

A  $P$ -ADIC CLASS FORMULA FOR ANDERSON  $t$ -MODULES

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ABSTRACT. In 2012, Taelman proved a class formula for  $L$ -series associated to Drinfeld  $\mathbb{F}_q[\theta]$  modules and considered it as a function field analog of the Birch and Swinnerton-Dyer conjecture. Since then, Taelman's class formula has been generalized to the setting of Anderson  $t$ -modules. Let  $P$  be a monic prime of  $\mathbb{F}_q[\theta]$ , we define the  $P$ -adic  $L$ -series associated with Anderson  $t$ -modules and prove a  $P$ -adic class formula à la Taelman linking a  $P$ -adic regulator, the class module and a local factor at  $P$ . Next, we extend this result to the multi-variable setting à la Pellarin. Finally, we give some applications to Drinfeld modules defined over  $\mathbb{F}_q[\theta]$  itself.

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## 1. INTRODUCTION

**1.1. Class formula à la Taelman.** In 2010, Taelman introduced the notion of  $L$ -series associated to Drinfeld  $\mathbb{F}_q[\theta]$ -modules and conjectured a class formula, see [20, Conjecture 1]. He [21] proved this class formula in 2012, also considered as a function field analogue of the Birch and Swinnerton-Dyer conjecture. These results were generalised by Fang [13] and Demeslay [11] in the context of Anderson  $t$ -modules that are Drinfeld modules of higher dimension. Finally, Anglès, Tavares Ribeiro and Ngo Dac [4] proved the class formula for a general ring  $A$  in the context of admissible Anderson  $A$ -modules, including in particular all Drinfeld  $A$ -modules.

The objective of the present article is to construct  $P$ -adic analogs of these  $L$ -series associated to Anderson  $t$ -modules. We call them  $P$ -adic  $L$ -series, and prove a  $P$ -adic class formula à la Taelman. We then extend these results to the setting of variables following the work of Anglès, Pellarin and Tavares Ribeiro [5] and Pellarin [19].

The key ingredient will be the notion of  $z$ -deformation introduced by Anglès, Tavares Ribeiro [7] as well as the introduction of evaluation of  $z$  not only at  $z = 1$  but at elements of  $\overline{\mathbb{F}}_q$ .

We then study the vanishing of the  $P$ -adic  $L$ -series at  $z = 1$ . Finally, we investigate in detail the case of  $\mathbb{F}_q[\theta]$ -Drinfeld modules defined over  $\mathbb{F}_q[\theta]$ .

**1.2. Main results.** Let us give more precise statements of our results.

Let  $\mathbb{F}_q$  be a finite field with  $q$  elements and  $\theta$  an indeterminate over  $\mathbb{F}_q$ . Let us consider  $A = \mathbb{F}_q[\theta]$  and let  $K = \mathbb{F}_q(\theta)$  be the rational function field. Let  $L/K$  be a finite field extension of degree  $n$ . We denote by  $\mathcal{O}_L$  the integral closure of  $A$  in  $L$ . We consider the valuation  $v_\infty$  of  $K$  normalized such that  $v_\infty(\theta^{-1}) = 1$ . Let  $K_\infty$  be the completion of  $K$  with respect to this valuation and we set  $L_\infty = L \otimes_K K_\infty$ . Let  $\tau : x \mapsto x^q$  be the Frobenius map.

If  $M$  is an  $A$ -module having a finite number of elements, we denote by  $[M]_A$  the monic generator of the Fitting ideal of  $M$ .

An Anderson  $t$ -module  $E$  of dimension  $d$  defined over  $\mathcal{O}_L$  is a non-constant  $\mathbb{F}_q$ -algebra homomorphism  $E : A \rightarrow M_d(\mathcal{O}_L)\{\tau\}$  such that if  $a \in A$  and  $E_a = \sum_{i=0}^{r_a} E_{a,i} \tau^i$  then we require  $(E_{a,0} - aI_d)^d = 0$ . We denote by  $\partial_E : A \rightarrow M_d(\mathcal{O}_L)$  the homomorphism of  $\mathbb{F}_q$ -algebras  $\partial_E(a) = E_{a,0}$ . If  $B$  is an  $\mathcal{O}_L$ -algebra, then we can define two  $A$ -module structures on  $B^d$ : the first is denoted by  $E(B)$  where  $A$  acts on  $B^d$  via  $E$ , and the second is denoted by  $\text{Lie}_E(B)$  where  $A$  acts on  $B^d$  via  $\partial_E$ .

There exists a unique series  $\exp_E \in M_d(L)\{\{\tau\}\}$ , called the exponential series, such that  $\exp_E \partial_E(\theta) = E_\theta \exp_E$ . Moreover,  $\exp_E$  converges on  $\text{Lie}_E(L_\infty)$ .

The key notion will be the notion of  $z$ -deformation introduced by Anglès and Tavares Ribeiro [7]. Let  $z$  be a new variable such that  $\tau(z) = z$ . Set  $\tilde{A} = \mathbb{F}_q(z)A$ ,  $\tilde{K} = \mathbb{F}_q(z)K$ ,  $\tilde{\mathcal{O}}_L = \mathbb{F}_q(z)\mathcal{O}_L$  and  $\tilde{K}_\infty = \mathbb{F}_q(z)((\theta^{-1}))$ . We set  $\tilde{L}_\infty = L \otimes_K \tilde{K}_\infty$ . We consider  $\tilde{E}$  the  $z$ -twist of  $E$ , introduced in [7], that is the homomorphism of  $\mathbb{F}_q(z)$ -algebras  $\tilde{E} : \tilde{A} \rightarrow M_d(\tilde{\mathcal{O}}_L)\{\tau\}$  given by

$$\tilde{E}_a = \sum_{i=0}^{r_a} E_{a,i} z^i \tau^i, \text{ for all } a \in A.$$

We also have an exponential series  $\exp_{\tilde{E}}$  associated with  $\tilde{E}$  that converges on  $\text{Lie}_{\tilde{E}}(\tilde{L}_\infty)$ .

Taelman showed that the module of  $z$ -units

$$U(\tilde{E}; \tilde{\mathcal{O}}_L) = \{x \in \text{Lie}_{\tilde{E}}(L_\infty) \mid \exp_{\tilde{E}}(x) \in \tilde{E}(\tilde{\mathcal{O}}_L)\}$$

is an  $\tilde{A}$ -lattice in  $\text{Lie}_{\tilde{E}}(\tilde{L}_\infty)$  and that the Taelman unit module

$$U(E; \mathcal{O}_L) = \{x \in \text{Lie}_E(L_\infty) \mid \exp_E(x) \in E(\mathcal{O}_L)\}$$

is an  $A$ -lattice in  $\text{Lie}_E(L_\infty)$ . Moreover, he proved that the class module

$$H(E; \mathcal{O}_L) = \frac{E(L_\infty)}{E(\mathcal{O}_L) + \exp_E(\text{Lie}_E(\mathcal{O}_L))}$$

is finite. The local factors associated with  $\tilde{E}$  and  $E$  at a monic prime polynomial  $Q$  are respectively

$$z_Q(\tilde{E}/\tilde{\mathcal{O}}_L) = \frac{[\text{Lie}_{\tilde{E}}(\tilde{\mathcal{O}}_L/Q\tilde{\mathcal{O}}_L)]_{\tilde{A}}}{[\tilde{E}(\tilde{\mathcal{O}}_L/Q\tilde{\mathcal{O}}_L)]_{\tilde{A}}} \in \tilde{K}^* \text{ and } z_Q(E/\mathcal{O}_L) = \frac{[\text{Lie}_E(\mathcal{O}_L/Q\mathcal{O}_L)]_A}{[E(\mathcal{O}_L/Q\mathcal{O}_L)]_A} \in K^*.$$

Set  $m = dn$  and consider  $\mathcal{C}$  an  $A$ -basis of  $\text{Lie}_E(\mathcal{O}_L)$ . Fix  $(u_1(z), \dots, u_m(z))$  an  $\tilde{A}$ -basis of  $U(\tilde{E}; \tilde{\mathcal{O}}_L)$  and  $(u_1, \dots, u_m)$  an  $A$ -basis of  $U(E; \mathcal{O}_L)$ . Demeslay proved in [11] that the following product converges in  $\mathbb{T}_z(K_\infty)$ , the completion of  $K_\infty[z]$  with respect to the Gauss norm:

$$L(\tilde{E}/\tilde{\mathcal{O}}_L) = \prod_Q z_Q(\tilde{E}/\tilde{\mathcal{O}}_L)$$

where the product runs over all the monic irreducible polynomials  $Q$  of  $A$ . We call this product the  $L$ -series associated with  $\tilde{E}$  and  $\tilde{\mathcal{O}}_L$ . By evaluation at  $z = 1$  we obtain:

$$L(E/\mathcal{O}_L) = \text{ev}_{z=1} L(\tilde{E}, \tilde{\mathcal{O}}_L) = \prod_Q z_Q(E/\mathcal{O}_L) \in K_\infty^*.$$

We call this product the  $L$ -series associated with  $E$  and  $\mathcal{O}_L$ . Demeslay [11] proved the following class formulas à la Taelman

$$L(\tilde{E}/\tilde{\mathcal{O}}_L) = \frac{\det_{\mathcal{C}}(u_1(z), \dots, u_m(z))}{\text{sgn}(\det_{\mathcal{C}}(u_1(z), \dots, u_m(z)))}$$

and

$$L(E/\mathcal{O}_L) = \frac{\det_{\mathcal{C}}(u_1, \dots, u_m)}{\text{sgn}(\det_{\mathcal{C}}(u_1, \dots, u_m))} [H(E; \mathcal{O}_L)]_A.$$

We consider the  $P$ -adic setting. Let  $P$  be an irreducible monic polynomial of  $A$  and  $v_P$  its associated valuation on  $K$  such that  $v_P(P) = 1$ . We consider  $K_P \simeq \mathbb{F}_{q^{\deg(P)}}((P))$  (resp.  $A_P$ ) the completion of  $K$  (resp.  $A$ ) with respect to  $P$ . Consider the Tate algebra in the variable  $z$ ,  $\mathbb{T}_z(K_P)$ , that is the completion of  $K_P[z]$  with respect to the Gauss norm. Our first main result is the construction and the convergence of the following  $P$ -adic  $L$ -series. The key argument will be the evaluation at  $z = \zeta \in \mathbb{F}_q$ , see Subsections 3.4 and 4.3.

**Theorem A** (Theorem 4.10). *The following product converges in  $\mathbb{T}_z(K_P)$*

$$L_P(\tilde{E}/\tilde{\mathcal{O}}_L) = \prod_{Q \neq P} z_Q(\tilde{E}/\tilde{\mathcal{O}}_L)$$

where the product runs over all the monic irreducible polynomials  $Q$  of  $A$  different from  $P$ .

We then define a  $P$ -adic logarithm  $\text{Log}_{E,P}$  which converges on  $\{x \in \mathcal{O}_L^d \mid v_P(x) \geq 0\}$  and a  $P$ -adic regulator associated with the unit module as follows. Let  $(u_1, \dots, u_m)$  be an  $A$ -basis of the unit module. We then define the  $P$ -adic regulator of the unit module by:

$$R_P(U(E; \mathcal{O}_L)) = \frac{\det_{\mathcal{C}}(\text{Log}_{E,P}(\exp_E(u_1)), \dots, \text{Log}_{E,P}(\exp_E(u_m)))}{\text{sgn}(\det_{\mathcal{C}}(u_1, \dots, u_m))} \in K_P.$$

The construction does not depend on  $\mathcal{C}$  nor on the choice of the basis of the unit module. We do the same with the variable  $z$ . We then prove the following  $P$ -adic class formula à la Taelman.

**Theorem B** (Theorem 4.21). *Let  $E$  be an Anderson  $t$ -module defined over  $\mathcal{O}_L$ . Then we have the  $P$ -adic class formula for  $\widetilde{E}$ :*

$$z_P(\widetilde{E}/\widetilde{\mathcal{O}_L})L_P(\widetilde{E}/\widetilde{\mathcal{O}_L}) = R_P(U(\widetilde{E}; \widetilde{\mathcal{O}_L}))$$

and the class formula for  $E$ :

$$z_P(E/\mathcal{O}_L)L_P(E/\mathcal{O}_L) = R_P(U(E; \mathcal{O}_L)) [H(E; \mathcal{O}_L)]_A.$$

A major difference from the  $\infty$ -adic case is that our  $P$ -adic  $L$ -series obtained will vanish at  $z = 1$  in certain cases that we are able to characterize. Set  $U(E; P\mathcal{O}_L) = \{x \in \text{Lie}_E(L_\infty) \mid \exp_E(x) \in E(P\mathcal{O}_L)\}$  and  $\mathcal{U}(E; P\mathcal{O}_L) = \exp_E(U(E; P\mathcal{O}_L))$  which is provided with an  $A_P$ -structure module

**Theorem C** (Proposition 5.4 and Conjecture 4.7). *We have the following assertions.*

- (1) *If the exponential map  $\exp_E : L_\infty^d \rightarrow L_\infty^d$  is not injective, then  $L_P(E/\mathcal{O}_L) = 0$ .*
- (2) *If the  $A$ -rank of  $\exp_E(U(E; \mathcal{O}_L))$  and the  $A_P$ -rank of  $\mathcal{U}(E; P\mathcal{O}_L)$  coincide, then the two following assertions are equivalent:*
  - (a)  $L_P(E/\mathcal{O}_L) \neq 0$ ,
  - (b)  $\exp_E : L_\infty^d \rightarrow L_\infty^d$  is injective.

We extend the previous results to the multi-variable setting in the spirit of the work of Anglès, Pellarin and Tavares Ribeiro [5], Demeslay [11] and Pellarin [19]. Consider  $s \geq 1$  an integer and  $t_1, \dots, t_s$  new variables. Set  $k = \mathbb{F}_q(t_1, \dots, t_s)$ ,  $A_s = k[\theta]$ ,  $\mathcal{O}_{L,s} = k\mathcal{O}_L$  and  $K_{s,P} = \mathbb{F}_{q^{\deg(P)}}(t_1, \dots, t_s)((P))$ . Let  $s \geq 0$  be a non-negative integer and  $E$  be an Anderson  $A_s$ -module defined over  $\mathcal{O}_{L,s}$ . We consider the Tate algebra with multi-variable  $\mathbb{T}_s(K_P)$  that is the completion of  $K_P[t_1, \dots, t_s]$ . We prove the following result.

**Theorem D** (Theorem 5.8 and Theorem 5.3).

- (1) *The infinite product*

$$L_P(E/\mathcal{O}_{L,s}) = \prod_{Q \neq P} z_Q(E/\mathcal{O}_{L,s})$$

where  $Q$  runs through the monic primes of  $A$  different from  $P$ , converges in  $K_{s,P}$  and we have the class formula:

$$z_P(E/\mathcal{O}_{L,s})L_P(E/\mathcal{O}_{L,s}) = R_P(U(E; \mathcal{O}_{L,s})) [H(E; \mathcal{O}_{L,s})]_{A_s}.$$

- (2) *If  $E$  is defined over  $\mathbb{F}_q[t_1, \dots, t_s]\mathcal{O}_L$ , then  $L_P(E/\mathcal{O}_{L,s}) \in \mathbb{T}_s(K_P)$ .*

The key point for proving the second assertion will be the successive evaluation at  $t_i = \zeta_i \in \overline{\mathbb{F}_q}$  so using techniques from Subsections 3.4 and 4.3.

Finally, in Section 6, we give a detailed application of the previous theorems and obtain bounds on the vanishing order at  $z = 1$  of the  $P$ -adic  $L$ -series.

**Theorem E** (Proposition 6.7). *Consider  $\phi$  an  $A$ -Drinfeld module defined over  $A$  of rank  $r$ . Then the vanishing order at  $z = 1$  of the  $P$ -adic  $L$ -series is less than or equal to  $r$ .*

The  $P$ -adic  $L$ -series is studied in detail in a work of Caruso, Gazda and the author in the context of  $A$ -Drinfeld modules defined over  $A$ , see [9].

**1.3. Some remarks.** Recently, Anglès [3] defined the  $P$ -adic  $L$ -series  $L_P(\widetilde{C}/\widetilde{\mathcal{O}_L})$  associated with the Carlitz module defined over  $\mathcal{O}_L$ . He was able to prove that the  $P$ -adic  $L$ -series  $L_P(\widetilde{C}/\widetilde{\mathcal{O}_L})$  is a meromorphic series without pole at  $z = 1$ , see [3, Theorem 6.6]. In particular he defined the  $P$ -adic  $L$ -series as the limit at  $z = 1$  of this series which is an element of  $K_P$  and proved a  $P$ -adic class formula, see [3, Theorem 6.7]. The present work generalizes this result for Anderson  $t$ -modules, including all Drinfeld modules, by proving that the  $P$ -adic  $L$ -series is an element of  $\mathbb{T}_z(K_P)$  so that we can evaluate at  $z = 1$ .

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## 2. NOTATION AND BACKGROUND

**2.1. Notation.** We keep the notation in the Introduction and we introduce the following notation.

- $\mathbb{C}_\infty$ : the completion of a fixed algebraic closure of  $K_\infty$ ,
- $\tau : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$  the Frobenius endomorphism,
- $M_d(R) = M_{d \times d}(R)$ , for a ring  $R$  the left  $R$ -module of  $d \times d$  matrices,
- $I_d$ : the identity matrix of  $M_d(R)$ .

Let us fix an integer  $d \geq 1$  and  $B$  an  $\mathbb{F}_q$ -algebra. If  $M = (m_{i,j})$  is a matrix with coefficients in  $\mathbb{C}_\infty$  and  $k \in \mathbb{N}$ , we set  $\tau^k(M) = M^{(k)}$  to be the matrix whose  $ij$ -entry is given by  $\tau^k(m_{i,j})^{(k)} = m_{i,j}^{q^k}$ . We denote by  $M_d(B)\{\tau\}$  the non-commutative ring of twisted polynomials in  $\tau$  with coefficients in  $M_d(B)$  equipped with the usual addition and the commutation rule  $\tau^k M = M^{(k)} \tau^k$  for all  $k \in \mathbb{N}$  and all  $M \in M_d(B)$ . Let  $M_d(B)\{\{\tau\}\}$  be the non-commutative ring of twisted power series in  $\tau$  with coefficients in  $M_d(B)$ .

If  $k$  is a field containing  $\mathbb{F}_q$ , we set  $(kK)_\infty = k \hat{\otimes}_{\mathbb{F}_q} K_\infty = k((\frac{1}{\theta}))$ . If  $x \in (kK)_\infty^\times$ , we can write  $x$  uniquely as  $x = \sum_{n \geq N} x_n \frac{1}{\theta^n}$ ,  $x_n \in k$  and  $x_N \neq 0$ . Then we call  $x_N \in K$  the sign denoted by  $\text{sgn}(x)$ . We say that such an  $x \in (kK)_\infty$  is monic if  $\text{sgn}(x) = 1$ .

**2.2. Fitting ideals.** We recall here some definitions about Fitting ideals of modules over Dedekind rings. Let  $R$  be a Dedekind ring, and  $M$  be a finite and torsion  $R$ -module. By the structure theorem, there exists  $s \in \mathbb{N}$  and  $I_1, \dots, I_s$  ideals of  $R$  such that we have an isomorphism of  $R$ -modules

$$M \simeq R/I_1 \times \dots \times R/I_s.$$

We then define the Fitting ideal of  $M$  by

$$\text{Fitt}_R(M) = I_1 \dots I_s.$$

We have the following properties that can be found in the appendix of [17] except the second one which appears in [12, Corollary 20.5].

**Proposition 2.1.**

- (1) We have  $\text{Fitt}_R(M) \subseteq \text{Ann}_R(M) = \{x \in R \mid x.m = 0 \ \forall m \in M\}$ .
- (2) If  $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$  is exact, then  $\text{Fitt}_R(M_1) \text{Fitt}_R(M_2) = \text{Fitt}_R(M)$ .

**2.3. Lattices and Ratio of co-volumes.** We use the following notation from [22, Section 7.2.3]. We fix  $k$  a field containing  $\mathbb{F}_q$  and recall that  $(kK)_\infty = k \hat{\otimes}_{\mathbb{F}_q} K_\infty = k((\frac{1}{\theta}))$ . In what follows, we fix  $V$  to be a finite-dimensional  $(kK)_\infty$ -vector space endowed with the natural topology coming from  $(kK)_\infty$ .

**Definition 2.2.** A sub- $k[\theta]$ -module  $M$  of  $V$  is a  $k[\theta]$ -lattice in  $V$  if  $M$  is discrete in  $V$  and if  $M$  generates  $V$  over  $(kK)_\infty$ .

**Proposition 2.3.** Let  $M$  be a sub- $k[\theta]$ -module of  $V$ . The following are equivalent:

- (1)  $M$  is a  $k[\theta]$ -lattice in  $V$ .
- (2) There exists a  $(kK)_\infty$ -basis  $(v_1, \dots, v_n)$  of  $V$  such that  $M$  is the free  $k[\theta]$ -module of basis  $(v_1, \dots, v_n)$ .

*Proof.* See [22, Proposition 7.2.3].  $\square$

Let  $M$  and  $M'$  be two  $k[\theta]$ -lattices in  $V$ . Let  $\mathcal{B}$  and  $\mathcal{B}'$  be  $k[\theta]$ -basis of  $M$  and  $M'$ , respectively. The ratio of co-volumes of  $M$  in  $M'$  is then defined as

$$[M' : M]_{k[\theta]} = \frac{\det_{\mathcal{B}'} \mathcal{B}}{\text{sgn}(\det_{\mathcal{B}'} \mathcal{B})} \in (kK)_\infty^*.$$

Note that this is independent of the choices of  $\mathcal{B}$  and  $\mathcal{B}'$ . The definition immediately implies that if  $M_0, M_1$  and  $M_2$  are lattices in  $V$ , then

$$[M_0 : M_1]_{k[\theta]} [M_1 : M_2]_{k[\theta]} = [M_0 : M_2]_{k[\theta]}.$$

We also observe that for two lattices  $M, M'$  in  $V$  we have

$$[M' : M]_{k[\theta]} = [M : M']_{k[\theta]}^{-1}.$$

### 3. THE $\infty$ -ADIC CASE

From now on, let  $L/K$  be a finite fields extension. Recall that we denote by:  $\mathcal{O}_L$  the integral closure of  $A$  in  $L$ ,  $\mathcal{O}_L[z] \simeq \mathbb{F}_q[z] \otimes_{\mathbb{F}_q} \mathcal{O}_L$ ,  $\widetilde{\mathcal{O}}_L = \mathbb{F}_q(z) \otimes_{\mathbb{F}_q} \mathcal{O}_L$ ,  $\widetilde{L}_\infty = L \otimes_{\mathbb{F}_q} \widetilde{K}_\infty$ . In this section, we extend the notion of the Taelman unit module and class module by twisting with some elements  $\zeta \in \overline{\mathbb{F}_q}$ .

**3.1. Anderson modules.** An Anderson  $t$ -module (or shortly a  $t$ -module)  $E$  of dimension  $d$  defined over  $\mathcal{O}_L$  is an  $\mathbb{F}_q$ -algebra homomorphism  $E : A \rightarrow M_d(\mathcal{O}_L)\{\tau\}$  such that if  $a \in A$  and  $E_a = \sum_{i=0}^{r_a} E_{a,i} \tau^i$ , then we require  $(E_{a,0} - aI_d)^d = 0$  and that  $\deg_\tau(E_\theta) > 0$ . Let  $E : A \rightarrow M_d(\mathcal{O}_L)\{\tau\}$  be a  $t$ -module of dimension  $d \geq 1$ , completely determined by the value at  $\theta$ :

$$E_\theta = \sum_{i=0}^r E_{\theta,i} \tau^i$$

with  $E_{\theta,i} \in M_d(\mathcal{O}_L)$  and  $(E_{\theta,0} - \theta I_d)^d = 0$ . For  $a \in A$ , if  $E_a = \sum_{i=0}^{r_a} E_{a,i} \tau^i$ , we define  $\partial_E(a) = E_{a,0}$ . Then the map  $\partial_E : A \rightarrow M_d(A)$  is a homomorphism of  $\mathbb{F}_q$ -algebras.

We then consider the  $z$ -twist of  $E$ , introduced in [7], (remember that  $\tau$  acts as the identity over  $\mathbb{F}_q(z)$ ) denoted by  $\widetilde{E}$  to be the homomorphism of  $\mathbb{F}_q(z)$ -algebras  $\widetilde{E} : \widetilde{A} \rightarrow M_d(\widetilde{\mathcal{O}}_L)\{\tau\}$  given by:

$$\widetilde{E}_\theta = \sum_{i=0}^r E_{\theta,i} z^i \tau^i.$$

Recall the following notation taken from [4]. Let  $E$  be a  $t$ -module of dimension  $d$  over  $R$  an extension of  $\mathbb{F}_q$  and let  $B$  be an  $R$ -algebra. We can then define two  $A$ -module structures on  $B^d$ . The first is denoted  $E(B)$  where  $A$  acts on  $B^d$  via  $E$ :

$$a.x = E_a(x) \in B^d \text{ for all } a \in A, x \in B^d.$$

The second is  $\text{Lie}_E(B)$  where  $A$  acts on  $B^d$  via  $\partial_E$ :

$$a.x = \partial_E(a)x \text{ for all } a \in A, x \in B^d.$$

We have the following results that can be found in [1, Proposition 2.1.4].

**Proposition 3.1.** *There exists a unique element  $\exp_E \in M_d(L)\{\{\tau\}\}$  such that:*

- (1)  $\exp_E \partial_E(a) = E_a \exp_E$  hold in  $M_d(L)\{\{\tau\}\}$  for all  $a \in A$ ,

$$(2) \exp_E \equiv I_d \text{ mod } M_d(L)\{\{\tau\}\}\tau.$$

We call  $\exp_E$  the exponential map associated with the  $t$ -module  $E$ , and denote this element by  $\exp_E = \sum_{n=0}^{\infty} d_n \tau^n$ .

**Proposition 3.2.** *There exists a unique element  $\log_E \in M_d(L)\{\{\tau\}\}$  such that:*

- (1)  $\log_E E_a = \partial_E(a) \log_E$  hold in  $M_d(L)\{\{\tau\}\}$  for all  $a \in A$ ,
- (2)  $\log_E \equiv I_d \text{ mod } M_d(L)\{\{\tau\}\}\tau$ .

In addition, we have the equalities in  $M_d(L)\{\{\tau\}\}$ :

$$\log_E \exp_E = \exp_E \log_E = I_d.$$

We call  $\log_E$  the logarithm map associated to the  $t$ -module  $E$ , and we denote this element by  $\log_E = \sum_{n=0}^{\infty} l_n \tau^n$ . We also have exponential and logarithm series for the  $z$ -twist of the  $t$ -module  $\tilde{E}$  which we denote by  $\exp_{\tilde{E}}$  and  $\log_{\tilde{E}}$  and given by:

$$\exp_{\tilde{E}} = \sum_{n \geq 0} d_n z^n \tau^n \text{ and } \log_{\tilde{E}} = \sum_{n \geq 0} l_n z^n \tau^n.$$

**3.2. Unit module and class module.** We consider an over-additive valuation  $v_{\infty}$  on the finite dimensional  $K_{\infty}$ -vector space  $L_{\infty}$  (for example with respect to the choice of a basis of  $L/K$ ). The key point is the next result from [15, Theorem 4.6.9].

**Lemma 3.3.** *We have*

$$\lim_{i \rightarrow +\infty} \frac{v_{\infty}(d_i)}{q^i} = +\infty.$$

**Corollary 3.4.** *The exponential map  $\exp_E$  converges on  $\text{Lie}_E(L_{\infty})$  and induces a homomorphism of  $A$ -modules:*

$$\exp_E : \text{Lie}_E(L_{\infty}) \rightarrow E(L_{\infty}).$$

We also have the convergence of  $\exp_{\tilde{E}}$  on  $\text{Lie}_{\tilde{E}}(\widetilde{L_{\infty}})$  (resp.  $\text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_{\infty}))$ ) that induces a homomorphism of  $\tilde{A}$ -modules (resp.  $A[z]$ ):

$$\exp_{\tilde{E}} : \text{Lie}_{\tilde{E}}(\widetilde{L_{\infty}}) \rightarrow \tilde{E}(\widetilde{L_{\infty}}).$$

(resp.  $\exp_{\tilde{E}} : \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_{\infty})) \rightarrow \tilde{E}(\mathbb{T}_z(L_{\infty}))$ ). We can now define the Taelman unit module

$$U(E; \mathcal{O}_L) = \{x \in \text{Lie}_E(L_{\infty}) \mid \exp_E(x) \in E(\mathcal{O}_L)\}$$

provided with  $A$ -module structure, as well as the module of  $z$ -units:

$$U(\tilde{E}; \widetilde{\mathcal{O}_L}) = \left\{x \in \text{Lie}_{\tilde{E}}(\widetilde{L_{\infty}}) \mid \exp_{\tilde{E}}(x) \in \tilde{E}(\widetilde{\mathcal{O}_L})\right\}$$

provided with  $\tilde{A}$ -module structure, and the module of  $z$ -units at the integral level:

$$U(\tilde{E}; \mathcal{O}_L[z]) = \left\{x \in \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_{\infty})) \mid \exp_{\tilde{E}}(x) \in \tilde{E}(\mathcal{O}_L[z])\right\}$$

provided with  $A[z]$ -module structure. We also define the class module (introduced by Taelman in [21]):

$$H(E; \mathcal{O}_L) = \frac{E(L_{\infty})}{E(\mathcal{O}_L) + \exp_E(\text{Lie}_E(L_{\infty}))}$$

as well as the class module for the  $z$ -deformation

$$H(\tilde{E}; \widetilde{\mathcal{O}_L}) = \frac{\tilde{E}(\widetilde{L_{\infty}})}{\tilde{E}(\widetilde{\mathcal{O}_L}) + \exp_{\tilde{E}}(\text{Lie}_{\tilde{E}}(\widetilde{L_{\infty}}))}$$

and finally the class module at the integral level

$$H(\tilde{E}; \mathcal{O}_L[z]) = \frac{\tilde{E}(\mathbb{T}_z(L_\infty))}{\tilde{E}(\mathcal{O}_L[z]) + \exp_{\tilde{E}}(\text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)))}.$$

Consider one of the following cases:

- $k_0 = \mathbb{F}_q$ ,  $\varphi = E$  and  $(k_0 L)_\infty = L_\infty$ ,
- $k_0 = \mathbb{F}_q[z]$  and  $\varphi = \tilde{E}$ ,
- $k_0 = \mathbb{F}_q(z)$ ,  $\varphi = \tilde{E}$  and  $(k_0 L)_\infty = \widetilde{L_\infty}$ .

We have the following result from [11, Proposition 2.8].

**Proposition 3.5.**

- (1) *The class module  $H(\varphi; k_0 \mathcal{O}_L)$  is a finite-dimensional  $k_0$ -vector space, so a finite and torsion  $k_0 A$ -module.*
- (2) *If  $k_0$  is a field, then the module of units  $U(\varphi; k_0 \mathcal{O}_L)$  is a  $k_0 A$ -lattice in  $\text{Lie}_\varphi((k_0 L)_\infty)$ .*

We also have the following result in [7, Proposition 2.3].

**Proposition 3.6.** *We have the following equality:*

$$U(\tilde{E}; \widetilde{\mathcal{O}_L}) = \mathbb{F}_q(z)U(\tilde{E}; \mathcal{O}_L[z]).$$

Consider the evaluation morphism:

$$\text{ev}_{z=1} : \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \rightarrow \text{Lie}_E(L_\infty).$$

It induces an exact sequence of  $A$ -modules:

$$0 \longrightarrow (z-1) \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \longrightarrow \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \xrightarrow{\text{ev}_{z=1}} \text{Lie}_E(L_\infty) \longrightarrow 0.$$

For all  $x \in \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty))$  we have  $\text{ev}_{z=1}(\exp_{\tilde{E}}(x)) = \exp_E(\text{ev}_{z=1}(x))$ . Moreover, if  $f(z) \in \widetilde{L_\infty}$  belongs to the  $\infty$ -adic convergence domain of the logarithm map  $\log_{\tilde{E}}$ , then we have

$$\text{ev}_{z=1}(\log_{\tilde{E}}(f(z))) = \log_E(\text{ev}_{z=1}(f(z))).$$

We recall the notion of Stark units introduced by B. Anglès and F. Tavares Ribeiro in [7, section 2.5].

**Definition 3.7.** The module of Stark units  $U_{\text{St}}(E; \mathcal{O}_L)$  is defined by:

$$U_{\text{St}}(E; \mathcal{O}_L) = \text{ev}_{z=1} U(\tilde{E}; \mathcal{O}_L[z]).$$

Given the compatibility between the exponential and the evaluation morphism,  $U_{\text{St}}(E; \mathcal{O}_L)$  is a sub- $A$ -module of  $U(E, \mathcal{O}_L)$ . We have the following result from [7, Theorem 1].

**Theorem 3.8.** *The  $A$ -module  $U_{\text{St}}(E; \mathcal{O}_L)$  is an  $A$ -lattice in  $\text{Lie}_E(L_\infty)$ .*

**3.3. The  $L$  series.** For a monic prime  $P$  of  $A$ , we define the local factor at  $P$  associated with  $E$ :

$$z_P(E/\mathcal{O}_L) = \frac{[\text{Lie}_E(\mathcal{O}_L/P\mathcal{O}_L)]_A}{[E(\mathcal{O}_L/P\mathcal{O}_L)]_A} \in K$$

and the local factor at  $P$  associated with  $\tilde{E}$ :

$$z_P(\tilde{E}/\widetilde{\mathcal{O}_L}) = \frac{[\text{Lie}_{\tilde{E}}(\widetilde{\mathcal{O}_L}/P\widetilde{\mathcal{O}_L})]_{\tilde{A}}}{[\tilde{E}(\widetilde{\mathcal{O}_L}/P\widetilde{\mathcal{O}_L})]_{\tilde{A}}} \in \tilde{K}.$$

We then define the  $L$ -series associated with  $E$  and  $\mathcal{O}_L$  by the Eulerian product:

$$L(E/\mathcal{O}_L) = \prod_{P \in A} z_P(E/\mathcal{O}_L)$$



where  $P$  runs through the monic primes of  $A$ , and the  $L$ -series associated with  $\tilde{E}$  and  $\tilde{\mathcal{O}}_L$  by the Eulerian product:

$$L(\tilde{E}/\tilde{\mathcal{O}}_L) = \prod_{P \in A} z_P(\tilde{E}/\tilde{\mathcal{O}}_L).$$

We have the convergence of the  $L$ -series and the class formula for  $z$ -deformation from [11, Theorem 2.7].

**Theorem 3.9** (Class formula for the  $z$ -deformation). *The product defining  $L(\tilde{E}/\tilde{\mathcal{O}}_L)$  converges in  $\widetilde{K_\infty}^*$  and we have the formula*

$$(1) \quad L(\tilde{E}/\tilde{\mathcal{O}}_L) = \left[ \text{Lie}_{\tilde{E}}(\tilde{\mathcal{O}}_L) : U(\tilde{E}; \tilde{\mathcal{O}}_L) \right]_{\tilde{A}}.$$

Adapting the proof of [22, Corollary 7.5.6] in the higher dimensional case we obtain that the polynomial  $\left[ \text{Fitt}_{\tilde{A}}(\tilde{E}(\tilde{\mathcal{O}}_L/P\tilde{\mathcal{O}}_L)) \right]_{\tilde{A}} \in A[z]$  is a unit in  $\mathbb{T}_z(K_\infty)$ . We then obtain:

**Corollary 3.10.** *The  $L$ -series  $L(\tilde{E}/\tilde{\mathcal{O}}_L)$  converges in  $\mathbb{T}_z(K_\infty)^\times$ .*

We can evaluate the  $L$ -series at  $z = 1$ :

$$L(E/\mathcal{O}_L) = \text{ev}_{z=1} L(\tilde{E}/\tilde{\mathcal{O}}_L) = \prod_Q \frac{[\text{Lie}_E(\mathcal{O}_L/Q\mathcal{O}_L)]_A}{[E(\mathcal{O}_L/Q\mathcal{O}_L)]_A} \in K_\infty^*$$

where  $Q$  runs through the monic primes of  $A$ . We have the following class formula for  $t$ -modules obtained by Fang in [13], generalizing Taelman's class formula for Drinfeld modules.

**Theorem 3.11** (Class formula for Anderson  $t$ -modules). *The product defining  $L(E/\mathcal{O}_L)$  converges in  $K_\infty^*$ , and we have the equalities*

$$(2) \quad L(E/\mathcal{O}_L) = [\text{Lie}_E(\mathcal{O}_L) : U(E; \mathcal{O}_L)]_A [H(E; \mathcal{O}_L)]_A = [\text{Lie}_E(\mathcal{O}_L) : U_{\text{st}}(E; \mathcal{O}_L)]_A.$$

**3.4. Evaluation at  $z = \zeta \in \overline{\mathbb{F}}_q$ .** We want to extend the notion of Stark units by evaluating the variable  $z$  at  $z = \zeta$  for all  $\zeta \in \overline{\mathbb{F}}_q$ .

Let  $\zeta$  be an element of  $\overline{\mathbb{F}}_q$  and consider  $\mathbb{F}_q(\zeta)$  the finite field obtained by adding  $\zeta$  to  $\mathbb{F}_q$ . Let us define the ring  $A_\zeta = \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} A$ . We define a Frobenius  $\tau_\zeta = \text{id} \otimes \tau$  acting on  $A_\zeta$ . Let us define  $\widetilde{A}_\zeta = \mathbb{F}_q(z) \otimes_{\mathbb{F}_q} A_\zeta$  on which we extend the Frobenius  $\tau_\zeta$  by  $\tau_\zeta = \text{id} \otimes \tau_\zeta$ , still denoted by  $\tau_\zeta$  (i.e., the Frobenius  $\tau_\zeta$  acts as the identity on  $\mathbb{F}_q(z)$ ). Denote by  $A_\zeta[z] = \mathbb{F}_q[z] \otimes_{\mathbb{F}_q} A_\zeta$ . Set  $\mathcal{O}_{L,\zeta} = \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathcal{O}_L$ . It is also equipped with the following Frobenius  $\tau_\zeta = \text{id} \otimes \tau$ .

Similarly as the  $z$ -deformation, let us twist the  $t$ -module  $E$  into an Anderson  $A_\zeta$ -module  $E_\zeta$  defined over  $M_d(\mathcal{O}_{L,\zeta})$  by

$$(E_\zeta)_\theta = \sum_{i=0}^r E_{\theta,i} \zeta^i \tau_\zeta^i \in M_d(\mathcal{O}_{L,\zeta}) \{ \tau_\zeta \}$$

then extend to  $A_\zeta$  by  $\mathbb{F}_q(\zeta)$ -linearity.

Set  $M_w = \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} L_w$  where  $w = \infty$  or  $w = P$ . Consider  $\mathbb{F}_q[z] \otimes_{\mathbb{F}_q} \mathcal{O}_{L,\zeta} = \mathcal{O}_{L,\zeta}[z]$  then set  $\widetilde{\mathcal{O}}_{L,\zeta} = \mathbb{F}_q(z) \otimes_{\mathbb{F}_q} \mathcal{O}_{L,\zeta}$  and  $\widetilde{M}_w = \mathcal{O}_{L,\zeta}[z] \otimes_{\mathcal{O}_L[z]} \mathbb{T}_z(L_w) \simeq \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathbb{T}_z(L_w)$  and consider  $\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \widetilde{L}_w$ . We extend  $v_\infty$  to  $M_\infty$  as follows. Let's fix  $(f_1, \dots, f_m)$  a  $\mathbb{F}_q$ -basis of  $\mathbb{F}_q(\zeta)$ . We set

$$v_\infty \left( \sum_{i=1}^m f_i \otimes x_i \right) = \min_{i=1, \dots, h} v_\infty(x_i)$$

for  $x_i \in K_\infty$ . The topology over  $M_\infty$  does not depend on the choice of the basis  $(f_1, \dots, f_m)$ . We then consider  $v_\infty$  an over-additive valuation on the  $\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} K_\infty$ -vector space of finite dimension  $M_\infty$ . We then extend similarly  $v_\infty$  to  $\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \widetilde{L}_\infty$ . Remark that we cannot just replace  $v_\infty$  by  $v_P$  on  $\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} K_P$  with theses constructions, in fact we do not obtain a valuation over  $M_{v_P}$ . See Subsection 4.3 for more details.

Finally, we deform  $E$  into  $E^{(\zeta)}$  an Anderson  $A_\zeta$ -module on  $M_d(\mathcal{O}_{L,\zeta})$  by

$$E_\theta^{(\zeta)} = \sum_{i=0}^r E_{\theta,i} \tau_\zeta^i$$

and extend it to  $A_\zeta$  by  $\mathbb{F}_q(\zeta)$ -linearity. We finally extend it to an Anderson  $\widetilde{A}_\zeta$ -module  $\widetilde{E}^{(\zeta)}$  on  $M_d(\widetilde{\mathcal{O}_{L,\zeta}})$  in the usual way.

We have exponential maps associated with each of the Anderson modules. From the definitions we have the equalities

$$\exp_{E^{(\zeta)}} = \sum_{n \geq 0} d_n \tau_\zeta^n \text{ and } \log_{E^{(\zeta)}} = \sum_{n \geq 0} l_n \tau_\zeta^n,$$

and the map  $\exp_{E^{(\zeta)}}$  (resp.  $\exp_{\widetilde{E}^{(\zeta)}}$ ) converges on  $\text{Lie}_{E^{(\zeta)}}(M_\infty)$  (resp. on  $\text{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M}_\infty)$ ). Moreover, we have the following equalities in  $M_d(\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} L) \{\{\tau_\zeta\}\}$ :

$$\exp_{E_\zeta} = \sum_{n \geq 0} d_n \zeta^n \tau_\zeta^n \text{ and } \log_{E_\zeta} = \sum_{n \geq 0} l_n \zeta^n \tau_\zeta^n.$$

Consider the evaluation morphism at  $z = \zeta$ :

$$\text{ev}_\zeta = \text{ev}_{z=\zeta} : \widetilde{M}_\infty \rightarrow M_\infty$$

whose kernel is given by  $(z - \zeta)\widetilde{M}_\infty$ , then we consider the following evaluation morphism still denoted by  $\text{ev}_\zeta$ :

$$\text{ev}_\zeta : \text{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M}_\infty) \rightarrow \text{Lie}_{E^{(\zeta)}}(M_\infty).$$

For  $x \in \text{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M}_\infty)$ , we have in  $\text{Lie}_{E^{(\zeta)}}(M_\infty)$ :

$$\text{ev}_\zeta(\exp_{\widetilde{E}^{(\zeta)}}(x)) = \exp_{E_\zeta}(\text{ev}_\zeta(x)).$$

Let us consider the module of  $\zeta$ -units at the integral level:

$$U(\widetilde{E}^{(\zeta)}; \mathcal{O}_{L,\zeta}[z]) = \left\{ x \in \text{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M}_\infty) \mid \exp_{\widetilde{E}^{(\zeta)}}(x) \in \widetilde{E}(\mathcal{O}_{L,\zeta}[z]) \right\}$$

as well as the module of the  $\zeta$ -classes at the integral level:

$$H(\widetilde{E}^{(\zeta)}; \mathcal{O}_{L,\zeta}[z]) = \frac{\widetilde{E}^{(\zeta)}(\widetilde{M}_\infty)}{\widetilde{E}^{(\zeta)}(\mathcal{O}_{L,\zeta}[z]) + \exp_{\widetilde{E}^{(\zeta)}}(\text{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M}_\infty))} = A_\zeta[z] \otimes_{\mathbb{F}_q[z]} H(\widetilde{E}, \mathcal{O}_L[z])$$

provided with a structure of  $A_\zeta[z]$ -modules. Next, consider the  $\zeta$ -unit module:

$$U(E_\zeta; \mathcal{O}_{L,\zeta}) = \{x \in \text{Lie}_{E_\zeta}(M_\infty) \mid \exp_{E_\zeta}(x) \in E_\zeta(\mathcal{O}_{L,\zeta})\}$$

and the  $\zeta$ -class module

$$H(E_\zeta, \mathcal{O}_{L,\zeta}) = \frac{E_\zeta(M_\infty)}{E_\zeta(\mathcal{O}_{L,\zeta}) + \exp_{E_\zeta}(\text{Lie}_{E_\zeta}(M_\infty))}$$

provided with their  $A_\zeta$ -module structure via  $E_\zeta$ .

Results to come in this section are adapted from [11] and [7].

**Proposition 3.12.**

- (1) The exponential map  $\exp_{E_\zeta} : \text{Lie}_{E_\zeta}(M_\infty) \rightarrow E_\zeta(M_\infty)$  is locally an isometry.
- (2) The exponential map  $\exp_{\widetilde{E}^{(\zeta)}} : \text{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M}_\infty) \rightarrow \widetilde{E}^{(\zeta)}(\widetilde{M}_\infty)$  is locally an isometry.

*Proof.* The proof is a direct corollary of Lemma 3.3, we omit the proof.  $\square$

**Proposition 3.13.**

- (1) The module of  $\zeta$ -classes  $H(E_\zeta; \mathcal{O}_{L,\zeta})$  is a  $\mathbb{F}_q(\zeta)$ -vector space of finite dimension, hence a torsion  $A_\zeta$ -module of finite type.
- (2) The module of  $\zeta$ -units  $U(E_\zeta; \mathcal{O}_{L,\zeta})$  is an  $A_\zeta$ -lattice in  $M_\infty$ .

(3) The class module  $H(\tilde{E}^{(\zeta)}; \mathcal{O}_{L,\zeta}[z])$  is a  $\mathbb{F}_q(\zeta)[z]$ -module of finite type.

*Proof.* The proof follows the proof of [11, Proposition 2.6] for the two first assertions, and the proof of [7, Proposition 2] for the last one, by replacing  $A$  by  $A_\zeta$ ,  $\mathcal{O}_L$  by  $\mathcal{O}_{L,\zeta}$  and  $E$  by  $E_\zeta$ . We omit the details.  $\square$

Just as Stark's units consist of the evaluation at  $z = 1$  of the  $z$ -units, we define the evaluation at  $z = \zeta$  of the  $\zeta$ -units at the integral level:

$$U_\zeta(E; \mathcal{O}_L) = \text{ev}_\zeta U(\tilde{E}^{(\zeta)}; \mathcal{O}_{L,\zeta}[z]) \subseteq U(E_\zeta; \mathcal{O}_{L,\zeta})$$

provided with an  $A_\zeta$ -module structure via  $E_\zeta$ .

**Theorem 3.14.** *There exists an  $A_\zeta$ -module isomorphism:*

$$\frac{U(E_\zeta; \mathcal{O}_{L,\zeta})}{U_\zeta(E; \mathcal{O}_L)} \simeq H(\tilde{E}^{(\zeta)}; \mathcal{O}_{L,\zeta}[z])[z - \zeta]$$

where  $H(\tilde{E}^{(\zeta)}; \mathcal{O}_{L,\zeta}[z])[z - \zeta]$  is the  $(z - \zeta)$ -torsion of the  $\zeta$ -class module at the integral level.

In the following, we will denote by  $M = \mathcal{O}_{L,\zeta}$  and  $\tilde{M} = \mathcal{O}_{L,\zeta}[z]$ .

*Proof.* We follow the proof of [7, Proposition 2.6].

Consider the map

$$\begin{aligned} \alpha : M_\infty^d &\rightarrow \tilde{M}_\infty^d \\ x &\mapsto \frac{\exp_{\tilde{E}^{(\zeta)}}(x) - \exp_{E_\zeta}(x)}{z - \zeta}. \end{aligned}$$

We divide the proof into several steps.

Step 1: The map is well defined since

$$\text{ev}_\zeta(\exp_{\tilde{E}^{(\zeta)}}(x)) = \exp_{E_\zeta}(x)$$

for  $x \in M_\infty^d$ , thus  $(z - \zeta)$  divide  $\exp_{\tilde{E}^{(\zeta)}}(x) - \exp_{E_\zeta}(x)$  in  $\tilde{M}_\infty^d$ .

Step 2: We still denote  $\alpha$  to be the restriction:  $\alpha : U(E_\zeta, M) \rightarrow H(\tilde{E}^{(\zeta)}; \tilde{M})$ . Let us prove that it is a homomorphism of  $A_\zeta$ -modules. Let  $x \in U(E_\zeta; M)$  be a unit and  $a \in A_\zeta$ . Then:

$$\begin{aligned} (z - \zeta)\alpha(ax) &= \exp_{\tilde{E}^{(\zeta)}}(ax) - \exp_{E_\zeta}(ax) \\ &= \tilde{E}_a^{(\zeta)}(\exp_{\tilde{E}^{(\zeta)}}(x)) - (E_\zeta)_a(\exp_{E_\zeta}(x)) \\ &= \sum_{i=0}^{r_a} E_{a,i} z^i \tau_\zeta^i(\exp_{\tilde{E}^{(\zeta)}}(x)) - \sum_{i=0}^{r_a} E_{a,i} \zeta^i \tau_\zeta^i(\exp_{E_\zeta}(x)) \\ &= \sum_{i=0}^{r_a} E_{a,i} z^i \tau_\zeta^i(\exp_{\tilde{E}^{(\zeta)}}(x) - \exp_{E_\zeta}(x)) + \sum_{i=1}^{r_a} E_{a,i} (z^i - \zeta^i) \tau_\zeta^i(\exp_{E_\zeta}(x)). \end{aligned}$$

Thus

$$\alpha(ax) = \tilde{E}_a^{(\zeta)}(\alpha(x)) + \underbrace{\sum_{i=0}^h a_i \frac{z^i - \zeta^i}{z - \zeta} \tau_\zeta^i(\exp_{E_\zeta}(x))}_{\in \tilde{M}^d}.$$

We have proved that  $\alpha(ax) = \tilde{E}_a^{(\zeta)}(\alpha(x)) \bmod \left( \tilde{M}^d + \exp_{\tilde{E}^{(\zeta)}}(\text{Lie}_{\tilde{E}^{(\zeta)}}(\tilde{M}_{s,\infty})) \right)$ , so  $\alpha(ax) = \tilde{E}_a^{(\zeta)}(\alpha(x))$  in  $H(\tilde{E}^{(\zeta)}, \tilde{M})$ .

Step 3: We claim that the image of  $U(E_\zeta, M)$  is in the  $(z - \zeta)$ -torsion of the  $\zeta$ -class module at the integral level. In fact, let  $x \in U(E_\zeta; M)$  be a unit. We have:

$$(z - \zeta)\alpha(x) = \exp_{\tilde{E}^{(\zeta)}}(x) - \exp_{E_\zeta}(x) = 0 \bmod \left( E_\zeta(M) + \exp_{\tilde{E}^{(\zeta)}}(\text{Lie}_{\tilde{E}^{(\zeta)}}(\tilde{M}_\infty)) \right).$$

Step 4: Let us prove the surjectivity of  $\alpha$  on  $H(\tilde{E}^{(\zeta)}; \tilde{M})[z - \zeta]$ . Let  $x \in \tilde{E}^{(\zeta)}(\tilde{M}_\infty)$  be such that

$$(z - \zeta)x = \exp_{\tilde{E}^{(\zeta)}}(u) + v$$

with  $u \in \text{Lie}_{\tilde{E}^{(\zeta)}}(\tilde{M}_\infty)$  and  $v \in \tilde{E}^{(\zeta)}(\tilde{M})$ . We write  $u = u_1 + (z - \zeta)u_2$ ,  $u_1 \in M_\infty^d$ ,  $u_2 \in \tilde{M}_\infty^d$  and  $v = v_1 + (z - \zeta)v_2$ ,  $v_1 \in M^d$ ,  $v_2 \in \tilde{M}^d$ . We have:

$$(z - \zeta)x = \exp_{\tilde{E}^{(\zeta)}}(u_1) + v_1 + (z - \zeta)(v_2 + \exp_{\tilde{E}^{(\zeta)}}(u_2)).$$

By evaluating at  $z = \zeta$  yields  $\exp_{E_\zeta}(u_1) + v_1 = 0$ . Thus  $u_1 \in U(E_\zeta; M)$ . Moreover, we have:

$$\alpha(u_1) = x - \underbrace{(\exp_{\tilde{E}^{(\zeta)}}(u_2))}_{\in \exp_{\tilde{E}^{(\zeta)}}(\tilde{M}_\infty^d)} + \underbrace{v_2}_{\in \tilde{M}^d}$$

thus  $\alpha(u_1) = x \bmod (\tilde{M}^d + \exp_{\tilde{E}^{(\zeta)}}(\tilde{M}_\infty^d))$ .

Step 5: We now consider the kernel of  $\alpha : U(E_\zeta; M) \rightarrow H(\tilde{E}^{(\zeta)}; \tilde{M})$  denoted by  $\kappa$ . We want to prove that  $\kappa = U_\zeta(E, \mathcal{O}_L)$ . We proceed by double inclusion.

$\supseteq$  Let  $x \in U_\zeta(E, \mathcal{O}_L)$  be a unit and write  $x = \text{ev}_\zeta(u)$  with  $u \in U(\tilde{E}^{(\zeta)}; \tilde{M})$ . We have  $\text{ev}_\zeta(x - u) = 0$  thus we can find  $v \in \tilde{M}_\infty^d$  such that

$$x = u + (z - \zeta)v.$$

We have

$$\alpha(x) = \frac{\exp_{\tilde{E}^{(\zeta)}}(u) - \exp_{E_\zeta}(x)}{z - \zeta} + \exp_{\tilde{E}^{(\zeta)}}(v).$$

But  $\exp_{E_\zeta}(x) = \text{ev}_\zeta \exp_{\tilde{E}^{(\zeta)}}(u) \in M^d$  so  $\alpha(x) = 0 \bmod (\tilde{M}^d + \exp_{\tilde{E}^{(\zeta)}}(\tilde{M}_\infty^d))$ .

$\subseteq$  Let  $x \in U(E_\zeta; M)$  be such that  $\alpha(x) \in \tilde{M}^d + \exp_{\tilde{E}^{(\zeta)}}(\tilde{M}_\infty^d)$ . Let us express  $\alpha(x) = u + \exp_{\tilde{E}^{(\zeta)}}(v)$ . We have

$$(z - \zeta)\alpha(x) = \exp_{\tilde{E}^{(\zeta)}}(x) + \exp_{E_\zeta}(x) = (z - \zeta)u + \exp_{\tilde{E}^{(\zeta)}}((z - \zeta)v).$$

Thus  $\exp_{\tilde{E}^{(\zeta)}}(x - (z - \zeta)v) = (z - \zeta)u + \exp_{E_\zeta}(x) \in \tilde{M}^d$  so  $x - (z - \zeta)v \in U(\tilde{E}^{(\zeta)}; \tilde{M})$ . Finally we obtain

$$\text{ev}_\zeta(x - (z - \zeta)v) = x \in U_\zeta(E; \mathcal{O}_L).$$

□

**Corollary 3.15.** *The unit module  $U_\zeta(E; \mathcal{O}_L)$  is an  $A_\zeta$ -lattice in  $M_\infty^d$ . Moreover, we have the following equalities*

$$\left[ H(\tilde{E}^{(\zeta)}; \tilde{M})[z - \zeta] \right]_{A_\zeta} = [H(E_\zeta; M)]_{A_\zeta} = \left[ \frac{U(E_\zeta; M)}{U_\zeta(E; \mathcal{O}_L)} \right]_{A_\zeta}.$$

Let us start by proving the following result.

**Lemma 3.16.** *We have an exact sequence of  $A_\zeta$ -modules:*

$$0 \longrightarrow (z - \zeta)H(\tilde{E}^{(\zeta)}; \tilde{M}) \longrightarrow H(\tilde{E}^{(\zeta)}; \tilde{M}) \xrightarrow{\text{ev}_\zeta} H(E_\zeta; M) \longrightarrow 0.$$

*Proof of Lemma 3.16.* We apply the snake lemma to the following commutative diagram (where the lines are exact sequences of  $A_\zeta$ -modules) and the  $i_j$  represent natural injections:

$$\begin{array}{ccccc} (z - \zeta)(\exp_{\tilde{E}^{(\zeta)}}(\tilde{M}_\infty^d) + \tilde{M}^d) & \longrightarrow & \exp_{\tilde{E}^{(\zeta)}}(\tilde{M}_\infty^d) + \tilde{M}^d & \xrightarrow{\text{ev}_{z=\zeta}} & \exp_{E_\zeta}(M_\infty^d) + M^d \\ \downarrow i_1 & & \downarrow i_2 & & \downarrow i_3 \\ (z - \zeta)\tilde{M}_\infty^d & \longrightarrow & \tilde{M}_\infty^d & \xrightarrow{\text{ev}_{z=\zeta}} & M_\infty^d \end{array}$$

□

*Proof of Corollary 3.15.* We deduce by Lemma 3.16 an exact sequence of  $\mathbb{F}_q(\zeta)$ -vector spaces of finite dimension (and of finitely-generated  $A_\zeta$ -modules):

$$0 \longrightarrow H(\widetilde{E}^{(\zeta)}; \widetilde{M})[z - \zeta] \longrightarrow H(\widetilde{E}^{(\zeta)}; \widetilde{M}) \xrightarrow{(z - \zeta)} H(\widetilde{E}^{(\zeta)}, \widetilde{M}) \xrightarrow{\text{ev}_\zeta} H(E_\zeta; M) \longrightarrow 0.$$

By Proposition 2.1 we obtain:

$$\left[ H(\widetilde{E}^{(\zeta)}; \widetilde{M})[z - \zeta] \right]_{A_\zeta} = [H(E_\zeta; M)]_{A_\zeta} = \left[ \frac{U(E_\zeta; M)}{U_\zeta(E; \mathcal{O}_L)} \right]_{A_\zeta},$$

the last equality coming from Theorem 3.14. Since  $H(\widetilde{E}^{(\zeta)}; \widetilde{M})[z - \zeta]$  is a  $\mathbb{F}_q(\zeta)$ -vector space of finite dimension and  $U(E_\zeta; M)$  is an  $A_\zeta$ -lattice in  $M_\infty^d$ , the result follows. □

#### 4. THE $P$ -ADIC CASE

We keep the notation of Section 3. Recall that  $L$  is a finite extension of  $K$  of degree  $n$ ,  $\mathcal{O}_L$  denotes the integral closure of  $A$  in  $L$  and  $E$  is an Anderson  $t$ -module defined over  $\mathcal{O}_L$  of dimension  $d$ . The goal of this section is to define and study some  $P$ -adic  $L$ -series associated with Anderson  $t$ -modules by removing the local factor at  $P$  of the classical  $L$ -series.

**4.1. Introduction and notation.** Recall that the local factor at  $Q$  associated with  $E$  is defined by  $z_Q(\widetilde{E}/\widetilde{\mathcal{O}}_L) = \frac{[\text{Lie}_{\widetilde{E}}(\widetilde{\mathcal{O}}_L/Q\widetilde{\mathcal{O}}_L)]_{\widetilde{A}}}{[\widetilde{E}(\widetilde{\mathcal{O}}_L/Q\widetilde{\mathcal{O}}_L)]_{\widetilde{A}}} \in \widetilde{K}$  (resp.  $z_Q(E/\mathcal{O}_L) = \frac{[\text{Lie}_E(\mathcal{O}_L/Q\mathcal{O}_L)]_A}{[E(\mathcal{O}_L/Q\mathcal{O}_L)]_A} \in K$ ).

The goal of this section is to study the following infinite product of local factors  $z_Q(E/\mathcal{O}_L)$  (resp.  $z_Q(\widetilde{E}/\widetilde{\mathcal{O}}_L)$ ) that we call the  $P$ -adic  $L$ -series (resp. the  $z$ -twisted  $P$ -adic  $L$ -series):

$$L_P(\widetilde{E}/\widetilde{\mathcal{O}}_L) = \prod_{Q \neq P} z_Q(\widetilde{E}/\widetilde{\mathcal{O}}_L) \quad (\text{resp. } L_P(E/\mathcal{O}_L) = \prod_{Q \neq P} z_Q(E/\mathcal{O}_L))$$

where  $Q$  runs through the monic primes of  $A$  different from  $P$ .

More precisely, let us denote by  $v_P$  a finite place of  $K$  associated to an irreducible monic polynomial  $P$  of  $A$ . Let  $\mathbb{F}_P = \mathbb{F}_{q^{\deg(P)}}$  be the residue field associated to  $v_P$  and let  $K_P = \mathbb{F}_P((P))$  (resp.  $A_P = \mathbb{F}_P[[P]]$ ) be the completion of  $K$  (resp.  $A$ ) for  $v_P$ . Let  $\mathbb{C}_P$  be the completion of an algebraic closure of  $K_P$  and  $v_P$  the valuation on  $\mathbb{C}_P$  normalized such that  $v_P(P) = 1$ . Set  $\widetilde{K}_P = \mathbb{F}_q(z) \hat{\otimes}_{\mathbb{F}_q(z)} K_P = \mathbb{F}_P(z)((P))$  on which we extend the valuation  $v_P$ :

$$v_P\left(\sum_{n \geq N} \alpha_n(z) P^n\right) = N, N \in \mathbb{Z}, \alpha_n(z) \in \mathbb{F}_P(z), \alpha_N(z) \neq 0.$$

Let  $|\cdot|_P$  be the absolute value on  $\mathbb{C}_P$  defined by  $|x|_P = q^{-v_P(x)}$ . Let  $\mathcal{B} = (f_1, \dots, f_n)$  be an  $A$ -basis of  $\mathcal{O}_L$  (that is also a  $K$ -basis of  $L$ ). We set  $L_P = L \otimes_K K_P$  and  $\widetilde{L}_P = L \otimes_K \widetilde{K}_P$ . In what follows, the reader will be careful not to confuse the notation  $L_P(E/\mathcal{O}_L)$  for the  $P$ -adic  $L$ -series and  $L_P$  for the tensor product  $L \otimes_K K_P$ . Then  $L_P$  is a  $K_P$ -vector space with  $\mathcal{B}$  as a basis, and  $\widetilde{L}_P$  is a  $\widetilde{K}_P$ -vector space with  $\mathcal{B}$  as a basis. In particular on  $L_P$  all the norms of  $K_P$ -vector space of finite dimension are equivalent. Let us work with the following.

Consider the sup norm  $|\cdot|_P$  with respect to this basis. In other words, if  $x = \sum_{i=1}^n f_i \otimes x_i$

with the  $x_i \in K_P$ , then we set

$$|x|_P = \max_{i=1, \dots, n} |x_i|_P.$$

We obtain over  $L_P$  a norm of  $K_P$ -algebra. We then consider the over-additive valuation of  $K_P$ -vector spaces of finite dimension on  $L_P$  defined by:

$$v_P(x) = -\log_q |x|_P = \min_{i=1, \dots, d} v_P(x_i).$$

For all  $d \geq 1$ , we extend these definitions to  $L_P^d$ : if  $x = (x_1, \dots, x_d) \in L_P^d$ , then we set

$$|x| = \max_{i=1, \dots, d} |x_i|$$

or equivalently

$$v_P(x) = -\log_q |x| = \min_{i=1, \dots, d} v_P(x_i).$$

In particular for all  $x \in \mathcal{O}_L^d$  we get  $v_P(x) \geq 0$ . Consider

$$\mathbb{T}_z(K_P) = \left\{ f(z) = \sum_{n \geq 0} a_n z^n \mid a_n \in K_P \text{ and } \lim_{n \rightarrow +\infty} v_P(a_n) = +\infty \right\} \subset \widetilde{K_P}$$

and

$$\mathbb{T}_z(L_P) = \left\{ f(z) = \sum_{n \geq 0} a_n z^n \mid a_n \in L_P \text{ and } \lim_{n \rightarrow +\infty} v_P(a_n) = +\infty \right\} = L \otimes_K \mathbb{T}_z(K_P).$$

We define a  $K_P$ -vector space structure over  $\text{Lie}_E(L_P)$ . We take inspiration from the  $\infty$ -adic case in [13, Lemma 1.7] and [11, section 2.3].

**Proposition 4.1.** *We can extend the homomorphism  $\partial_E : A \rightarrow M_d(\mathcal{O}_L)$  into a homomorphism from  $K_P$  to  $M_d(L_P)$  in the following way:*

$$\begin{aligned} \partial_E : K_P &\rightarrow M_d(L_P), \\ \sum_{i \geq -N} \alpha_i P^i &\mapsto \sum_{i \geq -N} \alpha_i \partial_E(P)^i. \end{aligned}$$

Moreover, with respect to this action,  $L_P^d$  is a  $K_P$ -vector space of dimension  $m = d[L : K]$  denoted by  $\text{Lie}_E(L_P)$ .

*Proof.* Consider  $n \geq 0$ . There exists a unique integer  $t_n$  such that

$$q^d t_n \leq n < (t_n + 1)q^d.$$

We have:

$$\partial_E(P)^n = \partial_E(P)^{q^d t_n} \partial_E(P)^{n - q^d t_n} = P^{q^d t_n} \underbrace{\partial_E(P)^{n - q^d t_n}}_{\in M_d(\mathcal{O}_L)}.$$

We obtain that  $\lim_{n \rightarrow +\infty} v_P(\partial_E(P)^n) = +\infty$  thus the map  $\partial_E$  is well-defined. Denote by  $W_P = \mathbb{F}_{q^{\deg(P)}}((P^{q^d})) \subseteq K_P$ . Then for all  $x \in W_P$  we have  $\partial_E(x) = xI_d$ , thus we have an isomorphism of  $W_P$ -vector spaces:

$$\text{Lie}_E(L_P) \simeq L_P^d.$$

Hence  $\text{Lie}_E(L_P)$  is a  $K_P$ -vector space of finite dimension. We have:

$$\dim_{W_P}(\text{Lie}_E(L_P)) = \dim_{K_P}(\text{Lie}_E(L_P)) \dim_{W_P}(K_P).$$

But from the isomorphism of  $W_P$ -vector spaces  $\text{Lie}_E(L_P) \simeq L_P^d$  we have:

$$\dim_{W_P}(\text{Lie}_E(L_P)) = \dim_{W_P}(L_P^d) = \dim_{K_P}(L_P^d) \dim_{W_P}(K_P)$$

thus

$$\dim_{K_P}(\text{Lie}_E(L_P)) = \dim_{K_P}(L_P^d) = d[L : K].$$

□

We also have that  $\text{Lie}_E(\mathcal{O}_L)$  is an  $A$ -lattice in  $\text{Lie}_E(L_P)$ . Finally, everything is still valid by adding the variable  $z$ , in other words  $\text{Lie}_E(\widetilde{L_P})$  is a  $\widetilde{K_P}$ -vector space of dimension  $d[L : K]$  and  $\text{Lie}_{\widetilde{E}}(\widetilde{\mathcal{O}_L})$  is an  $\widetilde{A}$ -lattice in  $\text{Lie}_E(\widetilde{L_P})$ . In particular, we have:

$$\partial_{\widetilde{E}} : \mathbb{T}_z(K_P) \rightarrow M_d(\mathbb{T}_z(L_P)).$$

Remark that the topologies of  $L_P^d$  and  $\text{Lie}_E(L_P)$  are equivalent.

Consider the unique  $t$ -module  $F$  over  $\mathcal{O}_L$  satisfying  $PF_a = E_a P$  for all  $a \in A$ . If  $E_a = \sum_{i=0}^{r_a} E_{a,i} \tau^i$ , then  $F_a = \sum_{i=0}^{r_a} E_{a,i} P^{q^i-1} \tau^i$ . In particular for all  $a \in A$  we have:  $\partial_E(a) = \partial_F(a)$ . From [4, Section 3.2] we have the following equalities in  $M_d(L)\{\{\tau\}\}$ :

$$\log_F = P^{-1} \log_E P = \sum_{n \geq 0} l_n P^{q^n-1} \tau^n,$$

and

$$\exp_F = P^{-1} \exp_E P = \sum_{n \geq 0} d_n P^{q^n-1} \tau^n.$$

We now study the link between the local factors of  $E$  and  $F$ .

**Lemma 4.2.** *Let  $Q \in A$  be a monic prime. If  $Q \neq P$ , then we have the following equalities:  $z_Q(F/\mathcal{O}_L) = z_Q(E/\mathcal{O}_L)$  and  $z_Q(\tilde{F}/\tilde{\mathcal{O}}_L) = z_Q(\tilde{E}/\tilde{\mathcal{O}}_L)$ . Otherwise  $z_P(F/\mathcal{O}_L) = 1$  and  $z_P(\tilde{F}/\tilde{\mathcal{O}}_L) = 1$ .*

*Proof.* See [4, Lemma 3.7]. □

We then obtain the following result.

**Corollary 4.3.** *We have the following equality in  $K[[z]]$ :*

$$L(\tilde{F}/\tilde{\mathcal{O}}_L) = L_P(\tilde{E}/\tilde{\mathcal{O}}_L).$$

**4.2.  $P$ -adic exponential and  $P$ -adic logarithm.** We define  $(D_n)_{n \geq 0}$  and  $(L_n)_{n \geq 0}$  as the following sequences of elements of  $A$ :

$$\begin{cases} D_0 = 1, \\ D_n = \prod_{k=0}^{n-1} (\theta^{q^{n-k}} - \theta)^{q^k}, \end{cases} \text{ and } \begin{cases} L_0 = 1, \\ L_n = \prod_{k=1}^n (\theta - \theta^{q^k}). \end{cases}$$

We first estimate the  $P$ -adic valuation of  $D_n$  and  $L_n$  for all  $n \geq 0$ .

**Lemma 4.4.** *We have the following equalities for  $n \geq 1$ :*

$$\begin{aligned} (1) \quad v_P(D_n) &= q^n \frac{q^{-\deg(P)(\lfloor \frac{n}{\deg(P)} \rfloor + 1)} - q^{-\deg(P)}}{q^{-\deg(P)} - 1} = q^n \frac{q^{-\deg(P)\lfloor \frac{n}{\deg(P)} \rfloor} - 1}{1 - q^{\deg(P)}}, \\ (2) \quad v_P(L_n) &= \left\lfloor \frac{n}{\deg(P)} \right\rfloor. \end{aligned}$$

*Proof.* See [2, Section 2]. □

We recall that  $\partial_E(a) \in M_d(A)$  is the constant coefficient of  $E_a \in M_d(A)\{\tau\}$  for all  $a \in A$ , see Section 2. Set  $s \in \mathbb{N}$  the smallest integer such that  $(\partial_E(\theta) - \theta I_d)^{q^s} = 0$ . There exists because  $\partial_E(\theta) - \theta I_d$  is nilpotent. Then for all  $a \in A$  we have  $\partial_E(a^{q^s}) = a^{q^s} I_d$ .

Recall that  $\exp_E = \sum_{n \geq 0} d_n \tau^n \in M_d(L)\{\{\tau\}\}$  and  $\log_E = \sum_{n \geq 0} l_n \tau^n \in M_d(L)\{\{\tau\}\}$ . Following [15, Theorem 4.6.9], using functional equation of the logarithm map (resp. the exponential map)  $\log_E E_{\theta^{q^s}} = \partial_E(\theta^{q^s}) \log_E$ , an immediate induction tells us that  $l_n$  has the form

$$l_n = \frac{a_n}{L_n^{q^s}}$$

with  $a_n \in M_d(\mathcal{O}_L)$ .

Reasoning in a similar way for the exponential map and by Lemma 4.4 we obtain the following result.

**Proposition 4.5.** *We have the following inequalities for all  $n \geq 0$ :*

$$\begin{aligned}
(1) \quad v_P(l_n) &\geq -q^s \left\lfloor \frac{n}{\deg(P)} \right\rfloor, \\
(2) \quad v_P(d_n) &\geq -q^{s+n} \frac{q^{-\deg(P) \lfloor \frac{n}{\deg(P)} \rfloor} - 1}{1 - q^{\deg(P)}}.
\end{aligned}$$

So far, we have considered the exponential and logarithm series as functions of  $L_\infty^d$ , but now we want to look at them as functions of  $L_P^d$ , which we denote by  $\exp_{E,P}$  and  $\log_{E,P}$ . Note that formally (i.e., in  $M_d(L)\{\{\tau\}\}$ ), these are always the same series. We do the same for  $z$ -twist.

Let us denote by  $\text{ev}_{z=1,P} : \mathbb{T}_z(L_P)^d \rightarrow L_P^d$  the  $P$ -adic evaluation morphism at  $z = 1$ , whose kernel is given by  $(z-1)\mathbb{T}_z(L_P)^d$ .

We can first study the  $P$ -adic convergence domain of the  $P$ -adic logarithms maps associated with  $\tilde{F}$  and  $\tilde{E}$ . We consider the following sets:

- $\Omega_z = \{x \in \mathbb{T}_z(L_P)^d \mid v_P(x) \geq 0\}$  and  $\Omega_z^+ = \{x \in \mathbb{T}_z(L_P)^d \mid v_P(x) > 0\}$ ,
- $\Omega = \{x \in L_P^d \mid v_P(x) \geq 0\}$  and  $\Omega^+ = \{x \in L_P^d \mid v_P(x) > 0\}$ ,
- $\mathcal{D}_z = \left\{x \in \mathbb{T}_z(L_P)^d \mid v_P(x) > -1 + \frac{q^s}{q^{\deg(P)} - 1}\right\}$ ,
- $\mathcal{D}_z^+ = \left\{x \in \mathbb{T}_z(L_P)^d \mid v_P(x) > \frac{q^s}{q^{\deg(P)} - 1}\right\}$ ,
- $\mathcal{D} = \left\{u \in (L_P)^d \mid v_P(u) > -1 + \frac{q^s}{q^{\deg(P)} - 1}\right\}$ ,
- $\mathcal{D}^+ = \left\{u \in (L_P)^d \mid v_P(u) > \frac{q^s}{q^{\deg(P)} - 1}\right\}$ .

**Proposition 4.6.**

(1) We have the  $P$ -adic convergences:

$$\log_{\tilde{E},P} : \Omega_z^+ \rightarrow \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_P))$$

and

$$\log_{E,P} : \Omega^+ \rightarrow \text{Lie}_E((L_P)).$$

Moreover,  $\log_{\tilde{E},P} : \mathcal{D}_z^+ \rightarrow \mathcal{D}_z^+$  is an isometry and  $\log_{E,P} : \mathcal{D}^+ \rightarrow \mathcal{D}^+$  is an isometry.

(2) The first assertion remains true by replacing  $E$  by  $F$  and deleting the “+”. As a particular case, we have the convergence

$$\log_{\tilde{F},P} : \tilde{E}(\mathcal{O}_L[z]) \rightarrow \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_P)).$$

*Proof.* We give the proof only for  $\log_{\tilde{E},P}$ , the arguments are similar in the other cases. Consider  $f(z) \in \mathbb{T}_z(L_P)^d$ . We have (first formally):

$$\log_{\tilde{E},P} f(z) = \sum_{n \geq 0} L_n z^n \tau^n(f(z)).$$

For all  $n \geq 0$  we have:

$$v_P(L_n \tau^n(f(z))) \geq v_P(L_n) + v_P(\tau^n(f(z))) \geq -q^s \left\lfloor \frac{n}{\deg(P)} \right\rfloor + q^n v_P(f(z))$$

and this last quantity tends to  $\infty$  when  $n$  tends to  $\infty$  if  $v_P(f(z)) > 0$ . Moreover, if  $v_P(f(z)) > \frac{q^s}{q^{\deg(P)} - 1}$ , then we have for all  $n \geq 1$ :

$$v_P(L_n \tau^n(f(z))) - v_P(f(z)) \geq (q^n - 1)v_P(f(z)) + v(L_n) > \frac{q^s}{q^{\deg(P)} - 1}(q^n - 1) - q^s \left\lfloor \frac{n}{\deg(P)} \right\rfloor.$$

Write  $n = b \deg(P) + i \geq 1$  with  $b \in \mathbb{N}$  and  $0 \leq i < \deg(P)$ . Then:

$$\frac{q^s}{q^{\deg(P)} - 1}(q^n - 1) - q^s \left\lfloor \frac{n}{\deg(P)} \right\rfloor = q^s \left( \frac{q^{b \deg(P) + i} - 1}{q^{\deg(P)} - 1} - b \right).$$



But we have:

$$\frac{q^{b \deg(P)+i} - 1}{q^{\deg(P)} - 1} - b \geq \frac{q^{b \deg(P)} - 1}{q^{\deg(P)} - 1} - b = 1 + q^{\deg(P)} + \dots + (q^{\deg(P)})^{b-1} - b \geq 0.$$

□

We have results for the  $P$ -adic convergences of the exponentials series using similar arguments.

**Proposition 4.7.**

(1) *We have the  $P$ -adic convergences:*

$$\exp_{\tilde{E},P} : \mathcal{D}_z^+ \rightarrow \mathbb{T}_z(L_P)^d$$

and

$$\exp_{E,P} : \mathcal{D}^+ \rightarrow L_P^d.$$

Moreover,  $\exp_{\tilde{E},P} : \mathcal{D}_z^+ \rightarrow \mathcal{D}_z^+$  is an isometry and  $\exp_{E,P} : \mathcal{D}^+ \rightarrow \mathcal{D}^+$  is an isometry.

(2) *The first assertion remains true by replacing  $E$  by  $F$  and deleting “+”.*

In particular for all  $x \in \mathcal{D}_z$  we have the following  $P$ -adic equality:

$$(3) \quad \exp_{F,P}(\text{ev}_{z=1,P}(x)) = \text{ev}_{z=1,P}(\exp_{\tilde{F}}(x)).$$

Similarly for all  $x \in \Omega_z$  we have the following  $P$ -adic equality:

$$(4) \quad \log_{F,P}(\text{ev}_{z=1,P}(x)) = \text{ev}_{z=1,P}(\log_{\tilde{F}}(x)).$$

Similarly in their convergence domain, all of the  $P$ -adic exponential and logarithm maps verify the functional identities of the exponential and the logarithm maps:

$$\forall(a, x) \in A \times \Omega_z, \log_{\tilde{F},P} \partial_{\tilde{F}}(a)x = \tilde{F}_a \log_{\tilde{F},P} x,$$

$$\forall(a, x) \in A \times \mathcal{D}_z, \exp_{\tilde{F},P} \partial_{\tilde{F}}(a)x = \tilde{E}_a \log_{\tilde{F},P} x.$$

Moreover, for all  $x \in \mathcal{D}_z^+$  we have

$$\exp_{\tilde{E},P}(\log_{\tilde{E},P}(x)) = \log_{\tilde{E},P}(\exp_{\tilde{E},P}(x)) = x.$$

The same goes without the variable  $z$ , and the same goes for  $\tilde{E}$  (resp.  $E$ ) over  $\Omega_z^+$  and  $\mathcal{D}_z^+$  (resp. over  $\Omega^+$  and  $\mathcal{D}^+$ ).

**4.3. Evaluation at  $z = \zeta \in \overline{\mathbb{F}_q}$ : the  $P$ -adic setting.** Consider  $\mathbb{F}_P$  to be the residual field associated with  $P$ . Set  $\mathbb{F} = \mathbb{F}_q(\zeta) \cap \mathbb{F}_P$  and  $G = \text{Gal}(\mathbb{F}/\mathbb{F}_q)$ . Let us first remark that the valuation  $w$  defined in 3.4 is not a valuation over  $\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} K_P$ .

We have an isomorphism of  $\mathbb{F}$ -vector spaces:

$$\mathbb{F} \otimes_{\mathbb{F}_q} \mathbb{F} \simeq \prod_{g \in G} \mathbb{F} \simeq \prod_{g \in G} (\mathbb{F} \otimes_{\mathbb{F}} \mathbb{F})$$

given by

$$\eta : x \otimes y \mapsto (g(x)y, g \in G).$$

In particular, through this isomorphism, the Frobenius  $\tau_\zeta$  is identified with  $(\text{id} \otimes \tau, \dots, \text{id} \otimes \tau)$ . First, we extend the scalars from  $\mathbb{F}$  to  $\mathbb{F}_q(\zeta)$ . We obtain a (canonical) isomorphism (of  $\mathbb{F}_q(\zeta)$ -vector spaces)  $\eta' : \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathbb{F} \rightarrow \prod_{g \in G} \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} \mathbb{F}$ , given by the following. Let  $(f_1, \dots, f_l)$  be an  $\mathbb{F}$ -basis of  $\mathbb{F}_q(\zeta)$  and  $a_1, \dots, a_l \in \mathbb{F}$ . We set

$$\eta' \left( \sum_{i=1}^l a_i f_i \otimes_{\mathbb{F}_q} x_i \right) = \left( \sum_{i=1}^l g(a_i) f_i \otimes_{\mathbb{F}} x_i, g \in G \right).$$

Note that the isomorphism is canonical, but not the topologies that will appear. We then naturally extend the scalars (on the right) from  $\mathbb{F}$  to  $L_P$ . We obtain an isomorphism (of  $\mathbb{F}_q(\zeta)$ -vector spaces on the left and  $L_P$ -modules on the right) induced by  $\eta$ , also denoted  $\eta$ :

$$\eta : \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} L_P \simeq \prod_{g \in G} (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} L_P).$$

In particular we obtain  $L_P$ -vector spaces of dimension  $[\mathbb{F}_q(\zeta) : \mathbb{F}]$  on each component of the product, so an  $L_P$ -vector space of dimension  $[\mathbb{F}_q(\zeta) : \mathbb{F}_q]$ . For  $g \in G$ , we set  $H_g = \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} K_P \simeq \mathbb{F}_q(\zeta)((P))$  (where the action on the left of  $\mathbb{F}$  is determined by  $g \in G$ ).

For  $x = \sum_{i=1}^l f_i \otimes x_i \in H_g$ , we consider the usual valuation on  $H_g$ :

$$v_g(x) = \min_{i=1, \dots, m} v_P(x_i).$$

For all  $g \in G$  we provide  $L \otimes_K H_g$  with the topology  $v_g$  induced by its structure of  $H_g$ -vector space of finite dimension with respect to the choice of the basis  $\mathcal{B}$  of  $L/K$ . In particular, if we set  $\text{pr}_g$  the projection on the  $g$ -component of the product, then we obtain  $v_g(\text{pr}_g(\eta((\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} \mathcal{O}_L)))) \geq 0$ .

Let  $v_P$  be the over-additive valuation on the product  $\prod_{g \in G} (L \otimes_K H_g)$ :

$$v_P(x_g, g \in G) = \min_{g \in G} (v_g(x_g))$$

verifying  $v_P(\eta(1 \otimes P)) = 1$ . Remark that the Frobenius  $\tau_\zeta$  is equal to  $(\text{id} \otimes \tau, \dots, \text{id} \otimes \tau)$  on  $\prod_{g \in G} (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} L_P)$ .

**Remark 4.8.** Following exactly the same ideas, by extending the scalars from  $\mathbb{F}$  to  $\mathbb{T}_z(L_P)$  or  $\widetilde{L}_P$  we obtain the isomorphisms of  $\mathbb{F}_q(\zeta)$ -vector spaces on the left and  $\widetilde{L}_P$  (resp.  $\mathbb{T}_z(L_P)$ ) on the right:

$$\eta_z : \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathbb{T}_z(L_P) \simeq \prod_{g \in G} \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} \mathbb{T}_z(L_P)$$

and

$$\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \widetilde{L}_P \simeq \prod_{g \in G} \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} \widetilde{L}_P.$$

We are now interested in the case of the higher dimension  $d$ . We extend  $v_P$  onto  $(\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} L_P)^d$  (the same goes with  $z$ ) (topology of finite-dimensional vector spaces, for example with respect to the canonical basis). Then set

$$\Omega_{\zeta, d} = \{x \in (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} L_P)^d \mid v_P(x) \geq 0\} \supseteq (\mathbb{F}_q(\zeta) \otimes \mathcal{O}_L)^d$$

and

$$\Omega_{\zeta, d, z} = \{x \in (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathbb{T}_z(L_P))^d \mid v_P(x) \geq 0\} \supseteq (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathcal{O}_L[z])^d.$$

**Proposition 4.9.** *We have the following convergences:*

$$\log_{F(\zeta), P} : \Omega_{\zeta, d} \rightarrow (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} L_P)^d$$

and

$$\log_{F(\zeta), P} : \Omega_{\zeta, d, z} \rightarrow (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathbb{T}_z(L_P))^d.$$

*Proof.* It follows from Proposition 4.5 and the definitions of the objects. We omit the proof.  $\square$

**4.4. The  $P$ -adic  $L$ -series.** Recall that  $m = d[L : K]$  where  $d$  is the dimension of the  $t$ -module  $E$  and that  $F$  is the  $t$ -module given by  $F = P^{-1}EP$ . Let  $\mathcal{C} = (g_1, \dots, g_m)$  be an  $A$ -basis of  $\text{Lie}_E(\mathcal{O}_L)$ , it is also a  $\widetilde{K}_\infty$ -basis of  $\text{Lie}_{\widetilde{E}}(L_\infty)$ , a  $\widetilde{K}_P$ -basis of  $\text{Lie}_{\widetilde{E}}(L_P)$  and a  $\mathbb{T}_z(K_P)$ -basis of  $\text{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_P))$ . The same goes by replacing  $E$  by  $F$  since  $\partial_E = \partial_F$ .

Let us remark, from Corollary 4.6, that for any  $z$ -unit  $y(z) \in U(\widetilde{F}, \mathcal{O}_L[z])$  we have  $\exp_{\widetilde{F}}(y(z)) \in \widetilde{E}(\mathcal{O}_L[z]) \subseteq \Omega_z$  and therefore

$$\log_{\widetilde{F}, P}(\exp_{\widetilde{F}}(y(z))) \in \text{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_P)).$$

Moreover, for a family  $(x_1(z), \dots, x_m(z))$  of elements of  $\text{Lie}_{\widetilde{E}}(\mathbb{T}_z(L_P))$  we have

$$\text{Mat}_{\mathcal{C}}(x_1(z), \dots, x_m(z)) \in M_m(\mathbb{T}_z(K_P))$$

thus

$$\det_{\mathcal{C}}(x_1(z), \dots, x_m(z)) \in \mathbb{T}_z(K_P).$$

Next, formally in  $(L[[z]])^d$  we have the following equality for all  $f(z) \in (L[[z]])^d$ :

$$\log_{\widetilde{F}, P}(\exp_{\widetilde{F}}(f(z))) = f(z).$$

Let  $(v_1(z), \dots, v_m(z)) \subset U(\widetilde{F}; \mathcal{O}_L[z])$  be an  $\widetilde{A}$ -basis of  $U(\widetilde{F}; \widetilde{\mathcal{O}}_L)$ . Remark that the family  $(1 \otimes v_1(z), \dots, 1 \otimes v_m(z)) \subseteq \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathbb{T}_z(L_\infty)^d$  is also an  $\widetilde{A}_\zeta$ -basis of  $U(\widetilde{F}^{(\zeta)}; \widetilde{M})$ . Set

$$w(z) = \det_{\mathcal{C}}(v_1(z), \dots, v_m(z)) \in \mathbb{T}_z(K_\infty)$$

and

$$w_P(z) = \det_{\mathcal{C}}(\log_{\widetilde{F}, P}(\exp_{\widetilde{F}}(v_1(z))), \dots, \log_{\widetilde{F}, P}(\exp_{\widetilde{F}}(v_m(z)))) \in \mathbb{T}_z(K_P).$$

By the above discussions and the class formula, we have the following equality in  $K[[z]]$ :

$$L_P(\widetilde{F}/\widetilde{\mathcal{O}}_L) = \frac{w_P(z)}{\text{sgn}(w(z))}.$$

Since  $w_P(z) \in \mathbb{T}_z(K_P)$ , to study the  $P$ -adic convergence we want to prove that  $\text{sgn}(w(z))$  divides  $w_P(z)$  in  $\mathbb{T}_z(K_P)$ . Remark that the possible  $P$ -adic poles are the zeros of  $\text{sgn}(w(z)) \in \mathbb{F}_q[z]$  hence elements of  $\overline{\mathbb{F}}_q$ . We will prove that the meromorphic series  $\frac{w_P(z)}{\text{sgn}(w(z))}$  does not have a pole in  $\overline{\mathbb{F}}_q$ .

**Theorem 4.10.** *The meromorphic series  $\frac{w_P(z)}{\text{sgn}(w(z))}$  does not have a pole in  $\overline{\mathbb{F}}_q$ . In other words, we have the convergence  $\frac{w_P(z)}{\text{sgn}(w(z))} \in \mathbb{T}_z(K_P)$ .*

*Proof.* Let  $\zeta \in \overline{\mathbb{F}}_q$  be a root of  $\text{sgn}(w(z))$ . Recall that  $\mathcal{C}_\zeta = (1 \otimes g_1, \dots, 1 \otimes g_m) \subseteq \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathcal{O}_L$  if  $\mathcal{C} = (g_1, \dots, g_m)$ . Then  $\text{Lie}_{F_\zeta}(\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathcal{O}_L)$  is an  $A_\zeta$ -lattice in  $M_\infty$  and admits  $\mathcal{C}_\zeta$  as an  $A_\zeta$ -basis. Consider  $(w_1, \dots, w_m)$  an  $A_\zeta$ -basis of  $U_\zeta(F; \mathcal{O}_L) = \text{ev}_\zeta U(F^{(\zeta)}; \widetilde{M})$  and  $(w_1(z), \dots, w_m(z)) \subseteq U(F^{(\zeta)}; \widetilde{M})$  be such that  $\text{ev}_\zeta w_i(z) = w_i$  for  $i = 1, \dots, m$ . Set

$$W'(z) = \det_{\mathcal{C}_\zeta}(w_1(z), \dots, w_m(z)) \in \widetilde{M}_\infty \setminus (z - \zeta)\widetilde{M}_\infty$$

and

$$W'_P(z) = \det_{\mathcal{C}_\zeta}(\log_{\widetilde{F}^{(\zeta)}, P}(\exp_{\widetilde{F}^{(\zeta)}}(w_1(z))), \dots, \log_{\widetilde{F}^{(\zeta)}, P}(\exp_{\widetilde{F}^{(\zeta)}}(w_m(z)))) \in \widetilde{M}_v.$$

Recall that the family  $(1 \otimes v_1(z), \dots, 1 \otimes v_m(z))$  is an  $\widetilde{A}_\zeta$ -basis of  $U(\widetilde{F}^{(\zeta)}; \widetilde{M})$ . Let us set

$$W(z) = \det_{\mathcal{C}_\zeta}(1 \otimes v_1(z), \dots, 1 \otimes v_m(z)) = 1 \otimes w(z) \in \widetilde{M}_\infty,$$

$$W_P(z) = 1 \otimes w_P(z) \in \widetilde{M}_v,$$

and  $\Delta = \det_{(1 \otimes v_1(z), \dots, 1 \otimes v_m(z))}(w_1(z), \dots, w_m(z)) \in \tilde{A}_\zeta$ .  
From the equality

$$W'(z) = \Delta W(z)$$

we obtain

$$1 \otimes \underbrace{L(\tilde{F}/\tilde{\mathcal{O}}_L)}_{\in \mathbb{T}_z(K_\infty)} = 1 \otimes \frac{w(z)}{\text{sgn}(w(z))} = \frac{1 \otimes w(z)}{1 \otimes \text{sgn}(w(z))} = \frac{W(z)}{1 \otimes \text{sgn}(w(z))} = \frac{W'(z)}{\Delta(1 \otimes \text{sgn}(w(z)))}.$$

Since  $1 \otimes L(\tilde{F}/\tilde{\mathcal{O}}_L)$  does not have a pole in  $\overline{\mathbb{F}}_q$  and  $W'(z)$  is not divisible by  $z - \zeta$ , we obtain that  $\Delta(1 \otimes \text{sgn}(w(z)))$  is not divisible by  $z - \zeta$ . From the equality

$$\frac{W_P(z)}{1 \otimes \text{sgn}(w(z))} = \frac{W'_P(z)}{\Delta(1 \otimes \text{sgn}(w(z)))}$$

we can evaluate at  $z = \zeta$  so  $\zeta$  is not a pole of  $\frac{w_P(z)}{\text{sgn}(w(z))}$ .

Finally, the  $P$ -adic  $L$ -series is a meromorphic series without any pole in  $\overline{\mathbb{F}}_q$ : it is an element of  $\mathbb{T}_z(K_P)$ .  $\square$

**Corollary 4.11** ( $P$ -adic  $L$ -series for  $P^{-1}\tilde{E}P$ ). *Consider the  $t$ -module  $F = P^{-1}EP$ . Consider  $(v_1(z), \dots, v_r(z)) \subseteq U(\tilde{F}, \mathcal{O}_L[z])$  an  $\tilde{A}$ -basis of  $U(\tilde{F}; \tilde{\mathcal{O}}_L)$ ,  $\mathcal{C}$  an  $\tilde{A}$ -basis of  $\text{Lie}_{\tilde{F}}(\tilde{\mathcal{O}}_L)$ . Then the following product converges in  $\mathbb{T}_z(K_P)$*

$$L_P(\tilde{E}/\tilde{\mathcal{O}}_L) = \prod_{Q \neq P} \frac{[\text{Lie}_{\tilde{E}}(\tilde{\mathcal{O}}_L/Q\tilde{\mathcal{O}}_L)]_{\tilde{A}}}{[E(\tilde{\mathcal{O}}_L/Q\tilde{\mathcal{O}}_L)]_{\tilde{A}}}$$

where the product runs over all the monic irreducible polynomials  $Q$  of  $A$  different from  $P$ . Further, we have the equality:

$$L_P(\tilde{E}/\tilde{\mathcal{O}}_L) = \frac{\det_{\mathcal{C}} \left( \log_{\tilde{F}, P} \exp_{\tilde{F}} v_1(z), \dots, \log_{\tilde{F}, P} \exp_{\tilde{F}} v_m(z) \right)}{\text{sgn}(\det_{\mathcal{C}}(v_1(z), \dots, v_m(z)))}.$$

We can then define the  $P$ -adic  $L$ -series associated with  $E$  and  $\mathcal{O}_L$ :

$$L_P(E/\mathcal{O}_L) = \text{ev}_{z=1, P} L_P(\tilde{E}, \tilde{\mathcal{O}}_L) = \prod_{Q \neq P} \frac{[\text{Lie}_E(\mathcal{O}_L/Q\mathcal{O}_L)]_A}{[E(\mathcal{O}_L/Q\mathcal{O}_L)]_A} \in K_P.$$

We also have the following equalities from [4, Proposition 3.3]:

$$L_P(\tilde{E}/\tilde{\mathcal{O}}_L) = \prod_{\mathfrak{P} \nmid P} \frac{[\text{Lie}_{\tilde{E}}(\tilde{\mathcal{O}}_L/\mathfrak{P}\tilde{\mathcal{O}}_L)]_{\tilde{A}}}{[E(\tilde{\mathcal{O}}_L/\mathfrak{P}\tilde{\mathcal{O}}_L)]_{\tilde{A}}} \in \mathbb{T}_z(K_P)$$

where the product runs over all the primes of  $\mathcal{O}_L$  not dividing  $P$ , and

$$L_P(E/\mathcal{O}_L) = \prod_{\mathfrak{P} \nmid P} \frac{[\text{Lie}_E(\mathcal{O}_L/\mathfrak{P}\mathcal{O}_L)]_A}{[E(\mathcal{O}_L/\mathfrak{P}\mathcal{O}_L)]_A} \in K_P.$$

**4.5. A  $P$ -adic class formula associated with the  $t$ -module  $P^{-1}EP$ .** The next step is to introduce a  $P$ -adic regulator and obtain a  $P$ -adic class formula. We begin with  $P$ -twisted  $t$ -modules. Recall that  $\mathcal{C}$  is a fixed  $A$ -basis of  $\text{Lie}_E(\mathcal{O}_L) = \text{Lie}_F(\mathcal{O}_L)$ .

**Definition 4.12.** Consider  $V \subseteq U(F; \mathcal{O}_L)$  a sub- $A$ -lattice and let  $(v_1, \dots, v_m)$  be an  $A$ -basis of  $V$ . Then we define the  $P$ -adic regulator associated with  $V$  by

$$R_P(V) = \frac{\det_{\mathcal{C}}(\log_{F, P}(\exp_F(v_1)), \dots, \log_{F, P}(\exp_F(v_m)))}{\text{sgn}(\det_{\mathcal{C}}(v_1, \dots, v_m))} \in K_P$$

which is independent of the choice of the basis of  $V$  and of  $\text{Lie}_F(\mathcal{O}_L)$ .

**Theorem 4.13.** *[ $P$ -adic class formula for  $P^{-1}EP$ ] We have the following equality in  $K_P$ :*

$$L_P(E/\mathcal{O}_L) = R_P(U(F; \mathcal{O}_L)) [H(F; \mathcal{O}_L)]_A = R_P(U_{\text{st}}(F; \mathcal{O}_L)).$$

*Proof.* We use notation as in the proof of Theorem 4.10 with  $\zeta = 1$ . In particular  $(v_1, \dots, v_m)$  is an  $A$ -basis of  $U(F; \mathcal{O}_L)$ . Consider  $(u_1, \dots, u_m)$  an  $A$ -basis of  $U_{\text{st}}(F; \mathcal{O}_L)$ . Denote by  $b_i = \exp_F(u_i) \in F(\mathcal{O}_L)$  for  $i = 1, \dots, m$  and by  $a_i = \exp_F(v_i) \in F(\mathcal{O}_L)$  for  $i = 1, \dots, m$ . We have

$$L(\tilde{F}/\tilde{\mathcal{O}}_L) = \frac{w'(z)}{f(z)\Delta}.$$

As the  $L$ -series  $L(F/\mathcal{O}_L)$  (at infinity) is equal to

$$\frac{\det_{\mathcal{C}}(u_1, \dots, u_m)}{\text{sgn}(\det_{\mathcal{C}}(u_1, \dots, u_m))} = \frac{\text{ev}_{z=1}(w'(z))}{\text{sgn}(\det_{\mathcal{C}}(u_1, \dots, u_m))} = \text{ev}_{z=1} \frac{w'(z)}{f(z)\Delta}$$

we first have

$$(5) \quad \text{ev}_{z=1}(f(z)\Delta) = \text{sgn}(\det_{\mathcal{C}}(u_1, \dots, u_m)) \in \mathbb{F}_q^*.$$

Let us consider  $P_1 = \text{Mat}_{(v_1, \dots, v_m)}(u_1, \dots, u_m) \in M_m(A)$ . By [7, Theorem 1] we have

$$\frac{\det(P_1)}{\text{sgn}(\det(P_1))} = [H(F; \mathcal{O}_L)]_A. \text{ Moreover, we have:}$$

$$P_1 = \text{Mat}_{(\log_{F,P}(a_1), \dots, \log_{F,P}(a_m))}(\log_{F,P}(b_1), \dots, \log_{F,P}(b_m)).$$

Then we have in  $K_P$ :

$$(6) \quad \det(P_1) \det_{\mathcal{C}}(\log_{F,P}(a_1), \dots, \log_{F,P}(a_m)) = \det_{\mathcal{C}}(\log_{F,P}(b_1), \dots, \log_{F,P}(b_m)).$$

From the following equality in  $K_{\infty}$ :

$$\det(P_1) \det_{\mathcal{C}}(v_1, \dots, v_m) = \det_{\mathcal{C}}(u_1, \dots, u_m)$$

we deduce by comparing signs:

$$(7) \quad \text{sgn}(\det(P_1)) \text{sgn}(\det_{\mathcal{C}}(v_1, \dots, v_m)) = \text{sgn}(\det_{\mathcal{C}}(u_1, \dots, u_m)).$$

We finally have:

$$\begin{aligned} L_P(E/\mathcal{O}_L) &= \text{ev}_{z=1,P} \left( \frac{w'_P(z)}{f(z)\Delta} \right) \\ &= \frac{\det_{\mathcal{C}}(\log_{F,P}(b_1), \dots, \log_{F,P}(b_m))}{\text{sgn}(\det_{\mathcal{C}}(u_1, \dots, u_m))} \text{ by Equality (5) ,} \\ &= \det(P_1) \frac{\det_{\mathcal{C}}(\log_{F,P}(a_1), \dots, \log_{F,P}(a_m))}{\text{sgn}(\det_{\mathcal{C}}(u_1, \dots, u_m))} \text{ by Equality (6),} \\ &= \frac{\det(P_1)}{\text{sgn}(\det(P_1))} \frac{\det_{\mathcal{C}}(\log_{F,P}(a_1), \dots, \log_{F,P}(a_m))}{\text{sgn}(\det_{\mathcal{C}}(v_1, \dots, v_m))} \text{ by Equality (7),} \\ &= R_P(U(F; \mathcal{O}_L)) [H(F; \mathcal{O}_L)]_A \end{aligned}$$

and the second line equals  $R_P(U_{\text{st}}(E; \mathcal{O}_L))$ . □

**4.6. A  $P$ -adic class formula.** So far we've worked mainly with the  $t$ -module  $F$ , and now we want to link everything to the  $t$ -module  $E$ .

Set  $h(z) = \left[ \tilde{E}(\tilde{\mathcal{O}}_L/P\tilde{\mathcal{O}}_L) \right]_{\tilde{A}} \in A[z]$  and  $h(1) = [E(\mathcal{O}_L/P\mathcal{O}_L)]_A \in A \setminus \{0\}$ . Consider  $s \in \mathbb{N}$  such that  $\partial_{\tilde{E}}(h(z)^{q^s}) = h(z)^{q^s} I_d$  (e.g.,  $s$  such that  $q^s \geq d$ ) and denote by  $g(z) = h(z)^{q^s} \in A[z]$ . By Proposition 2.1, we have for all  $b(z) \in \mathcal{O}_L[z]^d$ :

$$\tilde{E}_{g(z)}(b(z)) \in P\mathcal{O}_L[z]^d$$

and for all  $b \in \mathcal{O}_L^d$ :

$$E_{g(1)}(b) \in P\mathcal{O}_L^d.$$

Then by Corollary 4.6 and the above discussion, we can define the following maps:

$$\begin{aligned} \text{Log}_{\tilde{E},P} : \Omega_z &\rightarrow \frac{1}{g(z)} \mathbb{T}_z(L_P)^d \\ x &\mapsto \frac{1}{g(z)} \log_{\tilde{E},P}(\tilde{E}_{g(z)}(x)) \end{aligned}$$

and

$$\begin{aligned} \text{Log}_{E,P} : \Omega &\rightarrow L_P^d \\ x &\mapsto \frac{1}{g(1)} \log_{E,P}(E_{g(1)}(x)). \end{aligned}$$

Moreover, if  $x \in \Omega_z^+$ , we have  $\log_{\tilde{E},P}(\tilde{E}_{g(z)}(x)) = g(z) \log_{\tilde{E},P}(x)$  in  $\mathbb{T}_z(L_P)^d$ . We obtain the following equality in  $\mathbb{T}_z(L_P)^d$  for such  $x$ :

$$\log_{\tilde{E},P}(x) = \text{Log}_{\tilde{E},P}(x)$$

thus the map  $\text{Log}_{\tilde{E},P}$  extends the map  $\log_{\tilde{E},P}$  from  $\Omega_z^+$  to  $\Omega_z$ . The same applies without  $z$ .

**Lemma 4.14.** *We have the following properties.*

(1) *For all  $a \in A[z]$  and  $x \in \Omega_z$  we have the following equality in  $\frac{1}{g(z)} \mathbb{T}_z(L_P)^d$ :*

$$\partial_{\tilde{E}}(a) \text{Log}_{\tilde{E},P}(x) = \text{Log}_{\tilde{E},P}(\tilde{E}_a(x)).$$

(2) *For all  $x \in \Omega_z$  we have the equality in  $L_P^d$ :*

$$\text{ev}_{z=1,P}(\text{Log}_{\tilde{E},P}(x)) = \text{Log}_{E,P}(\text{ev}_{z=1,P}(x)).$$

(3) *For all  $a \in A$  and  $x \in \Omega$  we have the following equality in  $L_P^d$ :*

$$\partial_E(a) \text{Log}_{E,P}(x) = \text{Log}_{E,P}(E_a(x)).$$

*Proof.*

(1) We have the following equalities in  $\mathbb{T}_z(L_P)^d$  for all  $x \in \Omega_z$  and  $a \in A[z]$ :

$$\begin{aligned} g(z) \text{Log}_{\tilde{E},P}(\tilde{E}_a(x)) &= \log_{\tilde{E},P}(\underbrace{\tilde{E}_{g(z)}(\tilde{E}_a(x))}_{v_P > 0}) = \partial_{\tilde{E}}(a) \log_{\tilde{E},P}(\tilde{E}_{g(z)}(x)) \\ &= \partial_{\tilde{E}}(a) g(z) \text{Log}_{\tilde{E},P}(x). \end{aligned}$$

(2) We have the following equality in  $\Omega^+$  for all  $x \in \Omega_z$ :

$$\text{ev}_{z=1,P}(\tilde{E}_{g(z)}(x)) = E_{g(1)}(\text{ev}_{z=1,P}(x)).$$

Then we have the following equalities:

$$\begin{aligned} \text{ev}_{z=1,P}(g(z) \text{Log}_{\tilde{E},P}(x)) &= \text{ev}_{z=1,P}(\log_{\tilde{E},P}(\underbrace{\tilde{E}_{g(z)}(x)}_{v_P > 0})) = \log_{E,P}(E_{g(1)}(\text{ev}_{z=1,P}(x))) \\ &= g(1) \text{Log}_{E,P}(\text{ev}_{z=1,P}(x)). \end{aligned}$$

(3) The proof is similar as for the first assertion. □

**Proposition 4.15.** *The logarithm map  $\log_{\tilde{E},P}$  is injective on  $\Omega_z^+$ .*

We begin with the following lemma.

**Lemma 4.16.** *For all  $a \in A \setminus \{0\}$  and for all  $x \in \mathbb{T}_z(L_P)^d \setminus \{0\}$  we have  $\tilde{E}_a(x) \neq 0$ .*

*Proof.* Fix  $a \in A \setminus \{0\}$  and  $x \in \mathbb{T}_z(K_P)^d$  such that  $E_a(x) = 0$ . We can suppose without loss of generality that  $\partial_E(a) = aI_d$ , even if it means applying  $\tilde{E}_{a^{q^s-1}}$  to  $\tilde{E}_a(x)$ . We can also assume that  $z$  does not divide  $x$  in  $\mathbb{T}_z(L_P)^d$ . Denote by  $\tilde{E}_a = a + \sum_{i=1}^{r_a} \tilde{E}_{a,i} \tau^i$  with  $\tilde{E}_{a,i} \in zM_d(\mathcal{O}_L[z])$  for all  $i = 1, \dots, r_a$ , and  $x = \sum_{n \geq 0} y_n z^n$  with  $a_n \in L_P^d$  and  $y_0 \neq 0$ . We have  $E_a(x) = ay_0 \bmod z\mathbb{T}_z(L_P)^d \neq 0 \bmod z\mathbb{T}_z(L_P)^d$ . Hence  $E_a(x) \neq 0$ .  $\square$

*Proof of Proposition 4.15.* Let  $x$  be in  $\Omega_z^+$  such that  $\log_{\tilde{E},P}(x) = 0$ . Then for all  $s \in \mathbb{N}$  we have

$$\partial_{\tilde{E}}(P^{q^s}) \log_{\tilde{E},P}(x) = 0 = \log_{\tilde{E},P}(\tilde{E}_{P^{q^s}}(x)).$$

Since  $v_P(x) > 0$ , we consider  $s \in \mathbb{N}$  large enough such that  $\tilde{E}_{P^{q^s}}(x)$  belongs to  $\mathcal{D}_z^+$ . For such an integer  $s$  we obtain:

$$\tilde{E}_{P^{q^s}}(x) = 0$$

which implies that  $x = 0$  by Lemma 4.16.  $\square$

**Proposition 4.17.** *The kernel of  $\text{Log}_{E,P} : \mathcal{O}_L^d \rightarrow L_P^d$  consists exactly of the torsion points of  $E(\mathcal{O}_L)$ , denoted by  $E(\mathcal{O}_L)_{\text{Tors}}$ .*

*Proof.* Consider first  $x \in \mathcal{O}_L^d$  such that  $\text{Log}_{E,P}(x) = 0$ . Then we have  $\log_{E,P}(E_{g(1)}(x)) = 0$ . Thus for all  $n \geq 0$  we have:

$$\log_{E,P}(E_{P^{q^n}g(1)}(x)) = \partial_E(P^{q^n}) \log_{E,P}(E_{g(1)}(x)) = 0.$$

Since  $v_P(E_{g(1)}(x)) > 0$ , we can consider  $n$  large enough such that  $E_{P^{q^n}g(1)}(x) \in \mathcal{D}^+$ . Then we find by applying the exponentiel map  $\exp_{E,P}$  to  $\log_{E,P}(E_{P^{q^n}g(1)}(x))$  that  $0 = E_{P^{q^n}g(1)}(x)$  so  $x$  is a torsion point.

Conversely, suppose that there is a non-zero polynomial  $a \in A$  such that  $E_a(x) = 0$ . We also have  $E_{a^{q^s}}(x) = 0$ . Then we have

$$a^{q^s} \log_{E,P}(E_{g(1)}(x)) = \log_{E,P}(E_{a^{q^s}g(1)}(x)) = \log_{E,P}(E_{g(1)}(E_{a^{q^s}}(x))) = 0.$$

Since  $a$  is non-zero, we obtain  $\log_{E,P}(E_{g(1)}(x)) = 0$ .  $\square$

Set  $\mathcal{O}_{L,P} = \mathcal{O}_L \otimes_A A_P$  and  $\widetilde{\mathcal{O}_{L,P}} = \widetilde{\mathcal{O}_L} \otimes_A A_P$ . By [7, Lemma 3.21], we can extend  $E$  by continuity to a homomorphism of  $\mathbb{F}_q$ -algebras

$$E : A_P \rightarrow M_d(\mathcal{O}_{L,P})\{\{\tau\}\}.$$

We can also extend  $\tilde{E}$  by continuity to a homomorphism of  $\mathbb{F}_q(z)$ -algebras

$$\tilde{E} : \tilde{A}_P \rightarrow M_d(\widetilde{\mathcal{O}_{L,P}})\{\{\tau\}\}.$$

In particular, the  $A$ -module  $E(P\mathcal{O}_{L,P})$  inherits a structure of  $A_P$ -module.

Set  $U' = g(1)U(E; \mathcal{O}_L)$  and  $U'_z = g(z)U(\tilde{E}; \widetilde{\mathcal{O}_L})$  (the multiplication is of course in  $\text{Lie}_E(L_\infty)$  (resp.  $\text{Lie}_{\tilde{E}}(\tilde{L}_\infty)$ )).

**Lemma 4.18.** *We have the following properties.*

- (1) *We have that  $U'$  and  $U'_z$  are sub-lattices of  $U(E; \mathcal{O}_L)$  and  $U(\tilde{E}; \widetilde{\mathcal{O}_L})$  respectively.*
- (2) *We have that  $U'$  and  $U'_z$  are sub-lattices of  $U(F; \mathcal{O}_L)$  and  $U(\tilde{F}; \widetilde{\mathcal{O}_L})$  respectively.*

*Proof.*

- (1) The first point is clear.

- (2) We have to prove the inclusions  $U' \subseteq U(F; \mathcal{O}_L)$  and  $U'_z \subseteq U(\tilde{F}; \tilde{\mathcal{O}}_L)$ . We do it for  $F$ , the same arguments apply to  $\tilde{F}$ . Let  $x \in \text{Lie}_E(L_\infty)$  be such that  $\exp_E(x) \in \mathcal{O}_L^d$ , then:

$$\exp_F(g(1)x) = P^{-1} \exp_E(P \partial_E(g(1))x) = P^{-1} E_{g(1)}(\exp_E(Px)) \in P^{-1} \cdot P \mathcal{O}_L^d = \mathcal{O}_L^d.$$

□

By Lemma 4.18,  $U'$  (resp.  $U'_z$ ) is a common sub- $A$ -lattice (resp. sub- $\tilde{A}$ -lattice) for  $U(E; \mathcal{O}_L)$  and  $U(F; \mathcal{O}_L)$  (resp.  $U(\tilde{E}; \tilde{\mathcal{O}}_L)$  and  $U(\tilde{F}; \tilde{\mathcal{O}}_L)$ ). We then have:

$$[U(F; \mathcal{O}_L) : U(E; \mathcal{O}_L)]_A = [U(F; \mathcal{O}_L) : U']_A [U' : U(E; \mathcal{O}_L)]_A = \frac{[U(F; \mathcal{O}_L) : U']_A}{[U(E; \mathcal{O}_L) : U']_A} \in K^*$$

and

$$\left[ U(\tilde{F}; \tilde{\mathcal{O}}_L) : U(\tilde{E}; \tilde{\mathcal{O}}_L) \right]_{\tilde{A}} = \left[ U(\tilde{F}; \tilde{\mathcal{O}}_L) : U'_z \right]_{\tilde{A}} \left[ U'_z : U(\tilde{E}; \tilde{\mathcal{O}}_L) \right]_{\tilde{A}} = \frac{\left[ U(\tilde{F}; \tilde{\mathcal{O}}_L) : U'_z \right]_{\tilde{A}}}{\left[ U(\tilde{E}; \tilde{\mathcal{O}}_L) : U'_z \right]_{\tilde{A}}} \in \tilde{K}^*.$$

Let us define  $P$ -adic regulators associated to  $U'$  as follows. Let  $(w_1, \dots, w_m)$  be an  $A$ -basis of  $U'$ . We set:

$$R_{P,E}(U') = \frac{\det_{\mathcal{C}}(\text{Log}_{E,P}(\exp_E(w_1)), \dots, \text{Log}_{E,P}(\exp_E(w_m)))}{\text{sgn}(\det_{\mathcal{C}}(w_1, \dots, w_m))} \in K_P$$

and

$$R_{P,F}(U') = \frac{\det_{\mathcal{C}}(\log_{F,P}(\exp_F(w_1)), \dots, \log_{F,P}(\exp_F(w_m)))}{\text{sgn}(\det_{\mathcal{C}}(w_1, \dots, w_m))} \in K_P.$$

These regulators do not depend of the choice of the basis  $(w_1, \dots, w_m)$ . We can also define a  $P$ -adic regulator  $R_P(U_{\text{st}}(E; \mathcal{O}_L))$  for  $U_{\text{st}}(E; \mathcal{O}_L)$  and we have the following equality from the proof of Theorem 4.13:

$$(8) \quad R_P(U(E; \mathcal{O}_L)) [H(E; \mathcal{O}_L)]_A = R_P(U_{\text{st}}(E; \mathcal{O}_L)).$$

Similarly, we define the  $P$ -adic regulators  $R_{P,\tilde{E}}(U'_z)$  and  $R_{P,\tilde{F}}(U'_z)$  associated with  $U'_z$  that are elements in  $\mathbb{T}_z(K_P)$  from Theorem 4.10.

**Lemma 4.19.** *We have the following equalities:*

$$(9) \quad R_{P,\tilde{E}}(U'_z) = R_{P,\tilde{F}}(U'_z) \text{ and } R_{P,E}(U') = R_{P,F}(U').$$

*Proof.* We prove the first equality. We want to prove that for all  $u(z) \in U(\tilde{E}; \mathcal{O}_L[z])$  we have the following equality in  $\mathbb{T}_z(L_P)^d$ :

$$\log_{\tilde{F},P}(\exp_{\tilde{F}}(u(z))) = \text{Log}_{\tilde{E},P}(\exp_{\tilde{E}}(u(z))).$$

Formally (i.e., in  $L[[z]]^d$ ) it is true, and the two quantities belong to  $\mathbb{T}_z(L_P)^d$ . Then the first equality of the lemma is clear.

To prove the second equality, consider  $(u_1, \dots, u_m)$  an  $A$ -basis of  $U_{\text{st}}(E; \mathcal{O}_L)$ . From the first equality of the lemma, we have the following equality in  $\mathbb{T}_z(K_P)$ :

$$\det_{\mathcal{C}}(\log_{\tilde{F},P}(\exp_{\tilde{F}}(g(z)u_i(z))), i = 1, \dots, m) = \det_{\mathcal{C}}(\text{Log}_{\tilde{E},P}(\exp_{\tilde{E}}(g(z)u_i(z))), i = 1, \dots, m).$$

By evaluating at  $z = 1$  we obtain

$$\det_{\mathcal{C}}(\log_{F,P}(\exp_F(g(1)u_i)), i = 1, \dots, m) = \det_{\mathcal{C}}(\text{Log}_{E,P}(\exp_E(g(1)u_i)), i = 1, \dots, m).$$



Consider now  $(v_1, \dots, v_m)$  an  $A$ -basis of  $U(E, \mathcal{O}_L)$ , then  $(g(1)v_1, \dots, g(1)v_m)$  is an  $A$ -basis of  $U'$ . Write  $Q = \text{Mat}_{(v_1, \dots, v_m)}(u_1, \dots, u_m) \in \text{Gl}_m(A)$ . We then have

$$\begin{aligned} \det_{\mathcal{C}}(\log_{F,P}(\exp_F(g(1)v_i)), i = 1, \dots, m) \\ = \det(Q)^{-1} \det_{\mathcal{C}}(\log_{F,P}(\exp_F(g(1)u_i)), i = 1, \dots, m) \\ = \det(Q)^{-1} \det_{\mathcal{C}}(\text{Log}_{E,P}(\exp_E(g(1)u_i)), i = 1, \dots, m) \\ = \det_{\mathcal{C}}(\text{Log}_{E,P}(\exp_E(g(1)v_i)), i = 1, \dots, m). \end{aligned}$$

We proved that the equality is true for one basis of  $U'$  and since the  $P$ -adic regulators does not depend of the choice of the basis, the equality is true.  $\square$

**Lemma 4.20.**

(1) We have the following equalities:

$$R_{P,\tilde{F}}(U'_z) = R_P(U(\tilde{F}; \widetilde{\mathcal{O}_L})) \left[ U(\tilde{F}; \widetilde{\mathcal{O}_L}) : U'_z \right]_{\tilde{A}}$$

and

$$R_{P,\tilde{E}}(U'_z) = R_P(U(\tilde{E}; \widetilde{\mathcal{O}_L})) \left[ U(\tilde{E}; \widetilde{\mathcal{O}_L}) : U'_z \right]_{\tilde{A}}$$

(2) We have the following equalities:

$$R_{P,F}(U') = R_P(U(F; \mathcal{O}_L)) [U(F; \mathcal{O}_L) : U']_A$$

and

$$R_{P,E}(U') = R_P(U(E; \widetilde{\mathcal{O}_L})) [U(E; \mathcal{O}_L) : U']_A$$

*Proof.* We only need to prove one of the equalities; all the others can be proven in a similar way. Let us prove the third one. By the structure theorem for finitely generated modules over a principal ideal domain, let us pick an  $A$ -basis  $(v_1, \dots, v_m)$  of  $U(F; \mathcal{O}_L)$  and  $a_1, \dots, a_m \in A$  such that  $(a_1v_1, \dots, a_mv_m)$  is an  $A$ -basis of  $U'$ . Then

$$\begin{aligned} R_{P,F}(U') &= \frac{\det_{\mathcal{C}}(\log_{F,P}(\exp_F(a_1w_1)), \dots, \log_{F,P}(\exp_F(a_mw_m)))}{\text{sgn}(\det_{\mathcal{C}}(\log_{F,P}(\exp_F(a_1w_1)), \dots, \log_{F,P}(\exp_F(a_mw_m)))})} \\ &= \frac{a_1 \dots a_m}{\text{sgn}(a_1 \dots a_m)} R_P(U(F; \mathcal{O}_L)) \\ &= [U(F; \mathcal{O}_L) : U']_A R_P(U(F; \mathcal{O}_L)). \end{aligned}$$

$\square$

The link between objects associated to  $F$  and those associated to  $E$  is contained in the local factor at  $P$ , and given by the following equalities from [4, Lemma 3.4]:

$$(10) \quad z_P(E/\mathcal{O}_L) = [U(F; \mathcal{O}_L) : U(E; \mathcal{O}_L)]_A \frac{[H(E; \mathcal{O}_L)]_A}{[H(F; \mathcal{O}_L)]_A}.$$

and

$$(11) \quad z_P(\tilde{E}/\widetilde{\mathcal{O}_L}) = \left[ U(\tilde{F}; \widetilde{\mathcal{O}_L}) : U(\tilde{E}; \widetilde{\mathcal{O}_L}) \right]_{\tilde{A}}.$$

We can state one of the main results of this work.

**Theorem 4.21** ( $P$ -adic class formula). *We have the  $P$ -adic class formula for  $\tilde{E}$ :*

$$z_P(\tilde{E}/\widetilde{\mathcal{O}_L}) L_P(\tilde{E}/\widetilde{\mathcal{O}_L}) = R_P(U(\tilde{E}; \widetilde{\mathcal{O}_L}))$$

and the class formula for  $E$ :

$$z_P(E/\mathcal{O}_L) L_P(E/\mathcal{O}_L) = R_P(U(E; \mathcal{O}_L)) [H(E; \mathcal{O}_L)]_A = R_P(U_{\text{st}}(E; \mathcal{O}_L)).$$

*Proof.* Let us start with the following equality from Corollary 4.11:

$$L_P(\tilde{E}/\tilde{\mathcal{O}}_L) = R_P(U(\tilde{F}; \tilde{\mathcal{O}}_L)).$$

Then:

$$\begin{aligned} z_P(\tilde{E}/\tilde{\mathcal{O}}_L)L_P(\tilde{E}/\tilde{\mathcal{O}}_L) &= \frac{[U(\tilde{F}; \tilde{\mathcal{O}}_L) : U'_z]_{\tilde{A}}}{[U(\tilde{E}; \tilde{\mathcal{O}}_L) : U'_z]_{\tilde{A}}} R_P(U(\tilde{F}; \tilde{\mathcal{O}}_L)) \text{ by Equality (11),} \\ &= \frac{R_{P,\tilde{F}}(U'_z)}{[U(\tilde{E}; \tilde{\mathcal{O}}_L) : U'_z]_{\tilde{A}}} \text{ by Lemma 4.20,} \\ &= \frac{R_{P,\tilde{E}}(U'_z)}{[U(\tilde{E}; \tilde{\mathcal{O}}_L) : U'_z]_{\tilde{A}}} \text{ by Equality (9),} \\ &= R_P(U(\tilde{E}; \tilde{\mathcal{O}}_L)) \text{ by Lemma 4.20.} \end{aligned}$$

Recall Theorem 4.13:

$$L_P(E/\mathcal{O}_L) = R_P(U(F; \mathcal{O}_L)) [H(F; \mathcal{O}_L)]_A.$$

Then:

$$\begin{aligned} z_P(E/\mathcal{O}_L)L_P(E/\mathcal{O}_L) &= \frac{[U(F; \mathcal{O}_L) : U']_A}{[U(E; \mathcal{O}_L) : U']_A} \frac{[H(E; \mathcal{O}_L)]_A}{[H(F; \mathcal{O}_L)]_A} R_P(U(F; \mathcal{O}_L)) [H(F; \mathcal{O}_L)]_A \text{ by Equality (10),} \\ &= \frac{R_{P,F}(U')}{[U(E; \mathcal{O}_L) : U']_A} [H(E; \mathcal{O}_L)]_A \text{ by Lemma 4.20,} \\ &= \frac{R_{P,E}(U')}{[U(E; \mathcal{O}_L) : U']_A} [H(E; \mathcal{O}_L)]_A \text{ by Equality (9),} \\ &= R_P(U(E; \mathcal{O}_L)) [H(E; \mathcal{O}_L)]_A \text{ by Lemma 4.20.} \end{aligned}$$

□

**4.7. Vanishing of the  $P$ -adic  $L$ -series.** We keep the notation as in Theorem 4.10. In particular,  $(v_1, \dots, v_m)$  is an  $A$ -basis of  $U_{\text{St}}(E; \mathcal{O}_L)$  and  $(u_1, \dots, u_m)$  is an  $A$ -basis of  $U(E; \mathcal{O}_L)$ .

**Proposition 4.22.** *Suppose that there exists a non-zero element  $x \in U_{\text{St}}(E; \mathcal{O}_L)$  such that  $\exp_E(x) = 0$ . Then*

$$L_P(E/\mathcal{O}_L) = 0.$$

*Proof.* Write  $x = \sum_{i=1}^m a_i v_i$  with  $a_i \in A$  and suppose without loss of generality that  $a_1 \neq 0$ .

Then  $x = x(1)$  with  $x(z) = \sum_{i=1}^m a_i v_i(z) \in U(\tilde{E}, \mathcal{O}_L[z])$ . We have:

$$\begin{aligned} &\det_{\mathcal{C}}(\text{Log}_{\tilde{E},P}(\exp_{\tilde{E}}(v_i(z))), i = 1, \dots, m) \\ &= \frac{1}{a_1} \det_{\mathcal{C}}(\text{Log}_{\tilde{E},P}(\exp_{\tilde{E}}(x(z))), \text{Log}_{\tilde{E},P}(\exp_{\tilde{E}}(v_i(z))), i = 2, \dots, m). \end{aligned}$$

Since  $\exp_E(x) = 0$ , we have:

$$\text{ev}_{z=1,P}(\text{Log}_{\tilde{E},P}(\exp_{\tilde{E}}(x(z)))) = \text{Log}_{E,P}(\exp_E(x)) = 0.$$

We then conclude that  $R_P(U_{\text{St}}(E; \mathcal{O}_L)) = 0$ , so  $L_P(E/\mathcal{O}_L) = 0$  from the  $P$ -adic class formula 4.21.

□

**Theorem 4.23.** *If the exponential map  $\exp_E : L_\infty^d \rightarrow L_\infty^d$  is not injective, then we have*

$$L_P(E/\mathcal{O}_L) = 0.$$

*Proof.* Let  $x \in L_\infty^d$  be non-zero such that  $\exp_E(x) = 0$ . There exists  $a \in A \setminus \{0\}$  such that  $ax \in U_{\text{st}}(E; \mathcal{O}_L)$  and we have  $\exp_E(ax) = 0$ . By Proposition 4.22 we have  $L_P(E/\mathcal{O}_L) = 0$ .  $\square$

We believe that the converse statement holds.

**Conjecture.** *The  $P$ -adic  $L$ -series is non-zero if and only if the exponential map  $\exp_E : L_\infty^d \rightarrow L_\infty^d$  is injective.*

By [7, Corollary 3.24], it is true when  $d = 1$  (i.e., in the Drinfeld module case) and  $L = K$ . Remark that in the case  $\exp_E : L_\infty^d \rightarrow L_\infty^d$  is injective, which we will call the totally real case, then  $\mathcal{U}(E; \mathcal{O}_L) = \exp_E(U(E; \mathcal{O}_L)) \subseteq E(\mathcal{O}_L)$  is a free  $A$ -module of rank  $m$ , and the family  $(\text{Log}_{E,P}(\exp_E(u_i)), i = 1, \dots, m)$  is  $A$ -free. We would like to have that this family is  $A_P$ -free to obtain the non-vanishing of the  $P$ -adic  $L$ -series. Set:  $U(E; P\mathcal{O}_L) = \{x \in \text{Lie}_E(L_\infty) \mid \exp_E(x) \in E(P\mathcal{O}_L)\}$  and  $\mathcal{U}(E; P\mathcal{O}_L) = \exp_E(U(E; P\mathcal{O}_L))$ . Then we can state an equivalent of the Leopoldt's conjecture in [16], introduced recently by Anglès in [3, Section 6.3] for the Carlitz module.

**Conjecture (Conjecture A).** *The  $A_P$ -rank of  $\mathcal{U}(E; P\mathcal{O}_L)$  is equal to the  $A$ -rank of  $\mathcal{U}(E; \mathcal{O}_L)$ .*

This conjecture is clear in the case  $d = 1$  and  $L = K$ . For further discussion of this conjecture, the reader may wish to see the paper by Anglès, Bosser and Taelman [6] where this conjecture is proved in the case of the Carlitz module defined on the  $P$ th cyclotomic extension.

In the totally real case, the non-vanishing of the  $P$ -adic  $L$ -series  $L_P(\widetilde{E}/\widetilde{\mathcal{O}_L})$  at  $z = 1$  is equivalent to the previous Leopoldt conjecture. This result can be seen as an analog to the following result from [10].

**Theorem 4.24.** *Let  $F$  be a totally real extension of  $\mathbb{Q}$ . Then the  $p$ -adic zeta function  $\zeta_{F,p}(s)$  has a simple pole at  $s = 1$  if and only if the (usual) Leopoldt conjecture is true for  $(F, p)$ .*

**Definition 4.25.** We call order of vanishing of the  $P$ -adic  $L$ -series and denote by  $\text{ord}_{z=1} L_P(\widetilde{E}/\widetilde{\mathcal{O}_L})$ , the greatest integer  $n$  such that  $(z - 1)^n$  divides  $L_P(\widetilde{E}/\widetilde{\mathcal{O}_L})$ .

For example if the exponential map  $\exp_E : L_\infty^d \rightarrow L_\infty^d$  is not injective, then for all  $P$  we have  $\text{ord}_{z=1} L_P(\widetilde{E}/\widetilde{\mathcal{O}_L}) \geq 1$  and the previous conjecture tells us that  $\text{ord}_{z=1} L_P(\widetilde{E}/\widetilde{\mathcal{O}_L}) = 0$  if and only if  $\exp_E$  is injective.

Here is a list of conjectures.

**Conjecture (Conjecture B).** *The vanishing order of the  $P$ -adic  $L$ -series at  $z = 1$  is independent of  $P$ .*

Caruso and Gazda [8] have already conjectured this in the context of Anderson motives. Caruso, Gazda and the author proved this conjecture in the case  $L = K$  and  $d = 1$ , see [9, Theorem 2.17].

**Conjecture (Conjecture C).** *We have  $\text{ord}_{z=1} L_P(\widetilde{E}/\widetilde{\mathcal{O}_L}) \leq [L : K]r_{\Omega_E}d$  where  $r_{\Omega_E}$  is the rank of the period lattices  $\Omega_E$  associated with  $E$ .*

We prove conjecture C in section 6 in the case  $d = 1$  and  $L = K$ .

## 5. THE MULTI-VARIABLE SETTING

We keep the notation from Sections 2, 3, 4 and from the Introduction. In particular  $L/K$  is a finite field extension of degree  $n$  and  $\mathcal{O}_L$  denotes the integral closure of  $A$  in  $L$ .

The aim of this section is to extend the previous constructions to the case where the constant field is no longer  $\mathbb{F}_q$  but  $\mathbb{F}_q(t_1, \dots, t_s)$  where the  $t_i$  are new variables. One of the

interests of these constructions is that in many cases, we can reduce the study of certain  $t$ -modules  $E : \mathbb{F}_q[\theta] \rightarrow M_d(\mathcal{O}_L)\{\tau\}$  to the study of Drinfeld modules  $\phi : \mathbb{F}_q(t_1, \dots, t_s)[\theta] \rightarrow \mathcal{O}_L(t_1, \dots, t_s)\{\tau\}$  simpler to understand. For an application to the study of the tensor power of the Carlitz module  $C^{\otimes n}$  reduced to the study of the Carlitz module  $C$ , see the work of Anglès, Pellarin and Tavares Ribeiro in [5].

**5.1. Setup.** The goal of this section is to extend the developed theory to the multi-variable setting by replacing  $\mathbb{F}_q$  by  $k = \mathbb{F}_q(t_1, \dots, t_s)$ . Recall that the Frobenius map acts as the identity on  $k$ . We keep the notation in the Introduction and we introduce the following notation.

- $A_s[z] \simeq k[z] \otimes_k A_s$ ,
- $\widehat{A}_s = k(z) \otimes_k A_s$ ,
- $w$ : a place of  $K$  ( $w = v_P$  a finite place or  $w = v_\infty$  the infinite place),
- $\pi_w$ : a uniformiser of  $w$  ( $\pi = P$  if  $w = v_P$  and  $\pi = \frac{1}{\theta}$  if  $w = v_\infty$ ),
- $K_w = \mathbb{F}_w((\pi_w))$  denoted by  $K_w = K_\infty$  if  $w = v_\infty$  and  $K_w = K_P$  if  $w = v_P$ ,
- $\mathbb{F}_w$ : the residue field associated with  $w$  i.e.,  $\mathbb{F}_w = \mathbb{F}_P$  if  $w = v_P$  and  $\mathbb{F}_w = \mathbb{F}_q$  if  $w = v_\infty$ ,
- $L_w = L \otimes_K K_w$  i.e.,  $L_w = L_P$  if  $w = v_P$  and  $L_w = L_\infty$  else,
- $k_w = \mathbb{F}_w(t_1, \dots, t_s)$ ,
- $\widetilde{K_{s,w}} = k_w((\pi_w))$  denoted by  $K_{s,P}$  if  $w = v_P$  and  $K_{s,\infty}$  if  $w = v_\infty$ ,
- $\widetilde{K_{s,w}} = k_w(z)((\pi_w))$ ,
- $L_s = kL$ ,
- $L_{s,w} = L \otimes_K \widetilde{K_{s,w}}$  denoted by  $L_{s,P}$  if  $w = v_P$ ,
- $\widetilde{L_{s,w}} = L \otimes_K \widetilde{K_{s,w}}$  denoted by  $\widetilde{L_{s,P}}$  if  $w = v_P$ ,
- $\mathcal{O}_{L,s}[z] \simeq k[z] \otimes_k \mathcal{O}_{L,s}$ ,
- $\mathcal{O}_{L,s} = k(z) \otimes_k \mathcal{O}_{L,s}$ .

We recall that every  $x \in \widetilde{K_{s,\infty}}^*$  (resp.  $\in K_{s,\infty}$ ) can be written uniquely as  $x = \sum_{n \geq N} x_n \frac{1}{\theta^n}$

with  $N \in \mathbb{Z}$ ,  $x_n \in k(z)$  (resp.  $x_n \in K$ ) and  $x_N \neq 0$ . We call  $x_N \in k(z)$  (resp.  $k$ ) the sign of  $x$  denoted by  $\text{sgn}(x)$ . We define the Tate algebra in variables  $\underline{t} = (t_1, \dots, t_s)$ :

$$\mathbb{T}_s(K_w) = \left\{ \sum_{n \in \mathbb{N}^s} a_n \underline{t}^n \in K_w[[\underline{t}]] \mid a_n \in K_w, \lim_{n \rightarrow +\infty} w(a_n) = +\infty \right\}$$

where  $\underline{t}^n = t_1^{n_1} \dots t_s^{n_s}$  if  $n = (n_1, \dots, n_s) \in \mathbb{N}^s$ . This is the completion of  $K[t_1, \dots, t_s]$  with respect to the Gauss norm associated with  $w$ . We set:

$$\mathbb{T}_s(L_w) = L \otimes_K \mathbb{T}_s(K_w).$$

An Anderson  $t$ -module  $E$  of dimension  $d$  over  $\mathcal{O}_{L,s}$  is a non-constant  $k$ -algebra homomorphism  $E : A_s \rightarrow M_d(\mathcal{O}_{L,s})$ ,  $a \mapsto E_a = \sum_{i=0}^{r_a} E_{a,i} \tau^i \in M_d(\mathcal{O}_{L,s})\{\tau\}$  such that  $(E_{\theta,0}^d - \theta I_d)^d = 0$ .

We can consider  $\widetilde{E}$ , the  $z$ -twist of  $E$ , as in Section 3. Following notation from Section 2, we denote by  $[M]_{A_s}$  the monic generator of  $\text{Fitt}_{A_s}(M)$  where  $M$  is a torsion  $A_s$ -module of finite type, e.g.,  $M = E(\mathcal{O}_{L,s}/P\mathcal{O}_{L,s})$  and  $M = \text{Lie}_E(\mathcal{O}_{L,s}/P\mathcal{O}_{L,s})$ .

As in Proposition 3.1 there exists a unique element  $\exp_E \in M_d(L_s)\{\{\tau\}\}$  called the exponential map associated with  $E$  and converging over  $L_{s,\infty}^d$ . Similarly, there exists a logarithm map  $\log_E \in M_d(L_s)\{\{\tau\}\}$  as in Proposition 3.2.

**5.2. The  $\infty$ -case.** We can now define the module of units and class module in the multi-variable setting:

$$U(E; \mathcal{O}_{L,s}) = \{x \in \text{Lie}_E(L_{s,\infty}) \mid \exp_E(x) \in E(\mathcal{O}_{L,s})\}$$

and the class module

$$H(E; \mathcal{O}_{L,s}) = \frac{E(L_{s,\infty})}{E(\mathcal{O}_{L,s}) + \exp_E(\text{Lie}_E(L_{s,\infty}))}$$

both provided with  $A_s$ -module structure. We define the module of  $z$ -units:

$$U(\tilde{E}; \widetilde{\mathcal{O}_{L,s}}) = \left\{ x \in \text{Lie}_{\tilde{E}}(\widetilde{L_{s,\infty}}) \mid \exp_{\tilde{E}}(x) \in \tilde{E}(\widetilde{\mathcal{O}_{L,s}}) \right\}$$

and the class module for the  $z$ -deformation:

$$H(\tilde{E}; \widetilde{\mathcal{O}_{L,s}}) = \frac{\tilde{E}(\widetilde{L_{s,\infty}})}{\tilde{E}(\widetilde{\mathcal{O}_{L,s}}) + \exp_{\tilde{E}}(\text{Lie}_{\tilde{E}}(\widetilde{L_{s,\infty}}))}$$

both provided with  $\widetilde{A_s}$ -module structure. We define the module of  $z$ -units at the integral level:

$$U(\tilde{E}; \mathcal{O}_{L,s}[z]) = \left\{ x \in \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_{s,\infty})) \mid \exp_{\tilde{E}}(x) \in \tilde{E}(\mathcal{O}_{L,s}[z]) \right\}$$

and finally the class module at the integral level

$$H(\tilde{E}; \mathcal{O}_{L,s}[z]) = \frac{\tilde{E}(\mathbb{T}_z(L_{s,\infty}))}{\tilde{E}(\mathcal{O}_{L,s}[z]) + \exp_{\tilde{E}}(\text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_{s,\infty})))}$$

both provided with  $A_s[z]$ -module structure. We have the following result from [11, Proposition 2.8].

**Proposition 5.1.** *The unit module  $U(E; \mathcal{O}_{L,s})$  is an  $A_s$ -lattice in  $\text{Lie}_E(L_{s,\infty})$  and the module of  $z$ -units  $U(\tilde{E}, \widetilde{\mathcal{O}_{L,s}})$  is an  $\widetilde{A_s}$ -lattice in  $\text{Lie}_{\tilde{E}}(\widetilde{L_{s,\infty}})$ .*

Denote by

$$z_P(\tilde{E}/\widetilde{\mathcal{O}_{L,s}}) = \frac{[\text{Lie}_{\tilde{E}}(\widetilde{\mathcal{O}_{L,s}/P\mathcal{O}_{L,s}})]_{\widetilde{A_s}}}{[\tilde{E}(\widetilde{\mathcal{O}_{L,s}/P\mathcal{O}_{L,s}})]_{\widetilde{A_s}}}$$

the local factor associated with  $\tilde{E}$  at  $P$  and

$$z_P(E/\mathcal{O}_{L,s}) = \frac{[\text{Lie}_E(\mathcal{O}_{L,s}/P\mathcal{O}_{L,s})]_{A_s}}{[E(\mathcal{O}_{L,s}/P\mathcal{O}_{L,s})]_{A_s}}$$

the local factor associated with  $E$  at  $P$ . We have the following class formula for  $t$ -modules defined over  $\mathcal{O}_{L,s}$ , see [11, Theorem 2.9].

**Theorem 5.2.** *The following product*

$$L(\tilde{E}/\widetilde{\mathcal{O}_{L,s}}) = \prod_P z_P(\tilde{E}/\widetilde{\mathcal{O}_{L,s}})$$

where  $P$  runs through the monic primes of  $A$ , converges in  $\widetilde{K_{s,\infty}}$  and we have the class formula:

$$L(\tilde{E}/\widetilde{\mathcal{O}_{L,s}}) = \left[ \text{Lie}_{\tilde{E}}(\widetilde{\mathcal{O}_{L,s}}) : U(\tilde{E}; \widetilde{\mathcal{O}_{L,s}}) \right]_{\widetilde{A_s}}.$$

**5.3. The  $P$ -adic case.** We define the  $P$ -adic  $L$ -series in the multi-variable setting.

**5.3.1. The  $P$ -adic class formula.** All results from Section 4 remain valid by replacing  $\mathbb{F}_q$  by  $k$ . In particular, we have the following  $P$ -adic class formula.

**Theorem 5.3** ( $P$ -adic class formula). *We have the following assertions.*

(1) *The infinite product*

$$L_P(\widetilde{E}/\widetilde{\mathcal{O}_{L,s}}) = \prod_{Q \neq P} z_Q(\widetilde{E}/\widetilde{\mathcal{O}_{L,s}})$$

where  $Q$  runs through the monic primes of  $A$  different from  $P$ , converges in  $\mathbb{T}_z(K_{s,P})$  and we have the class formula:

$$z_P(\widetilde{E}/\widetilde{\mathcal{O}_{L,s}}) L_P(\widetilde{E}/\widetilde{\mathcal{O}_{L,s}}) = R_P(U(\widetilde{E}; \widetilde{\mathcal{O}_{L,s}})).$$

(2) *The infinite product*

$$L_P(E/\mathcal{O}_{L,s}) = \prod_{Q \neq P} z_Q(E/\mathcal{O}_{L,s})$$

where  $Q$  runs through the monic primes of  $A$  different from  $P$ , converges in  $K_{s,P}$  and we have the class formula:

$$z_P(E/\mathcal{O}_{L,s}) L_P(E/\mathcal{O}_{L,s}) = R_P(U(E; \mathcal{O}_{L,s})) [H(E; \mathcal{O}_{L,s})]_{A_s}.$$

*Proof.* The proof follows the same lines as the proof of 4.10 by replacing  $\mathbb{F}_q$  by  $k$ . We omit the details.  $\square$

Denote by  $U(E; P\mathcal{O}_{L,s}) = \{x \in \text{Lie}_E(L_{s,\infty}) \mid \exp_E(x) \in E(P\mathcal{O}_{L,s})\}$  and consider the  $A_s$ -module  $\mathcal{U}(E; \mathcal{O}_{L,s}) = \exp_E(U(E; \mathcal{O}_{L,s}))$ . Consider also the  $A_{P,s}$ -module  $\mathcal{U}(E; P\mathcal{O}_{L,s}) = \exp_E(U(E; P\mathcal{O}_{L,s}))$ . Then the proof of Theorem 4.23 is still valid in the multi-variable setting by replacing  $\mathbb{F}_q$  by  $k$ .

**Proposition 5.4.** *We have the following assertions.*

- (1) *If the exponential map  $\exp_E : L_{s,\infty}^d \rightarrow L_{s,\infty}^d$  is not injective, then  $L_P(E/\mathcal{O}_{L,s}) = 0$ .*
- (2) *Assume that the  $A_s$ -rank of  $\mathcal{U}(E; \mathcal{O}_{L,s})$  and the  $A_{P,s}$ -rank of  $\mathcal{U}(E; P\mathcal{O}_{L,s})$  are equal. Then  $L_P(E/\mathcal{O}_{L,s}) \neq 0$  if and only if the exponential map  $\exp_E : L_{s,\infty}^d \rightarrow L_{s,\infty}^d$  is injective.*

5.3.2. *The integral level.* In the work of [5], given a  $t$ -module  $E : A_s \rightarrow M_d(A_s)\{\tau\}$ , they want to evaluate the variables  $(t_1, \dots, t_s)$  at some  $\zeta \in \overline{\mathbb{F}_q^s}$ . In this case, they need that all of the coefficients  $E_{\theta,i}$  of  $E_\theta$ , for  $i = 0, \dots, r$ , can be evaluated at  $\zeta$ . This is possible if all the  $E_{i,\theta}$  belong to  $M_d(\mathbb{F}_q[t_1, \dots, t_s]\mathcal{O}_L)$ . This is what we call the integral level.

We suppose now that:  $E_\theta \in M_d(\mathcal{O}_L[t_1, \dots, t_s])\{\tau\}$  i.e., we want to work at the integral level.

**Theorem 5.5.** *The  $L$ -series  $L(\widetilde{E}/\widetilde{\mathcal{O}_{L,s}})$  converges in  $\mathbb{T}_{s,z}(K_\infty)$  and we have the class formula:*

$$L(\widetilde{E}/\widetilde{\mathcal{O}_{L,s}}) = \frac{\det_{\mathcal{C}}(u_1(z), \dots, u_m(z))}{\text{sgn}(\det_{\mathcal{C}}(u_1(z), \dots, u_m(z)))}$$

where  $(u_1(z), \dots, u_m(z)) \in U(\widetilde{E}; \mathcal{O}_L[t_1, \dots, t_s, z])$  is an  $\widetilde{A}_s$ -basis of the  $z$ -unit module.

*Proof.* The proof of [22, Corollary 7.5.6] is still valid in the multi-variable setting at the integral level. We omit the details.  $\square$

The objective of this section is to prove that the  $P$ -adic  $L$ -series  $L_P(\widetilde{E}/\widetilde{\mathcal{O}_{L,s}})$  converges in  $\mathbb{T}_{s,z}(K_P)$ .

Set  $\Omega_{s,z} = \{x \in \mathbb{T}_{z,s}(L_P)^d \mid v_P(x) \geq 0\}$  and  $\Omega_{s,z}^+ = \{x \in \mathbb{T}_{z,s}(L_P)^d \mid v_P(x) > 0\}$ .

Following the proof of Proposition 4.5 we have the two following convergences:

$$\log_{\widetilde{E},P} : \Omega_{s,z}^+ \rightarrow \mathbb{T}_{s,z}(L_P)^d$$

and

$$\log_{\widetilde{F},P} : \Omega_{s,z} \rightarrow \mathbb{T}_{s,z}(L_P)^d.$$

We deduce that the  $P$ -adic  $L$ -series  $L_P(\widetilde{E}/\widetilde{\mathcal{O}_{L,s}})$  is written in the form  $\frac{w}{f}$  with  $w \in \mathbb{T}_{z,s}(K_P)$  and  $f \in \mathbb{F}_q[t_1, \dots, t_s, z]$ . We then consider  $\zeta = (\zeta_1, \dots, \zeta_s) \in \overline{\mathbb{F}_q}^s$  and we want to prove that we can evaluate the  $P$ -adic  $L$ -series at  $t_i = \zeta_i$  for all  $i = 1, \dots, s$  and at  $z = \zeta \in \overline{\mathbb{F}_q}$  (simultaneously).

We use arguments very similar to those used for the convergence of the  $P$ -adic  $L$ -series, so we omit some of the details.

We set  $\mathcal{K}(s) = \mathbb{F}_q(\zeta_1) \otimes_{\mathbb{F}_q} \dots \otimes_{\mathbb{F}_q} \mathbb{F}_q(\zeta_s)$ . We then consider the following notation for  $j = 0, \dots, s$ :

- $k_j = \mathbb{F}_q(t_{j+1}, \dots, t_s)$ , e.g.,  $k_0 = k = \mathbb{F}_q(t_1, \dots, t_s)$  and  $k_s = \mathbb{F}_q$ ,
- $k_j A = k_j \otimes_{\mathbb{F}_q} A \simeq \mathbb{F}_q(t_{j+1}, \dots, t_s)[\theta]$ ,
- $k_j K = k_j \otimes_{\mathbb{F}_q} K \simeq \mathbb{F}_q(t_{j+1}, \dots, t_s, \theta)$  and  $\widetilde{k_j K} = \mathbb{F}_q(z) \otimes_{\mathbb{F}_q} k_j K \simeq \mathbb{F}_q(z, t_{j+1}, \dots, t_s, \theta)$ ,
- $k_j \mathcal{O}_L = k_j \otimes_{\mathbb{F}_q} \mathcal{O}_L$ ,
- $A_{s,j} = \mathcal{K}(s) \otimes_{\mathbb{F}_q} k_j A$ ,
- $\widetilde{A_{s,j}} = \mathcal{K}(s) \otimes_{\mathbb{F}_q} k_j \widetilde{A}$ ,
- $\mathcal{O}_{L,s,j} = \mathcal{K}(s) \otimes_{\mathbb{F}_q} k_j \mathcal{O}_L$ ,
- $\widetilde{\mathcal{O}_{L,s,j}} = \mathcal{K}(s) \otimes_{\mathbb{F}_q} k_j \widetilde{\mathcal{O}_L}$ ,
- For a place  $w$  of  $K$  extended to  $k_j K$ ,  $\widetilde{K(j)}_w$  is the completion of  $\widetilde{k_j K}$  with respect to  $w$ .
- $\widetilde{L(j)}_w = L \otimes_K K(j)_w$ ,
- $\widetilde{M_{s,j,w}} = \mathcal{K}(s) \otimes_{\mathbb{F}_q} \widetilde{\mathbb{T}_{t_j}(L(j)_w)}$ ,
- $\widetilde{L_{j,w}} = L \otimes_K \widetilde{K_{j,w}}$ .
- For a place  $w$  of  $K$ ,  $\mathbb{T}_{z,j}(L_w) = \mathbb{T}_{z,t_{j+1}, \dots, t_s}(L_w)$ , e.g.,  $\mathbb{T}_{z,0}(L_w) = \mathbb{T}_{z,t_1, \dots, t_s}(L_w)$  and  $\mathbb{T}_{z,s}(L_w) = \mathbb{T}_z(L_w)$ .

For all  $j = 0, \dots, s$ , we extend the Frobenius  $\tau$  into  $\tau_s$  on  $A_{s,j}$  by  $\tau_s = \text{id} \otimes \tau$  where  $\text{id}$  is the identity on  $\mathcal{K}(s)$ . We do the same for  $\widetilde{A_{s,j}}$ , for  $\mathcal{O}_{L,s,j}$  and for  $\widetilde{\mathcal{O}_{L,s,j}}$ .

For  $j = 1, \dots, s$  we define  $E^{(j)}$  the homomorphism of  $\mathcal{K}(s) \otimes_{\mathbb{F}_q} k_j$ -algebras  $E^{(j)} : A_{s,j} \rightarrow M_d(\mathcal{O}_{L,s,j})\{\tau_s\}$ , that we call Anderson  $A_{s,j}$ -module defined over  $\mathcal{O}_{L,s,j}$ , by:

$$E_\theta^{(j)} = \sum_{i=0}^r \text{ev}_{t_1=\zeta_1, \dots, t_j=\zeta_j}(a_i) \tau_s^i$$

if  $E_\theta = \sum_{i=0}^r a_i \tau_s^i \in M_d(\mathcal{O}_{L,s})\{\tau\}$ . We also set  $E^{(0)} = E$  where we identify  $a_i$  with  $1 \otimes a_i \in \mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathcal{O}_L[t_1, \dots, t_s]$  and replace  $\tau$  with  $\tau_s$ .

Similarly, we define  $\widetilde{E^{(j)}}$  the  $z$ -twist of  $E^{(j)}$ , that is the homomorphism of  $\mathcal{K}(z) \otimes_{\mathbb{F}_q} k_j(z)$ -algebras  $\widetilde{E^{(j)}} : \widetilde{A_{s,j}} \rightarrow M_d(\widetilde{\mathcal{O}_{L,s,j}})\{\tau_s\}$ , that we call Anderson  $\widetilde{A_{s,j}}$ -module defined over  $\widetilde{\mathcal{O}_{L,s,j}}$ , by:

$$\widetilde{E^{(j)}}_\theta = \sum_{i=0}^r \text{ev}_{t_1=\zeta_1, \dots, t_j=\zeta_j}(a_i) z^i \tau_s^i$$

Finally, we consider  $F = P^{-1}EP$  and construct  $\widetilde{F^{(j)}}$  and  $F^{(j)}$  in the same way.

**Lemma 5.6.** *Consider  $j \in \{1, \dots, s\}$ .*

(1) *For all  $a \in A_{s,j}$  we have the following equalities in  $M_d(\mathcal{O}_{L,s,j})\{\tau_s\}$ :*

$$E_a^{(j)} = \text{ev}_{t_1=\zeta_1, \dots, t_j=\zeta_j} E_a = \text{ev}_{t_j=\zeta_j} E_a^{(j-1)}.$$

(2) *We have the following equalities in  $M_d(\mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{F}_q[t_{j+1}, \dots, t_s]L)\{\{\tau_s\}\}$ :*

$$\exp_{E^{(j)}} = \text{ev}_{t_1=\zeta_1, \dots, t_j=\zeta_j} \exp_E = \text{ev}_{t_j=\zeta_j} \exp_{E^{(j-1)}}.$$

(3) We have the following equalities for all  $x$  in  $(\mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{t_j} ((\mathbb{F}_q(t_{j+1}, \dots, t_s)L)_\infty))^d$ :

$$\exp_{E^{(j)}}(x) = \text{ev}_{t_1=\zeta_1, \dots, t_j=\zeta_j} \exp_E(x) = \text{ev}_{t_j=\zeta_j} \exp_{E^{(j-1)}}(x).$$

*Proof.* It follows from definitions of the objects, we omit the proof.  $\square$

We then define for all  $j = 1, \dots, s$ :

$$U(j) = \text{ev}_{t_j=\zeta_j} U\left(\widetilde{E}^{(j-1)}; \widetilde{\mathcal{O}_{L,s,j}}[t_j]\right) \subseteq U\left(\widetilde{E}^{(j)}; \widetilde{\mathcal{O}_{L,s,j}}\right).$$

Following the same arguments we used to prove Theorem 3.14, we have the following result.

**Theorem 5.7.**

(1) For all  $j = 1, \dots, s$ , we have an  $\widetilde{A_{s,j}}$ -module isomorphism:

$$\frac{U(j)}{U\left(\widetilde{E}^{(j)}; \widetilde{\mathcal{O}_{L,s,j}}\right)} \simeq H\left(\widetilde{E}^{(j-1)}; \widetilde{\mathcal{O}_{L,s,j}}[t_j]\right) [t_j - \zeta_j]$$

given by

$$f_j(x) = \frac{\exp_{\widetilde{E}^{(j-1)}} x - \exp_{\widetilde{E}^{(j)}} x}{t_j - \zeta_j}$$

where  $H\left(\widetilde{E}^{(j-1)}; \widetilde{\mathcal{O}_{L,s,j}}[t_j]\right) [t_j - \zeta_j]$  is the  $(t_j - \zeta_j)$ -torsion of the class module

$$H\left(\widetilde{E}^{(j-1)}; \widetilde{\mathcal{O}_{L,s,j}}[t_j]\right) = \frac{\widetilde{E}^{(j-1)}\left(\widetilde{M_{s,j,\infty}}\right)}{\widetilde{E}^{(j-1)}\left(\widetilde{\mathcal{O}_{L,s,j}}[t_j]\right) + \exp_{\widetilde{E}^{(j-1)}}\left(\widetilde{E}^{(j-1)}\left(\widetilde{M_{s,j,\infty}}\right)\right)}.$$

(2) The module  $U(j)$  is a sub- $\widetilde{A_{s,j}}$  lattice of  $U\left(\widetilde{E}^{(j)}; \widetilde{\mathcal{O}_{L,s,j}}\right)$ .

We are now able to prove the main theorem of this section.

**Theorem 5.8.** The  $P$ -adic  $L$ -series does not have a pole in  $\overline{\mathbb{F}_q^s}$ . In other words we have:

$$L_P(\widetilde{E}/\mathcal{O}_{L,s}) \in \mathbb{T}_{z,s}(K_P).$$

*Proof.* We closely follow the proof of Theorem 4.10. We identify  $\mathcal{C} = (g_1, \dots, g_m)$  with  $(1 \otimes g_1, \dots, 1 \otimes g_m) \subseteq \mathcal{K}(s) \otimes \text{Lie}_E(\mathcal{O}_L)$ . Consider  $(u_{i,1})_{i=1,\dots,m} \subseteq U(\widetilde{F}; \mathcal{O}_L[t_1, \dots, t_s, z])$  an  $\widetilde{A_{s,k}}$ -basis of  $U(\widetilde{F}; \mathcal{O}_{L,s})$ . Set

$$w_1 = \det_{\mathcal{C}}(u_{1,1}, \dots, u_{m,1}) \in \mathbb{T}_{z,s}(K_\infty)$$

with sign

$$f_1 \in \mathbb{F}_q[z, t_1, \dots, t_s]$$

and set

$$w_{1,P} = \det_{\mathcal{C}}(\log_{\widetilde{F},P}(\exp_{\widetilde{F}}(u_{1,1})), \dots, \log_{\widetilde{F},P}(\exp_{\widetilde{F}}(u_{m,1}))) \in \mathbb{T}_{z,s}(K_P).$$

We want to prove that the quotient  $\frac{w_{1,P}}{f_1}$  is an element of  $\mathbb{T}_{s,z}(K_P)$ . We will prove that we can evaluate the last quantity at every  $\zeta = (\zeta_1, \dots, \zeta_s) \in \overline{\mathbb{F}_q^s}$  and at  $z = \zeta \in \overline{\mathbb{F}_q}$ .

We will prove by induction that for all  $k = 1, \dots, s$ , there exists  $(v_{i,k+1}, i = 1, \dots, m)$  an  $\widetilde{A_{s,k}}$ -basis of  $U(k)$  and  $x_{k+1} \in \mathcal{K}(s) \otimes_{\mathbb{F}_q} \widetilde{A_{s,k}}$  such that

$$\text{ev}_{t_1=\zeta_1, \dots, t_k=\zeta_k}(1 \otimes L(\widetilde{F}/\mathcal{O}_{L,s})) = \frac{\det_{\mathcal{C}}(v_{i,k+1}, i = 1, \dots, m)}{x_{k+1}}$$



and

$$\begin{aligned} \text{ev}_{t_1=\zeta_1, \dots, t_k=\zeta_k} \left( 1 \otimes \frac{w_{1,P}}{f_1} \right) &= \frac{\det_{\mathcal{C}}(\log_{\tilde{F}^{(k)},P} \exp_{\tilde{F}^{(k)}} v_{i,k+1}, i = 1, \dots, m)}{x_{k+1}} \\ &\in \frac{1}{x_{k+1}} \mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_z(t_{k+1}, \dots, t_s)(L_P). \end{aligned}$$

Step 1: evaluation at  $t_1 = \zeta_1$ .

By Theorem 5.5 we have

$$L(\tilde{F}/\widetilde{\mathcal{O}_{L,s}}) = \frac{w_1}{f_1} \in \mathbb{T}_{z,s}(K_\infty).$$

Consider  $(v_{i,2})_{i=1,\dots,m}$  an  $\widetilde{A_{s,1}}$  basis of  $U(1)$  that can be assumed to be at the entire level, i.e.,  $(v_{i,2}) \subseteq (\mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_2,\dots,t_s}(L_\infty))^d$  and let  $(u_{i,2}) \subseteq (\mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,s}(L_\infty))^d$  be above (i.e.,  $\text{ev}_{t_1=\zeta_1} u_{i,2} = v_{i,2}$ ) for all  $i$ ). Set

$$w_2 = \det_{\mathcal{C}}(u_{1,2}, \dots, u_{m,2}) \in \mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,s}(K_\infty)$$

that is not divisible by  $t_1 - \zeta_1$  and set

$$w_{2,P} = \det_{\mathcal{C}}(\log_{\tilde{F},P}(\exp_{\tilde{F}}(u_{1,2})), \dots, \log_{\tilde{F},P}(\exp_{\tilde{F}}(u_{m,2}))) \in \mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,s}(K_P)$$

the  $P$ -adic analog of  $w_2$ . Set  $\delta_2 = \det_{(1 \otimes u_{1,1}, \dots, 1 \otimes u_{m,1})}(u_{1,2}, \dots, u_{m,2}) \in \mathcal{K}(s) \otimes_{\mathbb{F}_q} \tilde{A}_s$ . We have:

$$1 \otimes_{\mathbb{F}_q} w_1 = \frac{w_2}{\delta_2}$$

and

$$1 \otimes w_{1,P} = \frac{w_{2,P}}{\delta_2}.$$

We deduce the following equality from the class formula:

$$1 \otimes L(\tilde{F}/\widetilde{\mathcal{O}_{L,s}}) = \frac{w_2}{\delta_2(1 \otimes f_1)}.$$

Since  $\zeta_1$  is not a pole of the  $L$ -series and  $t_1 - \zeta_1$  does not divide  $w_2$ , we deduce that we can evaluate  $\delta_2(1 \otimes f_1)$  at  $t_1 = \zeta_1$  in a non-zero element  $x_2$  of  $\mathcal{K}(s) \otimes_{\mathbb{F}_q} \tilde{A}_{s-1}$ , in other words

$$\text{ev}_{t_1=\zeta_1} 1 \otimes L(\tilde{F}/\widetilde{\mathcal{O}_{L,s}}) = \frac{\det_{\mathcal{C}}(v_{i,2})}{x_2} \in \frac{1}{x_2} \mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_2,\dots,t_s}(L_\infty).$$

Next, from the equality

$$1 \otimes \frac{w_{1,P}}{f_1} = \frac{w_{2,P}}{\delta_2(1 \otimes f_1)}$$

we deduce that we can evaluate the  $P$ -adic  $L$ -series at  $t_1 = \zeta_1$ :

$$\begin{aligned} \text{ev}_{t_1=\zeta_1} 1 \otimes \frac{w_{1,P}}{f_1} &= \frac{\det_{\mathcal{C}}(\log_{\tilde{F}^{(1)},P}(\exp_{\tilde{F}^{(1)}}(v_{i,2})), i = 1, \dots, m)}{x_2} \\ &\in \frac{1}{x_2} \mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_2,\dots,t_s}(L_P). \end{aligned}$$

Step  $k$ : Assume the result to be true up to rank  $s-1 \geq k-1 \geq 1$  and we prove it at rank  $k$ .

Consider  $(v_{i,k+1})_{i=1,\dots,m}$  a  $\widetilde{A_{s,k}}$  basis of  $U(k)$  that can be assumed to be to the entire level, i.e.,  $v_{i,k+1} \subseteq (\mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_{k+1},\dots,t_s}(L_\infty))^d$  and let  $(u_{i,k+1}) \subseteq (\mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_k,\dots,t_s}(L_\infty))^d$  be above (i.e.,  $\text{ev}_{t_k=\zeta_k} u_{i,k+1} = v_{i,k+1}$ ).

Set

$$w_{k+1} = \det_{\mathcal{C}}(u_{1,k+1}, \dots, u_{m,k+1}) \in \mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_k,\dots,t_s}(K_\infty)$$

that is not divisible by  $t_k - \zeta_k$  and set

$$w_{k+1,P} = \det_{\mathcal{C}}(\log_{\tilde{F}^{(k-1)}}(\exp_{\tilde{F}^{(k-1)}}(u_{i,k+1})), i = 1, \dots, m) \in \mathcal{K}_s \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_k,\dots,t_s}(K_P)$$

the  $P$ -adic analog of  $w_{k+1}$ . Set  $\delta_{k+1} = \det_{(v_{1,k}), \dots, v_{m,k}}(u_{1,k+1}, \dots, u_{m,k+1}) \in \mathcal{K}(s) \otimes \widetilde{A_{s,k-1}}$ . We have:

$$\det_{\mathcal{C}}(v_{1,k}, \dots, v_{m,k}) = \frac{\det_{\mathcal{C}}(u_{1,k+1}, \dots, u_{m,k+1})}{\delta_{k+1}}$$

and

$$\det_{\mathcal{C}}(\log_{\widetilde{F}^{(k-1),P}}(\exp_{\widetilde{F}^{(k-1)}}(v_{1,k})), \dots, \log_{\widetilde{F}^{(k-1),P}}(\exp_{\widetilde{F}^{(k-1)}}(v_{m,k}))) = \frac{w_{k+1,P}}{\delta_{k+1}}.$$

Then we have the following equalities:

$$\text{ev}_{t_1=\zeta_1, \dots, t_k=\zeta_k} \left( 1 \otimes L(\widetilde{F}/\widetilde{\mathcal{O}_{L,s}}) \right) = \frac{w_{k+1}}{x_k \delta_{k+1}}$$

and

$$\text{ev}_{t_1=\zeta_1, \dots, t_k=\zeta_k} \left( 1 \otimes \frac{w_{1,P}}{f_1} \right) = \frac{w_{k+1,P}}{x_k \delta_{k+1}}.$$

Since we can evaluate at  $t_k = \zeta_k$  the  $L$ -series and  $t_k - \zeta_k$  does not divide  $w_{k+1}$ , we can evaluate  $x_k \delta_{k+1}$  at  $t_k = \zeta_k$  into a non-zero element  $x_{k+1} \in \mathcal{K}(s) \otimes \widetilde{A_{s,k}}$ . We have:

$$\text{ev}_{t_k=\zeta_k, \dots, t_z=\zeta_1} (1 \otimes L(\widetilde{F}/\widetilde{\mathcal{O}_{L,s}})) = \text{ev}_{t_k=\zeta_k} \frac{w_{k+1}}{x_k \delta_{k+1}} = \frac{\det_{\mathcal{C}}(v_{i,k+1}, i=1, \dots, m)}{x_{k+1}}$$

and

$$\begin{aligned} \text{ev}_{t_k=\zeta_k, \dots, t_1=\zeta_1} 1 \otimes \frac{w_{1,P}}{f_1} &= \frac{\det_{\mathcal{C}}(\log_{\widetilde{F}^{(k),P}}(\exp_{\widetilde{F}^{(k)}}(v_{i,k+1})))}{x_{k+1}} \\ &\in \frac{1}{x_{k+1}} \mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z, t_{k+1}, \dots, t_s}(K_P). \end{aligned}$$

Last step: evaluation at  $z$ . We write  $z = t_{s+1}$  and use similar arguments to conclude.  $\square$

**Remark 5.9.** If at some step  $j \in \{1, \dots, s\}$  we have for all  $i = 1, \dots, r$ :

$$\text{ev}_{t_1=\zeta_1, \dots, t_j=\zeta_j} A_i = 0,$$

then we have for all  $k \geq j$ :

$$E_{\theta}^{(k)} = \theta I_d + N_k$$

with  $N_k \in M_d(\mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{F}_q[t_{k+1}, \dots, t_s] \mathcal{O}_L)$  a nilpotent matrix. Then we have:

$$\partial_{E^{(k)}}(\theta^{q^d}) = E_{\theta^{q^d}}^{(k)} = \theta^{q^d} I_d.$$

Hence, if  $\exp_{E^{(k)}} = \sum_{n \geq 0} d_{n,k} \tau_s^k$ , then from the functional equation of the exponential map we obtain for all  $n \geq 0$ :

$$(\theta^{q^d})^{q^n} d_{n,k} = \theta^{q^d} d_{n,k}.$$

Thus  $d_{0,k} = I_d$  and  $d_{n,k} = 0$  for all  $n \geq 1$  so finally  $\exp_{E^{(k)}} = I_d \tau^0$  for all  $k \geq j$ . Then we have  $U(\widetilde{E}^{(j)}; \widetilde{\mathcal{O}_{L,s,j}}) = \text{Lie}_{\widetilde{E}^{(j)}}(\widetilde{\mathcal{O}_{L,s,j}})$  for all  $j \geq k$ , and the previous proof is still valid.

## 6. APPLICATIONS

In this section we investigate the case where  $L = K$  and  $d = 1$ , i.e., the case of Drinfeld modules defined over the ring  $A$  itself.

**6.1. Preliminaries.** We consider a Drinfeld module  $\phi : A \rightarrow A\{\tau\}$  of rank  $r \geq 1$  and we denote by  $\exp_\phi = \sum_{n \geq 0} d_n \tau^n \in K\{\{\tau\}\}$  its associated exponential map. We call a period of  $\phi$  any element  $\lambda \in \mathbb{C}_\infty$  such that  $\exp_\phi(\lambda) = 0$ . We denote by  $\Lambda_\phi$  its period lattice which is a free  $A$ -module of rank  $r$ . Let  $NP(\phi)$  be the Newton polygon associated with  $\exp_\phi$  which is defined as the lower convex hull of the points  $P_n = (q^n - 1, v_\infty(d_n))$ . Remark that the zeros of  $\exp_\phi$  are all simple since  $\frac{d}{dx} \exp_\phi x = 1$ .

We have the following property about the edges of  $NP(\phi)$  that can be found in [18, Theorem 2.5.2].

**Proposition 6.1.** *Consider  $\lambda$  a non-zero period of  $\phi$  of valuation  $x$ , and let  $N$  be the number of periods of valuation equal to  $x$ . Then  $NP(\phi)$  has a single edge with slope  $\lambda$  and length  $N$ .*

By Lemma 3.3, we know that  $\lim_{n \rightarrow +\infty} v_\infty(d_n) = +\infty$ , then we define  $N_0$  as the smallest  $n$  such that  $v_\infty(d_n)$  is minimal, and  $N_1$  as the largest  $n$  such that  $v_\infty(d_n)$  is minimal. Another way of looking at  $N_0$  and  $N_1$  is that the edge of slope equals to 0 of  $NP(\phi)$  has endpoints  $P_{N_0}$  and  $P_{N_1}$ .

We will use the concept of successive minimum basis from [14, section 3].

**Definition 6.2.** An ordered  $A$ -basis  $(\lambda_1, \dots, \lambda_r)$  of the  $A$ -lattice  $\Lambda_\phi$  in  $\mathbb{C}_\infty$  is a successive minimum basis (shortly an SMB) if for each  $1 \leq i \leq r$ , the vector  $\lambda_i$  has minimal valuation  $v_\infty(\lambda_i)$  among all  $w \in \Lambda_\phi$  not in the span  $\sum_{1 \leq j < i} \lambda_j A$  of  $\{\lambda_1, \dots, \lambda_{i-1}\}$ .

Gekeler proved the following result, see [14, Proposition 3.1]

**Proposition 6.3.**

- (1) *The period lattice  $\Lambda_\phi$  admits an SMB.*
- (2) *The sequence  $(v_\infty(\lambda_i))_{i=1, \dots, r}$  is independent of the choice of the SMB.*
- (3) *Consider  $\{\lambda_1, \dots, \lambda_r\}$  an SMB for  $\Lambda_\phi$ . Then for all  $\lambda = \sum_{i=1}^r a_i \lambda_i \in \Lambda_\phi$  we have:*

$$v_\infty(\lambda) = \min\{v_\infty(a_i \lambda_i) \mid i = 1, \dots, r\}.$$

We consider  $s \in \{1, \dots, r\} \cup \emptyset$  be such that  $v_\infty(\lambda_i) \geq 0$  for  $i = 1, \dots, s$  and  $v_\infty(\lambda_i) < 0$  for  $i > s$ . Denote also by  $x_i = v_\infty(\lambda_i)$  for all  $i = 1, \dots, r$ . Then we consider  $n_1, \dots, n_t \in \{1, \dots, s\}$  be such that  $x_{n_i} \in \mathbb{N}$  for  $i = 1, \dots, t$ , and we denote by  $S_\phi = \{n_1, \dots, n_t\}$ .

**Proposition 6.4.** *We have the following equality:*

$$N_0 - N_1 = t.$$

*Proof.* First we calculate  $N_0$ . To do this, we need to count the total length of the strictly negative slopes (which is also equal to  $q^{N_0} - 1$  by definition), in other words the number of non-zero periods of strictly positive valuation. Set  $\lambda = \sum_{i=1}^r a_i \lambda_i \in \Lambda_\phi$ . We have the equivalence by Proposition 6.3:

$$v_\infty(\lambda) > 0 \Leftrightarrow \begin{cases} a_i = 0 \text{ if } i > s, \\ \deg(a_i) \leq \lfloor x_i \rfloor \text{ if } i \leq s \text{ and } i \notin S_\phi, \\ \deg(a_i) < x_i \text{ if } i \in S_\phi. \end{cases}$$

We finally exclude the case  $\lambda = 0$ . We obtain that the total number of non-zero elements of  $\Lambda_\phi$  with strictly positive valuation is equal to

$$q^{N_0} - 1 = \prod_{j=1}^t q^{x_{n_j}} \prod_{i \leq s, i \notin S_\phi} q^{\lfloor x_i \rfloor + 1} - 1.$$

By applying the logarithm we obtain:

$$N_0 = \sum_{j=1}^t x_{n_j} + \sum_{i \leq s, i \notin S_\phi} (\lfloor x_j \rfloor + 1).$$

We then calculate  $q^{N_1} - 1$  that is equivalent to counting the number of periods with positive valuation in a similar way, and we obtain

$$N_1 = \sum_{j=1}^t (x_{n_j} + 1) + \sum_{i \leq s, i \notin S_\phi} (\lfloor x_j \rfloor + 1) = N_0 + t.$$

□

Note that we only worked with periods of  $\phi$  and never used the fact that  $\phi$  is defined over  $A$ . We can therefore generalize the previous result to any  $\phi : A \rightarrow \mathbb{C}_\infty\{\tau\}$  of rank  $r$  since the concept of SMB is defined in full generality.

We want to apply this result to the study of the vanishing order of  $L_P(\tilde{\phi}/\tilde{A})$  at  $z = 1$ .

**6.2. An application to the vanishing of the  $P$ -adic  $L$ -series.** Let  $P \in A$  be irreducible monic and set  $u_\phi(z) = \exp_{\tilde{\phi}} L(\tilde{\phi}/\tilde{A}) \in A[z]$ . We set:

$$g_{P,\phi}(z) = [\tilde{\phi}(\tilde{A}/P\tilde{A})]_{\tilde{A}} \in A[z]$$

and

$$g_{P,\phi}(1) = [\phi(A/PA)]_A \in A \setminus \{0\}.$$

We recall the definition of local factor associated with  $\tilde{\phi}$  and  $P$ :

$$z_P(\tilde{\phi}/\tilde{A}) = \frac{P}{g_{P,\phi}(z)}.$$

Let us recall that the  $P$ -adic  $L$ -series associated with  $\phi$  is defined as follows:

$$L_P(\tilde{\phi}/\tilde{A}) = \frac{1}{P} \log_{\tilde{\phi},P} \tilde{\phi}_{g_{P,\phi}(z)}(u_\phi(z)) \in \mathbb{T}_z(K_P).$$

By the proof of [22, Corollary 7.5.6], we deduce that  $L(\tilde{\phi}/\tilde{A}) \in \mathbb{T}_z(K_\infty)$  is a unit in  $\mathbb{T}_z(K_\infty)$  whose valuation is equal to 0 and whose constant coefficient is equal to 1.

From now on, we say that  $\phi$  does not have  $A$ -torsion if the  $A$ -module  $\phi(A)$  is torsion-free.

**Proposition 6.5.** *Let  $\phi$  be an  $A$ -Drinfeld module defined over  $A$  of rank  $r \geq 1$  without  $A$ -torsion. Then for all  $k \geq 0$  the following assertions are equivalent:*

- (1)  $(z - 1)^k |_{\mathbb{T}_z(K_P)} L_P(\tilde{\phi}/\tilde{A})$ ,
- (2)  $(z - 1)^k |_{A[z]} u_\phi(z)$ .

*Proof.* By the definition of the  $P$ -adic  $L$ -series:

$$L_P(\tilde{\phi}/\tilde{A}) = \frac{1}{P} \log_{\tilde{\phi},P}(\tilde{\phi}_{g_{P,\phi}(z)}(u_\phi(z))),$$

we see that  $2 \Rightarrow 1$  is clear.

Let us prove  $1 \Rightarrow 2$ . We have:

$$P^2 L_P(\tilde{\phi}/\tilde{A}) = \log_{\tilde{\phi},P}(\tilde{\phi}_{g_{P,\phi}(z)}(u_\phi(z))).$$

Since  $v_P(\tilde{\phi}_{g_{P,\phi}(z)}(u_\phi(z))) \geq 2$ , we have  $\tilde{\phi}_{g_{P,\phi}(z)}(u_\phi(z)) \in \mathcal{D}_z^+$ . By applying the  $P$ -adic exponential map we obtain:

$$\exp_{\tilde{\phi},P}(P^2 L_P(\tilde{\phi}/\tilde{A})) = \tilde{\phi}_{g_{P,\phi}(z)}(u_\phi(z)) \in A[z].$$

If  $(z-1)^k |_{\mathbb{T}_z(K_P)} L_P(\tilde{\phi}/\tilde{A})$ , then we have  $(z-1)^k |_{A[z]} \tilde{\phi}_{P, g_{P, \phi}(z)}(u_\phi(z))$ . Since  $\phi$  does not have  $A$ -torsion, we deduce that

$$(z-1)^k |_{A[z]} u_\phi(z).$$

□

**Proposition 6.6.** *Let  $m \in A \setminus \{0\}$  be a non zero polynomial and consider the Drinfeld module  $\psi = m^{-1}\phi m$ . Then the vanishing order at  $z = 1$  of  $L_P(\tilde{\phi}/\tilde{A})$  and  $L_P(\tilde{\psi}/\tilde{A})$  are equal.*

*Proof.* By Lemma 4.2, we have the following equality in  $\mathbb{T}_z(K_P)$ :

$$g_{P, \phi}(z) L_P(\tilde{\psi}/\tilde{A}) = g_{P, \psi}(z) L_P(\tilde{\phi}/\tilde{A}) \prod_{Q|m} \frac{g_{Q, \phi}(z)}{Q}.$$

Since  $g_{Q, \phi}(1) \neq 0$  for all  $Q$ , we obtain the result. □

**Proposition 6.7.** *If  $\phi : A \rightarrow A\{\tau\}$  has rank  $r$ , then the vanishing order at  $z = 1$  of the  $P$ -adic  $L$ -series  $L_P(\tilde{\phi}/\tilde{A})$  is lower than or equal to  $r$ .*

*Proof.* Let us first twist  $\phi$  into  $\psi = m^{-1}\phi m$  without  $A$ -torsion. By Proposition 6.5 we consider the vanishing order at  $z = 1$  of  $u_\psi(z) \in A[z]$ . We can compute its leading coefficient seen as a polynomial in the variable  $\theta$ . We have  $u_\psi(z) = \exp_{\tilde{\psi}} L(\tilde{\psi}/\tilde{A}) = \sum_{n \geq 0} d_n z^n \tau^n(L(\tilde{\psi}/\tilde{A}))$ . We know that  $L(\tilde{\psi}/\tilde{A})$  has the form  $1 + \sum_{n \geq 1} a_n z^n \in \mathbb{T}_z(K_\infty)$  with  $v_\infty(a_n) > 0$ . Let  $N_0, m_1, \dots, m_l, N_1$  be the integers  $n$  such that  $v_\infty(d_n)$  is minimal. Let  $\beta_n \in \mathbb{F}_q^*$  be the sign of  $d_n$ , we obtain:

$$\text{sgn}(u_\psi(z)) = z^{N_0} (\beta_{N_0} + \dots + \beta_{N_1} z^{N_1 - N_0}) \in \mathbb{F}_q[z]$$

that has at most  $r$  non-zero roots by Proposition 6.4. Thus,  $u_\psi(z)$  is divisible at most by  $(z-1)^r$ . □

Note that we have proved more precisely:

$$\text{ord}_{z=1} L_P(\tilde{\phi}/\tilde{A}) \leq \#\{i = 1 \dots, r \mid v_\infty(\lambda_i) \in \mathbb{Z}\}.$$

**Proposition 6.8.** *The previous inequality is not an equality in general.*

*Proof.* Consider the Drinfeld module given by  $\phi_\theta = \theta + \theta\tau^2$  with  $q = 3$ . One can prove that the Newton polygon of the associated exponential map is the polygon beginning at the point  $(0, 0)$  and has successive slopes of length  $(q^{2k+2} - q^{2k})$  and equal to  $k + 1$ . Thus, the number of periods of an SMB having valuation  $\in \mathbb{Z}$  is equal to 2. By [9, Proposition 2.21] we have  $u_\phi(1) = 1$ . One can prove that  $\phi$  does not have  $A$ -torsion. Then by Proposition 6.5 we obtain  $\text{ord}_{z=1} L_P(\tilde{\phi}/\tilde{A}) = 0$ . □

**Remark 6.9.** For any  $r \geq 1$ , we can construct explicit Drinfeld modules of rank  $r$  whose vanishing order of the associated  $P$ -adic  $L$ -series equals  $r$ . In fact, denote by  $(-1)^r (z-1)^r = 1 + \sum_{i=1}^r \alpha_i z^i$ , with  $\alpha_i \in \mathbb{F}_q$  and consider the Drinfeld module given by  $\phi_\theta = \theta + \sum_{i=1}^r \alpha_i \theta^{q^i} \tau^i$ .

By [9, Proposition 2.21], we have:  $u_\phi(z) = 1 + \sum_{i=1}^r \alpha_i z^i = (-1)^r (z-1)^r$ . Then the vanishing order at  $z = 1$  is greater than or equal to  $r$ , so equals  $r$ .

To conclude the text, we would like to ask the following question from a personal communication with Xavier Caruso and Quentin Gazda.

**Problem 6.10.** *Do we have the following equality*

$$\text{ord}_{z=1} L_P(\tilde{\phi}/\tilde{A}) = \# \left\{ i \in \{1, \dots, r\} \mid \lambda_i \in \bigcup_{n \geq 0} \mathbb{F}_{q^{p^n}} \left( \left( \frac{1}{\theta} \right) \right) \right\}?$$

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