Research Statement

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November 25, 2025

1 Current state of the art

The theory of Drinfeld modules, introduced by Drinfeld [15] in 1974, has seen many advances over the last 20 years and has led to a new arithmetic understanding of global function fields.

Let us begin by recalling some important points from the case of number fields, which have guided mathematicians, by analogy, in the function fields case.

As a broad generalization of the special values studied by Euler, zeta functions are introduced for number fields, called Dedekind zeta functions, and more generally L-functions. Class number formulas relate these functions to arithmetic information about the field, such as the Dedekind class formula; see [21, Chapter 7, Section 5]

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}R_K}{w_L\sqrt{|D_K|}}h_K.$$

In the context of function fields, let θ be an indeterminate, $A = \mathbb{F}_q[\theta]$, and $K = \mathbb{F}q(\theta)$. Fix a finite extension L/K, whose ring of integers is denoted by $\mathscr{O}_L$. In 2010, Taelman [24] introduced the notion of L-series associated with a Drinfeld module ϕ, given by the Euler product

$$L(\phi, \mathscr{O}_L) = \prod_Q \frac{Q^{[L:K]}}{g_{Q,E(1)}}$$

where the product ranges over all monic irreducible polynomials of A, and where $g_{Q,E}(1) \in A$ is a certain generator of a Fitting ideal. Let $z_Q(\phi, \mathscr{O}_L)$ denote the local factor at Q. Taelman proved [25, Theorem 1] a class formula

$$L(\phi, \mathscr{O}_L) = R(U(\phi; \mathscr{O}_L))[H(\phi; \mathscr{O}_L)]_A,$$

where $U(\phi; \mathscr{O}_L)$ is the Taelman unit module, $R(U(\phi; \mathscr{O}_L))$ the associated regulator (a determinant of units, analogous to the unit group of number fields), and $H(\phi; \mathscr{O}_L)$ is the class module associated with ϕ, for which $[H(\phi; \mathscr{O}_L)]_A$ plays the role of the class number of number fields. Since his seminal work, several developments have been undertaken, particularly in the setting of Anderson t-modules introduced in [1], which are generalization of Drinfeld modules in higher dimension.

Anglès and Tavares Ribeiro [6] introduced the notion of the z-deformation of an Anderson module E, allowing us to view the L-series $L(E, \mathscr{O}_L)$ as the value at $z = 1$ of certain L-functions, denoted by $L(E, \mathscr{O}_L, z)$. Finally, Anglès, Ngo Dac, and Tavares Ribeiro [4] proved a class formula in the case where the ring A is "general" and E is an "admissible" Anderson module, including in particular all Drinfeld modules and all abelian t-modules.

Let us return to number fields. Some p-adic analogs of L-functions have been constructed and studied, called p-adic L-functions, and these functions unexpectedly provide more arithmetic information than classical L-functions, for example, through Colmez's p-adic analytic class number formula [13]:

$$\lim_{s\to 1}(s-1)\zeta_{p,F}(s) = \prod_{\mathfrak{P}|p}\left(1 - \frac{1}{N_{F/\mathbb{Q}}(\mathfrak{P})}\right)\frac{2^{[F:\mathbb{Q}]-1}R_{p,F}}{\sqrt{D_F}}h_F$$

where F is a totally real extension of $\mathbb{Q}$, the product ranges over the maximal ideals of $\mathscr{O}_F$ dividing p, and $R_{p,F}$ is a certain p-adic regulator attached to the extension.

In view of the numerous analogies between number fields and function fields, we aim to construct P-adic analogs of the Anglès-Ngo Dac-Tavares Ribeiro L-functions, called P-adic L-functions, whose analytic would yield arithmetic or geometric information, through P-adic class formulas. This has been the main objective of my work. During my PhD, for P a monic irreducible polynomial of A, I defined the P-adic L-function $L_P(E, \mathscr{O}_L, z)$ in the context of Anderson t-modules, given by the Euler product

$$L_P(E, \mathscr{O}_L, z) = \prod_{Q \neq P} z_Q(E, \mathscr{O}_L, z),$$

where the product ranges over the monic irreducible polynomials of A not divisible by P, and I proved a P-adic class formula- à la Taelman, with the variable z

$$z_P(E, \mathscr{O}_L, z)\, L_P(E, \mathscr{O}_L, z) = R_P\left(U(\widetilde{E}; \widetilde{\mathscr{O}_L})\right),$$

and similarly without the variable z:

$$L_P(E, \mathscr{O}_L, 1) = R_P(U(E; \mathscr{O}_L))\, [H(E; \mathscr{O}_L)]_A,$$

where $R_P\left(U(\widetilde{E}; \widetilde{\mathscr{O}_L})\right)$ (resp. $R_P(U(E; \mathscr{O}_L))$) is a certain P-adic regulator defined as a determinant of P-adic logarithms of units. Moreover, I proved that this function lies in the following Tate algebra:

$$\mathbb{T}_z(K_P) = \left\{ \sum_{n \geq 0} a_n z^n \mid a_n \in K_P, \lim_{n \to +\infty} v_P(a_n) = +\infty \right\},$$

and is even a power series in the variable z. In particular, I studied the vanishing of this L-function at $z = 1$ and stated the following two conjectures.

Conjecture. *Let E be an Anderson t-module defined over $\mathscr{O}_L$.*

1. *The vanishing at $z = 1$ of the P-adic L-function $L_P(E, \mathscr{O}_L, z)$ does not depend on P.*
2. *The vanishing order at $z = 1$ of the P-adic L-function ${}_P(E, \mathscr{O}_L, z)L$ does not depend on P.*

I showed that the L-function vanishes at $z = 1$ when certain periods associated with E lie in certain fields L_u, the u-adic completion of L where u is a place of L above∞, and I only conjectured the converse.

In the case of Drinfeld modules defined over A, I gave an upper bound for the vanishing order at $z = 1$ of the P-adic L-function by relating it to certain periods of the Drinfeld module.

For the past three years, I have also begun a collaboration with Xavier Caruso and Quentin Gazda [9], which started with the study of L-functions in the context of Drinfeld modules defined over A. On the one hand, we proved that the vanishing order at $z = 1$ of the P-adic L-function is independent of P, and is related to the special element $u_\phi(z) = \exp_{\widetilde{\phi}}(L(\phi, A, z)) \in A[z]$.

We introduced a family of Drinfeld modules for which this special element is equal to 1, called "small" modules, which satisfy

$$\deg_\theta \phi_{\theta,i} < q^i, \forall i = 1, \ldots, r,$$

if $\phi_\theta = \theta + \sum_{i=1}^r \phi_{\theta,i} \tau^i$.

On the other hand, for a prime P of A, we introduce the notion of being Wieferich in base ϕ, and we establish a surprising connection between this notion and the P-adic valuation of the special value of the P-adic L-series $L_P(\phi, A, 1)$. In particular, we prove the following equivalence:

$$P \text{ is Wieferich in base } \phi \Leftrightarrow v_P(L_P(\phi, A, 1)) > 0.$$

This work raises questions completely analogous to those concerning classical Wieferich primes, particularly the question of whether there exist infinitely many Wieferich primes P in this setting.

Finally, in the case of small modules, which provides a suitable framework for statistical investigations, we proved that if one fixes P, then there exist infinitely many Drinfeld modules ϕ for which P is Wieferich in base ϕ, and infinitely many for which P is not Wieferich in base ϕ.

2 Proposed projects

In general, my areas of interest are related to number theory, particularly function fields. I plan to pursue the following directions:

1. Study the zeros of P-adic L-functions in general in the context of Anderson modules: multiplicity and localization of zeros, formulation of analogs of Birch and Swinnerton-Dyer conjectures for Anderson t-modules, relations between different L-functions.
2. Construct v-adic L-functions in the general case, i.e. for Anderson A-modules, and try to understand and highlight the major differences from the case of PID.
3. Fully understand the case of Carlitz's P-adic zeta functions, or, in other words, fully and explicitly study P-adic L-functions in the case of tensor powers of the Carlitz module $C^{\otimes n}$. In particular, study P-adic Riemann hypotheses stated in the positive characteristic setting.
4. Introduce effective methods and algorithms for conjecturing, studying, and verifying certain results.

1 Extension to a general base A

1.1 Reduction of the study

In the more general setting of Anderson t-modules defined over $\mathscr{O}_L$, the L-function is generally not a z-unit, and previous techniques are too naive to study these questions.

Continuing our collaboration with Xavier Caruso and Quentin Gazda, which is still ongoing [11], [10], we have resolved the case of Anderson t-modules defined on A, i.e., studied the vanishing of the P-adic L-function $L_P(E, A, z)$ at $z = 1$, as well as the valuation of its special value. With the objective of generalizing this latest work, let us begin with the following remark.

On the one hand, by setting $A = \mathbb{F}_q[\theta]$, we must consider Anderson t-modules defined not only over A, but also over extensions $\mathscr{O}_L$. On the other hand, we want to consider more general rings than $\mathbb{F}_q[\theta]$ as base rings, which is important for applications (e.g. to explicit class field theory, Drinfeld-Hayes modules, etc.), but this complicates the technical aspects.

Recently [10], we established general tools, called Weyl restriction to reduce to the case $\mathscr{O}_L = A$, and some localization of motives, which allow us to reduce the general case A to the basic case: $A = \mathbb{F}_q[\theta]$.

To get off to a good start and show the effectiveness of these methods, we apply them to the class formula (classical and v-adic) and, in the following, the work can always be studied initially in the case where $A = R = \mathbb{F}_q[\theta]$ and then, by applying the same techniques, the general case should follow.

1.2 A v-adic class formula

Let $K/\mathbb{F}_q$ be a global function field, ∞ a fixed point (called infinity), A the ring of regular functions outside ∞, v a finite place of K and E an Anderson A-module. We would like to extend the constructions of the author [20], which correspond to the case $K = \mathbb{F}_q(\theta)$ and $\infty = \dfrac{1}{\theta}$.

In [10], we proved a class formula in the case where A is a general ring, without any Anglès-Ngo Dac-Tavares Ribeiro-type 'admissibility' hypothesis on the Anderson module. An accessible project would then be, based on this latter construction, to construct the v-adic function L and prove a v-adic class formula.

In particular, the steps to be considered would be as follows, adapting the ideas of [20] and simultaneously applying the techniques of [10] to reduce the following questions to the case $A = \mathbb{F}_q[\theta]$ and t-modules defined on A.

Problem 1. *Let E be an Anderson A-module defined on $\mathscr{O}_L$.*

1. *Study the v-adic convergence of $\exp_{E,v}$ and $\log_{E,v}$. In particular, show that $\log_{E,v}$ converges on*

$$\{x \in \mathbb{C}_v \mid v(x) > 0\}.$$

2. *Construct, from the class formula of the L function and the concept of z-deformation, the v-adic L-function $L_v(E, \mathscr{O}_L, z)$ “formally”.*

3. *Via a v-adic study of Stark units and z-deformed Taelman units, show that this function does not have any pole and converges everywhere.*

4. *Introduce the v-adic regulator $R_v(U(\widetilde{E}; \mathscr{O}_L[z]))$ of Taelman units for general A, and prove a v-adic class formula à la Taelman*

$$L_v(E, \mathscr{O}_L, z) z_v(E, \mathscr{O}_L, z) = R_v(U(\widetilde{E}, \mathscr{O}_L[z])).$$

2 Study of the zeros and special values of P-adic L-functions

This section is devoted to the study of P-adic L-functions of Anderson t-modules, in particular the zeros and special values.

2.1 Study of the zeros of the P-adic L-function

Since the function $L_P(E, \mathscr{O}_L, z)$ of an Anderson t-module is an power series that converges everywhere, I would like to study its zeros in general. Let us assume, according to the reduction step, that $L = K$.

In this case, in the project [11], we have completely linked the vanishing order at $z = 1$ and at $z = \gamma \in \overline{\mathbb{F}}_q$ of $L_P(E, A, z)$ to certain submodules of the period lattice, which can be seen as an equivalent of the Birch and Swinnerton-Dyer conjectures for Anderson t-modules, as follows. Let $R_\infty = \bigcup_{n \geq 0} \mathbb{F}_{q^{p^n}}((\frac{1}{\theta}))$ be the maximal unramified $\mathbb{Z}_p$-extension of K_∞, and let

$$\Omega_d = \{x \in \mathrm{Lie}_E(R_\infty) \mid \exp_E(x) = 0\}, \ \overline{\Omega}_d = \left\{x \in \mathrm{Lie}_E\left(\overline{\mathbb{F}}_q((\theta^{-1}))\right) \mid \exp_E(x) = 0\right\}$$

two sub-A-modules of the period lattice of E.

Theorem 2 (Caruso, Gazda, L, 2025)**.** *In a z-deformed form of Leopoldt's conjecture, we have*

1. $$\mathrm{ord}_{z=1} L_P(E, A, z) = \mathrm{rank}_A \Omega_d,$$

2. $$\sum_{\gamma \in \overline{\mathbb{F}}_q} \mathrm{ord}_{z=\gamma} L_P(E, A, z) = \mathrm{rank}_A \overline{\Omega}_d.$$

According to the Weyl restriction, we obtain in particular, under these Leopoldt's conjectures, the following general formula:

$$\mathrm{ord}_{z=1} L_P(E, \mathscr{O}_L, z) = \mathrm{rank}_A \{x \in \mathrm{Lie}_E(L \otimes_K R_\infty) \mid \exp_E(x) = 0\}. \tag{1}$$

A first long-term project is precisely the study of this z-deformed Leopoldt conjecture, which is a generalization of the Leopoldt conjecture stated in [20]. It compares the A-rank and the A_P-rank of families of units, in a manner analogous to the classical case, as follows. Let h be a sufficiently large integer, $V(E; A) = \exp_E U(E; A) \subseteq E(A)$, and $V^{\mathrm{sat}}(E; A)$ be the evaluation at $z = 1$ of a certain “$(z-1)$-saturation” of $\exp_{\widetilde{E}} U(\widetilde{E}; A[z])$.

Conjecture (Weak and strong Leopoldt's conjectures)**.** *Let E be an Anderson t-module defined over A.*

1. *The map $A_P \otimes_A g_{P,E}(1) V(E; A) \to E(P^h A_P)$ is injective (weak conjecture).*

2. *The map $A_P \otimes_A g_{P,E}(1) V^{\mathrm{sat}}(E; A) \to E(P^h A_P)$ is injective (strong conjecture).*

Note that these conjectures are true in the case $d = 1$ and $L = K$, and according to [20], the weak Leopoldt conjecture is equivalent to the fact that the vanishing at $z = 1$ of the P-adic L-function does not depend on P.

A first approach to proving Leopoldt's conjectures is via transcendence tools, as in the classical case. But surprisingly (see, for example, Section 3), the introduction of this variable z, which does not exist at all for number fields, seems to allow these conjectures to be approached in a purely algebraic manner. It turns out that sufficient knowledge of the class module $H(\widetilde{E}; A[z])$ allows Leopoldt's conjectures to be proven via the following questions.

Problem 3. *For all $n \in \mathbb{N}$, denote by E_n the Anderson t-module defined over A by $E_n = \left(P^{q^n}\right)^{-1} EP^{q^n}$.*

1. *Prove that for all n big enough, $\left[\mathrm{ev}_{z=1} H\left(\widetilde{E_n}; A[z]\right)\right]_A$ does not depend on n.*
2. *Prove that for all n big enough, $\left[\mathrm{ev}_{z=1} H^{\mathrm{sat}}\left(\widetilde{E_n}; A[z]\right)\right]_A$ does not depend on n.*

A natural question then arises concerning the other zeros of the P-adic L-function $L_P(E, A, z)$, and not just the zeros in $\overline{\mathbb{F}}_q$. In particular, we would like to answer the following open question, using the techniques developed from the evaluation at $z = \zeta \in \overline{\mathbb{F}}_q$.

Problem 4. *Give an interpretation of the zeros of the P-adic L-function. In particular, what can be said about those living in K and in $\overline{K}$?*

I would then like to focus, in a longer-term project, still related to the zeros of the P-adic L-functions, on the following questions inspired by [18].

Problem 5. *Let K_P be the completion of K for the place P, and let E be an Anderson t-module defined over A.*

1. *Is there a finite extension $K_{E,P}$ of K_P such that all the zeros of the P-adic L-function live in $K_{E,P}$?*
2. *If the answer to the previous question is yes, let $d_{E,P}$ be the degree of the extension $K_{E,P}/K_P$. Can we give a uniform bound (relative to P) for $d_{E,P}$? And if we fix P and vary the t-modules E, how does $d_{E,P}$ vary?*

2.2 Extension of the domain of definition

In connection with the previous questions, we would like to be able, without considering the Weyl restriction, which increases the dimension,to study Drinfeld modules defined on rings of integers $\mathscr{O}_L$ by reducing the study to several Drinfeld modules defined on rings of integers $\mathscr{O}_F \subset \mathscr{O}_L$, with the aim of reducing to the case $\mathscr{O}_F = A$, in which case all the questions raised above are resolved. This leads to the following open question.

Problem 6. *Characterisze all of the Drinfeld modules $\phi : A \to \mathscr{O}_L\{\tau\}$ such that there exists a family of Drinfeld modules $\phi_i : A \to A\{\tau\}$ satisfying for all P*

$$L_P(\phi, \mathscr{O}_L, z) = \prod_{i=1}^{n} L_P(\phi_i, A, z).$$

A first step would be to consider the following "inverse" situation, which is more accessible, with strategies already established in the case of number fields. Let ϕ be an A-Drinfeld module defined on $\mathscr{O}_L$, and let M/L be a finite extension of the ring of integers (over A) denoted by $\mathscr{O}_M$, in particular $\mathscr{O}_L \subseteq \mathscr{O}_M$. We can then consider ϕ as an A-module of Drinfeld defined on $\mathscr{O}_M$, and ask what are the links between $L_P(\phi, \mathscr{O}_L, z)$ and $L_P(\phi, \mathscr{O}_M, z)$. A first natural question is the following.

Problem 7. *Prove that $L_P(\phi, \mathscr{O}_L, z)$ divides $L_P(\phi, \mathscr{O}_M, z)$ in$\mathbb{T}_z(K_P)$.*

Following calculations with the Carlitz module defined on cyclotomic extensions, we can state the following problem in the Galois case.

Problem 8. *Let M/L be a finite Galois extension of degree n, and let ϕ be an A-Drinfeld module defined over $\mathscr{O}_L$. Show that there exists a family $(\phi_1, \ldots, \phi_n)$ of Drinfeld A-modules defined on $\mathscr{O}_L$, with $\phi_1 = \phi$, satisfying for every irreducible unitary polynomial P*

$$L_P(\phi, \mathscr{O}_M, z) = \prod_{i=1}^{n} L_P(\phi_i, \mathscr{O}_L, z). \tag{2}$$

In particular, we would obtain a positive answer to Problem 7. One strategy would be to introduce and develop the notion of Artin formalism, drawing inspiration, for example, from [21, Chapter 7].

2.3 Special values of the P-adic L-functions

Denote by

$$L_P(E, \mathscr{O}_L, z) = (z-1)^k L_P^*(E, A, z)$$

and

$$L_P^*(E, \mathscr{O}_L, 1) := \mathrm{ev}_{z=1} L_P^*(E, A, z) \in K_P^*$$

the special value of the P-adic L-function. I would like to focus on the valuation of the P-adic L-function $L_P(E, \mathscr{O}_L, 1)$ and the special value of the P-adic L-function $L_P^*(E, \mathscr{O}_L, 1)$, in projects that seem unrelated to the previous ones, but are in fact completely connected.

Let $\phi : A \to A\{\tau\}$ be a Drinfeld module. A long-standing open problem, which seems quite out of reach with current tools, is to prove the existence (or non-existence) of an infinite number of Wieferich primes in base ϕ introduced in [9]. On the other hand, continuing the work carried out in the context of Drinfeld modules defined on A, I would like to answer the following questions, directly inspired by the work of Silvermann [22], to prove the infinity of non-Wieferich primes in base ϕ when $L_P(\phi, A, 1) \neq 0$.

Problem 9.

1. *State an ABC conjecture in the context of Drinfeld modules defined over A.*
2. *Assume that $L_P(\phi, A, 1) \neq 0$. Show that, under this ABC conjecture, there are infinitely many primes P such that P is not Wieferich in base ϕ.*
3. *Prove, under this conjecture, that there are infinitely many primes P such that*

$$v_P(L_P^*(\phi, A, 1)) = 0.$$

In the current project [11], we have extended the work of [9] to the more general setting of Anderson t-modules defined over A. For any sub-A-module $X \subseteq E(A)$, we have defined an A-module, called the Wieferich module and denoted $W_X(E; A) \subseteq E(A)$, whose length (i.e., the P-adic valuation of its Fitting) generalizes the P-ordic valuation of [9].

In particular, by taking successively $X = V(E; A) = \exp_E U(E; A)$ and $X = V^{\mathrm{sat}}(E; A)$, under Leopoldt's conjectures, we have shown that

$$v_P(L_P(E, A, 1)) = \mathrm{length}_A \left(W_{V(E;A)}(E, P) \right)$$

and

$$v_P(L_P^*(E; A)) = \mathrm{length}_A \left(W_{V^{\mathrm{sat}}(E;A)}(E, P) \right).$$

In an immediately accessible project, we would like to generalize these results in the case where E is defined over $\mathscr{O}_L$. Let S_P be the set of primes of $\mathscr{O}_L$ above P, and for $\mathfrak{P}$ a prime of $\mathscr{O}_L$, let P be the unique prime of A below $\mathfrak{P}$.

Problem 10. *Let E be an Anderson t-module defined over $\mathscr{O}_L$.*

1. *Let $\mathfrak{p}$ be a prime of $\mathscr{O}_L$. Define, for $X \subseteq E(\mathscr{O}_L)$, the Wieferich module $W_X(E, \mathfrak{P})$.*
2. *Under the "weak" Leopoldt's conjecture, relate $v_P(L_P(E, \mathscr{O}_L, 1))$ and $W_{U(E;\mathscr{O}_L))}(E, \mathfrak{P})$ for all $\mathfrak{P} \in S_P$.*
3. *Under the "strong" Leopoldt's conjecture, relate $V_P(L_P^*(E, \mathscr{O}_L, 1))$ and $W_{U(E;\mathscr{O}_L)^{\mathrm{sat}}}(E, \mathfrak{P})$ for all $\mathfrak{P} \in S_P$.*

Note that answering the previous questions seems accessible in the short term, using the techniques from the case $L = K$. However, we could try instead to use the Weyl restriction of E, allowing us to use all the results proven on A, then give an interpretation of the results by "going back" through the restriction. Finally, it would be interesting to compare the results with a recent study of Wieferich prime numbers in the number fields case, see [16].

3 Tensor power of the Carlitz module and extension

The most important and studied example, but still largely misunderstood in the P-adic case, is the study of the P-adic L-functions associated with the tensor powers of the Carlitz module $C^{\otimes n}$ defined over A, which are nothing other than the P-adic Carlitz zeta functions

$$L_P(C^{\otimes n}, A, z) = \zeta_P(A, n, z) = \sum_{\substack{a \in A \text{ unitary} \\ P \nmid a}} \frac{z^{\deg(a)}}{a^n}.$$

Following the work of Anderson-Thakur [2], it turned out, in disagreement with the classical case, that the Carlitz zeta function, as well as its P-adic analogs, are realized as "values" and not only as determinants.

Indeed, Anderson and Thakur exhibited a special vector $z_n(z) \in U\left(\widetilde{C^{\otimes n}}; A[z]\right)$, whose image under the exponential is denoted $Z_n(z)$, such that if we denote $Z_{n,P}(z) = \widetilde{C^{\otimes n}}_{P^m g_{P,C^{\otimes n}(z)}}(Z_n(z))$ and $z_{n,P}(z) = \log_{\widetilde{C^{\otimes n}},P}(Z_{n,P}(z))$ (where m is a sufficiently large integer), then we have

$$\mathrm{pr}_d(z_{n,P}(z)) = \Gamma_n P^{m+d} \zeta_P(A, n, z)$$

where $\mathrm{pr}_d : \mathbb{T}_z(K_P)^d \to \mathbb{T}_z(K_P)$ is the projection onto the last coordinate of the canonical basis.

It turns out that studying this special vector, and especially its image $Z_{d,P}(z) \in A[z]^n$, gives us all the arithmetic and analytical information about the P-adic zeta function.

Goss [18] stated the following conjecture.

Conjecture (P-adic Riemann hypothesis in positive characteristic)**.** *The zeros of $\zeta_P(A, n, z)$ are all simple.*

This has been proven thanks to the work of Wan [27], Thakur [26] and Diaz-Vargas and Polanco-Chi [14] in the case where P is of degree 1, but remains open in the general case.

In a current work, studying this special vector $Z_{n,P}(z)$, I have been able to prove this conjecture for any evaluation at elements of $\overline{\mathbb{F}}_q$ for any P, under the assumption that n is of the form $p^k(q-1)$, $k \geq 0$. This gives us a completely elementary and algebraic proof of Leopoldt's conjecture in these specific cases. By refining the study, I hope to answer the following question.

Problem 11. *Prove that for all $n \geq 1$ and $\zeta \in \overline{\mathbb{F}}_q$, we have*

$$\mathrm{ev}_{z=1}\, \zeta_P(A, n, z) = \begin{cases} 1 \textit{ if } \zeta = 1 \textit{ and } (q-1)|n, \\ 0 \textit{ otherwise.} \end{cases}$$

In connection with the study of the special vector of Anderson-Thakur, the reason for the presence of the polynomial Γ_n in the last coordinate of $z_n(z)$ is still an open question, for which the following is a very serious avenue to explain it.

First, let $u \in K_\infty^d = \mathrm{Lie}_{C^{\otimes n}}(K_\infty)$. We denote by u^∂ the vector u, decomposed in the canonical basis, but for the action of ∂. Then the answer seems to be hidden in the following first general question and its consequences on questions 2 and 3 below.

Problem 12. *1. Compare u and u^∂.*

2. Let $M \in M_n(K)$. We can view this matrix as the matrix of a map acting on a vector u. Then give M^∂, the matrix of this same application acting on u^∂.

3. Deduce d_k^∂ (resp. l_k^∂), where d_k (resp. l_k) is the k-th coefficient of $\exp_E$ (resp. $\log_E$).

Secondly, in [10], we proved that the family $(\log_{C^{\otimes n}}(e_1), \ldots, \log_{C^{\otimes n}}(e_n))$ is an A-basis of the module of units as well as the module of Stark units. We want to compare

$$\det_{\mathscr{B}}(\log_{C^{\otimes n}}(e_1), \ldots, \log_{C^{\otimes n}}(e_n)) \text{ and } \det_{\mathscr{B}}(\log_{C^{\otimes n}}(e_1)^\partial, \ldots, \log_{C^{\otimes n}}(e_n)^\partial).$$

According to the class formula, the second determinant is exactly the Carlitz zeta function $\zeta(A, n, z)$, and I would particularly like to answer the following question.

Problem 13. *Is it this comparison between these two determinants that explains the presence of the factor Γ_n in the special vector of Anderson-Thakur z_n?*

Still in connection with this special vector, numerical simulations also seem to unexpectedly link the valuation of the special value of the P-adic Carlitz zeta function and the Wieferich module associated with this special vector. In particular, I would like to answer the following question, the study of which could provide a new approach to Problem 13 on the one hand, and on the other hand, enable progress in the proof of Leopoldt's conjecture associated with the P-adic Carlitz zeta functions.

Problem 14. *Prove the following two equalities.*

1. $v_P(L_P(C^{\otimes n}, A, 1) = \mathrm{length}\, W_{Z_{n,P}(1)}(C^{\otimes n}, P) - n - m.$
2. $v_P(L_P^*(C^{\otimes n}, A, 1) = \mathrm{length}\, W_{Z_{n,P}^*(1)}(C^{\otimes n}, P) - n - m.$

The surprising fact that all the arithmetic and analytical information of the L-function is contained in a subspace of dimension 1 is not an isolated case, and has been conjectured by Taelman [23] and studied by Anglès-Ngo Dac-Tavares Ribeiro [3]. Under appropriate assumptions, they constructed a K_∞-vector space W of dimension equal to the rank of E, such that: $U(E; A) \cap W$ and $\mathrm{Lie}_E(A) \cap W$ are A-lattices in W and

$$[U(E; A) \cap W : \mathrm{Lie}_E(A) \cap W]_A = \alpha L(E, A, 1) \tag{3}$$

for a certain $\alpha \in K^\times$. In particular, if E is a t-module of rank 1 satisfying these hypotheses, we obtain trivially that the weak Leopoldt conjecture is true for E, in other words, the vanishing at 1 of the P-adic L-function does not depend on P.

Daniel Krell Calvo generalized these results with the variable z in a forthcoming paper, introducing a space W_z, then re-demonstrated a P-adic version of the previous result by Anglès-Ngo Dac-Tavares Ribeiro under appropriate assumptions, which in particular encompasses the case $C^{\otimes n}$. His work then allows us to study the special value of the P-adic L-series in this context.

Let us denote $W' = \exp_E(W \cap U(E; A))$, $W'_z = \exp_{\widetilde{E}}(U(\widetilde{E}; A[z] \cap W_z))$ and $W'^{,\mathrm{sat}} = \mathrm{ev}_{z=1} W_z'^{,\mathrm{sat}}$. We can then generalize problem 14 to the following Problem.

Problem 15. *Prove the following two equalities, where $m_1 \in \mathbb{Z}$ is a constant that will depend on α in Equation (3).*

1. $$v_P(L_P(E, A, 1)) = \mathrm{length}\, W_{W'}(E, P) + m_1$$

2. $$v_P(L_P^*(E, A, 1) = \mathrm{length}\, W_{W'^{,\mathrm{sat}}}(E, P) + m_1$$

4 Effective aspects

During the last years, and in particular thanks to the work of Caruso, Leudière, Ayotte, Musleh [7], Caruso and Leudière [12], as well as Caruso and Gazda [8], the implementation of Drinfeld module theory has begun. In connection with my work, here is a list of algorithmic projects that I would like to carry out, in particular to study some of the problems mentioned above.

1. To try to answer to Problem 3, given an Anderson t-module E defined on A, I would like to develop an algorithm to calculate the Taelman class modules $H(E;A)$, as well as the class modules for the z-deformation $H(\widetilde{E};A[z])$.

2. An important invariant of a Drinfeld module is its spectrum. This notion appears in an essential way in Drinfeld's theory of modular forms, see for example the work of Gekeler [17], and surprisingly gives us information about the vanishing at $z=1$ of the P-adic L-function, see [20, Proposition 6.7]. A short-term project would be to write an algorithm that, given a Drinfeld module, returns its spectrum. One possible approach would be to calculate it from the successive Newton polygons of $\phi_{\theta^n}(X)$ for $n \geq 0$, based on the work of [19].

3. In connection with the previous point, I would also like to focus on statistical questions concerning the vanishing of the P-adic L-functions in the context of Drinfeld modules. Let ϕ be an A-module of Drinfeld defined on A of rank r, denoted by $\phi_\theta = \theta + \sum_{i=1}^{r} a_i \tau^i$. I have shown that the vanishing order at $z=1$ of $L_P(\phi, A, z)$ is bounded by r and does not depend on P, denoted by o_ϕ. Let

   $$\Omega_{x,r} = \{\phi \text{ of rank } r \mid \deg(a_i) < x, i = 1, \dots, r\}, x \in \mathbb{N}^*,$$

   we would like to estimate, for all $0 \leq k \leq r$,

   $$n(r,x,k) = \{\phi \in \Omega_{r,x} \mid o_\phi = k\}$$

   and then let $x \to +\infty$. Note that contrary to what one might expect when comparing with the case of elliptic curves, initial computations seem to show that in general, one expects the order of vanishing of the P-adic L-series at 1 to be zero.

   A first strategy for studying $n(r,x,k)$ would be to consider $u_\phi(z) \in A[z]$, but it is not clear how this polynomial varies when the coefficients a_i are varied.

   Another strategy would be as follows. We have proved the bound

   $$o_\phi \leq \#\{i = 1, \dots, r \mid v_\infty(\lambda_i) \in \mathbb{Z}\}$$

   where $(\lambda_1, \dots, \lambda_r)$ is a "good basis" for the period lattice of ϕ, and the quantity $(v_\infty(\lambda_1), \dots, v_\infty(\lambda_r))$ is the spectrum of ϕ. We therefore want to be able to estimate the number of Drinfeld modules of rank r whose spectrum has exactly k integer values for $k = 0, \dots, r$, and we would then need the algorithm from Project 2 to study the problem.

4. In connection with the tensor powers of the Carlitz module and Problems 11,12,13,14, I would like to implement an algorithm for computing special vector of Anderson-Thakur, as well as algorithms for decomposing vectors for the action of ∂, and finally for studying the class module $H(\widetilde{C^{\otimes n}};A[z])$.

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