Anderson modules and L-series : a P-adic study

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Contents

Introduction (en français)

Au XVIII$^{\text{e}}$ siècle, Euler a étudié les valeurs $\zeta(s)$ où s est un entier au moins égal à 2. Ces valeurs zêta s'interpolent en une fonction méromorphe sur $\mathbb{C}$ appelée la fonction zêta de Riemann. De manière plus générale, lorsque l'on peut associer à un objet X de nature arithmétique, ou géométrique, des séries L, une généralisation des valeurs zêta, on espère construire une fonction méromorphe sur $\mathbb{C}$, appelée fonction L et qui interpole ces séries L. L'idée globale est alors que les propriétés analytiques de cette fonction et les propriétés arithmétiques, ou géométriques, de X doivent être étroitement liées. Le concept de fonction L p-adique a émergé en parallèle de celui de série L, fournissant de manière surprenante plus d'informations arithmétiques sur X que les fonctions L. C'est en particulier le cas dans les corps de nombres, où l'on dispose de nombreux exemples depuis le XVIII$^{\text{e}}$ siècle et les travaux d'Euler. Les analogies avec les corps de fonctions globaux étant nombreuses, des séries L puis des fonctions L ont été introduites ces dernières années dans ce contexte. Ce travail consiste à construire des séries L P-adiques pour les corps de fonctions.

Le cas des corps de nombres

Nous commençons par rappeler le cas classique des corps de nombres.

Pour tout nombre complexe s de partie réelle $\Re(s) > 1$, on définit la fonction zêta de Riemann par

$$\zeta(s) = \sum_{n \geq 0} \frac{1}{n^s}.$$

Étudiées par Euler au XVIII$^{\text{e}}$ siècle, les valeurs aux entiers de la fonction zêta, appelées valeurs spéciales, recèlent une quantité importante d'applications arithmétiques. Euler obtient en particulier le produit dit “eulérien” suivant pour tout réel $k > 1$:

$$\zeta(k) = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^k}\right)^{-1}$$

où le produit porte sur l'ensemble des nombres premiers. Cette écriture révèle un lien profond entre les valeurs zêta et l'arithmétique des nombres premiers, dont une quantité extraordinaire serait une conséquence de la fameuse hypothèse de Riemann.

Introduites par Richard Dedekind [20, Chapitre 5], les fonctions zêta de Dedekind sont des généralisations de la fonction zêta de Riemann aux corps de nombres, c'est à dire aux extensions finies de $\mathbb{Q}$. Soit $K/\mathbb{Q}$ une telle extension. Notons $N_{K/\mathbb{Q}}$ l'application norme et soit $\mathscr{O}_K$ la fermeture intégrale de $\mathbb{Z}$ dans K. On définit alors la fonction zêta de Dedekind pour tout complexe s de partie réelle $\Re(s) > 1$ par

$$\zeta_K(s) = \sum_{I \subseteq \mathcal{O}_K} \frac{1}{N_{K/\mathbb{Q}}(I)^s}$$

où la somme porte sur les idéaux non nuls de $\mathcal{O}_K$. Elle admet également un produit eulérien

$$\zeta_K(s) = \prod_{\mathfrak{P} \subseteq \mathcal{O}_K} \left(1 - \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{P})^s}\right)^{-1}$$

où $\mathfrak{P}$ parcourt l'ensemble des idéaux premiers non nuls de $\mathcal{O}_K$. En considérant $K = \mathbb{Q}$ on retrouve la fonction zêta de Riemann. On peut attacher plusieurs invariants à l'extension $K/\mathbb{Q}$:

- r_1 le nombre de plongements réels de K,
- r_2 le nombre de plongements complexes de K,
- h_K le nombre de classes,
- Reg_K le régulateur de K,
- w_K le nombre de racines de l'unité contenues dans K,
- D_K le discriminant de l'extension $K/\mathbb{Q}$.

On dispose de la formule analytique du nombre de classes de Richard Dedekind suivante, reliant les invariants précédemment introduits à la fonction zêta de Dedekind. On renvoie à [37, Chapitre 7, section 5] pour plus de détails.

Théorème (Formule analytique du nombre de classes)**.** *Soit K un corps de nombres. Alors $\zeta_K(s)$ converge absolument pour tout nombre complexe s tel que $\Re(s) > 1$. Elle se prolonge en une fonction méromorphe sur $\mathbb{C}$, avec un seul pôle simple en $s = 1$. De plus, nous avons la formule analytique du nombre de classes:*

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} R_K}{w_L \sqrt{|D_K|}} h_K$$

Soit p un nombre premier et notons v_p la valuation p-adique sur $\mathbb{Q}$. Soit $\mathbb{Q}_p$ le complété de $\mathbb{Q}$ pour v_p, appelé corps des nombres p-adiques. On peut le voir comme l'analogue de $\mathbb{R}$ pour la valeur absolue usuelle. Mais dans ce cadre p-adique, il s'avère qu'une clôture algébrique de $\mathbb{Q}_p$ n'est pas complète. Notons alors $\mathbb{C}_p$ le complété d'une clôture algébrique de $\mathbb{Q}_p$, qui est de plus algébriquement clos. On note $\mathbb{Z}_p$ le complété de $\mathbb{Z}$ pour v_p, appelé anneau des entiers p-adiques. Au XX[e] siècle, les premiers analogues p-adiques des fonctions zêta ont vues le jour.

L'idée de leurs constructions peut-être vue comme une réponse à la question suivante:

"Peut-on construire une fonction p-adique analytique $\zeta_p : \mathbb{Z}_p \to \mathbb{C}_p$ qui coïncide avec la fonction ζ sur les entiers strictement négatifs, à un certain facteur correctif près ? "

Leopoldt [34], puis Leopoldt et Kubota [32] via les congruences de Kummer et les nombres de Bernoulli, introduisent les fonctions zêta p-adiques. Ils prouvent le résultat suivant.

Théorème. *Soit $i \in \mathbb{Z}/(p-1)\mathbb{Z}$ ($i \in \mathbb{Z}/2\mathbb{Z}$ si $p = 2$). Il existe une unique fonction $\zeta_{p,i}$ continue sur $\mathbb{Z}_p$ (resp. $\mathbb{Z}_p \backslash \{1\}$) si $i \neq 1$ (resp. si $i = 1$) telle que la fonction $(s-1)\zeta_{p,i}$ soit analytique sur $\mathbb{Z}_p$ et que l'on ait*

$$\zeta_{p,i}(-n) = (1-p^n)\zeta(-n), \text{ si } n \in \mathbb{N} \text{ vérifie } -n \equiv i \pmod{p-1}.$$

Ils [32] ont en fait généralisé cette construction pour construire des fonctions L p-adiques sur des extensions réelles abéliennes finies de $\mathbb{Q}$.

Dans le cas d'une extension $F/\mathbb{Q}$ totalement réelle, c'est-à-dire lorsque $r_2 = 0$ avec les notations précédentes, Serre [41], via l'utilisation de formes modulaires p-adiques, définit au sens de Kubota et Leopoldt une famille de fonctions zêta p-adique $\zeta_{p,F}(s,i)$ pour $i \in \mathbb{Z}/(p-1)\mathbb{Z}$. En particulier pour tout entier pair $k \geq 2$, il [41, Théorème 20] obtient une formule d'interpolation avec la fonction ζ_F de Dedekind, en supprimant les facteurs locaux correspondant à p:

$$\zeta_{p,F}(1-k,i) = \zeta_F(1-k)\prod_{\mathfrak{P}|p}\left(1 - N_{F/\mathbb{Q}}(\mathfrak{P})^{k-1}\right) \text{ si } 1-k \equiv i \pmod{p-1}$$

où le produit porte sur les idéaux maximaux non nuls de $\mathscr{O}_F$ divisant p. On retrouve en particulier les fonctions zêta p-adiques de Kubota et Leopoldt si l'extension est abélienne. Cela lui permet alors de définir une fonction zêta p-adique $\zeta_{p,F}$. Colmez [15] obtient alors une formule analytique p-adique du nombre de classe.

Théorème. *Soit p un nombre premier, on a*

$$\lim_{s\to 1}(s-1)\zeta_{p,F}(s) = \prod_{\mathfrak{P}|p}\left(1 - \frac{1}{N_{F/\mathbb{Q}}(\mathfrak{P})}\right)\frac{2^{[F:\mathbb{Q}]-1}R_{p,F}}{\sqrt{D_F}}h_F$$

où le produit porte sur les idéaux maximaux de $\mathscr{O}_F$ divisant p et $R_{p,F}$ est un certain régulateur p-adique associé à l'extension.

Notons que la notion de régulateur p-adique $R_{p,F}$ du théorème précédent est définie (voir [34]) pour un corps de nombres quelconque, et la non-nullité de ce dernier est ce qu'on appelle conjecture de Leopoldt. Le théorème précédent nous donne un lien dans le cas totalement réel entre la conjecture de Leopolt et le pôle simple en $s = 1$ de la fonction zêta p-adique.

Le cas des corps de fonctions

Nous nous intéressons maintenant au cas des corps de fonctions. Commençons par introduire le contexte.

Soit $q = p^e$ une puissance d'un nombre premier et soit $\mathbb{F}_q$ le corps fini à q éléments. On fixe θ une indéterminée, et on considère $A = \mathbb{F}_q[\theta]$ l'anneau des polynômes à coefficients dans $\mathbb{F}_q$, qui jouera le rôle d'anneau de base. Notons A^+ l'ensemble des polynôme unitaires de A. Soit $K = \mathbb{F}_q(\theta)$ le corps des fractions rationnelles. On considère la valuation v_∞ de K normalisée telle que $v_\infty(\theta^{-1}) = 1$. Soit K_∞ la complétion de K par rapport à v_∞. On considère une clôture algébrique $\overline{K}_\infty$ de K_∞. Comme dans le cas p-adique, il s'avère que $\overline{K}_\infty$ n'est pas complet pour v_∞. On complète à nouveau $\overline{K}_\infty$ par rapport à v_∞ et on note $\mathbb{C}_\infty$ le corps obtenu qui est algébriquement clos. On dispose ainsi des analogies suivantes:

$$\begin{array}{lccccccccc}
\text{Corps de nombres:} & \mathbb{N}\backslash\{0\} & \subset & \mathbb{Z} & \subset & \mathbb{Q} & \subset & \mathbb{R} & \subset & \mathbb{C} \\
 & | & & | & & | & & | & & | \\
\text{Corps de fonctions} & A^+ & \subset & A & \subset & K & \subset & K_\infty & \subset & \mathbb{C}_\infty
\end{array}$$

L'analogue du cardinal sera l'idéal de Fitting dont on rappelle la définition. Soit M un A-module fini, c'est-à-dire de type fini et de torsion. Il existe alors des polynômes unitaires $f_1, \ldots, f_m$ tels que l'on ait l'isomorphisme de A-modules:

$$M \simeq \bigoplus_{i=1}^{m} A/f_iA.$$

On définit alors l'idéal de Fitting de M, noté $\mathrm{Fitt}_A(M)$, par $\mathrm{Fitt}_A(M) = \prod_{i=1}^{m} f_iA$ et on note $[M]_A = \prod_{i=1}^{m} f_i \in A$ le générateur unitaire de $\mathrm{Fitt}_A(M)$.

Carlitz [12] introduit l'analogue en caractéristique positive de la fonction zêta de Riemann, appelé fonction zêta de Carlitz, défini pour n un entier strictement positif par

$$\zeta_A(n) = \sum_{a \in A^+} \frac{1}{a^n} \in K_\infty$$

et admettant le produit eulérien

$$\zeta_A(n) = \prod_P (1 - \frac{1}{P^n})^{-1}$$

où le produit porte sur l'ensemble des polynômes irréductibles unitaires de A.

Il introduit également dans le même article le module de Carlitz, le premier exemple de module de Drinfeld défini comme suit. Soit $\tau : \mathbb{C}_\infty \to \mathbb{C}_\infty, x \mapsto x^q$ le Frobenius. On considère $K\{\tau\}$ l'anneau des polynômes tordus, qui est l'anneau des polynômes en la variable τ vérifiant la règle de multiplication

$$\tau a = a^q \tau, \forall a \in K.$$

On définit alors le module de Carlitz, noté C, comme le morphisme de $\mathbb{F}_q$-algèbres $C : A \to A\{\tau\}$ donné par $\theta \mapsto C_\theta = \theta\tau^0 + \tau$. Remarquons que la donnée de C_θ détermine complètement C. Si B est une A-algèbre, on peut munir B d'une nouvelle structure de A-module, notée $C(B)$, via

$$a.x = C_a(x), a \in A, x \in B.$$

En particulier d'après [29, Theorem 3.6.3], pour tout polynôme irréductible unitaire P, on dispose de l'égalité suivante:

$$\frac{[A/PA]_A}{[C(A/PA)]_A} = \frac{P}{P-1}$$

d'où

$$\zeta_A(1) = \prod_P \frac{[A/PA]_A}{[C(A/PA)]_A}$$

où le produit porte sur l'ensemble des polynômes irréductible unitaires de A. C'est de l'égalité précédente que viendra la notion de série L pour les modules de Drinfeld. C'est ce qu'on introduit maintenant.

Soit L/K une extension finie. On note par $\mathscr{O}_L$ la fermeture intégrale de A dans L. Drinfeld [21] introduit la notion de modules de Drinfeld (qu'il appela initialement "module elliptique") que nous définissons comme suit.

Un A-module de Drinfeld défini sur $\mathscr{O}_L$ est un morphisme de $\mathbb{F}_q$-algèbres $\phi : A \to \mathscr{O}_L\{\tau\}$ non constant et vérifiant

$$\phi_\theta = \theta + \sum_{i=1}^{r} \phi_{\theta,i}\tau^i \in \mathscr{O}_L\{\tau\}$$

avec les $\phi_{\theta,i} \in \mathscr{O}_L$ pour $i = 1, \ldots, r$. Remarquons que l'image de θ définit entièrement ϕ. Si B est une $\mathscr{O}_L$-algèbre, on peut alors munir B d'uns structure de A-module via ϕ, que nous notons $\phi(B)$.

L'analogue de la fonction zêta de Carlitz pour un module de Drinfeld général sera la série L suivante, introduite par Taelman dans [42] et définie par le produit eulérien

$$L(\phi/\mathscr{O}_L) = \prod_P \frac{[\mathscr{O}_L/P\mathscr{O}_L]_A}{[\phi(\mathscr{O}_L/P\mathscr{O}_L)]_A}$$

où le produit porte sur les polynômes irréductibles unitaires $P \in A$. Dans le même article, il introduit le module des classes $H(\phi; \mathscr{O}_L)$ et le module des unités $U(\phi; \mathscr{O}_L)$ et conjecture une formule reliant ces objets à la série L précédemment définie. Il [43, Theorem 1] démontre cette formule deux ans plus tard en utilisant la formule des traces d'Anderson [2], résultat connu aujourd'hui sous le nom de formule des classes de Taelman.

Théorème (Formule des classes de Taelman pour les modules de Drinfeld)**.** *On a l'égalité*

$$L(\phi/A) = [\mathscr{O}_L : U(\phi; \mathscr{O}_L)]_A \, [H(\phi; \mathscr{O}_L)]_A \in K_\infty^\times.$$

Dans le théorème précédent, la quantité $[\mathscr{O}_L : U(\phi; \mathscr{O}_L)]_A$ joue le rôle du régulateur des corps de nombres, et $[H(\phi; \mathscr{O}_L)]_A$ l'analogue du nombre de classes en caractéristique positive.

Depuis le travail fondateur de Taelman, des développements ont été effectués dans plusieurs directions.

Anderson [1] introduit la notion de t-modules d'Anderson, généralisation en dimension supérieure des modules de Drinfeld. Fang [24, Theorem 1.10], utilisant la théorie des Chtoucas considérée dans [33] et la théorie de Fontaine, démontre alors la formule des classes dans le cas des t-modules d'Anderson.

Si l'on s'intéresse aux t-modules d'Anderson $C^{\otimes n}$, appelés puissances tensorielles du module de Carlitz et introduits par Anderson et Thakur [3], nous obtenons par [18, Section 4.2] l'égalité

$$L(C^{\otimes n}/A) = \prod_{P \text{ premier}} \frac{P^n}{P^n - 1} = \sum_{a \in A^+} \frac{1}{a^n} = \zeta_A(n).$$

Anglès et Tavares Ribeiro [10] introduisent la notion de z-déformation des modules d'Anderson ainsi que la notion d'unité de Stark, notion qu'ils développent avec Ngo Dac dans [4, 5] et qui leur permettra de nouvelles approches pour étudier les séries L

et obtenir des formules de classes. Finalement, Anglès, Ngo Dac et Tavares Ribeiro [7] prouvent une formule de classe dans le cas où l'anneau A est "général" et E est un module d'Anderson "admissible", comprenant en particulier tous les modules de Drinfeld.

Demeslay [18] a adapté les idées développées par Taelman pour les modules de Drinfeld et démontré la formule des classes pour les t-modules d'Anderson à plusieurs variables. Un des intérêts de cette construction est que, pour étudier certains t-modules d'Anderson, on peut en fait se ramener à l'étude de certains modules de Drinfeld avec des variables, dont l'étude est plus "simple" à effectuer. C'est par exemple la méthode suivie par Anglès, Pellarin et Tavares Ribeiro dans [8] pour étudier les puissances tensorielles du module de Carlitz $C^{\otimes n}$ en se ramenant à l'étude du module de Carlitz C.

De plus, en considérant certains t-modules d'Anderson avec des variables, Demeslay retrouve certaines séries L, appellées série L de Pellarin, introduites par ce dernier dans [39] et définies pour un entier $n > 0$ par

$$L(\chi_t, n) = \sum_{a \in A^+} \frac{\chi_t(a)}{a^n} \in \mathbb{F}_q[t] \otimes_{\mathbb{F}_q} K_\infty$$

où t est une nouvelle variable et $\chi_t(a) = \sum_{j=0}^{m} a_j t^j$ si $a = \sum_{j=0}^{m} a_j \theta^j$.

Plan de la thèse

Le but de cette thèse est de construire et d'étudier une série L P-adique dans le contexte des t-modules d'Anderson. On les construit à partir des séries L déjà construites associées à la z-déformations des t-modules d'Anderson et en utilisant la notion d'unité de Stark. En voici les grandes lignes.

Le premier chapitre consiste principalement en une revue générale de la théorie connue pour la place ∞ mais à partir de laquelle nous définirons le cas P-adique dans les chapitres suivants. Dans les sections 1.1, 1.2, 1.3 et 1.4, nous introduisons les objets au coeur de la théorie : les modules d'Anderson, la z-déformation des modules d'Anderson, le module des unités de Taelman, le module des classes de Taelman et la série L. L'une des nouveautés est la section 1.4. Nous évaluons la nouvelle variable z non seulement en $z = 1$ comme le font Anglès, Tavares Ribeiro et Ngo Dac [4, 5, 7, 10], mais en $z = \zeta \in \overline{\mathbb{F}_q}$. Nous considérons le cas des modules de Drinfeld et du réseau des périodes associé dans la section 1.6. Ces résultats seront utilisés au Chapitre 4 pour étudier l'annulation de la série L P-adique dans le cadre des modules de Drinfeld. Enfin, dans la section 1.7, nous introduisons une classe spéciale de modules de Drinfeld appelés modules de Drinfeld très petits pour lesquels nous ferons une étude explicite de la série L P-adique.

Dans le chapitre 2, nous fixons un polynôme irréductible unitaire P et nous définissons et étudions certaines séries L P-adiques associées aux t-modules d'Anderson en supprimant le facteur local en P de la série L classique. Après quelques sections préliminaires qui sont les sections 2.1, 2.2 et 2.3, nous prouvons dans la section 2.4 le premier théorème principal (voir le Théorème 2.4.1), c'est-à-dire la convergence de la série L P-adique. Dans les sections 2.5 et 2.6, nous prouvons une formule de classe P-adique à la Taelman (voir le Théorème 2.6.9) reliant la série L P-adique, le module de classe et un régulateur P-adique. Dans la section 2.7, nous étudions l'annulation de la série L P-adique nouvellement introduite.

Le but du chapitre 3 est d'étendre les constructions du chapitre 2 au cas où le corps des constantes n'est plus $\mathbb{F}_q$ mais $\mathbb{F}_q(t_1, \ldots, t_s)$ où les t_i sont de nouvelles variables. Dans la section 3.1, nous introduisons les objets dans ce cadre multi-variable. On s'intéresse au cas P-adique dans la section 3.2. Tout d'abord, nous remarquons que tous les résultats du chapitre 2 restent valables en remplaçant $\mathbb{F}_q$ par $\mathbb{F}_q(t_1, \ldots, t_s)$. La difficulté est de considérer le cas "entier", c'est-à-dire lorsque $E_\theta \in M_d(\mathscr{O}_L[t_1, \ldots, t_s])\{\tau\}\}$. Nous adaptons les techniques du chapitre 1 et du chapitre 2 dans ce cadre pour prouver (voir le Théorème 3.2.6) que la série L P-adique n'admet pas de pôle dans aucune des variables $t_1, \ldots, t_s$,, c'est-à-dire que $L_P(\widetilde{E}/\widetilde{\mathscr{O}_{L,s}}) \in \mathbb{T}_{s,z}(K_P)$. Nous introduisons en particulier les séries L P-adiques de Pellarin.

Dans le chapitre 4, nous étudions la série L P-adique dans le cadre des A-modules de Drinfeld définis sur A. Dans la section 4.1, nous établissons un lien surprenant entre la propriété pour un polynôme irréductible unitaire P d'être de ϕ-Wieferich et la valuation P-adique de la valeur spéciale de la série L P-adique du module de Drinfeld ϕ correspondant (voir le Théorème 4.1.8 et le Théorème 4.1.10), un lien qui avait été établi par Thakur [47] uniquement pour le module de Carlitz. Dans la section 4.2, nous étudions l'ordre d'annulation en $z = 1$ de la série L P-adique. Le résultat principal (voir le Théorème 4.2.3) est une relation entre cet ordre et certaines propriétés des bases minimales successives du réseau des périodes associé à ϕ introduites dans la section 1.6. Nous donnons enfin une manière explicite de calculer l'ordre d'annulation dans le cas où ϕ est un module de Drinfeld très petit dans la section 4.3 (voir le Théorème 4.3.3), en ne connaissant que les coefficients de ϕ_θ.

Introduction (in english)

In the 18th century, Euler studied the values $\zeta(s)$ where s is an integer at least equal to 2. These zeta values are interpolated into a meromorphic function on $\mathbb{C}$, called the Riemann zeta function. More generally, when we can associate an arithmetic, or geometric, object X to an L-series, which generalizes the zeta values, we hope to construct a meromorphic function on $\mathbb{C}$, called L-function, which interpolates these L-series. The global idea is that the analytic properties of this function and the arithmetic, or geometric, properties of X should be closely linked. The concept of p-adic L-function has emerged alongside that of L-series, surprisingly providing more arithmetic information about X than the L-functions. This is particularly true for number fields, where we have had numerous examples since the 18th century and the work of Euler. As there are many analogies with global function fields, L-series and then L-functions have been introduced in this context in recent years. This work consists of constructing P-adic L-series for function fields.

The number fields case

We begin by recalling the classic case of number fields.

For any complex number s with real part $\Re(s) > 1$, the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n \geq 0} \frac{1}{n^s}.$$

Studied by Euler in the 18th century, the values of this function at positive integers, known as special values, have a large number of arithmetical consequences. In particular, Euler obtained the following "Eulerian" product for any real $k > 1$:

$$\zeta(k) = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^k}\right)^{-1}$$

where the product runs through the set of prime numbers. This product reveals a deep link between zeta values and the arithmetic of prime numbers, an extraordinary amount of which would be a consequence of the famous Riemann hypothesis.

Introduced by Richard Dedekind, Dedekind's zeta functions are generalizations of the zeta function to number fields, i.e. finite extensions of $\mathbb{Q}$. Let $K/\mathbb{Q}$ be a finite field extension. Let $N_{K/\mathbb{Q}}$ be the norm map, and let $\mathcal{O}_K$ be the integral closure of $\mathbb{Z}$ in K. We then define the Dedekind zeta function for any complex s with real part

$\Re(s) > 1$ by

$$\zeta_K(s) = \sum_{I \subseteq \mathscr{O}_L} \frac{1}{N_{K/\mathbb{Q}}(I)^s}$$

where the sum is taken over the non-zero ideals of $\mathscr{O}_K$. It also has an Euler product

$$\zeta_K(s) = \prod_{\mathfrak{P} \subseteq \mathscr{O}_L} \left(1 - \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{P})^s}\right)^{-1}$$

where $\mathfrak{P}$ runs through the non-zero prime ideals of $\mathscr{O}_K$. Considering $K = \mathbb{Q}$, we obtain the classical Riemann zeta function. We can associate several invariants to the extension $K/\mathbb{Q}$:
• r_1 the number of real embeddings of K,
• r_2 the number of complex embeddings of K,
• h_K the class number,
• Reg_K the regulator of K,
• w_K the number of roots of the unity contained in K,
• D_K the discriminant of the field extension $K/\mathbb{Q}$.
We have the following analytic class formula from Richard Dedekind, relating the invariants previously introduced, to the Dedekind zeta function. See [37, Chapter 7, section 5] for more details.

Theorem (Analytic class number formula)**.** *Let K be a number field. Then $\zeta_K(s)$ converges absolutly for all complex number s such that $\Re(s) > 1$. Furthermore, it extends into a meromorphic function defined for any complex number s with a single simple pole at $s = 1$. In addition, we have the analytic class number formula:*

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} R_K}{w_L \sqrt{|D_K|}} h_K$$

Let p be a prime number and let v_p be the p-adic valuation on $\mathbb{Q}$. Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ for v_p, called the field of p-adic numbers. This can be seen as the analogue of $\mathbb{R}$ for the usual absolute value. But in this p-adic setting, it turns out that an algebraic closure of $\mathbb{Q}_p$ is not complete for v_p. Let us then denote by $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$, which is moreover algebrically closed. Let $\mathbb{Z}_p$ be the completion of $\mathbb{Z}$ for v_p, called the ring of p-adic integers. In the 20th century, the first p-adic analogues of zeta functions and L-series appeared.

The idea of their constructions can be seen as an answer to the following question:

"Can we construct an p-adic "analytic" function $\zeta_p : \mathbb{Z}_p \to \mathbb{C}_p$ that coincides with the ζ function over strictly negative integers, up to a certain correcting factor? "

Leopoldt [34], then Leopoldt and Kubota [32], through Kummer congruences and Bernoulli numbers, introduce the classical p-adic zeta functions. They prove the following result.

Theorem. *Let $i \in \mathbb{Z}/(p-1)\mathbb{Z}$ ($i \in \mathbb{Z}/2\mathbb{Z}$ if $p = 2$). There exists a unique function $\zeta_{p,i}$, continuous on $\mathbb{Z}_p$ (resp. $\mathbb{Z}_p \backslash \{1\}$) if $i \neq 1$ (resp. if $i = 1$), such that the function $(s-1)\zeta_{p,i}$ is analytic on $\mathbb{Z}_p$ and verifying*

$$\zeta_{p,i}(-n) = (1-p^n)\zeta(-n), \text{ if } n \in \mathbb{N} \text{ is such that } -n \equiv i \pmod{p-1}.$$

They [32] in fact generalized this construction to define p-adic L-functions over finite abelian real extensions of $\mathbb{Q}$.

In the case of a totally real $F/\mathbb{Q}$ extension, i.e., when $r_2 = 0$ with the previous notation, Serre [41], via the use of p-adic modular forms, defines in the sense of Kubota and Leopoldt, a family of p-adic zeta functions $\zeta_{p,F}(s,i)$ for $i \in \mathbb{Z}/(p-1)\mathbb{Z}$. In particular, for any even integer $k \geq 2$, he [41, Theorem 20] obtains an interpolation formula with the Dedekind zeta function $\zeta_F(s)$, by removing the local factors corresponding to p:

$$\zeta_{p,F}(1-k,i) = \zeta_F(1-k)\prod_{\mathfrak{P}|p}\left(1 - N_{F/\mathbb{Q}}(\mathfrak{P})^{k-1}\right) \text{ si } 1-k \equiv i \pmod{p-1}$$

where the product is taken over the non zero maximal ideals of $\mathscr{O}_F$ dividing p. In particular, we obtain the p-adic zeta functions of Kubota and Leopoldt if the extension is abelian. This allows him to define a p-adic zeta function $\zeta_{p,F}$. Colmez [15] then obtained a p-adic analytic class number formula.

Theorem. *Let p be a prime number, we have*

$$\lim_{s\to 1}(s-1)\zeta_{p,F}(s) = \prod_{\mathfrak{P}|p}\left(1 - \frac{1}{N_{F/\mathbb{Q}}(\mathfrak{P})}\right)\frac{2^{[F:\mathbb{Q}]-1}R_{p,F}}{\sqrt{D_F}}h_F$$

where the product is taken over the non zero maximal ideals of $\mathscr{O}_F$ dividing p and $R_{p,F}$ is a certain p-adic regulator associated to the extension.

Note that the notion of p-adic regulator $R_{p,F}$ in the previous theorem is defined (see [34]) for any field of numbers, and the non-vanishing of the latter is what we call the Leopoldt conjecture. The previous theorem gives us a link in the totally real case between the Leopoldt conjecture and the simple pole at $s=1$ of the p-adic zeta function.

The function fields case

We now turn our attention to the case of function fields. We start by introducing the context.

Let $q = p^e$ be the power of a prime p and let $\mathbb{F}_q$ be the finite field with q elements. Let θ be an indeterminate and consider $A = \mathbb{F}_q[\theta]$ the ring of polynomials with coefficients in $\mathbb{F}_q$, which will act as the base ring. Let A^+ be the set of monic polynomials of A. Let $K = \mathbb{F}_q(\theta)$ be the rational function field. Consider the valuation v_∞ of K normalized such that $v_\infty(\theta^{-1}) = 1$. Let K_∞ be the completion of K with respect to v_∞. Consider an algebraic closure $\overline{K}_\infty$ of K_∞. As for the p-adic case, it turns out that $\overline{K}_\infty$ is not complete for v_∞. We again complete $\overline{K}_\infty$ with respect to v_∞, and we denote by $\mathbb{C}_\infty$ the field obtained, which is algebraically closed. This gives us the following analogies:

$$\begin{array}{lccccccccc}
\text{Number fields:} & \mathbb{N}\backslash\{0\} & \subset & \mathbb{Z} & \subset & \mathbb{Q} & \subset & \mathbb{R} & \subset & \mathbb{C} \\
 & | & & | & & | & & | & & | \\
\text{Function fields} & A^+ & \subset & A & \subset & K & \subset & K_\infty & \subset & \mathbb{C}_\infty
\end{array}$$

The analogue of the cardinal in positive characteristic will be the Fitting ideal, whose definition is as follows. Let M be a finite A-module i.e., of finite type and torsion. Then there exist monics polynomials $f_1, \dots, f_m \in A^+$ such that we have the isomorphism of A-modules

$$M \simeq \bigoplus_{i=1}^{m} A/f_i A.$$

We then define the Fitting ideal of M, denoted $\mathrm{Fitt}_A(M)$, by $\mathrm{Fitt}_A(M) = \prod_{i=1}^{m} f_i A$ and let $[M]_A = \prod_{i=1}^{m} f_i \in A$ be the unit generator of $\mathrm{Fitt}_A(M)$.

Carlitz [12] introduces the positive charectiristic analogue of the Riemann zeta function, called the Carlitz zeta function, defined for n a strictly positive integer by

$$\zeta_A(n) = \sum_{a \in A^+} \frac{1}{a^n} \in K_\infty$$

and admitting the following eulerian product

$$\zeta_A(n) = \prod_{P} (1 - \frac{1}{P^n})^{-1}$$

where P runs through the monic irreducible polynomial of A.

In the same article, Carlitz also introduced the Carlitz module, the first example of Drinfeld modules, defined as follows. Let $\tau : \mathbb{C}_\infty \to \mathbb{C}_\infty, x \mapsto x^q$ be the Frobenius map. Consider $K\{\tau\}$ the ring of twisted polynomials, which is the ring of polynomials in the variable τ satisfying the multiplication rule

$$\tau a = a^q \tau, \forall a \in K.$$

We then define the Carlitz module, denoted by C, as the homomorphism of $\mathbb{F}_q$-algebras $C : A \to A\{\tau\}$ given by $\theta \mapsto C_\theta = \theta \tau^0 + \tau$. Note that C_θ completely determines C. If B is an A-algebra, we can give B a new A-module structure, denoted $C(B)$, by

$$a.x = C_a(x), a \in A, x \in B.$$

In particular, according to [29, Theorem 3.6.3], for any irreducible unit polynomial P we have the following equality:

$$\frac{[A/PA]_A}{[C(A/PA)]_A} = \frac{P}{P-1}$$

thus,

$$\zeta_A(1) = \prod_{P} \frac{[A/PA]_A}{[C(A/PA)]_A}$$

where the product runs through the monic irreducible polynomials of A. The generalization to Drinfeld modules comes from the previous equality. This is what we are now introducing.

Let L/K be a finite field extension. We denote by $\mathscr{O}_L$ the integral closure of A in L. Drinfeld [21] introduced the Drinfeld modules (which he initially called "elliptic modules") as follows. A Drinfeld A-module defined over $\mathscr{O}_L$ is a non constant $\mathbb{F}_q$-algebra homomorphism $\phi : A \to \mathscr{O}_L\{\tau\}$ verifying

$$\phi_\theta = \theta\tau^0 + \sum_{i=1}^{r} \phi_{\theta,i}\tau^i \in \mathscr{O}_L\{\tau\}$$

with $\phi_{\theta,i} \in \mathscr{O}_L$ for $i = 1, \dots, r$. Remark that ϕ_θ completely determines ϕ. If B is an $\mathscr{O}_L$-algebra, then it provieds B with an A-module structure via ϕ, which we denote by $\phi(B)$.

The analogue of the Carlitz zeta function for a general Drinfeld module will be the following L-series, introduced by Taelman in [42] and defined by the Eulerian product

$$L(\phi/\mathscr{O}_L) = \prod_P \frac{[\mathscr{O}_L/P\mathscr{O}_L]_A}{[\phi(\mathscr{O}_L/P\mathscr{O}_L)]_A}$$

where P runs through the monic irreducible polynomial of A. In the same article, he introduced the classe $H(\phi; \mathscr{O}_L)$ module and the unit module $U(\phi; \mathscr{O}_L)$, and conjectured a formula linking these objects to the previously defined L-series. He [43, Theorem 1] proved this formula two years later using Anderson's trace formula from [2], a result now known as the Taelman class formula.

Theorem (Taelman class formula for Drinfeld modules)**.** *We have the following equality:*

$$L(\phi/\mathscr{O}_L) = [\mathscr{O}_L : U(\phi; \mathscr{O}_L)]_A \, [H(\phi; \mathscr{O}_L)]_A \in K_\infty^\times.$$

In the previous theorem, the quantity $[\mathscr{O}_L : U(\phi; \mathscr{O}_L)]_A$ plays the role of the regulator of number fields, and $[H(\phi; \mathscr{O}_L)]_A$ the analogue of the classe number in positive characteristic.

Since Taelman's seminal work, developments have been made in several directions.

Anderson [1] introduced the notion of Anderson t-modules, a higher-dimensional generalization of Drinfeld modules. Fang [24, Theorem 1.10], using Chtoukas considered in [33] and Fontaine's theory, proved the class formula for Anderson t-modules.

If we are interested in Anderson t-modules $C^{\otimes n}$, called tensor powers of the Carlitz modulus and introduced by Anderson and Thakur [3], we obtain by [18, Section 4.2] the equality

$$L(C^{\otimes n}/A) = \prod_{P \text{ prime}} \frac{P^n}{P^n - 1} = \sum_{a \in A^+} \frac{1}{a^n} = \zeta_A(n).$$

Anglès and Tavares Ribeiro [10] introduced the notion of z-deformation of Anderson modules as well as the notion of Stark units, which they developed with Ngo Dac in [4, 5] and which would allow them new approaches to study L-series and obtain class formulas. Finally, Anglès, Ngo Dac and Tavares Ribeiro[7] proved a class formula in the case where the ring A is "general" and E is an "admissible" Anderson module, including in particular all Drinfeld modules.

Demeslay [18] has adapted the ideas developed by Taelman for Drinfeld modules to Anderson t-modules with multivariable.

One of the interests of this construction is that we can reduce the study of certain Anderson t-modules to the study certain Drinfeld modules with variables, which are easier to study. This is, for example, the method followed by Anglès, Pellarin and Tavares Ribeiro in [8] to study the tensor powers of the Carlitz module $C^{\otimes n}$ reducing to the study of the Carlitz module C.

Moreover, considering certain Anderson t-modules with variables, Demeslay obtained certain L-series, called Pellarin L-series, introduced by the latter in [39] and defined for positive integers n by

$$L(\chi_t, n) = \sum_{a \in A^+} \frac{\chi_t(a)}{a^n} \in \mathbb{F}_q[t] \otimes_{\mathbb{F}_q} K_\infty$$

where t is a new variable and $\chi_t(a) = \sum_{j=0}^{m} a_j t^j$ if $a = \sum_{j=0}^{m} a_j \theta^j$.

Plan of the thesis

The aim of this thesis is to construct and study a P-adic L-series in the context of Anderson t-modules. We construct them from the z-twisted L-series and Stark units. Here is the outline.

The first chapter consists mainly of a general review of the theory known for the ∞ place but from which we will define the P-adic case in the next chapters. In Section 1.1, Section 1.2, Section 1.3 and Section 1.4, we introduce the objects at the heart of the theory: Anderson modules, z-deformation of Anderson modules, the Taelman unit module, the Taelman class module and the L-series. One of the new features is Section 1.5. We will evaluate the new variable z not only at $z = 1$ as Anglès, Tavares Ribeiro and Ngo Dac [4, 5, 7, 10] do, but at $z = \zeta \in \overline{\mathbb{F}_q}$, so we extend the definitions and properties of the objects previously introduced. We consider the case of Drinfeld modules and the associated period lattice in Section 1.6. These results will be used in Chapter 4 to study the vanishing of the P-adic L-series in the Drinfeld module setting. Finally, in Section 1.7 we introduce a special class of Drinfeld modules called very small Drinfeld modules for which we can make an explicit study of the P-adic L-series .

In Chapter 2, we fix a monic irreducible polynomial P and we define and study some P-adic L-series associated with Anderson t-modules by removing the local factor at P of the classical L-series. After some preliminaries sections that are Section 2.1, Section 2.2 and Section 2.3, we prove in Section 2.4 the first main theorem (see Theorem 2.4.1), that is, the convergence of the P-adic L-series. In Section 2.5 and Section 2.6, we prove a P-adic class formula à la Taelman (see Theorem 2.6.9) linking the P-adic L series, the class module and a P-adic regulator. In Section 2.7, we investigate the vanishing of the newly introduced P-adic L-series.

The aim of chapter 3 is to extend the constructions of chapter 2 to the case where the constant field is no longer $\mathbb{F}_q$ but $\mathbb{F}_q(t_1, \ldots, t_s)$ where the t_i are new variables. In section 3.1, we introduce objects in this multi-variable setting. We turn to the P-adic case in section 3.2. First of all, we note that all the results of chapter 2 remain valid by replacing $\mathbb{F}_q$ by $\mathbb{F}_q(t_1, \ldots, t_s)$. The difficulty is to consider the 'integral' case, i.e. when $E_\theta \in M_d(\mathscr{O}_L[t_1, \ldots, t_s])\{\tau\}$. We adapt the techniques of chapter 1 and chapter 2 in this framework to prove (see Theorem 3.2.6) that the P-adic L-series has no pole in any of the variables $t_1, \dots, t_s$, i.e., that $L_P(\widetilde{E}/\widetilde{\mathscr{O}_{L,s}}) \in \mathbb{T}_{s,z}(K_P)$. In particular, we introduce Pellarin P-adic L-series.

In Chapter 4 we study the P-adic L-series in the setting of A-Drinfeld module defined over A. In Section 4.1 we establish a surprising connection between the property of a monic irreducible polynomial P to be Wieferich and the P-adic valuation of the special value of the P-adic L-series of Drinfeld modules (see Theorem 4.1.8 and Theorem 4.1.10), a link that had been established by Thakur [47] only for the Carlitz module. In Section 4.2, we study the vanishing order at $z = 1$ of the P-adic L-series. The main result (see Theorem 4.2.3) is a relation between this order and some properties of successive minimum basis of the period lattice associated with ϕ introduced in Section 1.6. We finally give an explicit way to compute the vanishing order in the case ϕ is a very small Drinfeld module in Section 4.3 (see Theorem 4.3.3), knowing only the coefficients of ϕ_θ.

Chapter 1

Class formula à la Taelman

In this chapter, we first focus on objects related with the infinity place, on which the P-adic constructions of Chapter 2 will depend. In Section 1.1, Section 1.2, Section 1.3 and Section 1.4, we introduce the objects at the heart of the theory: Anderson modules, z-deformation introduced by Anglès and Tavares Ribeiro in [10], the Taelman unit module, the Taelman class module, and the L-series. One of the new features is Section 1.4. We will evaluate the new variable z not only at $z = 1$ as Anglès, Tavares Ribeiro and Ngo Dac [4, 5, 7, 10] do, but at $z = \zeta \in \overline{\mathbb{F}_q}$, so we extend the definitions and properties of the objects previously introduced. Next, we consider the case of Drinfeld modules and the associated period lattice in Section 1.6. These results will be used in Chapter 4 to study the vanishing of the P-adic L-series in this setting. Finally, we introduce in Section 1.7 a special class of Drinfeld modules called very small Drinfeld modules.

1.1. Notation and background

1.1.1. Notation. Let p be a fixed prime number and q a power of p. Let $\mathbb{F}_q$ be the finite field with q elements and θ an indeterminate over $\mathbb{F}_q$. We set $A = \mathbb{F}_q[\theta]$ and let $K = \mathbb{F}_q(\theta)$ be the rational function field. Let L/K be a finite field extension of degree n. We denote by $\mathscr{O}_L$ the integral closure of A in L. We consider the valuation v_∞ of K normalized such that $v_\infty(\theta^{-1}) = 1$. Let K_∞ be the completion of K with respect to this valuation and we set $L_\infty = L \otimes_K K_\infty$. We also consider the following notation.

- $\mathbb{C}_\infty$: the completion of a fixed algebraic closure of K_∞,
- $\tau : \mathbb{C}_\infty \to \mathbb{C}_\infty$ the Frobenius endomorphism,
- $M_d(R) = M_{d\times d}(R)$, for a ring R the left R-module of $d \times d$ matrices,
- I_d: the identity matrix of $M_d(R)$.

Let us fix an integer $d \geq 1$ and B an $\mathbb{F}_q$-algebra. If $M = (m_{i,j})$ is a matrix with coefficients in $\mathbb{C}_\infty$ and $k \in \mathbb{N}$, then we set $\tau^k(M) = M^{(k)}$ to be the matrix whose ij-entry is given by $\tau^k(m_{i,j}) = m_{i,j}^{q^k}$. We denote by $M_d(B)\{\tau\}$ the non-commutative ring of twisted polynomials in τ with coefficients in $M_d(B)$ equipped with the usual addition and the commutation rule $\tau^k M = M^{(k)}\tau^k$ for all $k \in \mathbb{N}$ and all $M \in M_d(B)$. Let $M_d(B)\{\{\tau\}\}$ be the non-commutative ring of twisted power series in τ with coefficients in $M_d(B)$.

If k is a field containing $\mathbb{F}_q$, then we set $(kK)_\infty = k\hat{\otimes}_{\mathbb{F}_q} K_\infty = k((\frac{1}{\theta}))$. If $x \in (kK)_\infty^\times$, then we can write x uniquely as $x = \displaystyle\sum_{n\geq N} x_n \frac{1}{\theta^n}$, $x_n \in k$ and $x_N \neq 0$. Then we call $x_N \in K$ the sign denoted by $\operatorname{sgn}(x)$. We say that such an $x \in (kK)_\infty$ is monic if $\operatorname{sgn}(x) = 1$.

1.1.2. Fitting ideals. We recall here some definitions about Fitting ideals of modules over Dedekind rings. Let R be a Dedekind ring, and M be a finite and torsion R-module. By the structure theorem, there exists $s \in \mathbb{N}$ and $I_1, \ldots, I_s$ ideals of R such that we have an isomorphism of R-modules

$$M \simeq R/I_1 \times \ldots \times R/I_s.$$

We then define the Fitting ideal of M by

$$\operatorname{Fitt}_R(M) = I_1 \ldots I_s.$$

We have the following properties that can be found in the appendix of [36] except the second one which appears in [22, Corollary 20.5].

Proposition 1.1.1.

(1) *We have* $\operatorname{Fitt}_R(M) \subseteq \operatorname{Ann}_R(M) = \{x \in R \mid x.m = 0 \ \forall m \in M\}$.

(2) *If* $0 \to M_1 \to M \to M_2 \to 0$ *is exact, then* $\operatorname{Fitt}_R(M_1)\operatorname{Fitt}_R(M_2) = \operatorname{Fitt}_R(M)$.

In this thesis, we will work mainly with rings which will be of the form $R = k[\theta]$ with k a field. In this case, we have an explicit way to compute $[M]_R$:

$$[M]_R = \det_{k[X]} (X - \theta \mid M)_{|X=\theta}\,.$$

1.1.3. Lattices and Ratio of co-volumes. We use the following notation from [45, Section 7.2.3]. We fix k a field containing $\mathbb{F}_q$ and recall that $(kK)_\infty = k\hat{\otimes}_{\mathbb{F}_q} K_\infty = k\left(\left(\frac{1}{\theta}\right)\right)$. In what follows, we fix V to be a finite-dimensional $(kK)_\infty$-vector space endowed with the natural topology coming from $(kK)_\infty$.

Definition 1.1.2. A sub-$k[\theta]$-module M of V is a $k[\theta]$-lattice in V if M is discrete in V and if M generates V over $(kK)_\infty$.

Proposition 1.1.3. *Let M be a sub-$k[\theta]$-module of V. The following are equivalent:*

(1) *M is a $k[\theta]$-lattice in V.*

(2) *There exists a $(kK)_\infty$-basis $(v_1, \ldots, v_n)$ of V such that M is the free $k[\theta]$-module of basis $(v_1, \ldots, v_n)$.*

Proof. See [45, Proposition 7.2.3]. □

Let M and M' be two $k[\theta]$-lattices in V. Let $\mathscr{B}$ and $\mathscr{B}'$ be $k[\theta]$-basis of M and M', respectively. The ratio of co-volumes of M in M' is then defined as

$$[M' : M]_{k[\theta]} = \frac{\det_{\mathscr{B}'} \mathscr{B}}{\operatorname{sgn}(\det_{\mathscr{B}'} \mathscr{B})} \in (kK)_\infty^*.$$

Note that this is independent of the choices of $\mathscr{B}$ and $\mathscr{B}'$. The definition immediately implies that if M_0, M_1 and M_2 are lattices in V, then

$$[M_0 : M_1]_{k[\theta]}[M_1 : M_2]_{k[\theta]} = [M_0 : M_2]_{k[\theta]}.$$

We also observe that for two lattices M, M' in V we have

$$[M' : M]_{k[\theta]} = [M : M']_{k[\theta]}^{-1}.$$

1.2. Anderson modules

From now on, let L/K be a finite fields extension. Recall that we denote by $\mathscr{O}_L$ the integral closure of A in L, z is a new variable on which the Frobenius acts as the identity, $\mathscr{O}_L[z] \simeq \mathbb{F}_q[z] \otimes_{\mathbb{F}_q} \mathscr{O}_L$, $\widetilde{\mathscr{O}_L} = \mathbb{F}_q(z) \otimes_{\mathbb{F}_q} \mathscr{O}_L$, $\widetilde{L_\infty} = L \otimes_{\mathbb{F}_q} \widetilde{K_\infty}$. In this section, we extend the notion of the Taelman unit module and class module by twisting with some elements $\zeta \in \overline{\mathbb{F}_q}$.

An Anderson t-module (or shortly a t-module) E of dimension d defined over $\mathscr{O}_L$ is an $\mathbb{F}_q$-algebra homomorphism $E : A \to M_d(\mathscr{O}_L)\{\tau\}$ such that if $a \in A$ and $E_a = \sum_{i=0}^{r_a} E_{a,i}\tau^i$, then we require $(E_{a,0} - aI_d)^d = 0$ and that $\deg_\tau(E_\theta) > 0$. Let $E : A \to M_d(\mathscr{O}_L)\{\tau\}$ be a t-module of dimension $d \geq 1$, completely determined by the value at θ:

$$E_\theta = \sum_{i=0}^{r} E_{\theta,i}\tau^i$$

with $E_{\theta,i} \in M_d(\mathscr{O}_L)$ and $(E_{\theta,0} - \theta I_d)^d = 0$. For $a \in A$, if $E_a = \sum_{i=0}^{r_a} E_{a,i}\tau^i$, then we define $\partial_E(a) = E_{a,0}$. Then the map $\partial_E : A \to M_d(A)$ is a homomorphism of $\mathbb{F}_q$-algebras.

The key notion through this work will be the notion of z-deformation introduced by Anglès and Tavares Ribeiro [10]. Let z be a new variable such that $\tau(z) = z$. Set $\widetilde{A} = \mathbb{F}_q(z)A$, $\widetilde{K} = \mathbb{F}_q(z)K$, $\widetilde{\mathscr{O}_L} = \mathbb{F}_q(z)\mathscr{O}_L$ and $\widetilde{K_\infty} = \mathbb{F}_q(z)((\theta^{-1}))$. We set $\widetilde{L_\infty} = L \otimes_K \widetilde{K}_\infty$. We then consider the z-twist of E, denoted by $\widetilde{E}$, to be the homomorphism of $\mathbb{F}_q(z)$-algebras $\widetilde{E} : \widetilde{A} \to M_d(\widetilde{\mathscr{O}_L})\{\tau\}$ given by:

$$\widetilde{E}_\theta = \sum_{i=0}^{r} E_{\theta,i} z^i \tau^i.$$

Recall the following notation taken from [7].

Definition 1.2.1. Let E be a t-module of dimension d over R an extension of $\mathbb{F}_q$ and let B be an R-algebra. We can then define two A-module structures on B^d. The first is denoted $E(B)$ where A acts on B^d via E:

$$a.x = E_a(x) \in B^d \text{ for all } a \in A, x \in B^d.$$

The second is $\mathrm{Lie}_E(B)$ where A acts on B^d via ∂_E:

$$a.x = \partial_E(a)x \text{ for all } a \in A, x \in B^d.$$

We have the following results that can be found in [1, Proposition 2.1.4].

Proposition 1.2.2. *There exists a unique element* $\exp_E \in M_d(L)\{\{\tau\}\}$ *such that:*

(1) $\exp_E \partial_E(a) = E_a \exp_E$ *holds in* $M_d(L)\{\{\tau\}\}$ *for all* $a \in A$,

(2) $\exp_E \equiv I_d \pmod{M_d(L)\{\{\tau\}\}\tau}$.

We call $\exp_E$ the exponential map associated with the t-module E, and denote this element by $\exp_E = \sum_{n=0}^{\infty} d_n \tau^n$.

Proposition 1.2.3. *There exists a unique element* $\log_E \in M_d(L)\{\{\tau\}\}$ *such that:*

(1) $\log_E E_a = \partial_E(a) \log_E$ *holds in* $M_d(L)\{\{\tau\}\}$ *for all* $a \in A$,

(2) $\log_E \equiv I_d \pmod{M_d(L)\{\{\tau\}\}\tau}$.

In addition, we have the equalities in $M_d(L)\{\{\tau\}\}$*:*

$$\log_E \exp_E = \exp_E \log_E = I_d.$$

We call $\log_E$ the logarithm map associated to the t-module E, and we denote this element by $\log_E = \sum_{n=0}^{\infty} l_n \tau^n$. We also have exponential and logarithm series for the z-twist of the t-module $\widetilde{E}$ which we denote by $\exp_{\widetilde{E}}$ and $\log_{\widetilde{E}}$ and given by:

$$\exp_{\widetilde{E}} = \sum_{n \geq 0} d_n z^n \tau^n \text{ and } \log_{\widetilde{E}} = \sum_{n \geq 0} l_n z^n \tau^n.$$

1.3. Unit module and class module

We consider an over-additive valuation v_∞ over the finite-dimensional K_∞-vector space L_∞ (for example with respect to the choice of a basis of L/K). The key point is the next result from [29, Theorem 5.9.6].

Lemma 1.3.1. *We have*

$$\lim_{i \to +\infty} \frac{v_\infty(d_i)}{q^i} = +\infty.$$

Corollary 1.3.2. *The exponential map* $\exp_E$ *converges on* $\mathrm{Lie}_E(L_\infty)$ *and induces a homomorphism of* A*-modules:*

$$\exp_E : \mathrm{Lie}_E(L_\infty) \to E(L_\infty).$$

We can also consider $\mathbb{T}_z(K_\infty)$, the completion of $K_\infty[z]$ with respect to the Gauss norm:

$$\mathbb{T}_z(K_\infty) = \left\{ \sum_{n \geq 0} a_n z^n \mid a_n \in K_\infty, \lim_{n \to +\infty} v_\infty(a_n) = +\infty \right\}.$$

We set:

$$\mathbb{T}_z(L_\infty) = \left\{ \sum_{n\geq 0} a_n z^n \mid a_n \in L_\infty, \lim_{n\to+\infty} v_\infty(a_n) = +\infty \right\} = L \otimes_K \mathbb{T}_z(K_\infty).$$

We also have the convergence of $\exp_{\widetilde{E}}$ on $\mathrm{Lie}_{\widetilde{E}}(\widetilde{L_\infty})$ (resp. $\mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_\infty))$) that induces a homomorphism of $\widetilde{A}$-modules (resp. $A[z]$):

$$\exp_{\widetilde{E}} : \mathrm{Lie}_{\widetilde{E}}(\widetilde{L_\infty}) \to \widetilde{E}(\widetilde{L_\infty}).$$

(resp. $\exp_{\widetilde{E}} : \mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_\infty)) \to \widetilde{E}(\mathbb{T}_z(L_\infty))$). We can now define the Taelman unit module

$$U(E; \mathscr{O}_L) = \{x \in \mathrm{Lie}_E(L_\infty) \mid \exp_E(x) \in E(\mathscr{O}_L)\}$$

provided with A-module structure, as well as the module of z-units

$$U(\widetilde{E}; \widetilde{\mathscr{O}_L}) = \left\{ x \in \mathrm{Lie}_{\widetilde{E}}(\widetilde{L_\infty}) \mid \exp_{\widetilde{E}}(x) \in \widetilde{E}(\widetilde{\mathscr{O}_L}) \right\}$$

provided with $\widetilde{A}$-module structure, and the module of z-units at the integral level

$$U(\widetilde{E}; \mathscr{O}_L[z]) = \left\{ x \in \mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_\infty)) \mid \exp_{\widetilde{E}}(x) \in \widetilde{E}(\mathscr{O}_L[z]) \right\}$$

provided with $A[z]$-module structure. We also define the class module (introduced by Taelman in [43])

$$H(E; \mathscr{O}_L) = \frac{E(L_\infty)}{E(\mathscr{O}_L) + \exp_E(\mathrm{Lie}_E(L_\infty))}$$

as well as the class module for the z-deformation

$$H(\widetilde{E}; \widetilde{\mathscr{O}_L}) = \frac{\widetilde{E}(\widetilde{L_\infty})}{\widetilde{E}(\widetilde{\mathscr{O}_L}) + \exp_{\widetilde{E}}(\mathrm{Lie}_{\widetilde{E}}(\widetilde{L_\infty}))}$$

and finally the class module at the integral level

$$H(\widetilde{E}; \mathscr{O}_L[z]) = \frac{\widetilde{E}(\mathbb{T}_z(L_\infty))}{\widetilde{E}(\mathscr{O}_L[z]) + \exp_{\widetilde{E}}(\mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_\infty)))}.$$

Consider one of the following cases:
- $k_0 = \mathbb{F}_q$, $\varphi = E$ and $(k_0L)_\infty = L_\infty$,
- $k_0 = \mathbb{F}_q[z]$ and $\varphi = \widetilde{E}$,
- $k_0 = \mathbb{F}_q(z)$, $\varphi = \widetilde{E}$ and $(k_0L)_\infty = \widetilde{L_\infty}$.

We have the following result from [18, Proposition 2.8].

Proposition 1.3.3.

(1) *The class module $H(\varphi; k_0\mathscr{O}_L)$ is a finite-dimensional k_0-vector space, so a finite and torsion k_0A-module.*

(2) *If k_0 is a field, then the module of units $U(\varphi; k_0\mathscr{O}_L)$ is a k_0A-lattice in $\mathrm{Lie}_\varphi((k_0L)_\infty)$.*

We also have the following result in [10, Proposition 2.3].

Proposition 1.3.4. *We have the following equality:*

$$U(\widetilde{E}; \widetilde{\mathscr{O}_L}) = \mathbb{F}_q(z)U(\widetilde{E}; \mathscr{O}_L[z]).$$

Consider the evaluation morphism:

$$\mathrm{ev}_{z=1} : \mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_\infty)) \to \mathrm{Lie}_E(L_\infty).$$

It induces an exact sequence of A-modules:

$$0 \longrightarrow (z-1)\,\mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_\infty)) \longrightarrow \mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_\infty)) \xrightarrow{\mathrm{ev}_{z=1}} \mathrm{Lie}_E(L_\infty) \longrightarrow 0.$$

For all $x \in \mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_\infty))$ we have $\mathrm{ev}_{z=1}(\exp_{\widetilde{E}}(x)) = \exp_E(\mathrm{ev}_{z=1}(x))$. Moreover, if $f(z) \in \widetilde{L_\infty}$ belongs to the ∞-adic convergence domain of the logarithm map $\log_{\widetilde{E}}$, then we have

$$\mathrm{ev}_{z=1}(\log_{\widetilde{E}}(f(z))) = \log_E(\mathrm{ev}_{z=1}(f(z))).$$

We recall the notion of Stark units introduced by B. Anglès and F. Tavares Ribeiro in [10, section 2.5].

Definition 1.3.5. The module of Stark units $U_{\mathrm{St}}(E; \mathscr{O}_L)$ is defined by:

$$U_{\mathrm{St}}(E; \mathscr{O}_L) = \mathrm{ev}_{z=1}\, U(\widetilde{E}; \mathscr{O}_L[z]).$$

Given the compatibility between the exponential and the evaluation morphism, $U_{\mathrm{St}}(E; \mathscr{O}_L)$ is a sub-A-module of $U(E; \mathscr{O}_L)$. We have the following result from [10, Theorem 1].

Theorem 1.3.6. *The A-module $U_{\mathrm{St}}(E; \mathscr{O}_L)$ is an A-lattice in $\mathrm{Lie}_E(L_\infty)$.*

1.4. The L-series

For a monic irreducible polynomial P of A, we define the local factor at P associated with E

$$z_P(E/\mathscr{O}_L) = \frac{[\mathrm{Lie}_E(\mathscr{O}_L/P\mathscr{O}_L)]_A}{[E(\mathscr{O}_L/P\mathscr{O}_L)]_A} \in K$$

and the local factor at P associated with $\widetilde{E}$

$$z_P(\widetilde{E}/\widetilde{\mathscr{O}_L}) = \frac{\left[\mathrm{Lie}_{\widetilde{E}}(\widetilde{\mathscr{O}_L}/P\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}}{\left[\widetilde{E}(\widetilde{\mathscr{O}_L}/P\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}} \in \widetilde{K}.$$

We then define the L-series associated with E and $\mathscr{O}_L$ by the Eulerian product

$$L(E/\mathscr{O}_L) = \prod_{P \in A} z_P(E/\mathscr{O}_L)$$

where P runs through the monic irreducible polynomials of A, and the L-series associated with $\widetilde{E}$ and $\widetilde{\mathscr{O}_L}$ by the Eulerian product

$$L(\widetilde{E}/\widetilde{\mathscr{O}_L}) = \prod_{P\in A} z_P(\widetilde{E}/\widetilde{\mathscr{O}_L}).$$

We have the convergence of the L-series and the class formula for z-deformation from [18, Theorem 2.7].

Theorem 1.4.1 (Class formula for the z-deformation)**.** *The product defining $L(\widetilde{E}/\widetilde{\mathscr{O}_L})$ converges in $\widetilde{K_\infty}^*$ and we have the formula*

$$L(\widetilde{E}/\widetilde{\mathscr{O}_L}) = \left[\mathrm{Lie}_{\widetilde{E}}\left(\widetilde{\mathscr{O}_L}\right) : U(\widetilde{E};\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}. \tag{1.1}$$

Proposition 1.4.2. *For all monic irreducible polynomial P, the local factor $z_P(\widetilde{E}/\widetilde{A})$ belongs to $\mathbb{T}_z(K_\infty)$ and*

$$z_P(\widetilde{E}/\widetilde{\mathscr{O}_L}) \equiv 1 \pmod{z\pi\widetilde{K_\infty}}$$

where $\pi = \dfrac{1}{\theta}$.

Proof. We adapt the proof of [45, Corollary 7.5.6]. We begin with $\left[\widetilde{E}(\widetilde{\mathscr{O}_L}/P\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}$. Let us recall that for all non-zero prime P in A, we have the following formula:

$$\begin{aligned}\left[\mathrm{Fitt}_{\widetilde{A}}(\widetilde{E}(\widetilde{\mathscr{O}_L}/P\widetilde{\mathscr{O}_L}))\right]_{\widetilde{A}} &= \det_{\mathbb{F}_q(z)[X]}\left(X-\theta \mid \widetilde{E}(\widetilde{\mathscr{O}_L}/P\widetilde{\mathscr{O}_L})\right)_{|X=\theta} \\ &= \det_{\mathbb{F}_q(z)[X]}\left(X-\widetilde{E}_\theta \mid \widetilde{\mathscr{O}_L}^d/P\widetilde{\mathscr{O}_L}^d\right)_{|X=\theta}.\end{aligned}$$

Since $\mathscr{O}_L[z]^d/P\mathscr{O}_L[z]^d$ is a finitely generated and free $\mathbb{F}_q[z]$-module, we deduce that

$$\left[\widetilde{E}(\widetilde{\mathscr{O}_L}/P\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}} = x_P(z) \in A[z].$$

Moreover, we can see that $x_P(z)$ is monic (see as a polynomial in the variable θ) of degree

$$\dim_{\mathbb{F}_q(z)}(\widetilde{\mathscr{O}_L}^d/P\widetilde{\mathscr{O}_L}^d) = d\deg(P)[L:K].$$

In particular it is an element of $\mathbb{T}_z(K_\infty)^\times$. We apply the same reasoning to $\left[\mathrm{Lie}_{\widetilde{E}}(\widetilde{\mathscr{O}_L}/P\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}$. We obtain that $\left[\mathrm{Lie}_{\widetilde{E}}(\widetilde{\mathscr{O}_L}/P\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}} \in A[z]$ and is a monic polynomial (see as a polynomial in the variable θ) of degree

$$\dim_{\mathbb{F}_q(z)}(\widetilde{\mathscr{O}_L}^d/P\widetilde{\mathscr{O}_L}^d) = d\deg(P)[L:K].$$

Finally, we deduce that the quotient $\dfrac{\left[\mathrm{Lie}_{\widetilde{E}}(\widetilde{\mathscr{O}_L}/P\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}}{\left[\widetilde{E}(\widetilde{\mathscr{O}_L}/P\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}}$ belongs to $\mathbb{T}_z(K_\infty)^\times$ and has the form $1+g(z)$ with $g(z) \in z\mathbb{T}_z(K_\infty)$ and $v_\infty(g(z)) > 0$.

□

We then obtain the following result.

Corollary 1.4.3. *The L-series* $L(\widetilde{E}/\widetilde{\mathscr{O}_L})$ *converges in* $\mathbb{T}_z(K_\infty)^\times$ *and*

$$L(\widetilde{E}/\widetilde{\mathscr{O}_L}) \equiv 1 \pmod{\pi\mathbb{T}_z(K_\infty)}.$$

We can evaluate the L-series at $z=1$:

$$L(E/\mathscr{O}_L) = \mathrm{ev}_{z=1} L(\widetilde{E}/\widetilde{\mathscr{O}_L}) = \prod_Q \frac{[\mathrm{Lie}_E(\mathscr{O}_L/Q\mathscr{O}_L)]_A}{[E(\mathscr{O}_L/Q\mathscr{O}_L))]_A} \in K_\infty^*$$

where Q runs through the monic irreducible polynomials of A. We have the following class formula for t-modules obtained by Fang in [24], generalizing Taelman's class formula for Drinfeld modules.

Theorem 1.4.4 (Class formula for Anderson t-modules)**.** *The product defining* $L(E/\mathscr{O}_L)$ *converges in* K_∞^**, and we have the equalities*

$$L(E/\mathscr{O}_L) = [\mathrm{Lie}_E(\mathscr{O}_L) : U(E;\mathscr{O}_L)]_A \, [H(E;\mathscr{O}_L)]_A = [\mathrm{Lie}_E(\mathscr{O}_L) : U_{\mathrm{st}}(E;\mathscr{O}_L)]_A \,. \tag{1.2}$$

1.5. Evaluation at $z=\zeta\in\overline{\mathbb{F}}_q$.

We want to extend the notion of Stark units by evaluating the variable z at $z=\zeta$ for all $\zeta\in\overline{\mathbb{F}}_q$.
Let ζ be an element of $\overline{\mathbb{F}}_q$ and consider $\mathbb{F}_q(\zeta)$ the finite field obtained by adding ζ to $\mathbb{F}_q$. Let us define the ring $A_\zeta = \mathbb{F}_q(\zeta)\otimes_{\mathbb{F}_q} A$. We define a Frobenius $\tau_\zeta = \mathrm{id}\otimes\tau$ acting on A_ζ. Let us define $\widetilde{A_\zeta} = \mathbb{F}_q(z)\otimes_{\mathbb{F}_q} A_\zeta$ on which we extend the Frobenius τ_ζ by $\tau_\zeta = \mathrm{id}\otimes\tau_\zeta$, still denoted by τ_ζ (i.e., the Frobenius τ_ζ acts as the identity on $\mathbb{F}_q(z)$). Denote by $A_\zeta[z] = \mathbb{F}_q[z]\otimes_{\mathbb{F}_q} A_\zeta$. Set $\mathscr{O}_{L,\zeta} = \mathbb{F}_q(\zeta)\otimes_{\mathbb{F}_q}\mathscr{O}_L$. It is also equiped with the following Frobenius $\tau_\zeta = \mathrm{id}\otimes\tau$.

Similarly as the z-deformation, let us twist the t-module E into an Anderson A_ζ-module E_ζ defined over $M_d(\mathscr{O}_{L,\zeta})$, that is an homomorphism of $\mathbb{F}_q(\zeta)$-algebra $E_\zeta = A_\zeta \to M_d(\mathscr{O}_{L,\zeta})\{\{\tau_\zeta\}\}$, by

$$(E_\zeta)_\theta = \sum_{i=0}^r E_{\theta,i}\zeta^i\tau_\zeta^i \in M_d(\mathscr{O}_{L,\zeta})\{\tau_\zeta\}$$

then extend to A_ζ by $\mathbb{F}_q(\zeta)$-linearity.

Set $M_w = \mathbb{F}_q(\zeta)\otimes_{\mathbb{F}_q} L_w$ where $w=\infty$ or $w=P$. Consider $\mathbb{F}_q[z]\otimes_{\mathbb{F}_q}\mathscr{O}_{L,\zeta} = \mathscr{O}_{L,\zeta}[z]$ then set $\widetilde{\mathscr{O}_{L,\zeta}} = \mathbb{F}_q(z)\otimes_{\mathbb{F}_q}\mathscr{O}_{L,\zeta}$ and $\widetilde{M_w} = \mathscr{O}_{L,\zeta}[z]\otimes_{\mathscr{O}_L[z]}\mathbb{T}_z(L_w) \simeq \mathbb{F}_q(\zeta)\otimes_{\mathbb{F}_q}\mathbb{T}_z(L_w)$ and consider $\mathbb{F}_q(\zeta)\otimes_{\mathbb{F}_q}\widetilde{L_w}$. We extend v_∞ to M_∞ as follows. Let's fix $(f_1,\dots,f_m)$ a $\mathbb{F}_q$-basis of $\mathbb{F}_q(\zeta)$. We set

$$v_\infty\left(\sum_{i=1}^m f_i\otimes x_i\right) = \min_{i=1,\dots,h} v_\infty(x_i)$$

for $x_i\in K_\infty$. The topology over M_∞ does not depend on the choice of the basis $(f_1,\dots,f_m)$. We then consider v_∞ an over-additive valuation on the $\mathbb{F}_q(\zeta)\otimes_{\mathbb{F}_q} K_\infty$-vector space of finite dimension M_∞. We then extend similarly v_∞ to $\mathbb{F}_q(\zeta)\otimes_{\mathbb{F}_q}$

$\widetilde{L_\infty}$. Remark that we cannot just replace v_∞ by v_P on $\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} K_P$ with theses constructions, in fact we do not obtain a valuation over M_{v_P}. See Subsection 2.3 for more details.

Finally, we deform E into $E^{(\zeta)}$ an Anderson A_ζ-module on $M_d(\mathscr{O}_{L,\zeta})$ by

$$E^{(\zeta)}_\theta = \sum_{i=0}^{r} E_{\theta,i}\tau_\zeta^i$$

and extend it to A_ζ by $\mathbb{F}_q(\zeta)$-linearity. We finally extend it to an Anderson $\widetilde{A_\zeta}$-module $\widetilde{E}^{(\zeta)}$ on $M_d(\widetilde{\mathscr{O}_{L,\zeta}})$ in the usual way.

We have exponential maps associated with each of the Anderson modules. From the definitions we have the equalities

$$\exp_{E^{(\zeta)}} = \sum_{n\geq 0} d_n\tau_\zeta^n \text{ and } \log_{E^{(\zeta)}} = \sum_{n\geq 0} l_n\tau_\zeta^n,$$

and the map $\exp_{E^{(\zeta)}}$ (resp. $\exp_{\widetilde{E}^{(\zeta)}}$) converges on $\mathrm{Lie}_{E^{(\zeta)}}(M_\infty)$ (resp. converges on $\mathrm{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M_\infty})$). Moreover, we have the following equalities in $M_d(\mathbb{F}_q(\zeta)\otimes_{\mathbb{F}_q} L)\{\{\tau_\zeta\}\}$:

$$\exp_{E_\zeta} = \sum_{n\geq 0} d_n\zeta^n\tau_\zeta^n \text{ and } \log_{E_\zeta} = \sum_{n\geq 0} l_n\zeta^n\tau_\zeta^n.$$

Consider the evaluation morphism at $z=\zeta$:

$$\mathrm{ev}_\zeta = \mathrm{ev}_{z=\zeta} : \widetilde{M_\infty} \to M_\infty$$

whose kernel is given by $(z-\zeta)\widetilde{M_\infty}$, then we consider the following evaluation morphism still denoted by ev_ζ:

$$\mathrm{ev}_\zeta : \mathrm{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M_\infty}) \to \mathrm{Lie}_{E^{(\zeta)}}(M_\infty).$$

For $x \in \mathrm{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M_\infty})$, we have in $\mathrm{Lie}_{E^{(\zeta)}}(M_\infty)$:

$$\mathrm{ev}_\zeta(\exp_{\widetilde{E}^{(\zeta)}}(x)) = \exp_{E_\zeta}(\mathrm{ev}_\zeta(x)).$$

Let us consider the module of ζ-units at the integral level:

$$U(\widetilde{E}^{(\zeta)};\mathscr{O}_{L,\zeta}[z]) = \left\{x \in \mathrm{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M_\infty}) \mid \exp_{\widetilde{E}^{(\zeta)}}(x) \in \widetilde{E}(\mathscr{O}_{L,\zeta}[z])\right\}$$

as well as the module of the ζ-classes at the integral level:

$$H(\widetilde{E}^{(\zeta)};\mathscr{O}_{L,\zeta}[z]) = \frac{\widetilde{E}^{(\zeta)}(\widetilde{M_\infty})}{\widetilde{E}^{(\zeta)}(\mathscr{O}_{L,\zeta}[z]) + \exp_{\widetilde{E}^{(\zeta)}}(\mathrm{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M_\infty}))} = A_\zeta[z]\otimes_{\mathbb{F}_q[z]} H(\widetilde{E},\mathscr{O}_L[z])$$

provided with a structure of $A_\zeta[z]$-modules. Next, consider the ζ-unit module:

$$U(E_\zeta;\mathscr{O}_{L,\zeta}) = \{x \in \mathrm{Lie}_{E_\zeta}(M_\infty) \mid \exp_{E_\zeta}(x) \in E_\zeta(\mathscr{O}_{L,\zeta})\}$$

and the ζ-class module

$$H(E_\zeta,\mathscr{O}_{L,\zeta}) = \frac{E_\zeta(M_\infty)}{E_\zeta(\mathscr{O}_{L,\zeta}) + \exp_{E_\zeta}(\mathrm{Lie}_{E_\zeta}(M_\infty))}$$

provided with their A_ζ-module structure via E_ζ.

Results to come in this section are adapted from [18] and [10].

Proposition 1.5.1.

(1) *The exponential map* $\exp_{E_\zeta} : \mathrm{Lie}_{E_\zeta}(M_\infty) \to E_\zeta(M_\infty)$ *is locally an isometry.*

(2) *The exponential map* $\exp_{\widetilde{E}^{(\zeta)}} : \mathrm{Lie}_{\widetilde{E^{(\zeta)}}}(\widetilde{M}_\infty) \to \widetilde{E}^{(\zeta)}(\widetilde{M}_\infty)$ *is locally an isometry.*

Proof. The proof is a direct corollary of Lemma 1.3.1, we omit the proof. □

Proposition 1.5.2.

(1) *The module of* ζ*-classes* $H(E_\zeta; \mathscr{O}_{L,\zeta})$ *is a* $\mathbb{F}_q(\zeta)$*-vector space of finite dimension, hence a torsion* A_ζ*-module of finite type.*

(2) *The module of* ζ*-units* $U(E_\zeta; \mathscr{O}_{L,\zeta})$ *is an* A_ζ*-lattice in* $\mathrm{Lie}_{E_\zeta}(M_\infty)$*.*

(3) *The class module* $H(\widetilde{E}^{(\zeta)}; \mathscr{O}_{L,\zeta}[z])$ *is a* $\mathbb{F}_q(\zeta)[z]$*-module of finite type.*

Proof. The proof follows the proof of [18, Proposition 2.6] for the two first assertions, and the proof of [10, Proposition 2] for the last one, by replacing A by A_ζ, $\mathscr{O}_L$ by $\mathscr{O}_{L,\zeta}$ and E by E_ζ. We omit the details. □

Just as Stark's units consist of the evaluation at $z = 1$ of the z-units, we define the evaluation at $z = \zeta$ of the ζ-units at the integral level:

$$U_\zeta(E; \mathscr{O}_L) = \mathrm{ev}_\zeta\, U(\widetilde{E}^{(\zeta)}; \mathscr{O}_{L,\zeta}[z]) \subseteq U(E_\zeta; \mathscr{O}_{L,\zeta})$$

provided with an A_ζ-module structure via E_ζ.

Theorem 1.5.3. *There exists an* A_ζ*-module isomorphism:*

$$\frac{U(E_\zeta; \mathscr{O}_{L,\zeta})}{U_\zeta(E; \mathscr{O}_L)} \simeq H(\widetilde{E}^{(\zeta)}; \mathscr{O}_{L,\zeta}[z])[z-\zeta]$$

where $H(\widetilde{E}^{(\zeta)}; \mathscr{O}_{L,\zeta}[z])[z-\zeta]$ *is the* $(z-\zeta)$*-torsion of the* ζ*-class module at the integral level.*

In the following, we will denote by $M = \mathscr{O}_{L,\zeta}$ and $\widetilde{M} = \mathscr{O}_{L,\zeta}[z]$.

Proof. We follow the proof of [10, Proposition 2.6].
Consider the map

$$\begin{aligned} \alpha : M_\infty^d &\to \widetilde{M}_\infty^d \\ x &\mapsto \frac{\exp_{\widetilde{E}^{(\zeta)}}(x) - \exp_{E_\zeta}(x)}{z-\zeta}. \end{aligned}$$

We divide the proof into several steps.
Step 1: The map is well defined since

$$\mathrm{ev}_\zeta(\exp_{\widetilde{E}^{(\zeta)}}(x)) = \exp_{E_\zeta}(x)$$

for $x \in M_\infty^d$, thus $(z-\zeta)$ divide $\exp_{\widetilde{E}^{(\zeta)}}(x) - \exp_{E_\zeta}(x)$ in $\widetilde{M}_\infty^d$.
Step 2: We still denote α to be the restriction: $\alpha : U(E_\zeta, M) \to H(\widetilde{E}^{(\zeta)}; \widetilde{M})$. Let us

prove that it is a homomorphism of A_ζ-modules. Let $x \in U(E_\zeta; M)$ be a unit and $a \in A_\zeta$. Then:

$$\begin{aligned}(z-\zeta)\alpha(ax) &= \exp_{\widetilde{E}^{(\zeta)}}(ax) - \exp_{E_\zeta}(ax) \\ &= \widetilde{E}_a^{(\zeta)}(\exp_{\widetilde{E}^{(\zeta)}}(x)) - (E_\zeta)_a(\exp_{E_\zeta}(x)) \\ &= \sum_{i=0}^{r_a} E_{a,i} z^i \tau_\zeta^i(\exp_{\widetilde{E}^{(\zeta)}}(x)) - \sum_{i=0}^{r_a} E_{a,i}\zeta^i\tau_\zeta^i(\exp_{E_\zeta}(x)) \\ &= \sum_{i=0}^{r_a} E_{a,i} z^i \tau_\zeta^i(\exp_{\widetilde{E}^{(\zeta)}}(x) - \exp_{E_\zeta}(x)) + \sum_{i=1}^{r_a} E_{a,i}(z^i-\zeta^i)\tau_\zeta^i(\exp_{E_\zeta}(x)).\end{aligned}$$

Thus

$$\alpha(ax) = \widetilde{E}_a^{(\zeta)}(\alpha(x)) + \underbrace{\sum_{i=0}^{h} a_i \frac{z^i-\zeta^i}{z-\zeta}\tau_\zeta^i(\underbrace{\exp_{E_\zeta}(x)}_{\in M^d})}_{\in \widetilde{M}^d}.$$

We have proved that $\alpha(ax) = \widetilde{E}_a^{(\zeta)}(\alpha(x)) \pmod{\widetilde{M}^d + \exp_{\widetilde{E}^{(\zeta)}}(\mathrm{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M}_{s,\infty}))}$, so $\alpha(ax) = \widetilde{E}_a^{(\zeta)}(\alpha(x))$ in $H(\widetilde{E}^{(\zeta)}, \widetilde{M})$.

Step 3: We claim that the image of $U(E_\zeta, M)$ is in the $(z-\zeta)$-torsion of the ζ-class module at the integral level. In fact, let $x \in U(E_\zeta; M)$ be a unit. We have:

$$(z-\zeta)\alpha(x) = \exp_{\widetilde{E}^{(\zeta)}}(x) - \exp_{E_\zeta}(x) = 0 \pmod{E_\zeta(M) + \exp_{\widetilde{E}^{(\zeta)}}(\mathrm{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M}_\infty))}.$$

Step 4: Let us prove the surjectivity of α on $H(\widetilde{E}^{(\zeta)}; \widetilde{M})[z-\zeta]$. Let $x \in \widetilde{E}^{(\zeta)}(\widetilde{M}_\infty)$ be such that

$$(z-\zeta)x = \exp_{\widetilde{E}^{(\zeta)}}(u) + v$$

with $u \in \mathrm{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M}_\infty)$ and $v \in \widetilde{E}^{(\zeta)}(\widetilde{M})$. We write $u = u_1 + (z-\zeta)u_2$, $u_1 \in M_\infty^d$, $u_2 \in \widetilde{M}_\infty^d$ and $v = v_1 + (z-\zeta)v_2$, $v_1 \in M^d$, $v_2 \in \widetilde{M}^d$. We have:

$$(z-\zeta)x = \exp_{\widetilde{E}^{(\zeta)}}(u_1) + v_1 + (z-\zeta)(v_2 + \exp_{\widetilde{E}^{(\zeta)}}(u_2)).$$

By evaluating at $z=\zeta$ yields $\exp_{E_\zeta}(u_1) + v_1 = 0$. Thus $u_1 \in U(E_\zeta; M)$. Moreover, we have:

$$\alpha(u_1) = x - (\underbrace{\exp_{\widetilde{E}^{(\zeta)}}(u_2)}_{\in \exp_{\widetilde{E}^{(\zeta)}}(\widetilde{M}_\infty^d)} + \underbrace{v_2}_{\in \widetilde{M}^d})$$

thus $\alpha(u_1) = x \pmod{\widetilde{M}^d + \exp_{\widetilde{E}^{(\zeta)}}(\widetilde{M}_\infty^d)}$.

Step 5: We now consider the kernel of $\alpha : U(E_\zeta; M) \to H(\widetilde{E}^{(\zeta)}, \widetilde{M})$ denoted by κ. We want to prove that $\kappa = U_\zeta(E, \mathscr{O}_L)$. We proceed by double inclusion.

$\boxed{\supseteq}$ Let $x \in U_\zeta(E, \mathscr{O}_L)$ be a unit and write $x = \mathrm{ev}_\zeta(u)$ with $u \in U(\widetilde{E}^{(\zeta)}; \widetilde{M})$. We have $\mathrm{ev}_\zeta(x-u) = 0$ thus we can find $v \in \widetilde{M}_\infty^d$ such that

$$x = u + (z-\zeta)v.$$

We have
$$\alpha(x)=\frac{\exp_{\widetilde{E}^{(\zeta)}}(u)-\exp_{E_\zeta}(x)}{z-\zeta}+\exp_{\widetilde{E}^{(\zeta)}}(v).$$
But $\exp_{E_\zeta}(x)=\mathrm{ev}_\zeta \exp_{\widetilde{E}^{(\zeta)}}(u)\in M^d$ so $\alpha(x)=0 \pmod{\widetilde{M}^d+\exp_{\widetilde{E}^{(\zeta)}}(\widetilde{M}^d_\infty)}$.

$\boxed{\subseteq}$ Let $x\in U(E_\zeta;M)$ be such that $\alpha(x)\in \widetilde{M}^d+\exp_{\widetilde{E}^{(\zeta)}}(\widetilde{M}^d_\infty)$. Let us express $\alpha(x)=u+\exp_{\widetilde{E}^{(\zeta)}}(v)$. We have
$$(z-\zeta)\alpha(x)=\exp_{\widetilde{E}^{(\zeta)}}(x)+\exp_{E_\zeta}(x)=(z-\zeta)u+\exp_{\widetilde{E}^{(\zeta)}}((z-\zeta)v).$$
Thus $\exp_{\widetilde{E}^{(\zeta)}}(x-(z-\zeta)v)=(z-\zeta)u+\exp_{E_\zeta}(x)\in\widetilde{M}^d$ so $x-(z-\zeta)v\in U(\widetilde{E}^{(\zeta)};\widetilde{M})$. Finally, we obtain
$$\mathrm{ev}_\zeta(x-(z-\zeta)v)=x\in U_\zeta(E;\mathscr{O}_L).$$
□

Corollary 1.5.4. *The unit module $U_\zeta(E;\mathscr{O}_L)$ is an A_ζ-lattice in $\mathrm{Lie}_{E_\zeta}(M_\infty)$. Moreover, we have the following equalities*
$$\left[H(\widetilde{E}^{(\zeta)};\widetilde{M})[z-\zeta]\right]_{A_\zeta}=[H(E_\zeta;M)]_{A_\zeta}=\left[\frac{U(E_\zeta;M)}{U_\zeta(E;\mathscr{O}_L)}\right]_{A_\zeta}.$$

Let us start by proving the following result.

Lemma 1.5.5. *We have an exact sequence of A_ζ-modules:*
$$0\longrightarrow (z-\zeta)H(\widetilde{E}^{(\zeta)};\widetilde{M})\longrightarrow H(\widetilde{E}^{(\zeta)};\widetilde{M})\xrightarrow{\mathrm{ev}_\zeta} H(E_\zeta;M)\longrightarrow 0.$$

Proof of Lemma 1.5.5. We apply the snake lemma to the following commutative diagram (where the lines are exact sequences of A_ζ-modules) and the i_j represent natural injections:

$$\begin{tikzcd}
(z-\zeta)(\exp_{\widetilde{E}^{(\zeta)}}(\widetilde{M}^d_\infty)+\widetilde{M}^d) \arrow[r] \arrow[d, "i_1"] & \exp_{\widetilde{E}^{(\zeta)}}(\widetilde{M}^d_\infty)+\widetilde{M}^d \arrow[r, "\mathrm{ev}_{z=\zeta}"] \arrow[d, "i_2"] & \exp_{E_\zeta}(M^d_\infty)+M^d \arrow[d, "i_3"] \\
(z-\zeta)\widetilde{M}^d_\infty \arrow[r] & \widetilde{M}^d_\infty \arrow[r, "\mathrm{ev}_{z=\zeta}"] & M^d_\infty
\end{tikzcd}$$

□

Proof of Corollary 1.5.4. We deduce by Lemma 1.5.5 an exact sequence of $\mathbb{F}_q(\zeta)$-vector spaces of finite dimension (and of finitely-generated A_ζ-modules):

$$0\longrightarrow H(\widetilde{E}^{(\zeta)};\widetilde{M})[z-\zeta]\longrightarrow H(\widetilde{E}^{(\zeta)};\widetilde{M})\xrightarrow{\cdot(z-\zeta)} H(\widetilde{E}^{(\zeta)},\widetilde{M})\xrightarrow{\mathrm{ev}_\zeta} H(E_\zeta;M)\longrightarrow 0.$$
By Proposition 1.1.1 we obtain:
$$\left[H(\widetilde{E}^{(\zeta)};\widetilde{M})[z-\zeta]\right]_{A_\zeta}=[H(E_\zeta;M)]_{A_\zeta}=\left[\frac{U(E_\zeta;M)}{U_\zeta(E;\mathscr{O}_L)}\right]_{A_\zeta},$$
the last equality coming from Theorem 1.5.3. Since $H(\widetilde{E}^{(\zeta)};\widetilde{M})[z-\zeta]$ is a $\mathbb{F}_q(\zeta)$-vector space of finite dimension and $U(E_\zeta;M)$ is an A_ζ-lattice in M^d_∞, the result follows. □

1.6. Periods of Drinfeld modules

In this section, we focus on the case where ϕ is a Drinfeld module A-module defined over A. In this case, we know that the kernel of the associated exponential $\exp_\phi$ is a free A-module of rank r, denoted by Λ_ϕ and called the period lattice. A period λ of ϕ is any $\lambda \in \Lambda_\phi$. We will interest in valuation of special basis of the period lattice, called successive minimum basis, introduced in [44, Section 4] and used by Gekeler in [25, 26, 27, 28].

1.6.1. Successive minimum basis.

Definition 1.6.1. An ordered A-basis $(\lambda_1, \ldots, \lambda_r)$ (ordered in the sense $v_\infty(\lambda_1) \geq \ldots \geq v_\infty(\lambda_r)$) of the A-lattice Λ_ϕ in $\mathbb{C}_\infty$ is a successive minimum basis (shortly an SMB) if for each $1 \leq i \leq r$, the vector λ_i has minimal valuation $v_\infty(\lambda_i)$ among all $w \in \Lambda_\phi$ not in the span $\sum_{1\leq j<i} \lambda_i A$ of $\{\lambda_1, \ldots, \lambda_{i-1}\}$.

Gekeler proved the following result, see [27, Proposition 3.1]

Proposition 1.6.2.

(1) *The period lattice Λ_ϕ admits an SMB.*

(2) *The sequence $(v_\infty(\lambda_i))_{i=1,\ldots,r}$ is independent of the choice of the SMB.*

(3) *Consider $\{\lambda_1, \ldots, \lambda_r\}$ an SMB for Λ_ϕ. Then for all $\lambda = \sum_{i=1}^{r} a_i\lambda_i \in \Lambda_\phi$ we have:*

$$v_\infty(\lambda) = \min\{v_\infty(a_i\lambda_i) \mid i = 1, \ldots, r\}.$$

The invariant sequence $(v_\infty(\lambda_1), \ldots, v_\infty(\lambda_r))$ is called the **spectrum** of ϕ and denoted by $\mathrm{Spec}(\phi)$.

In particular, an SMB of Λ_ϕ is an A-basis of Λ_ϕ. We have an analytic expression for the exponential $\exp_\phi(x)$ associated with ϕ:

$$\exp_\phi(x) = x \prod_{\lambda\in\Lambda_\phi\setminus\{0\}} \left(1 - \frac{x}{\lambda}\right). \tag{1.3}$$

The reader could read [40, Chapter 13] or the original work of [21, Section 3] to understand the link between the lattices of $\mathbb{C}_\infty$ and the Drinfeld modules.

Let $NP_{\exp_\phi}$ be the Newton polygon associated with $\exp_\phi$ which is defined as the lower convex hull of the points $Q_n = (q^n, v_\infty(d_n)))$ for all $n \geq 0$. Remark that the zeros of $\exp_\phi$ are all simple since $\frac{d}{dx}(\exp_\phi(x)) = 1$. We have the following property about the edges of $NP_{\exp_\phi}$ that can be found in [38, Theorem 2.5.2].

Proposition 1.6.3. *Consider λ a non-zero period of ϕ of valuation x, and let N be the number of periods of valuation equal to x. Then $NP_{\exp_\phi}$ has a single edge with slope x and length N.*

By Lemma 1.3.1, we know that $\lim_{n\to+\infty} v_\infty(d_n) = +\infty$, then we define N_0 as the smallest n such that $v_\infty(d_n)$ is minimal, and N_1 as the largest n such that $v_\infty(d_n)$ is minimal. Another way of looking at N_0 and N_1 is that the edge of slope equals to 0 of $NP_{\exp_\phi}$ has endpoints Q_{N_0} and Q_{N_1}.

We denote by $\lambda_1, \dots, \lambda_s$ $(s \leq r)$ be such that $x_i = v_\infty(\lambda_i) \geq 0$. Then we consider $n_1, \dots, n_t \in \{1, \dots, s\}$ such that $x_{n_i} \in \mathbb{N}$ for $i = 1, \dots, t$ and we denote by $S_\phi = \{n_1, \dots, n_t\}$.

Proposition 1.6.4. *We have the following equality:*

$$N_1 - N_0 = t.$$

Proof. First we calculate N_0. To do this, we need to count the total length of the strictly negative slopes (which is also equal to $q^{N_0} - 1$ by definition), in other words the number of non-zero periods of strictly positive valuation. Set $\lambda = \sum_{i=1}^{r} a_i \lambda_i \in \Lambda_\phi$. We have the equivalence by Proposition 1.6.2:

$$v_\infty(\lambda) > 0 \Leftrightarrow \begin{cases} a_i = 0 \text{ if } i > s, \\ \deg(a_i) \leq \lfloor x_i \rfloor \text{ if } i \leq s \text{ and } i \notin S_\phi, \\ \deg(a_i) < x_i \text{ if } i \in S_\phi. \end{cases}$$

We finally exclude the case $\lambda = 0$. We obtain that the total number of non-zero elements of Λ_ϕ with strictly positive valuation is equal to

$$q^{N_0} - 1 = \prod_{j=1}^{t} q^{x_{n_j}} \prod_{i \leq s, i \notin S_\phi} q^{\lfloor x_i \rfloor + 1} - 1.$$

By applying the logarithm we obtain:

$$N_0 = \sum_{j=1}^{t} x_{n_j} + \sum_{i \leq s, i \notin S_\phi} (\lfloor x_i \rfloor + 1).$$

We then calculate $q^{N_1} - 1$ that is equivalent to counting the number of periods with positive valuation. Set $\lambda = \sum_{i=1}^{r} a_i \lambda_i \in \Lambda_\phi$. We have the equivalence by Proposition 1.6.2:

$$v_\infty(\lambda) \geq 0 \Leftrightarrow \begin{cases} a_i = 0 \text{ if } i > s, \\ \deg(a_i) \leq \lfloor x_i \rfloor \text{ if } i \leq s \text{ and } i \notin S_\phi, \\ \deg(a_i) \leq x_i \text{ if } i \in S_\phi. \end{cases}$$

We finally exclude the case $x = 0$. We obtain that the total number of non-zero elements of Λ_ϕ with positive valuation is equal to

$$q^{N_1} - 1 = \prod_{j=1}^{t} q^{x_{n_j}+1} \prod_{i \leq s, i \notin S_\phi} q^{\lfloor x_i \rfloor + 1} - 1.$$

By applying the logarithm we obtain:

$$N_1 = \sum_{j=1}^{t} (x_{n_j} + 1) + \sum_{i \leq s, i \notin S_\phi} (\lfloor x_i \rfloor + 1) = N_0 + t.$$

$\square$

Note that we only worked with periods of ϕ and never used the fact that ϕ is defined over A. We can therefore generalize the previous result to any $\phi : A \to \mathbb{C}_\infty\{\tau\}$ of rank r since the concept of SMB is defined in full generality.

1.6.2. Convergence domain of the logarithm . We keep the notation of Section 1.5. In particular $\phi : A \to A\{\tau\}$ is an A-Drinfeld module defined over A, of rank $r \geq 1$, completely defined by the image of θ denoted by

$$\phi_\theta = \theta + \sum_{i=1}^{r} \phi_{\theta,i}\tau^i$$

with $\phi_{\theta,i} \in A$ for $i = 1, \ldots, r$ and $\phi_{\theta,r} \neq 0$. Denote by $\log_\phi = \sum_{n\geq 0} l_n\tau^n \in K\{\{\tau\}\}$ its associated logarithm and $\exp_\phi = \sum_{n\geq 0} d_n\tau^n \in \mathbb{C}_\infty\{\{\tau\}\}$ its associated exponential. We fix $(\lambda_1, \ldots, \lambda_r)$ an SMB of Λ_ϕ.

There is a number $R_\phi \in \mathbb{R}$ such that $\log_\phi(x)$ converges exactly for $x \in \mathbb{C}_\infty$ verifying $v_\infty(x) > R_\phi$. This is called the convergence radius of $\log_\phi$. We have the following result from [29, Proposition 4.14.2].

Proposition 1.6.5. *The power series* $\log_\phi(x)$ *converges exactly in*

$$\mathscr{N} = \{x \in \mathbb{C}_\infty \mid v_\infty(x) > v_\infty(\lambda_1)\}.$$

Moreover, $\log_\phi(x)$ *is a bijective isometry over* $\mathscr{N}$ *with inverse* $\exp_\phi(x)$*.*

In particular, we have $R_\phi = v_\infty(\lambda_1)$. We will give an explicit expression of R_ϕ as a function of the coefficients of ϕ_θ.

Example 1.6.6. *Consider the Drinfeld A-module $C : A \to \mathbb{C}_\infty\{\tau\}$ given by $C_\theta = \theta + \tau$. That is called the Carlitz module. By [3, Section 2] we have*

$$R_C = -\frac{q}{q-1} = -\frac{q^1 - \deg(C_{\theta,1})}{q^1 - 1}.$$

Let us define the following quantities:

$$\begin{aligned} p_i &:= \frac{v_\infty(\phi_{\theta,i}) + 1}{q^i - 1}, \forall i = 2, \ldots, r, \\ \beta &= \min_{i=2,\ldots,r}\{p_i\}, \\ B &= \{i \in \{2, \ldots, r\} \mid p_i = \beta\}, \\ i_{\max} &= \max\{i = 2, \ldots, r \mid p_i = \beta\} = \max\{i, i \in B\}. \end{aligned}$$

We have the following result from [31, Corollary 4.5].

Theorem 1.6.7. *We have the equality*

$$R_\phi = -\frac{q^{i_{\max}} + v_\infty(\phi_{\theta,i_{\max}})}{q^{i_{\max}} - 1}.$$

In the following, we explain a proof of this result which is similar to the proof of [31, Corollary 4.5].

Consider X a new variable and set $\phi_\theta(X) = \theta X + \sum_{i=1}^{r} \phi_{\theta,i} X^{q^i} \in A[X]$. Set

$$f_\phi(X) = \frac{\phi_\theta(X)}{X} = \theta + \phi_{\theta,1} X^{q-1} + ... + \phi_{\theta,r} X^{q^r-1} \in A[X].$$

Let NP_{f_ϕ} be the Newton polygon associated with $\dfrac{\phi_\theta(X)}{X}$. Recall that it is defined as follows. For $i = 1, ..., r$, denote by $P_i = (q^i - 1, v_\infty(\phi_{\theta,i}))$ and $P_0 := (0, -1)$. Then NP_{f_ϕ} is the lower convex envelope of the points $P_0, ..., P_r$. Denote by (D_i) the straight line through $(0, -1)$ and P_i, and denote by C_1 the first edge of NP_{f_ϕ}. Remark that the slope of (D_i) is equal to $p_i = \dfrac{v_\infty(\phi_{\theta,i}) + 1}{q^i - 1}$. We deduce that

(1) β equals the first slope of NP_{f_ϕ},

(2) We have the equivalence for $i \in \{1, \dots, r\}$: $i \in B \Leftrightarrow P_i \in C_1$,

(3) $P_{i_{\max}}$ is the first breakpoint of NP_{f_ϕ}.

We denote by

$$\phi[\theta] = \{x \in \mathbb{C}_\infty \mid \phi_\theta(x) = 0\}.$$

This is a finite $\mathbb{F}_q$-vector space of dimension r. We set $\mu_i = \exp_\phi\left(\dfrac{\lambda_i}{\theta}\right) \in \phi[\theta]$ for $i = 1, \dots, r$.

Lemma 1.6.8. *The family $(\mu_1, \dots, \mu_r)$ is an $\mathbb{F}_q$-basis of $\phi[\theta]$. Moreover, we have*

$$v_\infty\left(\sum_{i=1}^{r} \alpha_i \mu_i\right) = \min\{v_\infty(\alpha_i \mu_i), i = 1, \dots, r\}$$

Proof. See [30, Theorem 3.3]. □

We deduce, from the theory of Newton polygon, that $-v_\infty(\mu_1)$ equals the first slope of NP_{f_ϕ} thus

$$v_\infty(\mu_1) = -\frac{1 + v_\infty(\phi_{\theta,i_{\max}})}{q^{i_{\max}} - 1}. \tag{1.4}$$

Lemma 1.6.9. *We have $v_\infty(\lambda_1) = v_\infty(\mu_1) - 1$.*

Proof. By Equation 1.3, we have:

$$v_\infty(\mu_1) = v_\infty(\lambda_1) + 1 + \sum_{\lambda \in \Lambda_\phi \setminus \{0\}} v_\infty\left(1 + \frac{\lambda_1}{\theta\lambda}\right).$$

If $v_\infty\left(\dfrac{\lambda_1}{\theta\lambda}\right) > 0$, then we have

$$v_\infty\left(1 + \frac{\lambda_1}{\theta\lambda}\right) = 0.$$

If $v_\infty\left(\frac{\lambda_1}{\theta\lambda}\right) = 0$, then we have

$$v_\infty\left(1+\frac{\lambda_1}{\theta\lambda}\right) = v_\infty\left(\frac{\lambda_1+\lambda\theta}{\theta\lambda}\right) = 0$$

by Proposition 1.6.2. Thus,

$$v_\infty(\mu_1) = v_\infty(\lambda_1) + 1 + \sum_{\substack{\lambda\in\Lambda_\phi\setminus\{0\} \\ v_\infty(\lambda)-1\geq v_\infty(\lambda_1)}} v_\infty\left(\frac{\lambda_1}{\lambda\theta}\right).$$

By definition of SMB and Proposition 1.6.2, we have

$$\{\lambda \in \Lambda_\phi\setminus\{0\} \mid v_\infty(\lambda) - 1 \geq v_\infty(\lambda_1)\} = \emptyset.$$

Thus,

$$v_\infty(\lambda_1) = v_\infty(\mu_1) - 1.$$

$\square$

By Lemma 1.6.9 combined with Equation (1.4) and Proposition 1.6.5, we obtain

$$v_\infty(\lambda_1) = v_\infty(\mu_1) - 1 = -\frac{q^{i_{\max}} + v_\infty(\phi_{\theta,i_{\max}})}{q^{i_{\max}} - 1}. \tag{1.5}$$

so Theorem 1.6.7:

$$R_\phi = -\frac{q^{i_{\max}} + v_\infty(\phi_{\theta,i_{\max}})}{q^{i_{\max}} - 1}.$$

We can give more information on NP_{f_ϕ} and periods, coming from the following result from [26, Proposition 4.3].

Proposition 1.6.10. *For all $i = 1,\dots,r-1$, we have $v_\infty(\lambda_i) = v_\infty(\lambda_{i+1})$ if and only if $v_\infty(\mu_i) = v_\infty(\mu_{i+1})$.*

We deduce the following. Let s be the number of edges of NP_{ϕ_θ}, of lengths $l_1,\dots,l_s$ respectively. By Lemma 1.6.8 and Proposition 1.6.10, we can write

$$\{1,\dots,r\} = B_1 \sqcup \ldots \sqcup B_s$$

where

$$B_k = \{i_{k-1}+1, i_{k-1}+2, \dots, i_k\}$$

where $(i_k)_{k=0,\dots,s}$ is a strictly increasing sequence of integers with $i_0 = 0$ and $i_s = r$, and all the element of B_k have same valuation $v_\infty(\lambda_{i_k})$. We deduce that $\mathrm{Spec}(\phi)$ has the form

$$(v_\infty(\lambda_{i_1}),\dots,v_\infty(\lambda_{i_1}),\dots,v_\infty(\lambda_{i_s}),\dots,v_\infty(\lambda_{i_s}))$$

where each $v_\infty(\lambda_{i_k})$ appears $\#B_k$ times. Moreover, for all $k = 1,\dots,s$, we have

$$\#B_k = l_k = q^{i_k} - q^{i_{k-1}}.$$

In particular, we have $i_1 = i_{\max}$ so

$$v_\infty(\lambda_1) = \ldots = v_\infty(\lambda_{i_{\max}}) > v_\infty(\lambda_{i_{\max}+1}). \tag{1.6}$$

Remark 1.6.11. We never used the fact that ϕ is defined over A. Then all results remain valid by considering ϕ an A-Drinfeld module defined over $\mathbb{C}_\infty$.

1.7. Small Drinfeld modules

We keep the notation of the previous sections. In particular ϕ is a Drinfeld A-module defined over A of rank r. We apply the results in Section 1.5 to characterize a special class of Drinfeld modules.

Definition 1.7.1. We say that ϕ is *small* (resp. *very small*) if $v_\infty(\phi_{\theta,i}) \geq -q^i$ (resp. $v_\infty(\phi_{\theta,i}) > -q^i$) for all $i \in \{1, \ldots, r\}$.

For example the Carlitz module is very small.

We can now characterize the very smallness condition with a more conceptual definition.

Theorem 1.7.2. *The following assertions are equivalent.*

(1) *The Drinfeld module ϕ is very small,*

(2) *The logarithm $\log_\phi$ converges at 1.*

Proof. Assume first that ϕ is very small. Then $q^{i_{\max}} + v_\infty(\phi_{\theta,i_{\max}}) > 0$. By Theorem 1.6.7 we have $R_\phi < 0$ so $\log_\phi$ converges at 1.

To prove $2 \Rightarrow 1$, remark first that for all $i = \{1, \ldots, r\}$ we have by definition of $i_{\max}$:

$$\frac{v_\infty(\phi_{\theta,i}) + q^i}{q^i - 1} \geq \frac{v_\infty(\phi_{\theta,i_{\max}}) + q^{i_{\max}}}{q^{i_{\max}} - 1}.$$

Hence, if $R_\phi < 0$, then by Theorem 1.6.7 we have

$$q^{i_{\max}} + v_\infty(\phi_{\theta,i_{\max}}) > 0$$

thus

$$q^i + v_\infty(\phi_{\theta,i}) > 0, \forall i = 1, \ldots, r$$

so ϕ is very small. $\square$

Remark that $\log_\phi$ converges at 1 if and only if $\lim_{n\to+\infty} v_\infty(l_n) = +\infty$. Thus, $\log_\phi$ converges at 1 if and only if $\log_{\widetilde{\phi}}$ converges at 1.

Remark 1.7.3. The previous proof tells us that ϕ is very small if and only if $R_\phi < 0$. Adapting the same proof, we obtain that ϕ is small if and only if $R_\phi \leq 0$. In particular, if ϕ is a small Drinfeld module, then $\log_\phi$ (resp. $\log_{\widetilde{\phi}}$) is an isometry over $\{x \in K_\infty \mid v_\infty(x) > 0\}$ (resp. $\{x \in \mathbb{T}_z(K_\infty) \mid v_\infty(x) > 0\}$).

We denote by $u_\phi(z) = \exp_{\widetilde{\phi}}(L(\widetilde{\phi}/\widetilde{A}) \in A[z]$.

Proposition 1.7.4. *We assume that ϕ is small and, for $i \in \{1, \ldots, r\}$, we let $\alpha_i \in \mathbb{F}_q$ denote the coefficient of $\phi_{\theta,i}$ in front of θ^{q^i}. Then*

$$u_\phi(z) = 1 + \sum_{i=1}^{r} \alpha_i z^i \in \mathbb{F}_q[z].$$

In particular, if ϕ is very small, then we have $u_\phi(z) = 1$.

Proof. Denote by $\exp_E = \sum_{n\geq 0} d_n\tau^n \in K\{\{\tau\}\}$.

We complete the sequence (α_i) by letting $\alpha_i = 0$ for $i = 0$ and $i > r$.

We first prove, by induction on n, that $v_\infty(d_n) \geq 0$ and $d_n \equiv \alpha_n \pmod{\pi}$ where $\pi := 1/\theta$ is the uniformizer at infinity. The formula clearly holds for $n = 0$ since $d_0 = \alpha_0 = 1$. Let us assume that it holds for all $i < n$.

Then, from the forml equality $\exp_\phi \theta = \phi_\theta \exp_\phi$, we obtain the induction formula for the coefficients of the exponential

$$d_n = \frac{1}{\theta^{q^n} - \theta} \cdot \sum_{i=1}^{\min(r,n)} \phi_{\theta,i} d_{n-i}^{q^i}. \tag{1.7}$$

For $1 \leq i \leq \min(r, n)$, we have

$$v_\infty\left(\frac{\phi_{\theta,i} d_{n-i}^{q^i}}{\theta^{q^n} - \theta}\right) = q^n - \deg(\phi_{\theta,i}) + q^i v_\infty(d_{n-i}) \geq q^n - q^i \geq 0.$$

Thus $v_\infty(d_n) \geq 0$. The previous computation shows also that all the summands in the right hand side of Equation (1.7) have positive valuation, expect maybe the one corresponding to $i = n$. Therefore, reducing Equation (1.7) modulo π, we obtain $d_n \equiv \alpha_n \pmod{\pi}$ as claimed.

We now write

$$u_\phi(z) = \exp_{\widetilde{\phi}}(L(\widetilde{\phi}/\widetilde{A})) = \sum_{n\geq 0} d_n z^n \tau^n(L(\widetilde{\phi}/\widetilde{A})). \tag{1.8}$$

We know from [10, Lemma 2] that $v_\infty(L(\widetilde{\phi}/\widetilde{A})) = 0$ and $L(\widetilde{\phi}/\widetilde{A}) \equiv 1 \pmod{\pi}$; hence reducing Equation (1.8) modulo π yields

$$u_\phi(z) \equiv \sum_{n\geq 0} \alpha_n z^n \pmod{\pi}.$$

Finally, given that $\exp_{\widetilde{\phi}}(L(\widetilde{\phi}/\widetilde{A})) \in A[z]$ according to [10, Proposition 5], we obtain the result. □

Proposition 1.7.5. *Assume that ϕ is very small. Then*

$$L(\widetilde{\phi}/\widetilde{A}) = \log_{\widetilde{\phi}}(1).$$

Proof. According to [10, Proposition 5], we have

$$U(\widetilde{\phi}; A[z]) = L(\widetilde{\phi}/\widetilde{A})A[z].$$

Then there exists a non-zero element $a \in A[z]$ such that

$$\log_{\widetilde{\phi}}(1) = aL(\widetilde{\phi}/\widetilde{A}).$$

By taking the exponential, we obtain $1 = \widetilde{\phi}_a(u_\phi(z))$. Denote by $\widetilde{\phi}_a = \sum_{i=0}^{r_a} \phi_{a,i} z^i \tau^i$. We obtain

$$0 = \deg_z(\widetilde{\phi}_a(u_\phi(z))) = r_a + \deg_z(u_\phi(z)).$$

Thus $r_a = 0$, and so $a \in \mathbb{F}_q^*$. Looking at the constant coefficient, we finally derive $a = 1$ and so $\log_{\widetilde{\phi}}(1) = L(\widetilde{\phi}/\widetilde{A})$ as claimed. □

Finally, in the very small case, we can easily compute the class module and the module of Stark units.

Proposition 1.7.6. *Assume that ϕ is a small Drinfeld A-module defined over A. Then*

$$H(\phi; A) = H(\widetilde{\phi}; A[z]) = \{0\}$$

and

$$U_{\mathrm{st}}(\phi; A) = U(\phi; A).$$

Proof. Let $x \in \mathbb{T}_z(K_\infty)$. We can decompose x into

$$x = \underbrace{\sum_{n\geq 0} \frac{\alpha_n(z)}{\theta^n}}_{x_1(z)} + \underbrace{\sum_{n<0} \frac{\alpha_n(z)}{\theta^n}}_{x_2(z)}$$

with $\alpha_n(z) \in \mathbb{F}_q[z]$ for all n. We remark that $x_1(z) \in A[z]$ and by Remark 1.7.3, we have that $x_2(z)$ belongs to the convergence domain of $\log_{\widetilde{\phi}}$. Thus

$$x = x_1(z) + \exp_{\widetilde{\phi}}\left(\log_{\widetilde{\phi}}(x_2(z))\right) \in A[z] + \exp_{\widetilde{\phi}}(\mathbb{T}_z(K_\infty))$$

so

$$H(\widetilde{\phi}/A[z]) = \frac{\mathbb{T}_z(K_\infty)}{A[z] + \exp_{\widetilde{\phi}}(\mathbb{T}_z(K_\infty))} = \{0\}.$$

The same goes for $H(\phi/A)$. Moreover, by [10, Proposition 3], we have the following isomorphism of A-modules

$$\frac{U(\phi; A)}{U_{\mathrm{st}}(\phi; A)} \simeq H(\widetilde{\phi}; A[z])[z-1]$$

where $H(\widetilde{\phi}; A[z])[z-1]$ is the $(z-1)$-torsion of $H(\widetilde{\phi}; A[z])$. This concludes the proof. $\square$

Chapter 2

A P-adic class formula for Anderson t-modules

We keep the notation of Chapter 1. Recall that L is a finite extension of K of degree n, $\mathscr{O}_L$ denotes the integral closure of A in L and E is an Anderson t-module defined over $\mathscr{O}_L$ of dimension d. The aim of this chapter is to define and study some P-adic L-series associated with Anderson t-modules by removing the local factor at P of the classical L-series. After some preliminaries sections, we prove in Section 2.4 the first main theorem, that is the convergence of the P-adic L-series. In Section 2.6, we prove the second main theorem, that is the P-adic class formula à la Taelman and in Section 2.7 we investigate the vanishing of the newly introduced P-adic L-series. This chapter is the subject of an article [35, Section 4].

2.1. Introduction and notation

Recall that the local factor at Q associated with $\widetilde{E}$ is defined by

$$z_Q(\widetilde{E}/\widetilde{\mathscr{O}_L}) = \frac{\left[\mathrm{Lie}_{\widetilde{E}}(\widetilde{\mathscr{O}_L}/Q\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}}{\left[\widetilde{E}(\widetilde{\mathscr{O}_L}/Q\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}} \in \widetilde{K}$$

and the local factor associated with E is

$$z_Q(E/\mathscr{O}_L) = \frac{[\mathrm{Lie}_E(\mathscr{O}_L/Q\mathscr{O}_L)]_A}{[E(\mathscr{O}_L/Q\mathscr{O}_L)]_A} \in K.$$

The goal of this section is to study the following infinite products of local factors that we call the P-adic L-series and the z-twisted P-adic L-series:

$$L_P(\widetilde{E}/\widetilde{\mathscr{O}_L}) = \prod_{Q \neq P} z_Q(\widetilde{E}/\widetilde{\mathscr{O}_L})$$

where Q runs through the monic irreducible polynomials of A different from P, and

$$L_P(E/\mathscr{O}_L) = \prod_{Q \neq P} z_Q(E/\mathscr{O}_L)$$

where Q runs through themonic irreducible polynomials of A different from P.

More precisely, let us denote by v_P a finite place of K associated with an irreducible monic polynomial P of A. Let $\mathbb{F}_P = \mathbb{F}_{q^{\deg(P)}}$ be the residue field associated to v_P and let $K_P = \mathbb{F}_P((P))$ (resp. $A_P = \mathbb{F}_P[[P]]$) be the completion of K (resp. A) for

v_P. Let $\mathbb{C}_P$ be the completion of an algebraic closure of K_P and v_P the valuation on $\mathbb{C}_P$ normalized such that $v_P(P) = 1$. Set $\widetilde{K_P} = \mathbb{F}_q(z)\hat{\otimes}_{\mathbb{F}_q(z)} K_P = \mathbb{F}_P(z)((P))$ on which we extend the valuation v_P:

$$v_P\left(\sum_{n\geq N}\alpha_n(z)P^n\right) = N, N \in \mathbb{Z}, \alpha_n(z) \in \mathbb{F}_P(z), \alpha_N(z) \neq 0.$$

Let $|\cdot|_P$ be the absolute value on $\mathbb{C}_P$ defined by $|x|_P = q^{-v_P(x)}$. Let $\mathscr{B} = (f_1, \dots, f_n)$ be an A-basis of $\mathscr{O}_L$ (that is also a K-basis of L). We set $L_P = L \otimes_K K_P$ and $\widetilde{L_P} = L \otimes_K \widetilde{K_P}$. In what follows, the reader will be careful not to confuse the notation $L_P(E/\mathscr{O}_L)$ for the P-adic L-series and L_P for the tensor product $L \otimes_K K_P$. Then L_P is a K_P-vector space with $\mathscr{B}$ as a basis, and $\widetilde{L_P}$ is a $\widetilde{K_P}$-vector space with $\mathscr{B}$ as a basis. In particular, on L_P, all the norms of K_P-vector space of finite dimension are equivalent. Let us work with the following.

Consider the sup norm $|.|_P$ with respect to this basis. In other words, if $x = \sum_{i=1}^{n} f_i \otimes x_i$ with the $x_i \in K_P$, then we set

$$|x|_P = \max_{i=1,\dots,n} |x_i|_P.$$

We obtain over L_P a norm of K_P-algebra. We then consider the over-additive valuation of K_P-vector spaces of finite dimension on L_P defined by:

$$v_P(x) = -\log_q |x|_P = \min_{i=1,\dots,d} v_P(x_i).$$

For all $d \geq 1$, we extend these definitions to L_P^d: if $x = (x_1, \dots, x_d) \in L_P^d$, then we set

$$|x| = \max_{i=1,\dots,d} |x_i|$$

or equivalently

$$v_P(x) = -\log_q |x| = \min_{i=1,\dots,d} v_P(x_i).$$

In particular, for all $x \in \mathscr{O}_L^d$ we obtain $v_P(x) \geq 0$. Consider the completion of $K_P[z]$ with respect to the Gauss norm,

$$\mathbb{T}_z(K_P) = \left\{ f(z) = \sum_{n\geq 0} a_n z^n \mid a_n \in K_P \text{ and } \lim_{n\to+\infty} v_P(a_n) = +\infty \right\} \subset \widetilde{K_P}$$

and

$$\mathbb{T}_z(L_P) = \left\{ f(z) = \sum_{n\geq 0} a_n z^n \mid a_n \in L_P \text{ and } \lim_{n\to+\infty} v_P(a_n) = +\infty \right\} = L\otimes_K \mathbb{T}_z(K_P).$$

We define a K_P-vector space structure over $\mathrm{Lie}_E(L_P)$. We take inspiration from the ∞-adic case in [24, Lemma 1.7] and [18, section 2.3].

Proposition 2.1.1. *We can extend the homomorphism $\partial_E : A \to M_d(\mathscr{O}_L)$ into a homomorphism from K_P to $M_d(L_P)$ in the following way:*

$$\partial_E : K_P \to M_d(L_P),$$
$$\sum_{i\geq -N} \alpha_i P^i \mapsto \sum_{i\geq -N} \alpha_i \partial_E(P)^i.$$

Moreover, with respect to this action, L_P^d is a K_P-vector space of dimension $m = d[L:K]$ denoted by $\mathrm{Lie}_E(L_P)$*.*

Proof. Consider $n \geq 0$. There exists a unique integer t_n such that

$$q^d t_n \leq n < (t_n+1)q^d.$$

We have:

$$\partial_E(P)^n = \partial_E(P)^{q^d t_n}\partial_E(P)^{n-q^d t_n} = P^{q^d t_n}\underbrace{\partial_E(P)^{n-q^d t_n}}_{\in M_d(\mathscr{O}_L)}.$$

We obtain that $\lim_{n\to+\infty} v_P(\partial_E(P)^n) = +\infty$ thus the map ∂_E is well-defined. Denote by $W_P = \mathbb{F}_{q^{\deg(P)}}((P^{q^d})) \subseteq K_P$. Then for all $x \in W_P$ we have $\partial_E(x) = xI_d$,thus we have an isomorphism of W_P-vector spaces:

$$\mathrm{Lie}_E(L_P) \simeq L_P^d.$$

Hence $\mathrm{Lie}_E(L_P)$ is a K_P-vector space of finite dimension. We have:

$$\dim_{W_P}(\mathrm{Lie}_E(L_P)) = \dim_{K_P}(\mathrm{Lie}_E(L_P))\dim_{W_P}(K_P).$$

But from the isomorphism of W_P-vector spaces $\mathrm{Lie}_E(L_P) \simeq L_P^d$ we have:

$$\dim_{W_P}(\mathrm{Lie}_E(L_P)) = \dim_{W_P}(L_P^d) = \dim_{K_P}(L_P^d)\dim_{W_P}(K_P)$$

thus

$$\dim_{K_P}(\mathrm{Lie}_E(L_P)) = \dim_{K_P}(L_P^d) = d[L:K].$$

□

We also have that $\mathrm{Lie}_E(\mathscr{O}_L)$ is an A-lattice in $\mathrm{Lie}_E(L_P)$. Finally, everything is still valid by adding the variable z, in other words $\mathrm{Lie}_E(\widetilde{L_P})$ is a $\widetilde{K_P}$-vector space of dimension $d[L:K]$ and $\mathrm{Lie}_{\widetilde{E}}(\widetilde{\mathscr{O}_L})$ is an $\widetilde{A}$-lattice in $\mathrm{Lie}_E(\widetilde{L_P})$. In particular, we have:

$$\partial_{\widetilde{E}} : \mathbb{T}_z(K_P)) \to M_d(\mathbb{T}_z(L_P)).$$

Remark that the topologies of L_P^d and $\mathrm{Lie}_E(L_P)$ are equivalent.

Consider the unique t-module F over $\mathscr{O}_L$ satisfying $PF_a = E_aP$ for all $a \in A$. If $E_a = \sum_{i=0}^{r_a} E_{a,i}\tau^i$, then $F_a = \sum_{i=0}^{r_a} E_{a,i}P^{q^i-1}\tau^i$. In particular for all $a \in A$ we have: $\partial_E(a) = \partial_F(a)$. From [7, Section 3.2] we have the following equalities in $M_d(L)\{\{\tau\}\}$:

$$\log_F = P^{-1}\log_E P = \sum_{n\geq 0} l_n P^{q^n-1}\tau^n,$$

and

$$\exp_F = P^{-1}\exp_E P = \sum_{n\geq 0} d_n P^{q^n-1}\tau^n.$$

We now study the link between the local factors of E and F.

Lemma 2.1.2. *Let $Q \in A$ be a monic irreducible polynomial. If $Q \neq P$, then we have the following equalities: $z_Q(F/\mathscr{O}_L) = z_Q(E/\mathscr{O}_L)$ and $z_Q(\widetilde{F}/\widetilde{\mathscr{O}_L}) = z_Q(\widetilde{E}/\widetilde{\mathscr{O}_L})$. Otherwise $z_P(F/\mathscr{O}_L) = 1$ and $z_P(\widetilde{F}/\widetilde{\mathscr{O}_L}) = 1$.*

Proof. See [7, Lemma 3.7]. □

We then obtain the following result.

Corollary 2.1.3. *We have the following equality in $K[[z]]$:*

$$L(\widetilde{F}/\widetilde{\mathscr{O}_L}) = L_P(\widetilde{E}/\widetilde{\mathscr{O}_L}).$$

2.2. P-adic exponential and P-adic logarithm

We define $(D_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$ as the following sequences of elements of A:

$$\begin{cases} D_0 = 1, \\ D_n = \displaystyle\prod_{k=0}^{n-1}(\theta^{q^{n-k}} - \theta)^{q^k}, \end{cases} \quad \text{and} \quad \begin{cases} L_0 = 1, \\ L_n = \displaystyle\prod_{k=1}^{n}(\theta - \theta^{q^k}). \end{cases}$$

We first recall the P-adic valuation of D_n and L_n for all $n \geq 0$. We will also recall their infinite valuation. We give a detailed proof, but the reader could find this result in [3, Section 2].

Lemma 2.2.1. *We have the following equalities for $n \geq 1$:*

(1) $v_P(D_n) = q^n \dfrac{q^{-\deg(P)\left(\left\lfloor \frac{n}{\deg(P)}\right\rfloor + 1\right)} - q^{-\deg(P)}}{q^{-\deg(P)} - 1} = q^n \dfrac{q^{-\deg(P)\left\lfloor \frac{n}{\deg(P)}\right\rfloor} - 1}{1 - q^{\deg(P)}}$,

(2) $v_P(L_n) = \left\lfloor \dfrac{n}{\deg(P)} \right\rfloor$.

(3) $v_\infty(D_n) = nq^n$.

(4) $v_\infty(L_n) = \dfrac{q^{n+1} - q}{q - 1}$.

Proof. For all $k \geq 1$, we recall the following equality that can be found in [17, Proposition 9.22]:

$$\theta^{q^k} - \theta = \prod_{d\mid k}\prod_{Q\in G_q(d)} Q \tag{2.1}$$

where $G_q(d)$ denotes the set of irreducible polynomials of degree d over $\mathbb{F}_q$. We deduce the following equalities for all $n \geq 1$ and $0 \leq k \leq n-1$:

$$v_P(\theta^{q^{n-k}} - \theta) = v_P\left(\prod_{d|n-k}\prod_{Q\in G_q(d)} Q\right) = \begin{cases} 1 \text{ if } \deg(P)|n-k, \\ 0 \text{ otherwise.}\end{cases}$$

We obtain the following equalities for all $n \geq 1$:

(1)

$$\begin{aligned} v_P(D_n) &= v_P\left(\prod_{k=0}^{n-1}(\theta^{q^{n-k}} - \theta)^{q^k}\right) \\ &= \sum_{\substack{k=0 \\ \deg(P)|n-k}}^{n-1} q^k \\ &= \sum_{j=1}^{\left\lfloor \frac{n}{\deg(P)}\right\rfloor} q^{n-j\deg(P)} \\ &= q^n\left(\frac{q^{-\deg(P)(\left\lfloor \frac{n}{\deg(P)}\right\rfloor+1)} - q^{-\deg(P)}}{q^{-\deg(P)}-1}\right). \end{aligned}$$

(2)

$$v_P(L_n) = \sum_{k=1}^{n} v_P(\theta^{q^k} - \theta) = \sum_{\substack{k=1 \\ \deg(P)|k}}^{n} 1 = \left\lfloor \frac{n}{\deg(P)}\right\rfloor.$$

(3)

$$v_\infty(D_n) = \sum_{k=0}^{n-1} q^n = nq^n$$

(4)

$$v_\infty(L_n) = \sum_{k=1}^{n} q^k = \frac{q^{n+1}-q}{q-1}.$$

□

We recall that $\partial_E(a) \in M_d(A)$ is the constant coefficient of $E_a \in M_d(A)\{\tau\}$ for all $a \in A$, see Section 1.1. Set $s \in \mathbb{N}$ the smallest integer such that $(\partial_E(\theta) - \theta I_d)^{q^s} = 0$. There exists because $\partial_E(\theta) - \theta I_d$ is nilpotent. Then for all $a \in A$ we have $\partial_E(a^{q^s}) = a^{q^s} I_d$.

Recall that $\exp_E = \sum_{n\geq 0} d_n\tau^n \in M_d(L)\{\{\tau\}\}$ and $\log_E = \sum_{n\geq 0} l_n\tau^n \in M_d(L)\{\{\tau\}\}$. Following [29, Theorem 4.6.9], using functional equation of the logarithm map (resp. the exponential map) $\log_E E_{\theta^{q^s}} = \partial_E(\theta^{q^s})\log_E$, an immediate induction tells us that l_n has the form

$$l_n = \frac{a_n}{L_n^{q^s}}$$

with $a_n \in M_d(\mathscr{O}_L)$.

Reasoning in a similar way for the exponential map and by Lemma 2.2.1 we obtain the following result.

Proposition 2.2.2. *We have the following inequalities for all $n \geq 0$:*

(1) $v_P(l_n) \geq -q^s \left\lfloor \dfrac{n}{\deg(P)} \right\rfloor$,

(2) $v_P(d_n) \geq -q^{s+n} \dfrac{q^{-\deg(P)\left\lfloor \frac{n}{\deg(P)} \right\rfloor} - 1}{1 - q^{\deg(P)}}$.

So far, we have considered the exponential and logarithm series as functions of L_∞^d, but now we want to look at them as functions of L_P^d, which we denote by $\exp_{E,P}$ and $\log_{E,P}$. Note that formally (i.e., in $M_d(L)\{\{\tau\}\}$), these are always the same series. We do the same for z-twist.

Let us denote by $\mathrm{ev}_{z=1,P} : \mathbb{T}_z(L_P)^d \to L_P^d$ the P-adic evaluation morphism at $z = 1$, whose kernel is given by $(z-1)\mathbb{T}_z(L_P)^d$.

We can first study the P-adic convergence domain of the P-adic logarithms maps associated with $\widetilde{F}$ and $\widetilde{E}$. We consider the following sets:
- $\Omega_z = \left\{x \in \mathbb{T}_z(L_P)^d \mid v_P(x) \geq 0\right\}$ and $\Omega_z^+ = \left\{x \in \mathbb{T}_z(L_P)^d \mid v_P(x) > 0\right\}$,
- $\Omega = \left\{x \in L_P^d \mid v_P(x) \geq 0\right\}$ and $\Omega^+ = \left\{x \in L_P^d \mid v_P(x) > 0\right\}$,
- $\mathcal{D}_z = \left\{x \in \mathbb{T}_z(L_P)^d \mid v_P(x) > -1 + \dfrac{q^s}{q^{\deg(P)} - 1}\right\}$,
- $\mathcal{D}_z^+ = \left\{x \in \mathbb{T}_z(L_P)^d \mid v_P(x) > \dfrac{q^s}{q^{\deg(P)} - 1}\right\}$,
- $\mathcal{D} = \left\{u \in (L_P)^d \mid v_P(u) > -1 + \dfrac{q^s}{q^{\deg(P)} - 1}\right\}$,
- $\mathcal{D}^+ = \left\{u \in (L_P)^d \mid v_P(u) > \dfrac{q^s}{q^{\deg(P)} - 1}\right\}$.

Proposition 2.2.3.

(1) *We have the P-adic convergences:*

$$\log_{\widetilde{E},P} : \Omega_z^+ \to \mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_P))$$

and

$$\log_{E,P} : \Omega^+ \to \mathrm{Lie}_E((L_P).$$

Moreover, $\log_{\widetilde{E},P} : \mathcal{D}_z^+ \to \mathcal{D}_z^+$ *is an isometry and* $\log_{E,P} : \mathcal{D}^+ \to \mathcal{D}^+$ *is an isometry.*

(2) *The first assertion remains true by replacing E by F and deleting the "+". As a particular case, we have the convergence*

$$\log_{\widetilde{F},P} : \widetilde{E}(\mathscr{O}_L[z]) \to \mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_P)).$$

Proof. We give the proof only for $\log_{\widetilde{E},P}$, the arguments are similar in the other cases. Consider $f(z) \in \mathbb{T}_z(L_P)^d$. We have (first formally):

$$\log_{\widetilde{E},P} f(z) = \sum_{n\geq 0} L_n z^n \tau^n(f(z)).$$

For all $n \geq 0$ we have:

$$v_P\left(L_n\tau^n(f(z))\right) \geq v_P(L_n) + v_P\left(\tau^n(f(z))\right) \geq -q^s \left\lfloor \frac{n}{\deg(P)} \right\rfloor + q^n v_P(f(z))$$

and this last quantity tends to ∞ when n tends to ∞ if $v(f(z)) > 0$. Moreover, if $v_P(f(z)) > \dfrac{q^s}{q^{\deg(P)}-1}$, then we have for all $n \geq 1$:

$$\begin{aligned} v_P\left(L_n\tau^n(f(z))\right) - v_P(f(z)) &\geq (q^n-1)v_P(f(z)) + v(L_n) \\ &> \frac{q^s}{q^{\deg(P)}-1}(q^n-1) - q^s \left\lfloor \frac{n}{\deg(P)} \right\rfloor . \end{aligned}$$

Write $n = b\deg(P) + i \geq 1$ with $b \in \mathbb{N}$ and $0 \leq i < \deg(P)$. Then:

$$\frac{q^s}{q^{\deg(P)}-1}(q^n-1) - q^s \left\lfloor \frac{n}{\deg(P)} \right\rfloor = q^s \left(\frac{q^{b\deg(P)+i}-1}{q^{\deg(P)}-1} - b \right).$$

We have:

$$\frac{q^{b\deg(P)+i}-1}{q^{\deg(P)}-1} - b \geq \frac{q^{b\deg(P)}-1}{q^{\deg(P)}-1} - b = 1 + q^{\deg(P)} + \ldots + (q^{\deg(P)})^{b-1} - b \geq 0.$$

□

We have results for the P-adic convergences of the exponentials series using similar arguments.

Proposition 2.2.4.

(1) *We have the P-adic convergences:*

$$\exp_{\widetilde{E},P} : \mathcal{D}_z^+ \to \mathbb{T}_z(L_P)^d$$

and

$$\exp_{E,P} : \mathcal{D}^+ \to L_P^d.$$

Moreover, $\exp_{\widetilde{E},P} : \mathcal{D}_z^+ \to \mathcal{D}_z^+$ *is an isometry and* $\exp_{E,P} : \mathcal{D}^+ \to \mathcal{D}^+$ *is an isometry.*

(2) *The first assertion remains true by replacing E by F and deleting "+".*

In particular for all $x \in \mathcal{D}_z$ we have the following P-adic equality:

$$\exp_{F,P}(\mathrm{ev}_{z=1,P}(x)) = \mathrm{ev}_{z=1,P}(\exp_{\widetilde{F}}(x)). \tag{2.2}$$

Similarly for all $x \in \Omega_z$ we have the following P-adic equality:

$$\log_{F,P}(\mathrm{ev}_{z=1,P}(x)) = \mathrm{ev}_{z=1,P}(\log_{\widetilde{F}}(x)). \tag{2.3}$$

Similarly in their convergence domain, all of the P-adic exponential and logarithm maps verify the functional identities of the exponential and the logarithm maps:

$$\forall (a,x) \in A \times \Omega_z,\ \log_{\widetilde{F},P} \partial_{\widetilde{F}}(a)x = \widetilde{F}_a \log_{\widetilde{F},P} x,$$
$$\forall (a,x) \in A \times \mathcal{D}_z,\ \exp_{\widetilde{F},P} \partial_{\widetilde{F}}(a)x = \widetilde{E}_a \log_{\widetilde{F},P} x.$$

Moreover, for all $x \in \mathcal{D}_z^+$ we have

$$\exp_{\widetilde{E},P}(\log_{\widetilde{E},P}(x)) = \log_{\widetilde{E},P}(\exp_{\widetilde{E},P}(x)) = x.$$

The same goes without the variable z, and the same goes for $\widetilde{E}$ (resp. E) over Ω_z^+ and $\mathcal{D}_z^+$ (resp. over Ω^+ and $\mathcal{D}^+$).

2.3. Evaluation at $z = \zeta \in \overline{\mathbb{F}}_q$: the P-adic setting

Consider $\mathbb{F}_P$ to be the residual field associated with P. Set $\mathbb{F} = \mathbb{F}_q(\zeta) \cap \mathbb{F}_P$ and $G = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. Let us first remark that the valuation w defined in 1.5 is not a valuation over $\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} K_P$. We can consider the following example.

Example 2.3.1. *Let P be an irreducible polynomial of degree 2 over $\mathbb{F}_q$ such that ζ is a root of P, which we write $P = (\theta - \zeta)(\theta - \zeta^q)$. We have $\mathbb{F} = \mathbb{F}_P = \mathbb{F}_{q^2} = \mathbb{F}_q(\zeta)$ and $G = \{\mathrm{id}, \tau\}$. The map w as defined is not multiplicative:*

$$\underbrace{1 \otimes P}_{w=1} = \underbrace{(1 \otimes \theta - \zeta \otimes 1)}_{w=0} \underbrace{(1 \otimes \theta - \zeta^q \otimes 1)}_{w=0}.$$

First, we explain how to define certain valuations to solve this problem.

Lemma 2.3.2. *We have an isomorphism of $\mathbb{F}$-vector spaces:*

$$\mathbb{F} \otimes_{\mathbb{F}_q} \mathbb{F} \simeq \prod_{g \in G} \mathbb{F} \simeq \prod_{g \in G} (\mathbb{F} \otimes_{\mathbb{F}} \mathbb{F})$$

given by

$$\eta : x \otimes y \mapsto (g(x)y, g \in G).$$

Proof. We are inspired by the Artin's trick, see for example [11, Theorem 14]. We identify $\prod_{g \in G} (\mathbb{F} \otimes_{\mathbb{F}} \mathbb{F})$ and $\prod_{g \in G} \mathbb{F}$ in the proof. If $\mathbb{F} = \mathbb{F}_q$, then we have nothing to prove. Assume that $\mathbb{F}_q \subsetneq \mathbb{F}$. It is clear that η is a morphism of $\mathbb{F}$-vector spaces. The cardinal of $\mathbb{F} \otimes_{\mathbb{F}_q} \mathbb{F}$ and the cardinal of $\prod_G \mathbb{F}$ are the same (and equal to $q^{[\mathbb{F}:\mathbb{F}_q]^2}$). Let us prove the injectivity of η to obtain the isomorphism. Let us assume by contradiction that the kernel of η is not trivial. Let $x = \sum_{i=1}^{n} \alpha_i \otimes_{\mathbb{F}_q} x_i \in \mathrm{Ker}(\eta)$ with n be minimal among all the elements of the kernel. In particular, the family $(x_1, \ldots, x_n)$ is free over $\mathbb{F}_q$, as well as the family $(\alpha_1, \ldots, \alpha_n)$. Moreover, without loss of generality, we assume that $\alpha_1 = 1$. In particular, all α_i for $i \geq 2$ belong to $\mathbb{F} \backslash \mathbb{F}_q$. Since $x \in \mathrm{Ker}(\eta)$, we have

$$\sum_{i=1}^{n} g(\alpha_i) x_i = 0, \forall g \in G. \tag{2.4}$$

Let $g \in G \backslash \{\mathrm{id}\}$ and set $\beta_i = g(\alpha_i) - \alpha_i \in \mathbb{F}$ for all $i = 1, \ldots, n$. We have

$$\begin{aligned} \eta\left(\sum_{i=2}^{n} \beta_i \otimes x_i\right) &= \left(\sum_{i=2}^{n} h \circ g(\alpha_i) x_i - \sum_{i=2}^{n} h(\alpha_i) x_i, h \in G\right), \\ &= (-x_1 + x_1, \ldots, -x_1 + x_1, h \in G) \text{ by(2.4)}, \\ &= 0. \end{aligned}$$

By minimality of n, we obtain that $\beta_i = 0$ for $i = 2, \ldots, n$, in other words $g(\alpha_i) = \alpha_i$ for $i = 2, \ldots, n$, and this for all $g \in G \backslash \{\mathrm{id}\}$. We then obtain $\alpha_i = 0$ for $i = 2, \ldots, n$ and then $\eta(1 \otimes x_1) = x_1 = 0$, that is a contradiction. So $x = 0$.

□

In particular, through this isomorphism, the Frobenius τ_ζ is identified with $(\mathrm{id} \otimes \tau, \ldots, \mathrm{id} \otimes \tau)$.
First, we extend the scalars from $\mathbb{F}$ to $\mathbb{F}_q(\zeta)$. We obtain a (canonical) isomorphism (of $\mathbb{F}_q(\zeta)$-vector spaces) $\eta' : \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathbb{F} \to \prod_{g \in G} \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} \mathbb{F}$, given by the following. Let $(f_1, \ldots, f_l)$ be an $\mathbb{F}$-basis of $\mathbb{F}_q(\zeta)$ and $a_1, \ldots, a_l \in \mathbb{F}$. We set

$$\eta'\left(\sum_{i=1}^{l} a_i f_i \otimes_{\mathbb{F}_q} x_i\right) = \left(\sum_{i=1}^{l} g(a_i) f_i \otimes_{\mathbb{F}} x_i, g \in G\right).$$

Note that the isomorphism is canonical, but not the topologies that will appear. We then naturally extend the scalars (on the right) from $\mathbb{F}$ to L_P. We obtain an isomorphism (of $\mathbb{F}_q(\zeta)$-vector spaces on the left and L_P-modules on the right) induced by η, also denoted η:

$$\eta : \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} L_P \simeq \prod_{g \in G} \left(\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} L_P\right).$$

In particular we obtain L_P-vector spaces of dimension $[\mathbb{F}_q(\zeta) : \mathbb{F}]$ on each component of the product, so an L_P-vector space of dimension $[\mathbb{F}_q(\zeta) : \mathbb{F}_q]$. For $g \in G$, we set $H_g = \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} K_P \simeq \mathbb{F}_q(\zeta)((P))$ (where the left action of $\mathbb{F}$ is determined by $g \in G$).

For $x = \sum_{i=1}^{l} f_i \otimes x_i \in H_g$, we set:

$$v_g(x) = \min_{i=1,\ldots,m} v_P(x_i).$$

Proposition 2.3.3. *The application v_g is well defined and is a valuation over H_g.*

Proof. Fix $g \in G$. For $x \in \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} K_P$, it is clear that $v_g(x) = +\infty$ if and only if $x = 0$. Moreover, it is clear that v_g satisfies the ultrametric triangle inequality. For

$i, j = 1, \ldots, m$ let us write:

$$f_i f_j = \sum_{k=1}^{m} g(\alpha_{i,j,k}) f_k \in \mathbb{F}_q(\zeta)$$

with $\alpha_{i,j,k} \in \mathbb{F}$ for all $k = 1, \ldots, m$. We consider $x = \sum_{i=1}^{m} f_i \otimes x_i \in H_g$ and $y = \sum_{i=1}^{m} f_i \otimes y_i \in H_g$.

Let us assume without loss of generality that there exist two integers $l \leq m$ and $h \leq m$ such that $x_1, \ldots, x_l$ are of minimal P-adic valuation for x and $y_1, \ldots, y_h$ are of minimal P-adic valuation for y. Let us write w_i (resp. z_j) the P-adic sign of x_j (resp. y_j) for $j = 1, \ldots, m$. Set $S_l = \{1, \ldots, l\}$ and $S_h = \{1, \ldots, h\}$. We have

$$xy = \sum_{k=1}^{m} f_k \otimes \sum_{i,j=1}^{m} \alpha_{i,j,k} x_i y_j.$$

We then clearly have $v(xy) \geq v(x) + v(y)$. We can write:

$$xy = \sum_{k=1}^{m} f_k \otimes \sum_{(i,j) \in S_l \times S_h} \alpha_{i,j,k} x_i y_j + t$$

with $v_g(t) > v(x) + v(y)$. Then, we have the equivalences:

$$v(xy) > v(x) + v(y) \Leftrightarrow \sum_{(i,j) \in S_l \times S_h} \alpha_{i,j,k} w_i z_j = 0, \ \forall k = 1, \ldots, m.$$

Assume that $v(xy) > v(x) + v(y)$. This implies that

$$\begin{aligned} 0 &= \sum_{k=1}^{m} f_k \sum_{(i,j) \in S_l \times S_h} \alpha_{i,j,k} w_i z_j \\ &= \sum_{(i,j) \in S_l \times S_h} \sum_{k=1}^{m} f_k \alpha_{i,j,k} w_i z_j \\ &= \sum_{(i,j) \in S_l \times S_h} w_i z_j f_i f_j \\ &= \left(\sum_{i \in S_l} f_i w_i \right) \left(\sum_{j \in S_h} f_j z_j \right) \end{aligned}$$

which is impossible because of the freedom of the family $(f_1, \ldots, f_m)$ on $\mathbb{F}$. Thus, $v(xy) = v(x) + v(y)$. $\square$

For example, if we consider the element $1 \otimes P \in \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} K_P$, then it is sent via η to the element $(1 \otimes P, \ldots, 1 \otimes P) \in \prod_{g \in G} H_g$ therefore has valuation 1 on each component.

For all $g \in G$ we provide $L \otimes_K H_g$ with the topology v_g induced by its structure of H_g-vector space of finite dimension with respect to the choice of the basis $\mathscr{B}$ of L/K. In particular, if we set pr_g the projection on the g-component of the product, then we obtain $v_g\left(\mathrm{pr}_g\left(\eta\left((\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} \mathscr{O}_L)\right)\right)\right) \geq 0$.

Let v_P be the over-additive valuation on the product $\displaystyle\prod_{g\in G}(L \otimes_K H_g)$:

$$v_P\left((x_g, g \in G)\right) = \min_{g\in G}(v_g(x_g))$$

verifying $v_P(\eta(1 \otimes P)) = 1$. Remark that the Frobenius τ_ζ is equal to $(\mathrm{id} \otimes \tau, \ldots, \mathrm{id} \otimes \tau)$ on $\displaystyle\prod_{g\in G}(\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} L_P)$.

Remark 2.3.4. Following exactly the same ideas, by extending the scalars from $\mathbb{F}$ to $\mathbb{T}_z(L_P)$ or $\widetilde{L_P}$ we obtain the isomorphisms of $\mathbb{F}_q(\zeta)$-vector spaces on the left and $\widetilde{L_P}$ (resp. $\mathbb{T}_z(L_P)$) on the right:

$$\eta_z : \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathbb{T}_z(L_P) \simeq \prod_{g\in G} \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} \mathbb{T}_z(L_P)$$

and

$$\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \widetilde{L_P} \simeq \prod_{g\in G} \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} \widetilde{L_P}.$$

We are now interested in the case of the higher dimension d. We extend v_P onto $(\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} L_P)^d$ (the same goes with z) (topology of finite-dimensional vector spaces, for example with respect to the canonical basis). Then set

$$\Omega_{\zeta,d} = \{x \in (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} L_P)^d \mid v_P(x) \geq 0\} \supseteq (\mathbb{F}_q(\zeta) \otimes \mathscr{O}_L)^d$$

and

$$\Omega_{\zeta,d,z} = \{x \in (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathbb{T}_z(L_P))^d \mid v_P(x) \geq 0\} \supseteq (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathscr{O}_L[z])^d.$$

Proposition 2.3.5. *We have the following convergences:*

$$\log_{F^{(\zeta)},P} : \Omega_{\zeta,d} \to (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} L_P)^d$$

and

$$\log_{F^{(\zeta)},P} : \Omega_{\zeta,d,z} \to (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathbb{T}_z(L_P))^d.$$

Proof. It follows from Proposition 2.2.2 and the definitions of the objects. We omit the proof. $\square$

2.4. The P-adic L-series

Recall that $m = d[L : K]$ where d in the dimension of the t-module E and that F is the t-module given by $F = P^{-1}EP$. Let $\mathscr{C} = (g_1, \ldots, g_m)$ be an A-basis of $\mathrm{Lie}_E(\mathscr{O}_L)$, it is also a $\widetilde{K_\infty}$-basis of $\mathrm{Lie}_{\widetilde{E}}(L_\infty)$, a $\widetilde{K_P}$-basis of $\mathrm{Lie}_{\widetilde{E}}(L_P)$ and a $\mathbb{T}_z(K_P)$-basis of $\mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_P))$. The same goes by replacing E by F since $\partial_E = \partial_F$.

Let us remark, from Corollary 2.2.3, that for any z-unit $y(z) \in U(\widetilde{F}, \mathscr{O}_L[z])$ we have $\exp_{\widetilde{F}}(y(z)) \in \widetilde{E}(\mathscr{O}_L[z]) \subseteq \Omega_z$ and therefore

$$\log_{\widetilde{F},P}(\exp_{\widetilde{F}}(y(z))) \in \mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_P)).$$

Moreover, for a family $(x_1(z), \dots, x_m(z))$ of elements of $\mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_P))$ we have

$$\mathrm{Mat}_{\mathscr{C}}(x_1(z), \dots, x_m(z)) \in M_m(\mathbb{T}_z(K_P))$$

thus

$$\det_{\mathscr{C}}(x_1(z), \dots, x_m(z)) \in \mathbb{T}_z(K_P).$$

Next, formally in $(L[[z]])^d$ we have the following equality for all $f(z) \in (L[[z]])^d$:

$$\log_{\widetilde{F},P}(\exp_{\widetilde{F}}(f(z))) = f(z).$$

Let $(v_1(z), \dots, v_m(z)) \subset U(\widetilde{F}; \mathscr{O}_L[z])$ be an $\widetilde{A}$-basis of $U(\widetilde{F}; \widetilde{\mathscr{O}_L})$. Remark that the family
$(1 \otimes v_1(z), \dots, 1 \otimes v_m(z)) \subseteq \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathbb{T}_z(L_\infty)^d$ is also an $\widetilde{A_\zeta}$-basis of $U(\widetilde{F^{(\zeta)}}; \widetilde{M})$. Set

$$w(z) = \det_{\mathscr{C}}(v_1(z), \dots, v_m(z)) \in \mathbb{T}_z(K_\infty)$$

and

$$w_P(z) = \det_{\mathscr{C}}\Big(\log_{\widetilde{F},P}(\exp_{\widetilde{F}}(v_1(z))), \dots, \log_{\widetilde{F},P}(\exp_{\widetilde{F}}(v_m(z)))\Big) \in \mathbb{T}_z(K_P).$$

By the above discussions and the class formula, we have the following equality in $K[[z]]$:

$$L_P(\widetilde{F}/\widetilde{\mathscr{O}_L}) = \frac{w_P(z)}{\mathrm{sgn}(w(z))}.$$

Since $w_P(z) \in \mathbb{T}_z(K_P)$, to study the P-adic convergence we want to prove that $\mathrm{sgn}(w(z))$ divides $w_P(z)$ in $\mathbb{T}_z(K_P)$. Remark that the possible P-adic poles are the zeros of $\mathrm{sgn}(w(z)) \in \mathbb{F}_q[z]$ hence elements of $\overline{\mathbb{F}}_q$. We will prove that the meromorphic series $\dfrac{w_P(z)}{\mathrm{sgn}(w(z))}$ does not have a pole in $\overline{\mathbb{F}}_q$.

Theorem 2.4.1. *The meromorphic series* $\dfrac{w_P(z)}{\mathrm{sgn}(w(z))}$ *does not have a pole in* $\overline{\mathbb{F}}_q$. *In other words, we have the convergence* $\dfrac{w_P(z)}{\mathrm{sgn}(w(z))} \in \mathbb{T}_z(K_P)$.

Proof. Let $\zeta \in \overline{\mathbb{F}}_q$ be a root of $\mathrm{sgn}(w(z))$. Recall that $\mathscr{C}_\zeta = (1 \otimes g_1, \dots, 1 \otimes g_m) \subseteq \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathscr{O}_L$ if $\mathscr{C} = (g_1, \dots, g_m)$. Then $\mathrm{Lie}_{F_\zeta}(\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathscr{O}_L)$ is an A_ζ-lattice in M_∞ and admits $\mathscr{C}_\zeta$ as an A_ζ-basis. Consider $(w_1, \dots, w_m)$ an A_ζ-basis of $U_\zeta(F; \mathscr{O}_L) = \mathrm{ev}_\zeta U(F^{(\zeta)}; \widetilde{M})$ and $(w_1(z), \dots, w_m(z)) \subseteq U(F^{(\zeta)}; \widetilde{M})$ be such that $\mathrm{ev}_\zeta w_i(z) = w_i$ for $i = 1, \dots, m$. Set

$$W'(z) = \det_{\mathscr{C}_\zeta}(w_1(z), \dots, w_m(z)) \in \widetilde{M}_\infty \backslash (z - \zeta)\widetilde{M}_\infty$$

and

$$W_P'(z) = \det_{\mathscr{C}_\zeta}\left(\log_{\widetilde{F}^{(\zeta)},P}(\exp_{\widetilde{F}^{(\zeta)}}(w_1(z))), \ldots, \log_{\widetilde{F}^{(\zeta)},P}(\exp_{\widetilde{F}^{(\zeta)}}(w_m(z)))\right) \in \widetilde{M_v}.$$

Recall that the family $(1 \otimes v_1(z), \ldots, 1 \otimes v_m(z))$ is an $\widetilde{A}_\zeta$-basis of $U(\widetilde{F}^{(\zeta)}; \widetilde{M})$. Let us set

$$W(z) = \det_{\mathscr{C}_\zeta}(1 \otimes v_1(z), \ldots, 1 \otimes v_m(z)) = 1 \otimes w(z) \in \widetilde{M_\infty},$$

$$W_P(z) = 1 \otimes w_P(z) \in \widetilde{M_v},$$

and $\Delta = \det_{(1\otimes v_1(z),\ldots,1\otimes v_m(z))}(w_1(z), \ldots, w_m(z)) \in \widetilde{A}_\zeta$.
From the equality

$$W'(z) = \Delta W(z)$$

we obtain

$$1 \otimes \underbrace{L(\widetilde{F}/\widetilde{\mathscr{O}_L})}_{\in \mathbb{T}_z(K_\infty)} = 1 \otimes \frac{w(z)}{\operatorname{sgn}(w(z))} = \frac{W(z)}{1 \otimes \operatorname{sgn}(w(z))} = \frac{W'(z)}{\Delta(1 \otimes \operatorname{sgn}(w(z))}.$$

Since $1 \otimes L(\widetilde{F}/\widetilde{\mathscr{O}_L})$ does not have a pole in $\overline{\mathbb{F}}_q$ and $W'(z)$ is not divisible by the element $z - \zeta$, we obtain that $\Delta(1 \otimes \operatorname{sgn}(w(z))$ is not divisible by $z - \zeta$. From the equality

$$\frac{W_P(z)}{1 \otimes \operatorname{sgn}(w(z))} = \frac{W_P'(z)}{\Delta(1 \otimes \operatorname{sgn}(w(z))}$$

we can evaluate at $z = \zeta$ so ζ is not a pole of $\dfrac{w_P(z)}{\operatorname{sgn}(w(z))}$.

Finally, the P-adic L-series is a meromorphic series without any pole in $\overline{\mathbb{F}}_q$: it is an element of $\mathbb{T}_z(K_P)$. □

Remark 2.4.2. Assume that the module of z-units at the integral level $U(\widetilde{E}; \mathscr{O}_L[z])$ is $A[z]$-free. Then, by Proposition 1.3.4 any $A[z]$-basis of $U(\widetilde{E}; \mathscr{O}_L[z])$ is an $\widetilde{A}$-basis of $U(\widetilde{E}; \widetilde{\mathscr{O}_L})$. Let $(v_1(z), \ldots, v_m(z))$ be an $A[z]$-basis of $U(\widetilde{E}; \mathscr{O}_L[z])$. With the notation of the proof of Theorem 2.4.1 (just replace F by E in the notation), we obtain $\Delta \in A_\zeta[z]$ and

$$\mathrm{ev}_{z=\zeta}\left(\Delta \operatorname{sgn}(w(z))\right) \neq 0.$$

Thus,

$$\mathrm{ev}_{z=\zeta}\left(\operatorname{sgn}(w(z))\right) \neq 0.$$

Since it is true for any $\zeta \in \overline{\mathbb{F}_q}$, we obtain $\operatorname{sgn}(w(z)) \in \mathbb{F}_q^\times$. We proved that if $U(\widetilde{E}; A[z])$ is $A[z]$-free, then any basis $(v_1(z), \ldots, v_m(z))$ satisfies

$$\operatorname{sgn}\left(\det_{\mathscr{C}}(v_1(z), \ldots, v_m(z))\right) \in \mathbb{F}_q^\times.$$

This give us a class formula over $A[z]$, and rather surprisingly the polynomial $[H(\widetilde{E}; \mathscr{O}_L[z])]_{A[z]} \in A[z]$ does not appear.

Corollary 2.4.3 (P-adic L-series for $P^{-1}\widetilde{E}P$)**.** *Consider the t-module $F = P^{-1}EP$. Consider $(v_1(z), \ldots, v_r(z)) \subseteq U(\widetilde{F}, \mathscr{O}_L[z])$ an $\widetilde{A}$-basis of $U(\widetilde{F}; \widetilde{\mathscr{O}_L})$, $\mathscr{C}$ an $\widetilde{A}$-basis of* $\mathrm{Lie}_{\widetilde{F}}(\widetilde{\mathscr{O}_L})$*. Then the following product converges in $\mathbb{T}_z(K_P)$*

$$L_P(\widetilde{E}/\widetilde{\mathscr{O}_L}) = \prod_{Q \neq P} \frac{\left[\mathrm{Lie}_{\widetilde{E}}(\widetilde{\mathscr{O}_L}/Q\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}}{\left[E(\widetilde{\mathscr{O}_L}/Q\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}}$$

where the product runs over all the monic irreducible polynomials Q of A different from P. Further, we have the equality:

$$L_P(\widetilde{E}/\widetilde{\mathscr{O}_L}) = \frac{\det_{\mathscr{C}}\left(\log_{\widetilde{F},P}\exp_{\widetilde{F}} v_1(z), \ldots, \log_{\widetilde{F},P}\exp_{\widetilde{F}} v_m(z)\right)}{\mathrm{sgn}\left(\det_{\mathscr{C}}\left(v_1(z), \ldots, v_m(z)\right)\right)}.$$

We can then define the P-adic L-series associated with E and $\mathscr{O}_L$:

$$L_P(E/\mathscr{O}_L) = \mathrm{ev}_{z=1,P}\, L_P(\widetilde{E}, \widetilde{\mathscr{O}_L}) = \prod_{Q \neq P} \frac{[\mathrm{Lie}_E(\mathscr{O}_L/Q\mathscr{O}_L)]_A}{[E(\mathscr{O}_L/Q\mathscr{O}_L)]_A} \in K_P.$$

We also have the following equalities from [7, Proposition 3.3]:

$$L_P(\widetilde{E}/\widetilde{\mathscr{O}_L}) = \prod_{\mathfrak{P} \nmid P} \frac{\left[\mathrm{Lie}_{\widetilde{E}}(\widetilde{\mathscr{O}_L}/\mathfrak{P}\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}}{\left[E(\widetilde{\mathscr{O}_L}/\mathfrak{P}\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}} \in \mathbb{T}_z(K_P)$$

where the product runs over all the primes of $\mathscr{O}_L$ not dividing P, and

$$L_P(E/\mathscr{O}_L) = \prod_{\mathfrak{P} \nmid P} \frac{[\mathrm{Lie}_E(\mathscr{O}_L/\mathfrak{P}\mathscr{O}_L)]_A}{[E(\mathscr{O}_L/\mathfrak{P}\mathscr{O}_L)]_A} \in K_P.$$

Example 2.4.4. *Consider $n \geq 1$ an integer and the following two matrices:*

$$E = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \in M_n(A) \text{ and } N = \begin{pmatrix} 0 & \ldots & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & \ldots & 0 \\ 1 & 0 & \ldots & 0 \end{pmatrix} \in M_n(A).$$

Then the n-th tensor power of the Carlitz module is the t-module given by

$$C_\theta^{\otimes n} = \theta I_n + E + N\tau.$$

By [18, Section 4.2], the local factor at P is equal to:

$$z_P(\widetilde{C^{\otimes n}}/\widetilde{A}) = \frac{P^n}{P^n - z^{\deg(P)}} = \left(1 - \frac{z^{\deg(P)}}{P^n}\right)^{-1}.$$

We then obtain:

$$L_P(\widetilde{C^{\otimes n}}/\widetilde{A}) = \sum_{\substack{a \in A^+ \\ P \nmid a}} \frac{z^{\deg(a)}}{a^n} := \zeta_P(A, n, z) \in \mathbb{T}_z(K_P)$$

that is called the P-adic Carlitz zeta function, and

$$L_P(C^{\otimes n}/A) = \sum_{\substack{a\in A^+ \\ P\nmid a}} \frac{1}{a^n} = \zeta_P(A,n,1) \in K_P$$

that is called the P-adic Carlitz zeta value at $z=1$.

2.5. A P-adic class formula associated with the t-module $P^{-1}EP$

The next step is to introduce a P-adic regulator and obtain a P-adic class formula. We begin with P-twisted t-modules. Recall that $\mathscr{C}$ is a fixed A-basis of $\mathrm{Lie}_E(\mathscr{O}_L) = \mathrm{Lie}_F(\mathscr{O}_L)$.

Definition 2.5.1. Consider $V \subseteq U(F;\mathscr{O}_L)$ a sub-A-lattice and let $(v_1,\dots,v_m)$ be an A-basis of V. Then we define the P-adic regulator associated with V by

$$R_P(V) = \frac{\det_{\mathscr{C}}\left(\log_{F,P}(\exp_F(v_1)),\dots,\log_{F,P}(\exp_F(v_m))\right)}{\mathrm{sgn}(\det_{\mathscr{C}}(v_1,\dots,v_m))} \in K_P$$

which is independent of the choice of the basis of V and of $\mathrm{Lie}_F(\mathscr{O}_L)$.

Theorem 2.5.2. *[P-adic class formula for $P^{-1}EP$] We have the following equality in K_P:*

$$L_P(E/\mathscr{O}_L) = R_P(U(F;\mathscr{O}_L))\,[H(F;\mathscr{O}_L)]_A = R_P(U_{\mathrm{st}}(F;\mathscr{O}_L)).$$

Proof. We use notation as in the proof of Theorem 2.4.1 with $\zeta = 1$. In particular $(v_1,\dots,v_m)$ is an A-basis of $U(F;\mathscr{O}_L)$. Consider $(u_1,\dots,u_m)$ an A-basis of $U_{\mathrm{st}}(F;\mathscr{O}_L)$. Denote by $b_i = \exp_F(u_i) \in F(\mathscr{O}_L)$ for $i=1,\dots,m$ and by $a_i = \exp_F(v_i) \in F(\mathscr{O}_L)$ for $i=1,\dots,m$. We have

$$L(\widetilde{F}/\widetilde{\mathscr{O}_L}) = \frac{w'(z)}{f(z)\Delta}.$$

As the L-series $L(F/\mathscr{O}_L)$ (at infinity) is equal to

$$\frac{\det_{\mathscr{C}}(u_1,\dots,u_m)}{\mathrm{sgn}(\det_{\mathscr{C}}(u_1,\dots,u_m))} = \frac{\mathrm{ev}_{z=1}(w'(z))}{\mathrm{sgn}(\det_{\mathscr{C}}(u_1,\dots,u_m))} = \mathrm{ev}_{z=1}\frac{w'(z)}{f(z)\Delta}$$

we first have

$$\mathrm{ev}_{z=1}(f(z)\Delta) = \mathrm{sgn}(\det_{\mathscr{C}}(u_1,\dots,u_m)) \in \mathbb{F}_q^*. \tag{2.5}$$

Let us consider $P_1 = \mathrm{Mat}_{(v_1,\dots,v_m)}(u_1,\dots,u_m) \in M_m(A)$. By [10, Theorem 1] we have $\dfrac{\det(P_1)}{\mathrm{sgn}(\det(P_1))} = [H(F;\mathscr{O}_L)]_A$. Moreover, we have:

$$P_1 = \mathrm{Mat}_{(\log_{F,P}(a_1),\dots,\log_{F,P}(a_m))}(\log_{F,P}(b_1),\dots,\log_{F,P}(b_m)).$$

Then we have in K_P:

$$\det(P_1)\det_{\mathscr{C}}(\log_{F,P}(a_1),\dots,\log_{F,P}(a_m) = \det_{\mathscr{C}}(\log_{F,P}(b_1),\dots,\log_{F,P}(b_m)). \tag{2.6}$$

From the following equality in K_∞:

$$\det(P_1)\det_{\mathscr{C}}(v_1,\dots,v_m)=\det_{\mathscr{C}}(u_1,\dots,u_m)$$

we deduce by comparing signs:

$$\operatorname{sgn}(\det(P_1))\operatorname{sgn}(\det_{\mathscr{C}}(v_1,\dots,v_m))=\operatorname{sgn}(\det_{\mathscr{C}}(u_1,\dots,u_m)). \tag{2.7}$$

We finally have:

$$\begin{aligned}
L_P(E/\mathscr{O}_L) &= \mathrm{ev}_{z=1,P}\left(\frac{w'_P(z)}{f(z)\Delta}\right)\\
&= \frac{\det_{\mathscr{C}}(\log_{F,P}(b_1),\dots,\log_{F,P}(b_m))}{\operatorname{sgn}(\det_{\mathscr{C}}(u_1,\dots,u_m))} \text{ by Equality (2.5) },\\
&= \det(P_1)\frac{\det_{\mathscr{C}}(\log_{F,P}(a_1),\dots,\log_{F,P}(a_m))}{\operatorname{sgn}(\det_{\mathscr{C}}(u_1,\dots,u_m))} \text{ by Equality (2.6)},\\
&= \frac{\det(P_1)}{\operatorname{sgn}(\det(P_1))}\frac{\det_{\mathscr{C}}(\log_{F,P}(a_1),\dots,\log_{F,P}(a_m))}{\operatorname{sgn}(\det_{\mathscr{C}}(v_1,\dots,v_m))} \text{ by Equality (2.7)},\\
&= R_P(U(F;\mathscr{O}_L))\left[H(F;\mathscr{O}_L)\right]_A
\end{aligned}$$

and the second line equals $R_P(U_{\mathrm{st}}(E;\mathscr{O}_L))$.

□

2.6. A P-adic class formula

So far we've worked mainly with the t-module F, and now we want to link everything to the t-module E.

Set $h(z)=\left[\widetilde{E}(\widetilde{\mathscr{O}_L}/P\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}\in A[z]$ and $h(1)=\left[E(\mathscr{O}_L/P\mathscr{O}_L)\right]_A\in A\backslash\{0\}$. Consider $s\in\mathbb{N}$ such that $\partial_{\widetilde{E}}(h(z)^{q^s})=h(z)^{q^s}I_d$ (e.g., s such that $q^s\geq d$) and denote by $g(z)=h(z)^{q^s}\in A[z]$. By Proposition 1.1.1, we have for all $b(z)\in\mathscr{O}_L[z]^d$:

$$\widetilde{E}_{g(z)}(b(z))\in P\mathscr{O}_L[z]^d$$

and for all $b\in\mathscr{O}_L^d$:

$$E_{g(1)}(b)\in P\mathscr{O}_L^d.$$

Then by Corollary 2.2.3 and the above discussion, we can define the following maps:

$$\begin{aligned}
\mathrm{Log}_{\widetilde{E},P}:\Omega_z &\to \frac{1}{g(z)}\mathbb{T}_z(L_P)^d\\
x &\mapsto \frac{1}{g(z)}\log_{\widetilde{E},P}(\widetilde{E}_{g(z)}(x))
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Log}_{E,P}:\Omega &\to L_P^d\\
x &\mapsto \frac{1}{g(1)}\log_{E,P}(E_{g(1)}(x))
\end{aligned}.$$

Moreover, if $x \in \Omega_z^+$, then we have $\log_{\widetilde{E},P}(\widetilde{E}_{g(z)}(x)) = g(z)\log_{\widetilde{E},P}(x)$ in $\mathbb{T}_z(L_P)^d$. We obtain the following equality in $\mathbb{T}_z(L_P)^d$ for such x:

$$\log_{\widetilde{E},P}(x) = \mathrm{Log}_{\widetilde{E},P}(x)$$

thus the map $\mathrm{Log}_{\widetilde{E},P}$ extends the map $\log_{\widetilde{E},P}$ from Ω_z^+ to Ω_z. The same applies without z.

Lemma 2.6.1. *We have the following properties.*

(1) *For all $a \in A[z]$ and $x \in \Omega_z$ we have the following equality in $\frac{1}{g(z)}\mathbb{T}_z(L_P)^d$:*

$$\partial_{\widetilde{E}}(a)\,\mathrm{Log}_{\widetilde{E},P}(x) = \mathrm{Log}_{\widetilde{E},P}(\widetilde{E}_a(x)).$$

(2) *For all $x \in \Omega_z$ we have the equality in L_P^d:*

$$\mathrm{ev}_{z=1,P}(\mathrm{Log}_{\widetilde{E},P}(x)) = \mathrm{Log}_{E,P}(\mathrm{ev}_{z=1,P}(x)).$$

(3) *For all $a \in A$ and $x \in \Omega$ we have the following equality in L_P^d:*

$$\partial_E(a)\,\mathrm{Log}_{E,P}(x) = \mathrm{Log}_{E,P}(E_a(x)).$$

Proof.

(1) We have the following equalities in $\mathbb{T}_z(L_P)^d$ for all $x \in \Omega_z$ and $a \in A[z]$:

$$\begin{aligned} g(z)\,\mathrm{Log}_{\widetilde{E},P}(\widetilde{E}_a(x)) &= \log_{\widetilde{E},P}(\underbrace{\widetilde{E}_{g(z)}(\widetilde{E}_a(x))}_{v_P>0}) \\ &= \partial_{\widetilde{E}}(a)\log_{\widetilde{E},P}(\widetilde{E}_{g(z)}(x)) \\ &= \partial_{\widetilde{E}}(a)g(z)\,\mathrm{Log}_{\widetilde{E},P}(x). \end{aligned}$$

(2) We have the following equality in Ω^+ for all $x \in \Omega_z$:

$$\mathrm{ev}_{z=1,P}(\widetilde{E}_{g(z)}(x)) = E_{g(1)}(\mathrm{ev}_{z=1,P}(x)).$$

Then we have the following equalities:

$$\begin{aligned} \mathrm{ev}_{z=1,P}(g(z)\,\mathrm{Log}_{\widetilde{E},P}(x)) &= \mathrm{ev}_{z=1,P}(\log_{\widetilde{E},P}\underbrace{(\widetilde{E}_{g(z)}(x))}_{v_P>0}) \\ &= \log_{E,P}(E_{g(1)}(\mathrm{ev}_{z=1,P}(x))) \\ &= g(1)\,\mathrm{Log}_{E,P}(\mathrm{ev}_{z=1,P}(x)). \end{aligned}$$

(3) The proof is similar as for the first assertion.

□

Proposition 2.6.2. *The logarithm map $\log_{\widetilde{E},P}$ is injective on Ω_z^+.*

We begin with the following lemma.

Lemma 2.6.3. *For all $a \in A \backslash \{0\}$ and for all $x \in \mathbb{T}_z(L_P)^d \backslash \{0\}$ we have $\widetilde{E}_a(x) \neq 0$.*

Proof. Fix $a \in A\backslash\{0\}$ and $x \in \mathbb{T}_z(K_P)^d$ be such that $E_a(x) = 0$. We can suppose without loss of generality that $\partial_E(a) = aI_d$, even if it means applying $\widetilde{E}_{a^{q^s-1}}$ to $\widetilde{E}_a(x)$. We can also assume that z does not divide x in $\mathbb{T}_z(L_P)^d$. Denote by $\widetilde{E}_a = a+\sum_{i=1}^{r_a}\widetilde{E}_{a,i}\tau^i$ with $\widetilde{E}_{a,i} \in zM_d(\mathscr{O}_L[z])$ for all $i = 1, \dots, r_a$, and $x = \sum_{n\geq 0} y_n z^n$ with $a_n \in L_P^d$ and $y_0 \neq 0$. We have $E_a(x) = ay_0 \pmod{z\mathbb{T}_z(L_P)^d} \neq 0 \pmod{z\mathbb{T}_z(L_P)^d}$. Hence $E_a(x) \neq 0$. □

Proof of Proposition 2.6.2. Let x be in Ω_z^+ such that $\log_{\widetilde{E},P}(x) = 0$. Then for all $s \in \mathbb{N}$ we have

$$\partial_{\widetilde{E}}(P^{q^s}) \log_{\widetilde{E},P}(x) = 0 = \log_{\widetilde{E},P}(\widetilde{E}_{P^{q^s}}(x)).$$

Since $v_P(x) > 0$, we consider $s \in \mathbb{N}$ large enough such that $\widetilde{E}_{P^{q^s}}(x)$ belongs to $\mathcal{D}_z^+$. For such an integer s we obtain:

$$\widetilde{E}_{P^{q^s}}(x) = 0$$

which implies that $x = 0$ by Lemma 2.6.3.

□

Proposition 2.6.4. *The kernel of* $\mathrm{Log}_{E,P} : \mathscr{O}_L^d \to L_P^d$ *consists exactly of the torsion points of $E(\mathscr{O}_L)$, denoted by $E(\mathscr{O}_L)_{\mathrm{tors}}$.*

Proof. Consider first $x \in \mathscr{O}_L^d$ such that $\mathrm{Log}_{E,P}(x) = 0$. Then

$$\log_{E,P}(E_{g(1)}(x)) = 0.$$

Thus for all $n \geq 0$ we have:

$$\log_{E,P}(E_{P^{q^n}g(1)}(x)) = \partial_E(P^{q^n}) \log_{E,P}(E_{g(1)}(x)) = 0.$$

Since $v_P(E_{g(1)}(x)) > 0$, we can consider n large enough such that $E_{P^{q^n}g(1)}(x) \in \mathcal{D}^+$. Then we find by applying the exponentiel map $\exp_{E,P}$ to $\log_{E,P}(E_{P^{q^n}g(1)}(x))$ that $0 = E_{P^{q^n}g(1)}(x)$ so x is a torsion point.

Conversely, suppose that there is a non-zero polynomial $a \in A$ such that $E_a(x) = 0$. We also have $E_{a^{q^s}}(x) = 0$. Then we have

$$a^{q^s} \log_{E,P}(E_{g(1)}(x)) = \log_{E,P}(E_{a^{q^s}g(1)}(x)) = \log_{E,P}(E_{g(1)}(E_{a^{q^s}}(x))) = 0.$$

Since a is non-zero, we obtain $\log_{E,P}(E_{g(1)}(x)) = 0$. □

The previous proof tells us the following result.

Corollary 2.6.5. *For all monic irreducible P and n large enough such that $q^n > \dfrac{q^s}{q^{\deg(P)}-1}$, we have*

$$E(\mathscr{O}_L)_{\mathrm{tors}} = E[P^{q^n}g(1)]$$

where

$$E[P^{q^n}g(1)] = \{x \in \mathbb{C}_\infty^d \mid E_{P^{q^n}g(1)}(x) = 0\}.$$

Set $\mathscr{O}_{L,P} = \mathscr{O}_L \otimes_A A_P$ and $\widetilde{\mathscr{O}_{L,P}} = \widetilde{\mathscr{O}_L} \otimes_A A_P$. By [10, Lemma 3.21], we can extend E by continuity to a homomorphism of $\mathbb{F}_q$-algebras

$$E : A_P \to M_d(\mathscr{O}_{L,P})\{\{\tau\}\}.$$

We can also extend $\widetilde{E}$ by continuity to a homomorphism of $\mathbb{F}_q(z)$-algebras

$$\widetilde{E} : \widetilde{A}_P \to M_d(\widetilde{\mathscr{O}_{L,P}})\{\{\tau\}\}.$$

In particular, the A-module $E(P\mathscr{O}_{L,P})$ inherits a structure of A_P-module.

Set $U' = g(1)U(E;\mathscr{O}_L)$ and $U'_z = g(z)U(\widetilde{E};\widetilde{\mathscr{O}_L})$ (the multiplication is of course in $\mathrm{Lie}_E(L_\infty)$ (resp. $\mathrm{Lie}_{\widetilde{E}}(\widetilde{L_\infty})$).

Lemma 2.6.6. *We have the following properties.*

(1) *We have that U' and U'_z are sub-lattices of $U(E;\mathscr{O}_L)$ and $U(\widetilde{E};\widetilde{\mathscr{O}_L})$ respectively.*

(2) *We have that U' and U'_z are sub-lattices of $U(F;\mathscr{O}_L)$ and $U(\widetilde{F};\widetilde{\mathscr{O}_L})$ respectively.*

Proof.

(1) The first point is clear.

(2) We have to prove the inclusions $U' \subseteq U(F;\mathscr{O}_L)$ and $U'_z \subseteq U(\widetilde{F};\widetilde{\mathscr{O}_L})$. We do it for F, the same arguments apply to $\widetilde{F}$. Let $x \in \mathrm{Lie}_E(L_\infty)$ be such that $\exp_E(x) \in \mathscr{O}_L^d$, then:

$$\exp_F(g(1)x) = P^{-1}E_{g(1)}\left(\exp_E(Px)\right) \in P^{-1}P\mathscr{O}_L^d = \mathscr{O}_L^d.$$

□

By Lemma 2.6.6, U' (resp. U'_z) is a common sub-A-lattice (resp. sub-$\widetilde{A}$-lattice) for $U(E;\mathscr{O}_L)$ and $U(F;\mathscr{O}_L)$ (resp. $U(\widetilde{E};\widetilde{\mathscr{O}_L})$ and $U(\widetilde{F};\widetilde{\mathscr{O}_L})$). We then have:

$$\begin{aligned}[U(F;\mathscr{O}_L) : U(E;\mathscr{O}_L)]_A &= \left[U(F;\mathscr{O}_L) : U'\right]_A \left[U' : U(E;\mathscr{O}_L)\right]_A \\ &= \frac{[U(F;\mathscr{O}_L) : U']_A}{[U(E;\mathscr{O}_L) : U']_A} \in K^*\end{aligned}$$

and

$$\begin{aligned}\left[U(\widetilde{F};\widetilde{\mathscr{O}_L}) : U(\widetilde{E};\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}} &= \left[U(\widetilde{F};\widetilde{\mathscr{O}_L}) : U'_z\right]_{\widetilde{A}} \left[U'_z : U(\widetilde{E};\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}} \\ &= \frac{\left[U(\widetilde{F};\widetilde{\mathscr{O}_L}) : U'_z\right]_{\widetilde{A}}}{\left[U(\widetilde{E};\widetilde{\mathscr{O}_L}) : U'_z\right]_{\widetilde{A}}} \in \widetilde{K}^*.\end{aligned}$$

Let us define P-adic regulators associated to U' as follows. Let $(w_1,\ldots,w_m)$ be an A-basis of U'. We set:

$$R_{P,E}(U') = \frac{\det_{\mathscr{C}}\left(\mathrm{Log}_{E,P}(\exp_E(w_1)),\ldots,\mathrm{Log}_{E,P}(\exp_E(w_m))\right)}{\mathrm{sgn}(\det_{\mathscr{C}}(w_1,\ldots,w_m))} \in K_P$$

and

$$R_{P,F}(U') = \frac{\det_{\mathscr{C}}\left(\log_{F,P}(\exp_F(w_1)), \dots, \log_{F,P}(\exp_F(w_m))\right)}{\operatorname{sgn}(\det_{\mathscr{C}}(w_1, \dots, w_m))} \in K_P.$$

These regulators do not depend of the choice of the basis $(w_1, \dots, w_m)$. We can also define a P-adic regulator $R_P(U_{\mathrm{st}}(E; \mathscr{O}_L))$ for $U_{\mathrm{St}}(E; \mathscr{O}_L)$ and we have the following equality from the proof of Theorem 2.5.2:

$$R_P(U(E; \mathscr{O}_L))\,[H(E; \mathscr{O}_L)]_A = R_P(U_{\mathrm{St}}(E; \mathscr{O}_L)). \tag{2.8}$$

Similarly, we define the P-adic regulators $R_{P,\widetilde{E}}(U'_z)$ and $R_{P,\widetilde{F}}(U'_z)$ associated with U'_z that are elements in $\mathbb{T}_z(K_P)$ from Theorem 2.4.1.

Lemma 2.6.7. *We have the following equalities:*

$$R_{P,\widetilde{E}}(U'_z) = R_{P,\widetilde{F}}(U'_z) \text{ and } R_{P,E}(U') = R_{P,F}(U'). \tag{2.9}$$

Proof. We prove the first equality. We want to prove that for all $u(z) \in U(\widetilde{E}; \mathscr{O}_L[z])$ we have the following equality in $\mathbb{T}_z(L_P)^d$:

$$\log_{\widetilde{F},P}(\exp_{\widetilde{F}}(u(z))) = \operatorname{Log}_{\widetilde{E},P}(\exp_{\widetilde{E}}(u(z))).$$

Formally (i.e., in $L[[z]]^d$) it is true, and the two quantities belong to $\mathbb{T}_z(L_P)^d$. Then the first equality of the lemma is clear.

To prove the second equality, consider $(u_1, \dots, u_m)$ an A-basis of $U_{\mathrm{St}}(E; \mathscr{O}_L)$. From the first equality of the lemma, we have the following equality in $\mathbb{T}_z(K_P)$:

$$\begin{aligned}
&\det_{\mathscr{C}}(\log_{\widetilde{F},P}(\exp_{\widetilde{F}}(g(z)u_i(z))), i = 1, \dots, m) \\
&= \det_{\mathscr{C}}(\operatorname{Log}_{\widetilde{E},P}(\exp_{\widetilde{E}}(g(z)u_i(z))), i = 1, \dots, m).
\end{aligned}$$

By evaluating at $z = 1$ we obtain

$$\det_{\mathscr{C}}(\log_{F,P}(\exp_F(g(1)u_i)), i = 1, \dots, m) = \det_{\mathscr{C}}(\operatorname{Log}_{E,P}(\exp_E(g(1)u_i)), i = 1, \dots, m).$$

Consider now $(v_1, \dots, v_m)$ an A-basis of $U(E, \mathscr{O}_L)$, then $(g(1)v_1, \dots, g(1)v_m)$ is an A-basis of U'. Write $Q = \operatorname{Mat}_{(v_1,\dots,v_m)}(u_1, \dots, u_m) \in \mathrm{Gl}_m(A)$. We then have

$$\begin{aligned}
&\det_{\mathscr{C}}(\log_{F,P}(\exp_F(g(1)v_i)), i = 1, \dots, m) \\
&\qquad = \det(Q)^{-1} \det_{\mathscr{C}}(\log_{F,P}(\exp_F(g(1)u_i)), i = 1, \dots, m) \\
&\qquad = \det(Q)^{-1} \det_{\mathscr{C}}(\operatorname{Log}_{E,P}(\exp_E(g(1)u_i)), i = 1, \dots, m) \\
&\qquad = \det_{\mathscr{C}}(\operatorname{Log}_{E,P}(\exp_E(g(1)v_i)), i = 1, \dots, m).
\end{aligned}$$

We proved that the equality is true for one basis of U' and since the P-adic regulators does not depend of the choice of the basis, the equality is true. $\square$

Lemma 2.6.8.

(1) *We have the following equalities:*

$$R_{P,\widetilde{F}}(U'_z) = R_P(U(\widetilde{F};\widetilde{\mathscr{O}_L}))\left[U(\widetilde{F};\widetilde{\mathscr{O}_L}) : U'_z\right]_{\widetilde{A}}$$

and

$$R_{P,\widetilde{E}}(U'_z) = R_P(U(\widetilde{E};\widetilde{\mathscr{O}_L}))\left[U(\widetilde{E};\widetilde{\mathscr{O}_L}) : U'_z\right]_{\widetilde{A}}$$

(2) *We have the following equalities:*

$$R_{P,F}(U') = R_P(U(F;\mathscr{O}_L))\left[U(F;\mathscr{O}_L) : U'\right]_A$$

and

$$R_{P,E}(U') = R_P(U(E;\widetilde{\mathscr{O}_L}))\left[U(E;\mathscr{O}_L) : U'\right]_A$$

Proof. We only need to prove one of the equalities; all the others can be proven in a similar way. Let us prove the third one. By the structure theorem for finitely generated modules over a principal ideal domain, let us pick an A-basis $(v_1,\dots,v_m)$ of $U(F;\mathscr{O}_L)$ and $a_1,\dots,a_m \in A$ such that $(a_1v_1,\dots,a_mv_m)$ is an A-basis of U'. Then

$$\begin{aligned}
R_{P,F}(U') &= \frac{\det_{\mathscr{C}}(\log_{F,P}(\exp_F(a_1w_1)),\dots,\log_{F,P}(\exp_F(a_mw_m)))}{\operatorname{sgn}\left(\det_{\mathscr{C}}(\log_{F,P}(\exp_F(a_1w_1)),\dots,\log_{F,P}(\exp_F(a_mw_m)))\right)}\\
&= \frac{a_1\dots a_m}{\operatorname{sgn}(a_1\dots a_m)}R_P(U(F;\mathscr{O}_L))\\
&= \left[U(F;\mathscr{O}_L) : U'\right]_A R_P(U(F;\mathscr{O}_L)).
\end{aligned}$$

□

The link between objects associated to F and those associated to E is contained in the local factor at P, and given by the following equalities from [7, Lemma 3.4]:

$$z_P(E/\mathscr{O}_L) = [U(F;\mathscr{O}_L) : U(E;\mathscr{O}_L)]_A \frac{[H(E;\mathscr{O}_L)]_A}{[H(F;\mathscr{O}_L)]_A}. \tag{2.10}$$

and

$$z_P(\widetilde{E}/\widetilde{\mathscr{O}_L}) = \left[U(\widetilde{F};\widetilde{\mathscr{O}_L}) : U(\widetilde{E};\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}. \tag{2.11}$$

We can state one of the main results of this work.

Theorem 2.6.9 (P-adic class formula)**.** *We have the P-adic class formula for $\widetilde{E}$:*

$$z_P(\widetilde{E}/\widetilde{\mathscr{O}_L})L_P(\widetilde{E}/\widetilde{\mathscr{O}_L}) = R_P(U(\widetilde{E};\widetilde{\mathscr{O}_L}))$$

and the class formula for E:

$$z_P(E/\mathscr{O}_L)L_P(E/\mathscr{O}_L) = R_P(U(E;\mathscr{O}_L))\,[H(E;\mathscr{O}_L))]_A = R_P(U_{\mathrm{st}}(E;\mathscr{O}_L)).$$

Proof. Let us start with the following equality from Corollary 2.4.3:

$$L_P(\widetilde{E}/\widetilde{\mathscr{O}_L}) = R_P(U(\widetilde{F};\widetilde{\mathscr{O}_L})).$$

Then:

$$\begin{aligned}
z_P(\widetilde{E}/\widetilde{\mathscr{O}_L})L_P(\widetilde{E}/\widetilde{\mathscr{O}_L}) &= \frac{\left[U(\widetilde{F};\widetilde{\mathscr{O}_L}):U'_z\right]_{\widetilde{A}}}{\left[U(\widetilde{E};\widetilde{\mathscr{O}_L}):U'_z\right]_{\widetilde{A}}} R_P(U(\widetilde{F};\widetilde{\mathscr{O}_L})) \text{ by Equality (2.11)},\\
&= \frac{R_{P,\widetilde{F}}(U'_z)}{\left[U(\widetilde{E};\widetilde{\mathscr{O}_L}):U'_z\right]_{\widetilde{A}}} \text{ by Lemma 2.6.8},\\
&= \frac{R_{P,\widetilde{E}}(U'_z)}{\left[U(\widetilde{E};\widetilde{\mathscr{O}_L}):U'_z\right]_{\widetilde{A}}} \text{ by Equality (2.9)},\\
&= R_P(U(\widetilde{E};\widetilde{\mathscr{O}_L})) \text{ by Lemma 2.6.8}.
\end{aligned}$$

Recall Theorem 2.5.2:

$$L_P(E/\mathscr{O}_L) = R_P(U(F;\mathscr{O}_L))\,[H(F;\mathscr{O}_L)]_A\,.$$

Then:

$$\begin{aligned}
&z_P(E/\mathscr{O}_L)L_P(E/\mathscr{O}_L)\\
&= \frac{[U(F;\mathscr{O}_L):U')]_A}{[U(E;\mathscr{O}_L):U']_A}\frac{[H(E;\mathscr{O}_L)]_A}{[H(F;\mathscr{O}_L)]_A} R_P(U(F;\mathscr{O}_L))\,[H(F;\mathscr{O}_L)]_A \text{ by Equality (2.10)},\\
&= \frac{R_{P,F}(U')}{[U(E;\mathscr{O}_L):U']_A}\,[H(E;\mathscr{O}_L)]_A \text{ by Lemma 2.6.8},\\
&= \frac{R_{P,E}(U')}{[U(E;\mathscr{O}_L):U']_A}\,[H(E;\mathscr{O}_L)]_A \text{ by Equality (2.9)},\\
&= R_P(U(E;\mathscr{O}_L))\,[H(E;\mathscr{O}_L)]_A \text{ by Lemma 2.6.8}.
\end{aligned}$$

□

2.7. Vanishing of the P-adic L series

We keep the notation as in Theorem 2.4.1. In particular, $(v_1,\dots,v_m)$ is an A-basis of $U_{\mathrm{st}}(E;\mathscr{O}_L)$ and $(u_1,\dots,u_m)$ is an A-basis of $U(E;\mathscr{O}_L)$.

Proposition 2.7.1. *Suppose that there exists a non-zero element $x \in U_{\mathrm{St}}(E;\mathscr{O}_L)$ such that $\exp_E(x) = 0$. Then*

$$L_P(E/\mathscr{O}_L) = 0.$$

Proof. Write $x = \sum_{i=1}^{m} a_i v_i$ with $a_i \in A$ and suppose without loss of generality that $a_1 \neq 0$. Then $x = x(1)$ with $x(z) = \sum_{i=1}^{m} a_i v_i(z) \in U(\widetilde{E},\mathscr{O}_L[z])$. We have:

$$\begin{aligned}
&\det_{\mathscr{C}}(\mathrm{Log}_{\widetilde{E},P}(\exp_{\widetilde{E}}(v_i(z))), i=1,\dots,m)\\
=&\frac{1}{a_1}\det_{\mathscr{C}}(\mathrm{Log}_{\widetilde{E},P}(\exp_{\widetilde{E}}(x(z))), \mathrm{Log}_{\widetilde{E},P}(\exp_{\widetilde{E}}(v_i(z))), i=2,\dots,m).
\end{aligned}$$

Since $\exp_E(x) = 0$, we have:

$$\mathrm{ev}_{z=1,P}(\mathrm{Log}_{\widetilde{E},P}(\exp_{\widetilde{E}} x(z))) = \mathrm{Log}_{E,P}(\exp_E(x)) = 0.$$

We then conclude that $R_P(U_{\mathrm{St}}(E;\mathscr{O}_L)) = 0$, so $L_P(E/\mathscr{O}_L) = 0$ from the P-adic class formula 2.6.9.

□

Theorem 2.7.2. *If the exponential map* $\exp_E : L_\infty^d \to L_\infty^d$ *is not injective, then we have*

$$L_P(E/\mathscr{O}_L) = 0.$$

Proof. Let $x \in L_\infty^d$ be non-zero such that $\exp_E(x) = 0$. There exists $a \in A\backslash\{0\}$ such that $ax \in U_{\mathrm{st}}(E;\mathscr{O}_L)$ and we have $\exp_E(ax) = 0$. By Proposition 2.7.1 we have $L_P(E/\mathscr{O}_L) = 0$. □

We believe that the converse statement holds.

Conjecture. *The P-adic L-series is non-zero if and only if the exponential map* $\exp_E : L_\infty^d \to L_\infty^d$ *is injective.*

By [10, Corollary 3.24], it is true when $d = 1$ (i.e., in the Drinfeld modules case) and $L = K$.
Remark that in the case $\exp_E : L_\infty^d \to L_\infty^d$ is injective, which we will call the totally real case, then $\mathscr{U}(E;\mathscr{O}_L) = \exp_E(U(E;\mathscr{O}_L)) \subseteq E(\mathscr{O}_L)$ in a free A-module of rank m, and the family $(\mathrm{Log}_{E,P}(\exp_E(u_i)), i = 1,\dots,m)$ is A-free. We would like to have that this family is A_P-free to obtain the non-vanishing of the P-adic L-series.

Set:

$$U(E;P\mathscr{O}_L) = \{x \in \mathrm{Lie}_E(L_\infty) \mid \exp_E(x) \in E(P\mathscr{O}_L)\}$$

and

$$\mathscr{U}(E;P\mathscr{O}_L) = \exp_E(U(E;P\mathscr{O}_L)) \subseteq E(P\mathscr{O}_L)$$

and finally we denote by $\overline{\mathscr{U}(E;P\mathscr{O}_L)}$ the A_P-module generated by $\mathscr{U}(E;P\mathscr{O}_L)$. Then we can state an equivalent of the Leopoldt's conjecture in [34], introduced recently by Anglès in [6, Section 6.3] for the Carlitz module.

Conjecture (Conjecture A)**.** *The A_P-rank of $\overline{\mathscr{U}(E;P\mathscr{O}_L)}$ is equal to the A-rank of $\mathscr{U}(E;\mathscr{O}_L)$.*

This conjecture is clear in the case $d = 1$ and $L = K$. For further discussion of this conjecture, the reader may wish to read the paper by Anglès, Bosser and Taelman [9] where this conjecture is proved in the case of the Carlitz module defined on the Pth cyclotomic extension.

In the totally real case, the non-vanishing of the P-adic L-series $L_P(\widetilde{E}/\widetilde{\mathscr{O}_L})$ at $z = 1$ is equivalent to the previous Leopoldt conjecture. This result can be seen as an analogue of the following result from [15].

Theorem 2.7.3. *Let F be a totally real extension of $\mathbb{Q}$. Then the p-adic zeta function $\zeta_{p,F}(s)$ has a pole at $s = 1$ if and only if the (usual) Leopoldt conjecture is true for (F,p).*

Definition 2.7.4. We call order of vanishing of the P-adic L-series and denote it by $\mathrm{ord}_{z=1} L_P(\widetilde{E}/\widetilde{\mathscr{O}_L})$ the greatest integer n such that $(z-1)^n$ divides $L_P(\widetilde{E}/\widetilde{\mathscr{O}_L})$.

Fox example, if the exponential map $\exp_E : L_\infty^d \to L_\infty^d$ is not injective, then for all P we have $\mathrm{ord}_{z=1} L_P(\widetilde{E}/\widetilde{\mathscr{O}_L}) \geq 1$ and the previous conjecture tells us that $\mathrm{ord}_{z=1} L_P(\widetilde{E}/\widetilde{\mathscr{O}_L}) = 0$ if and only if $\exp_E$ in injective.
Here is a list of conjectures.

Conjecture (Conjecture B)**.** *The vanishing order of the P-adic L-series at $z = 1$ is independent of P.*

Caruso and Gazda [13] have already conjectured this in the context of Anderson motives. Caruso, Gazda and the author proved this conjecture in the case $L = K$ and $d = 1$, see [14, Theorem 2.17].

Conjecture (Conjecture C)**.** *We have $\mathrm{ord}_{z=1} L_P(\widetilde{E}/\widetilde{\mathscr{O}_L}) \leq [L : K] r_{\Omega_E} d$ where r_{Ω_E} is the rank of the period lattices Ω_E associated with E.*

We prove conjecture C in section 4.3 in the case $d = 1$ and $L = K$.

Example 2.7.5. *We continue our study of example 2.4.4.*

From the work of Anderson and Thakur [3, Corollary 2.5.9], we know that the exponential map $\exp_{C^{\otimes n}} : K_\infty^d \to K_\infty^d$ is injective if and only if $q-1$ does not divide n. By Theorem 2.7.2, we obtain that $\zeta_P(A, n, 1) = 0$ if $q-1$ divides n.

Conversely, if $q - 1 \nmid n$, Yu proved in [49, Theorem 3.7] that the P-adic Carlitz zeta function at $z = 1$ is non zero. In fact, he proved a much more powerful result in this setting, that the P-adic zeta function at $z = 1$ is transcendental over K.

Thus, conjecture 2.7 is true in this setting.

Chapter 3

The multi-variable setting

We keep the notation from Chapters 1 and 2. In particular L/K is a finite field extension of degree n and $\mathscr{O}_L$ denotes the integral closure of A in L.

The aim of this chapter is to extend the previous constructions to the case where the constant field is no longer $\mathbb{F}_q$ but $\mathbb{F}_q(t_1,\dots,t_s)$ where the t_i are new variables. One of the interests of these constructions is that in many cases, we can reduce the study of certain t-modules $E:\mathbb{F}_q[\theta]\to M_d(\mathscr{O}_L)\{\tau\}$ to the study of Drinfeld modules $\phi:\mathbb{F}_q(t_1,\dots,t_s)[\theta]\to\mathscr{O}_L(t_1,\dots,t_s)\{\tau\}$ simpler to understand. For an application to the study of the tensor power of the Carlitz module $C^{\otimes n}$ reduced to the study of the Carlitz module C, see the work of Anglès, Pellarin and Tavares Ribeiro in [8]. This chapter is the subject of an article [35, Section 5].

3.1. Setup

The goal of this section is to extend the developed theory to the multi-variable setting by replacing $\mathbb{F}_q$ by $k=\mathbb{F}_q(t_1,\dots,t_s)$. Recall that the Frobenius map acts as the identity on k. We keep the notation in the Introduction and we introduce the following notation.

- $A_s[z]\simeq k[z]\otimes_k A_s$,
- $\widetilde{A_s}=k(z)\otimes_k A_s$,
- w: a place of K ($w=v_P$ a finite place or $w=v_\infty$ the infinite place),
- π_w: a uniformiser of w ($\pi=P$ if $w=v_P$ and $\pi=\frac{1}{\theta}$ if $w=v_\infty$),
- $K_w=\mathbb{F}_w((\pi_w))$ denoted by $K_w=K_\infty$ if $w=v_\infty$ and $K_w=K_P$ if $w=v_P$,
- $\mathbb{F}_w$: the residue field associated with w i.e., $\mathbb{F}_w=\mathbb{F}_P$ if $w=v_P$ and $\mathbb{F}_w=\mathbb{F}_q$ if $w=v_\infty$,
- $L_w=L\otimes_K K_w$ i.e., $L_w=L_P$ if $w=v_P$ and $L_w=L_\infty$ otherwise,
- $k_w=\mathbb{F}_w(t_1,\dots,t_s)$,
- $K_{s,\pi_w}=k_w((\pi_w))$ denoted by $K_{s,P}$ if $w=v_P$ and $K_{s,\infty}$ if $w=v_\infty$,
- $\widetilde{K_{s,w}}=k_w(z)((\pi_w))$,
- $L_s=kL$,
- $L_{s,w}=L\otimes_K K_{s,w}$ denoted by $L_{s,P}$ if $w=v_P$ and $L_{s,\infty}$ otherwise,
- $\widetilde{L_{s,w}}=L\otimes_K\widetilde{K_{s,w}}$ denoted by $\widetilde{L_{s,P}}$ if $w=v_P$,
- $\mathscr{O}_{L,s}[z]\simeq k[z]\otimes_k\mathscr{O}_{L,s}$,
- $\widetilde{\mathscr{O}_{L,s}}=k(z)\otimes_k\mathscr{O}_{L,s}$.

We recall that every $x\in\widetilde{K_{s,\infty}}$ (resp. $\in K_{s,\infty}$) can be written uniquely as $x=\sum_{n\geq N}x_n\frac{1}{\theta^n}$ with $N\in\mathbb{Z}$, $x_n\in k(z)$ (resp. $x_n\in K$) and $x_N\neq 0$. We call $x_N\in k(z)$

(resp. k) the sign of x denoted by $\mathrm{sgn}(x)$. We define the Tale algebra in variables $\underline{t} = (t_1, \dots, t_s)$:

$$\mathbb{T}_s(K_w) = \left\{ \sum_{n \in \mathbb{N}^s} a_n \underline{t}^n \in K_w[[\underline{t}]] \mid a_n \in K_w, \lim_{n \to +\infty} w(a_n) = +\infty \right\}$$

where $\underline{t}^n = t_1^{n_1} \dots t_s^{n_s}$ if $n = (n_1, \dots, n_s) \in \mathbb{N}^s$. This is the completion of $K[t_1, \dots, t_s]$ with respect to the Gauss norm associated to w We set:

$$\mathbb{T}_s(L_w) = \left\{ \sum_{n \in \mathbb{N}^s} a_n \underline{t}^n \in L_w[[\underline{t}]] \mid a_n \in L_w, \lim_{n \to +\infty} w(a_n) = +\infty \right\} = L \otimes_K \mathbb{T}_s(K_w).$$

An Anderson t-module E of dimension d over $\mathscr{O}_{L,s}$ is a non-constant k-algebra homomorphism $E : A_s \to M_d(\mathscr{O}_{L,s})$, $a \mapsto E_a = \sum_{i=0}^{r_a} E_{a,i}\tau^i \in M_d(\mathscr{O}_{L,s})\{\tau\}$ such that $(E_{\theta,0}^d - \theta I_d)^d = 0$. We can consider $\widetilde{E}$, the z-twist of E, as in Section 1.2. Following notation from Section 1.1, we denote by $[M]_{A_s}$ the monic generator of $\mathrm{Fitt}_{A_s}(M)$ where M is an A_s-module, e.g., $M = E(\mathscr{O}_{L,s}/P\mathscr{O}_{L,s})$ and $M = \mathrm{Lie}_E(\mathscr{O}_{L,s}/P\mathscr{O}_{L,s})$. As in Proposition 1.2.2 there exists a unique element $\exp_E \in M_d(L_s)\{\{\tau\}\}$ called the exponential map associated with E and converging over $L_{\infty,s}^d$. Similarly, there exists a logarithm map $\log_E \in M_d(L_s)\{\{\tau\}\}$ as in Proposition 1.2.3.

We can now define the module of units and class module in the multi-variable setting:

$$U(E; \mathscr{O}_{L,s}) = \{x \in \mathrm{Lie}_E(L_{s,\infty}) \mid \exp_E(x) \in E(\mathscr{O}_{L,s})\}$$

and the class module

$$H(E; \mathscr{O}_{L,s}) = \frac{E(L_{s,\infty})}{E(\mathscr{O}_{L,s}) + \exp_E(\mathrm{Lie}_E(L_{s,\infty}))}$$

both provided with A_s-module structure. We define the module of z-units:

$$U(\widetilde{E}; \widetilde{\mathscr{O}_{L,s}}) = \left\{ x \in \mathrm{Lie}_{\widetilde{E}}(\widetilde{L_{s,\infty}}) \mid \exp_{\widetilde{E}}(x) \in \widetilde{E}(\widetilde{\mathscr{O}_{L,s}}) \right\}$$

and the class module for the z-deformation:

$$H(\widetilde{E}; \widetilde{\mathscr{O}_{L,s}}) = \frac{\widetilde{E}(\widetilde{L_{s,\infty}})}{\widetilde{E}(\widetilde{\mathscr{O}_{L,s}}) + \exp_{\widetilde{E}}(\mathrm{Lie}_{\widetilde{E}}(\widetilde{L_{s,\infty}}))}$$

both provided with $\widetilde{A_s}$-module structure. We define the module of z-units at the integral level:

$$U(\widetilde{E}; \mathscr{O}_{L,s}[z]) = \left\{ x \in \mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_{s,\infty})) \mid \exp_{\widetilde{E}}(x) \in \widetilde{E}(\mathscr{O}_{L,s}[z]) \right\}$$

and finally the class module at the integral level

$$H(\widetilde{E}; \mathscr{O}_{L,s}[z]) = \frac{\widetilde{E}(\mathbb{T}_z(L_{s,\infty}))}{\widetilde{E}(\mathscr{O}_L[z]) + \exp_{\widetilde{E}}(\mathrm{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_{s,\infty})))}$$

both provided with $A_s[z]$-module structure. We have the following result from [18, Proposition 2.8].

Proposition 3.1.1. *The unit module $U(E;\mathscr{O}_{L,s})$ is an A_s-lattice in* $\mathrm{Lie}_E(L_{s,\infty})$ *and the module of z-units $U(\widetilde{E},\widetilde{\mathscr{O}_{L,s}})$ is an $\widetilde{A_s}$-lattice in* $\mathrm{Lie}_{\widetilde{E}}(\widetilde{L_{s,\infty}})$.

Denote by

$$z_P(\widetilde{E}/\widetilde{\mathscr{O}_{L,s}}) = \frac{\left[\mathrm{Lie}_{\widetilde{E}}(\widetilde{\mathscr{O}_{L,s}/P\mathscr{O}_{L,s}})\right]_{\widetilde{A_s}}}{\left[\widetilde{E}(\widetilde{\mathscr{O}_{L,s}/P\mathscr{O}_{L,s}})\right]_{\widetilde{A_s}}}$$

the local factor associated with $\widetilde{E}$ at P and

$$z_P(E/\mathscr{O}_{L,s}) = \frac{[\mathrm{Lie}_E(\mathscr{O}_{L,s}/P\mathscr{O}_{L,s})]_{A_s}}{[E(\mathscr{O}_{L,s}/P\mathscr{O}_{L,s})]_{A_s}}$$

the local factor associated with E at P. We have the following class formula for t-modules defined over $\mathscr{O}_{L,s}$, see [18, Theorem 2.9].

Theorem 3.1.2. *The following product*

$$L(\widetilde{E}/\widetilde{\mathscr{O}_{L,s}}) = \prod_P z_P(\widetilde{E}/\widetilde{\mathscr{O}_{L,s}})$$

where P runs through the monic irreducible polynomials of A, converges in $\widetilde{K_{s,\infty}}$ and we have the class formula:

$$L(\widetilde{E}/\widetilde{\mathscr{O}_{L,s}}) = \left[\mathrm{Lie}_{\widetilde{E}}\left(\widetilde{\mathscr{O}_{L,s}}\right) : U(\widetilde{E};\widetilde{\mathscr{O}_{L,s}})\right]_{\widetilde{A_s}}.$$

3.2. The P-adic case

We define the P-adic L-series in the multi-variable setting.

3.2.1. The P-adic class formula. All results from Chapter 2 remain valid by replacing $\mathbb{F}_q$ by k. In particular, we have the following P-adic class formula.

Theorem 3.2.1 (P-adic class formula)**.** *We have the following assertions.*

(1) *The infinite product*

$$L_P(\widetilde{E}/\widetilde{\mathscr{O}_{L,s}}) = \prod_{Q\neq P} z_Q(\widetilde{E}/\widetilde{\mathscr{O}_{L,s}})$$

where Q runs through the monic irreducible polynomials of A different from P, converges in $\mathbb{T}_z(K_{s,P})$ and we have the class formula:

$$z_P(\widetilde{E}/\widetilde{\mathscr{O}_{L,s}})L_P(\widetilde{E}/\widetilde{\mathscr{O}_{L,s}}) = R_P(U(\widetilde{E};\widetilde{\mathscr{O}_{L,s}})).$$

(2) *The infinite product*

$$L_P(E/\mathscr{O}_{L,s}) = \prod_{Q\neq P} z_Q(E/\mathscr{O}_{L,s})$$

where Q runs through the monic irreducible polynomials of A different from P, converges in $K_{s,P}$ and we have the class formula:

$$z_P(E/\mathscr{O}_{L,s})L_P(E/\mathscr{O}_{L,s}) = R_P(U(E;\mathscr{O}_{L,s}))\,[H(E;\mathscr{O}_{L,s})]_{A_s}\,.$$

Proof. The proof follows the same linas as the proof of 2.4.1 by replacing $\mathbb{F}_q$ by k. We omit the details. $\square$

Denote by

$$U(E;P\mathscr{O}_{L,s}) = \{x \in \mathrm{Lie}_E(L_{s,\infty}) \mid \exp_E(x) \in E(P\mathscr{O}_{L,s})\}$$

and consider the A_s-module

$$\mathscr{U}(E;\mathscr{O}_{L,s}) = \exp_E(U(E;\mathscr{O}_{L,s})).$$

Consider also the $A_{P,s}$-module $\mathscr{U}(E;P\mathscr{O}_{L,s}) = \exp_E(U(E;P\mathscr{O}_{L,s}))$. Then the proof of Theorem 2.7.2 is still valid in the multi-variable setting by replacing $\mathbb{F}_q$ by k.

Proposition 3.2.2. *We have the following assertions.*

(1) *If the exponential map $\exp_E : L_{s,\infty}^d \to L_{s,\infty}^d$ is not injective, then $L_P(E/\mathscr{O}_{L,s}) = 0$.*

(2) *Assume that the A_s-rank of $\mathscr{U}(E;\mathscr{O}_{L,s})$ and the $A_{P,s}$-rank of $\mathscr{U}(E;P\mathscr{O}_{L,s})$ are equal. Then $L_P(E/\mathscr{O}_L) \neq 0$ if and only if the exponential map $\exp_E : L_{s,\infty}^d \to L_{s,\infty}^d$ is injective.*

3.2.2. The integral level. In the work of [8], given a t-module $E : A_s \to M_d(A_s)\{\tau\}$, they want to evaluate the variables $(t_1,\dots,t_s)$ at some $\zeta \in \overline{\mathbb{F}}_q^s$. In this case, they need that all of the coefficients $E_{\theta,i}$ of E_θ, for $i = 0,\dots,r$, can be evaluated at ζ. This is possible if all the $E_{i,\theta}$ belong to $M_d(\mathbb{F}_q[t_1,\dots,t_s]\mathscr{O}_L)$. This is what we call the integral level.

We suppose now that: $E_\theta \in M_d(\mathscr{O}_L[t_1,\dots,t_s])\{\tau\}$ i.e., we want to work at the integral level.

Theorem 3.2.3. *The L-series $L(\widetilde{E}/\widetilde{\mathscr{O}_{L,s}})$ converges in $\mathbb{T}_{s,z}(K_\infty)$ and we have the class formula:*

$$L(\widetilde{E}/\widetilde{\mathscr{O}_{L,s}}) = \frac{\det_{\mathscr{C}}(u_1(z),\dots,u_m(z))}{\mathrm{sgn}(\det_{\mathscr{C}}(u_1(z),\dots,u_m(z)))}$$

where $(u_1(z),\dots,u_m(z)) \in U(\widetilde{E};\mathscr{O}_L[t_1,\dots,t_s,z])$ is an $\widetilde{A_s}$-basis of the z-unit module.

Proof. The proof of [45, Corollary 7.5.6] is still valid in the multi-variable setting at the integral level. We omit the details. $\square$

The objective of this subsection is to prove that the P-adic L-series $L_P(\widetilde{E}/\widetilde{\mathscr{O}_{L,s}})$ converges in $\mathbb{T}_{s,z}(K_P)$.
Set $\Omega_{s,z} = \{x \in \mathbb{T}_{z,s}(L_P)^d \mid v_P(x) \geq 0\}$ and $\Omega_{s,z}^+ = \{x \in \mathbb{T}_{z,s}(L_P)^d \mid v_P(x) > 0\}$. Following the proof of Proposition 2.2.2 we have the two following convergences:

$$\log_{\widetilde{E},P} : \Omega_{s,z}^+ \to \mathbb{T}_{s,z}(L_P)^d$$

and

$$\log_{\widetilde{F},P} : \Omega_{s,z} \to \mathbb{T}_{s,z}(L_P)^d.$$

We deduce that the P-adic L-series $L_P(\widetilde{E}/\widetilde{\mathscr{O}_{L,s}})$ is written in the form $\dfrac{w}{f}$ with $w \in \mathbb{T}_{z,s}(K_P)$ and $f \in \mathbb{F}_q[t_1, \ldots, t_s, z]$. We then consider $\zeta = (\zeta_1, \ldots, \zeta_s) \in \overline{\mathbb{F}_q}^s$ and we want to prove that we can evaluate the P-adic L-series at $t_i = \zeta_i$ for all $i = 1, \ldots, s$ and at $z = \zeta \in \overline{\mathbb{F}_q}$ (simultaneously).

We use arguments very similar to those used for the convergence of the P-adic L-series, so we omit some of the details.

We set $\mathscr{K}(s) = \mathbb{F}_q(\zeta_1) \otimes_{\mathbb{F}_q} \ldots \otimes_{\mathbb{F}_q} \mathbb{F}_q(\zeta_s)$. We then consider the following notation for $j = 0, \ldots, s$:
- $k_j = \mathbb{F}_q(t_{j+1}, \ldots, t_s)$, e.g., $k_0 = k = \mathbb{F}_q(t_1, \ldots, t_s)$ and $k_s = \mathbb{F}_q$,
- $k_j A = k_j \otimes_{\mathbb{F}_q} A \simeq \mathbb{F}_q(t_{j+1}, \ldots, t_s)[\theta]$,
- $k_j K = k_j \otimes_{\mathbb{F}_q} K \simeq \mathbb{F}_q(t_{j+1}, \ldots, t_s, \theta)$,
- $\widetilde{k_j K} = \mathbb{F}_q(z) \otimes_{\mathbb{F}_q} k_j K \simeq \mathbb{F}_q(z, t_{j+1}, \ldots, t_s, \theta)$,
- $k_j \mathscr{O}_L = k_j \otimes_{\mathbb{F}_q} \mathscr{O}_L$,
- $A_{s,j} = \mathscr{K}(s) \otimes_{\mathbb{F}_q} k_j A$,
- $\widetilde{A_{s,j}} = \mathscr{K}(s) \otimes_{\mathbb{F}_q} k_j \widetilde{A}$,
- $\mathscr{O}_{L,s,j} = \mathscr{K}(s) \otimes_{\mathbb{F}_q} k_j \mathscr{O}_L$,
- $\widetilde{\mathscr{O}_{L,s,j}} = \mathscr{K}(s) \otimes_{\mathbb{F}_q} k_j \widetilde{\mathscr{O}_L}$,
- For a place w of K extended to $k_j K$, $\widetilde{K(j)_w}$ is the completion of $\widetilde{k_j K}$ with respect to w.
- $\widetilde{L(j)_w} = L \otimes_K K(j)_w$,
- $\widetilde{M_{s,j,w}} = \mathscr{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{t_j}(L(j)_w)$,
- $\widetilde{L_{j,w}} = L \otimes_K \widetilde{K_{j,w}}$.
- For a place w of K, $\mathbb{T}_{z,j}(L_w) = \mathbb{T}_{z,t_{j+1},\ldots,t_s}(L_w)$, e.g;, $\mathbb{T}_{z,0}(L_w) = \mathbb{T}_{z,t_1,\ldots,t_s}(L_w)$ and $\mathbb{T}_{z,s}(L_w) = \mathbb{T}_z(L_w)$.

For all $j = 0, \ldots, s$, we extend the Frobenius τ into τ_s on $A_{s,j}$ by $\tau_s = \mathrm{id} \otimes \tau$ where id is the identity on $\mathscr{K}(s)$. We do the same for $\widetilde{A_{s,j}}$, for $\mathscr{O}_{L,s,j}$ and for $\widetilde{\mathscr{O}_{L,s,j}}$.

For $j = 1, \ldots, s$ we define $E^{(j)}$ the homomorphism of $\mathscr{K}(s) \otimes_{\mathbb{F}_q} k_j$-algebras $E^{(j)} : A_{s,j} \to M_d(\mathscr{O}_{L,s,j})\{\tau_s\}$, that we call Anderson $A_{s,j}$-module defined over $\mathscr{O}_{L,s,j}$, by:

$$E^{(j)}_\theta = \sum_{i=0}^{r} \mathrm{ev}_{t_1=\zeta_1,\ldots,t_j=\zeta_j}(a_i)\tau_s^i$$

if $E_\theta = \displaystyle\sum_{i=0}^{r} a_i \tau_s^i \in M_d(\mathscr{O}_{L,s})\{\tau\}$. We also set $E^{(0)} = E$ where we identify a_i with $1 \otimes a_i \in \mathscr{K}(s) \otimes_{\mathbb{F}_q} \mathscr{O}_L[t_1, \ldots, t_s]$ and replace τ with τ_s.
Similarly, we define $\widetilde{E^{(j)}}$ the z-twist of $E^{(j)}$, that is the homomohprism of $\mathscr{K}(z) \otimes_{\mathbb{F}_q} k_j(z)$-algebras $\widetilde{E^{(j)}} : \widetilde{A_{s,j}} \to M_d(\widetilde{\mathscr{O}_{L,s,j}})\{\tau_s\}$, that we call Anderson $\widetilde{A_{s,j}}$-module defined over $\widetilde{\mathscr{O}_{L,s,j}}$, by:

$$\widetilde{E^{(j)}}_\theta = \sum_{i=0}^{r} \mathrm{ev}_{t_1=\zeta_1,..,t_j=\zeta_j}(a_i) z^i \tau_s^i$$

Finally, we consider $F = P^{-1}EP$ and construct $\widetilde{F^{(j)}}$ and $F^{(j)}$ in the same way.

Lemma 3.2.4. *Consider $j \in \{1, \dots, s\}$.*

(1) *For all $a \in A_{s,j}$ we have the following equalities in $M_d(\mathscr{O}_{L,s,j})\{\tau_s\}$:*

$$E_a^{(j)} = \mathrm{ev}_{t_1=\zeta_1,\dots,t_j=\zeta_j} E_a = \mathrm{ev}_{t_j=\zeta_j} E_a^{(j-1)}.$$

(2) *We have the following equalities in $M_d\left(\mathscr{K}(s) \otimes_{\mathbb{F}_q} \mathbb{F}_q[t_{j+1}, \dots, t_s]L\right)\{\{\tau_s\}\}$:*

$$\exp_{E^{(j)}} = \mathrm{ev}_{t_1=\zeta_1,\dots,t_j=\zeta_j} \exp_E = \mathrm{ev}_{t_j=\zeta_j} \exp_{E^{(j-1)}}.$$

(3) *Consider x in $\left(\mathscr{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{t_j}\left((\mathbb{F}_q(t_{j+1}, \dots, t_s)L)_\infty\right)\right)^d$. Then we have:*

$$\exp_{E^{(j)}}(x) = \mathrm{ev}_{t_1=\zeta_1,\dots,t_j=\zeta_j} \exp_E(x) = \mathrm{ev}_{t_j=\zeta_j} \exp_{E^{(j-1)}}(x).$$

Proof. It follows from definitions of the onjects, we omit the proof. □

We then define for all $j = 1, \dots, s$:

$$U(j) = \mathrm{ev}_{t_j=\zeta_j} U\left(\widetilde{E}^{(j-1)}; \widetilde{\mathscr{O}_{L,s,j}}[t_j]\right) \subseteq U\left(\widetilde{E}^{(j)}; \widetilde{\mathscr{O}_{L,s,j}}\right).$$

Following the same arguments we used to prove Theorem 1.5.3, we have the following result.

Theorem 3.2.5.

(1) *For all $j = 1, \dots, s$, we have an $\widetilde{A_{s,j}}$-module isomorphism:*

$$\frac{U(j)}{U\left(\widetilde{E}^{(j)}; \widetilde{\mathscr{O}_{L,s,j}}\right)} \simeq H\left(\widetilde{E}^{(j-1)}; \widetilde{\mathscr{O}_{L,s,j}}[t_j]\right)[t_j - \zeta_j]$$

given by

$$f_j(x) = \frac{\exp_{\widetilde{E}^{(j-1)}} x - \exp_{\widetilde{E}^{(j)}} x}{t_j - \zeta_j}$$

where $H\left(\widetilde{E}^{(j-1)}; \widetilde{\mathscr{O}_{L,s,j}}[t_j]\right)[t_j - \zeta_j]$ is the $(t_j - \zeta_j)$-torsion of the class module

$$H\left(\widetilde{E}^{(j-1)}; \widetilde{\mathscr{O}_{L,s,j}}[t_j]\right) = \frac{\widetilde{E^{(j-1)}}\left(\widetilde{M_{s,j,\infty}}\right)}{\widetilde{E^{(j-1)}}\left(\widetilde{\mathscr{O}_{L,s,j}}[t_j]\right) + \exp_{\widetilde{E^{(j-1)}}}\left(\widetilde{E^{(j-1)}}(\widetilde{M_{s,j,\infty}})\right)}.$$

(2) *The module $U(j)$ is a sub-$\widetilde{A_{s,j}}$ lattice of $U\left(\widetilde{E}^{(j)}; \widetilde{\mathscr{O}_{L,s,j}}\right)$.*

We are now able to prove the main theorem of this section.

Theorem 3.2.6. *The P-adic L-series does not have a pole in $\overline{\mathbb{F}}_q^s$. In other words we have:*

$$L_P(\widetilde{E}/\widetilde{\mathscr{O}_{L,s}}) \in \mathbb{T}_{z,s}(K_P).$$

Proof. We closely follow the proof of Theorem 2.4.1. We identify $\mathscr{C} = (g_1, \dots, g_m)$ with $(1 \otimes g_1, \dots, 1 \otimes g_m) \subseteq \mathscr{K}(s) \otimes_{\mathbb{F}_q} \mathrm{Lie}_E(\mathscr{O}_L)$. Consider an $\widetilde{A}_s$-basis $(u_{i,1})_{i=1,\dots,m} \subseteq U(\widetilde{F}; \mathscr{O}_L[t_1, \dots, t_s, z])$ of $U(\widetilde{F}; \widetilde{\mathscr{O}_{L,s}})$. Set

$$w_1 = \det_{\mathscr{C}}(u_{1,1}, \dots, u_{m,1}) \in \mathbb{T}_{z,s}(K_\infty)$$

with sign

$$f_1 \in \mathbb{F}_q[z, t_1, \dots, t_s]$$

and set

$$w_{1,P} = \det_{\mathscr{C}}(\log_{\widetilde{F},P}(\exp_{\widetilde{F}}(u_{1,1})), \dots, \log_{\widetilde{F},P}(\exp_{\widetilde{F}}(u_{m,1}))) \in \mathbb{T}_{z,s}(K_P).$$

We want to prove that the quotient $\dfrac{w_{1,P}}{f_1}$ is an element of $\mathbb{T}_{s,z}(K_P)$. We will prove that we can evaluate the last quantity at every $\zeta = (\zeta_1, \dots, \zeta_s) \in \overline{\mathbb{F}}_q^s$ and at $z = \zeta \in \overline{\mathbb{F}_q}$.

We will prove by induction that for all $k = 1, \dots, s$, there exists $(v_{i,k+1}, i = 1, \dots, m)$ an $\widetilde{A_{s,k}}$-basis of $U(k)$ and $x_{k+1} \in \mathscr{K}(s) \otimes_{\mathbb{F}_q} \widetilde{A_{s,k}}$ such that

$$\mathrm{ev}_{t_1=\zeta_1,\dots,t_k=\zeta_k}(1 \otimes L(\widetilde{F}/\widetilde{\mathscr{O}_{L,s}}) = \frac{\det_{\mathscr{C}}(v_{i,k+1}, i = 1, \dots, m)}{x_{k+1}}$$

and

$$\mathrm{ev}_{t_1=\zeta_1,\dots,t_k=\zeta_k}(1 \otimes \frac{w_{1,P}}{f_1}) = \frac{\det_{\mathscr{C}}(\log_{\widetilde{F}^{(k)},P} \exp_{\widetilde{F}^{(k)}} v_{i,k+1}, i = 1, \dots, m)}{x_{k+1}}$$
$$\in \frac{1}{x_{k+1}} \mathscr{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_z(t_{k+1}, \dots, t_s)(L_P).$$

Step 1: evaluation at $t_1 = \zeta_1$.
By Theorem 3.2.3 we have

$$L(\widetilde{F}/\widetilde{\mathscr{O}_{L,s}}) = \frac{w_1}{f_1} \in \mathbb{T}_{z,s}(K_\infty).$$

Consider$(v_{i,2})_{i=1,\dots,m}$ an $\widetilde{A_{s,1}}$ basis of $U(1)$ that can be assumed to be at the entire level, i.e., $(v_{i,2}) \subseteq \left(\mathscr{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_2,\dots,t_s}(L_\infty)\right)^d$ and let $(u_{i,2}) \subseteq (\mathscr{K}(s) \otimes \mathbb{T}_{z,s}(L_\infty))^d$ be above (i.e., $\mathrm{ev}_{t_1=\zeta_1} u_{i,2} = v_{i,2}$) for all i). Set

$$w_2 = \det_{\mathscr{C}}(u_{1,2}, \dots, u_{m,2}) \in \mathscr{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,s}(K_\infty)$$

that is not divisible by $t_1 - \zeta_1$ and set

$$w_{2,P} = \det_{\mathscr{C}}(\log_{\widetilde{F},P}(\exp_{\widetilde{F}}(u_{1,2})), \dots, \log_{\widetilde{F},P}(\exp_{\widetilde{F}}(u_{m,2}))) \in \mathscr{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,s}(K_P)$$

the P-adic analogue of w_2. Set $\delta_2 = \det_{(1 \otimes u_{1,1}, \dots, 1 \otimes u_{m,1})}(u_{1,2}, \dots, u_{m,2}) \in \mathscr{K}(s) \otimes_{\mathbb{F}_q} \widetilde{A}_s$. We have:

$$1 \otimes_{\mathbb{F}_q} w_1 = \frac{w_2}{\delta_2}$$

and

$$1 \otimes w_{1,P} = \frac{w_{2,P}}{\delta_2}.$$

We deduce the following equality from the class formula:

$$1 \otimes L(\widetilde{F}/\widetilde{\mathscr{O}_{L,s}}) = \frac{w_2}{\delta_2(1 \otimes f_1)}.$$

Since ζ_1 is not a pole of the L-series and $t_1 - \zeta_1$ does not divide w_2, we deduce that we can evaluate $\delta_2(1 \otimes f_1)$ at $t_1 = \zeta_1$ in a non-zero element x_2 of $\mathscr{K}(s) \otimes_{\mathbb{F}_q} \widetilde{A}_{s-1}$, in other words

$$\mathrm{ev}_{t_1=\zeta_1} 1 \otimes L(\widetilde{F}/\widetilde{\mathscr{O}_{L,s}}) = \frac{\det_{\mathscr{C}}(v_{i,2})}{x_2} \in \frac{1}{x_2}\mathscr{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_2,\dots,t_s}(L_\infty).$$

Next, from the equality

$$1 \otimes \frac{w_{1,P}}{f_1} = \frac{w_{2,P}}{\delta_2(1 \otimes f_1)}$$

we deduce that we can evaluate the P-adic L-series at $t_1 = \zeta_1$:

$$\begin{aligned}\mathrm{ev}_{t_1=\zeta_1} 1 \otimes \frac{w_{1,P}}{f_1} &= \frac{\det_{\mathscr{C}} \left(\log_{\widetilde{F}^{(1)},P}(\exp_{\widetilde{F}^{(1)}}(v_{i,2})), i = 1, \dots, m\right)}{x_2} \\ &\in \frac{1}{x_2}\mathscr{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_2,\dots,t_s}(L_P).\end{aligned}$$

<u>Step k</u>: Assume the result to be true up to rank $s - 1 \geq k - 1 \geq 1$ and we prove it at rank k. Consider $(v_{i,k+1})_{i=1,\dots,m}$ a $\widetilde{A_{s,k}}$ basis of $U(k)$ that can be assumed to be to the entire level, i.e., $v_{i,k+1} \subseteq \left(\mathscr{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_{k+1},\dots,t_s}(L_\infty)\right)^d$ and let$(u_{i,k+1}) \subseteq \left(\mathscr{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_k,\dots,t_s}(L_\infty)\right)^d$ be above (i.e., $\mathrm{ev}_{t_k=\zeta_k} u_{i,k+1} = v_{i,k+1}$). Set

$$w_{k+1} = \det_{\mathscr{C}}(u_{1,k+1}, \dots, u_{m,k+1}) \in \mathscr{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_k,\dots,t_s}(K_\infty)$$

that is not divisible by $t_k - \zeta_k$ and set

$$w_{k+1,P} = \det_{\mathscr{C}}(\log_{\widetilde{F}^{(k-1)}}(\exp_{\widetilde{F}^{(k-1)}}(u_{i,k+1})), i = 1, \dots, m) \in \mathscr{K}_s \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_k,\dots,t_s}(K_P)$$

the P-adic analogue of w_{k+1}. Set $\delta_{k+1} = \det_{(v_{1,k}),\dots,v_{m,k}}(u_{1,k+1}, \dots, u_{m,k+1}) \in \mathscr{K}(s) \otimes \widetilde{A_{s,k-1}}$. We have:

$$\det_{\mathscr{C}}(v_{1,k}, \dots, v_{m,k}) = \frac{\det_{\mathscr{C}}(u_{1,k+1}, \dots, u_{m,k+1})}{\delta_{k+1}}$$

and

$$\det_{\mathscr{C}} \left(\log_{\widetilde{F}^{(k-1)},P}(\exp_{\widetilde{F}^{(k-1)}}(v_{1,k})), \dots, \log_{\widetilde{F}^{(k-1)},P}(\exp_{\widetilde{F}^{(k-1)}}(v_{m,k}))\right) = \frac{w_{k+1,P}}{\delta_{k+1}}.$$

Then we have the following equalities:

$$\mathrm{ev}_{t_1=\zeta_1,\dots,t_k=\zeta_k} \left(1 \otimes L(\widetilde{F}/\widetilde{\mathscr{O}_{L,s}})\right) = \frac{w_{k+1}}{x_k \delta_{k+1}}$$

and

$$\mathrm{ev}_{t_1=\zeta_1,\dots,t_k=\zeta_k}\left(1\otimes\frac{w_{1,P}}{f_1}\right)=\frac{w_{k+1,P}}{x_k\delta_{k+1}}.$$

Since we can evaluate at $t_k=\zeta_k$ the L-series and $t_k-\zeta_k$ does not divide $\widetilde{w_{k+1}}$, we can evaluate $x_k\delta_{k+1}$ at $t_k=\zeta_k$ into a non-zero element $x_{k+1}\in\mathscr{K}(s)\otimes\widetilde{A_{s,k}}$. We have:

$$\mathrm{ev}_{t_k=\zeta_k,\dots,t_z=\zeta_1}(1\otimes L(\widetilde{F}/\widetilde{\mathscr{O}_{L,s}}))=\mathrm{ev}_{t_k=\zeta_k}\frac{w_{k+1}}{x_k\delta_{k+1}}=\frac{\det_{\mathscr{C}}(v_{i,k+1},i=1,\dots,m)}{x_{k+1}}$$

and

$$\mathrm{ev}_{t_k=\zeta_k,\dots,t_1=\zeta_1}\,1\otimes\frac{w_{1,P}}{f_1}=\frac{\det_{\mathscr{C}}\left(\log_{\widetilde{F}^{(k)},P}(\exp_{\widetilde{F}^{(k)}}(v_{i,k+1}))\right)}{x_{k+1}}$$
$$\in\frac{1}{x_{k+1}}\mathscr{K}(s)\otimes_{\mathbb{F}_q}\mathbb{T}_{z,t_{k+1},\dots,t_s}(K_P).$$

Last step: evaluation at z. We write $z=t_{s+1}$ and use similar arguments to conclude.

$\square$

Remark 3.2.7. If at some step $j\in\{1,\dots,s\}$ we have for all $i=1,\dots,r$:

$$\mathrm{ev}_{t_1=\zeta_1,\dots,t_j=\zeta_j}\,A_i=0,$$

then we have for all $k\geq j$:

$$E_\theta^{(k)}=\theta I_d+N_k$$

with $N_k\in M_d(\mathscr{K}(s)\otimes_{\mathbb{F}_q}\mathbb{F}_q[t_{k+1},\dots,t_s]\mathscr{O}_L)$ a nilpotent matrix. Then we have:

$$\partial_{E^{(k)}}(\theta^{q^d})=E^{(k)}_{\theta^{q^d}}=\theta^{q^d}I_d.$$

Hence, if $\exp_{E^{(k)}}=\sum_{n\geq0}d_{n,k}\tau_s^k$, then from the functional equation of the exponential map we obtain for all $n\geq0$:

$$(\theta^{q^d})^{q^n}d_{n,k}=\theta^{q^d}d_{n,k}.$$

Thus $d_{0,k}=I_d$ and $d_{n,k}=0$ for all $n\geq1$ so finally $\exp_{E^{(k)}}=I_d\tau^0$ for all $k\geq j$. Then we have $U(\widetilde{E}^{(j)};\widetilde{\mathscr{O}_{L,s,j}})=\mathrm{Lie}_{\widetilde{E}^{(j)}}(\widetilde{\mathscr{O}_{L,s,j}})$ for all $j\geq k$, and the previous proof is still valid.

Remark 3.2.8. Assume that ϕ is a Drinfeld $A[t_1,\dots,t_s]$-module defined over $A[t_1,\dots,t_s]$. Then Proposition 1.4.3 is still valid in this setting, that is

$$L(\widetilde{\phi}/\widetilde{A}_s)\equiv1\pmod{\pi\mathbb{T}_{s,z}(K_\infty)}.$$

We conclude that Proposition 1.7.4 is still true.

We also have from [10, Proposition 5] the following equality

$$L(\widetilde{\phi}/\widetilde{A_s})A[t_1,\dots,t_s,z])=U(\widetilde{\phi};A[t_1,\dots,t_s,z]).$$

We deduce that Proposition 1.7.5 still works in this setting, that is

$$L(\widetilde{\phi}/\widetilde{A_s})=\log_{\widetilde{\phi}}(1).$$

Example 3.2.9. *We will twist the tensor power of the Carlitz module as follows. Consider* $t_1, \ldots, t_s$ *new variables and let* E *be the* t*-module given by:*

$$E_\theta = \theta I_d + E + \prod_{i=1}^{s}(t_i - \theta)N\tau \in M_d(A[t_1, \ldots, t_s])\{\tau\}.$$

Following similar computation as in [45, section 7.6.2], we obtain that the local factor at P *is:*

$$z_P(\widetilde{E}/\widetilde{A}) = \left(1 - \frac{z^{\deg(P)} P(t_1) \ldots P(t_s)}{P^d}\right)^{-1}$$

hence

$$L_P(\widetilde{E}/\widetilde{A}_s) = \sum_{\substack{a \in A^+ \\ P \nmid a}} \frac{z^{\deg(a)} a(t_1) \ldots a(t_s)}{a^d} \in \mathbb{T}_{z,s}(K_P)$$

and

$$L_P(E/A_s) = \sum_{\substack{a \in A^+ \\ P \nmid a}} \frac{a(t_1) \ldots a(t_s)}{a^d} \in \mathbb{T}_s(K_P)$$

where $a(t_i) = \sum_{j=1}^{n} a_j t_i^j$ *if* $a = \sum_{j=1}^{n} a_j \theta^j$.

We will write $L_P(d, s)$ *for the previous quantity. This is the* P*-adic analogue of the so-called Pellarin* L*-series defined in [39] by*

$$L(d, s) = \sum_{a \in A^+} \frac{a(t_1) \ldots a(t_s)}{a^d} \in \mathbb{T}_s(K_\infty).$$

Chapter 4

Drinfeld modules defined over A

In this last chapter, we study the P-adic L-series in the setting of A-Drinfeld module defined over A. In Section 4.1 we establish a surprising connection between the property of a monic irreducible polynomial P to be Wieferich and the P-adic valuation of the special value of the P-adic L-series of Drinfeld modules. This section is the subject of an article [14] in joint work with Xavier Caruso and Quentin Gazda.

In Section 4.2, we study the vanishing order at $z = 1$ of the P-adic L-series. The main result is a relation between this order and some properties of successive minimum basis of the period lattice associated with ϕ introduced in Section 1.6. This section is the subject of an article [35, Section 6]. We finally give an explicit way to compute the vanishing order in the case ϕ is a very small Drinfeld module in Section 4.3.

4.1. Wieferich primes

In this section we study the valuation of the P-adic L-series and the valuation of the special value associated with the P-adic L-series.

We consider $\phi : A \to A\{\tau\}$ a Drinfeld A-module defined over A of rank r. Let $P \in A$ be an irreducible monic polynomial and set $u_\phi(z) = \exp_{\widetilde{\phi}} L(\widetilde{\phi}/\widetilde{A}) \in A[z]$. We set:

$$g_{P,\phi}(z) = \left[\widetilde{\phi}(\widetilde{A}/P\widetilde{A})\right]_{\widetilde{A}} \in A[z]$$

and

$$g_{P,\phi}(1) = [\phi(A/PA)]_A \in A\backslash\{0\}.$$

We define the folloging hypothesis:

$(H)_P$: if $q = 2$, then $\deg(P) > 1$.

4.1.1. Ordic valuation. In this subsection, we fix a base-point $x \in A$. The next definition generalizes Thakur's notion of C-Wieferich primes to ϕ, see [47].

Definition 4.1.1. Given an ideal I of A, we define

$$\pi_x(\phi; I) = \ker(A \to A/I,\, a \mapsto \phi_a(x)).$$

A place P of A is said *ϕ-Wieferich in base x* if $\pi_x(\phi; P) = \pi_x(\phi; P^2)$.

We record some properties of π_x as a function of ideals.

Proposition 4.1.2. *Let I and J be two nonzero ideals in A.*

(1) *If $I \subset J$, then $\pi_x(\phi; I) \subset \pi_x(\phi; J)$.*

(2) *If I and J are coprime, then $\pi_x(\phi; IJ) = \pi_x(\phi; I) \cap \pi_x(\phi; J)$.*

(3) *If P is a maximal ideal satisfying $(H)_P$ and $k \geq 2$, then $\pi_x(\phi; P^k)$ either equals $\pi_x(\phi; P^{k-1})$ or $P\pi_x(\phi; P^{k-1})$. Moreover, if $\pi_x(\phi; P^k) = P\pi_x(\phi; P^{k-1})$ then*

$$\pi_x(\phi; P^{k+1}) = P\pi_x(\phi; P^k).$$

(4) $\mathrm{Fitt}_A(\phi(A/I)) \subseteq \pi_x(\phi; I)$.

Proof. Point (1) is deduced from the A-module map $\phi(A/I) \to \phi(A/J)$. By the Chinese remainder theorem, we have $\phi(A/IJ) \cong \phi(A/I) \times \phi(A/J)$ and point (2) follows.

We turn to (3). By (1), we have $\pi_x(\phi; P^k) \subset \pi_x(\phi; P^{k-1})$. We claim that $P\pi_x(\phi; P^{k-1}) \subset \pi_x(\phi; P^k)$. Indeed, for $a \in \pi_x(\phi; P^{k-1})$, we have $\phi_a(x) \in P^{k-1}$ and hence $\phi_{Pa}(x) = \phi_P(\phi_a(x)) \in P^k + (\phi_a(x))^q$; assumptions on k are such that $P^{q(k-1)} \subset P^k$, thus $\phi_{Pa}(x) \in P^k$.

This shows that $\pi_x(\phi; P^k)$ either equals $\pi_x(\phi; P^{k-1})$ or $P\pi_x(\phi; P^{k-1})$. It remains to prove that if the latter case holds, then $\pi_x(\phi; P^{k+1}) = P\pi_x(\phi; P^k)$; for that it is enough to show that the canonical map

$$\pi_x(\phi; P^{k+1})/P\pi_x(\phi; P^k) \longrightarrow \pi_x(\phi; P^k)/P\pi_x(\phi; P^{k-1})$$

is injective. By the Snake lemma, this amounts to showing that the multiplication by P

$$\pi_x(\phi; P^{k-1})/\pi_x(\phi; P^k) \xrightarrow{\times P} \pi_x(\phi; P^k)/\pi_x(\phi; P^{k+1})$$

is injective. Given $a \in \pi_x(\phi; P^{k-1})$ such that Pa belongs to $\pi_x(\phi; P^{k+1})$, we have $P\phi_a(x) \in (\phi_{Pa}(x)) + (\phi_P(x))^q \subset P^{k+1}$. If $q > 2$, then we deduce that $a \in \pi_x(\phi; P^k)$ which shows injectivity. If $q = 2$, then the assumption $(H)_P$ ensures that the degree of P is at least 2; hence by [25, 1.4.(ii)] the τ-valuation of ϕ_P mod P is at least 2 as well. We deduce that

$$\phi_{Pa}(x) = \phi_P(\phi_a(x)) \in P(\phi_a(x)) + P(\phi_a(x)^2) + (\phi_a(x))^4 \subset P^{k+1}$$

from what we obtain $a \in \pi_x(\phi; P^k)$ as before.

Point (4) is deduced from Proposition 1.1.1. □

Since the ring A is principal, those properties reduces the determination of the function $\pi_x(\phi; I)$ to that of its value on maximal ideals and the integers $c_p(\phi; x)$ which appear in the next definition.

Definition 4.1.3. Let P be a prime in A. The *P-ordic valuation* of x relatively to ϕ, denoted by $c_P(\phi; x)$, is the largest $c \in \{0, 1, \ldots, +\infty\}$ such that $\pi_x(\phi; P^{c+1}) = \pi_x(\phi; P)$.

In other words we have

$$\pi_x(\phi; P) = \ldots = \pi_x(\phi; P^{c_P(\phi,x)+1})$$

and

$$\pi_x(\phi; P^{c_P(\phi,x)+1+k}) = P^k \pi_x(\phi; P^{c_P(\phi,x)+1}), \forall k \geq 0.$$

That is, a prime P is ϕ-Wieferich if and only if $c_P(\phi; x) > 0$.

Lemma 4.1.4. *Besides, $c_P(\phi; x) = +\infty$ if and only if $x \in \phi(A)_{\text{tors}}$ i.e., there exists $a \in A\backslash\{0\}$ such that $\phi_a(x) = 0$.*

Proof. Assume that $c_P(\phi; x) = +\infty$. Then $\pi_x(\phi; P^k) = \pi_x(\phi; P)$ for all $k \geq 1$. In particular we have

$$\phi_{g_{P,\phi}(1)}(x) \in P^k A$$

for all $k \geq 1$, then $\phi_{g_{P,\phi}(1)}(x) = 0$ Thus $x \in \phi(A)_{\text{tors}}$. Conversely, assume that $x \in \phi(A)_{\text{tors}}$. Consider $a \in A\backslash\{0\}$ such that $\phi_a(x) = 0$. We have $a \in \pi_x(\phi; P^k)$ for all $k \geq 0$. In particular, we have $a \in \pi_x(\phi; P^{c_P(\phi,x)+1+k}) = P^k \pi_x(\phi; P^{c_P(\phi,x)+1})$ for all $k \geq 0$ which is possible only if $c_P(\phi, x) = +\infty$. $\square$

Remark 4.1.5. We deduce from Proposition 4.1.2 that

$$\pi_x(\phi; I) = \bigcap_{i=1}^{s} P_i^{\max(0,\, k_i - c_{P_i}(\phi;x)-1)} \pi_x(\phi; P_i)$$

if $I = P_1^{k_1} \cdots P_s^{k_s}$ is the prime ideal decomposition of $I \neq (0)$.

In the following, we will also write by $\pi_x(\phi; P)$ the monic generator of the ideal $\pi_x(\phi; P)$.

4.1.2. Connexion with special values of P-adic L-series. In this part, we present a set of theorems relating ordic valuations to the P-adic valuation of several values of interest attached to the P-adic L-series.

To start with, we focus on $L_P(\phi/A)$. We will show that its P-adic valuation is related to $c_P(\phi; x)$ when x is the Taelman unit $u_\phi(1) = \exp_\phi(L(\phi/A)) \in A$.

The key result is the following.

Proposition 4.1.6. *Let P be a place satisfying $(H)_P$ and let $x \in A$. Then*

$$c_P(\phi; x) = v_P(\phi_{g_{P,\phi}(1)}(x)) - 1. \tag{4.1}$$

Proof. First we deal with the case $x \in \phi(A)_{\text{tors}}$. We claim that $\phi_{g_{P,\phi}(1)}(x) = 0$ if and only if $x \in \phi(A)$ is a torsion point. The direct implication is clear. Let us prove the converse and assume that $x \in \phi(A)_{\text{tors}}$. We then have $\operatorname{Log}_{\phi,P}(x) = 0$ by Proposition 2.6.4 and

$$g_{P,\phi}(1) \operatorname{Log}_{\phi,P}(x) = \operatorname{Log}_{\phi,P}(\phi_{g_{P,\phi}(1)}(x)) = 0.$$

Since $\phi_{g_{P,\phi}(1)}(x) \in \mathcal{D}^+$, we obtain $\phi_{g_{P,\phi}(1)}(x) = 0$.

We now observe that, if $x \in \phi(A)_{\text{tors}}$, then both quantities in (4.1) are infinite by the above claim and Lemma 4.1.4, thus the result is clear.

From now on, we may assume that $x \notin \phi(A)_{\mathrm{tors}}$, so that both quantities in (4.1) are finite. As $\mathbb{F}_q$-vector spaces, we have $\phi(A/PA) = A/PA$. In particular,

$$\deg_\theta(g_{P,\phi}(1)) = \dim_{\mathbb{F}_q}(\phi(A/PA)) = \dim_{\mathbb{F}_q}(A/PA) = \deg_\theta(P).$$

We deduce that the quotient $\dfrac{g_{P,\phi}(1)}{\pi_x(\phi;P)}$ is of degree smaller than $\deg(P)$, hence prime to P.

We also note that, for a given nonnegative integer k, the condition "P^k divides $\phi_a(x)$" is equivalent to "$\pi_x(\phi;P^k)$ divides a". Therefore, $v_P(\phi_{g_{P,\phi}(1)}(x))$ is the largest integer k such that $\pi_x(\phi;P^k)$ divides $g_{P,\phi}(1)$. Since the quotient $\dfrac{g_{P,\phi}(1)}{\pi_x(\phi;P)}$ is prime to P, Property (3) of Proposition 4.1.2 gives Equality (4.1). $\square$

We deduce the following theorem linking the valuation of the P-adic L-series and the property of being Wieferich.

Theorem 4.1.7. *Let P be a monic irreducible polynomial of A satisfying $(H)_P$. Then*

$$v_P(L_P(\phi/A)) = c_P(\phi; u_\phi(1)).$$

In particular, P is ϕ-Wieferich in base $u_\phi(1)$ if and only if P divides $L_P(\phi/A)$.

Proof. By Proposition 2.2.3 and the P-adic class formula we have

$$v_P(L_P(\phi/A)) = v_P(\phi_{g_{P,\phi}(1)}(u_\phi(1)) - 1.$$

We apply Proposition 4.1.6 to $u_\phi(1)$ to obtain the desired equality. The second assertion follows directly. $\square$

We will see in the next section, see Theorem 4.2.3, that the vanishing order of the P-adic L-series does not depend on P by Theorem 4.2.3. Denote it by $o_\phi = \mathrm{ord}_{z=1} L_P(\widetilde{\phi}/\widetilde{A})$.

We define the special value of the P-adic L-series:

$$L_P^*(\widetilde{\phi}/\widetilde{A}) = \frac{L_P(\widetilde{\phi}/\widetilde{A})}{(z-1)^{o_\phi}} \in \mathbb{T}_z(K_P)\backslash(z-1)\mathbb{T}_z(K_P).$$

We define the special value at $z = 1$ of the P-adic L-series:

$$L_P^*(\phi/A) = \mathrm{ev}_{z=1,P}(L_P^*(\widetilde{\phi}/\widetilde{A})) \in K_P^*.$$

Similarly, we write

$$u_\phi(z) = (z-1)^k u_\phi^*(z)$$

with $u_\phi^*(z) \in A[z]\backslash(z-1)A[z] \neq 0$. The exponent k is then the vanishing order of $u_\phi(z)$ at $z = 1$. Set $u_\phi^*(1) = \mathrm{ev}_{z=1}(u_\phi^*(z)) \in A\backslash\{0\}$.

Theorem 4.1.8. *We assume that $\phi(A)_{\mathrm{tors}} = \{0\}$. For any monic irreducible polynomial P satisfying $(H)_P$, we have*

$$v_P(L_P^*(\phi/A)) = c_P(\phi; u_\phi^*(1)).$$

Proof. Let k denote as before the vanishing order of $u_\phi(z)$ at $z=1$. According to the P-adic class formula we have:

$$L_P(\widetilde{\phi}/\widetilde{A}) = \frac{1}{P}(z-1)^k \log_{\widetilde{\phi},P}\left(\widetilde{\phi}_{g_{P,\phi}(z)}(u_\phi^*(z))\right). \tag{4.2}$$

Using that $\phi(A)_{\mathrm{tors}} = \{0\}$ and by Proposition 2.2.3, we have $\log_{\phi,P}(\phi_{g_{P,\phi}(1)}(u_\phi^*(1)))$ is nonzero. Hence

$$L_P^*(\phi/A) = \frac{1}{P}\log_{\phi,P}(\phi_{g_{P,\phi}(1)}(u_\phi^*(1))).$$

Using again Proposition 2.2.3, we have

$$v_P(L_P^*(\phi/A)) = v_P(\phi_{g_{P,\phi}(1)}(u_\phi^*(1))) - 1 = c_P(\phi, u_\phi^*(1)),$$

the last equality coming from Proposition 4.1.6. □

We will study the torsion case by twisting the Drinfeld module ϕ into a Drinfeld module $\psi = m^{-1}\phi m$ without A-torsion, following the ideas of Subsection 4.2.1.

In order to obtain further information about the special values, it is convenient to focus on the special case where the twisting element m is a power of P. Set $\psi_m = P^{-m}\phi P^m$ for $m \geq 1$. By Lemma 2.1.2 we have

$$L(\widetilde{\phi}/\widetilde{A}) = L(\widetilde{\psi_m}/\widetilde{A})\frac{P}{g_{P,\phi}(z)},$$
$$L_P(\widetilde{\phi}/\widetilde{A})) = L_P(\widetilde{\psi_m}/\widetilde{A}).$$

It turns out that we have analoguous relations for Taelman units and ordic valuations.

Lemma 4.1.9. *For all positive integers m and all $x \in A$, we have*

(1) $u_{\psi_m}(z) = P^{-m}\widetilde{\phi}_{P^{m-1}g_{P,\phi}(z)}(u_\phi(z))$,

(2) $c_P(\psi_m; x) = c_P(\phi; P^m x) - m$.

Proof. The first formula follows from the next computation:

$$\begin{aligned}
u_{\psi_m}(z) &= P^{-m}\exp_{\widetilde{\phi}}(P^m L(\widetilde{\psi_m}/\widetilde{A})\\
&= P^{-m}\exp_{\widetilde{\phi}}\left(P^m L(\widetilde{\phi}/\widetilde{A})\frac{g_{P,\phi}(z)}{P}\right)\\
&= P^{-m}\widetilde{\phi}_{P^{m-1}g_{P,\phi}(z)}(u_\phi(z)).
\end{aligned}$$

For the second formula, we first remark that for $k \leq m$, we have $\pi_{P^m x}(\phi; P^k) = A$; hence $c_P(\phi; P^m x) \geq m$. Moreover, for any $k \geq 0$, the equality $\psi_a(x) = P^{-m}\phi_a(P^m x)$ shows that

$$\pi_x(\psi; P^k) = \pi_{P^m x}(\phi; P^{m+k}).$$

The conclusion follows. □

We can state the last result of this Subsection, connecting in the general case the P-adic valuation of the special value at $z=1$ of the P-adic L-series and the P-ordic valuation of a certain polynomial.

Theorem 4.1.10. *Let P be a monic irreducible polynomial satisyfing $(H)_P$. For a positive integer m, we let $Q_m(z) \in A[z]$ and $k_m \in \mathbb{N}$ the quantities defined by*

$$\widetilde{\phi}_{P^{m-1}g_{P,\phi}(z)}(u_\phi(z)) = (z-1)^{k_m} Q_m(z)$$

with $Q_m(1) \neq 0$. Then, for m large enough, we have

$$v_P(L_P^*(\phi/A)) = c_P(\phi; Q_m(1)) - m.$$

Proof. By Lemma 4.1.9 we have

$$u_{\psi_m}(z) = P^{-m}(z-1)^{k_m} Q_m(z)$$

from which we deduce that k_m is the vanishing order of $u_{\psi_m}(z)$ at $z = 1$ and that $u^*_{\psi_m}(z) = P^{-m} Q_m(z)$. For all m large enough, we have that $\psi_m(A)_{\mathrm{tors}} = \{0\}$. We can then apply Theorem 4.1.8 and obtain

$$v_P(L_P^*(\psi_m/A)) = c_P(\psi_m; P^{-m} Q_m(1)).$$

Now we conclude by observing that the left hand side in the above equality is equal to $v_P(L_P^*(\phi/A))$ since $L_P^*(\phi/A) = L_P^*(\psi_m/A)$, while the right hand side is $c_P(\phi; Q_m(1)) - m$ by Lemma 4.1.9. $\square$

In the very small case, we can replace $u_\phi(z)$ by 1 in Theorem 4.1.7 and Theorem 4.1.10. For example, we obtain.

Corollary 4.1.11. *Let P be a monic irreducible polynomial satisfying $(H)_P$. If ϕ is very small, then*

$$v_P(L_P(\phi/A)) = c_P(\phi, 1).$$

In particular, in this case, P is ϕ-Wieferich in base 1 if and only if P divides $L_P(\phi/A)$.

This generalizes the following result of Thakur, see [47, Theorem 5].

Theorem 4.1.12. *Let P a monic irreducible polynomial satisfying $(H)_P$. Then $v_P(L_P(C/A)) > 0$ if and only if P is a C-Wieferich prime in base 1.*

4.1.3. Statistics on Wieferich primes. In this section, we adopt a probabilistic viewpoint. Our objective is to give credit to the naive expectation that a place P is Wieferich with probability $q^{-\deg(P)}$, supporting eventually the fact that a given Drinfeld module admits an infinite number of Wieferich places; indeed, the number of places of degree d is roughly q^d/d and

$$\sum_{d=1}^{\infty} \frac{q^d}{d} \cdot q^{-d} = \sum_{d=1}^{\infty} \frac{1}{d} = +\infty.$$

It is unclear to us if this heuristic is reasonable for a single Drinfeld module. However, in what follows, we shall prove that it is somehow valid when we average over larger universes.

Precisely, we fix a positive integer r and let Ω_r denote the set of all small Drinfeld modules $\phi : A \to A\{\tau\}$ of rank at most r. It is a finite set, and we equip it with the uniform distribution. Given in addition a place P of A, we consider the Bernoulli variable

$$\begin{array}{rccll} W_{r,P}: & \Omega_r & \longrightarrow & \{0,1\} & \\ & \phi & \mapsto & 0 & \text{if } P \text{ is } \phi\text{-Wieferich in base } 1 \\ & & & 1 & \text{otherwise.} \end{array}$$

Our objective is to study those random variables and their relationships.

Remark 4.1.13. To define our universe, we retained the property of being small because it looks quite meaningful regarding our purpose after the results of Section 1.6. However, most of the results we shall prove in this section are not strongly dependant on this choice, and will continue to hold true for many other families of universes, as soon as they eventually allow for arbitrary large ranks and arbitrary large degrees in the coefficients of the Drinfeld modules.

To start with, we consider the case where P is a place of degree 1. In this situation, we have a simple criterion for recognizing when a place is Wieferich. Before stating it, we recall that if $F = f_0 + f_1\tau + \cdots + f_n\tau^n \in A\{\tau\}$ is a polynomial in τ and if $x \in A$, we write

$$F(x) = f_0x + f_1x^q + \cdots + f_nx^{q^n}.$$

It is an element of A, *i.e.* a polynomial in θ over $\mathbb{F}_q$, and we will denote by $\frac{df(x)}{d\theta}$ its derivative with respect to θ. A direct computation shows that

$$\frac{dF(x)}{d\theta} = f_0'x + f_0x' + f_1'x^q + \cdots + f_n'x^{q^n}.$$

Proposition 4.1.14. *Let $\phi : A \to A\{\tau\}$ be a Drinfeld module. Let also $\alpha \in \mathbb{F}_q$ and $x \in A$, $x \neq 0$. Then the place $P = \theta - \alpha$ is ϕ-Wieferich in base x if and only if $\frac{d\phi_\theta(1)}{d\theta}(\alpha) = 0$.*

Proof. Set $f = \phi_\theta(1) \in A$ and write $\beta = f(\alpha) := \mathrm{ev}_{\theta=\alpha}(f) \in \mathbb{F}_q$. We observe that

$$\mathrm{ev}_{\theta=\alpha}(\phi_{\theta-\beta}(1)) = \mathrm{ev}_{\theta=\alpha}(\phi_\theta(1)) - \beta = 0 \equiv 0 \pmod{P}$$

which shows that $\pi_1(\phi; P)$ is the principal ideal generated by $\theta - \beta$. It follows that P is ϕ-Wieferich in base 1 if and only if P^2 divides $\phi_{\theta-\beta}(1) = f - \beta$. Besides, using Taylor expansion, we obtain the congruence $f \equiv \beta + Pf'(\alpha) \pmod{P^2}$, from what we conclude that P is ϕ-Wieferich in base 1 if and only if $f'(\alpha)$ vanishes. $\square$

Corollary 4.1.15. *Let r be a positive integer.*

(i) For any place P of degree 1, the Bernoulli variable $W_{r,P}$ takes the value 1 with probability q^{-1}.

(ii) The variables $W_{r,P}$ are mutually independent when P runs over the set of places of degree 1.

Proof. Proposition 4.1.14 tells us that the places of degree 1 which are ϕ-Wieferich in base 1 and in one-to-one correspondence with the roots in $\mathbb{F}_q$ of the polynomial $\frac{d\phi_\theta(1)}{d\theta}$. When ϕ varies in Ω_r, the polynomial $\phi_\theta(1)$ is uniformly distributed in the space $A_{\leq q^r}$ consisting of polynomials over $\mathbb{F}_q$ of degree at most q^r. We now conclude by noticing that the map

$$\begin{array}{rcl} A_{\leq q^r} & \longrightarrow & \mathbb{F}_q^{\mathbb{F}_q} \\ f & \mapsto & \big(f(\alpha)\big)_{\alpha\in\mathbb{F}_q} \end{array}$$

is $\mathbb{F}_q$-linear and surjective, thanks to Lagrange interpolation. □

We now aim at extending Corollary 4.1.15 to places of higher degrees. We start by determining the parameters of the Bernoulli variables $W_{r,P}$ at least when the rank is large compared to the degree of P.

Theorem 4.1.16. *Let r and d be two positive integers with $r \geq d + \log_q(2d)$. For any place P of degree d, the Bernoulli variable $W_{r,P}$ takes the value 1 (*i.e. *P is Wieferich in base 1) with probability q^{-d}.*

The rest of this subsection is devoted to the proof of Theorem 4.1.16. Unfortunately, it seems difficult to follow the same strategy we used for places of degree 1. Indeed, although Wieferich places of higher degrees d can certainly be characterized by the vanishing of some polynomial, it looks difficult to study its distribution when the underlying Drinfeld module ϕ varies.

Instead, we will use another characterization of Wieferich places, that we explain now. Let P be a place of A of degree d and let $\mathbb{F}_P := A/P$ denote the corresponding residual field as before. We first partition the universe Ω_r according to the reduction modulo P: given a Drinfeld module $\overline{\phi} : A \to \mathbb{F}_P\{\tau\}$, we let $\Omega_r(\overline{\phi})$ be the subset of Ω_r consisting of Drinfeld modules ϕ which reduces to $\overline{\phi}$ modulo P. We are going to prove that the proportion of Drinfeld modules admitting P as a Wieferich place inside each nonempty $\Omega_r(\overline{\phi})$ is q^{-d}; this will be enough to conclude.

From now on, we fix $\overline{\phi}$ as above, together with a lifting $\phi \in \Omega_r(\overline{\phi})$. If ψ is a second Drinfeld module in $\Omega_r(\overline{\phi})$, we have an equality of the form $\psi_\theta = \phi_\theta + Pf$ with $f \in A\{\tau\}$. Moreover, f takes the form $f = f_1\tau + f_2\tau^2 + \cdots + f_r\tau^r$ with $\deg(f_i) \leq q^i - d$. We will denote by Ω'_r this set in which f varies.

Lemma 4.1.17. *Keeping the previous notation, we have*

$$\psi_{\theta^i} \equiv \phi_{\theta^i} + P \cdot \sum_{j=0}^{i-1} \theta^j f \phi_\theta^{i-j-1} \pmod{P^2}.$$

Proof. We proceed by induction on i. For $i = 1$, the equality we have to prove is just $\psi_\theta = \phi_\theta + Pf$, which is true by definition of f. We now assume that the equality holds for i. We compute

$$\begin{aligned} \psi_{\theta^{i+1}} = \psi_{\theta^i}\psi_\theta &= \left(\phi_{\theta^i} + P \cdot \sum_{j=0}^{i-1} \theta^j f \phi_\theta^{i-j-1}\right) \cdot \big(\phi_\theta + Pf\big) \\ &\equiv \phi_{\theta^{i+1}} + P \cdot \sum_{j=0}^{i-1} \theta^j f \phi_\theta^{i-j} + \phi_\theta^i P f \pmod{P^2}. \end{aligned}$$

To handle the last summand, we write $\phi_\theta^i = \phi_{\theta^i} = \theta^i + h\tau$ with $h \in A\{\tau\}$. From this we derive

$$\phi_\theta^i P f = \theta^i P f + h\tau P f = \theta^i P f + hP^q \tau f \equiv \theta^i P f \pmod{P^2}.$$

Injecting finally this is the first equality, we obtain the announced formula. $\square$

We notice now that the ideal $\pi_1(\psi; P)$ does not depend on $\psi \in \Omega_r(\overline{\phi})$, but only on $\overline{\phi}$. Let us simply write $a = a_0 + a_1\theta + \cdots + a_d t^\theta$ (with $a_i \in \mathbb{F}_q$) for the monic generator of this ideal.

Proposition 4.1.18. *We keep the previous notation and let further ξ (resp. μ_j) be the image of θ (resp. of $\phi_{\theta^j}(1)$) in $\mathbb{F}_P$. Then the place P is ψ-Wieferich in base 1 if and only if*

$$\sum_{i=1}^{d} \sum_{j=0}^{i-1} a_i \xi^j f(\mu_{i-j-1}) \equiv -\frac{\phi_a(1)}{P} \pmod{P}.$$

Proof. It follows from Lemma 4.1.17 that

$$\psi_a(1) \equiv \phi_a(1) + P \sum_{i=1}^{d} \sum_{j=0}^{i-1} a_i \cdot \theta^j \cdot (f \phi_\theta^{i-j-1})(1) \pmod{P^2}.$$

Therefore the condition of the proposition is equivalent to the vanishing of $\psi_a(1)$ modulo P^2 which is, by definition, also equivalent to the fact that P is ψ-Wieferich in base 1. $\square$

The main insight of Proposition 4.1.18 is that it provides a *linear* characterization of the property of being Wieferich. To fully exploit this fact, we introduce the mapping

$$\begin{array}{rccl} L_r : & \Omega'_r & \longrightarrow & \mathbb{F}_P \\ & f & \mapsto & \displaystyle\sum_{i=1}^{d} \sum_{j=0}^{i-1} a_i \xi^j f(\mu_{i-j-1}) \end{array}$$

It is $\mathbb{F}_q$-linear. Besides, Proposition 4.1.18 ensures that the Drinfeld modules $\psi \in \Omega_r(\overline{\phi})$ admitting P as a Wieferich place in base 1 are in one-to-one correspondence with the inverse image by L_r of the element $-\frac{\phi_a(1)}{P} \in \mathbb{F}_P$. Proving that this event holds with probability $q^{-d} = \frac{1}{\mathrm{Card}(\mathbb{F}_P)}$ then amounts to proving that L_r is surjective.

Lemma 4.1.19. *Let $j \geq 1$. At least one of the elements $L_r(\tau^j), \ldots, L_r(\tau^{j+d-1})$ does not vanish.*

Proof. We assume by contradiction that $L_r(\tau^j) = \cdots = L_r(\tau^{j+d-1}) = 0$. By definition, we have

$$L_r(\tau^k) = \sum_{i=1}^{d} \sum_{j=0}^{i-1} a_i \xi^j \mu_{i-j-1}^{q^k}.$$

By assumption, the latter vanishes for k varying between j and $j+d-1$. Nonetheless, given that $\mu_{i-j-1} \in \mathbb{F}_P$, we have $\mu_{i-j-1}^{q^d} = \mu_{i-j-1}$, from what we conclude that the vanishing holds for all $k \in \mathbb{Z}$.

Using algebraic transformations, we are going to prove that this implies other vanishings. Precisely, for $n, s \geq 0$, we set

$$\xi_{n,s} := \sum_{\substack{e_0,\dots,e_{n-1}\geq 0 \\ e_0+\cdots+e_{n-1}=s}} \xi^{e_0+qe_1+\cdots+q^{n-1}e_{n-1}}.$$

We observe that $\xi_{n,0} = 1$ for all n, and that $\xi_{1,s} = \xi^s$ for all s. Thus our assumption reads

$$\forall k \in \mathbb{Z}, \qquad \sum_{i=1}^{d} \sum_{j=0}^{i-1} a_i \cdot \xi_{1,j} \cdot \mu_{i-j-1}^{q^k} = 0.$$

We will prove by induction on n that

$$\forall k \in \mathbb{Z}, \qquad \sum_{i=n}^{d} \sum_{j=0}^{i-n} a_i \cdot \xi_{n,j} \cdot \mu_{i-j-n}^{q^k} = 0. \tag{4.3}$$

for all $n \in \{1, \dots, d\}$. Starting from the induction hypothesis for some $n < d$ (with k replaced by $k-1$) and raising it to the q-th power, we obtain

$$\sum_{i=n}^{d} \sum_{j=0}^{i-n} a_i \cdot \xi_{n,j}^q \cdot \mu_{i-j-n}^{q^k} = 0.$$

It follows that

$$\sum_{i=n}^{d} \sum_{j=1}^{i-n} a_i \cdot (\xi_{n,j}^q - \xi_{n,j}) \cdot \mu_{i-j-n}^{q^k} = 0. \tag{4.4}$$

Note that the sum over j could safely starts at 1 because the terms corresponding to $j = 0$ all vanish. We now claim that the following identity holds:

$$\xi_{n,j}^q - \xi_{n,j} = \xi_{n+1,j-1} \cdot (\xi^{q^n} - \xi). \tag{4.5}$$

Indeed, we first observe that

$$\xi_{n,j}^q - \xi_{n,j} = \sum_{\substack{e_1,\dots,e_n\geq 0 \\ e_1+\cdots+e_n=j}} \xi^{qe_1+q^2e_2+\cdots+q^ne_n} - \sum_{\substack{e_0,\dots,e_{n-1}\geq 0 \\ e_0+\cdots+e_{n-1}=j}} \xi^{e_0+qe_1+\cdots+q^{n-1}e_{n-1}}.$$

The terms with $e_n = 0$ in the first sum cancel with the terms with $e_0 = 0$ is the second sum. Therefore, we do not change the value of the difference if we remove those terms. Performing in addition the changes of variables $e_0 \mapsto e_0 - 1$ et $e_n \mapsto e_n - 1$, we end up with

$$\xi_{n,j}^q - \xi_{n,j} = \\ \sum_{\substack{e_1,\dots,e_n\geq 0 \\ e_1+\cdots+e_n=j-1}} \xi^{qe_1+q^2e_2+\cdots+q^n(e_n+1)} - \sum_{\substack{e_0,\dots,e_{n-1}\geq 0 \\ e_0+\cdots+e_{n-1}=j-1}} \xi^{(e_0+1)+qe_1+\cdots+q^{n-1}e_{n-1}}.$$

Similarly, we compute

$$\xi_{n+1,j-1} \cdot (\xi^{q^n} - \xi) = \xi_{n+1,j-1}\xi^{q^n} - \xi_{n+1,j-1}\xi = \\ \sum_{\substack{e_0,\dots,e_n\geq 0 \\ e_0+\cdots+e_n=j-1}} \xi^{e_0+qe_1+q^2e_2+\cdots+q^n(e_n+1)} - \sum_{\substack{e_0,\dots,e_n\geq 0 \\ e_0+\cdots+e_n=j-1}} \xi^{(e_0+1)+qe_1+\cdots+q^ne_n}.$$

Again the terms in the first sum with $e_0 > 0$ cancel with the terms in the second sum with $e_n > 0$, leading to

$$\xi_{n+1,j-1} \cdot (\xi^{q^n} - \xi) = \\ \sum_{\substack{e_1,\dots,e_n \geq 0 \\ e_1+\cdots+e_n=j-1}} \xi^{qe_1+q^2e_2+\cdots+q^n(e_n+1)} - \sum_{\substack{e_0,\dots,e_{n-1} \geq 0 \\ e_0+\cdots+e_{n-1}=j-1}} \xi^{(e_0+1)+qe_1+\cdots+q^{n-1}e_{n-1}}.$$

The equality (4.5) follows. Injecting it in Equation (4.4), we obtain

$$(\xi^{q^n} - \xi) \cdot \sum_{i=n}^{d} \sum_{j=1}^{i-n} a_i \cdot \xi_{n+1,j-1} \cdot \mu_{i-j-n}^{q^k} = 0.$$

The prefactor $\xi^{q^n} - \xi$ does not vanish because ξ generates $\mathbb{F}_P$ over $\mathbb{F}_q$ and $n < [\mathbb{F}_P : \mathbb{F}_q] = d$. We can then safely delete it. Performing finally the change of variables $j \mapsto j+1$, we find Equation (4.3) for $n+1$ and the induction goes.

We conclude by considering the system of equations (4.3) as a linear system on the a_i. For a fixed n (and $k = 0$), Equation (4.3) is of the form

$$a_n + \star\, a_{n+1} + \star\, a_{n+2} + \cdots + \star\, a_d = 0$$

where the symbols $\star$ hide some coefficients in $\mathbb{F}_P$. It follows that $a_1 = \cdots = a_d = 0$. The polynomial $a = \pi_1(\overline{\phi}; P)$ would then be constant, which is not possible and proves the lemma. $\square$

From this point, it is easy to prove the surjectivity of L_r (and then Theorem 4.1.16). Let j be the smallest positive integer such that $q^j \geq 2d - 1$; one has $j \leq 1 + \log_q(2d)$. On the one hand, by Lemma 4.1.19, there exists $k \in \{0, \dots, d-1\}$ such that $L_r(\tau^{j+k}) \neq 0$. On the other hand, thanks to our assumptions on j and r, the Ore polynomial $h(t) \cdot \tau^{j+k}$ lies in Ω'_r for all $h(t) \in A$ of degree at most $d-1$. Thus the image of L_r contains all the elements of the form $h(\xi) L_r(\tau^{j+k})$, that are all the elements of $\mathbb{F}_P$. The surjectivity follows.

Now we have determined the law of $W_{r,P}$, we focus on their relationships, looking for statements in line with Corollary 4.1.15.(ii). The theorem we shall prove is the following.

Theorem 4.1.20. *Let $P_1, \dots, P_n$ be n pairwise distinct places of A and set $d_m := \deg(P_m)$ for all $m \in \{1, \dots, n\}$. Then, the random variables $W_{r,P_1}, \dots, W_{r,P_n}$ are mutually independent as soon as*

$$r \geq \max(d_1, \dots, d_n) + \log_q\big(2(d_1 + \cdots + d_n)\big).$$

Proof. We follow the same method as for proving Theorem 4.1.16. For each $m \in \{1, \dots, n\}$, we write $\mathbb{F}_{P_m} := A/P_m A$. Given a family of Drinfeld modules $\overline{\phi}_m : A \to \mathbb{F}_{P_m}\{\tau\}$ $(1 \leq m \leq n)$, we consider the set $\Omega_r(\overline{\phi}_1, \dots, \overline{\phi}_n) \subset \Omega_r$ consisting of Drinfeld modules which reduces to $\overline{\phi}_m$ modulo P_m for all m. We assume that $\Omega_r(\overline{\phi}_1, \dots, \overline{\phi}_n)$ is not empty and we fix a Drinfeld module $\phi : A \to A\{\tau\}$ in it. Any other Drinfeld module $\psi \in \Omega_r(\overline{\phi}_1, \dots, \overline{\phi}_n)$ is such that $\psi_\theta = \phi_\theta + P_1 \cdots P_n f$ with f varying in the set

$$\Omega'_r = \big\{\, f_0 + f_1\tau + \cdots + f_r\tau^r \in A\{\tau\} \text{ s.t. } \deg(f_i) \leq q^i - (d_1 + \cdots + d_n) \text{ for all } i \,\big\}.$$

For a fixed index m, let ξ_m and $\mu_{m,j}$ $(0 \leq j \leq r)$ be the images in $\mathbb{F}_{P_m}$ of θ and $\phi_{\theta^j}(1)$ respectively. Let also $a_m = a_{m,0} + a_{m,1}\theta + \cdots + a_{m,d}\theta^d \in A$ be the monic generator of $\pi_1(\psi; P_m)$ and set

$$u_m := \prod_{\substack{1 \leq m' \leq n \\ m' \neq m}} P_{m'}(\xi_m) \in \mathbb{F}_{P_m}.$$

Since $P_{m'}$ is coprime with P_m for all $m' \neq m$, we have $u_m \neq 0$. Repeating the proof of Proposition 4.1.18, we find that the place P_m is ψ-Wieferich in base 1 if and only if

$$\sum_{i=1}^{d} \sum_{j=0}^{i-1} a_{m,i}\xi_m^j f(\mu_{m,i-j-1}) \equiv -u_m \cdot \frac{\phi_a(1)}{P} \pmod{P_m}.$$

We now consider the $\mathbb{F}_q$-linear map

$$\begin{array}{cccc} L_{r,m} : & \Omega'_r & \longrightarrow & \mathbb{F}_{P_m} \\ & f & \mapsto & \displaystyle\sum_{i=1}^{d} \sum_{j=0}^{i-1} a_i \xi_m^j f(\mu_{m,i-j-1}) \end{array}$$

and $L_r : \Omega'_r \to \mathbb{F}_{P_1} \times \cdots \times \mathbb{F}_{P_n}$, $f \mapsto \big(L_{r,m}(f)\big)_{1 \leq m \leq n}$. We prove that $L_{r,m}$ is surjective following the same ideas as Lemma 4.1.19 and the discussion thereafter.

More precisely, fixing $j \geq \log_q(2(d_1 + \cdots + d_n))$, any element $\alpha_m \in \mathbb{F}_{P_m}$ has a preimage of the form

$$f_m = f_{m,j}\tau^j + \cdots + f_{m,j+d_m-1}\tau^{j+d_m-1} \in A\{\tau\}.$$

This property implies the surjectivity of L_r as follows. We pick $\alpha_1 \in \mathbb{F}_{P_1}, \ldots, \alpha_n \in \mathbb{F}_{P_n}$ and, for each m, we choose a preimage f_m of α_m of the above form. Setting $d := \max(d_1, \ldots, d_n)$ and applying the chinese remainder theorem separately on each coefficient, we find $g = g_j\tau^j + \cdots + g_{j+d-1}\tau^{j+d-1} \in A\{\tau\}$ such that $g \equiv f_m \pmod{P_m}$ for all m. Moreover, we may assume that all the g_i have θ-degree less than $d_1 + \cdots + d_n$ because they are defined modulo $P_1 \cdots P_n$. Thanks to our assumption on r, we then have $g \in \Omega'_r$. Finally, noticing that $L_{r,m}$ depends only on the reduction of g modulo P_m, we conclude that $L_{r,m}(g) = (\alpha_1, \ldots, \alpha_n)$, proving the surjectivity.

It follows that the probability that L_r takes the value $(\alpha_1, \ldots, \alpha_n)$ is constant, namely

$$\frac{1}{\mathrm{Card}\big(\mathbb{F}_{P_1} \times \cdots \times \mathbb{F}_{P_n}\big)} = q^{-(d_1 + \cdots + d_n)}.$$

This proves the independence of the random variables $L_{r,m}$ which, in turn, implies the independence of the W_{r,P_m}. $\square$

Specializing Theorem 4.1.20 to the case where we consider all places of a fixed degree d (resp. of degree at most d), we obtain the following.

Corollary 4.1.21. *Let r and d be two positive integers.*

(i) If $r \geq 2d$, the random variables $W_{r,P}$, for P running over all places of degree d, are mutually independent.

(ii) If $r \geq 2d+1$, the random variables $W_{r,P}$, for P running over all places of degree at most d, are mutually independent.

Proof. Let $\mathcal{P}_d$ (resp. $\mathcal{P}_{\leq d}$) denote the set of places of degree d (resp. of degree at most d). After Theorem 4.1.20, it only remains to estimate the sum of $\deg(P)$ for P running in $\mathcal{P}_d$ (resp. in $\mathcal{P}_{\leq d}$). This is quite standard and follows from the observation that each place of degree d is a divisor of the polynomial $\theta^{q^d} - \theta$. Therefore their product also divides $\theta^{q^d} - \theta$ and we conclude that $\sum_{P \in \mathcal{P}_d} \deg(P) \leq q^d$. The statement (i) follows.

Similarly, we have

$$\sum_{P \in \mathcal{P}_{\leq d}} \deg(P) \leq \sum_{\delta=1}^{d} q^\delta = \frac{q^{d+1} - q}{q-1} < q^{d+1}$$

which eventually implies (ii). □

Theorems 4.1.16 and 4.1.20 provide very precise informations on the random variables $W_{r,P}$ when the rank r is large enough. On the contrary, the situation in small rank is far less clear.

In order to obtain a better feeling on this, we have conducted numerical simulations for various values of q, r (the rank) and d (the degree). The results are reported in the tables of Figures 4.1–4.4. The columns labelled "all" correspond to all small Drinfeld modules with the prescribed rank, while the columns labelled "n.t." correspond only to those Drinfeld modules for which 1 is not a torsion point. Indeed, when 1 is torsion, every place is Wieferich and it seems to us that this could distort the statistics.

For each choice of the pair (q, r), we sampled $10,000$ random Drinfeld modules (except if there were less than $10,000$, in which case, we have considered all of them) and reported in each cell the empiric value of

$$\frac{q^d}{\mathrm{Card}(\mathcal{P}_d)} \sum_{P \in \mathcal{P}_d} W_{r,P}$$

where $\mathcal{P}_d$ denotes the set of all places of A of degree d (as in the proof of Corollary 4.1.21). Those values are then expected to be close to 1, at least if our heuristic that a place of degree d is Wieferich with probability q^{-d} is correct.

We see in the tables that it is indeed the case when the degree remains small compared to the rank; this is in line with Theorem 4.1.16. On the contrary, when the degree gets higher, the behaviour looks more erratic with entries attaining very large values. We notice nonetheless that the bounds of Theorems 4.1.16 and 4.1.20 look pessimistic: the expected behaviour seems to occur much earlier than what they claim.

Despite all of this, it is still unclear to us if expecting an infinite number of Wieferich places for a given Drinfeld module is reasonable or not. In any case, we emphasize that Theorems 4.1.16 and 4.1.20 do not imply such a result in average. They actually even cannot ensure the existence of a single Drinfeld module admitting an infinite number of Wieferich places[1]. In order to have more evidences on this question, we have run further experiments in the special case of the Carlitz module. Here is what we found.

- For $q = 2$, all places are Wieferich except those of degree 1.
- For $q = 3$, we have looked for Wieferich places until the degree 24 (which corresponds to a total of $18,054,379,372$ places) and found 4 Wieferich places, namely
  (1) $\theta^6 + \theta^4 + \theta^3 + \theta^2 + 2\theta + 2$,
  (2) $\theta^9 + \theta^6 + \theta^4 + \theta^2 + 2\theta + 2$,
  (3) $\theta^{12} + 2\theta^{10} + \theta^9 + 2\theta^4 + 2\theta^3 + \theta^2 + 1$,
  (4) $\theta^{15} + \theta^{13} + \theta^{12} + \theta^{11} + 2\theta^{10} + 2\theta^7 + 2\theta^5 + 2\theta^4 + \theta^3 + \theta^2 + \theta + 1$.
- For $q = 4$, we have looked for Wieferich places until the degree 17 (which corresponds to a total of $1,376,854,004$ places) and found 2 Wieferich places, namely
  (1) $\theta^2 + \theta + \alpha$,
  (2) $\theta^2 + \theta + (\alpha + 1)$,

  where $\alpha \in \mathbb{F}_4$ is a solution of $\alpha^2 + \alpha + 1 = 0$.
- For $q = 5$, we have looked for Wieferich places until the degree 17 (which corresponds to a total of $57,005,914,349$ places) and found 2 Wieferich places, namely
  (1) $\theta^5 + 4\theta + 1$,
  (2) $\theta^{10} + 3\theta^6 + 4\theta^5 + \theta^2 + \theta + 1$.

Again, the conclusion is unclear.

[1] It is instructive to compare with the following situation. If P is a fixed place of degree d (over $\mathbb{F}_q$), a random polynomial of degree $r \geq d$ is a multiple of P with probability q^{-d} and those events are independant when r is large enough. However, obviously, a given polynomial cannot be divisible by an infinite number of places.

	rank 1		rank 2		rank 3		rank 4		rank 5	
	all	n.t.	all	n.t.	all	n.t.	all	n.t.	all	n.t.
deg 1	1.14	0.80	1.00	0.94	0.99	0.99	0.99	0.99	1.01	1.01
deg 2	1.71	0.80	1.24	1.08	1.00	0.99	0.99	0.99	0.97	0.97
deg 3	3.43	1.60	0.87	0.44	1.13	1.10	1.06	1.06	1.00	1.00
deg 4	6.86	3.20	2.06	1.23	0.98	0.93	1.03	1.03	1.03	1.03
deg 5	13.71	6.40	2.54	0.77	1.05	0.95	0.98	0.98	1.01	1.01
deg 6	27.43	12.80	5.22	1.70	1.17	0.96	1.02	1.02	0.97	0.97
deg 7	54.86	25.60	8.49	1.34	1.38	0.97	1.00	1.00	1.04	1.04
deg 8	>100	58.03	16.41	2.08	1.86	1.04	1.03	1.03	0.98	0.98
deg 9	>100	>100	29.86	1.02	2.65	1.01	1.00	1.00	1.01	1.01
deg 10	>100	>100	62.10	4.55	4.28	1.00	1.03	1.03	1.06	1.06
deg 11	>100	>100	>100	1.69	7.56	1.01	1.05	1.05	0.97	0.97
deg 12	>100	>100	>100	4.70	14.19	1.08	1.06	1.06	1.05	1.05
deg 13	>100	>100	>100	0.67	27.26	1.04	0.96	0.96	0.95	0.95
deg 14	>100	>100	>100	11.04	53.48	1.05	1.02	1.02	0.94	0.94
deg 15	>100	>100	>100	2.18	>100	1.04	1.03	1.03	0.97	0.97
deg 16	>100	>100	>100	20.18	>100	1.16	1.00	1.00	1.11	1.11
deg 17	>100	>100	>100	1.31	>100	1.04	0.98	0.98	1.06	1.06
deg 18	>100	>100	>100	37.62	>100	1.50	1.14	1.14	1.09	1.09
deg 19	>100	>100	>100	0.81	>100	0.94	1.05	1.05	0.97	0.97

Figure 4.1: Statistics of Wieferich places with $q = 2$

	rank 1		rank 2		rank 3		rank 4		rank 5	
	all	n.t.	all	n.t.	all	n.t.	all	n.t.	all	n.t.
deg 1	1.01	0.94	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
deg 2	0.90	0.58	1.00	1.00	1.01	1.01	1.02	1.02	1.03	1.03
deg 3	2.19	1.23	0.99	0.99	0.98	0.98	1.01	1.01	0.98	0.98
deg 4	4.56	1.58	1.00	1.00	0.98	0.98	0.98	0.98	1.01	1.01
deg 5	9.11	0.00	1.00	1.00	1.04	1.04	0.98	0.98	1.00	1.00
deg 6	28.59	1.31	1.12	1.12	1.04	1.04	1.00	1.00	0.98	0.98
deg 7	82.01	0.00	1.00	1.00	1.04	1.04	0.98	0.98	1.01	1.01
deg 8	>100	0.00	1.44	1.44	0.99	0.99	0.99	0.99	1.03	1.03
deg 9	>100	1.64	0.98	0.98	1.00	1.00	1.01	1.01	1.01	1.01
deg 10	>100	5.09	2.17	2.17	1.00	1.00	1.06	1.06	1.01	1.01
deg 11	>100	0.00	0.96	0.96	0.94	0.94	0.96	0.96	1.03	1.03
deg 12	>100	4.06	4.42	4.42	0.97	0.97	1.02	1.02	0.99	0.99

Figure 4.2: Statistics of Wieferich places with $q = 3$

	rank 1		rank 2		rank 3		rank 4		rank 5	
	all	n.t.	all	n.t.	all	n.t.	all	n.t.	all	n.t.
deg 1	1.00	0.99	0.99	0.99	1.00	1.00	1.00	1.00	1.00	1.00
deg 2	0.94	0.88	1.02	1.02	0.99	0.99	0.99	0.99	1.02	1.02
deg 3	1.54	1.29	1.02	1.02	1.00	1.00	1.00	1.00	1.02	1.02
deg 4	2.15	1.16	1.02	1.02	1.01	1.01	0.99	0.99	0.99	0.99
deg 5	5.06	1.06	0.98	0.98	1.00	1.00	1.00	1.00	0.98	0.98
deg 6	17.29	1.28	1.06	1.06	0.98	0.98	1.02	1.02	0.99	0.99
deg 7	64.47	0.41	0.99	0.99	1.00	1.00	1.00	1.00	0.98	0.98
deg 8	>100	0.85	1.20	1.20	0.98	0.98	1.05	1.05	1.04	1.04
deg 9	>100	0.53	0.97	0.97	1.04	1.04	0.97	0.97	0.97	0.97

Figure 4.3: Statistics of Wieferich places with $q = 4$

	rank 1		rank 2		rank 3		rank 4		rank 5	
	all	n.t.	all	n.t.	all	n.t.	all	n.t.	all	n.t.
deg 1	1.00	1.00	1.00	1.00	1.01	1.01	1.00	1.00	0.98	0.98
deg 2	0.96	0.95	0.99	0.99	1.00	1.00	0.99	0.99	0.99	0.99
deg 3	0.99	0.95	0.98	0.98	1.00	1.00	0.99	0.99	1.02	1.02
deg 4	1.09	0.89	0.99	0.99	0.98	0.98	0.98	0.98	1.02	1.02
deg 5	2.08	1.08	0.98	0.98	1.00	1.00	0.99	0.99	1.02	1.02
deg 6	6.06	1.06	0.98	0.98	0.98	0.98	1.04	1.04	1.03	1.03
deg 7	26.07	1.07	0.97	0.97	1.04	1.04	0.94	0.94	0.98	0.98
deg 8	>100	0.85	1.01	1.01	1.05	1.05	0.99	0.99	0.95	0.95

Figure 4.4: Statistics of Wieferich places with $q = 5$

4.2. Vanishing order at $z = 1$ of the P-adic L-series

4.2.1. A bound on the vanishing order. In this section we investigate the vanishing order at $z = 1$ of the P-adic L-series in the case where $L = K$ and $d = 1$, i.e., the case of Drinfeld A-modules defined over the ring A itself.

We recall the definition of local factor associated with $\widetilde{\phi}$ and P:

$$z_P(\widetilde{\phi}/\widetilde{A}) = \frac{P}{g_{P,\phi}(z)}$$

where $g_{P,\phi}(z) = \left[\widetilde{\phi}(\widetilde{A}/P\widetilde{A})\right]_{\widetilde{A}} \in A[z]$. Let us recall that the P-adic L-series associated with ϕ is defined as follows:

$$L_P(\widetilde{\phi}/\widetilde{A}) = \frac{1}{P} \log_{\widetilde{\phi},P} \widetilde{\phi}_{g_{P,\phi}(z)}(u_\phi(z)) \in \mathbb{T}_z(K_P). \tag{4.6}$$

By the proof of [45, Corollary 7.5.6], we deduce that $L(\widetilde{\phi}/\widetilde{A}) \in \mathbb{T}_z(K_\infty)$ is a unit in $\mathbb{T}_z(K_\infty)$ whose valuation is equal to 0 and whose constant coefficient is equal to 1.

From now on, we say that ϕ does not have A-torsion if the A-module $\phi(A)$ is torsion-free.

Proposition 4.2.1. *Let ϕ be an A-Drinfeld defined over A of rank $r \geq 1$ without A-torsion. Then for all $k \geq 0$ the following assertions are equivalent:*

(1) $(z-1)^k |_{\mathbb{T}_z(K_P)} L_P(\widetilde{\phi}/\widetilde{A})$,

(2) $(z-1)^k |_{A[z]} u_\phi(z)$.

Proof. By the definition of the P-adic L-series:

$$L_P(\widetilde{\phi}/\widetilde{A}) = \frac{1}{P} \log_{\widetilde{\phi},P}(\widetilde{\phi}_{g_{P,\phi}(z)}(u_\phi(z))),$$

we see that implication $2 \Rightarrow 1$ is clear.

Let us prove $1 \Rightarrow 2$. We have:

$$P^2 L_P(\widetilde{\phi}/\widetilde{A}) = \log_{\widetilde{\phi},P}(\widetilde{\phi}_{Pg_{P,\phi}(z)}(u_\phi(z))).$$

Since $v_P(\widetilde{\phi}_{g_{P,\phi}(z)}(u_\phi(z)) > 0$, we have $v_P(\widetilde{\phi}_{Pg_{P,\phi}(z)}(u_\phi(z))) \geq 2$, thus

$$\widetilde{\phi}_{Pg_{P,\phi}(z)}(u_\phi(z)) \in \mathcal{D}_z^+.$$

By applying the P-adic exponential map we obtain:

$$\exp_{\widetilde{\phi},P}(P^2 L_P(\widetilde{\phi}/\widetilde{A})) = \widetilde{\phi}_{Pg_{P,\phi}(z)}(u_\phi(z)) \in A[z].$$

If $(z-1)^k |_{\mathbb{T}_z(K_P)} L_P(\widetilde{\phi}/\widetilde{A})$, then by the previous equality we have

$$(z-1)^k |_{A[z]} \widetilde{\phi}_{Pg_{P,\phi}(z)}(u_\phi(z)).$$

Since ϕ does not have A-torsion, we have

$$(z-1)^k |_{A[z]} u_\phi(z).$$

□

Proposition 4.2.2. *Let m be any non zero polynomial of A, and consider the Drinfeld module $\psi = m^{-1}\phi m$. Then the vanishing order at $z = 1$ of $L_P(\widetilde{\phi}/\widetilde{A})$ and $L_P(\widetilde{\psi}/\widetilde{A})$ are equal.*

Proof. By Lemma 2.1.2, we have the following equality in $\mathbb{T}_z(K_P)$:

$$g_{P,\phi}(z)L_P(\widetilde{\psi}/\widetilde{A}) = g_{P,\psi}(z)L_P(\widetilde{\phi}/\widetilde{A})\prod_{Q|m}\frac{g_{Q,\phi}(z)}{Q}.$$

Since $g_{Q,\phi}(1) \neq 0$ for all Q, we obtain the result. $\square$

Theorem 4.2.3. *Let ϕ be an A-Drinfeld module defined over A itself of rank r. Let $\{\lambda_1, \ldots, \lambda_r\}$ be an SMB of the period lattice Λ_ϕ. Then the vanishing order at $z = 1$ of the P-adic L-series $L_P(\widetilde{\phi}/\widetilde{A})$ verifies*

$$\mathrm{ord}_{z=1} L_P(\widetilde{\phi}/\widetilde{\mathscr{O}_L}) \leq \#\{i = 1 \ldots, r \mid v_\infty(\lambda_i) \in \mathbb{Z}\} \leq r.$$

Moreover, it does not depend on P.

Proof. Let us first twist ϕ into $\psi = m^{-1}\phi m$ without A-torsion. By Proposition 4.2.2, the vanishing order at $z = 1$ of $L_P(\widetilde{\phi}/\widetilde{A})$ and the vanishing order at $z = 1$ of $L_P(\widetilde{\psi}/\widetilde{A})$ are equal. Moreover, this order equals the vanishing order at $z = 1$ of $u_\psi(z)$ by Proposition 4.2.1 and does not depend on P. We can compute its leading coefficient seen as a polynomial in the variable θ. We have

$$u_\psi(z) = \exp_{\widetilde{\psi}}\left(L(\widetilde{\psi}/\widetilde{A})\right) = \sum_{n\geq 0} d_n z^n \tau^n(L(\widetilde{\psi}/\widetilde{A})).$$

We know that $L(\widetilde{\psi}/\widetilde{A})$ has the form $1 + \sum_{n\geq 1} a_n z^n \in \mathbb{T}_z(K_\infty)$ with $v_\infty(a_n) > 0$. Let $N_0, m_1, \ldots, m_l, N_1$ be the integers n such that $v_\infty(d_n)$ is minimal. Let $\beta_n \in \mathbb{F}_q^*$ be the sign of d_n, we obtain:

$$\mathrm{sgn}(u_\psi(z)) = z^{N_0}(\beta_{N_0} + \ldots + \beta_{N_1} z^{N_1 - N_0}) \in \mathbb{F}_q[z].$$

By Proposition 1.6.4 we obtain the desired result. $\square$

Proposition 4.2.4. *The previous inequality is not an equality in general.*

Proof. Set $q = 3$ and consider the Drinfeld module given by

$$\phi_\theta = \theta + \theta\tau^2.$$

From the functional equation of the exponential map, we obtain for all $n \geq 0$:

$$d_n = \frac{1}{\theta^{q^n} - \theta}\sum_{k=1}^{\min\{n,2\}} \theta d_{n-2}^{q^2}.$$

An immediate induction tells us that $d_n = 0$ for n odd, and

$$v_\infty(d_{2n}) = nq^{2n} - \frac{q^{2n}-1}{q^2-1}, \forall n \geq 0.$$

One can prove that the Newton polygon of the associated exponential map is the polygon beginning at the point $(0,0)$ and has successive slopes equal to $k+1$ and of length $(q^{2k+2}-q^{2k})$. Thus, the number of periods of an SMB having valuation $\in \mathbb{Z}$ is equal to 2. By Proposition 1.7.4 we have $u_\phi(1)=1$. One can prove that ϕ does not have A-torsion (see Lemma 4.3.2). Then, by Proposition 4.2.1 we obtain

$$\operatorname{ord}_{z=1} L_P(\widetilde{\phi}/\widetilde{A}) = 0.$$

$\square$

Remark 4.2.5. For any $r \geq 1$, we can construct explicit Drinfeld modules of rank r whose vanishing order of the associated P-adic L-series equals r. In fact, denote by

$$(-1)^r(z-1)^r = 1 + \sum_{i=1}^{r} \alpha_i z^i,$$

with $\alpha_i \in \mathbb{F}_q$ and consider the Drinfeld module given by

$$\phi_\theta = \theta + \sum_{i=1}^{r} \alpha_i \theta^{q^i} \tau^i.$$

By Proposition 1.7.4, we have

$$u_\phi(z) = 1 + \sum_{i=1}^{r} \alpha_i z^i = (-1)^r (z-1)^r.$$

Then the vanishing order at $z=1$ is greater than or equal to r, so equals r by Theorem 4.2.3.

Remark 4.2.6. So far, we have looked at the evaluation at $z=1$ but since $L_P(\widetilde{\phi}/\widetilde{A}) \in \mathbb{T}_z(K_P)$, we can consider the evaluation at $z=\zeta \in \overline{\mathbb{F}}_q$.

Set $\overline{A} = \overline{\mathbb{F}}_q[\theta]$. It has a structure of A-module through ϕ, denoted by $\phi(\overline{A})$, and we denote its A-torsion by $\phi(\overline{A})_{\mathrm{tors}}$.

Firstly, we can see that we can also twist ϕ into $\psi = m^{-1}\phi m$ by some polynomial $m \in A$ with degree large enough such that $\psi(\overline{A})_{\mathrm{tors}} = \{0\}$. Moreover, since $g_{P,\phi}(z) \in \mathbb{T}_z(K_\infty)^\times$ by Proposition 1.4.2, we have

$$\operatorname{ev}_{z=\zeta}(g_{P,\phi}(z)) \neq 0$$

for all Drinfeld module ϕ and $\zeta \in \overline{\mathbb{F}_q}$. Thus, for all $\zeta \in \overline{\mathbb{F}_q}$ and $k \geq 0$, we obtain the following equivalence:

$$(z-\zeta)^k | L_P(\widetilde{\phi}/\widetilde{A}) \Leftrightarrow (z-\zeta) | u_\psi(z).$$

Then the vanishing order at $z=\zeta$ does not depend on P, and by the proof of Theorem 4.2.3 we obtain

$$\operatorname{ord}_{z=\zeta} L_P(\widetilde{E}/\widetilde{\mathscr{O}_L}) \leq \#\{i = 1\ldots, r \mid v_\infty(\lambda_i) \in \mathbb{Z}\} \leq r.$$

To conclude this subsection, we would like to ask the following question from a personal communication with Xavier Caruso and Quentin Gazda.

Problem 4.2.7. *Set* $L_\infty = \bigcup_{n\geq 0} \mathbb{F}_{q^{p^n}}((\frac{1}{\theta}))$ *and*

$$\Lambda_{\phi,\infty} = \{x \in L_\infty \mid \exp_\phi(x) = 0\} = \Lambda_\phi \cap L_\infty.$$

Do we have the following equality

$$\operatorname{ord}_{z=1} L_P(\widetilde{\phi}/\widetilde{A}) = \operatorname{rk}_A(\Lambda_{\phi,\infty})?$$

4.2.2. Vanishing order of the P-adic L-series and periods . In many cases, Theorem 4.2.3 gives us an easy criterium to verify when the P-adic L-series is non zero if we are able to compute the spectrum $(v_\infty(\lambda_1), \ldots, v_\infty(\lambda_r))$ of ϕ. For example, by Equation (1.4) we have

$$v_\infty(\lambda_1) = \ldots = v_\infty(\lambda_{i_{\max}}) = -\frac{q^{i_{\max}} - \deg(\phi_{\theta,i_{\max}})}{q^{i_{\max}} - 1}.$$

Thus, if $\deg(\phi_{\theta,i_{\max}}) \not\equiv 1 \pmod{q^{i_{\max}} - 1}$, then we have

$$\operatorname{ord}_{z=1} L_P(\widetilde{\phi}/\widetilde{A}) \leq r - i_{\max}.$$

Example 4.2.8. *Assume that NP_{ϕ_θ} has only one edge. This is equivalent to have $i_{\max} = r$, and that is equivalent by definition to have*

$$\deg(A_r) \geq \frac{q^r - 1}{q^i - 1}(\deg(A_i) - 1) + 1, \forall i = 1, \ldots, r.$$

By Equation (1.6) *we have*

$$v_\infty(\lambda_1) = \ldots = v_\infty(\lambda_r) = -\frac{q^r - \deg(\phi_{\theta,r})}{q^r - 1}.$$

Hence $L_P(\phi/A) \neq 0$ if $\deg(\phi_{\theta,r}) \not\equiv 1 \pmod{q^r - 1}$.

Example 4.2.9. *Assume that $r = 2$ so $\phi_\theta = \theta + \phi_{\theta,1}\tau + \phi_{\theta,2}\tau^2$ with $\phi_{\theta,2} \in A\backslash\{0\}$. We consider the j-invariant $j(\phi)$ associated with ϕ and defined by*

$$j(\phi) = \frac{\phi_{\theta,1}^{q+1}}{\phi_{\theta,2}}.$$

We have two choices: $i_{\max} = 1$ or $i_{\max} = 2$. We have the following equivalences:

$$\begin{aligned} i_{\max} = 2 &\Leftrightarrow \frac{1 + v_\infty(\phi_{\theta,1})}{q - 1} \geq \frac{1 + v_\infty(\phi_{\theta,2})}{q^2 - 1} \\ &\Leftrightarrow v_\infty\left(\frac{\phi_{\theta,1}^{q+1}}{\phi_{\theta,2}}\right) \geq -q \\ &\Leftrightarrow v_\infty(j(\phi)) \geq -q. \end{aligned}$$

Hence by Theorem 1.6.7 we obtain:

$$R_\phi = \begin{cases} -\dfrac{q + v_\infty(\phi_{\theta,1})}{q - 1} & \text{if } v_\infty(j(\phi)) \leq -q, \\[2ex] -\dfrac{q^2 + v_\infty(\phi_{\theta,2})}{q^2 - 1} & \text{otherwise .} \end{cases}$$

Remark that Papanikolas and El-Guindy proved this result, see [23, Corollary 4.2]. They used the concept of shadowed partitions. They also provide explicitely basis of the period lattice in [23, Theorem 5.3]. After many computations, it appears that the basis they give are in fact SBM and we can apply Theorem 4.2.3 to these basis.

Now, we want to study the spectrum of ϕ. We have two cases. If $i_{\max} = 2$, then we have seen in example 4.2.8 that $v_\infty(\lambda_1) = v_\infty(\lambda_2) = -\dfrac{q^2 + v_\infty(\phi_{\theta,2})}{q^2-1}$ and

$$L_P(\phi/A) \neq 0 \text{ if } \deg(\phi_{\theta,2}) \not\equiv 1 \pmod{q^2-1}.$$

If $i_{\max} = 1$, then we have the following result from [30, Proposition 5.1].

Theorem 4.2.10. *If $i_{\max} = 1$ ($\Leftrightarrow v_\infty(j) < -q$), then we have*

$$v_\infty(\lambda_2) = -m + \frac{\deg(\phi_{\theta,1})}{q-1} + \frac{\deg(\phi_{\theta,2}) - (q+1)\deg(\phi_{\theta,1})}{q^m(q-1)}$$

where m is the unique integer ≥ 1 such that $v_\infty(j) \in]-q^{m+1}, -q^m]$.

Hence in the case $i_{\max} = 1$, we have

$$v_\infty(\lambda_1) \in \mathbb{Z} \Leftrightarrow \deg(\phi_{\theta,1}) \equiv 1 \pmod{q-1}$$

and

$$v_\infty(\lambda_2) \in \mathbb{Z} \Leftrightarrow q^m \deg(\phi_{\theta,1}) + \deg(\phi_{\theta,2}) - (q+1)\deg(\phi_{\theta,1}) \equiv 0 \pmod{q^m(q-1)}.$$

We will consider the following example introduced by Gekeler [27, Section 4] that generalizes the rank 2 case.

Example 4.2.11. *Consider $r \geq 2$ an integer and fix $1 \leq k < r$. We say that ϕ is k-sparse if ϕ_θ has the form*

$$\phi_\theta = \theta + \phi_{\theta,k}\tau^k + \phi_{\theta,r}\tau^r$$

with $\phi_{\theta,k}, \phi_{\theta,r} \in A\backslash\{0\}$. To simplify the notation, we denote $A_k = \phi_{\theta,k}$ and $A_r = \phi_{\theta,r}$.

By Proposition 1.6.10, we have

$$v_\infty(\lambda_1) = \ldots = v_\infty(\lambda_k) \geq v_\infty(\lambda_{k+1}) = \ldots = v_\infty(\lambda_r).$$

Moreover, by Lemma 1.6.9 we have

$$v_\infty(\lambda_1) = -\frac{q^k - \deg(A_k)}{q^k - 1}.$$

Set

$$j(\phi) = \frac{A_k^{\frac{q^r-1}{q-1}}}{A_r^{\frac{q^k-1}{q-1}}} \in K^*.$$

We have two choices: $i_{\max} = k$ or $i_{\max} = r$. We have the following equivalences:

$$\begin{aligned} i_{\max} = r &\Leftrightarrow \frac{v_\infty(A_k)+1}{q^k-1} \geq \frac{v_\infty(A_r)+1}{q^r-1} \\ &\Leftrightarrow v_\infty(j(\phi)) \geq -q^k \frac{q^{r-k}-1}{q-1}. \end{aligned}$$

In this case we obtain

$$L_P(\phi/A) \neq 0 \ \textit{if} \deg(A_r) \not\equiv 1 \pmod{q^r-1}.$$

We have the following result by [30, Remark 5.2].

Theorem 4.2.12. *Assume that $v_\infty(j(\phi)) < -q^k \frac{q^{r-k}-1}{q-1}$ and let m be the unique integer such that*

$$v_\infty(j(\phi)) \in]-q^{(m+1)k}\frac{q^{r-k}-1}{q-1}, -q^{mk}\frac{q^{r-k}-1}{q-1}].$$

Then,

$$v_\infty(\lambda_r) = -m + \frac{(q-1)v_\infty(j(\phi))}{q^{km}(q^k-1)(q^{r-k}-1)} - \frac{v_\infty(A_k)}{q^k-1}.$$

Hence in the case $i_{\max} = k$, we have

$$v_\infty(\lambda_k) \in \mathbb{Z} \Leftrightarrow \deg(A_k) \equiv 1 \pmod{q^k-1}$$

and

$$\begin{aligned} &v_\infty(\lambda_r) \in \mathbb{Z} \\ &\Leftrightarrow v_\infty(j(\phi))(q-1) + \deg(A_k)q^{km}(q^{r-k}-1) \equiv 0 \pmod{q^{km}(q^k-1)(q^{r-k}-1)}. \end{aligned}$$

4.3. Vanishing order in the very small case

We continue the study of the vanishing order from Section 4.2 in the special case of very small Drinfeld modules introduced in Section 1.7. We give a simple result to compute this order. To simplify the notation, we will denote by

$$\phi_\theta = \theta + \sum_{i=1}^{r} A_i \tau^i \in A\{\tau\}$$

a Drinfeld A-module defined over A of rank r.

4.3.1. Derivation in positive characteristic p. To study the vanishing order at $z = 1$ of the P-adic L-series, we use the notion of hyperderivative that we will recall here.

Let w be a place of K and K_w be the completion of K with respect to w. For all $k \geq 0$, we define the K_w-linear operators ∂_z^k on $\mathbb{T}_z(K_w)$, which is called the k-th hyperderivative with respect to z, by

$$\partial_z^k\left(\sum_{i\geq 0} a_i z^i\right) = \sum_{i \geq 0} a_i \binom{i}{k} z^{i-k}$$

where $\binom{i}{k}$ are the binomial coefficients in $\mathbb{F}_p$, verifying $\binom{i}{k}=0$ if $k>i$. Remark that for $k=1$, ∂_z^1 is just the classical derivation with respect to z.

In particular, we have the following properties that can be found in [16, Theorem 4].

Proposition 4.3.1. *We have the following assertions.*

(1)
$$\partial_z^k(fg)=\sum_{i=0}^{k}\partial_z^i(f)\partial_z^{k-i}(g), \forall f,g\in \mathbb{T}_z(K_w) \text{ (Leibniz rule)},$$

(2) *Let $n\in\mathbb{N}$ and $f(z)\in\mathbb{T}_z(K_w)$. The following assertions are equivalent:*

(a) $(z-1)^{n+1}|f(z)$ *(in $\mathbb{T}_z(K_w)$),*

(b) $\mathrm{ev}_{z=1}\,\partial_z^k(g(z))=0$ *for all $k\in\{0,\dots,n\}$.*

We extend these operators over $\mathbb{T}_z(K_w\{\tau\})$ by

$$\partial_z^k(\tau)=\tau, \forall k\geq 0.$$

Then the Leibniz rule is still valid, and we have the following two equalities that we will use:

$$\partial_z^k(\widetilde{\phi}_a(f(z)))=\sum_{i=0}^{k}\big(\partial_z^i(\widetilde{\phi}_a)\big)\big(\partial_z^{k-j}(f(z)\big), \forall a, f(z)\in A[z], k\geq 0, \tag{4.7}$$

and

$$\partial_z^k\big(\log_{\widetilde{\phi},P}(g(z))\big)=\sum_{i=0}^{k}\big(\partial_z^i(\log_{\widetilde{\phi},P})\big)\big(\partial_z^{k-i}(g(z))\big), \forall k\geq 0, g(z)\in\Omega_z^+. \tag{4.8}$$

4.3.2. Vanishing in the case $\Omega_z^+=\mathcal{D}_z^+$. Let $\phi:A\to A\{\tau\}$ a very small Drinfeld module of rank r. The goal of this part is to explain how we can explicitly compute the vanishing order of the P-adic L-series at $z=1$. The key result will be the following one.

Lemma 4.3.2. *Let ϕ be a very small Drinfeld A-module defined over A. We assume furthemore that ϕ is not the Carlitz module if $q=2$. Then for all $x\in A$ the following assertions are equivalent.*

(1) *$x\in\phi(A)_{\mathrm{tors}}$,*

(2) *$x\in\mathbb{F}_q$ and $\phi_\theta(x)\in\mathbb{F}_q$.*

Proof. If $x=0$, then the result is clear. We now consider the case $x\neq 0$. Let us first prove that $2\Rightarrow 1$. Assume first that $x\in\mathbb{F}_q^\times$ and $\phi_\theta(x)\in\mathbb{F}_q$. In particular, we have $x\theta\in A\backslash\mathbb{F}_q$ thus $x\theta-\phi_\theta(x)\neq 0$. Then we have:

$$\phi_{x\theta-\phi_\theta(x)}(x)=x\phi_\theta(x)-\phi_\theta(x)x=0.$$

Let us prove the converse. We denote by $t := \deg_\theta(x)$ and assume that $t > 0$. Recall that we set

$$\phi_\theta(x) = \theta x + A_1 x^q + \ldots + A_r x^{q^r}.$$

We compare degrees of all terms to apply the ultrametric property. We first have

$$\deg(A_r x^{q^r}) = \deg(A_r) + q^r t$$

and the last quantity is strictly bigger than $1 + t = \deg(\theta x)$, except in the case $q = 2$, $r = 1$ and $\deg(A_r) = 0$ that is, if ϕ is the Calitz module and $q = 2$. Next we have for all $1 \le k < r$:

$$\begin{aligned}
\deg(A_r x^{q^r}) &\ge q^r t, \\
&\ge 2q^k t \text{ since } q \ge 2, \\
&> q^k t + \deg(A_k)t \text{ by the smallness hypothesis,} \\
&\ge q^k t + \deg(A_k), \\
&= \deg(A_k x^{q^k}).
\end{aligned}$$

Thus, under the hypothesis of the lemma, if $\deg(x) > 0$, then we have

$$\deg(\phi_\theta(x)) = q^r \deg(x) + \deg(A_r) > 0.$$

An immediate induction tells us that for all $n \ge 0$ we have the equality:

$$\deg(\phi_{\theta^n}(x)) = q^r \deg(\phi_{\theta^{n-1}}(x)) + \deg(A_r). \tag{4.9}$$

We conclude that the sequence $(\deg(\phi_{\theta^n}(x)))_{n\ge 0}$ is strictly increasing so x is not a torsion point. Thus, if $x \in \phi(A)_{\mathrm{tors}}$, then we must have $x \in \mathbb{F}_q$. Moreover, if $x \in \phi(A)_{\mathrm{tors}}$, then we also have $\phi_\theta(x) \in \phi(A)_{\mathrm{tors}}$, so we must have $\phi_\theta(x) \in \mathbb{F}_q$. $\square$

Theorem 4.3.3. *Assume that $(q, \deg(P)) \neq (2, 1)$ and that ϕ is very small. For all $k \ge 1$, the following assertions are equivalent:*

(1) $(z-1)^k | L_P(\widetilde{\phi}/\widetilde{A})$,

(2) $\phi_\theta^{(j)}(1) \in \mathbb{F}_q$ *for $j = 0, \ldots, k-1$, where* $\phi_\theta^{(j)}(1) = \mathrm{ev}_{z=1}\left(\partial_z^j \left(\widetilde{\phi}_\theta(1)\right)\right)$.

Proof. Let us start with the $k = 1$ case. We have the following equivalences:

$$\begin{aligned}
(z-1)|L_P(\widetilde{\phi}/\widetilde{A}) &\Leftrightarrow L_P(\phi/A) = 0, \\
&\Leftrightarrow \mathrm{Log}_{\phi,P}(1) = 0, \text{ since } \phi \text{ is very small }, \\
&\Leftrightarrow 1 \in \phi(A)_{\mathrm{tors}} \text{ by Proposition 2.6.4,} \\
&\Leftrightarrow \phi_\theta(1) \in \mathbb{F}_q \text{ by Lemma 4.3.2.}
\end{aligned}$$

We assume that the P-adic L series vanishes, so we have $\phi_\theta(1) \in \mathbb{F}_q$. We set $a = \theta - \phi_\theta(1) \in A \backslash \mathbb{F}_q$ satisfying

$$\phi_a(1) = \phi_\theta(1) - \phi_\theta(1) = 0.$$

We assume the equivalence of the Theorem to be true up to rank $k \geq 1$ and prove it at rank $k+1$. Let $k > 1$ be such that $(z-1)^k | L_P(\widetilde{\phi}/\widetilde{A})$. Consider first the following equality in $\mathbb{T}_z(K_P)$:

$$aPL_P(\widetilde{\phi}/\widetilde{A}) = \log_{\widetilde{\phi},P}(\widetilde{\phi}_{g_{P,\phi}(z)}(\widetilde{\phi}_a(1))).$$

We then have for all $k \geq 0$ (formally first i.e., in $K[[z]]$):

$$aP\partial_z^k\left(L_P(\widetilde{\phi}/\widetilde{A})\right) = \sum_{j=0}^{k}\left((\partial_z^{k-j}\left(\log_{\widetilde{\phi,P}}\right)\right)\left(\partial_z^j\left(\widetilde{\phi}_{g_{P,\phi}(z)}(\widetilde{\phi}_a(1)\right)\right).$$

Since $\widetilde{\phi}_{g_{P,\phi}(z)}(\widetilde{\phi}_a(1)) \in PA[z]$, we have that for all $j \geq 0$

$$\partial_z^j\left(\widetilde{\phi}_{g_{P,\phi}(z)}(\widetilde{\phi}_a(1))\right) \in PA[z]$$

and the previous equality is in fact an equality in $\mathbb{T}_z(K_P)$ and we can evaluate P-adically at $z=1$ this equality.

For $j < k$, we have by Equality (4.7)

$$\partial_z^j\left(\widetilde{\phi}_{g_{P,\phi}(z)}(\widetilde{\phi}_a(1))\right) = \sum_{l=0}^{j}\left(\partial_z^l\left((\widetilde{\phi}_a\right)\right)\left(\partial_z^{j-l}\left(\widetilde{\phi}_{g_{P,\phi}(z)}(1)\right)\right).$$

By induction hypothesis, we have

$$\mathrm{ev}_{z=1}\left(\partial_z^{j-l}\left(\widetilde{\phi}_{g_{P,\phi}(z)}(1)\right)\right) = 0, \forall l \leq j$$

since $(z-1)^k | L_P(\widetilde{\phi}/\widetilde{A})$. Thus for $j < k$ we have:

$$\mathrm{ev}_{z=1}\,\partial_z^j\left(\widetilde{\phi}_{g_{P,\phi}(z)}(\widetilde{\phi}_a(1))\right) = 0.$$

Next for $j = k$ we have by Equality (4.7)

$$\partial_z^k\left(\widetilde{\phi}_{g_{P,\phi}(z)}(\widetilde{\phi}_a(1))\right) = \sum_{h=0}^{k}\partial_z^{k-h}\left(\widetilde{\phi}_{g_{P,\phi}(z)}\left(\partial_z^h\widetilde{\phi}_a(1)\right)\right).$$

Recall that $a = \theta - \phi_\theta(1)$ with $\phi_\theta(1) \in \mathbb{F}_q$. Thus, for all $h = 0, \ldots, k$, we have

$$\partial_z^h\left(\widetilde{\phi}_a(1)\right) = \partial_z^h\left(\widetilde{\phi}_\theta(1)\right).$$

Thus, we have:

$$\mathrm{ev}_{z=1}\,\partial_z^h\left(\widetilde{\phi}_a(1)\right) = \begin{cases} \phi_a(1) = 0 \text{ if } h = 0, \\ \alpha_h \in \mathbb{F}_q \text{ if } 0 < h < k, \\ \mathrm{ev}_{z=1}\,\partial_z^k\left(\widetilde{\phi}_\theta(1)\right) \text{ if } h = k, \end{cases}$$

where $\alpha_h = \mathrm{ev}_{z=1}\,\partial_z^h\left(\widetilde{\phi}_\theta(1)\right) \in \mathbb{F}_q$ if $0 < h < k$ by recurrence hypothesis. Then

$$\begin{aligned}\mathrm{ev}_{z=1}\sum_{h=0}^{k}\partial_z^{k-h}\left(\widetilde{\phi}_{g_{P,\phi}(z)}\left(\partial_z^h\left(\widetilde{\phi}_a(1)\right)\right)\right) &= \sum_{h=0}^{k-1}\alpha_h\phi^{(h)}_{g_{P,\phi}(1)}(1) + \phi_{g_{P,\phi}(1)}(\phi_a^{(k)}(1)) \\ &= \phi_{g_{P,\phi}(1)}(\phi_a^{(k)}(1)).\end{aligned}$$

We proved that if $(z-1)^k | L_P(\widetilde{\phi}/\widetilde{A})$, then we have

$$\mathrm{ev}_{z=1,P}\left(aP\partial_z^k\left(L_P(\widetilde{\phi}/\widetilde{A})\right)\right) = g_{P,\phi}(1)\,\mathrm{Log}_{\phi,P}(\phi_a^{(k)}(1)).$$

We deduce the following equivalences:

$$\begin{aligned}(z-1)^{k+1}|L_P(\widetilde{\phi}/\widetilde{A}) &\Leftrightarrow \mathrm{Log}_{\phi,P}(\phi_a^{(k)}(1)) = 0, \\ &\Leftrightarrow \phi_a^{(k)}(1) \in \phi(A)_{\mathrm{tors}} \text{ by Proposition 2.6.4,} \\ &\Leftrightarrow \phi_\theta^{(k)}(1) \in \mathbb{F}_q \text{ by Lemma 4.3.2.}\end{aligned}$$

□

We will explain later why this result actually works in the case $(q, \deg(P)) = (2, 1)$ except for the Carlitz module. For now, we obtain another proof of Theorem 4.2.3 on the P-independence of the vanishing order at $z = 1$ of the P-adic L-series and a bound about this order.

Corollary 4.3.4. *Assume that ϕ is very small and that $(q, \deg(P)) \neq (2, 1)$. We have the following assertions.*

1. *The vanishing order of the P-adic L series does not depend on P.*
2. *$(z-1)^{r+1} \nmid L_P(\widetilde{\phi}/\widetilde{A})$.*

Proof. The first assertion is clear, as the hypothesis of Theorem 4.3.3 do not depend on P. Let us prove the second assertion. For the sake of argument, assume that $(z-1)^{r+1}|L_P(\widetilde{\phi}/\widetilde{A})$ in $\mathbb{T}_z(K_P)$. By Theorem 4.3.3 we have:

$$\phi_\theta^{(j)}(1) \in \mathbb{F}_q$$

for $j = 0, ..., r$, that is we have for $j = 0, ..., r$:

$$\sum_{k=0,\ k\geq j}^{r} A_k\binom{j}{k} \in \mathbb{F}_q.$$

Hence we have (by looking at $j = r$ then $j = r-1, \dots$):

$$\begin{cases} A_r \in \mathbb{F}_q \\ A_{r-1} + (r-1)A_r \in \mathbb{F}_q \\ A_{r-2} + \binom{r-1}{r-2}A_{r-1} + \binom{r}{r-2}A_{r-2} \in \mathbb{F}_q \\ \vdots \end{cases}$$

So all of the A_i are in $\mathbb{F}_q$, which is absurd because in this case we would have $\deg(\phi_\theta(1)) = 1$, which contradicts Theorem 4.3.3. □

4.3.3. The "special" case: $(q,\deg(P))=(2,1)$. Assume now that ϕ is a very small Drinfeld module of rank r, $q=2$ and $\deg(P)=1$. Without loss of generality, we assume until the end of the Section that $P=\theta$. The goal of this part is to prove that Theorem 4.3.3 is still valid in this setting, except if ϕ is the Carlitz module. We will explain the Carlitz module case in Section 4.4, see Theorem 4.4.1.

Recall that

$$\phi_\theta=\theta+\sum_{i=1}^r A_i\tau^i.$$

The main problem is that, in this setting, we do not have the equality $\Omega_z^+=\mathcal{D}_z^+$. As a particular case we do not have the following equivalence for all $k\geq 0$:

$$(z-1)^k|\log_{\widetilde{\phi},P}(\widetilde{\phi}_{g_{P,\phi}(z)}(1))\Leftrightarrow(z-1)^k|\widetilde{\phi}_{g_{P,\phi}(z)}(1).$$

For all $i=1,...,r$, let $\alpha_i\in\mathbb{F}_q$ be such that $A_i\equiv\alpha_i\pmod{\theta}$ (so $\alpha_i=0$ or 1). We have

$$\widetilde{\phi}_{\theta+\sum_{i=1}^r\alpha_i z^i}(1)=\theta+\sum_{i=1}^r(A_i+\alpha_i)z^i\equiv 0\pmod{\theta},$$

thus

$$g_{P,\phi}(z)=\theta+\sum_{i=1}^r\alpha_i z^i,$$

and

$$\widetilde{\phi}_{g_{P,\phi}(z)}=\theta+(A_1+\alpha_1)z+...+(A_r+\alpha_r)z^r.$$

Lemma 4.3.5. *Keep the hypothesis of the subsection. For all $k\geq 0$, the following assertions are equivalent:*

(1) $(z-1)^k|L_P(\widetilde{\phi}/\widetilde{A})$,

(2) $(z-1)^k|\widetilde{\phi}_{Pg_{P,\phi}(z)P}(1)$.

Proof. We start from the following equality in $\mathbb{T}_z(K_P)$:

$$P^2L_P(\widetilde{\phi}/\widetilde{A})=\log_{\widetilde{\phi},P}\left(\widetilde{\phi}_{Pg_{P,\phi}(z)}(1)\right).$$

Since $v_P(\widetilde{\phi}_{g_{P,\phi}(z)}(1))>0$, we have $v_P(\widetilde{\phi}_{Pg_{P,\phi}(z)}(1))>1$. It follows that $\widetilde{\phi}_{g_{P,\phi}(z)P}(1)\in\mathcal{D}_z^+$. Then we have for all $k\geq 0$:

$$(z-1)^k|L_P(\widetilde{\phi}/\widetilde{A})\Leftrightarrow(z-1)^k|P^2L_P(\widetilde{\phi}/\widetilde{A})\Leftrightarrow(z-1)^k|\widetilde{\phi}_{Pg_{P,\phi}(z)}(1).$$

□

Lemma 4.3.6. *Assume that ϕ is not the Carlitz module. The following assertions are equivalent:*

$$L_P(\phi/A)=0\Leftrightarrow\phi_{g_{P,\phi}(1)}(1)=0.$$

Proof. The implication $\Leftarrow$ is clear since $L_P(\phi/A) = \dfrac{1}{P}\log_{\phi,P}(\phi_{g_{P,\phi}(1)}(1))$.

To prove the converse, assume that $L_P(\phi/A) = 0$. By Lemma 4.3.2, we have $\phi_\theta(1) \in \mathbb{F}_q$. Remark that we have the equivalence:

$$\phi_\theta(1) \in \mathbb{F}_q \Leftrightarrow \phi_\theta(1) = \alpha_1 + \ldots + \alpha_r.$$

Thus, if $\phi_\theta(1) \in \mathbb{F}_q$, then we have

$$\phi_{g_{P,\phi}(1)} = \phi_\theta(1) + \sum_{i=1}^{r} \alpha_i = 2(\alpha_1 + ... + \alpha_r) = 0.$$

$\square$

The main result of this section is the following one, allowing us to detach ourselves from the assumptions about q and $\deg(P)$ in Theorem 4.3.3, except for the Carlitz module case.

Proposition 4.3.7. *Assume that ϕ is not the Carlitz module,* $\deg(P) = 1$ *and $q = 2$. For all $k \geq 0$, the following assertions are equivalent:*

(1) $\phi_\theta^{(j)}(1) \in \mathbb{F}_2$ *for* $j = 0, ..., k-1$,

(2) $(z-1)^k | \widetilde{\phi}_{Pg_{P,\phi}(z)}(1)$.

Proof. The $k = 0$ case is obvious and the case $k = 1$ is Lemma 4.3.6 with Lemma 4.3.2. Let us assume that the two assertions are equivalent up to rank $k \geq 1$ and prove that they are equivalent at rank $k+1$.

Consider $k \geq 1$ and assume that $(z-1)^k | \widetilde{\phi}_{Pg_{P,\phi}(z)}(1)$. By the induction hypothesis, we have $\phi_\theta^{(j)}(1) \in \mathbb{F}_q$ for $j = 0, ..., k-1$. First, let us note that we have the following equivalence for all $j \geq 0$:

$$\phi_\theta^{(j)}(1) \in \mathbb{F}_q \Leftrightarrow \mathrm{ev}_{z=1}\, \partial_z^j \left(\widetilde{\phi}_\theta(1)\right) = \mathrm{ev}_{z=1}\, \partial_z^j \left(\alpha_1 z + ... + \alpha_r z^r\right). \tag{4.10}$$

We have:

$$\partial_z^k \left(\widetilde{\phi}_{\theta g_{P,\phi}(z)}(1)\right) = \sum_{j=0}^{k} \left(\partial_z^{k-j}\left(\widetilde{\phi}_\theta\right)\right)\left(\partial_z^j\left(\widetilde{\phi}_{g_{P,\phi}(z)}(1)\right)\right).$$

Next for all $j < k$ we have

$$\partial_z^j \left(\widetilde{\phi}_{g_{P,\phi}(z)}(1)\right) = \partial_z^j \left(\widetilde{\phi}_{\theta+\alpha_1 z+...+\alpha_r z^r}(1)\right) = \partial_z^j \left(\widetilde{\phi}_\theta(1) + \alpha_1 z + ... + \alpha_r z^r\right).$$

By the induction hypothesis we obtain for all $j < k$:

$$\mathrm{ev}_{z=1}\, \partial_z^j \left(\widetilde{\phi}_{g_{P,\phi}(z)}(1)\right) = 2\, \mathrm{ev}_{z=1}\, \partial_z^j \left(\widetilde{\phi}_\theta(1)\right) = 0.$$

We deduce the equality:

$$\mathrm{ev}_{z=1}\, \partial_z^k \left(\widetilde{\phi}_{\theta g_{P,\phi}(z)}(1)\right) = \phi_\theta \left(\mathrm{ev}_{z=1}\, \partial_z^k \left(\widetilde{\phi}_{g_{P,\phi}(z)}(1)\right)\right). \tag{4.11}$$

Let us first prove $2 \Rightarrow 1$. Assume that $(z-1)^{k+1} | \widetilde{\phi}_{\theta g_{P,\phi}(z)}(1)$. By Lemma 4.3.2 and Equation (4.11), we have $\mathrm{ev}_{z=1} \partial_z^k \left(\widetilde{\phi}_{g_{\theta,\phi}(z)}(1) \right) \in \mathbb{F}_q$. But

$$\begin{aligned}
\mathrm{ev}_{z=1} \partial_z^k \left(\widetilde{\phi}_{g_{\theta,\phi}(z)}(1) \right) &= \mathrm{ev}_{z=1} \partial_z^k \left(\widetilde{\phi}_\theta(1) + \alpha_1 z + ... + \alpha_r z^r \right) \\
&= \mathrm{ev}_{z=1} \partial_z^k \left(\widetilde{\phi}_\theta(1) \right) + \underbrace{\mathrm{ev}_{z=1} \partial_z^k (\alpha_1 z + ... + \alpha_r z^r)}_{\in \mathbb{F}_q}
\end{aligned}$$

thus $\mathrm{ev}_{z=1} \partial_z^k \left(\widetilde{\phi}_\theta(1) \right) \in \mathbb{F}_q$.

Let us now prove $1 \Rightarrow 2$. To do this, assume that $\phi_\theta^{(k)}(1) \in \mathbb{F}_q$. By equivalence (4.10), we have $\mathrm{ev}_{z=1} \partial_z^k \left(\widetilde{\phi}_\theta(1) \right) = \mathrm{ev}_{z=1} \partial_z^k (\alpha_1 z + ... + \alpha_r z^r)$. Thus,

$$\begin{aligned}
\mathrm{ev}_{z=1} \partial_z^k \widetilde{\phi}_{g_{\theta,\phi}(z)}(1) &= \mathrm{ev}_{z=1} \partial_z^k \left(\widetilde{\phi}_\theta(1) + \alpha_1 z + ... + \alpha_r z^r \right) \\
&= \mathrm{ev}_{z=1} \partial_z^k \widetilde{\phi}_\theta(1) + \mathrm{ev}_{z=1} \partial_z^k (\alpha_1 z + ... + \alpha_r z^r) \\
&= 2 \, \mathrm{ev}_{z=1} \partial_z^k (\alpha_1 z + ... + \alpha_r z^r) \\
&= 0
\end{aligned}.$$

We conclude that $(z-1)^{k+1} | \widetilde{\phi}_{\theta g_{\theta,\phi}(z)}(1)$ by Equation (4.11).

□

4.4. A detailed example

In this section, we compute explicitly the P-adic L-series of a Drinfeld A-module defined over an extension of K.

Consider the Carlitz module C, recall that it is given by $C_\theta = \theta + \tau$.

Theorem 4.4.1. *For all monic irreducible polynomials P of A, we have*

$$\mathrm{ord}_{z=1} L_P(\widetilde{C}/\widetilde{A}) = \begin{cases} 0 & \text{if } q > 2, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. By [3, Corollary 2.5.9], we have $\mathrm{ord}_{z=1} L_P(\widetilde{C}/\widetilde{A}) = 0$ if and only if $q > 2$. If $q = 2$, then by Theorem 4.2.3 we have

$$\mathrm{ord}_{z=1} L_P(\widetilde{C}/\widetilde{A}) = 1.$$

□

Remark that in the case $P = \theta$ and $q = 2$, this result was already known from the work of Wan [48], Thakur [46], and Diaz-Vargas and Polanco-Chi [19]. Diaz-Vargas and Polanco-Chi, using computations of the Newton polygon of the P-adic zeta function $\zeta_P(A, n, z)$, proved the following result, see [19, Theorem 3].

Theorem 4.4.2. *Assume that $P = \theta$. Then all the zeros of $L_P(\widetilde{C^{\otimes n}}/\widetilde{A})$ are simple and are in $K_P = \mathbb{F}_q((\theta))$.*

We now consider the Carlitz module defined on an extension of K, the θ-th cyclotomic extension. We give first some preliminaries. Let λ_θ be a non-zero root of $C_\theta(X) = \theta X + X^q$ and consider $L = K_\theta = K(\lambda_\theta) = \mathbb{F}_q(\lambda_\theta)$ the field generated by λ_θ.

Theorem 4.4.3. *The field K_θ is a cyclic extension of K of degree $q-1$, called the θ-th cyclotomic extension, and we have $\mathscr{O}_L = A[\lambda_\theta]$.*

Proof. See [40, Chapter 12]. □

Theorem 4.4.4. *For all irreducible monic polynomial P of A, we have*

$$\operatorname{ord}_{z=1} L_P(\widetilde{C}/\widetilde{\mathscr{O}_L}) = 1.$$

Proof. For a monic irreducible polynomial P of A, recall that the local factor at P is

$$z_P(\widetilde{C}/\widetilde{\mathscr{O}_L}) = \frac{P^{[L:K]}}{\left[\widetilde{C}(\widetilde{\mathscr{O}_L}/Q\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}}.$$

We compute $\left[\widetilde{C}(\widetilde{\mathscr{O}_L}/Q\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}$ for a monic irreducible polynomial Q of A.

For $Q = \theta$, we look at the θ-action (through C) on the $\mathbb{F}_q(z)$-basis $(\lambda_\theta^i + \theta\widetilde{\mathscr{O}_L}, i = 0, ..., q-2)$.

By definition, we have $\widetilde{C}_\theta(\lambda_\theta^i) = \theta\lambda_\theta^i + z\tau(\lambda_\theta^i)$, thus

$$\widetilde{C}_\theta(\lambda_\theta^i + \theta\widetilde{\mathscr{O}_L}) = \begin{cases} z + \theta\widetilde{\mathscr{O}_L} \text{ if } i = 0, \\ 0 + \theta\widetilde{\mathscr{O}_L} \text{ if } i \neq 0. \end{cases}$$

Then:

$$\begin{aligned}\left[\widetilde{C}(\widetilde{\mathscr{O}_L}/\theta\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}} &= \det_{\mathbb{F}_q(z)[T]}\left(T - \theta \mid \widetilde{C}(\widetilde{\mathscr{O}_L}/\theta\widetilde{\mathscr{O}_L})\right)_{|T=\theta} \\ &= \det\begin{pmatrix}\theta - z & & & \\ & \theta & & \\ & & \ddots & \\ & & & \theta\end{pmatrix} \\ &= (\theta - z)\theta^{q-2}.\end{aligned}$$

We deal with the case $Q \neq \theta$. Notice first that $\mathscr{O}_L$, $\widetilde{\mathscr{O}_L}/Q\widetilde{\mathscr{O}_L}$ is a $\mathbb{F}_q(z)$-vector space of dimension $\deg(Q)(q-1)$. Next, we have the following equality, see for example [29, Theorem 3.6.13].

$$\widetilde{C}_Q \equiv z^{\deg(Q)}\tau^{\deg(Q)} \pmod{Q\widetilde{A}\{\tau\}}.$$

We then have the following equalities for all $i = 0, ..., q-2$:

$$\begin{aligned}\widetilde{C}_Q(\lambda_\theta^i) &\equiv z^{\deg(Q)}\tau^{\deg(Q)}(\lambda_\theta^i) \pmod{Q\widetilde{\mathscr{O}_L}} \\ &\equiv z^{\deg(Q)}(-\theta)^{i\frac{q^{\deg(Q)}}{q-1}} \pmod{Q\widetilde{\mathscr{O}_L}} \\ &\equiv z^{\deg(Q)}\lambda_\theta^i(-\theta)^{i\frac{q^{\deg(Q)}-1}{q-1}} \pmod{Q\widetilde{\mathscr{O}_L}} \\ &\equiv z^{\deg(Q)}\lambda_\theta^i Q(0)^i \pmod{Q\widetilde{\mathscr{O}_L}}.\end{aligned}$$

Hence, for all $i = 0, ..., q-2$ we have

$$\widetilde{C}_{Q - z^{\deg(Q)}Q(0)^i}(\lambda_\theta^i) \in Q\widetilde{\mathscr{O}_L}.$$

If we set

$$G_Q(\theta) = \prod_{i=0}^{q-2}(Q - z^{\deg(Q)}Q(0)^i) \in A[z],$$

then we obtain

$$G_Q(\widetilde{\mathscr{O}_L}) \in Q\widetilde{\mathscr{O}_L}.$$

Since $G_Q \in A[z]$ has degree $\deg(Q)(q-1)$, we obtain

$$G_Q = \left[\widetilde{C}(\widetilde{\mathscr{O}_L}/Q\widetilde{\mathscr{O}_L})\right]_{\widetilde{A}}.$$

Remark that this equality also holds for $Q = \theta$. Finally, we have the following equality for all monic irreducible polynomials P of A:

$$L_P(\widetilde{C}/\widetilde{\mathscr{O}_L}) = \prod_{Q \neq P} z_Q(\widetilde{C}/\widetilde{\mathscr{O}_L}) = \prod_{i=0}^{q-2}\prod_{Q\neq P}\left(1 - \frac{z^{\deg(Q)}Q(0)^i}{Q}\right)^{-1} \in \mathbb{T}_z(K_P).$$

Now, consider for all $i = 0, ..., q-2$ the Drinfeld A-module $\phi_i : A \to A\{\tau\}$ defined by

$$\phi_{i,\theta} = \theta + (-\theta)^i\tau.$$

Remark that ϕ_i is a very small Drinfeld module for all $i = 0, \ldots, q-2$. By [45, Page 301], we have the following equality for all monic irreducible polynomial Q of A:

$$z_Q(\widetilde{\phi_i}/\widetilde{A}) = \left(1 - \frac{z^{\deg(Q)}Q(0)^i}{Q}\right)^{-1}.$$

Thus

$$L_P(\widetilde{\phi_i}/\widetilde{A}) = \prod_{Q\neq P}\left(1 - \frac{z^{\deg(Q)}Q(0)^i}{Q}\right)^{-1} = \sum_{\substack{a\in A^+ \\ P|a}} \frac{a(0)^i z^{\deg(a)}}{a} \in \mathbb{T}_z(K_P).$$

and

$$L_P(\widetilde{C}/\widetilde{\mathscr{O}_L}) = \prod_{i=0}^{q-2} L_P(\widetilde{\phi_i}/\widetilde{A}) \in \mathbb{T}_z(K_P).$$

If $q = 2$, then we have $L_P(\widetilde{C}/\widetilde{\mathscr{O}_L}) = L_P(\widetilde{C}/\widetilde{A})$ and $\mathrm{ord}_{z=1} L_P(\widetilde{C}/\widetilde{\mathscr{O}_L}) = 1$.

If $q > 2$, then we have

$$\mathrm{ord}_{z=1} L_P(\widetilde{\phi_i}/\widetilde{A}) = 0$$

for $i \in \{0, \ldots, q-2\}\backslash\{1\}$ by Theorem 4.3.3, and

$$\mathrm{ord}_{z=1} L_P(\widetilde{\phi_1}/\widetilde{A}) = 1$$

by Theorem 4.2.3 and Theorem 4.3.3. Finally, we obtain

$$\mathrm{ord}_{z=1} L_P(\widetilde{C}/\widetilde{\mathscr{O}_L}) = 1.$$

□

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Modules d'Anderson et séries L: une étude P-adique

Résumé : Dans cette thèse, nous étudions des séries L P-adiques dans le contexte des modules d'Anderson, qui seront les analogues P-adiques des séries L de Taelman pour les modules de Drinfeld. Le point de départ est la construction des séries L avec une variable z introduite par Anglès et Tavares Ribeiro, et la formule des classes pour les t-modules d'Anderson avec cette variable z démontrée par Demeslay. Nous construisons d'abord ces séries L P-adiques, étudions leur convergence et démontrons une formule P-adique du nombre de classes. Nous étendons ensuite ces constructions au cas des modules d'Anderson sur des anneaux de base possédant plusieurs variables. Enfin, nous effectuons une étude approfondie de ces séries L P-adiques dans le cas des modules de Drinfeld.

Anderson modules and L-series: a P-adic study

Abstract : In this thesis, we study P-adic L-series in the context of Anderson modules, which are the P-adic analogues of Taelman's L-series for Drinfeld modules. The starting point is the construction of L-series with a variable z introduced by Anglès and Tavares Ribeiro, and the class formula for Anderson t-modules with this variable z proved by Demeslay. We first construct these P-adic L-series, study their convergence, and prove a P-adic class number formula. We then extend these constructions to the case of Anderson modules over base rings with several variables. Finally, we study the case of Drinfeld modules.