

A P -ADIC CLASS FORMULA FOR ANDERSON t -MODULES

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ABSTRACT. In 2012, Taelman proved a class formula for L -series associated to Drinfeld $\mathbb{F}_q[\theta]$ modules and considered it as a function field analog of the Birch and Swinnerton-Dyer conjecture. Since then, Taelman's class formula has been generalized to the setting of Anderson t -modules. Let P be a monic prime of $\mathbb{F}_q[\theta]$, we define the P -adic L -series associated with Anderson t -modules and prove a P -adic class formula à la Taelman linking a P -adic regulator, the class module and a local factor at P . Next, we extend this result to the multi-variable setting à la Pellarin. Finally, we give some applications to Drinfeld modules defined over $\mathbb{F}_q[\theta]$ itself.

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1. INTRODUCTION

1.1. Class formula à la Taelman. In 2010, Taelman introduced the notion of L -series associated to Drinfeld $\mathbb{F}_q[\theta]$ -modules and conjectured a class formula, see [20, Conjecture 1]. He [21] proved this class formula in 2012, also considered as a function field analogue of the Birch and Swinnerton-Dyer conjecture. These results were generalised by Fang [13] and Demeslay [11] in the context of Anderson t -modules that are Drinfeld modules of higher dimension. Finally, Anglès, Tavares Ribeiro and Ngo Dac [4] proved the class formula for a general ring A in the context of admissible Anderson A -modules, including in particular all Drinfeld A -modules.

The objective of the present article is to construct P -adic analogs of these L -series associated to Anderson t -modules. We call them P -adic L -series, and prove a P -adic class formula à la Taelman. We then extend these results to the setting of variables following the work of Anglès, Pellarin and Tavares Ribeiro [5] and Pellarin [19].

The key ingredient will be the notion of z -deformation introduced by Anglès, Tavares Ribeiro [7] as well as the introduction of evaluation of z not only at $z = 1$ but at elements of $\overline{\mathbb{F}}_q$.

We then study the vanishing of the P -adic L -series at $z = 1$. Finally, we investigate in detail the case of $\mathbb{F}_q[\theta]$ -Drinfeld modules defined over $\mathbb{F}_q[\theta]$.

1.2. Main results. Let us give more precise statements of our results.

Let \mathbb{F}_q be a finite field with q elements and θ an indeterminate over \mathbb{F}_q . Let us consider $A = \mathbb{F}_q[\theta]$ and let $K = \mathbb{F}_q(\theta)$ be the rational function field. Let L/K be a finite field extension of degree n . We denote by \mathcal{O}_L the integral closure of A in L . We consider the valuation v_∞ of K normalized such that $v_\infty(\theta^{-1}) = 1$. Let K_∞ be the completion of K with respect to this valuation and we set $L_\infty = L \otimes_K K_\infty$. Let $\tau : x \mapsto x^q$ be the Frobenius map.

If M is an A -module having a finite number of elements, we denote by $[M]_A$ the monic generator of the Fitting ideal of M .

An Anderson t -module E of dimension d defined over \mathcal{O}_L is a non-constant \mathbb{F}_q -algebra homomorphism $E : A \rightarrow M_d(\mathcal{O}_L)\{\tau\}$ such that if $a \in A$ and $E_a = \sum_{i=0}^{r_a} E_{a,i}\tau^i$ then we require $(E_{a,0} - aI_d)^d = 0$. We denote by $\partial_E : A \rightarrow M_d(\mathcal{O}_L)$ the homomorphism of \mathbb{F}_q -algebras $\partial_E(a) = E_{a,0}$. If B is an \mathcal{O}_L -algebra, then we can define two A -module structures on B^d : the first is denoted by $E(B)$ where A acts on B^d via E , and the second is denoted by $\text{Lie}_E(B)$ where A acts on B^d via ∂_E .

There exists a unique series $\exp_E \in M_d(L)\{\{\tau\}\}$, called the exponential series, such that $\exp_E \partial_E(\theta) = E_\theta \exp_E$. Moreover, \exp_E converges on $\text{Lie}_E(L_\infty)$.

The key notion will be the notion of z -deformation introduced by Anglès and Tavares Ribeiro [7]. Let z be a new variable such that $\tau(z) = z$. Set $\tilde{A} = \mathbb{F}_q(z)A$, $\tilde{K} = \mathbb{F}_q(z)K$, $\widetilde{\mathcal{O}}_L = \mathbb{F}_q(z)\mathcal{O}_L$ and $\widetilde{K}_\infty = \mathbb{F}_q(z)((\theta^{-1}))$. We set $\widetilde{L}_\infty = L \otimes_K \widetilde{K}_\infty$. We consider \widetilde{E} the z -twist of E , introduced in [7], that is the homomorphism of $\mathbb{F}_q(z)$ -algebras $\widetilde{E} : \tilde{A} \rightarrow M_d(\widetilde{\mathcal{O}}_L)\{\tau\}$ given by

$$\widetilde{E}_a = \sum_{i=0}^{r_a} E_{a,i}z^i\tau^i, \text{ for all } a \in A.$$

We also have an exponential series $\exp_{\widetilde{E}}$ associated with \widetilde{E} that converges on $\text{Lie}_{\widetilde{E}}(\widetilde{L}_\infty)$.

Taelman showed that the module of z -units

$$U(\tilde{E}; \widetilde{\mathcal{O}}_L) = \{x \in \text{Lie}_{\tilde{E}}(L_\infty) \mid \exp_{\tilde{E}}(x) \in \tilde{E}(\widetilde{\mathcal{O}}_L)\}$$

is an \tilde{A} -lattice in $\text{Lie}_{\tilde{E}}(\widetilde{L}_\infty)$ and that the Taelman unit module

$$U(E; \mathcal{O}_L) = \{x \in \text{Lie}_E(L_\infty) \mid \exp_E(x) \in E(\mathcal{O}_L)\}$$

is an A -lattice in $\text{Lie}_E(L_\infty)$. Moreover, he proved that the class module

$$H(E; \mathcal{O}_L) = \frac{E(L_\infty)}{E(\mathcal{O}_L) + \exp_E(\text{Lie}_E(\mathcal{O}_L))}$$

is finite. The local factors associated with \tilde{E} and E at a monic prime polynomial Q are respectively

$$z_Q(\tilde{E}/\widetilde{\mathcal{O}}_L) = \frac{[\text{Lie}_{\tilde{E}}(\widetilde{\mathcal{O}}_L/Q\widetilde{\mathcal{O}}_L)]_{\tilde{A}}}{[\tilde{E}(\widetilde{\mathcal{O}}_L/Q\widetilde{\mathcal{O}}_L)]_{\tilde{A}}} \in \tilde{K}^* \text{ and } z_Q(E/\mathcal{O}_L) = \frac{[\text{Lie}_E(\mathcal{O}_L/Q\mathcal{O}_L)]_A}{[E(\mathcal{O}_L/Q\mathcal{O}_L)]_A} \in K^*.$$

Set $m = dn$ and consider \mathcal{C} an A -basis of $\text{Lie}_E(\mathcal{O}_L)$. Fix $(u_1(z), \dots, u_m(z))$ an \tilde{A} -basis of $U(\tilde{E}; \widetilde{\mathcal{O}}_L)$ and (u_1, \dots, u_m) an A -basis of $U(E; \mathcal{O}_L)$. Demeslay proved in [11] that the following product converges in $\mathbb{T}_z(K_\infty)$, the completion of $K_\infty[z]$ with respect to the Gauss norm:

$$L(\tilde{E}/\widetilde{\mathcal{O}}_L) = \prod_Q z_Q(\tilde{E}/\widetilde{\mathcal{O}}_L)$$

where the product runs over all the monic irreducible polynomials Q of A . We call this product the L -series associated with \tilde{E} and $\widetilde{\mathcal{O}}_L$. By evaluation at $z = 1$ we obtain:

$$L(E/\mathcal{O}_L) = \text{ev}_{z=1} L(\tilde{E}, \widetilde{\mathcal{O}}_L) = \prod_Q z_Q(E/\mathcal{O}_L) \in K_\infty^*.$$

We call this product the L -series associated with E and \mathcal{O}_L . Demeslay [11] proved the following class formulas à la Taelman

$$L(\tilde{E}/\widetilde{\mathcal{O}}_L) = \frac{\det_{\mathcal{C}}(u_1(z), \dots, u_m(z))}{\text{sgn}(\det_{\mathcal{C}}(u_1(z), \dots, u_m(z)))}$$

and

$$L(E/\mathcal{O}_L) = \frac{\det_{\mathcal{C}}(u_1, \dots, u_m)}{\text{sgn}(\det_{\mathcal{C}}(u_1, \dots, u_m))} [H(E; \mathcal{O}_L)]_A.$$

We consider the P -adic setting. Let P be an irreducible monic polynomial of A and v_P its associated valuation on K such that $v_P(P) = 1$. We consider $K_P \simeq \mathbb{F}_{q^{\deg(P)}}((P))$ (resp. A_P) the completion of K (resp. A) with respect to P . Consider the Tate algebra in the variable z , $\mathbb{T}_z(K_P)$, that is the completion of $K_P[z]$ with respect to the Gauss norm. Our first main result is the construction and the convergence of the following P -adic L -series. The key argument will be the evaluation at $z = \zeta \in \overline{\mathbb{F}}_q$, see Subsections 3.4 and 4.3.

Theorem A (Theorem 4.10). *The following product converges in $\mathbb{T}_z(K_P)$*

$$L_P(\tilde{E}/\widetilde{\mathcal{O}}_L) = \prod_{Q \neq P} z_Q(\tilde{E}/\widetilde{\mathcal{O}}_L)$$

where the product runs over all the monic irreducible polynomials Q of A different from P .

We then define a P -adic logarithm $\text{Log}_{E,P}$ which converges on $\{x \in \mathcal{O}_L^d \mid v_P(x) \geq 0\}$ and a P -adic regulator associated with the unit module as follows. Let (u_1, \dots, u_m) be an A -basis of the unit module. We then define the P -adic regulator of the unit module by:

$$R_P(U(E; \mathcal{O}_L)) = \frac{\det_{\mathcal{C}}(\text{Log}_{E,P}(\exp_E(u_1)), \dots, \text{Log}_{E,P}(\exp_E(u_m)))}{\text{sgn}(\det_{\mathcal{C}}(u_1, \dots, u_m))} \in K_P.$$

The construction does not depend on \mathcal{C} nor on the choice of the basis of the unit module. We do the same with the variable z . We then prove the following P -adic class formula à la Taelman.

Theorem B (Theorem 4.21). *Let E be an Anderson t -module defined over \mathcal{O}_L . Then we have the P -adic class formula for \tilde{E} :*

$$z_P(\tilde{E}/\widetilde{\mathcal{O}}_L)L_P(\tilde{E}/\widetilde{\mathcal{O}}_L) = R_P(U(\tilde{E}; \widetilde{\mathcal{O}}_L))$$

and the class formula for E :

$$z_P(E/\mathcal{O}_L)L_P(E/\mathcal{O}_L) = R_P(U(E; \mathcal{O}_L)) [H(E; \mathcal{O}_L)]_A.$$

A major difference from the ∞ -adic case is that our P -adic L -series obtained will vanish at $z = 1$ in certain cases that we are able to characterize. Set $U(E; P\mathcal{O}_L) = \{x \in \text{Lie}_E(L_\infty) \mid \exp_E(x) \in E(P\mathcal{O}_L)\}$ and $\mathcal{U}(E; P\mathcal{O}_L) = \exp_E(U(E; P\mathcal{O}_L))$ which is provided with an A_P -structure module

Theorem C (Proposition 5.4 and Conjecture 4.7). *We have the following assertions.*

- (1) *If the exponential map $\exp_E : L_\infty^d \rightarrow L_\infty^d$ is not injective, then $L_P(E/\mathcal{O}_L) = 0$.*
- (2) *If the A -rank of $\exp_E(U(E; \mathcal{O}_L))$ and the A_P -rank of $\mathcal{U}(E; P\mathcal{O}_L)$ coincide, then the two following assertions are equivalent:*
 - (a) $L_P(E/\mathcal{O}_L) \neq 0$,
 - (b) $\exp_E : L_\infty^d \rightarrow L_\infty^d$ is injective.

We extend the previous results to the multi-variable setting in the spirit of the work of Anglès, Pellat and Tavares Ribeiro [5], Demeslay [11] and Pellat [19]. Consider $s \geq 1$ an integer and t_1, \dots, t_s new variables. Set $k = \mathbb{F}_q(t_1, \dots, t_s)$, $A_s = k[\theta]$, $\mathcal{O}_{L,s} = k\mathcal{O}_L$ and $K_{s,P} = \mathbb{F}_{q^{\deg(P)}}(t_1, \dots, t_s)((P))$. Let $s \geq 0$ be a non-negative integer and E be an Anderson A_s -module defined over $\mathcal{O}_{L,s}$. We consider the Tate algebra with multi-variable $\mathbb{T}_s(K_P)$ that is the completion of $K_P[t_1, \dots, t_s]$. We prove the following result.

Theorem D (Theorem 5.8 and Theorem 5.3).

- (1) *The infinite product*

$$L_P(E/\mathcal{O}_{L,s}) = \prod_{Q \neq P} z_Q(E/\mathcal{O}_{L,s})$$

where Q runs through the monic primes of A different from P , converges in $K_{s,P}$ and we have the class formula:

$$z_P(E/\mathcal{O}_{L,s})L_P(E/\mathcal{O}_{L,s}) = R_P(U(E; \mathcal{O}_{L,s})) [H(E; \mathcal{O}_{L,s})]_{A_s}.$$

- (2) *If E is defined over $\mathbb{F}_q[t_1, \dots, t_s]\mathcal{O}_L$, then $L_P(E/\mathcal{O}_{L,s}) \in \mathbb{T}_s(K_P)$.*

The key point for proving the second assertion will be the successive evaluation at $t_i = \zeta_i \in \overline{\mathbb{F}_q}$ so using techniques from Subsections 3.4 and 4.3.

Finally, in Section 6, we give a detailed application of the previous theorems and obtain bounds on the vanishing order at $z = 1$ of the P -adic L -series.

Theorem E (Proposition 6.7). *Consider ϕ an A -Drinfeld module defined over A of rank r . Then the vanishing order at $z = 1$ of the P -adic L -series is less than or equal to r .*

The P -adic L -series is studied in detail in a work of Caruso, Gazda and the author in the context of A -Drinfeld modules defined over A , see [9].

1.3. Some remarks. Recently, Anglès [3] defined the P -adic L -series $L_P(\widetilde{C}/\widetilde{\mathcal{O}}_L)$ associated with the Carlitz module defined over \mathcal{O}_L . He was able to prove that the P -adic L -series $L_P(\widetilde{C}/\widetilde{\mathcal{O}}_L)$ is a meromorphic series without pole at $z = 1$, see [3, Theorem 6.6]. In particular he defined the P -adic L -series as the limit at $z = 1$ of this series which is an element of K_P and proved a P -adic class formula, see [3, Theorem 6.7]. The present work generalizes this result for Anderson t -modules, including all Drinfeld modules, by proving that the P -adic L -series is an element of $\mathbb{T}_z(K_P)$ so that we can evaluate at $z = 1$.

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2. NOTATION AND BACKGROUND

2.1. Notation. We keep the notation in the Introduction and we introduce the following notation.

- \mathbb{C}_∞ : the completion of a fixed algebraic closure of K_∞ ,
- $\tau : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ the Frobenius endomorphism,
- $M_d(R) = M_{d \times d}(R)$, for a ring R the left R -module of $d \times d$ matrices,
- I_d : the identity matrix of $M_d(R)$.

Let us fix an integer $d \geq 1$ and B an \mathbb{F}_q -algebra. If $M = (m_{i,j})$ is a matrix with coefficients in \mathbb{C}_∞ and $k \in \mathbb{N}$, we set $\tau^k(M) = M^{(k)}$ to be the matrix whose ij -entry is given by $\tau^k(m_{i,j})^{(k)} = m_{i,j}^{q^k}$. We denote by $M_d(B)\{\tau\}$ the non-commutative ring of twisted polynomials in τ with coefficients in $M_d(B)$ equipped with the usual addition and the commutation rule $\tau^k M = M^{(k)} \tau^k$ for all $k \in \mathbb{N}$ and all $M \in M_d(B)$. Let $M_d(B)\{\{\tau\}\}$ be the non-commutative ring of twisted power series in τ with coefficients in $M_d(B)$.

If k is a field containing \mathbb{F}_q , we set $(kK)_\infty = k \hat{\otimes}_{\mathbb{F}_q} K_\infty = k((\frac{1}{\theta}))$. If $x \in (kK)_\infty^\times$, we can write x uniquely as $x = \sum_{n \geq N} x_n \frac{1}{\theta^n}$, $x_n \in k$ and $x_N \neq 0$. Then we call $x_N \in k$ the sign denoted by $\text{sgn}(x)$. We say that such an $x \in (kK)_\infty$ is monic if $\text{sgn}(x) = 1$.

2.2. Fitting ideals. We recall here some definitions about Fitting ideals of modules over Dedekind rings. Let R be a Dedekind ring, and M be a finite and torsion R -module. By the structure theorem, there exists $s \in \mathbb{N}$ and I_1, \dots, I_s ideals of R such that we have an isomorphism of R -modules

$$M \simeq R/I_1 \times \dots \times R/I_s.$$

We then define the Fitting ideal of M by

$$\text{Fitt}_R(M) = I_1 \dots I_s.$$

We have the following properties that can be found in the appendix of [17] except the second one which appears in [12, Corollary 20.5].

Proposition 2.1.

- (1) We have $\text{Fitt}_R(M) \subseteq \text{Ann}_R(M) = \{x \in R \mid x.m = 0 \ \forall m \in M\}$.
- (2) If $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$ is exact, then $\text{Fitt}_R(M_1) \text{Fitt}_R(M_2) = \text{Fitt}_R(M)$.

2.3. Lattices and Ratio of co-volumes. We use the following notation from [22, Section 7.2.3]. We fix k a field containing \mathbb{F}_q and recall that $(kK)_\infty = k \hat{\otimes}_{\mathbb{F}_q} K_\infty = k((\frac{1}{\theta}))$. In what follows, we fix V to be a finite-dimensional $(kK)_\infty$ -vector space endowed with the natural topology coming from $(kK)_\infty$.

Definition 2.2. A sub- $k[\theta]$ -module M of V is a $k[\theta]$ -lattice in V if M is discrete in V and if M generates V over $(kK)_\infty$.

Proposition 2.3. Let M be a sub- $k[\theta]$ -module of V . The following are equivalent:

- (1) M is a $k[\theta]$ -lattice in V .
- (2) There exists a $(kK)_\infty$ -basis (v_1, \dots, v_n) of V such that M is the free $k[\theta]$ -module of basis (v_1, \dots, v_n) .

Proof. See [22, Proposition 7.2.3]. \square

Let M and M' be two $k[\theta]$ -lattices in V . Let \mathcal{B} and \mathcal{B}' be $k[\theta]$ -basis of M and M' , respectively. The ratio of co-volumes of M in M' is then defined as

$$[M' : M]_{k[\theta]} = \frac{\det_{\mathcal{B}'} \mathcal{B}}{\det_{\mathcal{B}} \mathcal{B}} \in (kK)_\infty^*.$$

Note that this is independent of the choices of \mathcal{B} and \mathcal{B}' . The definition immediately implies that if M_0, M_1 and M_2 are lattices in V , then

$$[M_0 : M_1]_{k[\theta]} [M_1 : M_2]_{k[\theta]} = [M_0 : M_2]_{k[\theta]}.$$

We also observe that for two lattices M, M' in V we have

$$[M' : M]_{k[\theta]} = [M : M']_{k[\theta]}^{-1}.$$

3. THE ∞ -ADIC CASE

From now on, let L/K be a finite fields extension. Recall that we denote by: \mathcal{O}_L the integral closure of A in L , $\mathcal{O}_L[z] \simeq \mathbb{F}_q[z] \otimes_{\mathbb{F}_q} \mathcal{O}_L$, $\widetilde{\mathcal{O}}_L = \mathbb{F}_q(z) \otimes_{\mathbb{F}_q} \mathcal{O}_L$, $\widetilde{L}_\infty = L \otimes_{\mathbb{F}_q} \widetilde{K}_\infty$. In this section, we extend the notion of the Taelman unit module and class module by twisting with some elements $\zeta \in \mathbb{F}_q$.

3.1. Anderson modules. An Anderson t -module (or shortly a t -module) E of dimension d defined over \mathcal{O}_L is an \mathbb{F}_q -algebra homomorphism $E : A \rightarrow M_d(\mathcal{O}_L)\{\tau\}$ such that if $a \in A$ and $E_a = \sum_{i=0}^{r_a} E_{a,i} \tau^i$, then we require $(E_{a,0} - aI_d)^d = 0$ and that $\deg_\tau(E_\theta) > 0$. Let $E : A \rightarrow M_d(\mathcal{O}_L)\{\tau\}$ be a t -module of dimension $d \geq 1$, completely determined by the value at θ :

$$E_\theta = \sum_{i=0}^r E_{\theta,i} \tau^i$$

with $E_{\theta,i} \in M_d(\mathcal{O}_L)$ and $(E_{\theta,0} - \theta I_d)^d = 0$. For $a \in A$, if $E_a = \sum_{i=0}^{r_a} E_{a,i} \tau^i$, we define $\partial_E(a) = E_{a,0}$. Then the map $\partial_E : A \rightarrow M_d(A)$ is a homomorphism of \mathbb{F}_q -algebras.

We then consider the z -twist of E , introduced in [7], (remember that τ acts as the identity over $\mathbb{F}_q(z)$) denoted by \tilde{E} to be the homomorphism of $\mathbb{F}_q(z)$ -algebras $\tilde{E} : \tilde{A} \rightarrow M_d(\widetilde{\mathcal{O}}_L)\{\tau\}$ given by:

$$\tilde{E}_\theta = \sum_{i=0}^r E_{\theta,i} z^i \tau^i.$$

Recall the following notation taken from [4]. Let E be a t -module of dimension d over R an extension of \mathbb{F}_q and let B be an R -algebra. We can then define two A -module structures on B^d . The first is denoted $E(B)$ where A acts on B^d via E :

$$a.x = E_a(x) \in B^d \text{ for all } a \in A, x \in B^d.$$

The second is $\text{Lie}_E(B)$ where A acts on B^d via ∂_E :

$$a.x = \partial_E(a)x \text{ for all } a \in A, x \in B^d.$$

We have the following results that can be found in [1, Proposition 2.1.4].

Proposition 3.1. *There exists a unique element $\exp_E \in M_d(L)\{\{\tau\}\}$ such that:*

- (1) $\exp_E \partial_E(a) = E_a \exp_E$ hold in $M_d(L)\{\{\tau\}\}$ for all $a \in A$,

(2) $\exp_E \equiv I_d \text{ mod } M_d(L)\{\{\tau\}\}\tau.$

We call \exp_E the exponential map associated with the t -module E , and denote this element by $\exp_E = \sum_{i=0}^{\infty} d_i \tau^n$.

Proposition 3.2. *There exists a unique element $\log_E \in M_d(L)\{\{\tau\}\}$ such that:*

- (1) $\log_E E_a = \partial_E(a) \log_E$ hold in $M_d(L)\{\{\tau\}\}$ for all $a \in A$,
- (2) $\log_E \equiv I_d \text{ mod } M_d(L)\{\{\tau\}\}\tau.$

In addition, we have the equalities in $M_d(L)\{\{\tau\}\}$:

$$\log_E \exp_E = \exp_E \log_E = I_d.$$

We call \log_E the logarithm map associated to the t -module E , and we denote this element by $\log_E = \sum_{n=0}^{\infty} l_n \tau^n$. We also have exponential and logarithm series for the z -twist of the t -module \tilde{E} which we denote by $\exp_{\tilde{E}}$ and $\log_{\tilde{E}}$ and given by:

$$\exp_{\tilde{E}} = \sum_{n \geq 0} d_n z^n \tau^n \text{ and } \log_{\tilde{E}} = \sum_{n \geq 0} l_n z^n \tau^n.$$

3.2. Unit module and class module. We consider an over-additive valuation v_{∞} on the-finite dimensional K_{∞} -vector space L_{∞} (for example with respect to the choise of a basis of L/K). The key point is the next result from [15, Theorem 4.6.9].

Lemma 3.3. *We have*

$$\lim_{i \rightarrow +\infty} \frac{v_{\infty}(d_i)}{q^i} = +\infty.$$

Corollary 3.4. *The exponential map \exp_E converges on $\text{Lie}_E(L_{\infty})$ and induces a homomorphism of A -modules:*

$$\exp_E : \text{Lie}_E(L_{\infty}) \rightarrow E(L_{\infty}).$$

We also have the convergence of $\exp_{\tilde{E}}$ on $\text{Lie}_{\tilde{E}}(\widetilde{L_{\infty}})$ (resp. $\text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_{\infty}))$) that induces a homomorphism of \tilde{A} -modules (resp. $A[z]$):

$$\exp_{\tilde{E}} : \text{Lie}_{\tilde{E}}(\widetilde{L_{\infty}}) \rightarrow \widetilde{E(L_{\infty})}.$$

(resp. $\exp_{\tilde{E}} : \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_{\infty})) \rightarrow \widetilde{E}(\mathbb{T}_z(L_{\infty}))$). We can now define the Taelman unit module

$$U(E; \mathcal{O}_L) = \{x \in \text{Lie}_E(L_{\infty}) \mid \exp_E(x) \in E(\mathcal{O}_L)\}$$

provided with A -module structure, as well as the module of z -units:

$$U(\tilde{E}; \widetilde{\mathcal{O}_L}) = \left\{ x \in \text{Lie}_{\tilde{E}}(\widetilde{L_{\infty}}) \mid \exp_{\tilde{E}}(x) \in \widetilde{E}(\widetilde{\mathcal{O}_L}) \right\}$$

provided with \tilde{A} -module structure, and the module of z -units at the integral level:

$$U(\widetilde{E}; \mathcal{O}_L[z]) = \left\{ x \in \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_{\infty})) \mid \exp_{\tilde{E}}(x) \in \widetilde{E}(\mathcal{O}_L[z]) \right\}$$

provided with $A[z]$ -module structure. We also define the class module (introduced by Taelman in [21]):

$$H(E; \mathcal{O}_L) = \frac{E(L_{\infty})}{E(\mathcal{O}_L) + \exp_E(\text{Lie}_E(L_{\infty}))}$$

as well as the class module for the z -deformation

$$H(\tilde{E}; \widetilde{\mathcal{O}_L}) = \frac{\widetilde{E(L_{\infty})}}{\widetilde{E}(\widetilde{\mathcal{O}_L}) + \exp_{\tilde{E}}(\text{Lie}_{\tilde{E}}(\widetilde{L_{\infty}}))}$$

and finally the class module at the integral level

$$H(\tilde{E}; \mathcal{O}_L[z]) = \frac{\tilde{E}(\mathbb{T}_z(L_\infty))}{\tilde{E}(\mathcal{O}_L[z]) + \exp_{\tilde{E}}(\text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)))}.$$

Consider one of the following cases:

- $k_0 = \mathbb{F}_q$, $\varphi = E$ and $(k_0 L)_\infty = L_\infty$,
- $k_0 = \mathbb{F}_q[z]$ and $\varphi = \tilde{E}$,
- $k_0 = \mathbb{F}_q(z)$, $\varphi = \tilde{E}$ and $(k_0 L)_\infty = \widetilde{L_\infty}$.

We have the following result from [11, Proposition 2.8].

Proposition 3.5.

- (1) *The class module $H(\varphi; k_0 \mathcal{O}_L)$ is a finite-dimensional k_0 -vector space, so a finite and torsion $k_0 A$ -module.*
- (2) *If k_0 is a field, then the module of units $U(\varphi; k_0 \mathcal{O}_L)$ is a $k_0 A$ -lattice in $\text{Lie}_\varphi((k_0 L)_\infty)$.*

We also have the following result in [7, Proposition 2.3].

Proposition 3.6. *We have the following equality:*

$$U(\tilde{E}; \widetilde{\mathcal{O}_L}) = \mathbb{F}_q(z) U(\tilde{E}; \mathcal{O}_L[z]).$$

Consider the evaluation morphism:

$$\text{ev}_{z=1} : \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \rightarrow \text{Lie}_E(L_\infty).$$

It induces an exact sequence of A -modules:

$$0 \longrightarrow (z - 1) \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \longrightarrow \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty)) \xrightarrow{\text{ev}_{z=1}} \text{Lie}_E(L_\infty) \longrightarrow 0.$$

For all $x \in \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_\infty))$ we have $\text{ev}_{z=1}(\exp_{\tilde{E}}(x)) = \exp_E(\text{ev}_{z=1}(x))$. Moreover, if $f(z) \in \widetilde{L_\infty}$ belongs to the ∞ -adic convergence domain of the logarithm map $\log_{\tilde{E}}$, then we have

$$\text{ev}_{z=1}(\log_{\tilde{E}}(f(z))) = \log_E(\text{ev}_{z=1}(f(z))).$$

We recall the notion of Stark units introduced by B. Anglès and F. Tavares Ribeiro in [7, section 2.5].

Definition 3.7. The module of Stark units $U_{\text{St}}(E; \mathcal{O}_L)$ is defined by:

$$U_{\text{St}}(E; \mathcal{O}_L) = \text{ev}_{z=1} U(\tilde{E}; \mathcal{O}_L[z]).$$

Given the compatibility between the exponential and the evaluation morphism, $U_{\text{St}}(E; \mathcal{O}_L)$ is a sub- A -module of $U(E; \mathcal{O}_L)$. We have the following result from [7, Theorem 1].

Theorem 3.8. *The A -module $U_{\text{St}}(E; \mathcal{O}_L)$ is an A -lattice in $\text{Lie}_E(L_\infty)$.*

3.3. The L series. For a monic prime P of A , we define the local factor at P associated with E :

$$z_P(E/\mathcal{O}_L) = \frac{[\text{Lie}_E(\mathcal{O}_L/P\mathcal{O}_L)]_A}{[E(\mathcal{O}_L/P\mathcal{O}_L)]_A} \in K$$

and the local factor at P associated with \tilde{E} :

$$z_P(\tilde{E}/\widetilde{\mathcal{O}_L}) = \frac{[\text{Lie}_{\tilde{E}}(\widetilde{\mathcal{O}_L}/P\widetilde{\mathcal{O}_L})]_{\tilde{A}}}{[\tilde{E}(\widetilde{\mathcal{O}_L}/P\widetilde{\mathcal{O}_L})]_{\tilde{A}}} \in \widetilde{K}.$$

We then define the L -series associated with E and \mathcal{O}_L by the Eulerian product:

$$L(E/\mathcal{O}_L) = \prod_{P \in A} z_P(E/\mathcal{O}_L)$$

where P runs through the monic primes of A , and the L -series associated with \tilde{E} and $\widetilde{\mathcal{O}}_L$ by the Eulerian product:

$$L(\tilde{E}/\widetilde{\mathcal{O}}_L) = \prod_{P \in A} z_P(\tilde{E}/\widetilde{\mathcal{O}}_L).$$

We have the convergence of the L -series and the class formula for z -deformation from [11, Theorem 2.7].

Theorem 3.9 (Class formula for the z -deformation). *The product defining $L(\tilde{E}/\widetilde{\mathcal{O}}_L)$ converges in \widetilde{K}_∞^* and we have the formula*

$$(1) \quad L(\tilde{E}/\widetilde{\mathcal{O}}_L) = [\text{Lie}_{\tilde{E}}(\widetilde{\mathcal{O}}_L) : U(\tilde{E}; \widetilde{\mathcal{O}}_L)]_{\tilde{A}}.$$

Adapting the proof of [22, Corollary 7.5.6] in the higher dimensional case we obtain that the polynomial $[\text{Fitt}_{\tilde{A}}(\tilde{E}(\widetilde{\mathcal{O}}_L/P\widetilde{\mathcal{O}}_L))]_{\tilde{A}} \in A[z]$ is a unit in $\mathbb{T}_z(K_\infty)$. We then obtain:

Corollary 3.10. *The L -series $L(\tilde{E}/\widetilde{\mathcal{O}}_L)$ converges in $\mathbb{T}_z(K_\infty)^\times$.*

We can evaluate the L -series at $z = 1$:

$$L(E/\mathcal{O}_L) = \text{ev}_{z=1} L(\tilde{E}/\widetilde{\mathcal{O}}_L) = \prod_Q \frac{[\text{Lie}_E(\mathcal{O}_L/Q\mathcal{O}_L)]_A}{[E(\mathcal{O}_L/Q\mathcal{O}_L)]_A} \in K_\infty^*$$

where Q runs through the monic primes of A . We have the following class formula for t -modules obtained by Fang in [13], generalizing Taelman's class formula for Drinfeld modules.

Theorem 3.11 (Class formula for Anderson t -modules). *The product defining $L(E/\mathcal{O}_L)$ converges in K_∞^* , and we have the equalities*

$$(2) \quad L(E/\mathcal{O}_L) = [\text{Lie}_E(\mathcal{O}_L) : U(E; \mathcal{O}_L)]_A [H(E; \mathcal{O}_L)]_A = [\text{Lie}_E(\mathcal{O}_L) : U_{\text{st}}(E; \mathcal{O}_L)]_A.$$

3.4. Evaluation at $z = \zeta \in \overline{\mathbb{F}}_q$. We want to extend the notion of Stark units by evaluating the variable z at $z = \zeta$ for all $\zeta \in \overline{\mathbb{F}}_q$.

Let ζ be an element of $\overline{\mathbb{F}}_q$ and consider $\mathbb{F}_q(\zeta)$ the finite field obtained by adding ζ to \mathbb{F}_q . Let us define the ring $A_\zeta = \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} A$. We define a Frobenius $\tau_\zeta = \text{id} \otimes \tau$ acting on A_ζ . Let us define $\widetilde{A}_\zeta = \mathbb{F}_q(z) \otimes_{\mathbb{F}_q} A_\zeta$ on which we extend the Frobenius τ_ζ by $\tau_\zeta = \text{id} \otimes \tau_\zeta$, still denoted by τ_ζ (i.e., the Frobenius τ_ζ acts as the identity on $\mathbb{F}_q(z)$). Denote by $A_\zeta[z] = \mathbb{F}_q[z] \otimes_{\mathbb{F}_q} A_\zeta$. Set $\mathcal{O}_{L,\zeta} = \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathcal{O}_L$. It is also equipped with the following Frobenius $\tau_\zeta = \text{id} \otimes \tau$.

Similarly as the z -deformation, let us twist the t -module E into an Anderson A_ζ -module E_ζ defined over $M_d(\mathcal{O}_{L,\zeta})$ by

$$(E_\zeta)_\theta = \sum_{i=0}^r E_{\theta,i} \zeta^i \tau_\zeta^i \in M_d(\mathcal{O}_{L,\zeta})\{\tau_\zeta\}$$

then extend to A_ζ by $\mathbb{F}_q(\zeta)$ -linearity.

Set $M_w = \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} L_w$ where $w = \infty$ or $w = P$. Consider $\mathbb{F}_q[z] \otimes_{\mathbb{F}_q} \mathcal{O}_{L,\zeta} = \mathcal{O}_{L,\zeta}[z]$ then set $\widetilde{\mathcal{O}}_{L,\zeta} = \mathbb{F}_q(z) \otimes_{\mathbb{F}_q} \mathcal{O}_{L,\zeta}$ and $\widetilde{M}_w = \mathcal{O}_{L,\zeta}[z] \otimes_{\mathcal{O}_{L,\zeta}[z]} \mathbb{T}_z(L_w) \simeq \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathbb{T}_z(L_w)$ and consider $\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \widetilde{L}_w$. We extend v_∞ to M_∞ as follows. Let's fix (f_1, \dots, f_m) a \mathbb{F}_q -basis of $\mathbb{F}_q(\zeta)$. We set

$$v_\infty \left(\sum_{i=1}^m f_i \otimes x_i \right) = \min_{i=1, \dots, m} v_\infty(x_i)$$

for $x_i \in K_\infty$. The topology over M_∞ does not depend on the choice of the basis (f_1, \dots, f_m) . We then consider v_∞ an over-additive valuation on the $\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} K_\infty$ -vector space of finite dimension M_∞ . We then extend similarly v_∞ to $\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \widetilde{L}_\infty$. Remark that we cannot just replace v_∞ by v_P on $\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} K_P$ with these constructions, in fact we do not obtain a valuation over M_{v_P} . See Subsection 4.3 for more details.

Finally, we deform E into $E^{(\zeta)}$ an Anderson A_ζ -module on $M_d(\mathcal{O}_{L,\zeta})$ by

$$E_\theta^{(\zeta)} = \sum_{i=0}^r E_{\theta,i} \tau_\zeta^i$$

and extend it to A_ζ by $\mathbb{F}_q(\zeta)$ -linearity. We finally extend it to an Anderson \widetilde{A}_ζ -module $\widetilde{E}^{(\zeta)}$ on $M_d(\widetilde{\mathcal{O}}_{L,\zeta})$ in the usual way.

We have exponential maps associated with each of the Anderson modules. From the definitions we have the equalities

$$\exp_{E^{(\zeta)}} = \sum_{n \geq 0} d_n \tau_\zeta^n \text{ and } \log_{E^{(\zeta)}} = \sum_{n \geq 0} l_n \tau_\zeta^n,$$

and the map $\exp_{E^{(\zeta)}}$ (resp. $\exp_{\widetilde{E}^{(\zeta)}}$) converges on $\text{Lie}_{E^{(\zeta)}}(M_\infty)$ (resp. on $\text{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M}_\infty)$). Moreover, we have the following equalities in $M_d(\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} L)\{\{\tau_\zeta\}\}$:

$$\exp_{E_\zeta} = \sum_{n \geq 0} d_n \zeta^n \tau_\zeta^n \text{ and } \log_{E_\zeta} = \sum_{n \geq 0} l_n \zeta^n \tau_\zeta^n.$$

Consider the evaluation morphism at $z = \zeta$:

$$\text{ev}_\zeta = \text{ev}_{z=\zeta} : \widetilde{M}_\infty \rightarrow M_\infty$$

whose kernel is given by $(z - \zeta)\widetilde{M}_\infty$, then we consider the following evaluation morphism still denoted by ev_ζ :

$$\text{ev}_\zeta : \text{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M}_\infty) \rightarrow \text{Lie}_{E^{(\zeta)}}(M_\infty).$$

For $x \in \text{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M}_\infty)$, we have in $\text{Lie}_{E^{(\zeta)}}(M_\infty)$:

$$\text{ev}_\zeta(\exp_{\widetilde{E}^{(\zeta)}}(x)) = \exp_{E_\zeta}(\text{ev}_\zeta(x)).$$

Let us consider the module of ζ -units at the integral level:

$$U(\widetilde{E}^{(\zeta)}; \mathcal{O}_{L,\zeta}[z]) = \left\{ x \in \text{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M}_\infty) \mid \exp_{\widetilde{E}^{(\zeta)}}(x) \in \widetilde{E}(\mathcal{O}_{L,\zeta}[z]) \right\}$$

as well as the module of the ζ -classes at the integral level:

$$H(\widetilde{E}^{(\zeta)}; \mathcal{O}_{L,\zeta}[z]) = \frac{\widetilde{E}^{(\zeta)}(\widetilde{M}_\infty)}{\widetilde{E}^{(\zeta)}(\mathcal{O}_{L,\zeta}[z]) + \exp_{\widetilde{E}^{(\zeta)}}(\text{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M}_\infty))} = A_\zeta[z] \otimes_{\mathbb{F}_q[z]} H(\widetilde{E}, \mathcal{O}_L[z])$$

provided with a structure of $A_\zeta[z]$ -modules. Next, consider the ζ -unit module:

$$U(E_\zeta; \mathcal{O}_{L,\zeta}) = \{x \in \text{Lie}_{E_\zeta}(M_\infty) \mid \exp_{E_\zeta}(x) \in E_\zeta(\mathcal{O}_{L,\zeta})\}$$

and the ζ -class module

$$H(E_\zeta, \mathcal{O}_{L,\zeta}) = \frac{E_\zeta(M_\infty)}{E_\zeta(\mathcal{O}_{L,\zeta}) + \exp_{E_\zeta}(\text{Lie}_{E_\zeta}(M_\infty))}$$

provided with their A_ζ -module structure via E_ζ .

Results to come in this section are adapted from [11] and [7].

Proposition 3.12.

- (1) The exponential map $\exp_{E_\zeta} : \text{Lie}_{E_\zeta}(M_\infty) \rightarrow E_\zeta(M_\infty)$ is locally an isometry.
- (2) The exponential map $\exp_{\widetilde{E}^{(\zeta)}} : \text{Lie}_{\widetilde{E}^{(\zeta)}}(\widetilde{M}_\infty) \rightarrow \widetilde{E}^{(\zeta)}(\widetilde{M}_\infty)$ is locally an isometry.

Proof. The proof is a direct corollary of Lemma 3.3, we omit the proof. \square

Proposition 3.13.

- (1) The module of ζ -classes $H(E_\zeta; \mathcal{O}_{L,\zeta})$ is a $\mathbb{F}_q(\zeta)$ -vector space of finite dimension, hence a torsion A_ζ -module of finite type.
- (2) The module of ζ -units $U(E_\zeta; \mathcal{O}_{L,\zeta})$ is an A_ζ -lattice in M_∞ .

(3) The class module $H(\tilde{E}^{(\zeta)}; \mathcal{O}_{L,\zeta}[z])$ is a $\mathbb{F}_q(\zeta)[z]$ -module of finite type.

Proof. The proof follows the proof of [11, Proposition 2.6] for the two first assertions, and the proof of [7, Proposition 2] for the last one, by replacing A by A_ζ , \mathcal{O}_L by $\mathcal{O}_{L,\zeta}$ and E by E_ζ . We omit the details. \square

Just as Stark's units consist of the evaluation at $z = 1$ of the z -units, we define the evaluation at $z = \zeta$ of the ζ -units at the integral level:

$$U_\zeta(E; \mathcal{O}_L) = \text{ev}_\zeta U(\tilde{E}^{(\zeta)}; \mathcal{O}_{L,\zeta}[z]) \subseteq U(E_\zeta; \mathcal{O}_{L,\zeta})$$

provided with an A_ζ -module structure via E_ζ .

Theorem 3.14. *There exists an A_ζ -module isomorphism:*

$$\frac{U(E_\zeta; \mathcal{O}_{L,\zeta})}{U_\zeta(E; \mathcal{O}_L)} \simeq H(\tilde{E}^{(\zeta)}; \mathcal{O}_{L,\zeta}[z])[z - \zeta]$$

where $H(\tilde{E}^{(\zeta)}; \mathcal{O}_{L,\zeta}[z])[z - \zeta]$ is the $(z - \zeta)$ -torsion of the ζ -class module at the integral level.

In the following, we will denote by $M = \mathcal{O}_{L,\zeta}$ and $\tilde{M} = \mathcal{O}_{L,\zeta}[z]$.

Proof. We follow the proof of [7, Proposition 2.6].

Consider the map

$$\begin{aligned} \alpha : M_\infty^d &\rightarrow \tilde{M}_\infty^d \\ x &\mapsto \frac{\exp_{\tilde{E}^{(\zeta)}}(x) - \exp_{E_\zeta}(x)}{z - \zeta}. \end{aligned}$$

We divide the proof into several steps.

Step 1: The map is well defined since

$$\text{ev}_\zeta(\exp_{\tilde{E}^{(\zeta)}}(x)) = \exp_{E_\zeta}(x)$$

for $x \in M_\infty^d$, thus $(z - \zeta)$ divide $\exp_{\tilde{E}^{(\zeta)}}(x) - \exp_{E_\zeta}(x)$ in \tilde{M}_∞^d .

Step 2: We still denote α to be the restriction: $\alpha : U(E_\zeta, M) \rightarrow H(\tilde{E}^{(\zeta)}; \tilde{M})$. Let us prove that it is a homomorphism of A_ζ -modules. Let $x \in U(E_\zeta; M)$ be a unit and $a \in A_\zeta$. Then:

$$\begin{aligned} (z - \zeta)\alpha(ax) &= \exp_{\tilde{E}^{(\zeta)}}(ax) - \exp_{E_\zeta}(ax) \\ &= \tilde{E}_a^{(\zeta)}(\exp_{\tilde{E}^{(\zeta)}}(x)) - (E_\zeta)_a(\exp_{E_\zeta}(x)) \\ &= \sum_{i=0}^{r_a} E_{a,i} z^i \tau_\zeta^i(\exp_{\tilde{E}^{(\zeta)}}(x)) - \sum_{i=0}^{r_a} E_{a,i} \zeta^i \tau_\zeta^i(\exp_{E_\zeta}(x)) \\ &= \sum_{i=0}^{r_a} E_{a,i} z^i \tau_\zeta^i(\exp_{\tilde{E}^{(\zeta)}}(x) - \exp_{E_\zeta}(x)) + \sum_{i=1}^{r_a} E_{a,i} (z^i - \zeta^i) \tau_\zeta^i(\exp_{E_\zeta}(x)). \end{aligned}$$

Thus

$$\alpha(ax) = \tilde{E}_a^{(\zeta)}(\alpha(x)) + \underbrace{\sum_{i=0}^h a_i \frac{z^i - \zeta^i}{z - \zeta} \tau_\zeta^i(\exp_{E_\zeta}(x))}_{\in \tilde{M}^d}.$$

We have proved that $\alpha(ax) = \tilde{E}_a^{(\zeta)}(\alpha(x)) \pmod{(\tilde{M}^d + \exp_{\tilde{E}^{(\zeta)}}(\text{Lie}_{\tilde{E}^{(\zeta)}}(\tilde{M}_{s,\infty})))}$, so $\alpha(ax) = \tilde{E}_a^{(\zeta)}(\alpha(x))$ in $H(\tilde{E}^{(\zeta)}, \tilde{M})$.

Step 3: We claim that the image of $U(E_\zeta, M)$ is in the $(z - \zeta)$ -torsion of the ζ -class module at the integral level. In fact, let $x \in U(E_\zeta; M)$ be a unit. We have:

$$(z - \zeta)\alpha(x) = \exp_{\tilde{E}^{(\zeta)}}(x) - \exp_{E_\zeta}(x) = 0 \pmod{(E_\zeta(M) + \exp_{\tilde{E}^{(\zeta)}}(\text{Lie}_{\tilde{E}^{(\zeta)}}(\tilde{M}_\infty)))}.$$

Step 4: Let us prove the surjectivity of α on $H(\tilde{E}^{(\zeta)}; \tilde{M})[z - \zeta]$. Let $x \in \tilde{E}^{(\zeta)}(\tilde{M}_\infty)$ be such that

$$(z - \zeta)x = \exp_{\tilde{E}^{(\zeta)}}(u) + v$$

with $u \in \text{Lie}_{\tilde{E}^{(\zeta)}}(\tilde{M}_\infty)$ and $v \in \tilde{E}^{(\zeta)}(\tilde{M})$. We write $u = u_1 + (z - \zeta)u_2$, $u_1 \in M_\infty^d$, $u_2 \in \tilde{M}_\infty^d$ and $v = v_1 + (z - \zeta)v_2$, $v_1 \in M^d$, $v_2 \in \tilde{M}^d$. We have:

$$(z - \zeta)x = \exp_{\tilde{E}^{(\zeta)}}(u_1) + v_1 + (z - \zeta)(v_2 + \exp_{\tilde{E}^{(\zeta)}}(u_2)).$$

By evaluating at $z = \zeta$ yields $\exp_{E_\zeta}(u_1) + v_1 = 0$. Thus $u_1 \in U(E_\zeta; M)$. Moreover, we have:

$$\alpha(u_1) = x - (\underbrace{\exp_{\tilde{E}^{(\zeta)}}(u_2)}_{\in \exp_{\tilde{E}^{(\zeta)}}(\tilde{M}_\infty^d)} + \underbrace{v_2}_{\in \tilde{M}^d})$$

thus $\alpha(u_1) = x \bmod (\tilde{M}^d + \exp_{\tilde{E}^{(\zeta)}}(\tilde{M}_\infty^d))$.

Step 5: We now consider the kernel of $\alpha : U(E_\zeta; M) \rightarrow H(\tilde{E}^{(\zeta)}, \tilde{M})$ denoted by κ . We want to prove that $\kappa = U_\zeta(E, \mathcal{O}_L)$. We proceed by double inclusion.

$\boxed{\supseteq}$ Let $x \in U_\zeta(E, \mathcal{O}_L)$ be a unit and write $x = \text{ev}_\zeta(u)$ with $u \in U(\tilde{E}^{(\zeta)}; \tilde{M})$. We have $\text{ev}_\zeta(x - u) = 0$ thus we can find $v \in \tilde{M}_\infty^d$ such that

$$x = u + (z - \zeta)v.$$

We have

$$\alpha(x) = \frac{\exp_{\tilde{E}^{(\zeta)}}(u) - \exp_{E_\zeta}(x)}{z - \zeta} + \exp_{\tilde{E}^{(\zeta)}}(v).$$

But $\exp_{E_\zeta}(x) = \text{ev}_\zeta \exp_{\tilde{E}^{(\zeta)}}(u) \in M^d$ so $\alpha(x) = 0 \bmod (\tilde{M}^d + \exp_{\tilde{E}^{(\zeta)}}(\tilde{M}_\infty^d))$.

$\boxed{\subseteq}$ Let $x \in U(E_\zeta; M)$ be such that $\alpha(x) \in \tilde{M}^d + \exp_{\tilde{E}^{(\zeta)}}(\tilde{M}_\infty^d)$. Let us express $\alpha(x) = u + \exp_{\tilde{E}^{(\zeta)}}(v)$. We have

$$(z - \zeta)\alpha(x) = \exp_{\tilde{E}^{(\zeta)}}(x) + \exp_{E_\zeta}(x) = (z - \zeta)u + \exp_{\tilde{E}^{(\zeta)}}((z - \zeta)v).$$

Thus $\exp_{\tilde{E}^{(\zeta)}}(x - (z - \zeta)v) = (z - \zeta)u + \exp_{E_\zeta}(x) \in \tilde{M}^d$ so $x - (z - \zeta)v \in U(\tilde{E}^{(\zeta)}; \tilde{M})$. Finally we obtain

$$\text{ev}_\zeta(x - (z - \zeta)v) = x \in U_\zeta(E; \mathcal{O}_L).$$

□

Corollary 3.15. *The unit module $U_\zeta(E; \mathcal{O}_L)$ is an A_ζ -lattice in M_∞^d . Moreover, we have the following equalities*

$$\left[H(\tilde{E}^{(\zeta)}; \tilde{M})[z - \zeta] \right]_{A_\zeta} = [H(E_\zeta; M)]_{A_\zeta} = \left[\frac{U(E_\zeta; M)}{U_\zeta(E; \mathcal{O}_L)} \right]_{A_\zeta}.$$

Let us start by proving the following result.

Lemma 3.16. *We have an exact sequence of A_ζ -modules:*

$$0 \longrightarrow (z - \zeta)H(\tilde{E}^{(\zeta)}; \tilde{M}) \longrightarrow H(\tilde{E}^{(\zeta)}; \tilde{M}) \xrightarrow{\text{ev}_\zeta} H(E_\zeta; M) \longrightarrow 0.$$

Proof of Lemma 3.16. We apply the snake lemma to the following commutative diagram (where the lines are exact sequences of A_ζ -modules) and the i_j represent natural injections:

$$\begin{array}{ccccccc} (z - \zeta)(\exp_{\tilde{E}^{(\zeta)}}(\tilde{M}_\infty^d) + \tilde{M}^d) & \longrightarrow & \exp_{\tilde{E}^{(\zeta)}}(\tilde{M}_\infty^d) + \tilde{M}^d & \xrightarrow{\text{ev}_{z=\zeta}} & \exp_{E_\zeta}(M_\infty^d) + M^d \\ \downarrow i_1 & & \downarrow i_2 & & \downarrow i_3 & & \\ (z - \zeta)\tilde{M}_\infty^d & \longrightarrow & \tilde{M}_\infty^d & \xrightarrow{\text{ev}_{z=\zeta}} & M_\infty^d & & \end{array}$$

□

Proof of Corollary 3.15. We deduce by Lemma 3.16 an exact sequence of $\mathbb{F}_q(\zeta)$ -vector spaces of finite dimension (and of finitely-generated A_ζ -modules):

$$0 \longrightarrow H(\tilde{E}^{(\zeta)}; \widetilde{M})[z - \zeta] \longrightarrow H(\tilde{E}^{(\zeta)}; \widetilde{M}) \xrightarrow{\cdot(z - \zeta)} H(\tilde{E}^{(\zeta)}, \widetilde{M}) \xrightarrow{\text{ev}_\zeta} H(E_\zeta; M) \longrightarrow 0.$$

By Proposition 2.1 we obtain:

$$\left[H(\tilde{E}^{(\zeta)}; \widetilde{M})[z - \zeta] \right]_{A_\zeta} = [H(E_\zeta; M)]_{A_\zeta} = \left[\frac{U(E_\zeta; M)}{U_\zeta(E; \mathcal{O}_L)} \right]_{A_\zeta},$$

the last equality coming from Theorem 3.14. Since $H(\tilde{E}^{(\zeta)}; \widetilde{M})[z - \zeta]$ is a $\mathbb{F}_q(\zeta)$ -vector space of finite dimension and $U(E_\zeta; M)$ is an A_ζ -lattice in M_∞^d , the result follows. □

4. THE P -ADIC CASE

We keep the notation of Section 3. Recall that L is a finite extension of K of degree n , \mathcal{O}_L denotes the integral closure of A in L and E is an Anderson t -module defined over \mathcal{O}_L of dimension d . The goal of this section is to define and study some P -adic L -series associated with Anderson t -modules by removing the local factor at P of the classical L -series.

4.1. Introduction and notation. Recall that the local factor at Q associated with E is defined by $z_Q(\tilde{E}/\widetilde{\mathcal{O}_L}) = \frac{[\text{Lie}_{\tilde{E}}(\widetilde{\mathcal{O}_L}/Q\widetilde{\mathcal{O}_L})]_{\tilde{A}}}{[\tilde{E}(\widetilde{\mathcal{O}_L}/Q\widetilde{\mathcal{O}_L})]_{\tilde{A}}} \in \tilde{K}$ (resp. $z_Q(E/\mathcal{O}_L) = \frac{[\text{Lie}_E(\mathcal{O}_L/Q\mathcal{O}_L)]_A}{[E(\mathcal{O}_L/Q\mathcal{O}_L)]_A} \in K$). The goal of this section is to study the following infinite product of local factors $z_Q(E/\mathcal{O}_L)$ (resp. $z_Q(\tilde{E}/\widetilde{\mathcal{O}_L})$) that we call the P -adic L -series (resp. the z -twisted P -adic L -series):

$$L_P(\tilde{E}/\widetilde{\mathcal{O}_L}) = \prod_{Q \neq P} z_Q(\tilde{E}/\widetilde{\mathcal{O}_L}) \quad (\text{resp. } L_P(E/\mathcal{O}_L) = \prod_{Q \neq P} z_Q(E/\mathcal{O}_L))$$

where Q runs through the monic primes of A different from P .

More precisely, let us denote by v_P a finite place of K associated to an irreducible monic polynomial P of A . Let $\mathbb{F}_P = \mathbb{F}_{q^{\deg(P)}}$ be the residue field associated to v_P and let $K_P = \mathbb{F}_P((P))$ (resp. $A_P = \mathbb{F}_P[[P]]$) be the completion of K (resp. A) for v_P . Let \mathbb{C}_P be the completion of an algebraic closure of K_P and v_P the valuation on \mathbb{C}_P normalized such that $v_P(P) = 1$. Set $\widetilde{K}_P = \mathbb{F}_q(z) \hat{\otimes}_{\mathbb{F}_q(z)} K_P = \mathbb{F}_P(z)((P))$ on which we extend the valuation v_P :

$$v_P\left(\sum_{n \geq N} \alpha_n(z) P^n\right) = N, \quad N \in \mathbb{Z}, \quad \alpha_n(z) \in \mathbb{F}_P(z), \quad \alpha_N(z) \neq 0.$$

Let $|\cdot|_P$ be the absolute value on \mathbb{C}_P defined by $|x|_P = q^{-v_P(x)}$. Let $\mathcal{B} = (f_1, \dots, f_n)$ be an A -basis of \mathcal{O}_L (that is also a K -basis of L). We set $L_P = L \otimes_K K_P$ and $\widetilde{L}_P = L \otimes_K \widetilde{K}_P$. In what follows, the reader will be careful not to confuse the notation $L_P(E/\mathcal{O}_L)$ for the P -adic L -series and L_P for the tensor product $L \otimes_K K_P$. Then L_P is a K_P -vector space with \mathcal{B} as a basis, and \widetilde{L}_P is a \widetilde{K}_P -vector space with \mathcal{B} as a basis. In particular on L_P all the norms of K_P -vector space of finite dimension are equivalent. Let us work with the following.

Consider the sup norm $|\cdot|_P$ with respect to this basis. In other words, if $x = \sum_{i=1}^n f_i \otimes x_i$ with the $x_i \in K_P$, then we set

$$|x|_P = \max_{i=1, \dots, n} |x_i|_P.$$

We obtain over L_P a norm of K_P -algebra. We then consider the over-additive valuation of K_P -vector spaces of finite dimension on L_P defined by:

$$v_P(x) = -\log_q |x|_P = \min_{i=1, \dots, d} v_P(x_i).$$

For all $d \geq 1$, we extend these definitions to L_P^d : if $x = (x_1, \dots, x_d) \in L_P^d$, then we set

$$|x| = \max_{i=1, \dots, d} |x_i|$$

or equivalently

$$v_P(x) = -\log_q |x| = \min_{i=1, \dots, d} v_P(x_i).$$

In particular for all $x \in \mathcal{O}_L^d$ we get $v_P(x) \geq 0$. Consider

$$\mathbb{T}_z(K_P) = \left\{ f(z) = \sum_{n \geq 0} a_n z^n \mid a_n \in K_P \text{ and } \lim_{n \rightarrow +\infty} v_P(a_n) = +\infty \right\} \subset \widetilde{K_P}$$

and

$$\mathbb{T}_z(L_P) = \left\{ f(z) = \sum_{n \geq 0} a_n z^n \mid a_n \in L_P \text{ and } \lim_{n \rightarrow +\infty} v_P(a_n) = +\infty \right\} = L \otimes_K \mathbb{T}_z(K_P).$$

We define a K_P -vector space structure over $\text{Lie}_E(L_P)$. We take inspiration from the ∞ -adic case in [13, Lemma 1.7] and [11, section 2.3].

Proposition 4.1. *We can extend the homomorphism $\partial_E : A \rightarrow M_d(\mathcal{O}_L)$ into a homomorphism from K_P to $M_d(L_P)$ in the following way:*

$$\begin{aligned} \partial_E : K_P &\rightarrow M_d(L_P), \\ \sum_{i \geq -N} \alpha_i P^i &\mapsto \sum_{i \geq -N} \alpha_i \partial_E(P)^i. \end{aligned}$$

Moreover, with respect to this action, L_P^d is a K_P -vector space of dimension $m = d[L : K]$ denoted by $\text{Lie}_E(L_P)$.

Proof. Consider $n \geq 0$. There exists a unique integer t_n such that

$$q^d t_n \leq n < (t_n + 1)q^d.$$

We have:

$$\partial_E(P)^n = \partial_E(P)^{q^d t_n} \partial_E(P)^{n-q^d t_n} = P^{q^d t_n} \underbrace{\partial_E(P)^{n-q^d t_n}}_{\in M_d(\mathcal{O}_L)}.$$

We obtain that $\lim_{n \rightarrow +\infty} v_P(\partial_E(P)^n) = +\infty$ thus the map ∂_E is well-defined. Denote by $W_P = \mathbb{F}_{q^{\deg(P)}}((P^{q^d})) \subseteq K_P$. Then for all $x \in W_P$ we have $\partial_E(x) = xI_d$, thus we have an isomorphism of W_P -vector spaces:

$$\text{Lie}_E(L_P) \simeq L_P^d.$$

Hence $\text{Lie}_E(L_P)$ is a K_P -vector space of finite dimension. We have:

$$\dim_{W_P}(\text{Lie}_E(L_P)) = \dim_{K_P}(\text{Lie}_E(L_P)) \dim_{W_P}(K_P).$$

But from the isomorphism of W_P -vector spaces $\text{Lie}_E(L_P) \simeq L_P^d$ we have:

$$\dim_{W_P}(\text{Lie}_E(L_P)) = \dim_{W_P}(L_P^d) = \dim_{K_P}(L_P^d) \dim_{W_P}(K_P)$$

thus

$$\dim_{K_P}(\text{Lie}_E(L_P)) = \dim_{K_P}(L_P^d) = d[L : K].$$

□

We also have that $\text{Lie}_E(\mathcal{O}_L)$ is an A -lattice in $\text{Lie}_E(L_P)$. Finally, everything is still valid by adding the variable z , in other words $\text{Lie}_E(\widetilde{L}_P)$ is a \widetilde{K}_P -vector space of dimension $d[L : K]$ and $\text{Lie}_{\widetilde{E}}(\widetilde{\mathcal{O}}_L)$ is an \widetilde{A} -lattice in $\text{Lie}_E(\widetilde{L}_P)$. In particular, we have:

$$\partial_{\widetilde{E}} : \mathbb{T}_z(K_P) \rightarrow M_d(\mathbb{T}_z(L_P)).$$

Remark that the topologies of L_P^d and $\text{Lie}_E(L_P)$ are equivalent.

Consider the unique t -module F over \mathcal{O}_L satisfying $PF_a = E_a P$ for all $a \in A$. If $E_a = \sum_{i=0}^{r_a} E_{a,i} \tau^i$, then $F_a = \sum_{i=0}^{r_a} E_{a,i} P^{q^i-1} \tau^i$. In particular for all $a \in A$ we have: $\partial_E(a) = \partial_F(a)$. From [4, Section 3.2] we have the following equalities in $M_d(L)\{\{\tau\}\}$:

$$\log_F = P^{-1} \log_E P = \sum_{n \geq 0} l_n P^{q^n-1} \tau^n,$$

and

$$\exp_F = P^{-1} \exp_E P = \sum_{n \geq 0} d_n P^{q^n-1} \tau^n.$$

We now study the link between the local factors of E and F .

Lemma 4.2. *Let $Q \in A$ be an monic prime. If $Q \neq P$, then we have the following equalities: $z_Q(F/\mathcal{O}_L) = z_Q(E/\mathcal{O}_L)$ and $z_Q(\tilde{F}/\widetilde{\mathcal{O}}_L) = z_Q(\tilde{E}/\widetilde{\mathcal{O}}_L)$. Otherwise $z_P(F/\mathcal{O}_L) = 1$ and $z_P(\tilde{F}/\widetilde{\mathcal{O}}_L) = 1$.*

Proof. See [4, Lemma 3.7]. □

We then obtain the following result.

Corollary 4.3. *We have the following equality in $K[[z]]$:*

$$L(\tilde{F}/\widetilde{\mathcal{O}}_L) = L_P(\tilde{E}/\widetilde{\mathcal{O}}_L).$$

4.2. P -adic exponential and P -adic logarithm. We define $(D_n)_{n \geq 0}$ and $(L_n)_{n \geq 0}$ as the following sequences of elements of A :

$$\begin{cases} D_0 = 1, \\ D_n = \prod_{k=0}^{n-1} (\theta^{q^{n-k}} - \theta)^{q^k}, \end{cases} \text{ and } \begin{cases} L_0 = 1, \\ L_n = \prod_{k=1}^n (\theta - \theta^{q^k}). \end{cases}$$

We first estimate the P -adic valuation of D_n and L_n for all $n \geq 0$.

Lemma 4.4. *We have the following equalities for $n \geq 1$:*

$$(1) \quad v_P(D_n) = q^n \frac{q^{-\deg(P)(\lfloor \frac{n}{\deg(P)} \rfloor + 1)} - q^{-\deg(P)}}{q^{-\deg(P)} - 1} = q^n \frac{q^{-\deg(P)\lfloor \frac{n}{\deg(P)} \rfloor} - 1}{1 - q^{\deg(P)}},$$

$$(2) \quad v_P(L_n) = \left\lfloor \frac{n}{\deg(P)} \right\rfloor.$$

Proof. See [2, Section 2]. □

We recall that $\partial_E(a) \in M_d(A)$ is the constant coefficient of $E_a \in M_d(A)\{\tau\}$ for all $a \in A$, see Section 2. Set $s \in \mathbb{N}$ the smallest integer such that $(\partial_E(\theta) - \theta I_d)^{q^s} = 0$. There exists because $\partial_E(\theta) - \theta I_d$ is nilpotent. Then for all $a \in A$ we have $\partial_E(a^{q^s}) = a^{q^s} I_d$.

Recall that $\exp_E = \sum_{n \geq 0} d_n \tau^n \in M_d(L)\{\{\tau\}\}$ and $\log_E = \sum_{n \geq 0} l_n \tau^n \in M_d(L)\{\{\tau\}\}$. Following

[15, Theorem 4.6.9], using functional equation of the logarithm map (resp. the exponential map) $\log_E E_{\theta^{q^s}} = \partial_E(\theta^{q^s}) \log_E$, an immediate induction tells us that l_n has the form

$$l_n = \frac{a_n}{L_n^{q^s}}$$

with $a_n \in M_d(\mathcal{O}_L)$.

Reasoning in a similar way for the exponential map and by Lemma 4.4 we obtain the following result.

Proposition 4.5. *We have the following inequalities for all $n \geq 0$:*

- $$(1) \quad v_P(l_n) \geq -q^s \left\lfloor \frac{n}{\deg(P)} \right\rfloor,$$
- $$(2) \quad v_P(d_n) \geq -q^{s+n} \frac{q^{-\deg(P)} \lfloor \frac{n}{\deg(P)} \rfloor - 1}{1 - q^{\deg(P)}}.$$

So far, we have considered the exponential and logarithm series as functions of L_∞^d , but now we want to look at them as functions of L_P^d , which we denote by $\exp_{E,P}$ and $\log_{E,P}$. Note that formally (i.e., in $M_d(L)\{\{\tau\}\}$), these are always the same series. We do the same for z -twist.

Let us denote by $\text{ev}_{z=1,P} : \mathbb{T}_z(L_P)^d \rightarrow L_P^d$ the P -adic evaluation morphism at $z = 1$, whose kernel is given by $(z - 1)\mathbb{T}_z(L_P)^d$.

We can first study the P -adic convergence domain of the P -adic logarithms maps associated with \tilde{F} and \tilde{E} . We consider the following sets:

- $\Omega_z = \{x \in \mathbb{T}_z(L_P)^d \mid v_P(x) \geq 0\}$ and $\Omega_z^+ = \{x \in \mathbb{T}_z(L_P)^d \mid v_P(x) > 0\}$,
- $\Omega = \{x \in L_P^d \mid v_P(x) \geq 0\}$ and $\Omega^+ = \{x \in L_P^d \mid v_P(x) > 0\}$,
- $\mathcal{D}_z = \left\{x \in \mathbb{T}_z(L_P)^d \mid v_P(x) > -1 + \frac{q^s}{q^{\deg(P)} - 1}\right\}$,
- $\mathcal{D}_z^+ = \left\{x \in \mathbb{T}_z(L_P)^d \mid v_P(x) > \frac{q^s}{q^{\deg(P)} - 1}\right\}$,
- $\mathcal{D} = \left\{u \in (L_P)^d \mid v_P(u) > -1 + \frac{q^s}{q^{\deg(P)} - 1}\right\}$,
- $\mathcal{D}^+ = \left\{u \in (L_P)^d \mid v_P(u) > \frac{q^s}{q^{\deg(P)} - 1}\right\}$.

Proposition 4.6.

- (1) We have the P -adic convergences:

$$\log_{\tilde{E},P} : \Omega_z^+ \rightarrow \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_P))$$

and

$$\log_{E,P} : \Omega^+ \rightarrow \text{Lie}_E((L_P)).$$

Moreover, $\log_{\tilde{E},P} : \mathcal{D}_z^+ \rightarrow \mathcal{D}_z^+$ is an isometry and $\log_{E,P} : \mathcal{D}^+ \rightarrow \mathcal{D}^+$ is an isometry.

- (2) The first assertion remains true by replacing E by F and deleting the "+". As a particular case, we have the convergence

$$\log_{\tilde{F},P} : \tilde{E}(\mathcal{O}_L[z]) \rightarrow \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_P)).$$

Proof. We give the proof only for $\log_{\tilde{E},P}$, the arguments are similar in the other cases. Consider $f(z) \in \mathbb{T}_z(L_P)^d$. We have (first formally):

$$\log_{\tilde{E},P} f(z) = \sum_{n \geq 0} L_n z^n \tau^n(f(z)).$$

For all $n \geq 0$ we have:

$$v_P(L_n \tau^n(f(z))) \geq v_P(L_n) + v_P(\tau^n(f(z))) \geq -q^s \left\lfloor \frac{n}{\deg(P)} \right\rfloor + q^n v_P(f(z))$$

and this last quantity tends to ∞ when n tends to ∞ if $v(f(z)) > 0$. Moreover, if $v_P(f(z)) > \frac{q^s}{q^{\deg(P)} - 1}$, then we have for all $n \geq 1$:

$$v_P(L_n \tau^n(f(z))) - v_P(f(z)) \geq (q^n - 1)v_P(f(z)) + v(L_n) > \frac{q^s}{q^{\deg(P)} - 1}(q^n - 1) - q^s \left\lfloor \frac{n}{\deg(P)} \right\rfloor.$$

Write $n = b \deg(P) + i \geq 1$ with $b \in \mathbb{N}$ and $0 \leq i < \deg(P)$. Then:

$$\frac{q^s}{q^{\deg(P)} - 1}(q^n - 1) - q^s \left\lfloor \frac{n}{\deg(P)} \right\rfloor = q^s \left(\frac{q^{b \deg(P) + i} - 1}{q^{\deg(P)} - 1} - b \right).$$

But we have:

$$\frac{q^{b\deg(P)+i}-1}{q^{\deg(P)}-1} - b \geq \frac{q^{b\deg(P)}-1}{q^{\deg(P)}-1} - b = 1 + q^{\deg(P)} + \dots + (q^{\deg(P)})^{b-1} - b \geq 0.$$

□

We have results for the P -adic convergences of the exponential series using similar arguments.

Proposition 4.7.

(1) *We have the P -adic convergences:*

$$\exp_{\tilde{E},P} : \mathcal{D}_z^+ \rightarrow \mathbb{T}_z(L_P)^d$$

and

$$\exp_{E,P} : \mathcal{D}^+ \rightarrow L_P^d.$$

Moreover, $\exp_{\tilde{E},P} : \mathcal{D}_z^+ \rightarrow \mathcal{D}_z^+$ is an isometry and $\exp_{E,P} : \mathcal{D}^+ \rightarrow \mathcal{D}^+$ is an isometry.

(2) *The first assertion remains true by replacing E by F and deleting "+".*

In particular for all $x \in \mathcal{D}_z$ we have the following P -adic equality:

$$(3) \quad \exp_{F,P}(\text{ev}_{z=1,P}(x)) = \text{ev}_{z=1,P}(\exp_{\tilde{F}}(x)).$$

Similarly for all $x \in \Omega_z$ we have the following P -adic equality:

$$(4) \quad \log_{F,P}(\text{ev}_{z=1,P}(x)) = \text{ev}_{z=1,P}(\log_{\tilde{F}}(x)).$$

Similarly in their convergence domain, all of the P -adic exponential and logarithm maps verify the functional identities of the exponential and the logarithm maps:

$$\begin{aligned} \forall(a, x) \in A \times \Omega_z, \quad & \log_{\tilde{F},P} \partial_{\tilde{F}}(a)x = \tilde{F}_a \log_{\tilde{F},P} x, \\ \forall(a, x) \in A \times \mathcal{D}_z, \quad & \exp_{\tilde{F},P} \partial_{\tilde{F}}(a)x = \tilde{E}_a \log_{\tilde{F},P} x. \end{aligned}$$

Moreover, for all $x \in \mathcal{D}_z^+$ we have

$$\exp_{\tilde{E},P}(\log_{\tilde{E},P}(x)) = \log_{\tilde{E},P}(\exp_{\tilde{E},P}(x)) = x.$$

The same goes without the variable z , and the same goes for \tilde{E} (resp. E) over Ω_z^+ and \mathcal{D}_z^+ (resp. over Ω^+ and \mathcal{D}^+).

4.3. Evaluation at $z = \zeta \in \overline{\mathbb{F}}_q$: the P -adic setting. Consider \mathbb{F}_P to be the residual field associated with P . Set $\mathbb{F} = \mathbb{F}_q(\zeta) \cap \mathbb{F}_P$ and $G = \text{Gal}(\mathbb{F}/\mathbb{F}_q)$. Let us first remark that the valuation w defined in 3.4 is not a valuation over $\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} K_P$.

We have an isomorphism of \mathbb{F} -vector spaces:

$$\mathbb{F} \otimes_{\mathbb{F}_q} \mathbb{F} \simeq \prod_{g \in G} \mathbb{F} \simeq \prod_{g \in G} (\mathbb{F} \otimes_{\mathbb{F}} \mathbb{F})$$

given by

$$\eta : x \otimes y \mapsto (g(x)y, g \in G).$$

In particular, through this isomorphism, the Frobenius τ_ζ is identified with $(\text{id} \otimes \tau, \dots, \text{id} \otimes \tau)$. First, we extend the scalars from \mathbb{F} to $\mathbb{F}_q(\zeta)$. We obtain a (canonical) isomorphism (of $\mathbb{F}_q(\zeta)$ -vector spaces) $\eta' : \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathbb{F} \rightarrow \prod_{g \in G} \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} \mathbb{F}$, given by the following. Let (f_1, \dots, f_l) be an \mathbb{F} -basis of $\mathbb{F}_q(\zeta)$ and $a_1, \dots, a_l \in \mathbb{F}$. We set

$$\eta' \left(\sum_{i=1}^l a_i f_i \otimes_{\mathbb{F}_q} x_i \right) = \left(\sum_{i=1}^l g(a_i) f_i \otimes_{\mathbb{F}} x_i, g \in G \right).$$

Note that the isomorphism is canonical, but not the topologies that will appear. We then naturally extend the scalars (on the right) from \mathbb{F} to L_P . We obtain an isomorphism (of $\mathbb{F}_q(\zeta)$ -vector spaces on the left and L_P -modules on the right) induced by η , also denoted η :

$$\eta : \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} L_P \simeq \prod_{g \in G} (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} L_P).$$

In particular we obtain L_P -vector spaces of dimension $[\mathbb{F}_q(\zeta) : \mathbb{F}]$ on each component of the product, so an L_P -vector space of dimension $[\mathbb{F}_q(\zeta) : \mathbb{F}_q]$. For $g \in G$, we set $H_g = \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} K_P \simeq \mathbb{F}_q(\zeta)((P))$ (where the action on the left of \mathbb{F} is determined by $g \in G$).

For $x = \sum_{i=1}^l f_i \otimes x_i \in H_g$, we consider the usual valuation on H_g :

$$v_g(x) = \min_{i=1,\dots,m} v_P(x_i).$$

For all $g \in G$ we provide $L \otimes_K H_g$ with the topology v_g induced by its structure of H_g -vector space of finite dimension with respect to the choice of the basis \mathcal{B} of L/K . In particular, if we set pr_g the projection on the g -component of the product, then we obtain $v_g(\text{pr}_g(\eta((\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} \mathcal{O}_L)))) \geq 0$.

Let v_P be the over-additive valuation on the product $\prod_{g \in G} (L \otimes_K H_g)$:

$$v_P(x_g, g \in G) = \min_{g \in G} (v_g(x_g))$$

verifying $v_P(\eta(1 \otimes P)) = 1$. Remark that the Frobenius τ_ζ is equal to $(\text{id} \otimes \tau, \dots, \text{id} \otimes \tau)$ on $\prod_{g \in G} (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} L_P)$.

Remark 4.8. Following exactly the same ideas, by extending the scalars from \mathbb{F} to $\mathbb{T}_z(L_P)$ or \widetilde{L}_P we obtain the isomorphisms of $\mathbb{F}_q(\zeta)$ -vector spaces on the left and \widetilde{L}_P (resp. $\mathbb{T}_z(L_P)$) on the right:

$$\eta_z : \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathbb{T}_z(L_P) \simeq \prod_{g \in G} \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} \mathbb{T}_z(L_P)$$

and

$$\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \widetilde{L}_P \simeq \prod_{g \in G} \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}} \widetilde{L}_P.$$

We are now interested in the case of the higher dimension d . We extend v_P onto $(\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} L_P)^d$ (the same goes with z) (topology of finite-dimensional vector spaces, for example with respect to the canonical basis). Then set

$$\Omega_{\zeta,d} = \{x \in (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} L_P)^d \mid v_P(x) \geq 0\} \supseteq (\mathbb{F}_q(\zeta) \otimes \mathcal{O}_L)^d$$

and

$$\Omega_{\zeta,d,z} = \{x \in (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathbb{T}_z(L_P))^d \mid v_P(x) \geq 0\} \supseteq (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathcal{O}_L[z])^d.$$

Proposition 4.9. *We have the following convergences:*

$$\log_{F(\zeta),P} : \Omega_{\zeta,d} \rightarrow (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} L_P)^d$$

and

$$\log_{F(\zeta),P} : \Omega_{\zeta,d,z} \rightarrow (\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathbb{T}_z(L_P))^d.$$

Proof. It follows from Proposition 4.5 and the definitions of the objects. We omit the proof. \square

4.4. The P -adic L -series. Recall that $m = d[L : K]$ where d is the dimension of the t -module E and that F is the t -module given by $F = P^{-1}EP$. Let $\mathcal{C} = (g_1, \dots, g_m)$ be an A -basis of $\text{Lie}_E(\mathcal{O}_L)$, it is also a \widetilde{K}_∞ -basis of $\text{Lie}_{\tilde{E}}(L_\infty)$, a \widetilde{K}_P -basis of $\text{Lie}_{\tilde{E}}(L_P)$ and a $\mathbb{T}_z(K_P)$ -basis of $\text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_P))$. The same goes by replacing E by F since $\partial_E = \partial_F$.

Let us remark, from Corollary 4.6, that for any z -unit $y(z) \in U(\tilde{F}, \mathcal{O}_L[z])$ we have $\exp_{\tilde{F}}(y(z)) \in \tilde{E}(\mathcal{O}_L[z]) \subseteq \Omega_z$ and therefore

$$\log_{\tilde{F}, P}(\exp_{\tilde{F}}(y(z))) \in \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_P)).$$

Moreover, for a family $(x_1(z), \dots, x_m(z))$ of elements of $\text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_P))$ we have

$$\text{Mat}_{\mathcal{C}}(x_1(z), \dots, x_m(z)) \in M_m(\mathbb{T}_z(K_P))$$

thus

$$\det_{\mathcal{C}}(x_1(z), \dots, x_m(z)) \in \mathbb{T}_z(K_P).$$

Next, formally in $(L[[z]])^d$ we have the following equality for all $f(z) \in (L[[z]])^d$:

$$\log_{\tilde{F}, P}(\exp_{\tilde{F}}(f(z))) = f(z).$$

Let $(v_1(z), \dots, v_m(z)) \subset U(\tilde{F}; \mathcal{O}_L[z])$ be an \tilde{A} -basis of $U(\tilde{F}; \widetilde{\mathcal{O}_L})$. Remark that the family $(1 \otimes v_1(z), \dots, 1 \otimes v_m(z)) \subseteq \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathbb{T}_z(L_\infty)^d$ is also an A_ζ -basis of $U(\tilde{F}^{(\zeta)}; \widetilde{M})$. Set

$$w(z) = \det_{\mathcal{C}}(v_1(z), \dots, v_m(z)) \in \mathbb{T}_z(K_\infty)$$

and

$$w_P(z) = \det_{\mathcal{C}}\left(\log_{\tilde{F}, P}(\exp_{\tilde{F}}(v_1(z))), \dots, \log_{\tilde{F}, P}(\exp_{\tilde{F}}(v_m(z)))\right) \in \mathbb{T}_z(K_P).$$

By the above discussions and the class formula, we have the following equality in $K[[z]]$:

$$L_P(\tilde{F}/\widetilde{\mathcal{O}_L}) = \frac{w_P(z)}{\text{sgn}(w(z))}.$$

Since $w_P(z) \in \mathbb{T}_z(K_P)$, to study the P -adic convergence we want to prove that $\text{sgn}(w(z))$ divides $w_P(z)$ in $\mathbb{T}_z(K_P)$. Remark that the possible P -adic poles are the zeros of $\text{sgn}(w(z)) \in \mathbb{F}_q[z]$ hence elements of $\overline{\mathbb{F}}_q$. We will prove that the meromorphic series $\frac{w_P(z)}{\text{sgn}(w(z))}$ does not have a pole in $\overline{\mathbb{F}}_q$.

Theorem 4.10. *The meromorphic series $\frac{w_P(z)}{\text{sgn}(w(z))}$ does not have a pole in $\overline{\mathbb{F}}_q$. In other words, we have the convergence $\frac{w_P(z)}{\text{sgn}(w(z))} \in \mathbb{T}_z(K_P)$.*

Proof. Let $\zeta \in \overline{\mathbb{F}}_q$ be a root of $\text{sgn}(w(z))$. Recall that $\mathcal{C}_\zeta = (1 \otimes g_1, \dots, 1 \otimes g_m) \subseteq \mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathcal{O}_L$ if $\mathcal{C} = (g_1, \dots, g_m)$. Then $\text{Lie}_{F_\zeta}(\mathbb{F}_q(\zeta) \otimes_{\mathbb{F}_q} \mathcal{O}_L)$ is an A_ζ -lattice in M_∞ and admits \mathcal{C}_ζ as an A_ζ -basis. Consider (w_1, \dots, w_m) an A_ζ -basis of $U_\zeta(F; \mathcal{O}_L) = \text{ev}_\zeta U(F^{(\zeta)}; \widetilde{M})$ and $(w_1(z), \dots, w_m(z)) \subseteq U(F^{(\zeta)}; \widetilde{M})$ be such that $\text{ev}_\zeta w_i(z) = w_i$ for $i = 1, \dots, m$. Set

$$W'(z) = \det_{\mathcal{C}_\zeta}(w_1(z), \dots, w_m(z)) \in \widetilde{M}_\infty \setminus (z - \zeta) \widetilde{M}_\infty$$

and

$$W'_P(z) = \det_{\mathcal{C}_\zeta}\left(\log_{\tilde{F}^{(\zeta)}, P}(\exp_{\tilde{F}^{(\zeta)}}(w_1(z))), \dots, \log_{\tilde{F}^{(\zeta)}, P}(\exp_{\tilde{F}^{(\zeta)}}(w_m(z)))\right) \in \widetilde{M}_v.$$

Recall that the family $(1 \otimes v_1(z), \dots, 1 \otimes v_m(z))$ is an \tilde{A}_ζ -basis of $U(\tilde{F}^{(\zeta)}; \widetilde{M})$. Let us set

$$W(z) = \det_{\mathcal{C}_\zeta}(1 \otimes v_1(z), \dots, 1 \otimes v_m(z)) = 1 \otimes w(z) \in \widetilde{M}_\infty,$$

$$W_P(z) = 1 \otimes w_P(z) \in \widetilde{M}_v,$$

and $\Delta = \det_{(1 \otimes v_1(z), \dots, 1 \otimes v_m(z))}(w_1(z), \dots, w_m(z)) \in \tilde{A}_\zeta$.
From the equality

$$W'(z) = \Delta W(z)$$

we obtain

$$1 \otimes \underbrace{L(\tilde{F}/\widetilde{\mathcal{O}_L})}_{\in \mathbb{T}_z(K_\infty)} = 1 \otimes \frac{w(z)}{\text{sgn}(w(z))} = \frac{1 \otimes w(z)}{1 \otimes \text{sgn}(w(z))} = \frac{W(z)}{1 \otimes \text{sgn}(w(z))} = \frac{W'(z)}{\Delta(1 \otimes \text{sgn}(w(z)))}.$$

Since $1 \otimes L(\tilde{F}/\widetilde{\mathcal{O}_L})$ does not have a pole in $\overline{\mathbb{F}}_q$ and $W'(z)$ is not divisible by $z - \zeta$, we obtain that $\Delta(1 \otimes \text{sgn}(w(z)))$ is not divisible by $z - \zeta$. From the equality

$$\frac{W_P(z)}{1 \otimes \text{sgn}(w(z))} = \frac{W'_P(z)}{\Delta(1 \otimes \text{sgn}(w(z)))}$$

we can evaluate at $z = \zeta$ so ζ is not a pole of $\frac{w_P(z)}{\text{sgn}(w(z))}$.

Finally, the P -adic L -series is a meromorphic series without any pole in $\overline{\mathbb{F}}_q$: it is an element of $\mathbb{T}_z(K_P)$. \square

Corollary 4.11 (P -adic L -series for $P^{-1}\tilde{E}P$). *Consider the t -module $F = P^{-1}EP$. Consider $(v_1(z), \dots, v_r(z)) \subseteq U(\tilde{F}, \mathcal{O}_L[z])$ an \tilde{A} -basis of $U(\tilde{F}; \widetilde{\mathcal{O}_L})$, \mathcal{C} an \tilde{A} -basis of $\text{Lie}_{\tilde{F}}(\widetilde{\mathcal{O}_L})$. Then the following product converges in $\mathbb{T}_z(K_P)$*

$$L_P(\tilde{E}/\widetilde{\mathcal{O}_L}) = \prod_{Q \neq P} \frac{[\text{Lie}_{\tilde{E}}(\widetilde{\mathcal{O}_L}/Q\widetilde{\mathcal{O}_L})]_{\tilde{A}}}{[E(\widetilde{\mathcal{O}_L}/Q\widetilde{\mathcal{O}_L})]_{\tilde{A}}}$$

where the product runs over all the monic irreducible polynomials Q of A different from P . Further, we have the equality:

$$L_P(\tilde{E}/\widetilde{\mathcal{O}_L}) = \frac{\det_{\mathcal{C}}(\log_{\tilde{E}, P} \exp_{\tilde{E}} v_1(z), \dots, \log_{\tilde{E}, P} \exp_{\tilde{E}} v_m(z))}{\text{sgn}(\det_{\mathcal{C}}(v_1(z), \dots, v_m(z)))}.$$

We can then define the P -adic L -series associated with E and \mathcal{O}_L :

$$L_P(E/\mathcal{O}_L) = \text{ev}_{z=1, P} L_P(\tilde{E}, \widetilde{\mathcal{O}_L}) = \prod_{Q \neq P} \frac{[\text{Lie}_E(\mathcal{O}_L/Q\mathcal{O}_L)]_A}{[E(\mathcal{O}_L/Q\mathcal{O}_L)]_A} \in K_P.$$

We also have the following equalities from [4, Proposition 3.3]:

$$L_P(\tilde{E}/\widetilde{\mathcal{O}_L}) = \prod_{\mathfrak{P} \nmid P} \frac{[\text{Lie}_{\tilde{E}}(\widetilde{\mathcal{O}_L}/\mathfrak{P}\widetilde{\mathcal{O}_L})]_{\tilde{A}}}{[E(\widetilde{\mathcal{O}_L}/\mathfrak{P}\widetilde{\mathcal{O}_L})]_{\tilde{A}}} \in \mathbb{T}_z(K_P)$$

where the product runs over all the primes of \mathcal{O}_L not dividing P , and

$$L_P(E/\mathcal{O}_L) = \prod_{\mathfrak{P} \nmid P} \frac{[\text{Lie}_E(\mathcal{O}_L/\mathfrak{P}\mathcal{O}_L)]_A}{[E(\mathcal{O}_L/\mathfrak{P}\mathcal{O}_L)]_A} \in K_P.$$

4.5. A P -adic class formula associated with the t -module $P^{-1}EP$. The next step is to introduce a P -adic regulator and obtain a P -adic class formula. We begin with P -twisted t -modules. Recall that \mathcal{C} is a fixed A -basis of $\text{Lie}_E(\mathcal{O}_L) = \text{Lie}_F(\mathcal{O}_L)$.

Definition 4.12. Consider $V \subseteq U(F; \mathcal{O}_L)$ a sub- A -lattice and let (v_1, \dots, v_m) be an A -basis of V . Then we define the P -adic regulator associated with V by

$$R_P(V) = \frac{\det_{\mathcal{C}}(\log_{F, P}(\exp_F(v_1)), \dots, \log_{F, P}(\exp_F(v_m)))}{\text{sgn}(\det_{\mathcal{C}}(v_1, \dots, v_m))} \in K_P$$

which is independent of the choice of the basis of V and of $\text{Lie}_F(\mathcal{O}_L)$.

Theorem 4.13. [P -adic class formula for $P^{-1}EP$] We have the following equality in K_P :

$$L_P(E/\mathcal{O}_L) = R_P(U(F; \mathcal{O}_L)) [H(F; \mathcal{O}_L)]_A = R_P(U_{\text{st}}(F; \mathcal{O}_L)).$$

Proof. We use notation as in the proof of Theorem 4.10 with $\zeta = 1$. In particular (v_1, \dots, v_m) is an A -basis of $U(F; \mathcal{O}_L)$. Consider (u_1, \dots, u_m) an A -basis of $U_{\text{st}}(F; \mathcal{O}_L)$. Denote by $b_i = \exp_F(u_i) \in F(\mathcal{O}_L)$ for $i = 1, \dots, m$ and by $a_i = \exp_F(v_i) \in F(\mathcal{O}_L)$ for $i = 1, \dots, m$. We have

$$L(\tilde{F}/\widetilde{\mathcal{O}_L}) = \frac{w'(z)}{f(z)\Delta}.$$

As the L -series $L(F/\mathcal{O}_L)$ (at infinity) is equal to

$$\frac{\det_{\mathcal{C}}(u_1, \dots, u_m)}{\text{sgn}(\det_{\mathcal{C}}(u_1, \dots, u_m))} = \frac{\text{ev}_{z=1}(w'(z))}{\text{sgn}(\det_{\mathcal{C}}(u_1, \dots, u_m))} = \text{ev}_{z=1} \frac{w'(z)}{f(z)\Delta}$$

we first have

$$(5) \quad \text{ev}_{z=1}(f(z)\Delta) = \text{sgn}(\det_{\mathcal{C}}(u_1, \dots, u_m)) \in \mathbb{F}_q^*.$$

Let us consider $P_1 = \text{Mat}_{(v_1, \dots, v_m)}(u_1, \dots, u_m) \in M_m(A)$. By [7, Theorem 1] we have $\frac{\det(P_1)}{\text{sgn}(\det(P_1))} = [H(F; \mathcal{O}_L)]_A$. Moreover, we have:

$$P_1 = \text{Mat}_{(\log_{F,P}(a_1), \dots, \log_{F,P}(a_m))}(\log_{F,P}(b_1), \dots, \log_{F,P}(b_m)).$$

Then we have in K_P :

$$(6) \quad \det(P_1) \det_{\mathcal{C}}(\log_{F,P}(a_1), \dots, \log_{F,P}(a_m)) = \det_{\mathcal{C}}(\log_{F,P}(b_1), \dots, \log_{F,P}(b_m)).$$

From the following equality in K_∞ :

$$\det(P_1) \det_{\mathcal{C}}(v_1, \dots, v_m) = \det_{\mathcal{C}}(u_1, \dots, u_m)$$

we deduce by comparing signs:

$$(7) \quad \text{sgn}(\det(P_1)) \text{sgn}(\det_{\mathcal{C}}(v_1, \dots, v_m)) = \text{sgn}(\det_{\mathcal{C}}(u_1, \dots, u_m)).$$

We finally have:

$$\begin{aligned} L_P(E/\mathcal{O}_L) &= \text{ev}_{z=1, P} \left(\frac{w'_P(z)}{f(z)\Delta} \right) \\ &= \frac{\det_{\mathcal{C}}(\log_{F,P}(b_1), \dots, \log_{F,P}(b_m))}{\text{sgn}(\det_{\mathcal{C}}(u_1, \dots, u_m))} \text{ by Equality (5),} \\ &= \det(P_1) \frac{\det_{\mathcal{C}}(\log_{F,P}(a_1), \dots, \log_{F,P}(a_m))}{\text{sgn}(\det_{\mathcal{C}}(u_1, \dots, u_m))} \text{ by Equality (6),} \\ &= \frac{\det(P_1)}{\text{sgn}(\det(P_1))} \frac{\det_{\mathcal{C}}(\log_{F,P}(a_1), \dots, \log_{F,P}(a_m))}{\text{sgn}(\det_{\mathcal{C}}(v_1, \dots, v_m))} \text{ by Equality (7),} \\ &= R_P(U(F; \mathcal{O}_L)) [H(F; \mathcal{O}_L)]_A \end{aligned}$$

and the second line equals $R_P(U_{\text{st}}(E; \mathcal{O}_L))$.

□

4.6. A P -adic class formula. So far we've worked mainly with the t -module F , and now we want to link everything to the t -module E .

Set $h(z) = [\tilde{E}(\widetilde{\mathcal{O}_L}/P\widetilde{\mathcal{O}_L})]_{\tilde{A}} \in A[z]$ and $h(1) = [E(\mathcal{O}_L/P\mathcal{O}_L)]_A \in A \setminus \{0\}$. Consider $s \in \mathbb{N}$ such that $\partial_{\tilde{E}}(h(z)^{q^s}) = h(z)^{q^s} I_d$ (e.g., s such that $q^s \geq d$) and denote by $g(z) = h(z)^{q^s} \in A[z]$. By Proposition 2.1, we have for all $b(z) \in \mathcal{O}_L[z]^d$:

$$\tilde{E}_{g(z)}(b(z)) \in P\mathcal{O}_L[z]^d$$

and for all $b \in \mathcal{O}_L^d$:

$$E_{g(1)}(b) \in P\mathcal{O}_L^d.$$

Then by Corollary 4.6 and the above discussion, we can define the following maps:

$$\begin{aligned} \text{Log}_{\tilde{E},P} : \Omega_z &\rightarrow \frac{1}{g(z)} \mathbb{T}_z(L_P)^d \\ x &\mapsto \frac{1}{g(z)} \log_{\tilde{E},P}(\tilde{E}_{g(z)}(x)) \end{aligned}$$

and

$$\begin{aligned} \text{Log}_{E,P} : \Omega &\rightarrow L_P^d \\ x &\mapsto \frac{1}{g(1)} \log_{E,P}(E_{g(1)}(x)). \end{aligned}$$

Moreover, if $x \in \Omega_z^+$, we have $\log_{\tilde{E},P}(\tilde{E}_{g(z)}(x)) = g(z) \log_{\tilde{E},P}(x)$ in $\mathbb{T}_z(L_P)^d$. We obtain the following equality in $\mathbb{T}_z(L_P)^d$ for such x :

$$\log_{\tilde{E},P}(x) = \text{Log}_{\tilde{E},P}(x)$$

thus the map $\text{Log}_{\tilde{E},P}$ extends the map $\log_{\tilde{E},P}$ from Ω_z^+ to Ω_z . The same applies without z .

Lemma 4.14. *We have the following properties.*

(1) *For all $a \in A[z]$ and $x \in \Omega_z$ we have the following equality in $\frac{1}{g(z)} \mathbb{T}_z(L_P)^d$:*

$$\partial_{\tilde{E}}(a) \text{Log}_{\tilde{E},P}(x) = \text{Log}_{\tilde{E},P}(\tilde{E}_a(x)).$$

(2) *For all $x \in \Omega_z$ we have the equality in L_P^d :*

$$\text{ev}_{z=1,P}(\text{Log}_{\tilde{E},P}(x)) = \text{Log}_{E,P}(\text{ev}_{z=1,P}(x)).$$

(3) *For all $a \in A$ and $x \in \Omega$ we have the following equality in L_P^d :*

$$\partial_E(a) \text{Log}_{E,P}(x) = \text{Log}_{E,P}(E_a(x)).$$

Proof.

(1) We have the following equalities in $\mathbb{T}_z(L_P)^d$ for all $x \in \Omega_z$ and $a \in A[z]$:

$$\begin{aligned} g(z) \text{Log}_{\tilde{E},P}(\tilde{E}_a(x)) &= \log_{\tilde{E},P}(\underbrace{\tilde{E}_{g(z)}(\tilde{E}_a(x))}_{v_P > 0}) = \partial_{\tilde{E}}(a) \log_{\tilde{E},P}(\tilde{E}_{g(z)}(x)) \\ &= \partial_{\tilde{E}}(a) g(z) \text{Log}_{\tilde{E},P}(x). \end{aligned}$$

(2) We have the following equality in Ω^+ for all $x \in \Omega_z$:

$$\text{ev}_{z=1,P}(\tilde{E}_{g(z)}(x)) = E_{g(1)}(\text{ev}_{z=1,P}(x)).$$

Then we have the following equalities:

$$\begin{aligned} \text{ev}_{z=1,P}(g(z) \text{Log}_{\tilde{E},P}(x)) &= \text{ev}_{z=1,P}(\log_{\tilde{E},P}(\underbrace{\tilde{E}_{g(z)}(x)}_{v_P > 0})) = \log_{E,P}(E_{g(1)}(\text{ev}_{z=1,P}(x))) \\ &= g(1) \text{Log}_{E,P}(\text{ev}_{z=1,P}(x)). \end{aligned}$$

(3) The proof is similar as for the first assertion.

□

Proposition 4.15. *The logarithm map $\log_{\tilde{E},P}$ is injective on Ω_z^+ .*

We begin with the following lemma.

Lemma 4.16. *For all $a \in A \setminus \{0\}$ and for all $x \in \mathbb{T}_z(L_P)^d \setminus \{0\}$ we have $\tilde{E}_a(x) \neq 0$.*

Proof. Fix $a \in A \setminus \{0\}$ and $x \in \mathbb{T}_z(K_P)^d$ such that $E_a(x) = 0$. We can suppose without loss of generality that $\partial_E(a) = aI_d$, even if it means applying $\tilde{E}_{a^{q^s-1}}$ to $\tilde{E}_a(x)$. We can also assume that z does not divide x in $\mathbb{T}_z(L_P)^d$. Denote by $\tilde{E}_a = a + \sum_{i=1}^{r_a} \tilde{E}_{a,i}\tau^i$ with $\tilde{E}_{a,i} \in zM_d(\mathcal{O}_L[z])$ for all $i = 1, \dots, r_a$, and $x = \sum_{n \geq 0} y_n z^n$ with $y_n \in L_P^d$ and $y_0 \neq 0$. We have $E_a(x) = ay_0 \bmod z\mathbb{T}_z(L_P)^d \neq 0 \bmod z\mathbb{T}_z(L_P)^d$. Hence $E_a(x) \neq 0$. \square

Proof of Proposition 4.15. Let x be in Ω_z^+ such that $\log_{\tilde{E},P}(x) = 0$. Then for all $s \in \mathbb{N}$ we have

$$\partial_{\tilde{E}}(P^{q^s}) \log_{\tilde{E},P}(x) = 0 = \log_{\tilde{E},P}(\tilde{E}_{P^{q^s}}(x)).$$

Since $v_P(x) > 0$, we consider $s \in \mathbb{N}$ large enough such that $\tilde{E}_{P^{q^s}}(x)$ belongs to \mathcal{D}_z^+ . For such an integer s we obtain:

$$\tilde{E}_{P^{q^s}}(x) = 0$$

which implies that $x = 0$ by Lemma 4.16. \square

Proposition 4.17. *The kernel of $\text{Log}_{E,P} : \mathcal{O}_L^d \rightarrow L_P^d$ consists exactly of the torsion points of $E(\mathcal{O}_L)$, denoted by $E(\mathcal{O}_L)_{\text{Tors}}$.*

Proof. Consider first $x \in \mathcal{O}_L^d$ such that $\text{Log}_{E,P}(x) = 0$. Then we have $\log_{E,P}(E_{g(1)}(x)) = 0$. Thus for all $n \geq 0$ we have:

$$\log_{E,P}(E_{P^{q^n}g(1)}(x)) = \partial_E(P^{q^n}) \log_{E,P}(E_{g(1)}(x)) = 0.$$

Since $v_P(E_{g(1)}(x)) > 0$, we can consider n large enough such that $E_{P^{q^n}g(1)}(x) \in \mathcal{D}^+$. Then we find by applying the exponentiel map $\exp_{E,P}$ to $\log_{E,P}(E_{P^{q^n}g(1)}(x))$ that $0 = E_{P^{q^n}g(1)}(x)$ so x is a torsion point.

Conversely, suppose that there is a non-zero polynomial $a \in A$ such that $E_a(x) = 0$. We also have $E_{a^{q^s}}(x) = 0$. Then we have

$$a^{q^s} \log_{E,P}(E_{g(1)}(x)) = \log_{E,P}(E_{a^{q^s}g(1)}(x)) = \log_{E,P}(E_{g(1)}(E_{a^{q^s}}(x))) = 0.$$

Since a is non-zero, we obtain $\log_{E,P}(E_{g(1)}(x)) = 0$. \square

Set $\mathcal{O}_{L,P} = \mathcal{O}_L \otimes_A A_P$ and $\widetilde{\mathcal{O}_{L,P}} = \widetilde{\mathcal{O}_L} \otimes_A A_P$. By [7, Lemma 3.21], we can extend E by continuity to a homomorphism of \mathbb{F}_q -algebras

$$E : A_P \rightarrow M_d(\mathcal{O}_{L,P})\{\{\tau\}\}.$$

We can also extend \tilde{E} by continuity to a homomorphism of $\mathbb{F}_q(z)$ -algebras

$$\tilde{E} : \tilde{A}_P \rightarrow M_d(\widetilde{\mathcal{O}_{L,P}})\{\{\tau\}\}.$$

In particular, the A -module $E(P\mathcal{O}_{L,P})$ inherits a structure of A_P -module.

Set $U' = g(1)U(E; \mathcal{O}_L)$ and $U'_z = g(z)U(\tilde{E}; \widetilde{\mathcal{O}_L})$ (the multiplication is of course in $\text{Lie}_E(L_\infty)$ (resp. $\text{Lie}_{\tilde{E}}(\widetilde{L}_\infty)$).

Lemma 4.18. *We have the following properties.*

- (1) *We have that U' and U'_z are sub-lattices of $U(E; \mathcal{O}_L)$ and $U(\tilde{E}; \widetilde{\mathcal{O}_L})$ respectively.*
- (2) *We have that U' and U'_z are sub-lattices of $U(F; \mathcal{O}_L)$ and $U(\tilde{F}; \widetilde{\mathcal{O}_L})$ respectively.*

Proof.

- (1) The first point is clear.

(2) We have to prove the inclusions $U' \subseteq U(F; \mathcal{O}_L)$ and $U'_z \subseteq U(\tilde{F}; \widetilde{\mathcal{O}}_L)$. We do it for F , the same arguments apply to \tilde{F} . Let $x \in \text{Lie}_E(L_\infty)$ be such that $\exp_E(x) \in \mathcal{O}_L^d$, then:

$$\exp_F(g(1)x) = P^{-1} \exp_E(P\partial_E(g(1))x) = P^{-1} E_{g(1)}(\exp_E(Px)) \in P^{-1} \cdot P\mathcal{O}_L^d = \mathcal{O}_L^d.$$

□

By Lemma 4.18, U' (resp. U'_z) is a common sub- A -lattice (resp. sub- \tilde{A} -lattice) for $U(E; \mathcal{O}_L)$ and $U(F; \mathcal{O}_L)$ (resp. $U(\tilde{E}; \widetilde{\mathcal{O}}_L)$ and $U(\tilde{F}; \widetilde{\mathcal{O}}_L)$). We then have:

$$[U(F; \mathcal{O}_L) : U(E; \mathcal{O}_L)]_A = [U(F; \mathcal{O}_L) : U']_A [U' : U(E; \mathcal{O}_L)]_A = \frac{[U(F; \mathcal{O}_L) : U']_A}{[U(E; \mathcal{O}_L) : U']_A} \in K^*$$

and

$$\left[U(\tilde{F}; \widetilde{\mathcal{O}}_L) : U(\tilde{E}; \widetilde{\mathcal{O}}_L) \right]_{\tilde{A}} = \left[U(\tilde{F}; \widetilde{\mathcal{O}}_L) : U'_z \right]_{\tilde{A}} \left[U'_z : U(\tilde{E}; \widetilde{\mathcal{O}}_L) \right]_{\tilde{A}} = \frac{\left[U(\tilde{F}; \widetilde{\mathcal{O}}_L) : U'_z \right]_{\tilde{A}}}{\left[U(\tilde{E}; \widetilde{\mathcal{O}}_L) : U'_z \right]_{\tilde{A}}} \in \tilde{K}^*.$$

Let us define P -adic regulators associated to U' as follows. Let (w_1, \dots, w_m) be an A -basis of U' . We set:

$$R_{P,E}(U') = \frac{\det_{\mathcal{C}}(\text{Log}_{E,P}(\exp_E(w_1)), \dots, \text{Log}_{E,P}(\exp_E(w_m)))}{\text{sgn}(\det_{\mathcal{C}}(w_1, \dots, w_m))} \in K_P$$

and

$$R_{P,F}(U') = \frac{\det_{\mathcal{C}}(\log_{F,P}(\exp_F(w_1)), \dots, \log_{F,P}(\exp_F(w_m)))}{\text{sgn}(\det_{\mathcal{C}}(w_1, \dots, w_m))} \in K_P.$$

These regulators do not depend of the choice of the basis (w_1, \dots, w_m) . We can also define a P -adic regulator $R_P(U_{\text{st}}(E; \mathcal{O}_L))$ for $U_{\text{st}}(E; \mathcal{O}_L)$ and we have the following equality from the proof of Theorem 4.13:

$$(8) \quad R_P(U(E; \mathcal{O}_L)) [H(E; \mathcal{O}_L)]_A = R_P(U_{\text{st}}(E; \mathcal{O}_L)).$$

Similarly, we define the P -adic regulators $R_{P,\tilde{E}}(U'_z)$ and $R_{P,\tilde{F}}(U'_z)$ associated with U'_z that are elements in $\mathbb{T}_z(K_P)$ from Theorem 4.10.

Lemma 4.19. *We have the following equalities:*

$$(9) \quad R_{P,\tilde{E}}(U'_z) = R_{P,\tilde{F}}(U'_z) \text{ and } R_{P,E}(U') = R_{P,F}(U').$$

Proof. We prove the first equality. We want to prove that for all $u(z) \in U(\tilde{E}; \mathcal{O}_L[z])$ we have the following equality in $\mathbb{T}_z(L_P)^d$:

$$\log_{\tilde{F},P}(\exp_{\tilde{F}}(u(z))) = \text{Log}_{\tilde{E},P}(\exp_{\tilde{E}}(u(z))).$$

Formally (i.e., in $L[[z]]^d$) it is true, and the two quantities belong to $\mathbb{T}_z(L_P)^d$. Then the first equality of the lemma is clear.

To prove the second equality, consider (u_1, \dots, u_m) an A -basis of $U_{\text{st}}(E; \mathcal{O}_L)$. From the first equality of the lemma, we have the following equality in $\mathbb{T}_z(K_P)$:

$$\det_{\mathcal{C}}(\log_{\tilde{F},P}(\exp_{\tilde{F}}(g(z)u_i(z))), i = 1, \dots, m) = \det_{\mathcal{C}}(\text{Log}_{\tilde{E},P}(\exp_{\tilde{E}}(g(z)u_i(z))), i = 1, \dots, m).$$

By evaluating at $z = 1$ we obtain

$$\det_{\mathcal{C}}(\log_{F,P}(\exp_F(g(1)u_i)), i = 1, \dots, m) = \det_{\mathcal{C}}(\text{Log}_{E,P}(\exp_E(g(1)u_i)), i = 1, \dots, m).$$

Consider now (v_1, \dots, v_m) an A -basis of $U(E; \mathcal{O}_L)$, then $(g(1)v_1, \dots, g(1)v_m)$ is an A -basis of U' . Write $Q = \text{Mat}_{(v_1, \dots, v_m)}(u_1, \dots, u_m) \in \text{Gl}_m(A)$. We then have

$$\begin{aligned} & \det_{\mathcal{C}}(\log_{F,P}(\exp_F(g(1)v_i)), i = 1, \dots, m) \\ &= \det(Q)^{-1} \det_{\mathcal{C}}(\log_{F,P}(\exp_F(g(1)u_i)), i = 1, \dots, m) \\ &= \det(Q)^{-1} \det_{\mathcal{C}}(\text{Log}_{E,P}(\exp_E(g(1)u_i)), i = 1, \dots, m) \\ &= \det(\text{Log}_{E,P}(\exp_E(g(1)v_i)), i = 1, \dots, m). \end{aligned}$$

We proved that the equality is true for one basis of U' and since the P -adic regulators does not depend of the choice of the basis, the equality is true. \square

Lemma 4.20.

(1) *We have the following equalities:*

$$R_{P,\tilde{F}}(U'_z) = R_P(U(\tilde{F}; \widetilde{\mathcal{O}}_L)) [U(\tilde{F}; \widetilde{\mathcal{O}}_L) : U'_z]_{\tilde{A}}$$

and

$$R_{P,\tilde{E}}(U'_z) = R_P(U(\tilde{E}; \widetilde{\mathcal{O}}_L)) [U(\tilde{E}; \widetilde{\mathcal{O}}_L) : U'_z]_{\tilde{A}}$$

(2) *We have the following equalities:*

$$R_{P,F}(U') = R_P(U(F; \mathcal{O}_L)) [U(F; \mathcal{O}_L) : U']_A$$

and

$$R_{P,E}(U') = R_P(U(E; \widetilde{\mathcal{O}}_L)) [U(E; \widetilde{\mathcal{O}}_L) : U']_A$$

Proof. We only need to prove one of the equalities; all the others can be proven in a similar way. Let us prove the third one. By the structure theorem for finitely generated modules over a principal ideal domain, let us pick an A -basis (v_1, \dots, v_m) of $U(F; \mathcal{O}_L)$ and $a_1, \dots, a_m \in A$ such that $(a_1 v_1, \dots, a_m v_m)$ is an A -basis of U' . Then

$$\begin{aligned} R_{P,F}(U') &= \frac{\det_{\mathcal{C}}(\log_{F,P}(\exp_F(a_1 w_1)), \dots, \log_{F,P}(\exp_F(a_m w_m)))}{\text{sgn}(\det_{\mathcal{C}}(\log_{F,P}(\exp_F(a_1 w_1)), \dots, \log_{F,P}(\exp_F(a_m w_m))))} \\ &= \frac{a_1 \dots a_m}{\text{sgn}(a_1 \dots a_m)} R_P(U(F; \mathcal{O}_L)) \\ &= [U(F; \mathcal{O}_L) : U']_A R_P(U(F; \mathcal{O}_L)). \end{aligned}$$

\square

The link between objects associated to F and those associated to E is contained in the local factor at P , and given by the following equalities from [4, Lemma 3.4]:

$$(10) \quad z_P(E/\mathcal{O}_L) = [U(F; \mathcal{O}_L) : U(E; \mathcal{O}_L)]_A \frac{[H(E; \mathcal{O}_L)]_A}{[H(F; \mathcal{O}_L)]_A}.$$

and

$$(11) \quad z_P(\tilde{E}/\widetilde{\mathcal{O}}_L) = [U(\tilde{F}; \widetilde{\mathcal{O}}_L) : U(\tilde{E}; \widetilde{\mathcal{O}}_L)]_{\tilde{A}}.$$

We can state one of the main results of this work.

Theorem 4.21 (P -adic class formula). *We have the P -adic class formula for \tilde{E} :*

$$z_P(\tilde{E}/\widetilde{\mathcal{O}}_L) L_P(\tilde{E}/\widetilde{\mathcal{O}}_L) = R_P(U(\tilde{E}; \widetilde{\mathcal{O}}_L))$$

and the class formula for E :

$$z_P(E/\mathcal{O}_L) L_P(E/\mathcal{O}_L) = R_P(U(E; \mathcal{O}_L)) [H(E; \mathcal{O}_L)]_A = R_P(U_{\text{st}}(E; \mathcal{O}_L)).$$

Proof. Let us start with the following equality from Corollary 4.11:

$$L_P(\tilde{E}/\widetilde{\mathcal{O}}_L) = R_P(U(\tilde{F}; \widetilde{\mathcal{O}}_L)).$$

Then:

$$\begin{aligned} z_P(\tilde{E}/\widetilde{\mathcal{O}}_L)L_P(\tilde{E}/\widetilde{\mathcal{O}}_L) &= \frac{[U(\tilde{F}; \widetilde{\mathcal{O}}_L) : U'_z]_{\tilde{A}}}{[U(\tilde{E}; \widetilde{\mathcal{O}}_L) : U'_z]_{\tilde{A}}} R_P(U(\tilde{F}; \widetilde{\mathcal{O}}_L)) \text{ by Equality (11),} \\ &= \frac{R_{P,\tilde{F}}(U'_z)}{[U(\tilde{E}; \widetilde{\mathcal{O}}_L) : U'_z]_{\tilde{A}}} \text{ by Lemma 4.20,} \\ &= \frac{R_{P,\tilde{E}}(U'_z)}{[U(\tilde{E}; \widetilde{\mathcal{O}}_L) : U'_z]_{\tilde{A}}} \text{ by Equality (9),} \\ &= R_P(U(\tilde{E}; \widetilde{\mathcal{O}}_L)) \text{ by Lemma 4.20.} \end{aligned}$$

Recall Theorem 4.13:

$$L_P(E/\mathcal{O}_L) = R_P(U(F; \mathcal{O}_L)) [H(F; \mathcal{O}_L)]_A.$$

Then:

$$\begin{aligned} z_P(E/\mathcal{O}_L)L_P(E/\mathcal{O}_L) &= \frac{[U(F; \mathcal{O}_L) : U']_A}{[U(E; \mathcal{O}_L) : U']_A} \frac{[H(F; \mathcal{O}_L)]_A}{[H(E; \mathcal{O}_L)]_A} R_P(U(F; \mathcal{O}_L)) [H(F; \mathcal{O}_L)]_A \text{ by Equality (10),} \\ &= \frac{R_{P,F}(U')}{[U(E; \mathcal{O}_L) : U']_A} [H(E; \mathcal{O}_L)]_A \text{ by Lemma 4.20,} \\ &= \frac{R_{P,E}(U')}{[U(E; \mathcal{O}_L) : U']_A} [H(E; \mathcal{O}_L)]_A \text{ by Equality (9),} \\ &= R_P(U(E; \mathcal{O}_L)) [H(E; \mathcal{O}_L)]_A \text{ by Lemma 4.20.} \end{aligned}$$

□

4.7. Vanishing of the P -adic L -series. We keep the notation as in Theorem 4.10. In particular, (v_1, \dots, v_m) is an A -basis of $U_{\text{st}}(E; \mathcal{O}_L)$ and (u_1, \dots, u_m) is an A -basis of $U(E; \mathcal{O}_L)$.

Proposition 4.22. *Suppose that there exists a non-zero element $x \in U_{\text{St}}(E; \mathcal{O}_L)$ such that $\exp_E(x) = 0$. Then*

$$L_P(E/\mathcal{O}_L) = 0.$$

Proof. Write $x = \sum_{i=1}^m a_i v_i$ with $a_i \in A$ and suppose without loss of generality that $a_1 \neq 0$.

Then $x = x(1)$ with $x(z) = \sum_{i=1}^m a_i v_i(z) \in U(\tilde{E}, \mathcal{O}_L[z])$. We have:

$$\begin{aligned} &\det_{\mathcal{C}}(\text{Log}_{\tilde{E}, P}(\exp_{\tilde{E}}(v_i(z))), i = 1, \dots, m) \\ &= \frac{1}{a_1} \det_{\mathcal{C}}(\text{Log}_{\tilde{E}, P}(\exp_{\tilde{E}}(x(z))), \text{Log}_{\tilde{E}, P}(\exp_{\tilde{E}}(v_i(z))), i = 2, \dots, m). \end{aligned}$$

Since $\exp_E(x) = 0$, we have:

$$\text{ev}_{z=1, P}(\text{Log}_{\tilde{E}, P}(\exp_{\tilde{E}} x(z))) = \text{Log}_{E, P}(\exp_E(x)) = 0.$$

We then conclude that $R_P(U_{\text{St}}(E; \mathcal{O}_L)) = 0$, so $L_P(E/\mathcal{O}_L) = 0$ from the P -adic class formula 4.21.

□

Theorem 4.23. *If the exponential map $\exp_E : L_\infty^d \rightarrow L_\infty^d$ is not injective, then we have*

$$L_P(E/\mathcal{O}_L) = 0.$$

Proof. Let $x \in L_\infty^d$ be non-zero such that $\exp_E(x) = 0$. There exists $a \in A \setminus \{0\}$ such that $ax \in U_{\text{st}}(E; \mathcal{O}_L)$ and we have $\exp_E(ax) = 0$. By Proposition 4.22 we have $L_P(E/\mathcal{O}_L) = 0$. \square

We believe that the converse statement holds.

Conjecture. *The P -adic L -series is non-zero if and only if the exponential map $\exp_E : L_\infty^d \rightarrow L_\infty^d$ is injective.*

By [7, Corollary 3.24], it is true when $d = 1$ (i.e., in the Drinfeld module case) and $L = K$. Remark that in the case $\exp_E : L_\infty^d \rightarrow L_\infty^d$ is injective, which we will call the totally real case, then $\mathcal{U}(E; \mathcal{O}_L) = \exp_E(U(E; \mathcal{O}_L)) \subseteq E(\mathcal{O}_L)$ is a free A -module of rank m , and the family $(\text{Log}_{E,P}(\exp_E(u_i)), i = 1, \dots, m)$ is A -free. We would like to have that this family is A_P -free to obtain the non-vanishing of the P -adic L -series. Set:

$U(E; P\mathcal{O}_L) = \{x \in \text{Lie}_E(L_\infty) \mid \exp_E(x) \in E(P\mathcal{O}_L)\}$ and $\mathcal{U}(E; P\mathcal{O}_L) = \exp_E(U(E; P\mathcal{O}_L))$. Then we can state an equivalent of the Leopoldt's conjecture in [16], introduced recently by Anglès in [3, Section 6.3] for the Carlitz module.

Conjecture (Conjecture A). *The A_P -rank of $\mathcal{U}(E; P\mathcal{O}_L)$ is equal to the A -rank of $\mathcal{U}(E; \mathcal{O}_L)$.*

This conjecture is clear in the case $d = 1$ and $L = K$. For further discussion of this conjecture, the reader may wish to see the paper by Anglès, Bosser and Taelman [6] where this conjecture is proved in the case of the Carlitz module defined on the P th cyclotomic extension.

In the totally real case, the non-vanishing of the P -adic L -series $L_P(\tilde{E}/\tilde{\mathcal{O}}_L)$ at $z = 1$ is equivalent to the previous Leopoldt conjecture. This result can be seen as an analog to the following result from [10].

Theorem 4.24. *Let F be a totally real extension of \mathbb{Q} . Then the p -adic zeta function $\zeta_{F,p}(s)$ has a simple pole at $s = 1$ if and only if the (usual) Leopoldt conjecture is true for (F, p) .*

Definition 4.25. We call order of vanishing of the P -adic L -series and denote by $\text{ord}_{z=1} L_P(\tilde{E}/\tilde{\mathcal{O}}_L)$, the greatest integer n such that $(z - 1)^n$ divides $L_P(\tilde{E}/\tilde{\mathcal{O}}_L)$.

Fox example if the exponential map $\exp_E : L_\infty^d \rightarrow L_\infty^d$ is not injective, then for all P we have $\text{ord}_{z=1} L_P(\tilde{E}/\tilde{\mathcal{O}}_L) \geq 1$ and the previous conjecture tells us that $\text{ord}_{z=1} L_P(\tilde{E}/\tilde{\mathcal{O}}_L) = 0$ if and only if \exp_E is injective.

Here is a list of conjectures.

Conjecture (Conjecture B). : *The vanishing order of the P -adic L -series at $z = 1$ is independent of P .*

Caruso and Gazda [8] have already conjectured this in the context of Anderson motives. Caruso, Gazda and the author proved this conjecture in the case $L = K$ and $d = 1$, see [9, Theorem 2.17].

Conjecture (Conjecture C). *We have $\text{ord}_{z=1} L_P(\tilde{E}/\tilde{\mathcal{O}}_L) \leq [L : K]r_{\Omega_E}d$ where r_{Ω_E} is the rank of the period lattices Ω_E associated with E .*

We prove conjecture C in section 6 in the case $d = 1$ and $L = K$.

5. THE MULTI-VARIABLE SETTING

We keep the notation from Sections 2, 3, 4 and from the Introduction. In particular L/K is a finite field extension of degree n and \mathcal{O}_L denotes the integral closure of A in L .

The aim of this section is to extend the previous constructions to the case where the constant field is no longer \mathbb{F}_q but $\mathbb{F}_q(t_1, \dots, t_s)$ where the t_i are new variables. One of the

interests of these constructions is that in many cases, we can reduce the study of certain t -modules $E : \mathbb{F}_q[\theta] \rightarrow M_d(\mathcal{O}_L)\{\tau\}$ to the study of Drinfeld modules $\phi : \mathbb{F}_q(t_1, \dots, t_s)[\theta] \rightarrow \mathcal{O}_L(t_1, \dots, t_s)\{\tau\}$ simpler to understand. For an application to the study of the tensor power of the Carlitz module $C^{\otimes n}$ reduced to the study of the Carlitz module C , see the work of Anglès, Pellarin and Tavares Ribeiro in [5].

5.1. Setup. The goal of this section is to extend the developed theory to the multi-variable setting by replacing \mathbb{F}_q by $k = \mathbb{F}_q(t_1, \dots, t_s)$. Recall that the Frobenius map acts as the identity on k . We keep the notation in the Introduction and we introduce the following notation.

- $A_s[z] \simeq k[z] \otimes_k A_s$,
- $\widetilde{A}_s = k(z) \otimes_k A_s$,
- w : a place of K ($w = v_P$ a finite place or $w = v_\infty$ the infinite place),
- π_w : a uniformiser of w ($\pi = P$ if $w = v_P$ and $\pi = \frac{1}{\theta}$ if $w = v_\infty$),
- $K_w = \mathbb{F}_w((\pi_w))$ denoted by $K_w = K_\infty$ if $w = v_\infty$ and $K_w = K_P$ if $w = v_P$,
- \mathbb{F}_w : the residue field associated with w i.e., $\mathbb{F}_w = \mathbb{F}_P$ if $w = v_P$ and $\mathbb{F}_w = \mathbb{F}_q$ if $w = v_\infty$,
- $L_w = L \otimes_K K_w$ i.e., $L_w = L_P$ if $w = v_P$ and $L_w = L_\infty$ else,
- $k_w = \mathbb{F}_w(t_1, \dots, t_s)$,
- $K_{s,w} = k_w((\pi_w))$ denoted by $K_{s,P}$ if $w = v_P$ and $K_{s,\infty}$ if $w = v_\infty$,
- $\widetilde{K}_{s,w} = k_w(z)((\pi_w))$,
- $L_s = kL$,
- $L_{s,w} = L \otimes_K K_{s,w}$ denoted by $L_{s,P}$ if $w = v_P$,
- $\widetilde{L}_{s,w} = L \otimes_K \widetilde{K}_{s,w}$ denoted by $\widetilde{L}_{s,P}$ if $w = v_P$,
- $\mathcal{O}_{L,s}[z] \simeq k[z] \otimes_k \mathcal{O}_{L,s}$,
- $\widetilde{\mathcal{O}}_{L,s} = k(z) \otimes_k \mathcal{O}_{L,s}$.

We recall that every $x \in \widetilde{K}_{s,\infty}^*$ (resp. $\in K_{s,\infty}$) can be written uniquely as $x = \sum_{n \geq N} x_n \frac{1}{\theta^n}$ with $N \in \mathbb{Z}$, $x_n \in k(z)$ (resp. $x_n \in K$) and $x_N \neq 0$. We call $x_N \in k(z)$ (resp. k) the sign of x denoted by $\text{sgn}(x)$. We define the Tate algebra in variables $\underline{t} = (t_1, \dots, t_s)$:

$$\mathbb{T}_s(K_w) = \left\{ \sum_{n \in \mathbb{N}^s} a_n \underline{t}^n \in K_w[[\underline{t}]] \mid a_n \in K_w, \lim_{n \rightarrow +\infty} w(a_n) = +\infty \right\}$$

where $\underline{t}^n = t_1^{n_1} \dots t_s^{n_s}$ if $n = (n_1, \dots, n_s) \in \mathbb{N}^s$. This is the completion of $K[t_1, \dots, t_s]$ with respect to the Gauss norm associated with w . We set:

$$\mathbb{T}_s(L_w) = L \otimes_K \mathbb{T}_s(K_w).$$

An Anderson t -module E of dimension d over $\mathcal{O}_{L,s}$ is a non-constant k -algebra homomorphism $E : A_s \rightarrow M_d(\mathcal{O}_{L,s})$, $a \mapsto E_a = \sum_{i=0}^{r_a} E_{a,i} \tau^i \in M_d(\mathcal{O}_{L,s})\{\tau\}$ such that $(E_{\theta,0}^d - \theta I_d)^d = 0$.

We can consider \tilde{E} , the z -twist of E , as in Section 3. Following notation from Section 2, we denote by $[M]_{A_s}$ the monic generator of $\text{Fitt}_{A_s}(M)$ where M is a torsion A_s -module of finite type, e.g., $M = E(\mathcal{O}_{L,s}/P\mathcal{O}_{L,s})$ and $M = \text{Lie}_E(\mathcal{O}_{L,s}/P\mathcal{O}_{L,s})$.

As in Proposition 3.1 there exists a unique element $\exp_E \in M_d(L_s)\{\{\tau\}\}$ called the exponential map associated with E and converging over $L_{s,\infty}^d$. Similarly, there exists a logarithm map $\log_E \in M_d(L_s)\{\{\tau\}\}$ as in Proposition 3.2.

5.2. The ∞ -case. We can now define the module of units and class module in the multi-variable setting:

$$U(E; \mathcal{O}_{L,s}) = \{x \in \text{Lie}_E(L_{s,\infty}) \mid \exp_E(x) \in E(\mathcal{O}_{L,s})\}$$

and the class module

$$H(E; \mathcal{O}_{L,s}) = \frac{E(L_{s,\infty})}{E(\mathcal{O}_{L,s}) + \exp_E(\text{Lie}_E(L_{s,\infty}))}$$

both provided with A_s -module structure. We define the module of z -units:

$$U(\tilde{E}; \widetilde{\mathcal{O}_{L,s}}) = \left\{ x \in \text{Lie}_{\tilde{E}}(\widetilde{L_{s,\infty}}) \mid \exp_{\tilde{E}}(x) \in \tilde{E}(\widetilde{\mathcal{O}_{L,s}}) \right\}$$

and the class module for the z -deformation:

$$H(\tilde{E}; \widetilde{\mathcal{O}_{L,s}}) = \frac{\tilde{E}(\widetilde{L_{s,\infty}})}{\tilde{E}(\widetilde{\mathcal{O}_{L,s}}) + \exp_{\tilde{E}}(\text{Lie}_{\tilde{E}}(\widetilde{L_{s,\infty}}))}$$

both provided with \widetilde{A}_s -module structure. We define the module of z -units at the integral level:

$$U(\tilde{E}; \mathcal{O}_{L,s}[z]) = \left\{ x \in \text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_{s,\infty})) \mid \exp_{\tilde{E}}(x) \in \tilde{E}(\mathcal{O}_{L,s}[z]) \right\}$$

and finally the class module at the integral level

$$H(\tilde{E}; \mathcal{O}_{L,s}[z]) = \frac{\tilde{E}(\mathbb{T}_z(L_{s,\infty}))}{\tilde{E}(\mathcal{O}_L[z]) + \exp_{\tilde{E}}(\text{Lie}_{\tilde{E}}(\mathbb{T}_z(L_{s,\infty})))}$$

both provided with $A_s[z]$ -module structure. We have the following result from [11, Proposition 2.8].

Proposition 5.1. *The unit module $U(E; \mathcal{O}_{L,s})$ is an A_s -lattice in $\text{Lie}_E(L_{s,\infty})$ and the module of z -units $U(\tilde{E}; \widetilde{\mathcal{O}_{L,s}})$ is an \widetilde{A}_s -lattice in $\text{Lie}_{\tilde{E}}(\widetilde{L_{s,\infty}})$.*

Denote by

$$z_P(\tilde{E}/\widetilde{\mathcal{O}_{L,s}}) = \frac{[\text{Lie}_{\tilde{E}}(\widetilde{\mathcal{O}_{L,s}}/P\widetilde{\mathcal{O}_{L,s}})]_{\widetilde{A}_s}}{[\tilde{E}(\widetilde{\mathcal{O}_{L,s}}/P\widetilde{\mathcal{O}_{L,s}})]_{\widetilde{A}_s}}$$

the local factor associated with \tilde{E} at P and

$$z_P(E/\mathcal{O}_{L,s}) = \frac{[\text{Lie}_E(\mathcal{O}_{L,s}/P\mathcal{O}_{L,s})]_{A_s}}{[E(\mathcal{O}_{L,s}/P\mathcal{O}_{L,s})]_{A_s}}$$

the local factor associated with E at P . We have the following class formula for t -modules defined over $\mathcal{O}_{L,s}$, see [11, Theorem 2.9].

Theorem 5.2. *The following product*

$$L(\tilde{E}/\widetilde{\mathcal{O}_{L,s}}) = \prod_P z_P(\tilde{E}/\widetilde{\mathcal{O}_{L,s}})$$

where P runs through the monic primes of A , converges in $\widetilde{K_{s,\infty}}$ and we have the class formula:

$$L(\tilde{E}/\widetilde{\mathcal{O}_{L,s}}) = \left[\text{Lie}_{\tilde{E}}(\widetilde{\mathcal{O}_{L,s}}) : U(\tilde{E}; \widetilde{\mathcal{O}_{L,s}}) \right]_{\widetilde{A}_s}.$$

5.3. The P -adic case. We define the P -adic L -series in the multi-variable setting.

5.3.1. The P -adic class formula. All results from Section 4 remain valid by replacing \mathbb{F}_q by k . In particular, we have the following P -adic class formula.

Theorem 5.3 (P -adic class formula). *We have the following assertions.*

(1) *The infinite product*

$$L_P(\widetilde{E}/\widetilde{\mathcal{O}_{L,s}}) = \prod_{Q \neq P} z_Q(\widetilde{E}/\widetilde{\mathcal{O}_{L,s}})$$

where Q runs through the monic primes of A different from P , converges in $\mathbb{T}_z(K_{s,P})$ and we have the class formula:

$$z_P(\widetilde{E}/\widetilde{\mathcal{O}_{L,s}})L_P(\widetilde{E}/\widetilde{\mathcal{O}_{L,s}}) = R_P(U(\widetilde{E}; \widetilde{\mathcal{O}_{L,s}})).$$

(2) *The infinite product*

$$L_P(E/\mathcal{O}_{L,s}) = \prod_{Q \neq P} z_Q(E/\mathcal{O}_{L,s})$$

where Q runs through the monic primes of A different from P , converges in $K_{s,P}$ and we have the class formula:

$$z_P(E/\mathcal{O}_{L,s})L_P(E/\mathcal{O}_{L,s}) = R_P(U(E; \mathcal{O}_{L,s})) [H(E; \mathcal{O}_{L,s})]_{A_s}.$$

Proof. The proof follows the same lines as the proof of 4.10 by replacing \mathbb{F}_q by k . We omit the details. \square

Denote by $U(E; P\mathcal{O}_{L,s}) = \{x \in \text{Lie}_E(L_{s,\infty}) \mid \exp_E(x) \in E(P\mathcal{O}_{L,s})\}$ and consider the A_s -module $\mathcal{U}(E; \mathcal{O}_{L,s}) = \exp_E(U(E; \mathcal{O}_{L,s}))$. Consider also the $A_{P,s}$ -module $\mathcal{U}(E; P\mathcal{O}_{L,s}) = \exp_E(U(E; P\mathcal{O}_{L,s}))$. Then the proof of Theorem 4.23 is still valid in the multi-variable setting by replacing \mathbb{F}_q by k .

Proposition 5.4. *We have the following assertions.*

- (1) *If the exponential map $\exp_E : L_{s,\infty}^d \rightarrow L_{s,\infty}^d$ is not injective, then $L_P(E/\mathcal{O}_{L,s}) = 0$.*
- (2) *Assume that the A_s -rank of $\mathcal{U}(E; \mathcal{O}_{L,s})$ and the $A_{P,s}$ -rank of $\mathcal{U}(E; P\mathcal{O}_{L,s})$ are equal. Then $L_P(E/\mathcal{O}_L) \neq 0$ if and only if the exponential map $\exp_E : L_{s,\infty}^d \rightarrow L_{s,\infty}^d$ is injective.*

5.3.2. *The integral level.* In the work of [5], given a t -module $E : A_s \rightarrow M_d(A_s)\{\tau\}$, they want to evaluate the variables (t_1, \dots, t_s) at some $\zeta \in \overline{\mathbb{F}}_q^s$. In this case, they need that all of the coefficients $E_{\theta,i}$ of E_θ , for $i = 0, \dots, r$, can be evaluated at ζ . This is possible if all the $E_{i,\theta}$ belong to $M_d(\mathbb{F}_q[t_1, \dots, t_s]\mathcal{O}_L)$. This is what we call the integral level.

We suppose now that: $E_\theta \in M_d(\mathcal{O}_L[t_1, \dots, t_s])\{\tau\}$ i.e., we want to work at the integral level.

Theorem 5.5. *The L -series $L(\widetilde{E}/\widetilde{\mathcal{O}_{L,s}})$ converges in $\mathbb{T}_{s,z}(K_\infty)$ and we have the class formula:*

$$L(\widetilde{E}/\widetilde{\mathcal{O}_{L,s}}) = \frac{\det_{\mathcal{C}}(u_1(z), \dots, u_m(z))}{\text{sgn}(\det_{\mathcal{C}}(u_1(z), \dots, u_m(z)))}$$

where $(u_1(z), \dots, u_m(z)) \in U(\widetilde{E}; \mathcal{O}_L[t_1, \dots, t_s, z])$ is an \widetilde{A}_s -basis of the z -unit module.

Proof. The proof of [22, Corollary 7.5.6] is still valid in the multi-variable setting at the integral level. We omit the details. \square

The objective of this section is to prove that the P -adic L -series $L_P(\widetilde{E}/\widetilde{\mathcal{O}_{L,s}})$ converges in $\mathbb{T}_{s,z}(K_P)$.

Set $\Omega_{s,z} = \{x \in \mathbb{T}_{z,s}(L_P)^d \mid v_P(x) \geq 0\}$ and $\Omega_{s,z}^+ = \{x \in \mathbb{T}_{z,s}(L_P)^d \mid v_P(x) > 0\}$. Following the proof of Proposition 4.5 we have the two following convergences:

$$\log_{\widetilde{E},P} : \Omega_{s,z}^+ \rightarrow \mathbb{T}_{s,z}(L_P)^d$$

and

$$\log_{\widetilde{F},P} : \Omega_{s,z} \rightarrow \mathbb{T}_{s,z}(L_P)^d.$$

We deduce that the P -adic L -series $L_P(\widetilde{E}/\widetilde{\mathcal{O}_{L,s}})$ is written in the form $\frac{w}{f}$ with $w \in \mathbb{T}_{z,s}(K_P)$ and $f \in \mathbb{F}_q[t_1, \dots, t_s, z]$. We then consider $\zeta = (\zeta_1, \dots, \zeta_s) \in \overline{\mathbb{F}_q}^s$ and we want to prove that we can evaluate the P -adic L -series at $t_i = \zeta_i$ for all $i = 1, \dots, s$ and at $z = \zeta \in \overline{\mathbb{F}_q}$ (simultaneously).

We use arguments very similar to those used for the convergence of the P -adic L -series, so we omit some of the details.

We set $\mathcal{K}(s) = \mathbb{F}_q(\zeta_1) \otimes_{\mathbb{F}_q} \dots \otimes_{\mathbb{F}_q} \mathbb{F}_q(\zeta_s)$. We then consider the following notation for $j = 0, \dots, s$:

- $k_j = \mathbb{F}_q(t_{j+1}, \dots, t_s)$, e.g., $k_0 = k = \mathbb{F}_q(t_1, \dots, t_s)$ and $k_s = \mathbb{F}_q$,
- $k_j A = k_j \otimes_{\mathbb{F}_q} A \simeq \mathbb{F}_q(t_{j+1}, \dots, t_s)[\theta]$,
- $k_j K = k_j \otimes_{\mathbb{F}_q} K \simeq \mathbb{F}_q(t_{j+1}, \dots, t_s, \theta)$ and $\widetilde{k_j K} = \mathbb{F}_q(z) \otimes_{\mathbb{F}_q} k_j K \simeq \mathbb{F}_q(z, t_{j+1}, \dots, t_s, \theta)$,
- $k_j \mathcal{O}_L = k_j \otimes_{\mathbb{F}_q} \mathcal{O}_L$,
- $A_{s,j} = \mathcal{K}(s) \otimes_{\mathbb{F}_q} k_j A$,
- $\widetilde{A_{s,j}} = \mathcal{K}(s) \otimes_{\mathbb{F}_q} k_j \widetilde{A}$,
- $\mathcal{O}_{L,s,j} = \mathcal{K}(s) \otimes_{\mathbb{F}_q} k_j \mathcal{O}_L$,
- $\widetilde{\mathcal{O}_{L,s,j}} = \mathcal{K}(s) \otimes_{\mathbb{F}_q} k_j \widetilde{\mathcal{O}_L}$,
- For a place w of K extended to $k_j K$, $\widetilde{K(j)_w}$ is the completion of $\widetilde{k_j K}$ with respect to w .
- $\widetilde{L(j)_w} = L \otimes_K K(j)_w$,
- $\widetilde{M_{s,j,w}} = \mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{t_j}(L(j)_w)$,
- $\widetilde{L_{j,w}} = L \otimes_K \widetilde{K_{j,w}}$.
- For a place w of K , $\mathbb{T}_{z,j}(L_w) = \mathbb{T}_{z,t_{j+1}, \dots, t_s}(L_w)$, e.g.; $\mathbb{T}_{z,0}(L_w) = \mathbb{T}_{z,t_1, \dots, t_s}(L_w)$ and $\mathbb{T}_{z,s}(L_w) = \mathbb{T}_z(L_w)$.

For all $j = 0, \dots, s$, we extend the Frobenius τ into τ_s on $A_{s,j}$ by $\tau_s = \text{id} \otimes \tau$ where id is the identity on $\mathcal{K}(s)$. We do the same for $\widetilde{A_{s,j}}$, for $\mathcal{O}_{L,s,j}$ and for $\widetilde{\mathcal{O}_{L,s,j}}$.

For $j = 1, \dots, s$ we define $E^{(j)}$ the homomorphism of $\mathcal{K}(s) \otimes_{\mathbb{F}_q} k_j$ -algebras $E^{(j)} : A_{s,j} \rightarrow M_d(\mathcal{O}_{L,s,j})\{\tau_s\}$, that we call Anderson $A_{s,j}$ -module defined over $\mathcal{O}_{L,s,j}$, by:

$$E_\theta^{(j)} = \sum_{i=0}^r \text{ev}_{t_1=\zeta_1, \dots, t_j=\zeta_j}(a_i) \tau_s^i$$

if $E_\theta = \sum_{i=0}^r a_i \tau_s^i \in M_d(\mathcal{O}_{L,s})\{\tau\}$. We also set $E^{(0)} = E$ where we identify a_i with $1 \otimes a_i \in \mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathcal{O}_L[t_1, \dots, t_s]$ and replace τ with τ_s .

Similarly, we define $\widetilde{E^{(j)}}$ the z -twist of $E^{(j)}$, that is the homomorphism of $\mathcal{K}(z) \otimes_{\mathbb{F}_q} k_j(z)$ -algebras $\widetilde{E^{(j)}} : \widetilde{A_{s,j}} \rightarrow M_d(\widetilde{\mathcal{O}_{L,s,j}})\{\tau_s\}$, that we call Anderson $\widetilde{A_{s,j}}$ -module defined over $\widetilde{\mathcal{O}_{L,s,j}}$, by:

$$\widetilde{E^{(j)}}_\theta = \sum_{i=0}^r \text{ev}_{t_1=\zeta_1, \dots, t_j=\zeta_j}(a_i) z^i \tau_s^i$$

Finally, we consider $F = P^{-1}EP$ and construct $\widetilde{F^{(j)}}$ and $F^{(j)}$ in the same way.

Lemma 5.6. *Consider $j \in \{1, \dots, s\}$.*

(1) *For all $a \in A_{s,j}$ we have the following equalities in $M_d(\mathcal{O}_{L,s,j})\{\tau_s\}$:*

$$E_a^{(j)} = \text{ev}_{t_1=\zeta_1, \dots, t_j=\zeta_j} E_a = \text{ev}_{t_j=\zeta_j} E_a^{(j-1)}.$$

(2) *We have the following equalities in $M_d(\mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{F}_q[t_{j+1}, \dots, t_s]L)\{\{\tau_s\}\}$:*

$$\exp_{E^{(j)}} = \text{ev}_{t_1=\zeta_1, \dots, t_j=\zeta_j} \exp_E = \text{ev}_{t_j=\zeta_j} \exp_{E^{(j-1)}}.$$

(3) We have the following equalities for all x in $(\mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{t_j}((\mathbb{F}_q(t_{j+1}, \dots, t_s)L)_\infty))^d$:

$$\exp_{E^{(j)}}(x) = \text{ev}_{t_1=\zeta_1, \dots, t_j=\zeta_j} \exp_E(x) = \text{ev}_{t_j=\zeta_j} \exp_{E^{(j-1)}}(x).$$

Proof. It follows from definitions of the objects, we omit the proof. \square

We then define for all $j = 1, \dots, s$:

$$U(j) = \text{ev}_{t_j=\zeta_j} U\left(\widetilde{E}^{(j-1)}; \widetilde{\mathcal{O}_{L,s,j}}[t_j]\right) \subseteq U\left(\widetilde{E}^{(j)}; \widetilde{\mathcal{O}_{L,s,j}}\right).$$

Following the same arguments we used to prove Theorem 3.14, we have the following result.

Theorem 5.7.

(1) For all $j = 1, \dots, s$, we have an $\widetilde{A}_{s,j}$ -module isomorphism:

$$\frac{U(j)}{U\left(\widetilde{E}^{(j)}; \widetilde{\mathcal{O}_{L,s,j}}\right)} \simeq H\left(\widetilde{E}^{(j-1)}; \widetilde{\mathcal{O}_{L,s,j}}[t_j]\right)[t_j - \zeta_j]$$

given by

$$f_j(x) = \frac{\exp_{\widetilde{E}^{(j-1)}} x - \exp_{\widetilde{E}^{(j)}} x}{t_j - \zeta_j}$$

where $H\left(\widetilde{E}^{(j-1)}; \widetilde{\mathcal{O}_{L,s,j}}[t_j]\right)[t_j - \zeta_j]$ is the $(t_j - \zeta_j)$ -torsion of the class module

$$H\left(\widetilde{E}^{(j-1)}; \widetilde{\mathcal{O}_{L,s,j}}[t_j]\right) = \frac{\widetilde{E}^{(j-1)}\left(\widetilde{M}_{s,j,\infty}\right)}{\widetilde{E}^{(j-1)}\left(\widetilde{\mathcal{O}_{L,s,j}}[t_j]\right) + \exp_{\widetilde{E}^{(j-1)}}\left(\widetilde{E}^{(j-1)}(\widetilde{M}_{s,j,\infty})\right)}.$$

(2) The module $U(j)$ is a sub- $\widetilde{A}_{s,j}$ lattice of $U\left(\widetilde{E}^{(j)}; \widetilde{\mathcal{O}_{L,s,j}}\right)$.

We are now able to prove the main theorem of this section.

Theorem 5.8. The P -adic L -series does not have a pole in $\overline{\mathbb{F}_q^s}$. In other words we have:

$$L_P(\widetilde{E}/\mathcal{O}_{L,s}) \in \mathbb{T}_{z,s}(K_P).$$

Proof. We closely follow the proof of Theorem 4.10. We identify $\mathcal{C} = (g_1, \dots, g_m)$ with $(1 \otimes g_1, \dots, 1 \otimes g_m) \subseteq \mathcal{K}(s) \otimes \text{Lie}_E(\mathcal{O}_L)$. Consider $(u_{i,1})_{i=1, \dots, m} \subseteq U(\widetilde{F}; \mathcal{O}_L[t_1, \dots, t_s, z])$ an \widetilde{A}_s -basis of $U(\widetilde{F}; \widetilde{\mathcal{O}_{L,s}})$. Set

$$w_1 = \det_{\mathcal{C}}(u_{1,1}, \dots, u_{m,1}) \in \mathbb{T}_{z,s}(K_\infty)$$

with sign

$$f_1 \in \mathbb{F}_q[z, t_1, \dots, t_s]$$

and set

$$w_{1,P} = \det_{\mathcal{C}}(\log_{\widetilde{F},P}(\exp_{\widetilde{F}}(u_{1,1})), \dots, \log_{\widetilde{F},P}(\exp_{\widetilde{F}}(u_{m,1}))) \in \mathbb{T}_{z,s}(K_P).$$

We want to prove that the quotient $\frac{w_{1,P}}{f_1}$ is an element of $\mathbb{T}_{s,z}(K_P)$. We will prove that we can evaluate the last quantity at every $\zeta = (\zeta_1, \dots, \zeta_s) \in \overline{\mathbb{F}_q^s}$ and at $z = \zeta \in \overline{\mathbb{F}_q}$.

We will prove by induction that for all $k = 1, \dots, s$, there exists $(v_{i,k+1}, i = 1, \dots, m)$ an $\widetilde{A}_{s,k}$ -basis of $U(k)$ and $x_{k+1} \in \mathcal{K}(s) \otimes_{\mathbb{F}_q} \widetilde{A}_{s,k}$ such that

$$\text{ev}_{t_1=\zeta_1, \dots, t_k=\zeta_k}(1 \otimes L(\widetilde{F}/\widetilde{\mathcal{O}_{L,s}})) = \frac{\det_{\mathcal{C}}(v_{i,k+1}, i = 1, \dots, m)}{x_{k+1}}$$

and

$$\begin{aligned} \text{ev}_{t_1=\zeta_1, \dots, t_k=\zeta_k} (1 \otimes \frac{w_{1,P}}{f_1}) &= \frac{\det_{\mathcal{C}}(\log_{\tilde{F}^{(k)}, P} \exp_{\tilde{F}^{(k)}} v_{i,k+1}, i = 1, \dots, m)}{x_{k+1}} \\ &\in \frac{1}{x_{k+1}} \mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z(t_{k+1}, \dots, t_s)}(L_P). \end{aligned}$$

Step 1: evaluation at $t_1 = \zeta_1$.

By Theorem 5.5 we have

$$L(\tilde{F}/\widetilde{\mathcal{O}_{L,s}}) = \frac{w_1}{f_1} \in \mathbb{T}_{z,s}(K_\infty).$$

Consider $(v_{i,2})_{i=1, \dots, m}$ an $\widetilde{A}_{s,1}$ basis of $U(1)$ that can be assumed to be at the entire level, i.e., $(v_{i,2}) \subseteq (\mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_2, \dots, t_s}(L_\infty))^d$ and let $(u_{i,2}) \subseteq (\mathcal{K}(s) \otimes \mathbb{T}_{z,s}(L_\infty))^d$ be above (i.e., $\text{ev}_{t_1=\zeta_1} u_{i,2} = v_{i,2}$ for all i). Set

$$w_2 = \det_{\mathcal{C}}(u_{1,2}, \dots, u_{m,2}) \in \mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,s}(K_\infty)$$

that is not divisible by $t_1 - \zeta_1$ and set

$$w_{2,P} = \det_{\mathcal{C}}(\log_{\tilde{F},P}(\exp_{\tilde{F}}(u_{1,2})), \dots, \log_{\tilde{F},P}(\exp_{\tilde{F}}(u_{m,2}))) \in \mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,s}(K_P)$$

the P -adic analog of w_2 . Set $\delta_2 = \det_{(1 \otimes u_{1,1}, \dots, 1 \otimes u_{m,1})}(u_{1,2}, \dots, u_{m,2}) \in \mathcal{K}(s) \otimes_{\mathbb{F}_q} \widetilde{A}_s$. We have:

$$1 \otimes_{\mathbb{F}_q} w_1 = \frac{w_2}{\delta_2}$$

and

$$1 \otimes w_{1,P} = \frac{w_{2,P}}{\delta_2}.$$

We deduce the following equality from the class formula:

$$1 \otimes L(\tilde{F}/\widetilde{\mathcal{O}_{L,s}}) = \frac{w_2}{\delta_2(1 \otimes f_1)}.$$

Since ζ_1 is not a pole of the L -series and $t_1 - \zeta_1$ does not divide w_2 , we deduce that we can evaluate $\delta_2(1 \otimes f_1)$ at $t_1 = \zeta_1$ in a non-zero element x_2 of $\mathcal{K}(s) \otimes_{\mathbb{F}_q} \widetilde{A}_{s-1}$, in other words

$$\text{ev}_{t_1=\zeta_1} 1 \otimes L(\tilde{F}/\widetilde{\mathcal{O}_{L,s}}) = \frac{\det_{\mathcal{C}}(v_{i,2})}{x_2} \in \frac{1}{x_2} \mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_2, \dots, t_s}(L_\infty).$$

Next, from the equality

$$1 \otimes \frac{w_{1,P}}{f_1} = \frac{w_{2,P}}{\delta_2(1 \otimes f_1)}$$

we deduce that we can evaluate the P -adic L -series at $t_1 = \zeta_1$:

$$\begin{aligned} \text{ev}_{t_1=\zeta_1} 1 \otimes \frac{w_{1,P}}{f_1} &= \frac{\det_{\mathcal{C}}(\log_{\tilde{F}^{(1)}, P}(\exp_{\tilde{F}^{(1)}}(v_{i,2})), i = 1, \dots, m)}{x_2} \\ &\in \frac{1}{x_2} \mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_2, \dots, t_s}(L_P). \end{aligned}$$

Step k : Assume the result to be true up to rank $s-1 \geq k-1 \geq 1$ and we prove it at rank k .

Consider $(v_{i,k+1})_{i=1, \dots, m}$ a $\widetilde{A}_{s,k}$ basis of $U(k)$ that can be assumed to be at the entire level, i.e., $v_{i,k+1} \subseteq (\mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_{k+1}, \dots, t_s}(L_\infty))^d$ and let $(u_{i,k+1}) \subseteq (\mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_k, \dots, t_s}(L_\infty))^d$ be above (i.e., $\text{ev}_{t_k=\zeta_k} u_{i,k+1} = v_{i,k+1}$).

Set

$$w_{k+1} = \det_{\mathcal{C}}(u_{1,k+1}, \dots, u_{m,k+1}) \in \mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_k, \dots, t_s}(K_\infty)$$

that is not divisible by $t_k - \zeta_k$ and set

$$w_{k+1,P} = \det_{\mathcal{C}}(\log_{\tilde{F}^{(k-1)}}(\exp_{\tilde{F}^{(k-1)}}(u_{i,k+1})), i = 1, \dots, m) \in \mathcal{K}_s \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_k, \dots, t_s}(K_P)$$

the P -adic analog of w_{k+1} . Set $\delta_{k+1} = \det_{(v_{1,k}), \dots, v_{m,k}}(u_{1,k+1}, \dots, u_{m,k+1}) \in \mathcal{K}(s) \otimes \widetilde{A}_{s,k-1}$. We have:

$$\det_{\mathcal{C}}(v_{1,k}, \dots, v_{m,k}) = \frac{\det_{\mathcal{C}}(u_{1,k+1}, \dots, u_{m,k+1})}{\delta_{k+1}}$$

and

$$\det_{\mathcal{C}}(\log_{\tilde{F}^{(k-1),P}}(\exp_{\tilde{F}^{(k-1)}}(v_{1,k})), \dots, \log_{\tilde{F}^{(k-1),P}}(\exp_{\tilde{F}^{(k-1)}}(v_{m,k}))) = \frac{w_{k+1,P}}{\delta_{k+1}}.$$

Then we have the following equalities:

$$\text{ev}_{t_1=\zeta_1, \dots, t_k=\zeta_k} \left(1 \otimes L(\tilde{F}/\widetilde{\mathcal{O}}_{L,s}) \right) = \frac{w_{k+1}}{x_k \delta_{k+1}}$$

and

$$\text{ev}_{t_1=\zeta_1, \dots, t_k=\zeta_k} \left(1 \otimes \frac{w_{1,P}}{f_1} \right) = \frac{w_{k+1,P}}{x_k \delta_{k+1}}.$$

Since we can evaluate at $t_k = \zeta_k$ the L -series and $t_k - \zeta_k$ does not divide w_{k+1} , we can evaluate $x_k \delta_{k+1}$ at $t_k = \zeta_k$ into a non-zero element $x_{k+1} \in \mathcal{K}(s) \otimes \widetilde{A}_{s,k}$. We have:

$$\text{ev}_{t_k=\zeta_k, \dots, t_z=\zeta_1} (1 \otimes L(\tilde{F}/\widetilde{\mathcal{O}}_{L,s})) = \text{ev}_{t_k=\zeta_k} \frac{w_{k+1}}{x_k \delta_{k+1}} = \frac{\det_{\mathcal{C}}(v_{i,k+1}, i=1, \dots, m)}{x_{k+1}}$$

and

$$\begin{aligned} \text{ev}_{t_k=\zeta_k, \dots, t_1=\zeta_1} 1 \otimes \frac{w_{1,P}}{f_1} &= \frac{\det_{\mathcal{C}}(\log_{\tilde{F}^{(k)},P}(\exp_{\tilde{F}^{(k)}}(v_{i,k+1})))}{x_{k+1}} \\ &\in \frac{1}{x_{k+1}} \mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{T}_{z,t_{k+1}, \dots, t_s}(K_P). \end{aligned}$$

Last step: evaluation at z . We write $z = t_{s+1}$ and use similar arguments to conclude.

□

Remark 5.9. If at some step $j \in \{1, \dots, s\}$ we have for all $i = 1, \dots, r$:

$$\text{ev}_{t_1=\zeta_1, \dots, t_j=\zeta_j} A_i = 0,$$

then we have for all $k \geq j$:

$$E_{\theta}^{(k)} = \theta I_d + N_k$$

with $N_k \in M_d(\mathcal{K}(s) \otimes_{\mathbb{F}_q} \mathbb{F}_q[t_{k+1}, \dots, t_s] \mathcal{O}_L)$ a nilpotent matrix. Then we have:

$$\partial_{E^{(k)}}(\theta^{q^d}) = E_{\theta^{q^d}}^{(k)} = \theta^{q^d} I_d.$$

Hence, if $\exp_{E^{(k)}} = \sum_{n \geq 0} d_{n,k} \tau_s^n$, then from the functional equation of the exponential map we obtain for all $n \geq 0$:

$$(\theta^{q^d})^{q^n} d_{n,k} = \theta^{q^d} d_{n,k}.$$

Thus $d_{0,k} = I_d$ and $d_{n,k} = 0$ for all $n \geq 1$ so finally $\exp_{E^{(k)}} = I_d \tau^0$ for all $k \geq j$. Then we have $U(\tilde{E}^{(j)}; \widetilde{\mathcal{O}}_{L,s,j}) = \text{Lie}_{\tilde{E}^{(j)}}(\widetilde{\mathcal{O}}_{L,s,j})$ for all $j \geq k$, and the previous proof is still valid.

6. APPLICATIONS

In this section we investigate the case where $L = K$ and $d = 1$, i.e., the case of Drinfeld modules defined over the ring A itself.

6.1. Preliminaries. We consider a Drinfeld module $\phi : A \rightarrow A\{\tau\}$ of rank $r \geq 1$ and we denote by $\exp_\phi = \sum_{n \geq 0} d_n \tau^n \in K\{\{\tau\}\}$ its associated exponential map. We call a period of ϕ any element $\lambda \in \mathbb{C}_\infty$ such that $\exp_\phi(\lambda) = 0$. We denote by Λ_ϕ its period lattice which is a free A -module of rank r . Let $NP(\phi)$ be the Newton polygon associated with \exp_ϕ which is defined as the lower convex hull of the points $P_n = (q^n - 1, v_\infty(d_n))$. Remark that the zeros of \exp_ϕ are all simple since $\frac{d}{dx} \exp_\phi x = 1$.

We have the following property about the edges of $NP(\phi)$ that can be found in [18, Theorem 2.5.2].

Proposition 6.1. *Consider λ a non-zero period of ϕ of valuation x , and let N be the number of periods of valuation equal to x . Then $NP(\phi)$ has a single edge with slope λ and length N .*

By Lemma 3.3, we know that $\lim_{n \rightarrow +\infty} v_\infty(d_n) = +\infty$, then we define N_0 as the smallest n such that $v_\infty(d_n)$ is minimal, and N_1 as the largest n such that $v_\infty(d_n)$ is minimal. Another way of looking at N_0 and N_1 is that the edge of slope equals to 0 of $NP(\phi)$ has endpoints P_{N_0} and P_{N_1} .

We will use the concept of successive minimum basis from [14, section 3].

Definition 6.2. An ordered A -basis $(\lambda_1, \dots, \lambda_r)$ of the A -lattice Λ_ϕ in \mathbb{C}_∞ is a successive minimum basis (shortly an SMB) if for each $1 \leq i \leq r$, the vector λ_i has minimal valuation $v_\infty(\lambda_i)$ among all $w \in \Lambda_\phi$ not in the span $\sum_{1 \leq j < i} \lambda_j A$ of $\{\lambda_1, \dots, \lambda_{i-1}\}$.

Gekeler proved the following result, see [14, Proposition 3.1]

Proposition 6.3.

- (1) *The period lattice Λ_ϕ admits an SMB.*
 - (2) *The sequence $(v_\infty(\lambda_i))_{i=1, \dots, r}$ is independent of the choice of the SMB.*
 - (3) *Consider $\{\lambda_1, \dots, \lambda_r\}$ an SMB for Λ_ϕ . Then for all $\lambda = \sum_{i=1}^r a_i \lambda_i \in \Lambda_\phi$ we have:*
- $$v_\infty(\lambda) = \min\{v_\infty(a_i \lambda_i) \mid i = 1, \dots, r\}.$$

We consider $s \in \{1, \dots, r\} \cup \emptyset$ be such that $v_\infty(\lambda_i) \geq 0$ for $i = 1, \dots, s$ and $v_\infty(\lambda_i) < 0$ for $i > s$. Denote also by $x_i = v_\infty(\lambda_i)$ for all $i = 1, \dots, r$. Then we consider $n_1, \dots, n_t \in \{1, \dots, s\}$ be such that $x_{n_i} \in \mathbb{N}$ for $i = 1, \dots, t$, and we denote by $S_\phi = \{n_1, \dots, n_t\}$.

Proposition 6.4. *We have the following equality:*

$$N_0 - N_1 = t.$$

Proof. First we calculate N_0 . To do this, we need to count the total length of the strictly negative slopes (which is also equal to $q^{N_0} - 1$ by definition), in other words the number of non-zero periods of strictly positive valuation. Set $\lambda = \sum_{i=1}^r a_i \lambda_i \in \Lambda_\phi$. We have the equivalence by Proposition 6.3:

$$v_\infty(\lambda) > 0 \Leftrightarrow \begin{cases} a_i = 0 \text{ if } i > s, \\ \deg(a_i) \leq \lfloor x_i \rfloor \text{ if } i \leq s \text{ and } i \notin S_\phi, \\ \deg(a_i) < x_i \text{ if } i \in S_\phi. \end{cases}$$

We finally exclude the case $\lambda = 0$. We obtain that the total number of non-zero elements of Λ_ϕ with strictly positive valuation is equal to

$$q^{N_0} - 1 = \prod_{j=1}^t q^{x_{n_j}} \prod_{i \leq s, i \notin S_\phi} q^{\lfloor x_i \rfloor + 1} - 1.$$

By applying the logarithm we obtain:

$$N_0 = \sum_{j=1}^t x_{n_j} + \sum_{i \leq s, i \notin S_\phi} (\lfloor x_j \rfloor + 1).$$

We then calculate $q^{N_1} - 1$ that is equivalent to counting the number of periods with positive valuation in a similar way, and we obtain

$$N_1 = \sum_{j=1}^t (x_{n_j} + 1) + \sum_{i \leq s, i \notin S_\phi} (\lfloor x_j \rfloor + 1) = N_0 + t.$$

□

Note that we only worked with periods of ϕ and never used the fact that ϕ is defined over A . We can therefore generalize the previous result to any $\phi : A \rightarrow \mathbb{C}_\infty \setminus \{\tau\}$ of rank r since the concept of SMB is defined in full generality.

We want to apply this result to the study of the vanishing order of $L_P(\tilde{\phi}/\tilde{A})$ at $z = 1$.

6.2. An application to the vanishing of the P -adic L -series. Let $P \in A$ be irreducible monic and set $u_\phi(z) = \exp_{\tilde{\phi}} L(\tilde{\phi}/\tilde{A}) \in A[z]$. We set:

$$g_{P,\phi}(z) = [\tilde{\phi}(\tilde{A}/P\tilde{A})]_{\tilde{A}} \in A[z]$$

and

$$g_{P,\phi}(1) = [\phi(A/PA)]_A \in A \setminus \{0\}.$$

We recall the definition of local factor associated with $\tilde{\phi}$ and P :

$$z_P(\tilde{\phi}/\tilde{A}) = \frac{P}{g_{P,\phi}(z)}.$$

Let us recall that the P -adic L -series associated with ϕ is defined as follows:

$$L_P(\tilde{\phi}/\tilde{A}) = \frac{1}{P} \log_{\tilde{\phi}, P} \tilde{\phi}_{g_{P,\phi}(z)}(u_\phi(z)) \in \mathbb{T}_z(K_P).$$

By the proof of [22, Corollary 7.5.6], we deduce that $L(\tilde{\phi}/\tilde{A}) \in \mathbb{T}_z(K_\infty)$ is a unit in $\mathbb{T}_z(K_\infty)$ whose valuation is equal to 0 and whose constant coefficient is equal to 1.

From now on, we say that ϕ does not have A -torsion if the A -module $\phi(A)$ is torsion-free.

Proposition 6.5. *Let ϕ be an A -Drinfeld module defined over A of rank $r \geq 1$ without A -torsion. Then for all $k \geq 0$ the following assertions are equivalent:*

- (1) $(z-1)^k|_{\mathbb{T}_z(K_P)} L_P(\tilde{\phi}/\tilde{A})$,
- (2) $(z-1)^k|_{A[z]} u_\phi(z)$.

Proof. By the definition of the P -adic L -series:

$$L_P(\tilde{\phi}/\tilde{A}) = \frac{1}{P} \log_{\tilde{\phi}, P} (\tilde{\phi}_{g_{P,\phi}(z)}(u_\phi(z))),$$

we see that $2 \Rightarrow 1$ is clear.

Let us prove $1 \Rightarrow 2$. We have:

$$P^2 L_P(\tilde{\phi}/\tilde{A}) = \log_{\tilde{\phi}, P} (\tilde{\phi}_{P g_{P,\phi}(z)}(u_\phi(z))).$$

Since $v_P(\tilde{\phi}_{P g_{P,\phi}(z)}(u_\phi(z))) \geq 2$, we have $\tilde{\phi}_{P g_{P,\phi}(z)}(u_\phi(z)) \in \mathcal{D}_z^+$. By applying the P -adic exponential map we obtain:

$$\exp_{\tilde{\phi}, P} (P^2 L_P(\tilde{\phi}/\tilde{A})) = \tilde{\phi}_{P g_{P,\phi}(z)}(u_\phi(z)) \in A[z].$$

If $(z-1)^k|_{\mathbb{T}_z(K_P)} L_P(\tilde{\phi}/\tilde{A})$, then we have $(z-1)^k|_{A[z]} \tilde{\phi}_{P g_{P,\phi}(z)}(u_\phi(z))$. Since ϕ does not have A -torsion, we deduce that

$$(z-1)^k|_{A[z]} u_\phi(z).$$

□

Proposition 6.6. *Let $m \in A \setminus \{0\}$ be a non zero polynomial and consider the Drinfeld module $\psi = m^{-1}\phi m$. Then the vanishing order at $z = 1$ of $L_P(\tilde{\phi}/\tilde{A})$ and $L_P(\tilde{\psi}/\tilde{A})$ are equal.*

Proof. By Lemma 4.2, we have the following equality in $\mathbb{T}_z(K_P)$:

$$g_{P,\phi}(z) L_P(\tilde{\psi}/\tilde{A}) = g_{P,\psi}(z) L_P(\tilde{\phi}/\tilde{A}) \prod_{Q|m} \frac{g_{Q,\phi}(z)}{Q}.$$

Since $g_{Q,\phi}(1) \neq 0$ for all Q , we obtain the result. □

Proposition 6.7. *If $\phi : A \rightarrow A\{\tau\}$ has rank r , then the vanishing order at $z = 1$ of the P -adic L -series $L_P(\tilde{\phi}/\tilde{A})$ is lower than or equal to r .*

Proof. Let us first twist ϕ into $\psi = m^{-1}\phi m$ without A -torsion. By Proposition 6.5 we consider the vanishing order at $z = 1$ of $u_\psi(z) \in A[z]$. We can compute its leading coefficient seen as a polynomial in the variable θ . We have $u_\psi(z) = \exp_{\tilde{\psi}} L(\tilde{\psi}/\tilde{A}) = \sum_{n \geq 0} d_n z^n \tau^n (L(\tilde{\psi}/\tilde{A}))$. We know that $L(\tilde{\psi}/\tilde{A})$ has the form $1 + \sum_{n \geq 1} a_n z^n \in \mathbb{T}_z(K_\infty)$ with $v_\infty(a_n) > 0$. Let $N_0, m_1, \dots, m_l, N_1$ be the integers n such that $v_\infty(d_n)$ is minimal. Let $\beta_n \in \mathbb{F}_q^*$ be the sign of d_n , we obtain:

$$\text{sgn}(u_\psi(z)) = z^{N_0} (\beta_{N_0} + \dots + \beta_{N_1} z^{N_1 - N_0}) \in \mathbb{F}_q[z]$$

that has at most r non-zero roots by Proposition 6.4. Thus, $u_\psi(z)$ is divisible at most by $(z-1)^r$. □

Note that we have proved more precisely:

$$\text{ord}_{z=1} L_P(\tilde{\phi}/\tilde{A}) \leq \#\{i = 1, \dots, r \mid v_\infty(\lambda_i) \in \mathbb{Z}\}.$$

Proposition 6.8. *The previous inequality is not an equality in general.*

Proof. Consider the Drinfeld module given by $\phi_\theta = \theta + \theta\tau^2$ with $q = 3$. One can prove that the Newton polygon of the associated exponential map is the polygon beginning at the point $(0,0)$ and has successive slopes of length $(q^{2k+2} - q^{2k})$ and equal to $k+1$. Thus, the number of periods of an SMB having valuation $\in \mathbb{Z}$ is equal to 2. By [9, Proposition 2.21] we have $u_\phi(1) = 1$. One can prove that ϕ does not have A -torsion. Then by Proposition 6.5 we obtain $\text{ord}_{z=1} L_P(\tilde{\phi}/\tilde{A}) = 0$. □

Remark 6.9. For any $r \geq 1$, we can construct explicit Drinfeld modules of rank r whose vanishing order of the associated P -adic L -series equals r . In fact, denote by $(-1)^r (z-1)^r = 1 + \sum_{i=1}^r \alpha_i z^i$, with $\alpha_i \in \mathbb{F}_q$ and consider the Drinfeld module given by $\phi_\theta = \theta + \sum_{i=1}^r \alpha_i \theta^{q^i} \tau^i$. By [9, Proposition 2.21], we have: $u_\phi(z) = 1 + \sum_{i=1}^r \alpha_i z^i = (-1)^r (z-1)^r$. Then the vanishing order at $z = 1$ is greater than or equal to r , so equals r .

To conclude the text, we would like to ask the following question from a personal communication with Xavier Caruso and Quentin Gazda.

Problem 6.10. Do we have the following equality

$$\text{ord}_{z=1} L_P(\tilde{\phi}/\tilde{A}) = \# \left\{ i \in \{1, \dots, r\} \mid \lambda_i \in \bigcup_{n \geq 0} \mathbb{F}_{q^{pn}} \left(\left(\frac{1}{\theta} \right) \right) \right\}?$$

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