2.5 The Chain Rule

Let $f(x)$ and $g(x)$ be differentiable functions.

If $P(x) = f(x) \cdot g(x)$

and

$$Q(x) = \frac{f(x)}{g(x)}$$

then

$$P'(x) = g(x)f'(x) + f(x)g'(x)$$

and

$$Q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

EX: Which of the following functions do we know how to differentiate?

(a) $\sin(x) \cdot e^x$ ✓

(b) $\dfrac{\sin(x)}{e^x}$ ✓

(c) $e^{\sin(x)}$ ← how do we do this? The chain Rule

Notice this is a composition of two functions.

<u>Def</u>: Let $f(x)$ and $g(x)$ be functions. The composite function $(f \circ g)(x)$ is given by

$$(f \circ g)(x) = f(g(x))$$

<u>EX</u>: For the following composite functions identify the $\underbrace{\text{inner function}}_{g(x)}$ + $\underbrace{\text{outer function}}_{f(x)}$.

① $\sqrt{x^2+1}$ $g(x) = x^2+1$ $f(x) = \sqrt{x}$

$$f(g(x)) = \sqrt{x^2+1}$$

② $(x^3-1)^{100}$ $g(x) = x^3-1$ $f(x) = x^{100}$

$$f(g(x)) = (x^3-1)^{100}$$

③ $\cos(x^2)$ $g(x) = x^2$ $f(x) = \cos x$

$$f(g(x)) = \cos(x^2)$$

④ $e^{\sin x}$ $g(x) = \sin x$ $f(x) = e^x$

$f(g(x)) = e^{\sin x}$

⑤ $\cos^2 x$ $g(x) = \cos x$ $f(x) = x^2$

$\parallel$

$(\cos(x))^2$

$f(g(x)) = (\cos(x))^2$

The Chain Rule

Let $f(x)$ and $g(x)$ be functions and let $F(x) = f(g(x))$. Suppose g is differentiable at x and f is diff. at $g(x)$. Then

$$F'(x) = f'(g(x))\,g'(x).$$

"Proof": → single pt Full pf in book

Consider the limit definition of the derivative of $f(x)$ at $x=a$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

by substution

$$f'(g(a)) = \lim_{g(x) \to g(a)} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)}$$

Also

$$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

multiplying these together gives

$$f'(g(a))\, g'(a)$$

$$= \lim_{g(x) \to g(a)} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

Since $g(x)$ is continuous
$g(x) \longrightarrow g(a)$ as $x \longrightarrow a$

so

$$= \lim_{x \to a} \left(\frac{f(g(x)) - f(g(a))}{\cancel{(g(x) - g(a))}} \right) \cdot \frac{\cancel{(g(x) - g(a))}}{(x - a)}$$

$$= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}$$

$$= (f \circ g)'(a). \checkmark$$

Let's check what we know!

<u>Ex 1</u>: use power rule + chain rule to compute derivative of $(x^2)^2$

power rule:

$$\frac{d}{dx}(x^4) = 4x^3$$

Chain rule:

$$f(g(x)) = (x^2)^2$$

$g(x) = x^2 \quad f(x) = x^2$

$g'(x) = 2x \quad f'(x) = 2x$

$f'(g(x))g'(x) = 2(x^2)2x = 4x^3$

Ex 2: Use the product rule + chain rule to compute the derivative of $\cos^2 x$

product of $\cos x \cdot \cos x$

$$\cos x(-\sin x) + \cos x(-\sin x)$$

$$= -2\cos x \sin x$$

chain

$g(x) = \cos x \quad g'(x) = -\sin x$

$f(x) = x^2 \quad f'(x) = 2x$

$$f'(g(x))g'(x)$$

$$= f'(\cos x) \cdot (-\sin x)$$

$$= -2\cos x \sin x$$

Ex 3: Find derivative of $\sqrt{x^2+1}$

$g(x) = x^2+1 \quad g'(x) = 2x$

$f(x) = \sqrt{x} \quad f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$$f'(g(x))g'(x)$$
$$= f'(x^2+1)\cdot 2x$$
$$= \frac{1}{2\sqrt{x^2+1}}\cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

Ex 4: Let $h(x) = \sec(x^2e^x)$

compute $h'(x)$.

$$g(x) = x^2e^x \xrightarrow[\text{prod. rule}]{} g'(x) = 2xe^x + e^x x^2$$

$$f(x) = \sec(x) \qquad f'(x) = \sec x \tan x$$

$$h'(x) = f'(g(x))g'(x)$$
$$= f'(x^2e^x)\cdot(2xe^x + e^x x^2)$$
$$= \sec(x^2e^x)\tan(x^2e^x)(2xe^x + e^x x^2)$$

useful special case:

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$$

Ex:

$$\frac{d}{dx}\left(3(x^2+4)^{100}\right) = 3 \cdot 100(x^2+4)^{99}(2x)$$

or

$$\frac{d}{d\theta}(\cos(\theta))^4 = 4(\cos\theta)^3(-\sin\theta)$$