

Métodos Computacionales

Ec. Diferenciales Ordinarias

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Ecuaciones Diferenciales Ordinarias (vs Parciales)

↓ ↓
¿ $y(x)$? ¿ $u(x, y, \dots, z)$?

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ky}{dx^k}\right) = 0 \quad \text{¿Orden?} \quad \text{¿Grado?}$$

Lineales...

$$\sum_{i=0}^k a_k(x) \frac{d^k y}{dx^k} = b(x)$$

y ... | No Lineales !



EDO de cualquier orden!

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^k y}{dx^k}\right) = 0 \quad \text{← EDO de orden } k$$

$$\left. \begin{array}{l} z_i \equiv \frac{d^i y}{dx^i} \quad i = 1, \dots, k-1 \\ F\left(x, y, z_1, z_2, \dots, z_{k-1}, \frac{d^k y}{dx^k}\right) = 0 \end{array} \right\} k \text{ EDO de orden 1}$$

EXPLÍCITAS

$$\mathbf{F}\left(x, \mathbf{y}, \frac{d\mathbf{y}}{dx}\right) = 0 \quad \longrightarrow \boxed{\frac{d\mathbf{y}}{dx} = \mathbf{F}(x, \mathbf{y})}$$

Lotka-Volterra:

$$\frac{dx}{dt} = \alpha x - \beta xy$$

$$\frac{dy}{dt} = \delta xy - \gamma y$$



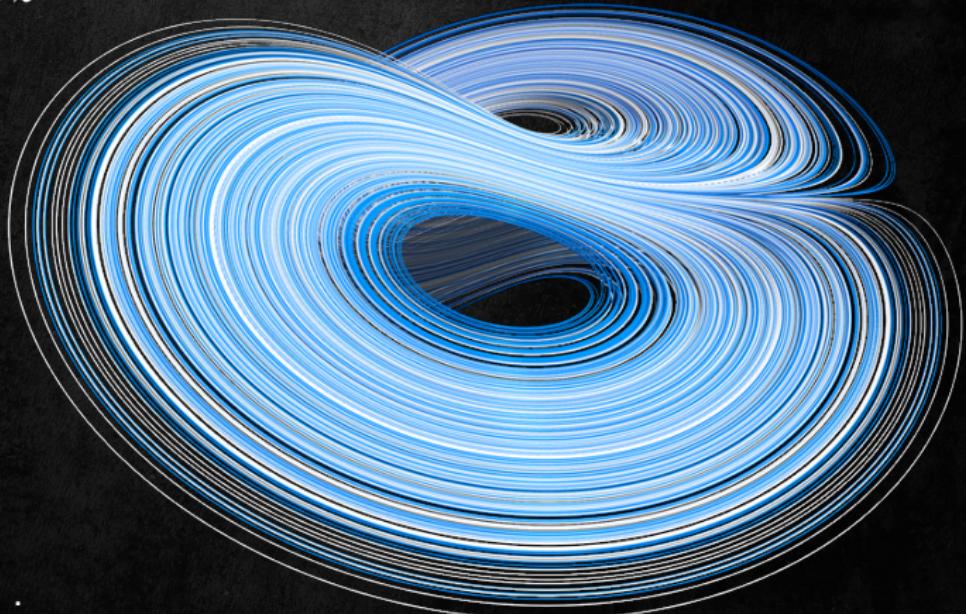
OJO notación!! Física \neq Matemática \neq Código

Atractor de Lorenz:

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$

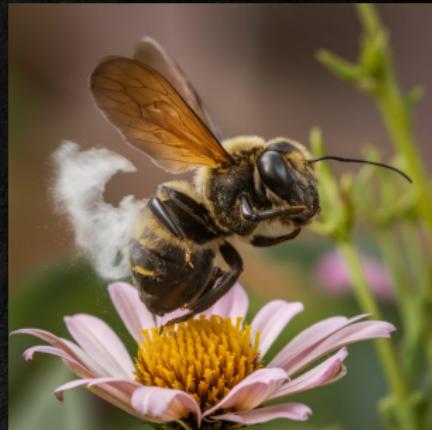


Es un caos...

pero . . . ¿Y las parciales?

Navier-Stokes: problema del milenio [+info](#)

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{f}$$



[+info](#)

hecho con ideogram



Fojas et al. (2013).
APCBEE Procedia. 7. 86–92.

Dado $\frac{dy}{dt} = f(t, y)$, $a \leq t \leq b$, $y(a) = \alpha$

Problema Bien Planteado (EDO1, PVI)

- Existe una única $y(x)$
- Existen constantes $\epsilon_0 > 0$ y $k > 0$, tales que para cualquier $\epsilon \in (0, \epsilon_0)$, siempre que $\delta(t)$ es continua con $|\delta(t)| < \epsilon$ para toda $t \in [a, b]$, y cuando $|\delta_0| < \epsilon$, el PVI

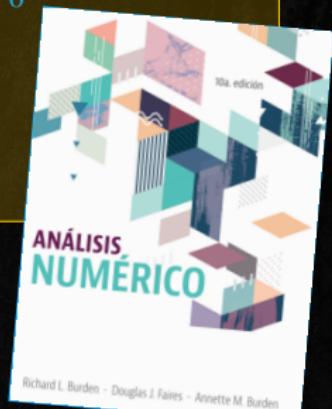
$$\frac{dz}{dt} = f(t, z) + \delta(t), \quad a \leq t \leq b, \quad z(a) = \alpha + \delta_0$$

tiene una única solución $z(t)$ que satisface

$$|z(t) - y(t)| < k\epsilon \text{ para toda } t \in [a, b]$$

Ejemplo, toda f que satisface Lipschitz:

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$



Richard L. Burden - Douglas J. Faires - Annette M. Burden

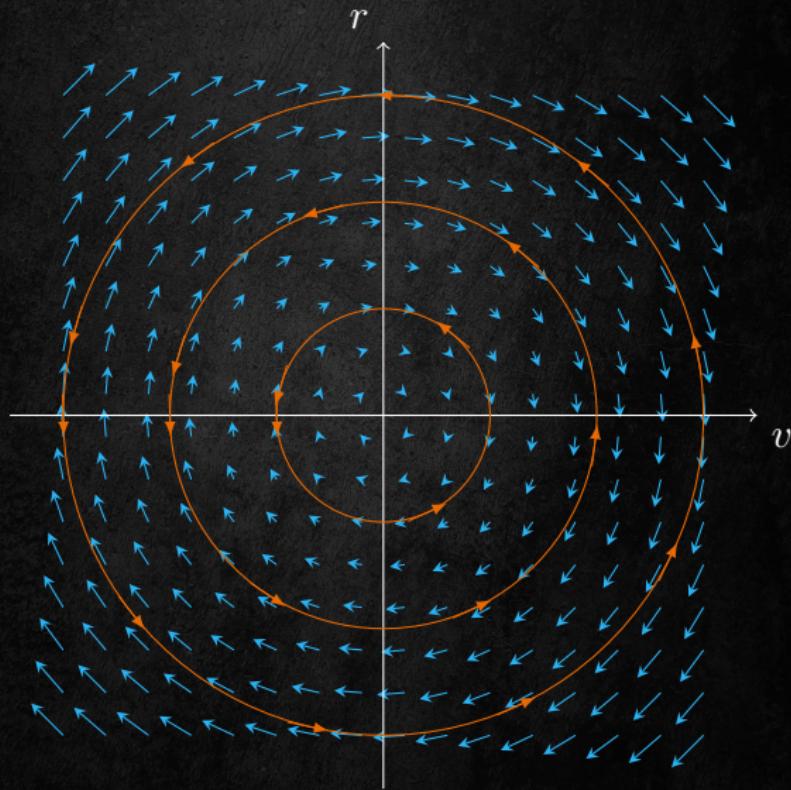
"REPASO"

Espacio de las fases

Osc. Armónico:

$$\frac{dr}{dt} = v$$

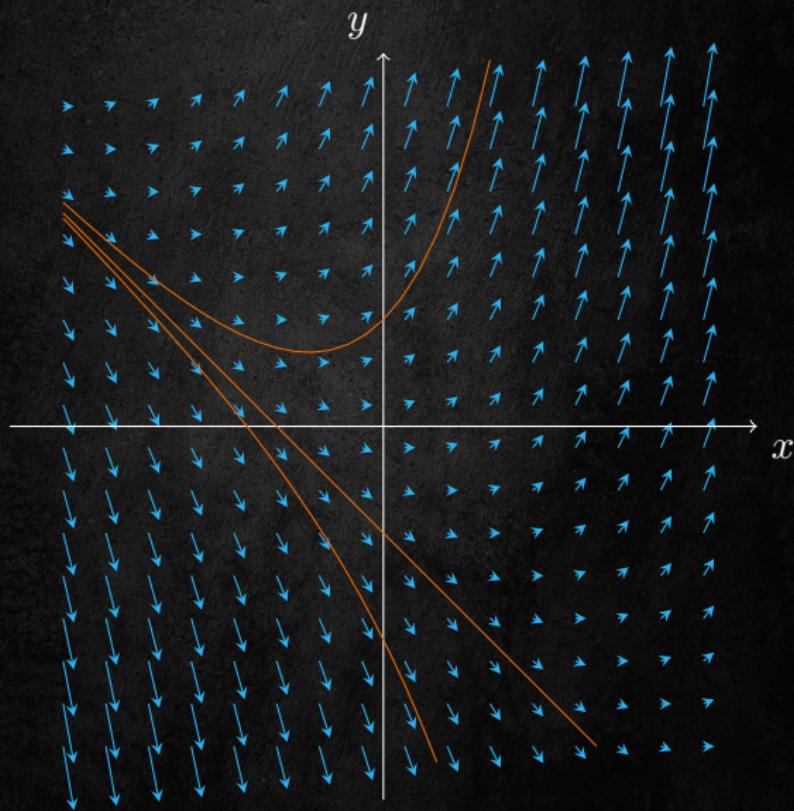
$$\frac{dv}{dt} = -kr$$



"REPASO"

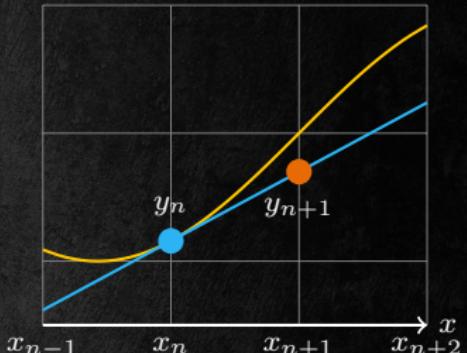
Trayectoria

$$\frac{dy}{dx} = y + x$$



Euler hacia adelante:

$$y_{n+1} = y_n + hF(x_n, y_n) + O(h^2)$$



Euler hacia atrás:

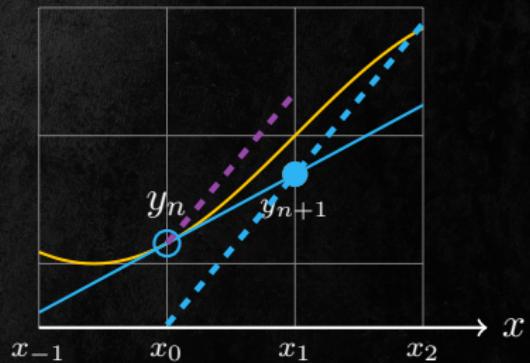
$$y_{n+1} = y_n + hF(x_{n+1}, y_{n+1}) + O(h^2)$$

MÉTODO IMPLÍCITO

Opción 1: Busco la raíz y_{n+1}
Bisección, Netwon-Rapshon.

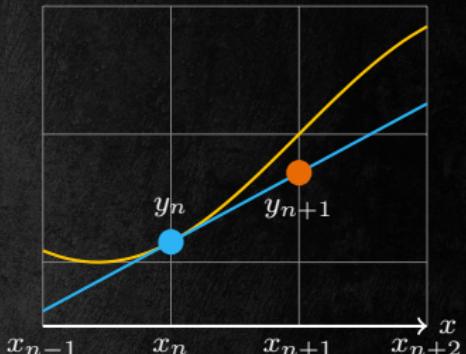
Opción 2: Itero (punto fijo) →

No siempre converge



Euler hacia adelante:

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Euler hacia atrás:

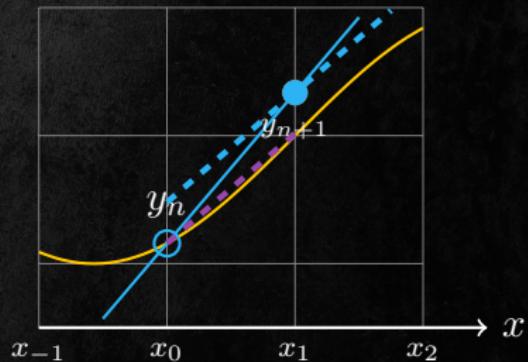
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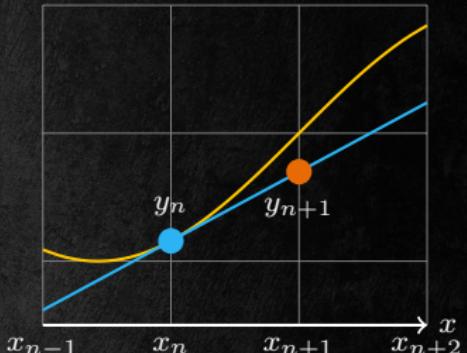
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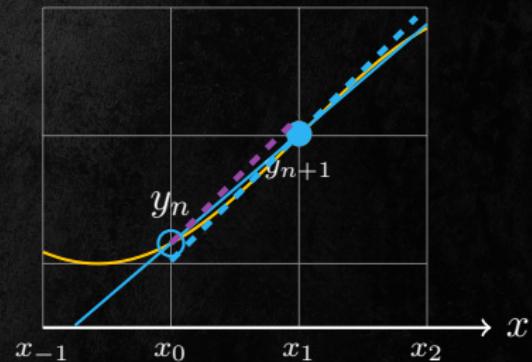
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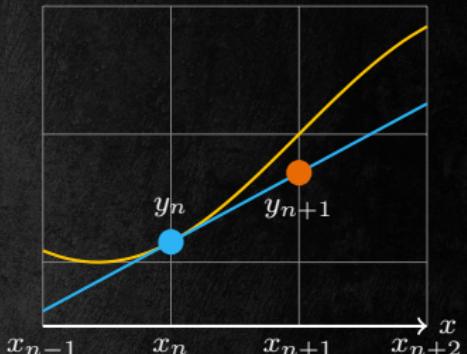
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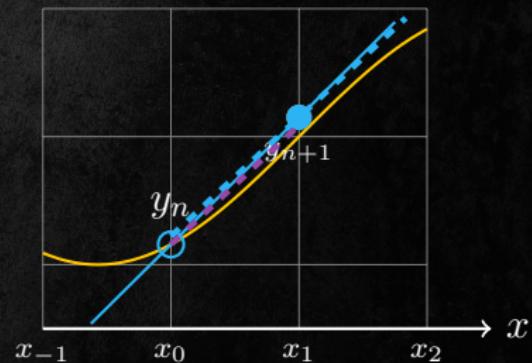
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MÉTODO IMPLÍCITO

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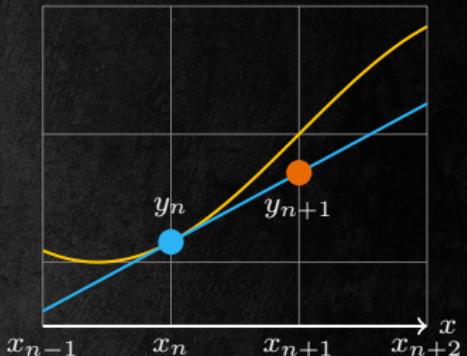
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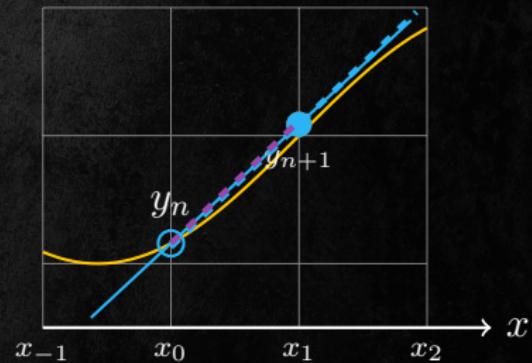
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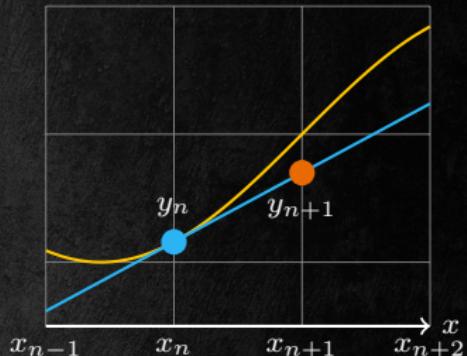
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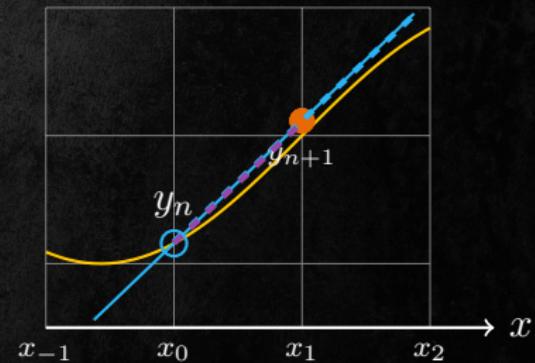
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MÉTODO IMPLÍCITO

Opción 1: Busco la raíz y_{n+1}
Bisección, Netwon-Rapshon.

Opción 2: Itero (punto fijo) →

No siempre converge



Ejemplo:

$$\frac{dy}{dx} = yx$$

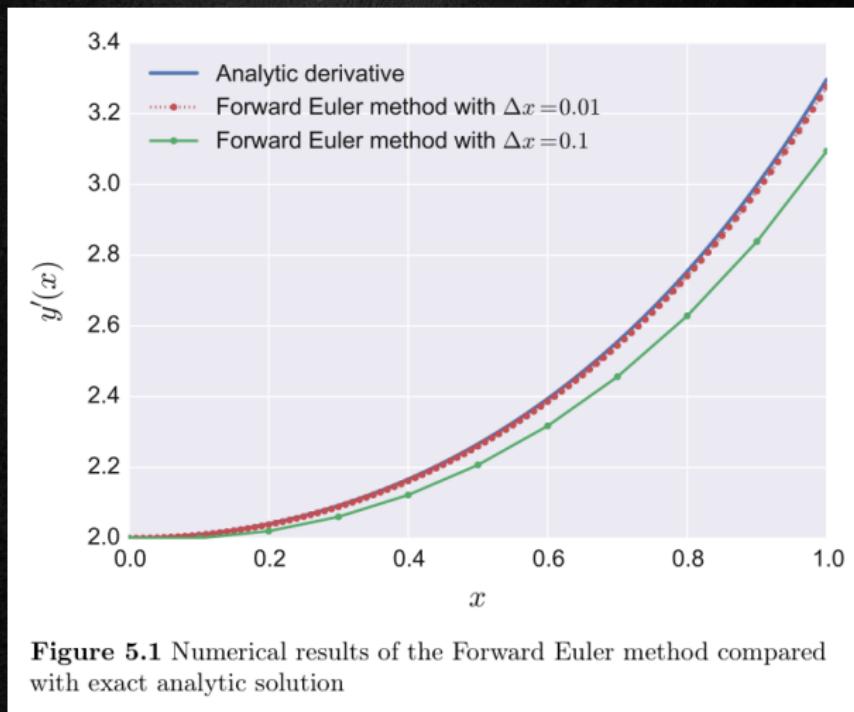


Figure 5.1 Numerical results of the Forward Euler method compared with exact analytic solution

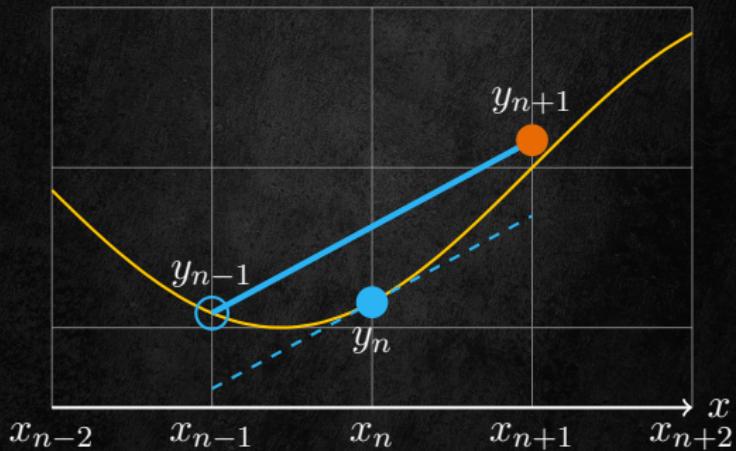
Taylor:

$$y_{n+1} = y_n + F(x_n, y_n)h + \frac{h^2}{2} \left[\frac{\partial F(x_n, y_n)}{\partial y} F(x_n, y_n) \right] + O(h^4)$$

Ejemplo:

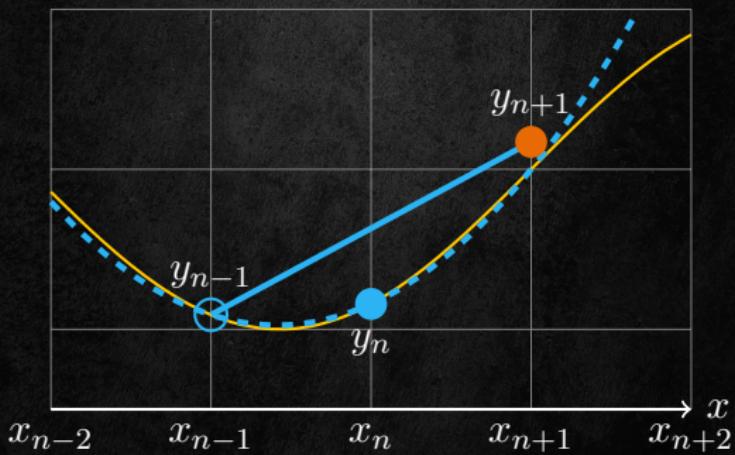
Leap-Frog:

$$y_{n+1} = y_{n-1} + 2hF(x_n, y_n) + O(h^3)$$



Verlet: $y_{n+1} = 2y_n - y_{n-1} + \frac{dy}{dx} \Big|_n h^2 + O(h^4)$

$$\left[\frac{\partial F(x_n, y_n)}{\partial x} + \frac{\partial F(x_n, y_n)}{\partial y} F(x_n, y_n) \right]$$



ERROR

... total de la iteracion n

ABSOLUTO: $\epsilon_n = |x_n - L|$

RELATIVO: $\tilde{\epsilon}_n = |x_n - L| / x_n$

... por usar una compu

PRECISIÓN / NUMÉRICO: $O(\epsilon_M)$

... por la aproximación

TRUNCAMIENTO: - LOCAL

$$\frac{df(x_n)}{dx} = \frac{f(x_{n+1}) - f(x_n)}{2} + O(h^3)$$

$$\int_{x_n}^{x_{n+1}} f(x) dx = f(x + h/2)h + O(h^2)$$

$$y_{n+1} = y_n + hF(x_n, y_n) + O(h^3)$$

GLOBAL:

$$(N-1)O(h^x) \\ = \frac{b-a}{h}O(h^x)$$

$$\times(N-1) \longrightarrow = O(h^{x-1})$$

$\times(N-1)$
... pero es recurrente!

CONVERGENCIA

Cuando buscamos aproximarnos a un valor (e.g. búsqueda de raíces), si el error disminuye en cada iteración. . .

CONVERGE: $\epsilon_{n+1}/\epsilon_n \leq 1$

ESTABILIDAD

Si hacemos $(N - 1)$ aproximaciones de forma recurrente, además del error global, hay que ver si una perturbación no se propaga por la recurrencia. . .

ESTABLE: $\delta y_{n+1}/\delta y_n \leq 1$

Condiciones de estabilidad

$$\left| 1 + h \frac{\partial F(x, y)}{\partial y} \right| \leq 1 \quad \text{Euler hacia adelante}$$

$$\left| 1 - h \frac{\partial F(x, y)}{\partial y} \right| \geq 1 \quad \text{Euler hacia atrás}$$

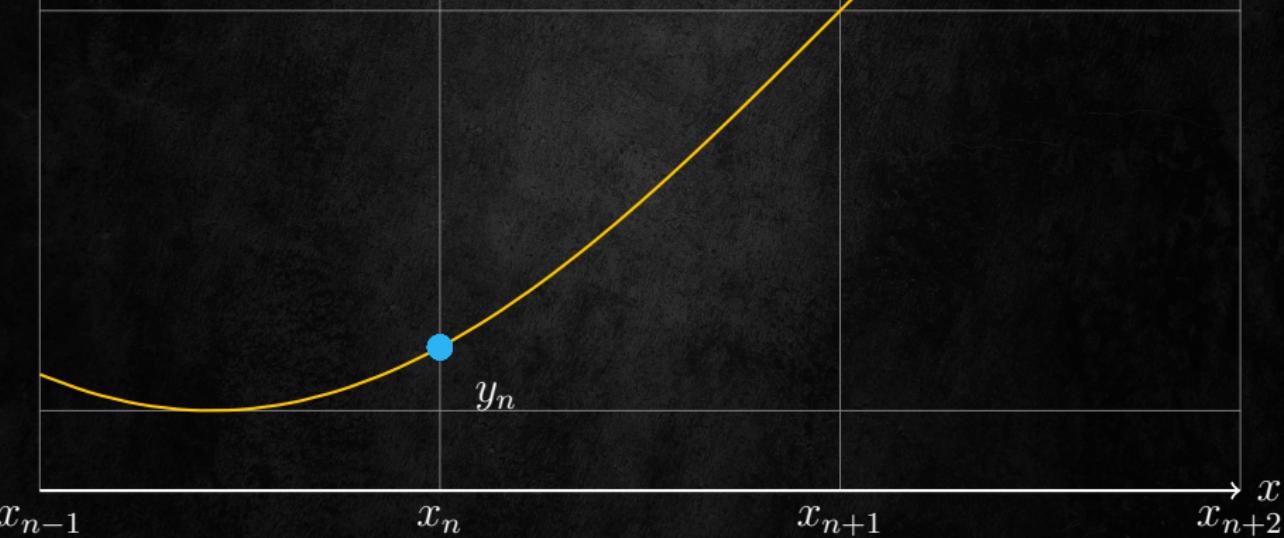
$$\text{Re}\left(\frac{\partial F(x, y)}{\partial y}\right) = 0 \wedge h \left| \text{Im}\left(\frac{\partial F(x, y)}{\partial y}\right) \right| \leq 1 \quad \text{Leap-Frog}$$

Euler de punto medio:

$$y_{n+1} = y_n + k_2 + O(h^3) \longrightarrow \text{Leap-Frog}$$

$$k_2 = hF(x_n + h/2, y_n + k_1/2)$$

$$k_1 = hF(x_n, y_n)$$



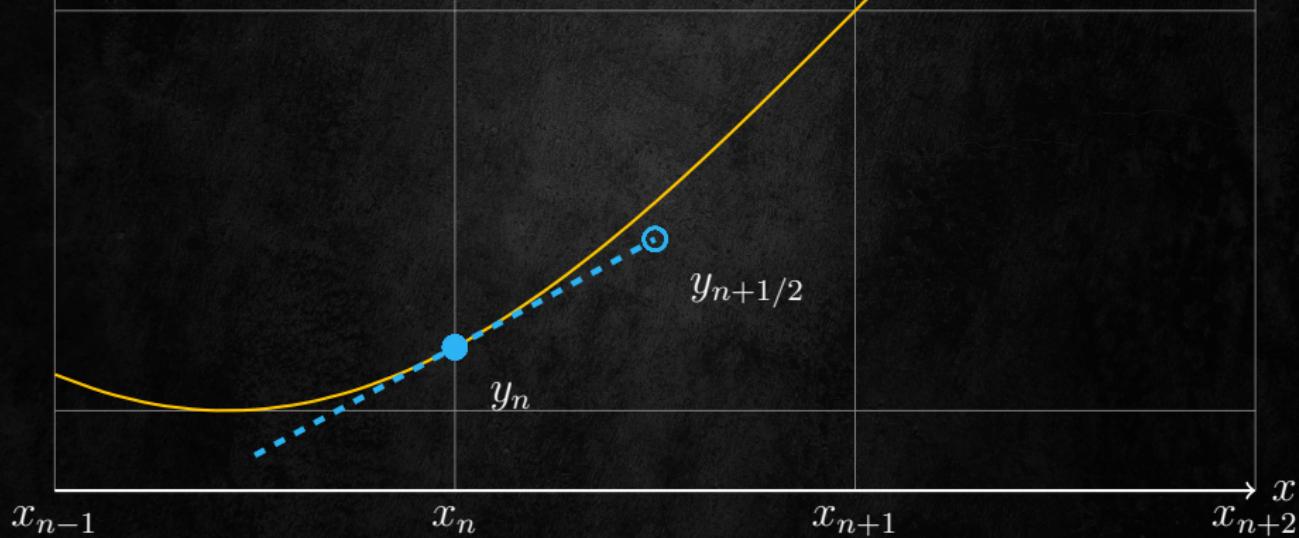
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Euler

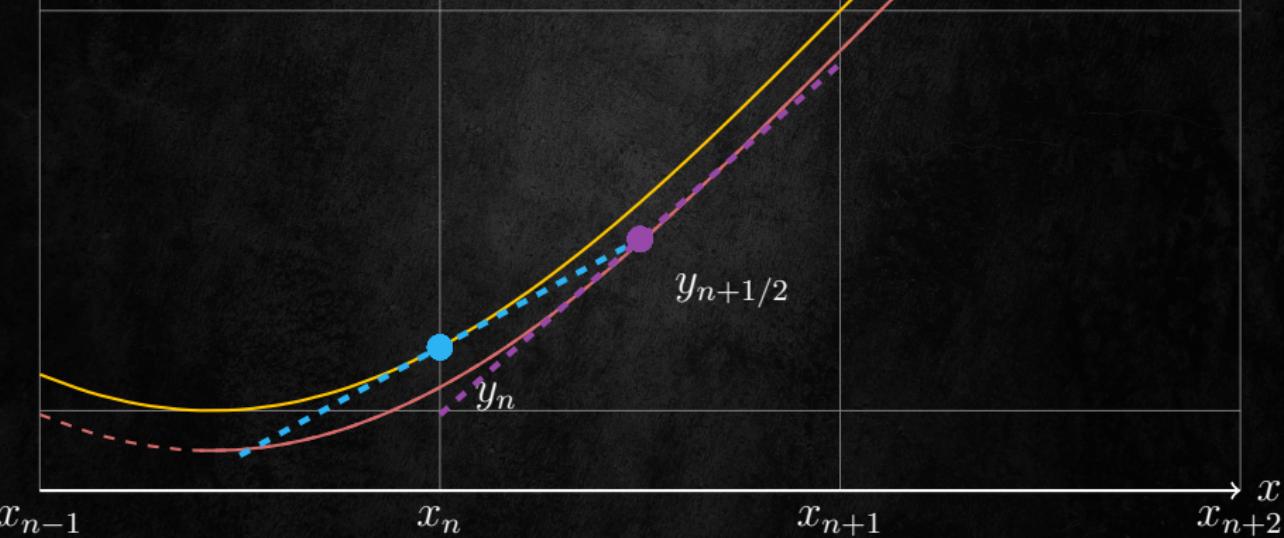


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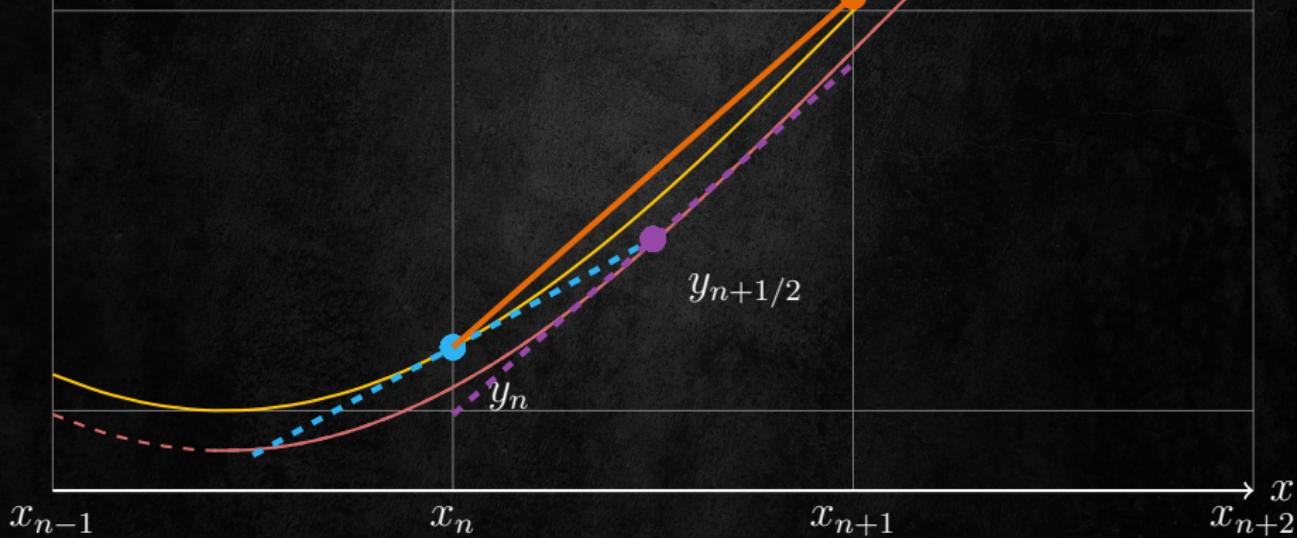
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Euler

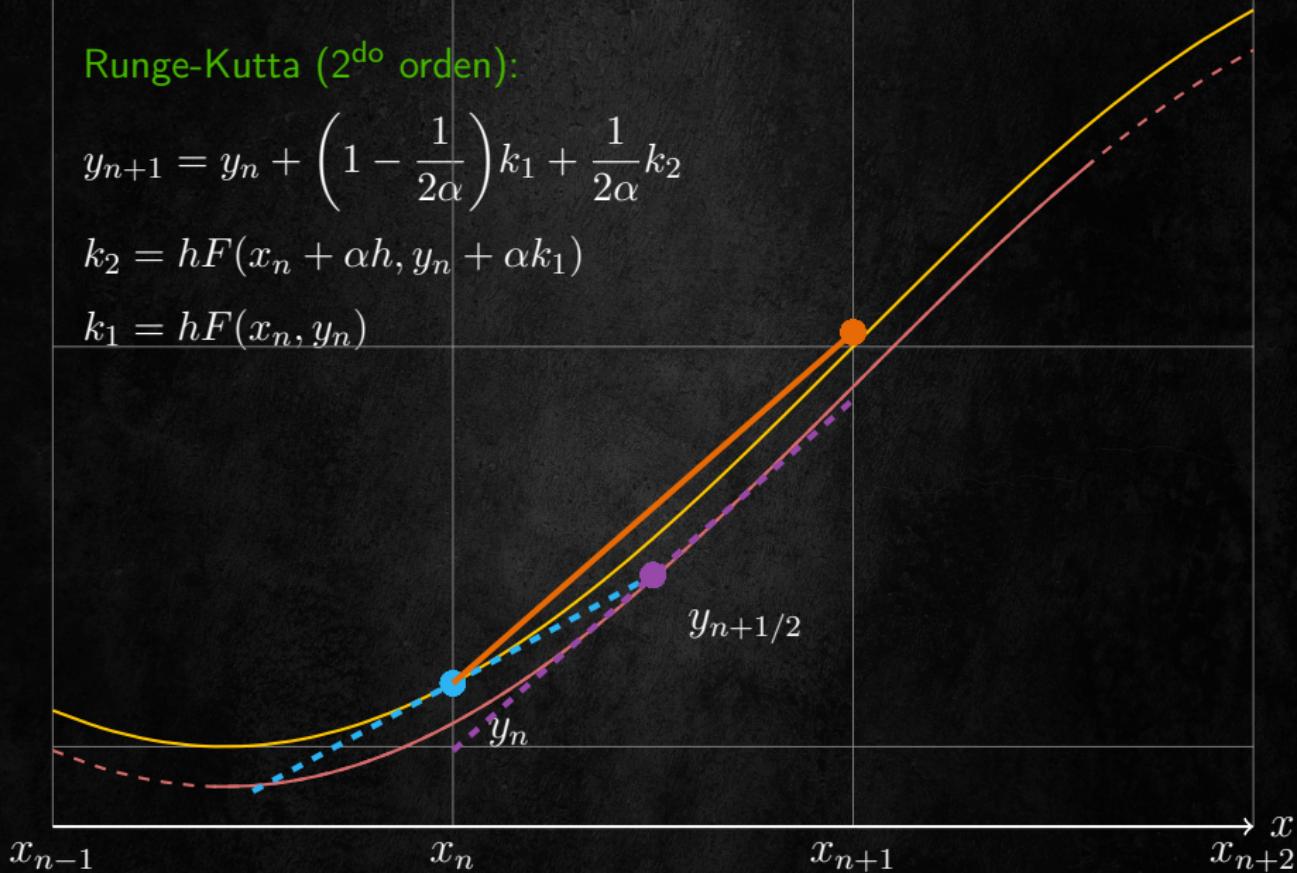


Runge-Kutta (2^{do} orden):

$$y_{n+1} = y_n + \left(1 - \frac{1}{2\alpha}\right)k_1 + \frac{1}{2\alpha}k_2$$

$$k_2 = hF(x_n + \alpha h, y_n + \alpha k_1)$$

$$k_1 = hF(x_n, y_n)$$



Runge-Kutta (4^{to} orden):

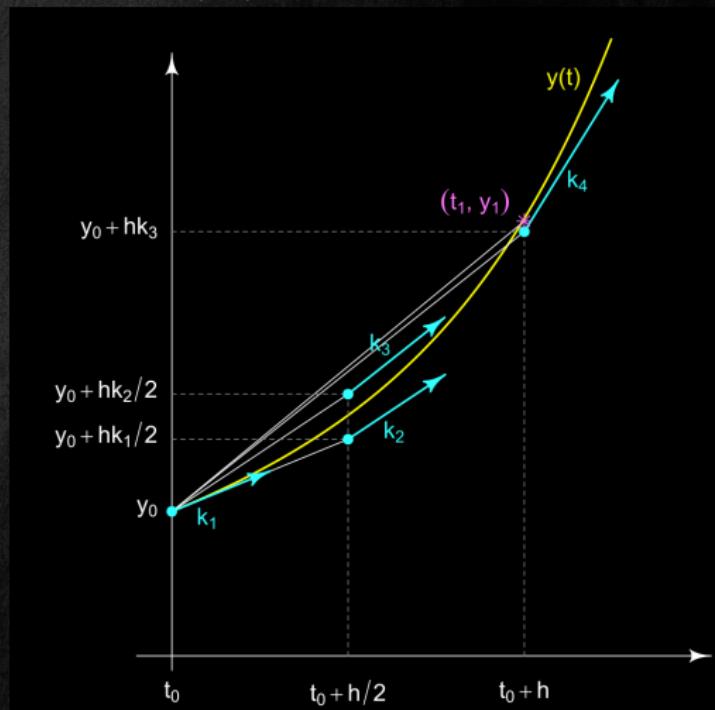
$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + O(h^5)$$

$$k_4 = hF(x_n + h, y_n + k_3)$$

$$k_3 = hF\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_2 = hF\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_1 = hF(x_n, y_n)$$



Algoritmos EDO1

$$y_{n+1} = y_n + hF(x_n, y_n) + O(h^2) \quad \text{Euler hacia adelante}$$

$$y_{n+1} = y_n + hF(x_{n+1}, y_{n+1}) + O(h^2) \quad \text{Euler hacia atrás}$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + O(h^5) \quad \text{RK4}$$

$$y_{n+1} = y_{n-1} + 2hF(x_n, y_n) + O(h^3) \quad \text{Leap-Frog}$$

Verlet

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \left[\frac{\partial F(x_n, y_n)}{\partial x} + \frac{\partial F(x_n, y_n)}{\partial y} F(x_n, y_n) \right] + O(h^4)$$