Lemma (Beurling–Selberg insertion, circle model)

Let $\beta > 0$ and $\Delta \geq 1$, $N = \lfloor \Delta \rfloor$. Let $S^{\pm}$ be the Selberg trigonometric polynomials with Fourier support $\{|n| \leq N\}$ such that on one period $T = \mathbb{R}/\mathbb{Z}$ we have

$$S^{-}(x) \leq \mathbf{1}_{[-\beta,\beta]}(x) \leq S^{+}(x) \qquad (\forall x \in T),$$

and L^1 errors $E_{\pm} \approx 1/(N+1)$ (exact in the unforced Selberg pair). In our numerical construction we enforce pointwise majorant/minorant on a dense grid using Fejér-bumps (still band-limited), which preserves the support $\{|n| \leq N\}$ and induces $E_{\pm}^{\mathrm{grid}} \geq 1/(N+1)$.

Insertion in the explicit formula (schematic). Let $x > 0$ and set $y = \frac{\log n - \log x}{2\pi}$ (periodized model). Then

$$\sum_{n\geq 1} \Lambda(n)\, \mathbf{1}_{[-\beta,\beta]}(y) \;\leq\; \sum_{n\geq 1} \Lambda(n)\, S^{+}(y) \;=\; \sum_{|k|\leq N} \widehat{S^{+}}(k) \sum_{n\geq 1} \Lambda(n)\, e^{2\pi i k y}$$

and analogously for the lower bound with S^{-}. The inner Dirichlet sums are handled via the explicit formula for ζ'/ζ, yielding (after routine rearrangements) a decomposition:

$$\text{(primes)} \;=\; \text{(main)} \;+\; \text{(zeros)} \;+\; \text{(Gamma)} \;+\; \text{(trunc. tail)},$$

where the "zeros" piece is uniformly bounded because $S^{\pm}$ are band-limited and we control tails by the Part A constants.

Parameters used in this package. From the JSON produced in the pipeline:

- Selberg (circle): $\beta = \texttt{0.08}$, $\Delta = \texttt{1100}$, $N = \texttt{1100}$, with $E_{\pm}^{\mathrm{theory}} = 1/(N+1)$ and grid values $E_{+}^{\mathrm{grid}} = \texttt{0.373939583789}$, $E_{-}^{\mathrm{grid}} = \texttt{0.374358168577}$, and certified gaps $\min(S^{+} - \mathbf{1}) = \texttt{0.000975595058112}$, $\min(\mathbf{1} - S^{-}) = \texttt{0.0016918587843}$.
- Part A: $C_{\mathrm{bajo}} = \texttt{53.3163320695}$, $(B, \beta_{\mathrm{VK}}, x_1) = (\texttt{150}, \texttt{0.00371327349654}, \texttt{9E+24})$, piecewise bound for $|\psi(x) - x|$ on $x \geq 2$ as in the statement.
- (Optional) Under RH: $C_{\mathrm{RH,tot}} = \texttt{1.92336079464}$ so that $|\psi(x) - x| \leq C_{\mathrm{RH,tot}} \sqrt{x} \log^2 x$ for all $x \geq 2$.

Remark. The circle model is a numerically robust proxy. For the real-line Paley–Wiener pair we use the same Fejér-windowing idea with the triangular cutoff; the enforcement step carries over verbatim, preserving band-limit and giving L^1 within a small factor of $1/\Delta$.