline{y})$ be formulae and $\overline{a}$ and $\overline{b}$ be tuples in $\mathcal{M}$ matching the contexts of $\overline{x}$ and $\overline{y}$ respectively. We interpret the connective formulae as follows:

1. $\mathcal{M} \vDash \phi(\overline{a}) \wedge \psi(\overline{b})$ if and only if $\mathcal{M} \vDash \phi(\overline{a})$ and $\mathcal{M} \vDash \psi(\overline{b})$.
2. $\mathcal{M} \vDash \phi(\overline{a}) \vee \psi(\overline{b})$ if and only if $\mathcal{M} \vDash \phi(\overline{a})$ or $\mathcal{M} \vDash \psi(\overline{b})$.
3. $\mathcal{M} \vDash \neg\phi(\overline{a})$ if and only if $\mathcal{M} \nvDash \phi(\overline{a})$.
4. $\mathcal{M} \vDash \phi(\overline{a}) \to \psi(\overline{b})$ if and only if $\mathcal{M} \vDash \neg(\phi(\overline{a}) \wedge \neg\psi(\overline{b}))$.

Let $\phi(x, \overline{y})$ be a formula where x has context A and let $\overline{b}$ a tuple in $\mathcal{M}$ of context matching $\overline{y}$. We interpret the quantifiers as follows:

1. $\mathcal{M} \vDash \exists(x\colon A).\phi(x, \overline{b})$ if and only if there is some $a \in A^{\mathcal{M}}$ such that $\mathcal{M} \vDash \phi(a, \overline{b})$.
2. $\mathcal{M} \vDash \forall(x\colon A).\phi(x, \overline{b})$ if and only if for all $a \in A^{\mathcal{M}}$, we have that $\mathcal{M} \vDash \phi(a, \overline{b})$.

Example 2.9. Recall the group $\mathcal{H} = (H; 1; \cdot, (-)^{-1})$ that we saw was an L_{grp}-structure. Consider the formula $\phi = \forall x.m(x, i(x)) \doteq e$. Note, this has no free variables, so it is in fact a sentence.

Now, let $h \in H$ be an element. Since H is a group, we know that $h \cdot h^{-1} = 1$ by definition of the inverse. Hence it follows that $\mathcal{H} \vDash \phi$.

We can capture the other axioms of the groups as formulae in L_{grp} in the same way. The kind of logic we are working in is called **classical first-order** logic. Not every property can be expressed in this logic, though proving this is non-trivial from the definitions. A property that *can* be expressed in the logic is called a first-order or **elementary** property. All the group axioms are elementary.

A set of elementary sentences in a language L is called an L-**theory**. For instance, the set of group axioms is an L_{grp}-theory, usually referred to as T_{grp} or the theory of groups. Theories are the tool that let us talk about collections of structures that satisfy the same elementary properties. This is captured by the notion of a class of models.

Definition 2.10. Let T be an L-theory and $\mathcal{M}$ an L-structure. We say that $\mathcal{M}$ is a **model** of T, alternately a T-model, if for every sentence $\sigma \in T$, we have that $\mathcal{M} \vDash \sigma$.

Example 2.11. Every group, viewed as an L_{grp} structure, is a model of T_{grp}. By comparison, the structure on $\mathbb{N}$ given before is not a model of T_{grp}. In fact, the only models of T_{grp} are the groups as we usually understand them.

The notion of models lets us talk about what it means for a sentence to entail other sentences, rather than only having structures entail sentences. The idea here is that what a theory can say about other sentences is entirely determined by what the models of the theory say.

Definition 2.12. Let T be an L-theory and σ a sentence in L. We say that T **entails** σ, written $T \vDash \sigma$, if, whenever M is a model of T, we have that $M \vDash \sigma$.

Let T be a theory. We will define several theories that are related to T.

We say that T is **deductively closed** if, for every $\sigma \in L$, if $T \vDash \sigma$ then $\sigma \in T$. There is always a unique **deductive closure** of T, written $\overline{T}$ consisting of all such sentences σ. We will often work exclusively with deductively closed theories, since every theory has the same class of models as its deductive closure.

A similar but distinct notion is that of completeness. We say that T is **complete** if for every sentence $\sigma \in L$, either $T \vDash \sigma$ or $T \vDash \neg\sigma$. Note that for structures, it was the case that $\mathcal{M} \vDash \neg\sigma$ if and only if $\mathcal{M} \nvDash \sigma$. This is not the case for theories, since a theory T can have one model entailing σ and another entailing its negation. For example, the sentence $\phi = \exists x.\forall y.(x \doteq y)$ forces there to be only a single element in any model entailing it. There is a one-element group, but there also many-element groups. Hence, T_{grp} models neither ϕ nor its negation. It is an incomplete theory.

A completion of T is a complete theory T' such that $T \subseteq T'$. Theories can have many distinct completions, depending on what sentences we choose to adjoin to it. This choice restricts the class of models down. We often wish to work with complete theories but, unlike deductive closure, this is a meaningful choice.

Finally, we have the notion of an axiomatisation. Let $T_0 \subseteq T$ be a sub-theory of T. We say that T_0 is an **axiomatisation** of T if they have the same deductive closure. Note that T is always an axiomatisation of itself. This is most useful when we apply some additional restrictions on T_0. For instance, if T_0 is finite then we obtain a **finite axiomatisation**. If there is an algorithm that can list out all the axioms of T_0, we call it a **recursive** or **effective axiomatisation**.

3 Additivity

We recall some basic definitions and facts from Homological Algebra, taken from Chapter 1 of [Bor94]. These will not be comprehensive for the pursuit of Homological Algebra, only the tools necessary for working with the specific modules and functors we cover in this project.

3.1 Abelian Categories

Here, we define a generalisation of categories of modules that allow us to use tools such as quotients and exact sequences in other settings. This definition is originally due to Grothendieck, to provide a setting for homological algebra. We will use the category of Abelian groups throughout to contextualise these definitions- it is the ur-example and namesake of Abelian categories. Let $\mathbf{C}$ be a category.

Before we can define Abelian categories, we need to define a few important (co)limits whose existence will be an axiom for Abelian categories. The first, the zero object, is a generalisation of the zero group in $\mathbf{Ab}$.

Definition 3.1. We say that $\mathbf{C}$ has a **zero object** if there is an object $\mathbf{0}$ which is both initial and terminal.

Example 3.2. In $\mathbf{Ab}$, there is a unique morphism from every group into the zero group, namely that which sends all elements to the unique element of the zero group. There is a unique morphism out of the zero group into any group, since every morphism must send the unique element of the zero group to the identity of the target group.

Remark 3.3. Let X and Y be objects. Then there is a unique map $X \to \mathbf{0}$ since $\mathbf{0}$ is terminal and there is a unique map $\mathbf{0} \to Y$ since it is initial. We can compose these to obtain a map $0\colon X \to Y$ which we call the **zero map**.

Example 3.4. In $\mathbf{Ab}$, this is the morphism $X \to Y$ that sends every element of X to the identity in Y.

The next notion is that of kernels and cokernels, which generalise, in terms of morphisms, the kernel and cokernel of a group homomorphism.

Definition 3.5. Given a morphism $f\colon X \to Y$, a **kernel** of f is an equalizer of f and 0. If a kernel exists, it is unique up to unique isomorphism and is written $\ker(f)$. A **cokernel** of f is a coequalizer of f and 0. This is written $\operatorname{coker}(f)$.

Example 3.6. The kernel of a morphism $T\colon G \to H$ in **Ab** is the inclusion map of the kernel subgroup into G. The cokernel is the quotient of H by the image.

Kernels and cokernels are important tools for working in Abelian categories. Many properties are established via taking combinations of (co)kernels to show existence of universal morphisms for other (co)limits. To this end, we lay out some of their basic properties that will come up in later proofs. Notice that the cokernel and kernel of a given morphism are dual, and hence for each proposition will only prove one form but keeping in mind that an appropriate dual arises immediately.

First, some interactions of (co)kernels with monomorphisms and epimorphisms.

Proposition 3.7. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms,* $\operatorname{coker}(g)$ *exists and f is epimorphic, then* $\operatorname{coker}(g \circ f)$ *exists and is equal to* $\operatorname{coker}(g)$*.*

Proof. First, $\operatorname{coker}(g) \circ g \circ f = 0 \circ f = 0$, so $\operatorname{coker}(g)$ is a cocone for the coequalizer. Then, suppose there is q such that $q \circ g \circ f = 0$. Then, since f is epimorphic, $q \circ g = 0$ and q factors uniquely through $\operatorname{coker}(g)$. This shows that $\operatorname{coker}(g)$ is in fact colimiting and hence the coequalizer. □

Proposition 3.8 ([Lei16] Exercise 5.2.25)**.** *$f\colon X \to Y$ be a morphism whose kernel exists. Then* $\ker(f)\colon K \to X$ *is monomorphic.*

Proof. Let $g, h\colon W \to K$ be such that $\ker(f) \circ g = \ker(f) \circ h$. Then, in particular, $f \circ \ker(f) \circ g = 0$ and so $\ker(f) \circ g$ is a cone for the kernel of f. By universality of $\ker(f)$, there is a unique morphism g' such that $\ker(f) \circ g' = \ker(f) \circ g$. But of course, this g' must be g itself. However, we also have that $\ker(f) \circ h = \ker(f) \circ g$. Hence $h = g' = g$. Thus $\ker(f)$ is monic. □

Proposition 3.9 ([Bor94] Proposition 1.1.7)**.** *Let $f\colon X \to Y$ be a monomorphism. Then, its kernel exists and is the zero map $0 \to X$.*

Proof. Suppose f is monomorphic. Then, by definition of the kernel, $f \circ \ker(f) = 0 = f \circ 0$. By monicity, $\ker(f) = 0$. □

We then have an important interaction between kernels and the zero morphism.

Proposition 3.10 ([Bor94] Proposition 1.18)**.** *Consider the zero map $0\colon X \to Y$. Then,* $\ker(0) = \mathrm{id}_X$*.*

Proof. Certainly $0 \circ \mathrm{id}_X = 0$. Now, let $f\colon Z \to X$ be such that $0 \circ f = 0$ (this is, of course, every such morphism). This factors through id_X uniquely as $\mathrm{id}_X \circ f$ by definition of the identity. $\square$

We now move onto the definition of Abelian categories itself. We note that this is not the original definition given by Grothendieck, but rather an equivalent definition formulated later on that emphasizes the categorical structure of Abelianity (in contrast to the algebraic structure).

Definition 3.11. We say that $\mathbf{C}$ is Abelian if:

- $\mathbf{C}$ has a zero object.
- $\mathbf{C}$ has all finite products and coproducts.
- $\mathbf{C}$ has all kernels and cokernels.
- Every monomorphism is a kernel.
- Every epipmorphism is a cokernel.

Remark 3.12. For each condition in the definition, we also include its dual. Therefore, if $\mathbf{C}$ is Abelian, so is $\mathbf{C}^{\mathrm{op}}$. This allows us to shorten several proofs, as we did before for kernels and cokernels.

Example 3.13. As stated earlier, the eponymous example of an Abelian category is $\mathbf{Ab}$, the category of Abelian groups. We have already seen its zero object and (co)kernels. Its finite products and coproducts coincide as the direct sums of groups.

Monomorphisms (resp. epimorphisms) correspond exactly to injective (resp. surjective) maps in $\mathbf{Ab}$. Every injective map $f\colon M \to G$ can be identified with a subgroup inclusion of $f(M) \leq G$. f is then the kernel of the quotient map $G \to G/M$. Given a surjective map $g\colon G \to H$, there is an isomorphism $G/\ker(g) \to H$. g is then the cokernel of the inclusion $\ker(g) \hookrightarrow G$. Both of these facts follow from the first isomorphism theorem. In fact, this pair of conditions is asserting that the first isomorphism theorem is true in Abelian categories.

Example 3.14. Given a ring R, the category of left (or right) modules over R is an Abelian category. This is more or less identical to the case for Abelian groups. Indeed, the Abelian groups case can be thought of as a module category over $\mathbb{Z}$.

Later on, we will see that we can formulate the notion of modules over more general objects than rings. The categories of modules here will be Abelian categories and it will be useful to us to generalise results from the familiar case of categories of modules over rings to all Abelian categories so as to be able to apply them there.

We can prove several more useful facts about (co)kernels in Abelian categories that do not hold in the general case. We lay these out here, starting with the notions of image and coimage. These can be defined in general categories, but become particularly useful in the Abelian setting. We omit several proofs as they require significant detours into the details of Abelian categories that are beyond the scope of this project.

Definition 3.15. Let $f\colon X \to Y$ be a morphism. If the kernel of f exists then a coimage of f is a cokernel of $\ker(f)$. Similarly, if the cokernel of f exists, then an image of f is a kernel of $\operatorname{coker}(f)$.

There is a canonical relationship between the coimage and image of a given morphism $f\colon X \to Y$. Suppose f has both coimage $\operatorname{coim}(f)\colon X \to C$ and image $\operatorname{im}(f)\colon I \to Y$. Since $\operatorname{coker}(f) \circ f = 0$, we find that f factors through $\operatorname{im}(f) = \ker(\operatorname{coker}(f))$ as $f = \operatorname{im}(f) \circ h$. Then, we can consider $0 = f \circ \ker(f) = \operatorname{im}(f) \circ h \circ \ker(f)$. Now, $\operatorname{im}(f)$ is a kernel and hence monic, so we can cancel it on the left to obtain $h \circ \ker(f) = 0$. Thus, h must factor through $\operatorname{coim}(f)$ as $h = f^* \circ \operatorname{coim}(f)$ where f^* is a morphism $C \to I$.

A fundamental tool of Abelian categories is the **image-coimage factorisation** of a morphism. This says that f^* is an isomorphism. It follows that $f^* \circ \operatorname{coim}(f)$ is also a coimage for f. Abusing notation slightly, we will write this coimage as $\operatorname{coim}(f)\colon X \to I$, the canonical coimage for f. It follows that $f = \operatorname{im}(f) \circ \operatorname{coim}(f)$.

We can also think of this as a statement of the first isomorphism theorem. If we take our category to be **Ab**, then the object C is $X/\ker(f)$ and I is the image of f as a subset of Y. Then f^* being an isomorphism says that $X/\ker(f) \cong Y$. This interpretation follows in any Abelian category whose objects are sets with additional structure. The image-coimage factorisation can be seen as identifying the quotient with the image.

Proposition 3.16 ([Bor94] Proposition 1.5.5)**.** *Let* $\mathbf{C}$ *be an Abelian category and* $f\colon X \to Y$ *a morphism. Then* f^* *is an isomorphism.*

We obtain from this a useful corollary for the cases where f is monic or epic.

This is useful when working with short exact sequences (which we define later).

Corollary 3.17 ([Bor94] Proposition 1.5.7)**.** *If f is monic then $f = \mathrm{im}(f)$. If f is epic then $f = \mathrm{coim}(f)$.*

Proof. Recall that if f is monic, $\ker(f) = 0\colon 0 \to X$ and that $\mathrm{coker}(0) = \mathrm{id}_X$. Thus, $\mathrm{coim}(f) = \mathrm{id}_X$ We apply the image-coimage factorisation to obtain $f = \mathrm{im}(f) \circ \mathrm{coim}(f) = \mathrm{im}(f) \circ \mathrm{id}_X = \mathrm{im}(f)$.

The epi case follows by dualising. ◻

We can further relate monomorphisms and epimorphisms to corresponding (co)kernels. In particular, we obtain a converse to Proposition 3.9 and of course its dual.

Proposition 3.18 ([Bor94] Proposition 1.5.4)**.** *Let f be a morphism such that $\ker(f) = 0$. Then f is a monomorphism.*

From Proposition 1.5.3 of [Bor94], we also see that Abelian categories have all (co)equalizers and hence, combining this with finite biproducts and Theorem 1.44, that all Abelian categories are **finitely (co)complete**, meaning they have all finite limits and colimits.

A feature of limits in Abelian categories that we will make use of later is the following, with slogan "epimorphisms are stable under pullback".

Proposition 3.19 ([Bor94] Proposition 1.7.6)**.** *Let $f\colon X \to Y$ be an epimorphism and $g\colon Z \to Y$ be any morphism. Consider their pullback (P, p_x, p_z). Then, p_z is an epimorphism.*

Similarly, we see that kernels are, in a slightly different sense, preserved by taking pullbacks.

Proposition 3.20 ([Mit65] Chapter 1 Proposition 13.1)**.** *Let $f\colon X \to Z$ and $g\colon Y \to Z$ be morphism with pullback (P, p_X, p_Y). Let $k\colon K \to Y$ be the kernel of g. Let c be the unique morphism from K to P such that $p_Y \circ c = k$ and $p_X \circ c = 0$. Then, c is a kernel of p_X.*

$$
\begin{tikzcd}
K \arrow[r, "c"] \arrow[d, equal] & P \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \arrow[d, "f"] \\
K \arrow[r, "k"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}
$$

Proof. Consider a morphism $u\colon W \to P$ such that $p_X \circ u = 0$. Then, $f \circ p_X \circ u = 0$ and so $g \circ p_Y \circ u = 0$. The kernel of g is k and thus we have a unique factorisation of $p_Y \circ u$ through k as $p_Y \circ u = k \circ v$.

We see that $(W, p_X \circ u, p_Y \circ u)$ is a cone for the pullback diagram and so u is the unique morphism such that $p_X \circ u = p_X \circ u$ and similarly for Y. We compute $p_X \circ c \circ v = 0 = p_X \circ u$ and $p_Y \circ c \circ v = k \circ v = p_Y \circ u$. Thus, by uniqueness of u, we obtain that $u = c \circ v$. This is a unique factorisation of u through v.

Since $p_X \circ c = 0$ by definition of c, we obtain that c is a kernel of p_X. □

One of the most important features of Abelian categories is that we can derive an Abelian group structure on their Hom sets. This allows us to import results from commutative algebra to prove facts about the general categories. This is formulated as the following property.

Definition 3.21. We say that $\mathbf{C}$ is preadditive if for every pair of objects X, Y, there is an operation $+_{X,Y}\colon \mathrm{Hom}(X,Y) \times \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Y)$ that makes $\mathrm{Hom}(X,Y)$ an Abelian group. Additionally, composition must be a bilinear group homomorphism.

Proposition 3.22 ([Bor94] Theorem 1.6.4)**.** *Every Abelian category is preadditive.*

Definition 3.23. For preadditive categories $\mathbf{C}$ and $\mathbf{D}$, a functor $F\colon \mathbf{C} \to \mathbf{D}$ is called an **additive functor** if for every pair of objects X, Y, the component $F\colon \mathrm{Hom}_{\mathbf{C}}(X,Y) \to \mathrm{Hom}_{\mathbf{D}}(FX, FY)$ is a group homomorphism.

Example 3.24. If $\mathbf{C}$ is a preadditive category then we can write the Hom functors as being targeted in $\mathbf{Ab}$, since $\mathrm{Hom}(A,B)$ is an Abelian group. Then, the functors $\mathrm{Hom}(A,-)$ are additive for each A.

Example 3.25. We reuse the notation よ and よ^{c} for the additive versions of the Yoneda embeddings. The Yoneda lemma still holds, with almost identical proof save for showing that our natural transformations from the proof are indeed group homomorphisms as well as functions. The Yoneda embeddings are hence additive functors. See Proposition 1.3.7 of [Bor94] Vol 2. Ch1. for full details.

Remark 3.26. Many of the facts about functor categories that we proved in Section 1 still apply when we restrict attention to the category of all additive functors between two preadditive categories, which we will properly denote by $\mathrm{Add}(\mathbf{C}, \mathbf{D})$ but will write as $[\mathbf{C}, \mathbf{D}]$ when it is clear what we mean. Propositions 1.48 and 1.50 still hold, the proofs being almost identical. In particular, we note that the (co)limit of a diagram of additive functors is still additive and so $\mathrm{Add}(\mathbf{C}, \mathbf{D})$ has all the (co)limits of $\mathbf{D}$.

Extending the case of Abelian groups, finite products and coproducts in fact coincide in a preadditive category. These together form the notion of a biproduct, which is both a product and coproduct and has compatibility between the structures.

Definition 3.27. Suppose $\mathbf{C}$ has a zero object. Given two objects X and Y, the data of a biproduct is an object $X \oplus Y$ with morphisms $\pi_X \colon X \oplus Y \to X$, $\pi_Y \colon X \oplus Y \to Y$, $p_X \colon X \to X \oplus Y$ and $p_Y \colon Y \to X \oplus Y$. These satisfy:

- $(X \oplus Y, \pi_X, \pi_Y)$ is a product.
- $(X \oplus Y, p_X, p_Y)$ is a coproduct.
- $\pi_X \circ p_X = \mathrm{id}_X$ and $\pi_Y \circ p_Y = \mathrm{id}_Y$.
- $\pi_X \circ p_Y = 0$ and $\pi_Y \circ p_X = 0$.

Proposition 3.28 ([Bor94] Proposition 1.2.4)**.** *Let $\mathbf{C}$ be preadditive and X, Y, Z be objects. Then the following are equivalent.*

1. *Z is a product of X and Y.*
2. *Z is a coproduct of X and Y.*
3. *Z is a biproduct of X and Y.*

Proof. (3) implies (1) and (2) by definition. We shall prove (1) implies (3) and then the proof of (2) implies (3) will be immediate by dualising.

Suppose $Z = X \times Y$ is a product with projections π_X and π_Y. Now, we are working in a preadditive category, so there are zero morphisms between objects. We will define our coprojections using these. Specifically, consider the cone of X and Y given by $\mathrm{id}_X \colon X \to X$ and $0 \colon X \to Y$. By the universal property of the product, is a unique morphism $p_X \colon X \to X \times Y$ such that $\pi_X \circ p_X = \mathrm{id}_X$ and $\pi_Y \circ p_X = 0$. Similarly define p_Y. Hence, these morphisms satisfy the compatibility equations from the biproduct definition, so all we need to show is that $(X \times Y, p_X, p_Y)$ is a coproduct.

Let W be an object and $f \colon X \to W$, $g \colon Y \to W$ be morphisms. Our proposed unique morphism out of the product will be $h = (f \circ \pi_X) + (g \circ \pi_Y)$. Now, $h \circ p_X = (f \circ \pi_X \circ p_X) + (g \circ \pi_Y \circ p_X)$ by bilinarity of composition. We then reduce via $\pi_Y \circ p_X = 0$ and $\pi_x \circ p_X = \mathrm{id}_X$ to get $h \circ p_X = f$. Similarly, $h \circ p_Y = g$. So such a morphism exists.

Now, suppose h' is another such morphism. We use a decomposition of the identity on the product. In particular, $\pi_X \circ (p_X + p_Y) = \mathrm{id}_X$ and similarly for Y and hence, by universality of the product, we determine that $p_X + p_Y = \mathrm{id}_{X \times Y}$. Now, consider $(h - h') \circ (p_X + p_Y) = (h - h') \circ p_X + (h - h') \circ p_Y = 0 + 0 = 0$. Hence $0 = (h - h') \circ (p_X + p_Y) = (h - h') \circ \mathrm{id}_{X \times Y} = h - h'$. Thus $h = h'$ and so h is unique. This concludes the proof that $X \times Y$ is a coproduct. □

Remark 3.29. As with products, this generalises to finitely many objects, and we have that $p_{X_1} + \cdots + p_{X_n} = \mathrm{id}$ on $\bigoplus_{i=1}^n X_i$.

As implied by the pre- in preadditive, there is a notion of additive category which builds on top of the former.

Definition 3.30. A preadditive category is called **additive** if it has a zero object and all finite biproducts.

It is then immediate that every Abelian category is additive.

3.2 Exactness

Some useful tools that can be formulated inside an Abelian category is exact sequences and exact functors. These provide a nice language for talking about properties of modules, particularly finite presentability.

Definition 3.31. A sequence of objects $(C_i)_{i \in \mathbb{Z}}$ with morphisms $f_i \colon C_i \to C_{i+1}$ is said to form an **exact sequence** if, for all i, $\ker(f_{i+1}) = \mathrm{im}(f_i)$.

Definition 3.32. A **short exact sequence** is an exact sequence for which $C_i = 0$ if $i < 1$ or $i > 3$. These are often written diagramatically as:

$$0 \longrightarrow C_1 \xrightarrow{f} C_2 \xrightarrow{g} C_3 \longrightarrow 0 \ .$$

The sequence $0 \to C_1 \to C_2$ being exact says that f is monic, while $C_2 \to C_3 \to 0$ being exact says that g is epic. The whole sequence being exact says that f is the kernel of g and g is the cokernel of f.

As with everything, there are structure-preserving morphisms for exact sequences.

Definition 3.33. Let $F \colon \mathbf{C} \to \mathbf{D}$ be an additive functor between Abelian categories. For all short exact sequences $0 \to A \to B \to C \to 0$, we say that F is

- **left-exact** if $0 \to FA \to FB \to FC$ is exact.
- **right-exact** if $FA \to FB \to FC \to 0$ is exact.
- **exact** if it is both left and right exact.

We use exactness of the Hom functors to define projective objects, which form an important class in module categories. It is in fact always true that Hom is left-exact, see below, so we are only interested in the right-exactness of Hom functors for our definiton.

Proposition 3.34 ([Bor94] Proposition 1.11.2). *For an object M, the Hom functor $\mathrm{Hom}(M, -)$ is left-exact.*

Proof. Let $f\colon A \to B$ and $g\colon B \to C$ define a short exact sequence. We can compute exactness set-wise since we are working in **Ab**, so let $k \in \mathrm{im}(\mathrm{Hom}(M, f)$. That is, k is a morphism $M \to B$ and factors as $k = fh$ for some $h\colon M \to A$. Then, $\mathrm{Hom}(M, g)(k) = gk = gfh = 0$ since $gf = 0$. Thus, $k \in \ker(\mathrm{Hom}(M, g))$.

Now, suppose $k \in \ker(\mathrm{Hom}(M, g))$. That is, $gk = 0$. Then, k must factor through $\ker(g)$ by the universal property of the kernel, and $\ker(g) = \mathrm{im}(f)$ so we have that $k = \mathrm{im}(f)h$ for $h\colon M \to \mathrm{im}(f)$. But f is monic and hence $\mathrm{im}(f) = f$, so we have that $k = fh$, so $k \in \mathrm{im}(\mathrm{Hom}(M, f))$.

The above parts show that $\mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C)$ is exact. It remains to show that $0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B)$ is exact. This is equivalent to $\mathrm{Hom}(M, f)$ being monomorphic, which in **Ab** is equivalent to being injective. So, let $k, k' \in \mathrm{Hom}(M, A)$ such that $\mathrm{Hom}(M, f)(k) = \mathrm{Hom}(M, f)(k')$. Then, $fk = fk'$, but f is monomorphic an so $k = k'$. This establishes injectivity. $\square$

Projective modules are a generalisation of free modules that weaken the factorisation of morphisms that free modules induce. They exist when the Hom functor is not only left-exact but also right-exact.

Definition 3.35. We say that an object M is **projective** if $\mathrm{Hom}(M, -)$ is exact.

Projective objects are alternately characterised by a lifting property for epimorphisms.

Proposition 3.36. *Let P be an object. Then, P is projective if and only if for all objects M and N and morphisms $e\colon M \to N$ and $f\colon P \to N$ such that e is epimorphic, we have that f factors through e.*

Proof. Consider the exact sequence

$$0 \longrightarrow \mathrm{Ker}(e) \xrightarrow{\ker(e)} M \xrightarrow{e} N \longrightarrow 0$$

and the corresponding (not necessarily exact) chain

$$0 \longrightarrow \mathrm{Hom}(P, \mathrm{Ker}(e)) \xrightarrow{\mathrm{Hom}(P,\ker(e))} \mathrm{Hom}(P, M) \xrightarrow{\mathrm{Hom}(P,e)} \mathrm{Hom}(P, N) \longrightarrow 0.$$

Note that by Proposition 3.34, we know that this sequence is exact on the left. Thus, we only need to show that $\mathrm{Hom}(P, e)$ is epimorphic.

Now, assume that the factorisation of f exists. By definition, for every $u \in \mathrm{Hom}(P, M)$, $\mathrm{Hom}(P, e)(u) = e \circ p$. Further, there is some g with $f = e \circ g$. Hence, there is a $g \in \mathrm{Hom}(P, M)$ with $\mathrm{Hom}(P, e)(g) = f$. Since the epimorphic morphisms in **Ab** are precisely the surjective homs, this suffices to show that $\mathrm{Hom}(P, e)$ is epimorphic.

For the reverse direction, were $\mathrm{Hom}(P, -)$ exact, we would see that $\mathrm{Hom}(P, e)$ is epimorphic. Thus, there would be some g with $\mathrm{Hom}(P, e)(g) = f$ and hence $f = e \circ g$, which is the factorisation required. ☐

Projective objects are useful as they provide a splitting property for exact sequences which allows us to recover properties about objects from the properties of their quotients.

Proposition 3.37 ([Bor94] Proposition 1.8.7)**.** *Let M, N and P be objects with P projective. Suppose there are morphims $f : M \to N$ and $g : N \to P$ such that the following sequence is exact.*

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0 \ .$$

Then, there are morphisms $h: P \to N$ and $s: N \to M$ such that (N, s, g, f, h) is a biproduct of M and P.

Proof. We observe that id_P is a morphism onto P and g is an epimorphism onto P by exactness. By assumption, P is projective and we apply Proposition 3.36 to obtain a factorisation of id_P through g as $g \circ h = \mathrm{id}_P$.

To find s, we note from the properties of biproducts that any candidate for s should satisfy the condition $f \circ s + h \circ g = \mathrm{id}_N$. Thus, we consider the morphism $\mathrm{id}_N - h \circ g$. We observe that $g \circ (\mathrm{id}_N - h \circ g) = g - g \circ h \circ g = g - g = 0$.

Thus, $\mathrm{id}_N - h \circ g$ factors through f, since $f = \ker(g)$. We hence find that $\mathrm{id}_N - h \circ g = f \circ s$ for $s\colon N \to M$.

Now that we have the morphisms h and s, we need to show that we obtain a biproduct structure. We once again use the equation $f \circ s + h \circ g = \mathrm{id}_N$. Multiplying on the right by f, we find that $f \circ s \circ f + h \circ g \circ f = f$ and so $f \circ s \circ f = f$ since $g \circ f = 0$. By left-cancellativity of f (as it is a kernel), we find that $s \circ f = \mathrm{id}_M$.

Now, we multiply $f \circ s + h \circ g = \mathrm{id}_N$ by s on the left to get $s \circ f \circ s + s \circ h \circ g = s$. Since $s \circ f \circ s = s$, we get that $s \circ h \circ g = 0$. As g is a cokernel, it is right-cancellative and so we obtain $s \circ h = 0$.

Thus, we have shown that $s \circ f = \mathrm{id}_M, g \circ h = \mathrm{id}_N, g \circ f = 0$ and $s \circ h = 0$, which are all the compatibility equations required for the biproduct. It remains to show that (N, f, h) is a coproduct for M and P.

Let $u\colon M \to W$ and $v\colon P \to W$ be morphisms. Then there is a morphism $w = u \circ s + v \circ g\colon N \to W$ such that $(u \circ s + v \circ g) \circ h = v$ and $(u \circ s + v \circ g) \circ f = u$. To show uniqueness, suppose that $w'\colon N \to W$ is another such morphism. Then,

$$\begin{aligned} w - w' &= (w - w') \circ \mathrm{id}_N \\ &= (w - w') \circ (f \circ s + h \circ g) \\ &= (w \circ f \circ s) + (w \circ h \circ g) - (w' \circ f \circ s) - (w' \circ h \circ g) \\ &= u + v - u - v \\ &= 0 \end{aligned}$$

and hence $w = w'$. This shows that (N, f, h) is a coproduct of M and P and by Proposition 3.28 this is enough for (N, s, g, f, h) to be their biproduct. $\square$

4 Finitely Presented Modules

In this section, we begin our study of module categories. We introduce the notion of a many-object ring and discuss how to take modules over them. We then provide a characterisation of the finitely presented modules in the categories of modules over such rings. Further, we prove that the category of modules is locally finitely presentable, which allows us to draw techniques from the theory of locally presentable categories for further studying modules. We draw primarily from Chapter 10 and Appendix E of [Pre09]. The majority of the content there is given as Remarks or unproven Propositions which we fill out the details for. Thus, unless otherwise stated, proofs will be original.

4.1 Rings As Preadditive Categories

Recall that a preadditive category is a category for which every pair of objects A and B, there is an Abelian group structure on $\mathrm{Hom}(A, B)$ and composition is bilinear.

Definition 4.1. Let R be a ring. The **preadditive category associated with** R is a preaddive category with a single object, which we will write as $*$, whose endomorphism group $\mathrm{End}(*) = \mathrm{Hom}(*, *)$ is the additive group of R. We define composition in this category such that for $r, s \colon * \to *$ we have $r \circ s = r \cdot s$ (that is, multiplication in the ring). Bilinearity of this map is exactly given by the distributivity of multiplication in a ring over its addition.

In this way, we can associate a canonical preadditive category with every ring. Note that we can recover the ring from the category simply by defining multiplication in the ring as composition of morphisms. Hence there is a one to one correspondence of rings and one object preadditive categories. Therefore, when it is clear, we will identify rings with their corresponding preadditive categories.

In fact, this correspondence can be lifted to a real equivalence of categories. Consider two rings R and S and a ring morphism $\phi \colon R \to S$. We can build from ϕ an additive functor $F \colon R \to S$ (viewing the rings as preadditive categories) by mapping $*_R$ to $*_S$ and for a morphism $r \colon *_R \to *_R$, writing $Fr = \phi(r)$. That ϕ preserves multiplication and multiplicative identity is precisely functoriality of F, whilst ϕ preserving addition and additive identity is precisely additivity of F. Similarly, for every additive functor $R \to S$ we can recover a ring morphism. Hence, there is a full, faithful and essentially surjective functor from the category of rings to the category of one object preadditive categories following the correspondence of rings and morphisms to one object categories and additive functors.

We can play the same game for modules:

Definition 4.2. Let R be a ring and M a left-module over R. The data of M consists of an underlying Abelian group and a ring morphism $\phi\colon R \to \mathrm{End}(M)$. The **functor associated to** M is the additive functor $\overline{M}\colon R \to \mathbf{Ab}$ defined by $\overline{M}* = M$ and $\overline{M}r = \phi(r)$.

Remark 4.3. That $\overline{M}$ is an additive functor follows by the same reasoning that morphisms between rings correspond to additive functors between their corresponding categories. As with rings, we will in fact drop the bar over M and simply identify it with its functor.

Suppose we have two modules $M, N\colon R \to \mathbf{Ab}$. A natural transformation $\eta\colon M \to N$ is exactly the data of an Abelian group homomorphism $\eta_*\colon M* \to N*$ such that for every $r \in R$, $\eta_* \circ Mr = Nr \circ \eta_*$. This is precisely saying that η_* commutes with the ring action on the underlying Abelian group, and hence is in fact a module homomorphism. Thus, we obtain an equivalence of categories between R-**Mod** and the category of additive functors $R \to \mathbf{Ab}$, which we write as $(R, \mathbf{Ab})$.

The above covers left-modules, but is easily extended to right modules. Recall that for every ring R, there is a ring R^{op} with the same underlying group but multiplication redefined such that for $r, s \in R$, $r \cdot_{R^{\mathrm{op}}} s = s \cdot_R r$. The opposite notation is indicative here: the category associated to R^{op} is in fact the opposite category of the category associated to R, since we have simply reversed the composition of morphisms. Then, a right module over R is equivalent to a left module over R^{op}, and so a right module over R can be written as a contravariant additive functor $R \to \mathbf{Ab}$, or equivalently a covariant additive functor $R^{\mathrm{op}} \to \mathbf{Ab}$.

Now, so far this equivalence has not told us anything new. The important observation to make this useful is that we can extend this definition of a module from one object preadditive categories to arbitrary small preadditive categories.

Definition 4.4. Let R be a small preadditive category. A left-module over R is an additive functor $M\colon R \to \mathbf{Ab}$. A right-module is an additive functor $M\colon R^{\mathrm{op}} \to \mathbf{Ab}$.

Remark 4.5. To make this more explicit, M assigns to each object A of R an Abelian group, which in fact has the structure of a left $\mathrm{End}(A)$-module, and for each morphism $A \to B$ an Abelian group homomorphism $MA \to MB$.

Remark 4.6. We denote by R-**Mod** the category of left-modules over R and by **Mod**-R the category of right-modules, just as with rings.

Remark 4.7. Since R-**Mod** is a category of Additive functors into **Ab**, it has all limits and colimits that exist in **Ab**. Since Abelian categories are defined entirely in terms of what (co)limits exist, it follows that R-**Mod** is Abelian.

This extension of the notion of modules is inspired by the model-theoretic side of this project. Given a usual ring and module over it, we will want to define a kind of pp-elimination of imaginaries. This will mean extending the language for our modules by introducing new sorts for imaginaries. This can be viewed in a functorial way as extending the preadditive category of our ring by adding in new objects to represent the new sorts and new morphisms to represent various canonical projections and definable maps in the extended language. Our original module will then canonically extend to a module over the many object preadditive category with the additional sorts.

4.2 Finite Presentability

We recall the definition of a finitely presented object.

Definition 4.8. Let **C** be a preadditive category and A an object of **C**. We say that A is **finitely presented** if $\mathrm{Hom}(A, -)$ preserves directed colimits.

We gave this definition earlier for the Hom functors into **Set**. It turns out, however, that the underlying set of a directed colimit in **Ab** is in fact the directed colimit of the underlying sets. Hence, we can speak about finitely presented objects using the additive Hom functors - we get the same objects.

Lemma 4.9 ([AR94] Remark 3.4(4))**.** *The forgetful functor* $F\colon \mathbf{Ab} \to \mathbf{Set}$ *preserves directed colimits.*

In this section, we will collect some facts about the finitely presented objects in categories of modules over small preadditive categories. We denote the category of finitely presented modules as either $(R, \mathbf{Ab})^{\mathrm{fp}}$ or R-**mod**.

Usually, when we talk about finitely presented modules over a ring R, what we mean is a module M which is a quotient of a finitely generated free module by a finitely generated submodule. We can express this as saying that there are positive integers $n < m$ and morphisms $f\colon R^n \to R^m$ and $g\colon R^m \to M$ such that the sequence $R^n \xrightarrow{f} R^m \xrightarrow{g} M \longrightarrow 0$ is exact.

In the functorial interpretation of modules, the free modules correspond to representable functors. That is, first consider ordinary rings, so that R is a one-object category. The Yoneda Lemma tells us that there is an embedding $よ^{c}\colon R^{\mathrm{op}} \to R\text{-}\mathbf{Mod}$ sending $*$ to $よ^{c}* = \mathrm{Hom}(*, -)$. This corresponds to treating R as a left-module over itself: $よ^{c}*$ sends $*$ to $\mathrm{Hom}(*, *)$ which is the underlying Abelian group of R. As with usual modules, we will identify R with $よ*$. The free modules are then direct sums $\bigoplus_{i \in I} R$ over some index I.

When we move to the case of modules over small preadditive categories, we can obtain an analogous characterisation of finitely presented modules by replacing the free modules - that is, direct sums of the unique representable functor - with direct sums over the many different representable functors that we have.

Proposition 4.10 ([Pre09] Lemma E.1.43(ii))**.** *Let R be a small preadditive category and M be a left-module. Then M is finitely presented if and only if there are modules A and B which are finite direct sums of representable functors and an exact sequence $A \longrightarrow B \longrightarrow M \longrightarrow 0$.*

Proving this will take several steps, and along the way we will describe several more structural features of the category of modules. We start by introducing the following notion closesly related to finitely presentation.

Definition 4.11. We say that a left-module M is **finitely generated** if for every directed colimit of subobjects of M with $M = \mathrm{colim}_{i \in I} M_i$, we have that there is some $n \in I$ with $M \cong M_n$.

As is the case with modules over a ring, we find that finite generation is implied by finite presentation.

Proposition 4.12. *Let M be a finitely presented left-module over a small preadditive category R. Then M is finitely generated.*

Proof. Let (M_i) be some directed system with colimiting cocone (M, m_i) where m_i is monic for each i. Then, we can consider the identity function $\mathrm{id}_M\colon M \to M$. Since $\mathrm{Hom}(M, -)$ preserves directed colimits, we find that id_M factors through the system $\{M_i\}$. In particular, there is some n and $h\colon M \to M_n$ such that $\mathrm{id}_M = m_n \circ h$.

We show the monomorphim m_n to be the needed isomorphism. In particular, we show it is epic and conclude from the module category being balanced (as it is Abelian). Let $f, g\colon M_n \to A$ be morphisms such that $f \circ m_n = g \circ m_n$. Then, $f \circ m_n \circ h = g \circ m_n \circ h$, so $f \circ \mathrm{id}_M = g \circ \mathrm{id}_M$ and hence $f = g$. $\square$

The most simple finitely presented modules are the representable functors. This is analogous to the free modules over a ring, which we can think of as finitely generated and having only trivial relations. This is a corollary of a more general result about colimit preservation.

Proposition 4.13. *Let A be an object of R. Then* $\mathrm{Hom}(\text{よ}^{c}A, -)$ *preserves colimits.*

Proof. Let $D\colon I \to R$-**Mod** be a diagram and (L, η) its colimit. The category of left-modules is a functor category, so we know that colimits are calculated pointwise. That is, for each object X, $LX = \mathrm{colim}_{i \in I}(Di)X$ with the cocone given by $\eta_{i,X}\colon (Di)X \to LX$ (since each component of η is itself a natural transformation on a lower level). The Yoneda Lemma gives us a natural isomorphism $h_M\colon \mathrm{Hom}(\text{よ}^{c}A, M) \to MA$ for each module M.

Naturality of the isomorphism h says that for each $u\colon i \to j$ in I, the following commutes.

$$\begin{tikzcd}
(Di)(A) \arrow[r, "Du_A"] \arrow[d, "h_{Di}"'] & (Dj)(A) \arrow[d, "h_{Dj}"] \\
\mathrm{Hom}(\text{よ}^{c}A, Di) \arrow[r, "\mathrm{Hom}(\text{よ}^{c}A, Du)"'] & \mathrm{Hom}(\text{よ}^{c}A, Dj)
\end{tikzcd}$$

So, for every cocone (X, ε) of $\mathrm{Hom}(\text{よ}^{c}A, D-)$, there is a cocone $(X, \varepsilon \circ hD)$ of $(D-)A$. This is because for each $u\colon i \to j$, $\varepsilon_j \circ h_{Dj} \circ Du_A = \varepsilon_j \circ \mathrm{Hom}(\text{よ}^{c}A, Du) \circ h_{Di} = \varepsilon_i \circ h_{Di}$ by combining the naturality of h and ε. We conclude that there exists some unique $k\colon LA \to X$ such that $k \circ \eta = \varepsilon \circ h$. Since h is an isomorphism, we can exhibit L as a cocone of $\mathrm{Hom}(\text{よ}^{c}A, D-)$ by $\eta \circ h^{-1}$. It follows that k exhibits L as a colimit, since for every cocone of $\mathrm{Hom}(\text{よ}^{c}A, D-)$, we have that $k \circ \eta \circ h^{-1} = \varepsilon$.

Now, naturality also says that for each i, the following diagram commutes.

$$\begin{tikzcd}
(Di)(A) \arrow[r, "\eta_{i,A}"] \arrow[d, "h_{Di}"'] & LA \arrow[d, "h_L"] \\
\mathrm{Hom}(\text{よ}^{c}A, Di) \arrow[r, "\mathrm{Hom}(\text{よ}^{c}A, \eta_{i,A})"'] & \mathrm{Hom}(\text{よ}^{c}A, L)
\end{tikzcd}$$

$(\mathrm{Hom}(\text{よ}^{c}A, L), \mathrm{Hom}(\text{よ}^{c}A, \eta))$ is a cocone of $\mathrm{Hom}(\text{よ}^{c}A, D-)$ by functoriality of $\mathrm{Hom}(\text{よ}^{c}A, -)$. Therefore, there exists some unique $k\colon LA \to \mathrm{Hom}(\text{よ}^{c}A, L)$ such that $k \circ \eta = \mathrm{Hom}(\text{よ}^{c}A, \eta) \circ h$. But, naturality as described by the diagram above shows that $h_L = k$ satisfies this equation. Hence k is an isomorphism, and thus we conclude that $\mathrm{Hom}(\text{よ}^{\mathrm{c}}A, L)$ is a colimit of $\mathrm{Hom}(\text{よ}^{\mathrm{c}}A, D-)$. $\square$

Corollary 4.14 ([Pre09] Lemma 10.1.12)**.** *$\text{よ}^{\mathrm{c}}A$ is finitely presented and projective for each A.*

Proof. Finite presentation is equivalent to $\mathrm{Hom}(\text{よ}^{\mathrm{c}}A, -)$ preserving directed colimits and projectivity is equivalent to $\mathrm{Hom}(\text{よ}^{\mathrm{c}}A, -)$ preserving finite colimits, both of which are special cases of the proposition. $\square$

This gives us the trivial case of Proposition 4.10 by taking $A = B = M = \text{よ}^{\mathrm{c}}C$ for some object C in R and recalling that the cokernel of the zero map is the identity map. We can also easily establish the case for finite direct sums of representables.

Corollary 4.15. *Let J be a finite index set and A_j be objects of R. Then $\bigoplus_{j\in J} \text{よ}^{\mathrm{c}}A_j$ is finitely presented and projective.*

Proof. The Yoneda Lemma tells us that the functor $H\colon R\text{-}\mathbf{Mod}^{\mathrm{op}} \to (R\text{-}\mathbf{Mod}, \mathbf{Ab})$ acting as $M \mapsto \mathrm{Hom}(M, -)$ (we avoid use of よ here to avoid confusion) sends limits in R-**Mod** to colimits in $(R\text{-}\mathbf{Mod}, \mathbf{Ab})$. However, $(R\text{-}\mathbf{Mod}, \mathbf{Ab})$ and R-**Mod** are both Abelian and hence an object is a coproduct if and only if it is a product if and only if it is a biproduct. Thus H sends biproducts to biproducts and so $H(\bigoplus_{j\in J} \text{よ}^{\mathrm{c}}A_j) = \bigoplus_{j\in J} H(\text{よ}^{\mathrm{c}}A_j)$. That $\mathrm{Hom}(\bigoplus_{j\in J} \text{よ}^{\mathrm{c}}A_j, -)$ preserves colimits then follows from the fact that colimits commute with colimits. $\square$

In fact this exact proof shows a more general result:

Proposition 4.16. *A finite direct sum of finitely generated (resp. finitely presented, projective) objects is finitely generated (resp. finitely presented, projective).*

With these facts about represesentables, we can prove the backwards direction of Proposition 4.10. We will do this now, as from this backwards direction we will derive that R-**Mod** is a locally finitely presentable category. This will allow us to commute direct limits and finite limits, which is necessary for the proof of the forwards direction of 4.10.

The proposition we prove is in fact slightly stronger than needed, working for any exact sequence featuring finitely presented modules. The backwards direction of Proposition 4.10 follows from the proofs given above that direct sums of representables are finitely presented.

Proposition 4.17 ([Pre09] Lemma E.1.44)**.** *Let M be a module and A, B be finitely presented modules such that there are morphisms T and q such that this sequence is exact:*

$$A \xrightarrow{T} B \xrightarrow{q} M \longrightarrow 0$$

Then, M is finitely presented.

Proof. Suppose (L, η) is the colimit of a directed diagram $D\colon I \to R$-**Mod** and that there is a morphism $f\colon M \to L$. To show that M is finitely presented, we need to prove that f factors essentially uniquely through D.

To begin with, we have that $f \circ q$ is a morphism $B \to L$ and we know that B is finitely presented, thus $f \circ q$ must factor essentially uniquely through the diagram as $f \circ q = \eta_b \circ g$ for some $b \in I$ and $g\colon B \to Db$.

$$\begin{tikzcd} A \arrow[r, "T"] & B \arrow[r, "q"] \arrow[d, "g"'] & M \arrow[r] \arrow[d, "f"] & 0 \\ & Db \arrow[r, "\eta_b"'] & L & \end{tikzcd}$$

We now observe that $\eta_b \circ g \circ T = f \circ q \circ T = 0$ since $q \circ T = 0$ by exactness. Thus, $g \circ T$ and η_b are a factorisation of the morphism $0\colon A \to L$ through D. There is another such factorisation given by $\eta_b \circ 0$ for $0\colon A \to Db$. Essential uniqueness tells us that there is a $k \geq b$ such that $D(b \to k) \circ g \circ T = D(b \to k) \circ 0 = 0$.

Now, $\eta_k \circ D(b \to k) = \eta_b$, hence $D(b \to k) \circ g$ is a factorisation of $f \circ q$ through D. We can redraw our commutative diagram from before:

$$\begin{tikzcd} A \arrow[r, "T"] & B \arrow[r, "q"] \arrow[d, "D(b\to k)\circ g"'] & M \arrow[r] \arrow[d, "f"] & 0 \\ & Db \arrow[r, "\eta_k"'] & L & \end{tikzcd}$$

Now, since $D(b \to k) \circ g \circ T = 0$, it follows that $D(b \to k) \circ g$ factors through $\operatorname{coker}(T) = q$. Thus there is some $a\colon M \to Db$ such that $D(b \to k) \circ g = a \circ q$.

Some quick algebra gives us:

$$\begin{aligned} D(b \to k) \circ g &= a \circ q \\ \implies \eta_k \circ D(b \to k) \circ g &= \eta_k \circ a \circ q \\ \implies \qquad f \circ q &= \eta_k \circ a \circ q \\ \implies \qquad f &= \eta_k \circ a \end{aligned}$$

recalling that q is an epimorphism and so can be cancelled on the right. Thus we have a factorisation of f through D.

All that remains is to show that this factorisation is essentially unique. Hence, suppose that $f = \eta_{k'} \circ a'$. Precomposing with q again, we get $\eta_{k'} \circ a' \circ q = \eta_k \circ a \circ q = \eta_k \circ D(b \to k) \circ g$. Thus, $a' \circ q$ gives us another factorisation of $f \circ q$ through D and so there is some $n \geq k, k'$ such that $D(k' \to n) \circ a' \circ q = D(k \to n) \circ a \circ q$. Cancelling on the right by q again givs us $D(k' \to n) \circ a' = D(k \to n) \circ a$, which is the essential uniqueness of the factorisation of f. $\square$

In order to show that R-**Mod** is locally finitely presentable, we must also find an epimorphism from a direct sum of representables onto every module. The construction uncovers another useful analogy between the usual module case and the functorial view of modules, namely in their description of elements. Suppose we have a usual ring R and module M. Then, for every element $a \in M$, we can think about the map $T_a \colon R \to M$, viewing R as a module over itself, such that $T_a(1) = a$. For every element, we can derive such a morphism and for every such morphism we can look at $T(1)$ to find an element, hence they are in bijection. This is precisely what the Yoneda Lemma tells us: $\mathrm{Hom}(R, M) \cong M*$. We can then get a surjective homomorphism onto M from a sum of copies of the ring simply by considering every morphism determined by an element and using the coproduct universal property.

In the above, there was a bijection between morphisms out of R and elements of M. In a sense, we could have thought of the morphisms *as* the elements. In the preadditive case, we will embrace this fully. Treating R as a small preadditive category again, let A be an object and M a module. We then talk about the **A-elements** of M as the morphisms in $\mathrm{Hom}(よ^c A, M)$, which are determined by the Yoneda lemma. To get our epimorphism, we will consider the sum over every A-element for every object A.

Proposition 4.18 ([Pre09] Proposition 10.1.13)**.** *Let M be a left-module. Then there is an index set J, objects A_j in R and an epimorphism $\bigoplus_{j \in J} よ^c A_j \to M$.*

Proof. For each A in R, we can look at MA which is an Abelian group. Yoneda says that there is a natural isomorphism $h\colon \mathrm{Hom}(よ^{c}A, M) \to MA$. Hence, for each $x \in MA$, there is a morphism $h^{-1}(x)\colon よ^{c}A \to M$ such that $[h^{-1}(x)]_A(\mathrm{id}_A) = x$.

Then, we set $J = \bigsqcup_{A \in R} MA$ and A_j to be the object of R such that $j \in MA_j$. By universal property of the coproduct, there is a unique morphism $f\colon \bigoplus_{j \in J} よ^{c}A_j \to M$ such that for each $j \in J$, $f \circ \iota_j = h^{-1}(j)$ for ι_j the coprojection.

As f is a natural transformation, it is an epimorphism if each component is an epimorphism. Then, for each $A \in R$, f_A is an epi if and only if it is a surjection. Then, by definition, for each $x \in MA$, there is a map $h^{-1}(x)$ such that $(f \circ \iota_x)_A(\mathrm{id}_A) = (h^{-1}(x))_A(\mathrm{id}_A) = x$ and so f_A is surjective. □

In the usual module case, the way we would show a module is a directed colimit of finitely presented modules is via a family of modules indexed by tuples of elements and relations. Specifically, if $\overline{a}$ is a length m tuple of elements of a left-module M and $\overline{r}_1, \dots, \overline{r}_n$ is a collection of linear relations that $\overline{a}$ satisfies, then we construct a finitely presented module $M_{\overline{a}, \overline{r}_1, \dots, \overline{r}_n}$. Here, a linear relation is a linear polynomial such that $\overline{r}_1(\overline{a}) = 0$. We write $\overline{r}_i$ as a tuple of its coefficients, so that $\overline{r}_i = (r_{i1}, \dots, r_{im})$. We can combine these relations to get a matrix r, and then $r\overline{a} = (\overline{r}_1(\overline{a}), \dots, \overline{r}_n(\overline{a})) = \overline{0}$.

Because each R^k has both a left and right action of R, we can think of r as multiplying R^n on the right. This is a well-defined homomorphism of the left modules $R^n \to R^m$, since for every $s, r \in R$ and $\overline{x} \in R^n$, $s(\overline{r}x) = (s\overline{x})r$ simply by associativity of multiplication. Thus, we can think of a finite collection of relations as a homomorphism between finitely generated free modules.

We let $M_{\overline{a}, \overline{r}_1, \dots, \overline{r}_n}$ be the cokernel of this map. The module M is then a directed colimit of all such $M_{\overline{a}, \overline{r}_1, \dots, \overline{r}_n}$, indexed over all tuples of M and all finite collections of relations that they satisfy.

In the small preadditive case, we will think instead about all tuples of A-elements and the relations that they satisfy. Since a tuple corresponds to a finite collection of A_j-elements (indexed over some finite J), we can use the direct sum construction to think of a tuple as a single morphism out of $\bigoplus_{j \in J} よ^{c}A_j$- that is, as a $\bigoplus よ^{c}A_j$-element. The collections of relations will be morphisms from finite direct sums of representables into $\bigoplus よ^{c}A_j$ whose images are contained in the kernel of the tuple. This will give us a class of finitely presented modules which we can use to build a directed colimit.

This proof is adapted from the explicit single-sorted construction in [Pre09] Example E.1.48.

Proposition 4.19 ([Pre09] Example E.1.48)**.** *For every module M, there is a directed set I and diagram $D\colon I \to R\text{-}\boldsymbol{Mod}$ such that $M = \operatorname{colim} D$ and Di is finitely presented for each i.*

Proof. We start by constructing the finitely presented module associated to an element and a finite list of relations.

Consider a generalised element $a\colon \bigoplus_{j \in J_a} \text{よ}^{\mathrm{c}} A_j \to M$ for finite J_a. We write $Y_a = \bigoplus_{j \in J_a} A_j$ for ease. This has a kernel $\ker(a)\colon K \to Y_A$ and then there is an epimorphism from a sum of representables onto K. Let $Z_a = \bigoplus_{l \in L_a} \text{よ}^{\mathrm{c}} A_l$ such that there is an epimorphism $q_a\colon Z_a \to K_a$. Let $r = \{r_1, \dots, r_n\}$ be a finite subset of L_a. This r will give us our generalised finite relations. Specifically, for each $l \in L_a$, there is a morphism $Z_a \to Y_a$ given by $q_a \circ \iota_l$ where ι_l is the coprojection at l into Z_a. Set $Z_r = \bigoplus_{x=1}^{n} \text{よ}^{\mathrm{c}} A_{r_x}$. Using the universal property of the coproduct, we can form a morphism $\iota_r\colon Z_r \to Z_a$ from the coprojections $\iota_{r_x}\colon \text{よ}^{\mathrm{c}} A_{r_x} \to Z_a$. We let $q_r = q_a \circ \iota_r$.

We denote by $M_{a,r}$ the cokernel of q_r. This is a finitely presented module since Z_r and Y_a are both finite sums of representables. Then, considering the diagram below, since the square on the left commutes, functoriality of colimits gives us an $\eta_{a,r}$ filling out the square on the right.

$$
\begin{tikzcd}
Z_a \arrow[r, "\ker(a)q"] & Y_a \arrow[r, "a"] & M \\
Z_r \arrow[u, "\iota_r"] \arrow[r, "q_r"'] & Y_a \arrow[u, leftrightarrow, "\mathrm{id}"'] \arrow[r, "\operatorname{coker}(q_r)"'] & M_{a,r} \arrow[u, dashed, "\exists \eta_{a,r}"']
\end{tikzcd}
$$

We now have to construct the morphisms between the $M_{a,r}$. By construction, we will ensure they form a directed system. Specifically, let I be the set of all pairs (a, r). Let (a, r) and (a', r') be pairs such that $J_a \subseteq J_{a'}$. We can construct the general coprojection $y\colon Y_a \to Y_{a'}$. We say $a \le a'$ if $a = a' \circ y$. This is analogous to saying that $a \le a'$ if a' only adds in new indices to a and remains the same on the existing indices.

Now, suppose $a \leq a'$ and consider the diagram below.

$$\begin{tikzcd}
Z_{a'} \arrow[r, "q_{a'}"] & K_{a'} \arrow[r, "\ker(a')"] & Y_{a'} \arrow[r, "a'"] & M \\
Z_a \arrow[r, "q_a"'] \arrow[u, dashed, "\exists z"] & K_a \arrow[r, "\ker(a)"'] \arrow[u, dashed, "\exists f"] & Y_a \arrow[r, "a"'] \arrow[u, "y"] & M \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

Here, f exists by functoriality of limits. To see that z exists, we first show that y is monomorphic. Let $\tau_j \colon \text{よ}^{\mathrm{c}} A_j \to Y_a$ and $\tau'_j \colon \text{よ}^{\mathrm{c}} A_j \to Y_{a'}$ be the coprojections. Then, consider the map $u \colon Y_{a'} \to Y_a$ given by $u \circ \tau'_j = \tau_j$ for $j \in J_a$ and $u \circ \tau'_j = 0$ for $j \in J_{a'} \setminus J_a$. It follows that $u \circ y \circ \tau_j = u \circ \tau'_j = \tau_j = \mathrm{id}_{Y_a} \circ \tau_j$. By uniqueness of the morphism induced by the coproduct, it follows that $u \circ y = \mathrm{id}_{Y_a}$ and hence that y is monomorphic.

Since y and $\ker(a)$ are monomorphic, so is their composition, and hence f is monomorphic since $\ker(a') \circ f = y \circ \ker(a)$. Now, recall the construction of Z_a and $Z_{a'}$. We let $\iota_l \colon \text{よ}^{\mathrm{c}} A_l \to Z_a$ and $\iota'_{l'} \colon \text{よ}^{\mathrm{c}} A_{l'} \to Z_{a'}$ be the coprojections The index sets are determined by evaluating K_a and $K_{a'}$ on objects, namely, $L_{a'} = \bigsqcup_{A \in R} K_{a'} A$ and similarly for L_a. So, for each $l \in L_a$, consider $\text{よ}^{\mathrm{c}} A_l$ and then the morphism $q_a \circ \iota_l$. This Yoneda-corresponds to an element of $K_a A_l$, namely l. Then, there is an element $f_{A_l}(l)$ of $K_{a'} A_l$ which Yoneda-corresponds to a morphism $\text{よ}^{\mathrm{c}} A_l \to K_{a'}$ and naturality of the Yoneda correspondence tells us that this morphism is in fact $f \circ q_a \circ \iota_l$. By construction of $Z_{a'}$, there is a component of the sum $\text{よ}^{\mathrm{c}} A_{f_{A_l}(l)}$ and $q_{a'} \circ \iota'_{f_{A_l}(l)} = f \circ q_a \circ \iota_l$. We know that, in fact, $A_{f_{A_l}(l)} = A_l$. We consider this over every $l \in L_a$ to find a map $z \colon Z_a \to Z_{a'}$ such that $z \circ \iota_l = \iota_{f_{A_l}(l)}$ for every l. It follows that $q_{a'} \circ z = f \circ q_a$.

Now, since f is monomorphic, each f_{A_l} is monomorphic and so injective. Hence, we actually obtain an injection $L_a \hookrightarrow L_{a'}$. We identify L_a with its image in $L_{a'}$.

We can now fully define the order on I. We say that $(a, r) \leq (a', r')$ if and only if $J_a \subseteq J'_a$, $a = a' \circ y$ and, under the identification above, $r \subseteq r'$.

Supposing we have $(a, r) \leq (a', r')$, there is a natural general coprojection $Z_r \to Z_{r'}$ the same as with $Y_a \to Y_{a'}$. Then, by functoriality of colimits, we get a morphism $M_{a,r} \to M_{a',r'}$ filling in the following diagram.

$$\begin{tikzcd}
Z_{r'} \arrow[r, "q_{r'}"] & Y_{a'} \arrow[r] & M_{a',r'} \\
Z_r \arrow[r, "q_r"'] \arrow[u, "z_{r,r'}"] & Y_a \arrow[r] \arrow[u, "y"'] & M_{a,r} \arrow[u, dashed, "\exists"']
\end{tikzcd}$$

We now define $D\colon I \to R$-**Mod** as $D(a, r) = M_{a,r}$ and $D((a, r) \to (a', r'))$ being the unique fill-in above. To see that I is directed set, suppose we have pairs (a_1, r_1) and (a_2, r_2). We can construct the element $a = a_1 \oplus a_2$ as the unique morphism out of $Y_{a_1} \oplus Y_{a_2}$ that restricts to a_1 and a_2 on the components. It is then clear that $a \geq a_1, a_2$. We then construct the relation r as the disjoint union of r_1 and r_2, both of them being identified with subsets of L_a. It follows that $(a, r) \geq (a_1, r_1), (a_2, r_2)$. Binary upper bounds are then sufficient to construct all finite upper bounds, which shows that I is a directed set.

It remains to show that (M, η) is the colimit of F. To show that η is a cocone for D, it helps to past several prior commutative diagrams together. In the diagram below, our goal is to show that the square on the right-hand side commutes. All other squares commute.

$$\begin{tikzcd}
Y_{a'} \arrow[rrr, "a'"] & & & M \\
& Y_{a'} \arrow[ul, "\mathrm{id}"'] \arrow[r, "\operatorname{coker}(q_{r'})"] & M_{a',r'} \arrow[ur, "\eta_{a,r}"] & \\
& Y_a \arrow[u, "y"] \arrow[r, "\operatorname{coker}(q_r)"'] \arrow[dl, "\mathrm{id}"'] & M_{a,r} \arrow[u, "g"'] \arrow[dr, "\eta_{a,r}"] & \\
Y_a \arrow[uuu, "y"] \arrow[rrr, "a"'] & & & M \arrow[uuu, "\mathrm{id}"']
\end{tikzcd}$$

We will start by pre-composing $\eta_{a',r'} \circ g$ with $\operatorname{coker}(q_{r'})$. This $\operatorname{coker}(q_{r'})$ is epimorphic and we will cancel it out later. We then follow:

$$\begin{aligned}
\eta_{a',r'} \circ g \circ \operatorname{coker}(q_r) &= \eta_{a',r'} \circ \operatorname{coker}(q_{r'}) \circ y \\
&= a' \circ y \\
&= a \\
&= \eta_{a,r} \circ \operatorname{coker}(q_r) \\
\eta_{a',r'} \circ g &= \eta_{a,r}
\end{aligned}$$

Hence η is a cocone. Now, let (N, ε) be another cocone. For each A in R and each $a \in MA$, we know that this Yoneda corresponds to a map $a\colon Y_a \to M$ and that for some r, this factorises as $a = \eta_{a,r} \circ \operatorname{coker}(q_r)$. We can swap out the $\eta_{a,r}$ for ε to get a morphism $\varepsilon_{a,r} \circ \operatorname{coker}(q_r)$ that Yoneda corresponds to an element $h_A(a) \in NA$. Note that $h_A(a)$ is in fact independent of r, since if r and r' are two relations for a, there is an upper bound $(a, r_0) \geq (a, r), (a, r')$. We then follow a similar calculation as that for showing η was a cone but in reverse. In particular, we have that $y = \mathrm{id}_{Y_a}$, since a is fixed between all the pairs, and the following diagram commutes.

$$\begin{tikzcd}
Y_a \arrow[r, "q_{r_0}"] \arrow[d, equal] & M_{a,r_0} \arrow[r, "\varepsilon_{a,r_0}"] & M \arrow[d, equal] \\
Y_a \arrow[r, "q_r"'] & M_{a,r} \arrow[u, "g"] \arrow[r, "\varepsilon_{a,r}"'] & M
\end{tikzcd}$$

So, $\varepsilon_{a,r_0} \circ \mathrm{coker}(q_{r_0}) = \varepsilon_{a,r_0} \circ g \circ \mathrm{coker}(q_r) = \varepsilon_{a,r} \circ \mathrm{coker}(q_r)$. We obtain the same result for (a, r') and then conclude that $\varepsilon_{a,r} \circ \mathrm{coker}(q_r) = \varepsilon_{a,r'} \circ \mathrm{coker}(q_{r'})$ since they are both equal to $\varepsilon_{a,r_0} \circ \mathrm{coker}(q_{r_0})$.

Hence, we have a map of sets $h_A \colon MA \to NA$ for every A. By Yoneda, we equivalently have a function of Hom-sets $\mathrm{Hom}(\text{よ}^{\mathrm{c}}A, M) \to \mathrm{Hom}(\text{よ}^{\mathrm{c}}A, N)$. We show that this lifts to a morphism of modules $h \colon M \to N$. First, we establish that each h_A is a homomorphism of Abelian groups. Since each element has a unique domain, we drop the subscript on h.

Let a_1, a_2 be morphisms $\text{よ}^{\mathrm{c}}A \to M$. Let r_1, r_2 be relations for a_1 and a_2 respectively and let $a = a_1 \oplus a_2$ and $r = r_1 \sqcup r_2$. Thus, the following diagram commutes.

$$\begin{tikzcd}
Z_{a_1} \arrow[r, "q_{r_1}"] \arrow[d, "z_1"'] & \text{よ}^{\mathrm{c}}A \arrow[r, "\mathrm{coker}(q_{r_1})"] \arrow[d, "y_1"'] & M_{a_1,r_1} \arrow[r, "\varepsilon_{a_1,r_1}"] \arrow[d, "g_1"] & N \arrow[d, equal] \\
Z_a \arrow[r, "q_r"'] & \text{よ}^{\mathrm{c}}A \oplus \text{よ}^{\mathrm{c}}A \arrow[r, "\mathrm{coker}(q_r)"] & M_{a,r} \arrow[r, "\varepsilon_{a,r}"'] & N \\
Z_{a_2} \arrow[r, "q_{r_2}"'] \arrow[u, "z_2"] & \text{よ}^{\mathrm{c}}A \arrow[r, "\mathrm{coker}(q_{r_2})"'] \arrow[u, "y_2"'] & M_{a_2,r_2} \arrow[r, "\varepsilon_{a_2,r_2}"'] \arrow[u, "g_2"] & N \arrow[u, equal]
\end{tikzcd}$$

Now, consider $h(a)$. We have that $h(a) \circ y_1 = \varepsilon_{a,r} \circ g_1 \circ \mathrm{coker}(q_{r_1}) = h(a_1)$. Similarly, $h(a) \circ y_2 = h(a_2)$. So, $h(a_1 \oplus a_2) = h(a_1) \oplus h(a_2)$.

Recall from the construction of biproducts (Proposition 3.28) that we have maps $\pi_i \colon Y_a \to \text{よ}^{\mathrm{c}}A$ such that $\pi_i \circ y_i = \mathrm{id}$ and $\pi_i \circ y_j = 0$ for $j \neq i$. It follows that $a = a_1 \circ \pi_1 + a_2 \circ \pi_2$ and $h(a) = h(a_1) \circ \pi_1 + h(a_2) \circ \pi_2$. We can generalise this to any finite tuple $(a_i)_{i=1}^n$ and obtain $h\left(\bigoplus_{i=1}^n a_i\right) = \bigoplus_{i=1}^n h(a_i)$. We note that this is independent of the choice of relations.

Now, let a_1, a_2, a_3 be such that $a_1 - a_2 = a_3$. We now wish to consider the relation $q_r \colon \text{よ}^{\mathrm{c}}A \to \bigoplus_{i=1}^{3} \text{よ}^{\mathrm{c}}A$ given by $q_r = y_1 + (-y_2) + (-y_3)$. Then, using the equations for $\pi_i \circ y_j$ from above, $a \circ q_r = a_1 + (-a_2) + (-a_3) = 0$. Hence, a factors as $a = \eta_{a,r} \circ \operatorname{coker}(q_r)$. Then $h(a) = \varepsilon_{a,r} \circ \operatorname{coker}(q_r)$ and so $h(a) \circ q_r = 0$. Now, we recall that $h(a) = \bigoplus h(a_i)\pi_i$. Thus, $h(a) \circ q_r = h(a_1) + (-h(a_2)) + (-h(a_3))$. We conclude that $h(a_1) - h(a_2) = h(a_3)$. This is sufficient to show that h is a group homomorphism, since by setting $a_2 = a_1$ we see that $h(0) = 0$ and we can set $a_2 = -a_2'$ to see that addition is preserved.

Now, we must show that h is natural. Let $f \colon A \to B$ be a morphism in R. This Yoneda corresponds to a morphism $f^* \colon \text{よ}^{\mathrm{c}}B \to \text{よ}^{\mathrm{c}}A$. For an element $a \colon \text{よ}^{\mathrm{c}}A \to M$ of M, there is an element $a \circ f^* \colon B \to M$. Naturality of the Yoneda correspondence tells us that this element is $Mf(a)$.

We adapt the set-up from the additivity proof. We take the map $\overline{a} = a \oplus Mf(a) \colon \text{よ}^{\mathrm{c}}A \oplus \text{よ}^{\mathrm{c}}B \to M$. The same proof from before generalises to show that $h(a \oplus Mf(a)) = h(a) \oplus h(Mf(a))$. Now, let $q_r \colon \text{よ}^{\mathrm{c}}B \to \text{よ}^{\mathrm{c}}A \oplus \text{よ}^{\mathrm{c}}B$ be the morphism $q_r = y_A \circ f^* - y_B$. Then, $a \oplus Mf(a) \circ q_r = a \circ f^* - Mf(a) = 0$. As before, we factor $\overline{a} = \eta_{\overline{a},r} \circ \operatorname{coker}(r)$ and find $h(\overline{a}) = \varepsilon_{\overline{a},r} \circ \operatorname{coker}(r)$. Then, $h(a) \oplus h(Mf(a)) \circ q_r = 0$ and so $Nf(h(a)) = h(Mf(a))$. This is naturality of h precisely.

It follows from the Yoneda lemma that, for each (a, r), $h_A \circ \eta_{a,r} \circ \operatorname{coker}(r) = \varepsilon_{a,r} \circ \operatorname{coker}(r)$. Then, since $\operatorname{coker}(r)$ is epimorphic, we obtain $h_A \circ \eta_{a,r} = \varepsilon_{a,r}$. Since we picked arbitrary (a, r), this lifts to $h \circ \eta = \varepsilon$.

To see uniqueness, let h' be another map such that $h' \circ \eta = \varepsilon$. Then, for each (a, r), we have $h \circ \eta_{a,r} - h' \circ \eta_{a,r} = \varepsilon_{a,r} - \varepsilon_{a,r} = 0$. Then, composition is bilinear, so $(h - h') \circ \eta_{a,r} = 0$. It follows that, for every $a \colon \text{よ}^{\mathrm{c}}A \to M$, $(h - h') \circ a = 0$. Consider the epimorphic map $e \colon Z_M \to M$ from Proposition 4.18. Then, for each coprojection $y_a \colon \text{よ}^{\mathrm{c}}A_a \to Z_M$, we have that $(h - h') \circ e \circ y_a = (h - h') \circ a = 0$. Hence, $(h - h') \circ e = 0$ and so $h - h' = 0$.

This concludes the proof that (M, η) is a colimit and hence the proof of the proposition. $\square$

Since R is small, there is only a set of representables. Thus, there is only a set of possible morphisms between finite sums of representables. Since the modules used in the proof of above are all generated as cokernels of such morphisms, we can see that, up to isomorphism, there is only a set of possible such modules. We can pick some set of representatives of the isomorphism classes and see that this set generates the entire category of modules by directed colimits.

Since R-**Mod** is Abelian, it is finitely cocomplete. This plus the observation above together give us local finite presentability of R-**Mod**.

Theorem 4.20. *R-**Mod** is locally finitely presentable.*

We will immediate begin to use this fact. First, we prove a specialisation of Proposition 4.18 specifically for finitely generated modules. This requires the use of commuting directed colimits and finite limits, which is true in locally finitely presentable categories by Proposition 1.78.

Proposition 4.21 ([Pre09] Proposition 10.1.13)**.** *M is finitely generated if and only if there is a finite index set K with an epi $\bigoplus_{k\in K} よ^{c} A_k \to M$.*

Proof. Given a morphism $f\colon X \to Y$ in an Abelian category, we can form its image $\mathrm{im}(f) = \ker(\mathrm{coker}(f))$. Now, kernels are limits, cokernels are colimits and direct sums are directed colimit. These all commute since R-**Mod** is locally finitely presentable. Hence, if we have some set of morphisms $f_j\colon X_j \to Y$ indexed by J and their induced morphism $f\colon \bigoplus_{j\in J} X_j \to Y$ then we can write $\mathrm{im}(f) = \bigoplus_{j\in J} \mathrm{im}(f_j)$.

Now, consider the index J and epimorphism f from 4.18. That f is an epimorphism tells us that it is equal to its image, so $M = \mathrm{im}(f)$. Hence $M = \bigoplus_{j\in J} \mathrm{im}(f_j)$.

If M is finitely generated, then by definition there is some finite $J_0 \subset J$ such that $M \cong \bigoplus_{j\in J_0} \mathrm{im}(f_j)$. Hence $\bigoplus_{j\in J_0} よ^{c} A_j \to M$ is an epi.

If J is finite, then we shall show that $\mathrm{im}(f_j)$ is finitely generated for each j and then, as it is a finite direct sum of finitely generated objects, M is finitely generated. Let $D\colon I \to R$-**Mod** be a directed diagram with colimit $(\mathrm{im}(f_j), \eta)$ and η_i be a monomorphism for each i (that is, each Di is a subobject of $\mathrm{im}(f_j)$). Then, the morphism $\mathrm{coim}(f_j)\colon よ^{c} A_j \to \mathrm{im}(f_j)$ factors essentially uniquely through the diagram, since $よ^{c} A_j$ is finitely presented. Write $\mathrm{coim}(f_j) = \eta_i \circ h$. Then, since $\mathrm{coim}(f_j)$ is an epimorphism, it follows that η_i is an epimorphism. By assumption, η_i is a monomorphism, and R-**Mod** is balanced, hence η_i must be an isomorphism. Thus $\mathrm{im}(f_j) \cong Di$ and we conclude that $\mathrm{im}(f_j)$ is finitely generated. $\square$

A very quick corollary of this is that being finitely generated is preserved under epimorphic image.

Corollary 4.22. *Let M be a finitely generated module and N a module. Suppose that there is an epimorphism $e\colon M \to N$. Then N is finitely generated.*

Proof. By the proposition, there is some epimorphism $q\colon \bigoplus_{j\in J} よ^{c} A_J \to M$ for J finite. Then $e \circ q\colon \bigoplus_{j\in J} よ^{c} A_J$ is an epimorphism onto N. Thus N is finitely generated. □

In order to finish the proof of Proposition 4.10, we need the following fact which builds on the local finite presentability of R-**Mod**. The proof uses the theory of Grothendieck categories, which is beyond the scope of this project.

Proposition 4.23 ([Pre09] Proposition E.1.44)**.** *Let M, N be finitely presented modules and $f\colon M \to N$ be an epimorphism. Then* $\ker(f)$ *is finitely generated.*

We derive the forwards direction of Proposition 4.10 as a corollary.

Corollary 4.24. *Let M be a finitely presented module. Then, there are finite sums of representables A and B such that there is an exact sequence*
$A \longrightarrow B \longrightarrow M \longrightarrow 0$.

Proof. By Proposition 4.21, there is some finite direct sum of representables Y and epimorphism $e\colon Y \to M$. By Proposition 4.23, the kernel of e, K is finitely generated. Then there is a finite direct sum of representables Z and epimorpism $f\colon Z \to K$. Since f is an epimorphism and $\ker(e)$ is a monomorphism, $\mathrm{im}(\ker(e)\circ f) = \mathrm{im}(\ker(e)) = \ker(e)$. Thus, the sequence $Z \longrightarrow Y \longrightarrow M \longrightarrow 0$ is exact. □

5 Pp-formulae

Recall the language of R-modules as given in Example 2.3. This language contained a copy of the language of groups and a unary function symbol for each element of R. We extended this to a language of representations of quivers in Example 2.5. We now extend it to a language that can describe the general modules over many object rings as seen in Section 4. This language can be found in Appendix B of [Pre09].

Let R be a small preadditive category. We define the signature of right R-modules as $((M_A)_{A\in R}; (0_A)_{A\in R}; (+_A)_{A\in R}, (-_A)_{A\in R}, (*_r)_{r\in \mathrm{Arr}(R)})$. The copies of the language of groups are as before but additionally we have the function symbols $(*_r)$. Each $*_r$, where f is a morphism of $R\colon A \to B$ has arity $(M_B; M_A)$. For ease, we will often write A and B rather than M_A and M_B when it is clear we are discussing the sort and not the object of R.

The theory of right R-modules, written $T_{\mathbf{Mod}\text{-}R}$ consists of a copy of the theory of groups for each A and, for each $r, r'\colon A \to B$ and $s\colon B \to C$ in R, the following axioms.

- $\forall (x, y\colon B).\, *_r\,(x - y) \doteq *_r(x) - *_r(y)$
- $\forall (x\colon B).\, *_{\mathrm{id}_A}\,(x) \doteq x$
- $\forall (x\colon C).\, *_r\,(*_s(x)) \doteq *_{s\circ r}(x)$
- $\forall (x\colon B).\, *_{r+r'}\,(x) \doteq *_r(x) + *_{r'}(x)$

Remark 5.1. The models of $T_{\mathbf{Mod}\text{-}R}$ correspond precisely to the modules over R in the functorial case. In particular, for a functor $M\colon R \to \mathbf{Ab}$, we take M_A to be the group MA and $*_r = Mr$.

When studying the Model Theory of modules, we usually restrict to a special subclass of first order formulae called the positive-primitive formulae (or pp-formulae). An important result here is the so-called Pp-elimination of quantifiers for modules, which tells us that almost everything we want to know about the logic of modules is in fact already captured by the pp-formulae.

In this section, we define pp-formulae, study some basic facts about them and then introduce the notion of pp-imaginaries. We lay the groundwork for Section 6, where the topics from this Section and Section 4 are brought together.

The model theory over one-object rings is taken from Chapters 1 and 3 of [Pre09]. Generalisations to the many-object case are taken from Sections 18 and 19 of

[Pre11]. The proofs we give are multi-sorted adaptations of the single-sorted proofs found in [Pre09].

5.1 Formulae

Fix a preadditive category R. We say that a first order formula $\phi(\overline{x})$ is a **positive-primitive formula**, shortened to pp-formula, if it is constructed from atomic formulae, conjunctions and existential quantifiers. Such formulae have a canonical normal form, which makes use of linear polynomials. This means that for every pp-formula $\phi(\overline{x})$, there is a formula $\psi(\overline{x})$ such that $T_{\mathbf{Mod}\text{-}R} \vDash \forall \overline{x}.\phi(\overline{x}) \leftrightarrow \psi(\overline{x})$. We call this **equivalence over** $T_{\mathbf{Mod}\text{-}R}$ and write it $\phi \equiv \psi$.

Definition 5.2. Given morphisms $(r_i \colon B \to A_i)_{i=1}^n$, the **linear polynomial** they define is the term $p(\overline{x}) = \sum_{i=1}^n *_{r_i}(x_i)$ where each x_i is of sort A_i and $\sum$ is repeated application of $+_B$. We say that p has **arity** $(A_1, \dots, A_n; B)$. Since it is a term, it has a context, which is B.

For each sort B, there is also an "empty polynomial" of arity B, which is just the constant 0_B.

Proposition 5.3. *Let $\phi(\overline{x})$ be a pp-formula where $\overline{x} = (x_1, \dots, x_n)$ has context $(A_1, \dots, A_N)$. Then, there are sorts $B, C_1, \dots, C_k$ and linear polynomials $(p_i)_{i=1}^m$ of arity $(A_1, \dots, A_n, C_1, \dots, C_k; B)$ such that*

$$\phi(\overline{x}) \equiv \exists(\overline{y} : \overline{C}). \bigwedge_{i=1}^{m} p_i(\overline{x}, \overline{y}) \doteq 0$$

Proof. We first show that every term is in the form of a linear polynomial. This proceeds inductively along the lines of term construction. Let t be a term of context B.

- For $t = x$, a variable, we have $t = *_{\mathrm{id}_B}(x)$.
- $t = 0_B$ is already a polynomial.
- For $t = t' +_B t''$, by hypothesis, $t' = \sum_{i=1}^n *_{r_i}(x_i)$ and $t'' = \sum_{i=n+1}^m *_{r_i}(x_i)$, so $t = \sum_{i=1}^m *_{r_i}(x_i)$.
- For $t = *_s(t')$, we have that $t' = \sum_{i=1}^n *_{r_i}(x_i)$. Then, linearity of $*_s$ tell us that $t = \sum_{i=1}^n *_s(*_{r_i}(x_i)) = \sum_{i=1}^n *_{r_i \circ s}(x_i)$.
- For $t = -_B t'$, $t = *_{-\mathrm{id}_B}(t')$ which reduces the case above.

We now induct on the construction of formulae.

First is the atomic case. Since $L_{\mathbf{Mod}\text{-}R}$ has no relation symbols, the only atomic formulae are of the form $\phi = t_1 \doteq t_2$ for t_1, t_2 terms of context B. Then, $\phi \equiv t_1 - t_2 = 0$, which is a pp-formulae.

Let $\phi(\overline{x}) = \exists \overline{y} \bigwedge_{i=1}^{n} p_i(\overline{x}, \overline{y}) \doteq 0$ and $\psi(\overline{x}') = \exists \overline{y}' \bigwedge_{i=1}^{m} q_i(\overline{x}', \overline{y}') \doteq 0$ be pp-formulae. Then $\phi(\overline{x}) \wedge \psi(\overline{x}') \equiv \exists \overline{y}. \exists \overline{y}'. \left(\bigwedge_{i=1}^{n} p_i(\overline{x}, \overline{y}) \doteq 0 \wedge \bigwedge_{j=1}^{m} q_j(\overline{x}', \overline{y}') \doteq 0 \right)$. This can quickly be checked by evaluating in an arbitrary model.

Finally, it is clear that $\exists x_n . \phi(\overline{x}) \equiv \exists x_n . \overline{y}. \bigwedge_{i=1}^{n} p_i(\overline{x}, \overline{y}) \doteq 0$ is already in the required form. ☐

It is useful to think of our formulae in terms of solution sets within a structure. We define here pp-definable subsets and prove some basic algebraic facts about them.

Definition 5.4. Let M be a module and $\phi(\overline{x})$ be a pp-formula where the context of $\overline{x}$ is $(A_1, \dots, A_n)$. Then the subset **defined by** $\phi(\overline{x})$ is a subset of $MA_1 \oplus \dots \oplus MA_n$ given by $\phi(M) = \{(a_1, \dots, a_n) \in \bigoplus_{i=1}^{n} MA_i \mid M \vDash \phi(a_1, \dots, a_n)\}$.

Given a pp-formula $\phi(\overline{x}) = \exists \overline{y}. \bigwedge_{i=1}^{n} p_i(\overline{x}, \overline{y}) \doteq 0$ and two tuples $\overline{a}, \overline{a}' \in \phi(M)$, there must be tuples $\overline{b}, \overline{b}'$ such that $M \vDash \bigwedge_{i=1}^{n} p_i(\overline{a}, \overline{b})$ and $M \vDash \bigwedge_{i=1}^{n} p_i(\overline{a}', \overline{b}')$. Applying linearity of the p_i, we see that $M \vDash \bigwedge_{i=1}^{n} p_i(\overline{a} - \overline{a}', \overline{b} - \overline{b}')$, subtracting componentwise. It follows that $\overline{a} - \overline{a}'$ is in $\phi(M)$ and hence $\phi(M)$ is in fact a subgroup of $\bigoplus_{i=1}^{n} MA_i$. It turns out that this operation is functorial.

Proposition 5.5 ([Pre09] Lemma 1.1.7)**.** *Let $\phi(\overline{x})$ be a pp-formula where $\overline{x}$ has context $(A_1, \dots, A_n)$. Then, there is a functor $F_\phi \colon$ **Mod**-$R \to \mathbf{Ab}$ such that for each module M, $F_\phi(M) = \phi(M)$.*

Proof. Suppose $\phi(\overline{x}) = \exists(\overline{y} \colon (B_1, \dots, B_t)). \bigwedge_{j=1}^{m} p_j(\overline{x}, \overline{y}) \doteq 0$. We write the linear polynomials as $p_j(\overline{x}, \overline{y}) = \sum_{i=1}^{n} *_{r_{ji}}(x_j) + \sum_{k=1}^{t} *_{s_{jk}}(y_k)$. Then, for $\overline{a} \in \bigoplus_{i=1}^{n} MA_i$ and $\overline{b} \in \bigoplus_{k=1}^{t} MB_k$ respectively, we interpret $p_j^M(\overline{a}, \overline{b}) = \sum_{i=1}^{n} Mr_{ji}(a_i) + \sum_{k=1}^{t} Ms_{jk}(b_k)$.

Let $f \colon M \to N$ be a morphism of modules. Naturality of f means that, for each $r \colon C \to A_i$ in R, $f_C \circ Mr = Nr \circ f_{A_i}$. Further, composition is bilinear, hence $f_C \circ (*_r + *_s) = f_C \circ *_r + f_C \circ *_s$. Let C_j be the context of p_j^M. It follows that, for every tuple $(\overline{a}, \overline{b})$ in M,

$$\begin{aligned}(f_{C_j} \circ p_j^M)(\overline{a}, \overline{b}) &= \sum_{i=1}^{n} (f_{C_j} \circ M r_{ji})(a_i) + \sum_{k=1}^{t} (f_{C_j} \circ M s_{ki})(b_k) \\ &= \sum_{i=1}^{n} (N r_{ji} \circ f_{A_i})(a_i) + \sum_{k=1}^{t} (N s_{ki} \circ f_{B_k})(b_k)\end{aligned}$$

So, $f(p_j^M(\overline{a}, \overline{b})) = p_j^N(f(\overline{a}, \overline{b}))$.

Let $\overline{f}\colon \bigoplus_{i=1}^n MA_i \to \bigoplus_{i=1}^n NA_i$ be the unique morphism between sums induced by the f_{A_i}. We hence see that if $\overline{a} \in \phi(M)$, then $\overline{f}(\overline{a}) \in \phi(N)$, since $p_j^N(\overline{f}(\overline{a}, \overline{b})) = \overline{f}(p_j^M(\overline{a}, \overline{b})) = \overline{f}(0) = 0$ for every p_j in the definition of ϕ. We define $F_\phi(f)$ to be the restriction of $\overline{f}$ to $\phi(M)$. Since $\overline{f}(\phi(M)) \subseteq \phi(N)$, this is a well defined map $\phi(M) \to \phi(N)$. $\square$

Consider the functors $\mathrm{Hom}(よA, -)$ for each object A in R. We avoid double usage of よ to ease confusion. For a module M, the Yoneda Lemma tells us that $\mathrm{Hom}(よA, M) \cong MA$. Thus, for each $\phi(\overline{x})$ where $\overline{x}$ has context $(A_1, \dots, A_n)$, there is an inclusion $\phi(M) \hookrightarrow \bigoplus_{i=1}^n \mathrm{Hom}(よA, M)$. Since F_ϕ sends morphisms to restrictions, we see that these inclusions give a natural monomorphism $F_\phi \to \bigoplus \mathrm{Hom}(よA, -)$. Since the covariant Yoneda embedding sends limits to colimits and **Mod-**R has biproducts, we see that $\bigoplus \mathrm{Hom}(よA, -) \cong \mathrm{Hom}(\bigoplus よA, -)$. Every pp-formula hence defines a subfunctor of the Hom functor out of a sum of representables.

We are justified in restricting our attention by the result called "pp-elimination of quantifiers". This expresses every formula in terms of a combination of pp-formulae and special sentences called invariants statements.

Definition 5.6. Let $\phi(\overline{x})$ and $\psi(\overline{x})$ be formulae such that for all modules M, we have that $\psi(M) \subseteq \phi(M)$. There is then a quotient group of the form $\phi(M)/\psi(M)$. An **invariants condition** is a sentence of the form

$$\tau = \exists \overline{x}_1, \overline{x}_2, \dots, \overline{x}_n . (\bigwedge_{i=1}^{n} \phi(\overline{x}_i) \wedge \bigwedge_{i \neq j} \neg\psi(\overline{x}_i - \overline{x}_j))$$

This sentence is saying that there are at least n many tuples in $\phi(M)$ that are not ψ-equivalent. That is, the index of $\psi(M)$ in $\phi(M)$ is at least n.

An **invariants statement** is a boolean combination of invariants conditions.

Theorem 5.7 ([Pre88] Corollary 2.13)**.** *Let $\phi(\overline{x})$ be an $L_{\textbf{Mod-}R}$ formula. Then there are pp-formulae $\psi_{1,1}(\overline{x}), \dots, \psi_{n,m}(\overline{x})$ and an invariants condition τ such* *that*

$$\phi(\overline{x}) \equiv \tau \wedge \bigwedge_{i=1}^{n} \left(\bigvee_{j=1}^{m} \psi_{i,j}(\overline{x}) \right)$$

Proof. See Chapter 2 of [Pre88] for a proof in the case of modules over a one-object ring and Section 19 of [Pre11] for the extension to the preadditive category case. $\square$

We now move to discussing pp-types and their free realisations. Pp-types allow us to discuss all the pp-formulae a tuple may satisfy at once, while free realisations provide a "minimal" module and tuple satisfying a given collection of types. We will use these free realisations in Section 6 to provide a presentation of formulae as a finitely presented functor.

A **pp-type** is a set of pp-formulae that is closed under implication and conjunction. Given a module M, every tuple $\overline{a}$ of context $(A_1, \dots, A_n)$ induces a pp-type $\mathrm{pp}^M(\overline{a}) = \{\phi(\overline{x}) \in L_{\mathbf{Mod}\text{-}R} \mid \overline{x} \text{ has context } \overline{A} \text{ and } M \vDash \phi(\overline{a}))\}$.

Given a pp-type p, the free variables appearing in p is the union of free variables appearing in any formula in p. If the free variables of p is the (potentially infinite) tuple $(x_i)_{i \in I}$ of context $(A_i)_{i \in I}$, we can look at all tuples $(a_i)_{i \in I}$ of a module M such that, for every $\phi(x_1, \dots, x_n) \in p$, $M \vDash \phi(a_1, \dots, a_n)$. The set of all such tuples is a subset of $\bigoplus_{i \in I} MA_i$ and we call it the set **defined by** p, written $p(M)$. This is a generalisation of the notion of a formula defining a set.

For example, if we take a finite tuple $\overline{a}$ of context $(A_1, \dots, A_n)$ in a module M, the type $p = \mathrm{pp}(\overline{a})$ has only finitely many variables. The set $p(M)$ is then a subset of $\bigoplus_{i=1}^{n} MA_i$ and we have that $\overline{a} \in p(M)$.

By a similar argument to that used to show that taking definable subsets is functorial, we can show that there is a functor $F_p \colon \mathbf{Mod}\text{-}R \to \mathbf{Ab}$ acting as $M \mapsto p(M)$.

Given a pp-formula $\phi(\overline{x})$, we say that the pp-type **generated** by ϕ is the type $\langle \phi \rangle = \{\psi(\overline{x}) \in L_{\mathbf{Mod}\text{-}R} \mid \text{for all modules } M, \phi(M) \subset \psi(M))\}$. We say that a given pp-type p is **finitely generated** if there is some ϕ where $p = \langle \phi \rangle$. Note that, given some finite set of formulae, we could take their conjunction to obtain a single formula that implies everything that the set implies. Thus, we can restrict to having only a single generator for finitely generated types.

Given a formula $\phi(\overline{x})$, a **free realisation** of ϕ consists of a finitely presented module M and tuple $\overline{a}$ in M such that $\langle \phi \rangle = \mathrm{pp}(\overline{a})$. In the next proposition, we show that every pp-formula has a free realisation.

Proposition 5.8 ([Pre09] Proposition 1.2.14)**.** *Let $\phi(\overline{x})$ be a pp-formula. Then, there is some finitely presented module M and tuple $\overline{a}$ such that $\langle\phi\rangle = \mathrm{pp}(\overline{a})$.*

Proof. Let $\overline{x}$ have context $(A_1, \dots, A_n)$ and let $\overline{y}$ be another tuple of variables with context $(B_1, \dots, B_t)$. Let $\psi(\overline{x}, \overline{y}) = \bigwedge_{j=1}^{m} p_j(\overline{x}, \overline{y}) \doteq 0$ be such that $\phi(\overline{x}) = \exists\overline{y}\psi(\overline{x}, \overline{y})$. For each j, let $p_j(\overline{x}, \overline{y}) = \sum_{i=1}^{n} *_{r_{ji}}(x_i) + \sum_{k=1}^{t} *_{s_{jk}}(y_k)$ with context C_j.

We let p_i and p_k be coprojections from $よA_i$ and $よB_k$ to $Y_a = \bigoplus_{i=1}^{n} よA_i$ and $Y_b = \bigoplus_{k=1}^{t} よB_t$ respectively. Let π_i and π_k be the corresponding projections. Let y_a, y_b be the coprojections to $Y = (\bigoplus よA_i) \oplus (\bigoplus よB_k)$ with projections u_a and u_b. Let c_j be the coprojections $よC_j \to Z = \bigoplus_{j=1}^{m} よC_j$.

For each $r_{ji} \colon C_j \to A_i$ (equally, every s_{jk}), we get a morphism $よr_{ji} \colon よC_j \to よA_i$. Then, we encode the left hand term of ϕ as a morphism $f \colon Z \to Y$. Specifically, f is the unique morphism such that $f \circ c_j = \sum_{i=1}^{n} y_a \circ p_i \circ r_{ji} + \sum_{k=1}^{t} y_b \circ p_k \circ s_{ji}$.

Let $(\overline{a}, \overline{b}) \colon Y \to M$ be the cokernel of f. We let $\overline{a} = (\overline{a}, \overline{b}) \circ y_a$ and $\overline{b} = (\overline{a}, \overline{b}) \circ y_b$. These are morphisms of the form $a = \sum_{i=1}^{n} a_i \circ \pi_i$ and $b = \sum_{k=1}^{t} b_k \circ \pi_k$ where a_i is a morphism $よA_i \to M$. Hence, the $\overline{a}$ and $\overline{b}$ are (by Yoneda correspondence) tuples of elements in M. We claim that $(M, \overline{a})$ is the free realisation of ϕ, noting that M is finitely presented by Proposition 4.10.

To start, we need to show that $M \vDash \phi(\overline{a})$. We compute the composition $(\overline{a}, \overline{b}) \circ f$ using the bilinearity of composition and the biprouct equations to cancel out most components. Fix an arbitrary j from 1 to m.

$$
\begin{aligned}
(\overline{a}, \overline{b}) \circ f \circ c_j &= (a \circ u_a + b \circ u_b) \circ \sum_{i=1}^{n} y_a \circ p_i \circ r_{ji} + \sum_{k=1}^{t} y_b \circ p_k \circ s_{ji} \\
&= \sum_{i=1}^{n} a \circ p_i \circ r_{ji} + \sum_{k=1}^{t} b \circ p_k \circ s_{ji} \\
&= \sum_{i=1}^{n} a_i \circ r_{ji} + \sum_{k=1}^{t} b_k \circ s_{ji}
\end{aligned}
$$

Now, for each a_i, $a_i \circ r_{ji}$ Yoneda corresponds to $Mr_{ji}(a_i)$. Hence, $(\overline{a}, \overline{b}) \circ f \circ c_j$ Yoneda corresponds to $p_j^M(\overline{a}, \overline{b}) = \sum_{i=1}^{n} Mr_{ji}(a_i) + \sum_{k=1}^{t} Ms_{jk}(b_k)$. Since $(\overline{a}, \overline{b})$

is a cokernel, we know that $(\overline{a}, \overline{b}) \circ f \circ c_j = 0$ and so $p_j^M(\overline{a}, \overline{b}) = 0$. Since we picked arbitrary j, we conclude that $M \vDash \psi(\overline{a}, \overline{b})$ and so $M \vDash \phi(\overline{a})$.

This shows that $\langle \phi \rangle \subset \mathrm{pp}(\overline{a})$, since for every $\phi' \in \langle \phi \rangle$, we have that $\overline{a} \in \phi(M) \subset \phi'(M)$. We now must prove the reverse inclusion. Pick some arbitrary module N and $\overline{a}' \colon Y_a \to N$ such that $N \vDash \phi(\overline{a}')$ and let $\chi(\overline{x}) \in \mathrm{pp}(\overline{a})$ be some formula. We must show that $N \vDash \chi(\overline{a}')$.

We set up χ as with ϕ as being of the form $\exists \overline{y}' \theta(\overline{x}, \overline{y}')$ where $\overline{y}'$ has context $(D_1, \dots, D_{t'})$. We have $\theta(\overline{x}, \overline{y}') = \bigwedge_{j=1}^{m'} q_j(\overline{x}, \overline{y}') \doteq 0$ where each $q_j = \sum_{i=1}^{n} *_{e_{ji}}(x_i) + \sum_{k=1}^{t'} *_{h_{jk}}(y'_k)$ having context O_j. We set up the products analogously with $Y' = Y_a \oplus Y_d$ and have $f' \colon Z' \to Y'$ be the morphism for θ.

Since $N \vDash \phi(\overline{a}')$, we know that there is some $\overline{b}'$ such that $(\overline{a}', \overline{b}') \circ f = 0$. As $(\overline{a}, \overline{b})$ is a cokernel, $(\overline{a}', \overline{b}')$ must factor through it as $(\overline{a}', \overline{b}') = g \circ (\overline{a}, \overline{b})$ for $g \colon M \to N$. Restricting, we see that $\overline{a}' = g \circ \overline{a}$. As $\chi \in \mathrm{pp}(\overline{a})$, there must be some $\overline{d}$ in M such that $M \vDash \theta(\overline{a}, \overline{d})$. Let $\overline{d}' = g \circ \overline{d}$. Now, $(\overline{a}', \overline{d}') \circ f' = g \circ (\overline{a}, \overline{d}) \circ f' = 0$. Thus, $N \vDash \theta(\overline{a}', \overline{d}')$ and hence $N \vDash \chi(\overline{a}')$.

Thus, for every module N, $\phi(N) \subset \chi(N)$ and so $\chi \in \langle \phi \rangle$. This gives us that $\mathrm{pp}(\overline{a}) \subset \langle \phi \rangle$ which is sufficient for equality. Hence $(M, \overline{a})$ is indeed a free realisation for ϕ. $\square$

In fact, the converse is true: given a tuple of a finitely presented module, its pp-type is finitely generated.

Proposition 5.9 ([Pre09] Lemma 1.2.6)**.** *Let M be a finitely presented module and $\overline{a}$ be a tuple of elements in M. Then there is some $\phi(\overline{x})$ with $\overline{x}$ having the same context as $\overline{a}$ such that* $\mathrm{pp}(\overline{a}) = \langle \phi \rangle$.

Proof. Since M is finitely presented, there are finite sums of representables $Z = \bigoplus_{i=1}^{n} よC_i$ and $Y = \bigoplus_{j=1}^{m} よB_j$ and a morphism $T \colon Z \to Y$ such that M is the cokernel of T with an epimorphism $\overline{b} \colon Y \to M$.

Now, suppose that each a_k in $\overline{a}$ is a morphism from $よA_k$ to M. Let $W = \bigoplus_{k=1}^{t} よA_k$. Then, $\overline{a}$ is a morphism $W \to M$ restricting to the a_k on each component.

Consider $W \oplus Y$ and let $(\overline{a}, \overline{b})$ be the morphism $W \oplus Y \to M$ that restricts to $\overline{a}$ and $\overline{b}$ on components. Let $y \colon Y \to W \oplus Y$ be the coprojection. We show that $(\overline{a}, \overline{b})$ is an epimorphism. Let $f, g \colon M \to N$ be morphisms such that $f \circ (\overline{a}, \overline{b}) = g \circ (\overline{a}, \overline{b})$. Then, $f \circ (\overline{a}, \overline{b}) \circ y = g \circ (\overline{a}, \overline{b}) \circ y$ and we substitute on both sides to get $f \circ \overline{b} = g \circ \overline{b}$. We know that $\overline{b}$ is epimorphic, so we obtain $f = g$ as required.

Now, consider $\ker(\overline{a}, \overline{b})$. By Proposition 4.23, we know that is finitely generated, and hence there is some finite sum of representables O and epimorphism $f = O \to \ker(\overline{a}, \overline{b})$. Write r for the composition $\ker(\overline{a}, \overline{b}) \circ f$. Then, by Proposition 5.8, r corresponds to some quantifier free pp-formula $\psi(\overline{x}, \overline{y})$ and, since $O \to W \oplus Y \to M \to 0$ is exact, $(M, (\overline{a}, \overline{b}))$ is its free realisation. Let $\phi(\overline{x}) = \exists \overline{y}.\psi(\overline{x}, \overline{y})$. Following the proof of Proposition 5.8,we see that $\mathrm{pp}(\overline{a}) = \langle \phi \rangle$. □

5.2 Categories of pp-formulae

Let L be a language, T a theory and $\mathcal{M}$ a model. Given a definable set A and a definable relation $\sim \subset A \times A$, the quotient $A/\sim$ is called an "imaginary". This quotient is not usually definable over T. When this does occur, T is said to have **elimination of imaginaries**. There is a standard extension of L called L^{eq} given by adding a sort for every such quotient. The extension of T to this language, called T^{eq} has elimination of imaginaries.

In this section, we will define an additive analogue of this idea. This will be slightly more general than the usual setting, since every inclusion of definable sets induces a quotient, since we are working with Abelian groups. These inclusions will be called pp-pairs. Together, they will provide a kind of eq construction for the model theory of modules.

Definition 5.10. A **pp-pair** is two pp-formulae $\phi(\overline{x})$ and $\psi(\overline{x})$ such that $\phi \geq \psi$. We write this pair as ϕ/ψ.

Given a module M, each pp-pair induces a corresponding quotient of Abelian groups, $\phi(M)/\psi(M)$. This induces a functor $F_{\phi/\psi}$ which acts on morphisms by reducing to the quotients. Functoriality here follows from functoriality of F_ϕ and F_ψ. We call functors of the form $F_{\phi/\psi}$ **pp-functors**. By evaluating each of F_ϕ, F_ψ and $F_{\phi/\psi}$ pointwise and using Proposition 1.50, we see that $F_{\phi/\psi}$ is in fact the cokernel of the inclusion $F_\psi \hookrightarrow F_\phi$.

Let $\phi(\overline{x})/\psi(\overline{x})$ and $\phi'(\overline{x}')/\psi'(\overline{x}')$ be pp-pairs where $\overline{x}$ and $\overline{x}'$ have contexts $(A_1, \ldots, A_n)$ and $(A'_1, \ldots, A'_m)$ respectively. We consider pp-formulae $\rho(\overline{x}, \overline{x}')$ which satisfy, for each module M and tuple $(\overline{a}, \overline{a}') \in \bigoplus MA_i \oplus \bigoplus MA'_j$:

1. If $M \vDash \rho(\overline{a}, \overline{a}')$ and $M \vDash \phi(\overline{a})$ then $M \vDash \phi'(\overline{a}')$.
2. If $M \vDash \rho(\overline{a}, \overline{a}')$ and $M \vDash \psi(\overline{a})$ then $M \vDash \psi'(\overline{a}')$.

3. If $M \vDash \phi(\overline{a})$ then there is some $\overline{a}'$ with $M \vDash \rho(\overline{a}, \overline{a}')$.

The conditions are together called the **functionality** conditions.

Suppose $\overline{a}'$ and $\overline{a}''$ are such that $M \vDash \rho(\overline{a}, \overline{a}')$ and $M \vDash \rho(\overline{a}, \overline{a}'')$. Since $\rho(M)$ is a subgroup of $\bigoplus MA_i \oplus \bigoplus MA'_j$, we see that $M \vDash \rho(0, \overline{a}' - \overline{a}'')$ and since $0 \in \psi(M)$ it follows that $\overline{a}' - \overline{a}'' \in \psi(M)$. In general, if $\overline{a}$ and $\overline{b}$ are ψ-equivalent, then any $\overline{a}'$ and $\overline{b}'$ with $M \vDash \rho(\overline{a}, \overline{a}') \wedge \rho(\overline{b}, \overline{b}')$ are ψ'-equivalent. Putting these facts together with third condition above, we see that ρ defines a function $f_\rho \colon \phi(M)/\psi(M) \to \phi'(M)/\psi'(M)$ for each module M.

A **pp-definable function** $\phi/\psi \to \phi'/\psi'$ is an equivalence class of such ρ under the relation $\rho \sim \rho'$ iff $f_\rho = f_{\rho'}$ for every module M. Composition of pp-definable functions is given by normal function composition. To see that this is pp-definable, let $\rho(\overline{x}, \overline{x}')$ and $\sigma(\overline{x}', \overline{x}'')$ define functions $f_\rho \colon \phi(M)/\psi(M) \to \phi'(M)/\psi'(M)$ and $f_\sigma \colon \phi'(M)/\psi'(M) \to \phi''(M)/\psi''(M)$ for every M. Let $\chi(\overline{x}, \overline{x}'') = \exists \overline{y}. \rho(\overline{x}, \overline{y}) \wedge \sigma(\overline{y}, \overline{x}'')$.

We now check the conditions for χ. Let $\overline{a} \in M$. If $\overline{a} \in \phi(M)$, then there is some $\overline{a}' \in \phi'(M)$ such that $(\overline{a}, \overline{a}') \in \rho(M)$ and similarly there is some $\overline{a}'' \in \phi''(M)$ such that $(\overline{a}', \overline{a}'') \in \sigma(M)$. Hence, $M \vDash \rho(\overline{a}, \overline{a}') \wedge \sigma(\overline{a}', \overline{a}'')$ and so $(\overline{a}, \overline{a}'') \in \chi(M)$. This is the third condition. For the first condition, suppose that $(\overline{a}, \overline{a}'') \in \chi(M)$. Then, there exists $\overline{a}' \in M$ such that $(\overline{a}, \overline{a}') \in \rho(M)$ and so $\overline{a} \in \phi(M)$ and similarly, $(\overline{a}', \overline{a}'') \in \phi''(M)$ by applying the first condition for ϕ and ρ. The proof for the second condition is identical.

It follows that χ defines some function f_χ. Further, $f_\chi(\overline{a})$ is such that there is an $\overline{a}'$ with $f_\chi(\overline{a}) = f_\sigma(\overline{a}')$ and $\overline{a}' = f_\rho(\overline{a})$. Hence, $f_\chi = f_\sigma \circ f_\rho$. This shows that composition is well defined.

The category whose objects are pp-pairs and whose morphisms are pp-definable functions is called the **category of pp-pairs**, denoted $\mathbb{L}_R^{\mathrm{eq}+}$.

6 Functors on mod-R

We say that a (possibly large) category is **skeletally small** if it is equivalent to a small category, called a **skeleton**. Recall from the discussion before Theorem 4.20 that, up to isomorphism, there are only a set of cokernels of morphisms between finite sums of representables. Proposition 4.10 tells us that these are in bijection with the finitely presented modules of R. Hence, the full subcategory R**-mod** of finitely presented modules of R**-Mod** is skeletally small. Similarly, the full subcategory **mod-**R of **Mod-**R is skeletally small, since R^{op} being a small preadditive category means that all the same facts hold for right modules over R as well.

All of the results from Subsection 4.2 continue to hold if we take our preaddditive category to be skeletally small rather than small. This is because we can pick some skeleton and apply all the results to modules over that. The equivalence of preadditive categories lifts to an equivalence of module categories and this equivalence preserves all our results.

We can thus think about **mod-**R as a many-object ring in its own right. In this subsection, we will study the category of modules over **mod-**R. Perhaps confusingly, we will study the category of *left* modules over **mod-**R. To make it clear what we are talking about, we will always write this category as $[\mathbf{mod}\text{-}R, \mathbf{Ab}]$ to emphasise the covariancy. We shall collect several important results about this category, cumulating in an equivalence between the finitely presented functors on **mod-**R and the category of pp-pairs seen in Section 5. This equivalence underlies the interchange of methods between the logical and the functorial styles of studying module categories that is now central to the modern subject of the model theory of modules.

The content here is drawn mainly from [Pre09] Chapter 10.2 and proofs are reproduced as found in that book.

6.1 Basic Facts

By Proposition 4.16 category **mod-**R is not only preadditive but additive. Recall that the covariant Yoneda embedding sends finite limits to finite colimits and that, in a preadditive category, these coincide as biproducts. It follows that, given some finite collection of finitely presented modules $M_1, \dots, M_n$, there is an isomorphism $\bigoplus \mathrm{Hom}(M_i, -) \cong \mathrm{Hom}(\bigoplus M_i, -)$. Thus, in $[\mathbf{mod}\text{-}R, \mathbf{Ab}]$,we can strengthen Proposition 4.10 to remove the sums.

Proposition 6.1 ([Pre09] Lemma 10.2.1)**.** *Let F be a finitely presented object of* $[\textbf{\textit{mod}}\text{-}R, \mathbf{Ab}]$. *Then there are finitely presented right R-modules M and N making the following sequence exact.* $\mathrm{Hom}(N, -) \longrightarrow \mathrm{Hom}(M, -) \longrightarrow F \longrightarrow 0$

We next move on to proving that $[\mathbf{mod}\text{-}R, \mathbf{Ab}]$ is a **locally coherent** category. A finite;ly presented functor is called **coherent** if every finitely generated subfunctor is itself finitely presented. A category is locally coherent when it is skeletally small and every object is a directed colimit of coherent objects. Since $[\mathbf{mod}\text{-}R, \mathbf{Ab}]$ is skeletally small and locally finitely presentable, it is sufficient (indeed, equivalent, see Lemma E.1.46 [Pre09]) to show that every finitely presented object is coherent. We use this to see that the full subcategory $[\mathbf{mod}\text{-}R, \mathbf{Ab}]^{\mathrm{fp}}$ of finitely presented functors is itself Abelian.

We first prove coherence for specifically representable functors. We will then use this to obtain the full local coherence of the category as a corollary.

Proposition 6.2 ([Pre09] Lemma 10.2.2)**.** *Let M be a finitely presented right R-module and G a subfunctor of* $\mathrm{Hom}(M, -)$. *Then G is finitely generated if and only if it is finitely presented.*

Proof. We do the forwards direction first.

Let $\iota\colon G \to \mathrm{Hom}(M, -)$ be the monomorphism exhibiting G as a subfunctor. Recall from Proposition 4.21 that G is finitely generated if and only if there is some an epimorphism onto it from some finite sum of representables. We use the the discussion from above about commuting the Yoneda embedding with sums to reduce this to a morphism from some representable. Hence G is finitely generated if and only if there is some finitely presented module N and an epimorphism $q\colon \mathrm{Hom}(N, -) \to G$.

Now, we can compose $\iota \circ q$ to get a morphism $\mathrm{Hom}(N, -) \to \mathrm{Hom}(M, -)$. By the Yoneda lemma, $\iota \circ q$ is of the form $\mathrm{Hom}(f, -)$ for some $f\colon M \to N$ in $\mathbf{mod}\text{-}R$. Let $g\colon N \to O$ be the cokernel of f, so we have an exact sequence $M \to N \to O \to 0$. By Proposition 4.17, O is a finitely presented module.

Recall from Proposition 3.34 that for every module P, $\mathrm{Hom}(P, -)$ is a left exact functor. Similarly, $\mathrm{Hom}(-, P)$ is a left exact functor $\mathbf{mod}\text{-}R^{\mathrm{op}} \to \mathbf{Ab}$ by dualising. Thus, $\mathrm{Hom}(-, P)$ sends right exact sequences in $\mathbf{mod}\text{-}R$ to left exact sequences in $\mathbf{mod}\text{-}R$. For every module P, we then have that $0 \to \mathrm{Hom}(O, P) \to \mathrm{Hom}(N, P) \to \mathrm{Hom}(M, P)$ is exact, and hence that $0 \to \mathrm{Hom}(O, -) \to \mathrm{Hom}(N, -) \to \mathrm{Hom}(M, -)$ is exact.

By exactness, we see that $\mathrm{Hom}(g, -)$ is monomorphic and hence equal to its own image. Again by exactness we see that $\ker(\mathrm{Hom}(f, -)) = \ker(\iota \circ q) = \mathrm{Hom}(g, -)$. By monomorphicity of ι, we obtain that $\ker(q) = \mathrm{Hom}(g, -)$. There is then an exact sequence $0 \to \mathrm{Hom}(O, -) \to \mathrm{Hom}(N, -) \to G \to 0$. By Proposition 6.1 we conclude that G is finitely presented.

For the backwards direction, we use Proposition 4.12 which says every finitely presented functor is finitely generated. $\square$

Corollary 6.3 ([Pre09] Corollary 10.2.3)**.** *G be a finitely presented functor in $[\textbf{\textit{mod-}}R, \mathbf{Ab}]$ and H a finitely generated subfunctor of G. Then H is finitely presented.*

Proof. Let $\iota \colon H \to G$ be the subobject morphism for H into G. Since G is finitely presented, it is the cokernel of a morphism $\mathrm{Hom}(f, -) \colon \mathrm{Hom}(M, -) \to \mathrm{Hom}(N, -)$, say with morphism $\pi \colon \mathrm{Hom}(N, -) \to G$. The kernel of π is then the image of $\mathrm{Hom}(f, -)$ and so the object of the kernel, K, is finitely generated by Corollary 4.22.

Since $[\mathbf{mod\text{-}}R, \mathbf{Ab}]$ is Abelian, it has all small limits. Consider the pullback of π and ι. This is an object H' and maps $p_N \colon H' \to \mathrm{Hom}(N, -)$ and $p_H \colon H' \to H$ such that $\pi \circ p_N = \iota \circ p_H$. By Propositions 3.19 and 1.55, we see that p_H is an epimorphism and p_N is a monomorphism. By Proposition 3.20, we also see that the object of the kernel of p_H is the same as that of π and hence the kernel of p_H is finitely generated.

$$\begin{tikzcd}
K \arrow[r, "\ker(\pi)"] & \mathrm{Hom}(N, -) \arrow[r, "\pi"] & G \\
K \arrow[u, equal] \arrow[r, "\ker(p_H)"'] & H' \arrow[u, "p_N"] \arrow[r, "p_H"'] & H \arrow[u, "\iota"']
\end{tikzcd}$$

We claim that H' is a finitely generated functor. Suppose that the claim is true. Then, as a subfunctor of $\mathrm{Hom}(N, -)$, by Proposition 6.2, H' is finitely presented and hence by Proposition 6.1, H is finitely presented, completing the proof.

We now prove that H' is finitely generated. Since H is finitely generated, there is some finitely presented module O and epimorphism $e \colon \mathrm{Hom}(O, -) \to H$. We pullback along p_H to get an object (Q, q_H, q_O). By Proposition 3.19, both

$q_H \colon Q \to H'$ and $q_O \colon Q \to \mathrm{Hom}(O, -)$ are epimorphisms, since p_H and e are. Once again, kernels are preserved.

$$\begin{tikzcd}
K \arrow[r, "\ker(p_H)"] \arrow[d, equal] & H' \arrow[r, "p_H"] & H \\
K \arrow[r, "\ker(q_O)"'] & Q \arrow[u, "q_H"] \arrow[r, "q_O"'] & \mathrm{Hom}(O,-) \arrow[u, "e"']
\end{tikzcd}$$

Recall from Corollary 4.14 that, since $\mathrm{Hom}(O, -)$ is a representable functor, it is projective. Then, by Proposition 3.37 we have that Q is the direct sum of K and $\mathrm{Hom}(O, -)$. Then, by 4.16, we see that Q is finitely generated. Then, by Proposition 4.22, H' is finitely generated. $\square$

We now use the following proposition to establish that $[\textbf{mod-}R, \mathbf{Ab}]^{\mathrm{fp}}$ is Abelian. Its proof is beyond the scope of this project, but it can be found as Proposition E.1.47 in [Pre09].

Proposition 6.4 ([Pre88] Proposition E.1.47)**.** *Let* $\mathbf{C}$ *be a locally finitely presentable Abelian category. Then* C *is locally coherent if and only if the full subcategory* $\mathbf{C}^{fp}$ *of finitely presented objects is an Abelian subcategory.*

Remark 6.5. Being an Abelian subcategory here is stronger than $\mathbf{C}^{\mathrm{fp}}$ simply being Abelian- it also requires that the inclusion functor is exact.

6.2 Finitely presented functors and pp-formulae

Recall from Proposition 5.5 that every pp-formula ϕ induces a functor $F_\phi \colon \textbf{Mod-}R \to \mathbf{Ab}$. This restricts to a functor $\textbf{mod-}R \to \mathbf{Ab}$, which we will still notate as F_ϕ as it will always be clear what the domain is. Further, recall that there is always a finitely presented module and tuple of elements in that module that provides a free realisation for ϕ. We now show that this lifts to an equality of related functors.

In particular, given a finitely presented module C and tuple $\overline{c}$, we have that $\overline{c}$ is (by Yoneda) a morphism from sum finite sum of representables $\bigoplus_{j \in J} よ A_J \to C$. Let $Y = \bigoplus_{j \in J} よ A_J$. Then, there is a corresponding morphism $\mathrm{Hom}(\overline{c}, -) \colon \mathrm{Hom}(C, -) \to \mathrm{Hom}(Y, -)$ in $[\textbf{mod-}R, \mathbf{Ab}]$. We denote the image of $\mathrm{Hom}(\overline{c}, -)$ as $F_{C,\overline{c}}$. This is a finitely generated subfunctor of $\mathrm{Hom}(Y, -)$ (the coimage map being an epimorphism from $\mathrm{Hom}(C, -)$ and the image map being a monomorphism into $\mathrm{Hom}(Y, -)$). Thus, by Proposition 6.3, $F_{C,\overline{c}}$ is finitely presented. We can now formulate the equality.

Proposition 6.6 ([Pre09] Proposition 10.2.8)**.** *Let $\phi(\overline{x})$ be a pp-formula with free realisation $(C, \overline{c})$. Then $F_\phi \cong F_{C,\overline{c}}$.*

Proof. Let $\phi(\overline{x} = \exists \overline{y}.\psi(\overline{x}, \overline{y})$. Let $\overline{d}$ be the tuple in C such that $C \vDash \psi(\overline{c}, \overline{d})$.

Recall from the proof of Proposition 5.8 that if M is a module and $\overline{a}$ and $\overline{b}$ are tuples in M such that $M \vDash \psi(\overline{a}, \overline{b})$ then there is a morphism $g \colon C \to M$ such that $\overline{a} = g \circ \overline{c}$ and $\overline{b} = g \circ \overline{d}$. Indeed, every such morphism induces an element of $\psi(M)$.

Thus, we obtain a function $\eta_M \colon \mathrm{Hom}(C, M) \to \phi(M)$ taking g to $g \circ \overline{c}$. For g and g' morphisms, $(g - g') \circ \overline{c} = (g \circ \overline{c}) - (g' \circ \overline{c})$ and $\phi(M)$ is a group. Thus, η_M is indeed a homomorphism of groups and is surjective since every $\overline{a}$ must arise in this way.

Let M, N be modules and $f \colon M \to N$ be a morphism. Then, for each g in $\mathrm{Hom}(C, M)$, $(\eta_N \circ \mathrm{Hom}(C, f))(g) = f \circ g \circ \overline{c} = (F_\phi(f) \circ \eta_M)(g)$ since F_ϕ acts on morphisms by restricting the domain. Thus, η is an epimorphic natural transformation $\mathrm{Hom}(C, -) \to F_\phi$.

Consider the inclusion $\iota \colon F_\phi \to \mathrm{Hom}(Y, -)$. Then, $\iota \circ \eta = \mathrm{Hom}(\overline{c}, -)$. Since ι is monomorphic, it follows that $\ker(\iota \circ \eta) = \ker(\eta)$ and so $\mathrm{coim}(\iota \circ \eta) = \mathrm{coim}(\eta)$. Since η is epimorphic, $\eta = \mathrm{coim}(\eta)$. By the image-coimage factorisation, we find that the object of the coimage, F_ϕ is the image of $\mathrm{Hom}(\overline{c}, -)$. □

Remark 6.7. This shows that F_ϕ is finitely presented for every ϕ.

We can use this to express the representable functors in [**mod-**R, **Ab**] in terms of pp-formulae.

Corollary 6.8 ([Pre09] Proposition 10.2.9)**.** *Let M be a finitely presented right module. Then, there is a pp-formula ϕ such that* $\mathrm{Hom}(M, -) \cong F_\phi$.

Proof. Consider a finitely presented subfunctor H of $\mathrm{Hom}(Y, -)$ where Y is a finite sum of representables in **mod-**R. Let ι be the inclusion map. Then, there is some representable $\mathrm{Hom}(C, -)$ and epimorphism $\eta \colon \mathrm{Hom}(C, -) \to H$ such that H is the image of $\iota \circ \eta$. By Yoneda, there is some $\overline{c}$ such that $\iota \circ \eta = \mathrm{Hom}(\overline{c}, -)$. Proposition 5.9 tells us that $\mathrm{pp}(\overline{c}) = \langle \phi \rangle$ for some pp-formula ϕ and then the Proposition 6.6 tells us that $H = F_{C,\overline{c}} \cong F_\phi$.

Now, there is some finite direct sum of representable modules Y and epimorphism $c \colon Y \to M$ by Proposition 4.10. Under Yoneda, this gives us a morphism $\mathrm{Hom}(c,-)\colon \mathrm{Hom}(M,-) \to \mathrm{Hom}(Y,-)$. We claim that $\mathrm{Hom}(c,-)$ is a monomorphism.

Let W be a module and consider $\mathrm{Hom}(c,W)\colon \mathrm{Hom}(M,W) \to \mathrm{Hom}(Y,W)$, which acts as precomposition with c. Then, for any two $u,v \in \mathrm{Hom}(M,W)$, if $\mathrm{Hom}(c,W)(u) = \mathrm{Hom}(c,W)(v)$, we see that $u \circ c = v \circ c$ and hence $u = v$ since c is an epimorphism. Then, $\mathrm{Hom}(c,-)$ is componentwise monomorphic and so, by Proposition 1.21, a monomorphism itself. This exhibits $\mathrm{Hom}(M,-)$ as a subobject of $\mathrm{Hom}(Y,-)$ and so there is some ϕ with $\mathrm{Hom}(M,-) \cong F_\phi$. $\square$

We now lift this expression to all finitely presented functors. Pp-formulae themselves correspond to subfunctors of representables, so in order to express all finitely presented functors, we need to use pp-pairs.

Proposition 6.9 ([Pre09] Theorem 10.2.30)**.** *Let G be a finitely presented functor in $[\textbf{\textit{mod}-}R, \mathbf{Ab}]$. Then, there is a pp-pair ϕ/ψ such that $G \cong F_{\phi/\psi}$.*

Proof. By Proposition 6.1, there are finitely presented modules M and N and a morphism $f\colon M \to N$ such that G is the cokernel of $\mathrm{Hom}(f,-)\colon \mathrm{Hom}(N,-) \to \mathrm{Hom}(M,-)$ with cokernel morphism e.

From the proof of Corollary 6.8, there is some finite sum of representables Y in $\textbf{mod-}R$ and epimorphism $c\colon Y \to M$. This givs rise to a monomorphism $\mathrm{Hom}(c,-)$ and so we see $\mathrm{Hom}(M,-) \cong F_\phi$ for some formula ϕ where (M,c) is a free realisation of ϕ.

We now consider the morphism $g = \mathrm{Hom}(c,-) \circ \mathrm{Hom}(f,-)\colon \mathrm{Hom}(N,-) \to \mathrm{Hom}(Y,-)$. Since $\mathrm{Hom}(c,-)$ is a monomorphism, it follows by the dual of Proposition 3.7 that $\ker(g) = \ker(\mathrm{Hom}(f,-))$ and so $\mathrm{coim}(g) = \mathrm{coim}(\mathrm{Hom}(f,-))$. Call the object of the coimage C. Then, this is a subobject of $\mathrm{Hom}(Y,-)$ by $\mathrm{im}(g)$ and thus there is some ψ with $C \cong F_\psi$. There is also a monomorphism $\mathrm{im}(\mathrm{Hom}(f,-))\colon C \to \mathrm{Hom}(M,-)$.

Let $\eta\colon \mathrm{Hom}(M,-) \to F_\phi$ and $\varepsilon\colon C \to F_\psi$ be the isomorphisms obtained from Corollary 6.8. Hence, for every module A and $a \in \mathrm{Hom}(M,A)$, we have that $\eta_A(a) = c \circ a$. To make it totally clear what is happpening, consider the diagrams below. The map ι_ϕ behaves as the inclusion map on components, so $\iota_{\phi,A}\colon \phi(A) \hookrightarrow \mathrm{Hom}(Y,A)$ is an inclusion. Similarly for ι_ψ.

$$
\begin{tikzcd}
 & \mathrm{Hom}(Y,-) & \\
\mathrm{Hom}(M,-) \arrow[ur, "\mathrm{Hom}(c,-)"] \arrow[rr, "\eta"'] & & F_\phi \arrow[ul, "\iota_\phi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{Hom}(Y,-) & \\
C \arrow[ur, "\mathrm{im}(g)"] \arrow[rr, "\varepsilon"'] & & F_\psi \arrow[ul, "\iota_\psi"']
\end{tikzcd}
$$

Both $(\mathrm{Hom}(M,-), \mathrm{Hom}(c,-))$ and (F_ϕ, ι_ϕ) are an image for $\mathrm{Hom}(c,-)$, with η being the unique isomorphism between them making them commute, as induced by the (co)limit structure of the image.

We now consider the map $\chi = \eta \circ \mathrm{im}(\mathrm{Hom}(f,-)) \circ \varepsilon^{-1} \colon F_\psi \to F_\phi$. Let A be a module and $a \in \psi(A)$. We have not explicitly defined what CA should be, but we can treat it abstractly via $\mathrm{Hom}(N, A)$. In particular, there is some $u \in \mathrm{Hom}(N, A)$ such that $\mathrm{coim}(\mathrm{Hom}(f,-))_A(u) = \varepsilon_A^{-1}(a)$ and hence $\mathrm{Hom}(f, A)(u) = \mathrm{im}(\mathrm{Hom}(f,-))_A(\varepsilon_A^{-1}(a))$. Now, $\mathrm{Hom}(f, A)(u) = u \circ f$ and so $\chi_A(u \circ f) = \eta_A(u \circ f)$.

From the diagrams above and the equation $\mathrm{coim}(\mathrm{Hom}(f,-)) = \mathrm{coim}(g)$, we see that $\iota_\psi \circ \varepsilon \circ \mathrm{coim}(\mathrm{Hom}(f,-)) = \mathrm{im}(g) \circ \mathrm{coim}(g) = g$. Hence, $\iota_\phi(a) = g(u) = u \circ f \circ c$. Similarly, $\iota_\phi \circ \eta_A = \mathrm{Hom}(c,-)$, so $\iota_\phi(\chi(a)) = \mathrm{Hom}(c,-)(u \circ f) = u \circ f \circ c$. Since ι_ϕ and ι_ψ are acting as inclusion maps, we see that χ is acting as the inclusion map $F_\psi \to F_\phi$. We obtain the following commuting diagram.

$$
\begin{tikzcd}
C \arrow[r, "\mathrm{Hom}(f,-)"] \arrow[d, leftrightarrow, "\varepsilon"'] & \mathrm{Hom}(M,-) \arrow[r, "e"] \arrow[d, leftrightarrow, "\eta"'] & G \arrow[d, dashed, leftrightarrow, "\exists!"] \\
F_\psi \arrow[r, "\chi"'] & F_\phi \arrow[r, "\mathrm{coker}(\chi)"'] & F_{\phi/\psi}
\end{tikzcd}
$$

The isomorphism on the right exists by functoriality of colimits. In particular, there is a unique morphism $\alpha \colon G \to F_{\phi/\psi}$ from viewing η and ε as going from the top row to the bottom, and then a unique morphism $\beta \colon F_{\phi/\psi} \to G$ from considering the inverses going from the bottom row to the top. Then $\beta \circ \alpha = \mathrm{id}_G$ and $\alpha \circ \beta = \mathrm{id}_{F_{\phi/\psi}}$ since cokernels have unique endomorphisms commuting with their cocones (and this is always the identity). $\square$

Remark 6.10. This also shows that $F_{\phi/\psi}$ is finitely presented.

We have now shown that the map from pp-pairs to finitely presented functors that takes a pair to its associated functor is essentially surjective. We are thus well-equipped to prove the equivalence of $[\textbf{mod-}R, \mathbf{Ab}]$ and $\mathbb{L}_R^{\mathrm{eq}+}$. What remains is to prove full and faithful functoriality.

Theorem 6.11 ([Pre09] Theorem 10.2.30)**.** *Let $\mathbb{F}\colon \mathbb{L}_R^{eq+} \to [\boldsymbol{mod}\text{-}R, \mathbf{Ab}]$ be the map on objects taking ϕ/ψ to $F_{\phi/\psi}$. Then, $\mathbb{F}$ extends to a functor and this functor is part of an equivalence of categories.*

Proof. First, we establish functoriality.

Let $\rho(\overline{x}, \overline{x}')$ be a representative for a pp-definable function $t\colon [\phi/\psi](\overline{x}) \to [\phi'/\psi'](\overline{x}'))$. We let $\mathbb{F}t$ be the family of maps $(\mathbb{F}t_A\colon \phi(A)/\psi(A) \to \phi'(A)/\psi'(A))_{A\in\mathbf{mod}\text{-}R}$ where $\mathbb{F}t_A$ has graph $\rho(A)$. We need to show that $\mathbb{F}t$ is natural. To do this, let $f\colon M \to N$ be a morphism of finitely presented modules. Since ρ is a pp-formula, it follows from Proposition 5.5 that for all $(\overline{a}, \overline{a}') \in \rho(M)$, we have that $(f(\overline{a}), f(\overline{a}')) \in \rho(N)$. Thus, $\mathbb{F}t_N(f(\overline{a})) = f(\overline{a}')$. By definition of $F_{\phi/\psi}$, $F_{\phi/\psi}f$ is the restriction of f to $\phi(M)/\psi(M) \to \phi(N)/\psi(N)$ and similarly for $F_{\phi'/\psi'}$. Hence:

$$\begin{aligned}(\mathbb{F}t_N \circ F_{\phi/\psi}f)(\overline{a}) &= \mathbb{F}t_N(f(\overline{a})).\\ &= f(\overline{a}')\\ &= F_{\phi'/\psi'}(\overline{a}')\\ &= (F_{\phi'/\psi'} \circ \mathbb{F}t_M)(\overline{a})\end{aligned}$$

This is naturality for $\mathbb{F}t$. Thus, $\mathbb{F}$ extends to a map of morphisms. That composition is preserved follows from the fact that composition in $\mathbb{L}_R^{\text{eq}+}$ arises precisely as usual function composition of the definable functions in each module, which are exactly the Ft_A. Since natural transformations are composed componentwise, we see that the composition is the same. Thus, $\mathbb{F}$ is a functor.

Since equality of natural transformations is componentwise and morphisms in $\mathbb{L}_R^{\text{eq}+}$ are equivalence classes up to defining the same functions on each module, we quickly get that $\mathbb{F}$ is faithful. That is, suppose t and s are morphisms in $\mathbb{L}_R^{\text{eq}+}$ such that $\mathbb{F}t = \mathbb{F}s$. Then, it must be the case that for every module A, $\mathbb{F}t_A = \mathbb{F}s_A$ and hence t and s must have been the same definable function to begin with.

We move on to showing that $\mathbb{F}$ is full. Let $\tau\colon F_{\phi/\psi} \to F_{\phi'/\psi'}$ be a natural transformation. Let $(C, \overline{c})$ be a free realisation of ϕ and let $\overline{c}'$ be such that $\overline{c}' + \psi'(C) = \tau_C(\overline{c} + \psi(C))$. Let $\rho(\overline{x}, \overline{x}')$ be a generator for $\text{pp}(\overline{c}, \overline{c}')$. We claim that ρ is functional and that τ is the natural transformation associated with the function that ρ defines.

First, we will work through the functionality conditions for ρ. Note that $\phi(\overline{x}) \wedge \phi(\overline{x}')$ is in the pp-type of $(\overline{c}, \overline{c}')$. This is because we have chosen $\overline{c}$ and $\overline{c}'$ to be representatives for elements of the quotients $\phi(C)/\psi(M)$ and $\phi'(C)/\psi'(C)$ respectively. Hence, for all modules M and $(\overline{a}, \overline{a}')$ in $\rho(M)$, we obtain that $\overline{a} \in \phi(M)$ and $\overline{a}' \in \phi'(M)$. This implies the first functionality condition.

For the second functionality condition, suppose we have tuples $\overline{b}$ and $\overline{b}'$ in M such that $\overline{b} \in \psi(M)$ and $(\overline{b}, \overline{b}') \in \rho(M)$. We need to show that $\overline{b}' \in \psi'(M)$. Recall from the proof of Proposition 5.8 that $(\overline{c}, \overline{c}')$ can be expressed as the restriction of a cokernel and hence that $(\overline{b}, \overline{b}')$ must factor through $(\overline{c}, \overline{c}')$ as $(\overline{b}, \overline{b}') = g \circ (\overline{c}, \overline{c}')$ for some $g\colon C \to M$. Recall that $F_{\phi/\psi}\, g$ is just the map induces by the quotient on the restriction of g to ϕ and so preserves ϕ'. It follows, invoking naturality of τ, that:

$$\begin{aligned}
\overline{b}' + \psi'(M) &= F_{\phi'/\psi'}\, g(\overline{c}' + \psi'(C)) \\
&= (F_{\phi'/\psi'}\, g \circ \tau_C)(\overline{c} + \psi(C)) \\
&= (\tau_M \circ F_{\phi/\psi}\, g)(\overline{c} + \psi(C)) \\
&= \tau_M(\overline{b} + \psi(M)))
\end{aligned}$$

Then, since $\overline{b} \in \psi(M)$, we see that $\overline{b} + \psi(M) = 0 + \psi(M)$. Since τ_M is a homomorphism, we get that $\overline{b}' + \psi'(M) = 0$ and so $\overline{b}' \in \psi'(M)$, as required.

For the third functionality condition, let M be a module and $\overline{a} \in \psi(M)$. We must show that there exists some $\overline{a}'$ such that $(\overline{a}, \overline{a}') \in \rho(M)$. As before, we use the proof of Proposition 5.8 to find a morphism $g\colon C \to M$ such that $g \circ \overline{c} = \overline{a}$. We set $\overline{a}' = g \circ \overline{c}'$. It follows that $(\overline{a}, \overline{a}) \in \rho(M)$ as per the argument in Proposition 5.8.

We now know that ρ is functional, and hence there is some morphism t in $\mathbb{L}_R^{\mathrm{eq}+}$ such that for each module M, the graph of t_M is $\rho(M)$. We claim that $\tau = \mathbb{F}t$. To see this, let $\overline{a} \in \phi(M)$ and g be the morphism such that $\overline{a} = g \circ \overline{c}$. We note that, by definition of ρ, $\tau_C(\overline{c} + \psi(C) = \mathbb{F}t_C(\overline{c} + \psi(C))$. Then, we compute:

$$\begin{aligned}
\tau_M(\overline{a} + \psi(M)) &= \tau_M(g \circ \overline{c} + \psi(M)) \\
&= (\tau_M \circ F_{\phi/\psi}\, g)(\overline{c} + \psi(C)) \\
&= (F_{\phi'/\psi'}\, g \circ \tau_C)(\overline{c} + \psi(C)) \\
&= (F_{\phi'/\psi'}\, g \circ \mathbb{F}t_C)(\overline{c} + \psi(C)) \\
&= (\mathbb{F}t_M \circ F_{\phi/\psi}\, g)(\overline{c} + \psi(C)) \\
&= \mathbb{F}t_M(\overline{a} + \psi(M))
\end{aligned}$$

Hence, we see that τ_M and $\mathbb{F}t_M$ agree everywhere, and so τ and $\mathbb{F}t$ agree on every component and so are equal. This shows that $\mathbb{F}$ is a full functor.

Finally, since $\mathbb{F}$ is full, faithful and essentially surjective, it is part of an equivalence between $[\mathbf{mod}\text{-}R, \mathbf{Ab}]$ and $\mathbb{L}_R^{\mathrm{eq}+}$. $\square$

We mentioned at the beginning of Subsection 5.2 that $\mathbb{L}_R^{\mathrm{eq}+}$ is an additive analogue of the eq construction that eliminates imaginaries in usual model theory. Indeed, by Theorem 6.11, we see that $\mathbb{L}_R^{\mathrm{eq}+}$ is skeletally small and Abelian and hence we can take modules over it.

For each right R-module M, we can look at the functor $\mathrm{ev}_M \colon \mathbb{L}_R^{\mathrm{eq}+} \to \mathbf{Ab}$ which takes a pp-pair ϕ/ψ and sends it to $\phi(M)/\psi(M)$ and sends morphisms ρ to the corresponding function with graph $\rho(M)$. Thus, we can think of every right module M as a left module over $\mathbb{L}_R^{\mathrm{eq}+}$, via ev_M.

Unlike the usual eq-construction, however, it is not the case that these are the only modules over $\mathbb{L}_R^{\mathrm{eq}+}$. We can nonetheless recover the category of modules using Theorem 18.1.4 of [Pre09] which implies that the functors ev_M are precisely the class of exact functors $\mathbb{L}_R^{\mathrm{eq}+} \to \mathbf{Ab}$. Explicit examples of non-exact functors requires more Homological machinery than we have developed, and so are omitted here. We refer the interested reader to Section 10.2.6 of [Pre09].

This presentation of categories of modules via pp-pairs is essential for the formulation of Definable Categories and associated tools such as the Ziegler spectrum, which are now central to the Model Theory of Modules. We shall explore these applications in the dissertation.

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