LASER Physics

Campus Life PRODUCED BY MooKEUK

□ 188mm X 260mm □ 8mm Ruled □ 28 Lines

People are like stained glass windows.
They sparkle and shine when the sun is out, but when the darkness sets in, their true beauty is revealed only if there is a light from within.

$$S_1 = C - DC^{-1}D = \begin{bmatrix} \gamma & -ig\sqrt{m+1} \\ -ig\sqrt{m+1} & \gamma \end{bmatrix} + \frac{g^2(m+1)}{\gamma^2+g^2(m+1)} \begin{bmatrix} \gamma & ig\sqrt{m+1} \\ ig\sqrt{n+1} & \gamma \end{bmatrix}$$

$$S_2 = C - DC^{-1}D$$

LASER Physics

$$\rightarrow S_1 = \begin{bmatrix} \gamma \cdot \left(\frac{\gamma^2+g^2(m+1+n+1)}{\gamma^2+g^2(m+1)}\right) & \gamma \cdot \left(\overline{\gamma^2+g^2(m+1)}\right) \\ & \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} S_1^{-1} & -C^{-1}DS_2^{-1} \\ -S_2^{-1}DC^{-1} & S_2^{-1} \end{bmatrix}$$

INDEX

PAGE	DESCRIPTION

$$\begin{bmatrix} \cancel{C^{-1}} & \cancel{0} \\ & C \end{bmatrix} \begin{bmatrix} C & D \\ D & C \end{bmatrix}$$

$$= \begin{bmatrix} C & D \\ D & C \end{bmatrix}$$

$$D = \begin{bmatrix} ig\sqrt{n+1} & 0 \\ 0 & ig\sqrt{n+1} \end{bmatrix}$$

$$C = \begin{bmatrix} \gamma & -ig\sqrt{m+1} \\ -ig\sqrt{m+1} & \gamma \end{bmatrix}$$

$$C^{-1} = \frac{1}{\gamma^2+g^2(m+1)} \begin{bmatrix} \gamma & ig\sqrt{m+1} \\ ig\sqrt{m+1} & \gamma \end{bmatrix}$$

$$D^{-1} = \frac{1}{-g^2(n+1)} \begin{bmatrix} ig(\sqrt{n+1}) & 0 \\ 0 & ig\sqrt{n+1} \end{bmatrix}$$

$$\cancel{M = \begin{bmatrix} \gamma & \\ & \gamma \end{bmatrix}}$$

$$M = \begin{bmatrix} \gamma & -ig\sqrt{m+1} & ig\sqrt{n+1} & 0 \\ -ig\sqrt{m+1} & \gamma & 0 & ig\sqrt{m+1} \\ ig\sqrt{n+1} & 0 & \gamma & -ig\sqrt{m+1} \\ 0 & ig\sqrt{n+1} & -ig\sqrt{m+1} & \gamma \end{bmatrix}$$

$$\cancel{\det(M) = \gamma \cdot \left(\gamma\left(\gamma^2 + g^2\sqrt{m+1}\sqrt{m+1}\right) + g\sqrt{n+1}\sqrt{m+1}\right)}$$

$$\mathrm{adj}(M)_{11} = (-1)^{1+1}$$

$$\begin{bmatrix} -\gamma & & & \\ & -\gamma & & \\ & & -\gamma & \\ & & & -\gamma \end{bmatrix}$$

$$D\cancel{C}^{-1}D =$$

$$\begin{bmatrix} \dot{\rho}_{an,am} \\ \rho_{an,bm+1} \\ \rho_{bn+1,am} \\ \rho_{bn+1,bm+1} \end{bmatrix} = \begin{bmatrix} -\gamma & ig\sqrt{m+1} & -ig\sqrt{n+1} & \cancel{ig\sqrt{m+1}}\ 0 \\ ig\sqrt{m+1} & -\gamma & 0 & -ig\sqrt{m+1} \\ -ig\sqrt{n+1} & 0 & -\gamma & ig\sqrt{m+1} \\ 0 & -ig\sqrt{n+1} & ig\sqrt{m+1} & -\gamma \end{bmatrix} \begin{bmatrix} \rho_{an,am} \\ \rho_{an,bm+1} \\ \rho_{bn+1,am} \\ \rho_{bn+1,bm+1} \end{bmatrix} + \begin{bmatrix} \frac{\gamma \lambda_a}{\lambda_a+1} \rho_{n,m} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\parallel$$
$$-M$$

박정현

010-3465-1538

$$\Gamma_0 = \frac{e^2 \omega_0^2}{6\pi\epsilon_0 m c^3}$$

$$= \frac{2}{3} \frac{r_e \omega_0^2}{c}$$

$$P = \frac{e^2}{6\pi\epsilon_0 c^3} \frac{1}{2} \cdot \mathrm{Re}\left[\ddot{\vec{x}} \cdot \ddot{\vec{x}}^*\right]$$

$$= \frac{e^2}{6\pi\epsilon_0 c^3} \times \frac{1}{2} \cdot \frac{(e/m)^2 \omega^4 |\vec{E}_0|^2}{(\omega_0^2-\omega^2)^2+(\Gamma_0\omega)^2}$$

$$= \frac{e^4\omega^4}{12\pi\epsilon_0 m^2 c^3} \frac{|\vec{E}_0|^2}{(\omega_0^2-\omega^2)^2+(\Gamma_0\omega)^2}$$

$$I_0 = \frac{1}{2}\epsilon_0 |\vec{E}_0|^2 \cdot c$$

$$\frac{1}{4\pi\epsilon_0} \to 1$$

$$\Rightarrow \sigma_{sc} = \frac{P}{I_0}$$

$$= \frac{e^4\omega^4}{6\pi\epsilon_0^2 m^2 c^4} \frac{1}{(\omega_0^2-\omega^2)^2+(\Gamma_0\omega)^2}$$

for $\omega = \omega_0$

$$\frac{e^2}{4\pi\epsilon_0 r_e} = mc^2$$

$$\sigma_{sc} = \frac{e^4\omega_0^2}{6\pi\epsilon_0^2 m^2 c^4 \Gamma_0^2}$$

$$= \frac{\cancel{e^4}\omega_0^2}{6\pi\cancel{\epsilon_0^2}\left(\frac{\cancel{e^2}}{4\pi\cancel{\epsilon_0} r_e}\right)^2 \Gamma_0^2}$$

$$= \frac{8\pi}{3} \cdot r_e^2 \cdot \frac{\omega_0^2}{\Gamma_0^2} = 6\pi\left(\frac{c}{\omega_0}\right)^2$$

1 Classical Theory of Absorption and Emission

1.1 Emission cross section

consider a radiatively-damped classical harmonic oscillator (= harmonically bound electron) driven by an EM wave

$$m(\ddot{\vec{x}} + \Gamma_0 \dot{\vec{x}} + \omega_0^2 \vec{x}) = e\vec{E}_0 e^{-i\omega t}$$

assume $\vec{x} = \vec{x}_0 e^{-i\omega t}$

$$m(-\omega^2 - i\Gamma_0\omega + \omega_0^2)\vec{x}_0 e^{-i\omega t} = e\vec{E}_0 e^{-i\omega t}$$

$$\vec{x}_0 = \frac{e\vec{E}_0}{-m(\omega^2 + i\Gamma_0\omega - \omega_0^2)}$$

we obtain

$$\vec{x} = \frac{(e/m)\vec{E}_0 e^{-i\omega t}}{-\omega^2 - i\Gamma_0\omega + \omega_0^2}$$

The induced electric dipole moment $\vec{p} = e\vec{x}$, which in turn acts as a source of radiation, The Lamor's formula gives the total radiated power P

$$P = \frac{2e^2|\dot{\vec{v}}|^2}{3c^3}$$

where $\vec{v}$ is real and we are using the convention $4\pi\epsilon_0 = 1$ (Gauss unit) from now on. Substituting $\vec{x}$ expression in P (using $\overline{A_{real}^2} = Re[A_{complex} A^*_{complex}]$

$$P = \frac{2e^2}{3c^3}\cdot\frac{1}{2} Re[\ddot{\vec{x}}_0 \ddot{\vec{x}}^*] = \frac{e^4}{3m^2c^3}\frac{|E_0|^2\omega^4}{(\omega_0^2-\omega^2)^2 + \Gamma_0^2\omega^2}$$

The total scattering (=emission) cross section is defined as

$$\sigma_{sc} = \frac{(\text{total scattered power})}{(\text{incident intensity})} = \frac{P}{I_0}$$

Incident intensity (i.e. Poynting vector), I_0 is (using $\overline{A_{real}^2} = \frac{1}{2}Re[A_{complex} A^*_{complex}]$

$$I_0 = \left|\left\langle \frac{c}{4\pi}\vec{E}\times\vec{B}\right\rangle_{time}\right| = \frac{c|\vec{E}_0|^2}{8\pi}$$

note that $\vec{S} = \left[\frac{1}{\mu_0}\vec{E}\times\vec{B}\right]_{SI} = \left[\frac{c}{4\pi}\vec{E}\times\vec{B}\right]_{Gaussian}$

$$\therefore \sigma_{sc} = P\cdot\frac{8\pi}{c|\vec{E}_0|^2}$$

$$= \frac{e^4}{3m^2c^3}\cdot\frac{|\vec{E}_0|^2\omega^4}{(\omega_0^2-\omega^2)^2+\Gamma_0^2\omega^2}\cdot\frac{8\pi}{c|\vec{E}_0|^2}$$

$$= \frac{8\pi e^4}{3m^2c^4}\cdot\frac{\omega^4}{(\omega_0^2-\omega^2)^2+\Gamma_0^2\omega^2}$$

$$= \underbrace{\frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2}\frac{\omega^4}{(\omega_0^2-\omega^2)^2+\Gamma_0^2\omega^2}$$

→ electron classical radius

No.

Date . .

And thus

$$\sigma_{sc} = \frac{8\pi}{3} r_e^2 \frac{\omega^4}{(\omega_0^2-\omega^2)^2+\Gamma_0^2\omega^2} \qquad (1-7)$$

where we use the definition of the elecrtron's classical radius r_e ($\sim 3\times10^{-13}$ cm)

$$mc^2 = \frac{e^2}{r_e} \qquad (1-8)$$

consider

- Low frequency limit, $\omega \ll \omega_0$ (Rayleigh scattering)

$$\sigma_{sc} \simeq \frac{8\pi}{3} r_e^2 \left(\frac{\omega}{\omega_0}\right)^4 \qquad (1-9)$$

→ explain why sky is blue

- High frequency limit, $\omega \gg \omega_0$ (Thomson scattering of free electron)

$$\sigma_{sc} \simeq \frac{8\pi}{3} r_e^2 \simeq 10^{-24}\ \text{cm}^2$$

- Near resonance $\omega \simeq \omega_0$ (Resonance fluorescence)

$$\sigma_{sc} \simeq 6\pi\left(\frac{c}{\omega_0}\right)^2 \frac{(\Gamma_0/2)^2}{(\omega-\omega_0)^2+(\Gamma_0/2)^2}$$

p.f.) $$\sigma_{sc} = \frac{8\pi}{3} r_e^2 \frac{\omega^4}{(\omega_0-\omega)^2(\omega_0+\omega)^2+\Gamma_0^2\omega^2}$$

$$\simeq \frac{8\pi}{3} r_e^2 \frac{\omega^4}{(\omega_0-\omega)^2 4\omega^2+\Gamma_0^2\omega^2}$$

$$= \frac{2\pi}{3} r_e^2 \frac{\omega^2}{(\omega_0-\omega)^2+\Gamma_0^2/4}$$

~~분자의 $\omega \simeq \omega_0+\Delta\omega$에서 $\Delta\omega$ 효과↓ (∵ 분모의 크기~~ $\sim \Delta\omega$)

radiative damping rate Γ_0 ($\ll \omega_0$)

$$\Gamma_0 = \frac{2e^2\omega_0^2}{3mc^3}, \quad r_e^2\omega^2 = \frac{e^4\omega^2}{m^2c^4} = \left(\frac{2e^2\omega_0^2}{3mc^3}\right)^2 \frac{9\omega^2c^2}{4\omega_0^4} \simeq \Gamma_0^2 \frac{9c^2}{4\omega_0^2}$$

$$\therefore \Rightarrow \sigma_{sc} \simeq 6\pi \left(\frac{c}{\omega_0}\right)^2 \frac{(\Gamma_0/2)^2}{(\omega-\omega_0)^2+(\Gamma_0/2)^2}$$

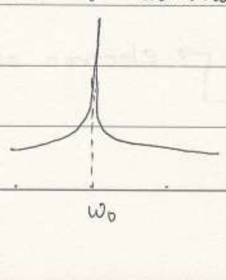
→ line width $= \Gamma_0$ = FWHM

- On resonance

radiative scattering cross section

$$\star\ \sigma_{sc}(\omega_0) = 6\pi\left(\frac{c}{\omega_0}\right)^2 = 6\pi \lambda\!\!\!^{-}{}_0^2 = \frac{3\lambda_0^2}{2\pi}$$

For optical wavelength ($\lambda_0 = 600$nm), $\sigma_{sc} \sim 2\times10^{-9}\ \text{cm}^2$, which is 10^{15} times larger than the off-resonance Thomson scattering cross section. This greatly enhanced emission on resonance is called "resonance fluorescence".



Suppose now that non-radiative damping also occurs such that total damping rate is given by

$$\Gamma_t = \Gamma' + \Gamma_0$$

where Γ' is the non-radiative damping rate (vibrational, collisional, etc).

The scattering cross section is then modified to

$$\sigma_{sc} = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \frac{\omega^4}{(\omega_0^2-\omega^2)^2+\Gamma_t^2\omega^2}$$

Particularly, on resonance

$$\sigma_{sc}(\omega_0) = 6\pi\lambda^2 (\Gamma_0/\Gamma_t)^2$$

→ repropagate in the medium

1.2 Absorption cross section

For purely radiative damping, the scattered power has to be equal to the power removed from the incident radiation, and thus the absorption cross section is the same as the scattering cross section.

In order to account for absorption, one can consider the imaginary part of the index of refraction of a medium composed of the harmonic oscillator as considered in Sec. 1.1. Macroscopic polarization $\vec{P}$ is given by

$$\vec{P} = ne\vec{x} = \frac{(ne^2/m)}{-\omega^2 - i\Gamma_t\omega + \omega_0^2}\vec{E} = \chi\vec{E} \rightarrow \text{for SI, } \vec{P} = \epsilon_0\chi\vec{E}$$

where n and χ are the density and the electric susceptibility of the medium, respectively. Imaginary part of the index of refraction comes from the imaginary part of $2\pi\chi$ in Gaussian unit $(\because \sqrt{\epsilon} = \sqrt{1+4\pi\chi} \simeq 1+2\pi\chi)$

$$\mathrm{Im}(\chi) = \left(\frac{ne^2}{m}\right)\cdot\frac{\Gamma_t\omega}{(\omega_0^2-\omega^2)^2+\Gamma_t^2\omega^2} \rightarrow \text{for SI, } \mathrm{Im}(\chi) = \frac{ne^2}{m\epsilon_0}\frac{\Gamma_t\omega}{(\omega_0^2-\omega^2)^2+\Gamma_t^2\omega^2}$$

The intensity attenuation of the incident EM field is governed by Beer's law,

$$|e^{i(n_r+in_i)(\frac{\omega}{c})x}|^2 = e^{-2n_i(\frac{\omega}{c})x}$$

$$\exp\left[-(4\pi\,\mathrm{Im}(\chi))\omega x/c\right] \equiv \exp(-n\sigma_{abs}x)$$

The absorption cross section is, therefore, $I_m(\chi) = \left(\frac{ne^2}{m}\right)\frac{\Gamma_t\omega}{(\omega_0^2-\omega^2)^2+\Gamma_t^2\omega^2}$

$$\sigma_{abs} = \frac{4\pi\omega I_m(\chi)}{nc} = \left(\frac{4\pi e^2}{mc}\right)\frac{\Gamma_t\omega^2}{(\omega_0^2-\omega^2)^2+\Gamma_t^2\omega^2}$$

Rewriting it for comparison with Eq. (1-15) $\Gamma_0 = \frac{2e^2\omega_0^2}{3mc^3}$

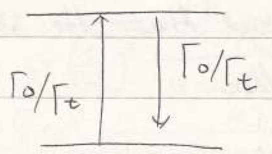
$$\sigma_{abs} = \frac{8\pi}{3}\Big(\underbrace{\frac{e^2}{mc^2}}_{}\Big)^2 \overset{r_e}{} \underbrace{\left(\frac{\Gamma_t}{\Gamma_0}\right)} \frac{\omega^2\omega_0^2}{(\omega_0^2-\omega^2)^2+\Gamma_t^2\omega^2}$$

($\gg = r_e$)

On resonance,

$$\sigma_{abs} = 6\pi\lambda^2\underbrace{(\Gamma_0/\Gamma_t)}_{\to \text{ reduction } \downarrow} \qquad \text{c.f. } \sigma_{sc} = 6\pi\lambda^2\underbrace{(\Gamma_0/\Gamma_t)^2}$$

Since absorption is involved with radiative excitation, σ_{abs} is proportional to Γ_0/Γ_t. Out of total decay only radiative decay results in emission, and thus σ_{sc} is reduced by Γ_0/Γ_t factor from σ_{abs}



emission → Γ_0/Γ_t만 ~~apsor~~ absorpt →

decay → 다른 채널로 decay

$\Gamma_0 = \frac{e^2 \omega_0^2}{6\pi\epsilon_0 m c^3} \Rightarrow \Gamma_0 6\pi\epsilon_0 m c^3 / \omega_0^2 = \frac{2\pi}{T} = \frac{2\pi\lambda}{T\lambda} = \frac{2\pi c}{\lambda}$ $\frac{\omega_0}{c}\frac{e^2}{m}\frac{1}{\Gamma_t \omega_0}$

Topic-1, HW. 3/16-20

No.

Date

$\frac{e^2}{\epsilon_0}\frac{1}{mc\Gamma_t} = 6\pi\left(\frac{c}{\omega_0}\right)^2$ $\therefore \frac{e^2}{mc\Gamma_t}$

$n = 10^6 / cm^3 \times \frac{1cm^3}{10^{-6}m^3} = 10^{12}/m^3$, $\sigma_{abs} = \frac{\omega}{nc}\cdot Im(\chi)$

$= \frac{\omega}{nc}\frac{ne^2}{m}\frac{\Gamma_t \omega}{(\omega_0^2-\omega^2)^2+\Gamma_t^2\omega^2}$

$L = 1cm = 1cm \times 10^{-2} m/cm = 10^{-2} m$

$\lambda = 553 nm \times \frac{10^{-9}m}{1nm} = 553\times10^{-9} m = \frac{2\pi c}{\omega} \Rightarrow \omega = \left(\frac{553\times10^{-9}}{2\pi c}\right)^{-1} [rad/s] = 3.41\times10^{15} [rad/s]$

$\Gamma_0 = 2\pi \times 20 MHz \times \frac{10^6 Hz\, rad}{1MHz} = 4\pi\times10^7 Hz \times rad = 4\pi\times10^7 rad/s$

$n = \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{1+\chi} \simeq 1 + \frac{1}{2}\chi$ (refraction index)

$\Rightarrow |e^{i(n_r + i n_c)\frac{\omega}{c}x}|^2 = e^{-2n_c\frac{\omega}{c}x} = e^{-Im(\chi)\frac{\omega}{c}x} = \exp(-n\sigma_{abs}\cdot x)$

we know $\chi = \frac{(ne^2/m)}{(\omega_0^2-\omega^2) - i\Gamma_t\omega}$, $Im(\chi) = \frac{(ne^2/m)\Gamma_t\omega}{(\omega_0^2-\omega^2)^2 + (\Gamma_t\omega)^2}$

due to purely radiative transition, $\Gamma_t = \Gamma_0$

$\Rightarrow OD = n\sigma_{abs}\cdot L = \frac{(ne^2/m)\Gamma_0\omega}{(\omega_0^2-\omega^2)^2+(\Gamma_0^2\omega^2)}\cdot\frac{\omega}{c}\cdot L$

$= \frac{L}{c}\frac{(ne^2/m)\Gamma_0\omega^2}{(\omega_0^2-\omega^2)^2+(\Gamma_0^2\omega^2)}$

$\Gamma_0 = \frac{e^2\omega_0^2}{6\pi m c^3\epsilon_0}$

$\Rightarrow \omega_0 \neq \sqrt{\frac{6\pi m c^3\epsilon_0}{e^2}\Gamma_0}$

LASER is tuned to $\lambda = 553nm$, ~~$\omega_0 - \omega = 0$~~

$= \sqrt{\frac{6\pi\times9.109\times10^{-31}\times(3\times10^8)^3\times8.854\times10^{-12}}{e^2}}$

~~$\Rightarrow OD = \frac{L}{c}\frac{(ne^2/m)\Gamma_0\omega_0^2}{\Gamma_0^2\omega_0^2}$~~ $= \frac{ne^2}{m\Gamma_0}\frac{L}{c}$

$\times 4\pi\times10^7 rad/s$

$= 4.48\times10^{15} rad/s$

$\vec{P} = \frac{N\vec{p}}{V}$ σ_{abs}

$\rightarrow \Gamma_0$ = radiative + nonradiative

$\frac{c}{\omega_0} = \frac{c}{\frac{2\pi}{T}} = \frac{\lambda}{2\pi}$ σ_{sc} $= ne\vec{x}$ $\Gamma_0 = \frac{e^2\omega_0^2}{6\pi\epsilon_0 mc^3}$ $\vec{P} = \epsilon_0\chi\vec{E}$

$\frac{2\pi}{3}$ $\vec{P} = \epsilon_0\vec{E}_0$ $OD = \frac{ne^2}{m\Gamma_0}\cdot\frac{L}{c}$

$P = \frac{e^2}{6\pi\epsilon_0 c^3 m}\frac{|E_0|^2\omega^4}{(\omega_0^2-\omega^2)+\frac{\Gamma_0^2\omega^2}{2}}\times\frac{1}{2}$

purely radiative

$\Rightarrow \omega_0 = \frac{2\pi c}{\lambda} = 3.41\times10^{15} [rad/s]$, $= \frac{2\pi c}{553 nm}$

$\Gamma_0 \neq \frac{e^2\omega_0^2}{6\pi m c^3\epsilon_0}$ → 다른 level 들의 transitions 있어서 그런 듯?

$\frac{1}{\epsilon_0} = \frac{F r^2}{e^2}$ $I_0 = \frac{1}{2}\epsilon_0 |E_0|^2$, $\frac{1}{2}\epsilon_0$ $\epsilon_0 = \frac{1}{4\pi}$ $\frac{e^2}{mc} = r_e\cdot c$, $\sigma_{sc} = \frac{P}{I_0}$

$P = \epsilon_0\chi E$ $r_e = \frac{e^2}{mc^2}$ $= \omega_0$

$N = m/s^2\cdot kg$ $D = (1+\chi_E)\epsilon_0 E$

$P = \frac{e^2}{6\pi\epsilon_0 m c^3}\frac{\omega^4}{(\omega_0^2-\omega^2)^2+\Gamma_0^2\omega^2}$

e^2 $\times n$ $n = \sqrt{1+\chi_E}$ L

$= \frac{m^3 kg}{s^2 C^2}$ $\frac{1}{2}\epsilon_0|\vec{E}|^2$ $\frac{1}{\mu_0}\vec{E}\times\vec{B} = \vec{S}$

$mc\Gamma_t$ $\frac{\omega}{c}$ $\frac{e^2}{mc}$ $\frac{\Gamma_t\omega^2}{(\omega_0^2-\omega^2)^2+\Gamma_t^2\omega^2}$ $\frac{e^2}{6\pi\epsilon_0^2 c^3(\omega_0^2-\omega^2)^2 m}$

σ_{abs} $= \lambda/T$ I_0 $\frac{2\pi c}{\lambda}$ $\langle\frac{1}{\mu_0}\vec{E}\times\vec{B}\rangle = \langle\frac{1}{2}c\epsilon_0\vec{E}\times\vec{B}\rangle$

$4c\epsilon_0$

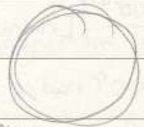
MooKeuk

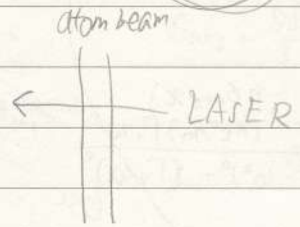
No.

Date

$P_0 = 1\times10^{-6}\,W$

$d_0 = 10^{-3}m$ $\quad I_0 = \frac{1}{2}\varepsilon_0 |\vec{E}_0|^2 c$





$I_0 = \frac{P_0}{\pi\left(\frac{d_0}{2}\right)^2} = \frac{4P_0}{\pi d_0^2}$

$= \frac{e^4\omega^4}{12\pi\epsilon_0 m^2 c^3} \frac{|\vec{E}_0|^2}{(\omega^2-\omega_0^2)^2 + (\Gamma_0\omega)^2}$

$P = \frac{e^2}{6\pi\epsilon_0 c^3} \cdot \frac{1}{2} Re[\ddot{\vec{x}}\cdot\ddot{\vec{x}}^*]$

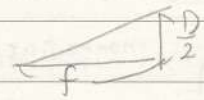
$Re[\ddot{\vec{x}}\cdot\ddot{\vec{x}}^*] = \frac{(e/m)^2|\vec{E}_0|^2\omega^4}{(\omega^2-\omega_0^2)^2+(\Gamma_0\omega)^2}$

$\cancel{R_{scatt} = \Gamma_0 \cdot}$

$R_{scatt} = P/\hbar\omega_0 = \frac{1}{\hbar\omega_0}\cdot 12$ $\quad = \frac{e^4\omega^4}{6\pi m^2 c^4} \frac{I_0}{(\omega^2-\omega_0^2)^2+(\Gamma_0\omega)^2}$

$= \boxed{\frac{e^4\omega^4}{6\pi m^2 c^4 \epsilon_0^2} \frac{I_0}{(\omega^2-\omega_0^2)^2+(\Gamma_0\omega)^2} \cdot \frac{1}{\hbar\omega_0}} n V$ $\quad V = \pi\left(\frac{d_0}{2}\right)^2 \cdot L$

$\Rightarrow f_\# = \frac{D}{f}$



$\Rightarrow \eta_L = \frac{2\pi}{4\pi}\left(1 - \sqrt{\frac{f^2}{\left(\frac{D}{2}\right)^2+f^2}}\right) = \frac{1}{2}\left(1-\sqrt{\frac{1}{\left(\frac{1}{2}f_\#\right)^2+1}}\right)$

$= \frac{1}{2}\left(1-\sqrt{\frac{4}{f_\#^2+4}}\right)$

$\eta_{PMT} = 0.1$

$\eta_{counter} = 0.5$

$\Rightarrow S = \eta_L \cdot \eta_{PMT}\, \eta_{counter}\, R_{scatt}$

for $\omega = \omega_0$, $R_{scatt} = \frac{e^4\omega_0^4}{6\pi m^2 c^4 \epsilon_0^2} \frac{I_0}{\Gamma_0^2\omega_0^2} \frac{1}{\hbar\omega_0} \cdot n \cdot V$

$= \frac{e^4\omega_0 \cdot I_0}{6\pi m^2 c^4 \Gamma_0^2 \hbar \epsilon_0^2} n V$

$I_0 \cdot N$ $\quad = \cancel{\frac{2e^4\omega_0 P_0}{3\pi^2 m^2 c^4 \Gamma_0^2 d_0^2 \epsilon_0^2}}$

I_0

$= \frac{e^4\omega_0 P_0 n L}{6\pi m^2 c^4 \Gamma_0^2 \hbar \epsilon_0^2}$ s^{-1}

$\cancel{\boxed{= 1.06\times10^{-13}}}$

1.332×10^9

$6.465\times10^8\ s^{-1}$

$$\approx \frac{\omega^4}{(\omega-\omega_0)^2(\omega+\omega_0)^2+(\Gamma_0\omega)^2} \simeq \frac{\omega^2/4}{(\omega-\omega_0)^2+(\Gamma_0/2)^2}$$

No.

Date . .

for $\lambda=\lambda_0+\Delta\lambda$, $g(\omega)=\dfrac{\omega^4}{(\omega^2-\omega_0^2)^2+(\Gamma_0\omega)^2}$

$$g(\omega_0)=\frac{\omega_0^2}{\Gamma_0^2}=\frac{4\pi^2c^2}{\lambda_0^2\Gamma_0^2}=\cancel{7.357534\times10^{14}} \quad \omega=\frac{2\pi c}{\lambda}$$

$$g(\omega_0+\Delta\omega)=\tilde{g}(\lambda_0+\Delta\lambda)$$

$$=\cancel{\frac{\left(\frac{2\pi}{\lambda_0+\Delta\lambda}\right)^4}{\left(\frac{2\pi}{\lambda}-\frac{2\pi}{\lambda_0}\right)^2+\left(\frac{\Gamma_0}{\lambda}\right)^2}} \quad 7.3$$

$$=\frac{c^4 4\pi^2/(\lambda_0+\Delta\lambda)^4}{c^4\left(\frac{4\pi^2\Delta\lambda^2}{(\lambda_0+\Delta\lambda)^2\lambda_0^2}\right)+\frac{\Gamma_0^2c^2}{(\lambda_0+\Delta\lambda)^2}}$$

$$\simeq\frac{(2\pi c/(\lambda_0+\Delta\lambda))^2/4}{\left(\frac{2\pi c}{\lambda_0+\Delta\lambda}-\frac{2\pi c}{\lambda_0}\right)^2+(\Gamma_0/2)^2}=\frac{\frac{1}{4}\left(\frac{1}{\lambda_0+\Delta\lambda}\right)^2}{\left(\frac{\Delta\lambda}{(\lambda_0+\Delta\lambda)\lambda_0}\right)^2+\left(\frac{\Gamma_0}{4\pi c}\right)^2}$$

$$=\cancel{7.35487\times10^{14}}$$

$$\cancel{7.645\times10^6}$$

$$R_{scatt}=\frac{e^4}{6\pi m^2c^4}\frac{1}{\hbar\omega_0}g(\omega)\cdot\frac{4P_0}{\pi d_0^2}\cdot\cancel{\frac{\pi d_0^2}{4}}\times L\,\frac{1}{\epsilon_0}$$

$$=\cancel{\frac{2P_0\cdot e^4}{8\pi^2m^2c^4d_0^2\hbar\omega_0}g(\omega)\frac{1}{\epsilon_0}}$$

$$=\frac{e^4}{6\pi m^2c^4}\frac{L P_0}{\hbar\omega_0}g(\omega)\frac{1}{\epsilon_0}$$

$$\Rightarrow \frac{g(\omega_0+\Delta\omega)}{g(\omega_0)}\simeq 0.9996$$

$$\frac{c}{\omega_0}=\frac{\lambda}{2\pi} \qquad =6\pi\left(\frac{c}{\omega_0}\right)^2\frac{P_0 n L}{\hbar\omega_0}$$

$$\Rightarrow R_{scatt}=\frac{6\pi c^2}{\omega_0^2}\frac{P_0}{\hbar\omega_0}\cdot n\cdot L$$

$$\frac{1}{\Gamma_0^2}=\frac{36\pi^2m^2c^6}{e^4\omega_0^4}$$

$$\frac{e^2\omega_0^2}{6\pi mc^3}=\Gamma_0$$

$$\frac{\quad^2}{\Gamma_0}=$$

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No.

Date . .

$$m(-\omega^2 - i\Gamma_0\omega + \omega_0^2)\vec{x} = e\vec{E}$$

$$\Rightarrow \vec{x} = \frac{e\vec{E}}{m(-\omega^2+\omega_0^2 - i\Gamma_0\omega)} \Rightarrow \vec{P} = \left(\frac{ne^2}{m}\right)\frac{\vec{E}}{(-\omega^2+\omega_0^2 - i\Gamma_0\omega)}$$

$$= \epsilon_0\chi\vec{E}$$

$$\Rightarrow \chi = \frac{ne}{m\epsilon_0}\frac{1}{(-\omega^2+\omega_0^2 - i\Gamma_0\omega)}$$

ne

$$P = \frac{e^2}{6\pi\epsilon_0 c^3}\cdot\frac{1}{2}Re[\ddot{\vec{x}}\cdot\ddot{\vec{x}}^*] \qquad \Gamma_0 = \frac{e^2\omega_0^2}{6\pi\epsilon_0 m c^2}$$

$$= \frac{e^4\omega^4}{6\pi\epsilon_0 m^2 c^3}\frac{1}{(-\omega^2+\omega_0^2)^2+\Gamma_0^2\omega^2}|\vec{E}_0|^2\cdot\frac{1}{2}$$

$$I_0 = \frac{1}{2}\epsilon_0|\vec{E}_0|^2\cdot c$$

$$\Rightarrow \sigma_{sc} = P/I_0 = \frac{e^4\omega^4}{6\pi\epsilon_0^2 m^2 c^4}\frac{1}{(-\omega^2+\omega_0^2)^2+\Gamma_0^2\omega^2}$$

$$= \frac{e^4\omega^4}{6\pi\epsilon_0^2 m^2 c^4}\frac{1}{(-\omega_0^2+\omega^2)^2+\Gamma_0^2\omega^2}$$

$$\omega_0 = \frac{2\pi c}{\lambda_0} \qquad Im(\chi) = \frac{ne^2}{m\epsilon_0}\frac{\Gamma_0\omega}{(-\omega^2+\omega_0^2)^2+\Gamma_0^2\omega^2}$$

$$\omega = \frac{2\pi c}{\lambda_0+\Delta\lambda}$$

$$\sigma_{sc}(\lambda_0+\Delta\lambda) = 5.06938\times10^{-22}m^2$$

$$\cancel{\tfrac{1}{2}}\,2\cdot\frac{\omega}{c}Im(\chi) = \sigma_{abs}\,L\,n$$

$$= \frac{\omega}{c}Im(\chi)L$$

$$= \frac{ne^2\omega}{m\epsilon_0 c}\frac{\Gamma_0\omega}{(-\omega^2+\omega_0^2)^2+\Gamma_0^2\omega^2}L$$

$$= \frac{ne^2\omega^2\Gamma_0}{m\epsilon_0 c}\frac{1}{(-\omega^2+\omega_0^2)^2+\Gamma_0^2\omega^2}L.$$

$$P = \cancel{\frac{ne}{6\pi\epsilon_0 c^3}}\ \frac{e^2}{6\pi\epsilon_0 c^3}\frac{1}{2}Re[\ddot{\vec{x}}\cdot\ddot{\vec{x}}^*] = \frac{e^2}{12\pi\epsilon_0 c^3}\frac{e^2|E_0|^2\omega^4}{(-\omega_0^2+\omega^2)^2+\Gamma_0^2\omega^2}$$

$$= \frac{e^4}{12\pi\epsilon_0 c^3 m^2}\frac{\omega^4|E_0|^2}{(-\omega_0^2+\omega^2)^2+\Gamma_0^2\omega^2} \qquad \frac{P_0}{S}$$

$$= \frac{e^4\omega^4}{6\pi\epsilon_0^2 c^4 m^2}\frac{I_0}{(-\omega_0^2+\omega^2)^2+\Gamma_0^2\omega^2}$$

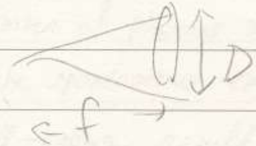
$$P_{tot} = nP\underbrace{V}_{SL}$$

$$P_{tot} = \frac{e^4\omega^4}{6\pi\epsilon_0^2 c^4 m^2}\cdot\frac{n\cdot L}{(-\omega_0^2+\omega^2)^2+\Gamma_0^2\omega^2}$$

No.
Date . .

$$R_{scatt} = \frac{e^4\omega^4 \, P_0}{6\pi\epsilon_0^2 c^4 m^2 \hbar\omega_0} \, \frac{n \cdot L}{(-\omega_0^2+\omega^2)^2+\Gamma_0^2\omega^2}$$

$$O, \ \underbrace{\tfrac{1}{2}}_{\eta_{peri}} \times 0.5 \times 5.28\times10^{-2}$$



$$\tan\theta = \frac{D}{2f}$$

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{1+\left(\frac{D}{2f}\right)^2}} = \frac{1}{\sqrt{1+\left(\frac{1}{2}\right)^2}}$$

$$= \sqrt{\frac{4}{5}}$$

No.

Date

2. Einstein's A & B coefficients

Albert Einstein (1917) showed that the blackbody radiation formula by Planck can be explained in terms of ensemble of atoms with discrete energy levels of various frequencies interacting with radiation field of continuum of frequencies. Consider two particular energy levels. Let N_a (N_b) the population in the upper (lower) energy level.

Einstein's rate equation.

$$\begin{cases} N_a + N_b = N = \text{constant} \\ \dot{N}_a = -AN_a - \overbrace{Bu(\omega)N_a}^{\text{stimulated emission}} + Bu(\omega) N_b = -\dot{N}_b \end{cases} \qquad (2-1)$$

AN_a: spontaneous emission; $Bu(\omega)N_b$: (stimulated) absorption — always stimulated.

$u(\omega)$: field energy density per unit frequency interval

In equilibrium $\dot{N}_a = \dot{N}_b = 0$

$$[A + Bu(\omega)]N_a = Bu(\omega) N_b \qquad (2-2)$$

Boltzmann distribution.

$$N_a/N_b = \exp(-\hbar\omega/k_B T) = \frac{Bu(\omega)}{A+Bu(\omega)} \qquad (2-3)$$

$$\rightarrow Bu(\omega) = \frac{Ae^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = \frac{A}{e^{\beta\hbar\omega}-1}$$

From Eqs (2-2) and (2-3),

$$u(\omega) = \frac{A/B}{e^{\hbar\omega/k_B T}-1} \qquad (2-4)$$

In order to be consitent with the Planck's formula

$$A/B = \underbrace{\hbar\omega}_{\text{energy per photon}} \underbrace{\frac{\omega^2}{\pi^2 c^3}}_{\text{density of modes (per unit volume per unit frequency interval) in free space}} \qquad (2-5)$$

Since the energy density is simply

$$u(\omega) = \hbar\omega \times (\#\text{ of photons per mode}) \times (\text{density of modes})$$

No.

Date . .

$\Rightarrow u(\omega) = \frac{A}{B} \times (\#$ number of photons per mode$) = \frac{A}{B} \cdot n$ (2-6)

$\Rightarrow n = \frac{1}{e^{\hbar\omega/k_BT} - 1}$

$\therefore \dot{N}_a = -AN_a - Bu(\omega)N_a + Bu(\omega)N_b, \quad Bu(\omega) = An$

Therefore, Einstein's rate equation can be written as

$$\dot{N}_a = -(n+1)AN_a + nAN_b \qquad (2-7)$$

where n is the number of photons in the mode, This "n+1" factor, with "1" accounting for the spontaneous emission and "n" for the the stimulated emission, is a well-known result of the quantum electrodynamics.

→ A. Einstein, "On the quantum theory of radiation", Physikalische Zeitung 18 (6), 1917

2.2. Einstein's "A" in quantum electrodynamics

Einstein's B coefficient can be derived from a simple consideration of excitation of a two-level atom by a single-mode radiation field. Although a complete description of this process requires quantum mechanics, here we just quote the ~~quant~~ equation of motion without being worried about quantum mechanics.

Consider a two-level atom with resonance frequency of ω_0 excited by a single-mode radiation field of frequency of ω and amplitude E_0. We quote the equation of motion below.

$$\dot{C}_a = i\frac{\Omega}{2}e^{-i\Delta t}C_b$$
$$\dot{C}_b = i\frac{\Omega}{2}e^{i\Delta t}C_a \qquad (2-8)$$

where $C_a(C_b)$ is excited (ground) state amplitude

$\Delta = \omega - \omega_0$, the field-atom frequency detuning

Ω is the Rabi frequency given by $\Omega = \frac{\mu E_0}{\hbar}$

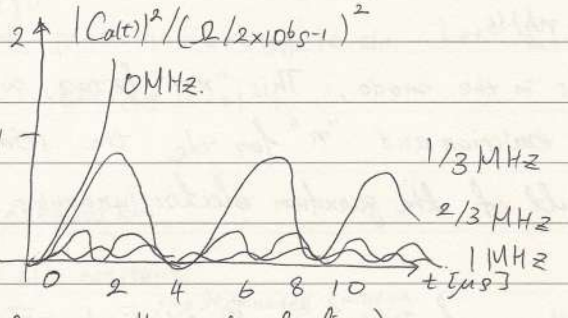
where μ is the induced dipole moment of atom along the direction of $\vec{E}_0$.

No.
Date . .

Here we are interested in the early time behavior ($\Delta t \ll 1$) when the field is turned on at $t=0$. Assume at $t=0$, $C_a(0)=0$, $C_b(0)=1$ (all in the ground state)

$$C_a(t) \simeq i\frac{\Omega}{2}\int_0^t e^{-i\Delta t}dt = \frac{\Omega}{2}\frac{e^{-i\Delta t}-1}{\Delta} \qquad (2-10)$$

$$|C_a(t)|^2 = \Omega^2\left[\frac{\sin(\Delta t/2)}{\Delta}\right]^2 \qquad (2-11)$$



→ valid for small t (early time)

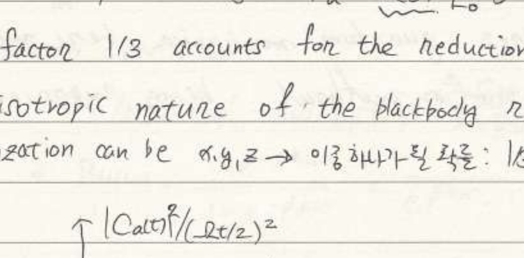
This result is for a single-frequency field of amplitude $\vec{E}_0$. To make connection with the blackbody radiation we make following substitution:

$$(\text{energy density}) = \frac{E_0^2}{8\pi} \rightarrow \int u(\omega)\,d\omega \qquad (2-12)$$

$$(\text{transition probability}) \equiv P_a = \left(\frac{1}{3}\right)\frac{8\pi}{E_0^2}\int d\omega\, u(\omega)\,|C_a(t)|^2 \qquad (2-13)$$

The factor 1/3 accounts for the reduction in the μ^2 factor due to the upolarized and isotropic nature of the blackbody radiation.

→ polarization can be x, y, z → 이중 하나가 될 확률: 1/3



$$\Rightarrow P_a = \frac{8\pi}{3\cancel{E_0^2}}\int d\omega\, u(\omega)\left[\frac{\mu^2 \cancel{E_0^2}}{\hbar^2}\frac{\sin^2\left(\frac{\Delta t}{2}\right)}{\Delta^2}\right]$$

$$\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx = \pi$$

Since $u(\omega)$ is slowly varying

$$P_a \simeq \frac{8\pi\mu^2}{3\hbar^2}u(\omega_0)\int d\Delta\,\frac{\sin^2(\Delta t/2)}{\Delta^2} = \frac{8\pi\mu^2}{3\hbar^2}u(\omega_0)(\pi t/2) \qquad (2-14)$$

$\Gamma_0 = \frac{2e^2\omega_0^2}{3mc^2} = \frac{4\mu^2\omega_0^3}{3\hbar c^3}$ $\quad m a_B^2 \omega_0^2 = \hbar\omega_0$ (energy) $\Rightarrow \frac{\omega_0}{\hbar} = \frac{1}{m a_B^2}$ $\quad \frac{1}{2}a_B^2 e^2 = \mu^2$

No. $\Rightarrow \frac{\omega_0}{\hbar} = \frac{1}{m a_B^2}$

Date $\quad e^2 = \frac{2\mu^2}{a_B^2}$

stimulated absorption rate $Bu(\omega_0)$ should be equal to P_a/t

$$\therefore \quad B = \frac{4\pi^2\mu^2}{3\hbar^2} \qquad (2\text{-}15)$$

Consequently, Einstein A coefficient or the radiative decay rate Γ_0 should be

$$A = \Gamma_0 = B\frac{\hbar\omega^3}{\pi^2 c^3} = \frac{4\mu^2\omega^3}{3\hbar c^3} \qquad (2\text{-}16)$$

This is the quantum mechanical counterpart to the classical formula, Eq (1-12).

from $A/B = \hbar\omega\frac{\omega^2}{\pi^2 c^3}$

$$\Gamma_0 = \frac{2e^2\omega_0^2}{3mc^3} = \frac{2e^2 a_B^2\omega_0^2}{3mc^3\left(\frac{\hbar^2}{mc^2}\right)^2} = \frac{2e^2 a_B^2\omega_0^2}{3c^3}\left(\frac{me^4}{\hbar^4}\right) = \frac{4(e^2 a_B^2/2)\omega_0^3}{3\hbar c^3}$$

using $\omega_0 = \frac{me^4}{\hbar^3}$

$\Rightarrow$ factor of 2 is different, but simmilar

$\mu \simeq$ charge $\times$ bohr radius

Due to ω^3 dependence, spontaneous emission is more important at optical frequencies that at microwave frequencies. If we substitute classical results,

$$\mu = e a_B = \frac{\hbar^2}{me}, \quad \hbar\omega_0 = \frac{me^4}{\hbar^2} = \frac{m a_B^2 e^2}{\hbar^2 a_B}\frac{e^2}{a_B} \qquad (2\text{-}17)$$

classical expression of atomic transition frequency.

$$A = \frac{4me^{10}}{3c^3\hbar^6} = \frac{4mc^2\alpha^5}{3\hbar} \simeq 10^9 s^{-1} \qquad (2\text{-}18)$$

for first excited hydrogen.

propagate medium gain $\rightarrow$ difficult

MASER (찰스 타운, 아더 섈로우) $\rightarrow$ LASER

$m^2 a_B^4\omega_0^2 = \hbar^2$

~~$m a_B^2\omega_0^2 = \hbar\omega_0$~~ $\quad \frac{e^2}{a_B^2}$ $\quad a_B^2 = \frac{\hbar}{m\omega_0}$ $\quad \frac{e^2}{a_B^2} = m a_B \omega_0^2$

$a_B^4 = \frac{m\omega_0}{\hbar}$ $\quad e^2 a_B^2 = \frac{\hbar^2}{m} = \frac{\hbar^2}{m a_B^2}$ $\quad m a_B = \frac{\hbar^2}{e^2}$

2.3. LASER rate equation

Suppose an ensemble of atoms are enclosed in a single-mode cavity when is resonant with the atoms. Einstein's rate equation,

$$\dot{N}_a = -nA(N_a - N_b) - AN_a \qquad (2\text{-}19)$$

$a_B = \frac{\hbar^2}{me^2}$ $\quad \omega_0 = \frac{\hbar}{m a_B^2} = \frac{me^4}{\hbar^3}$

is for the blackbody radiation of continuum of modes.

In the optical region, the stimulated emission/absorption due to the blackbody radiation is neglible since $n \ll 1$ in Eq. (2-19). For lasing, stimulated emission and absorption occur only at the cavity mode of lasing. Therefore, with a pumping term R_p, continuously pumping

$$\dot{N}_a = R_p - nK(N_a - N_b) - AN_b \qquad (2\text{-}20)$$

where n is now interpreted as the # of photons in the cavity mode and

instead of emitting to all direction $\quad (K \ll A)$

MooKEUK

No.

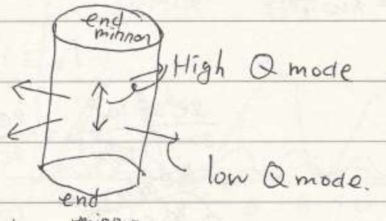
Date . .

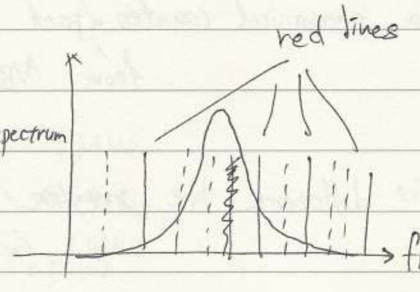
K is the laser coupling constant, which is defined as

for single mode radiation → opposite 1/3

$$K = \frac{3^* A}{p}$$ → p could be million or more

where p is the # of all cavity modes in the atomic ~~flou~~ fluorescence linewidth. The factor 3^* accounts for atomic orietation and polarization of field, ranging from 1 to 3.





↳ Simple cavity formed ~~form~~ by two mirrors. On the right red lines indicate high Q modes while blue (dotted) lines low Q modes.

In this example $p = 4$.

⇒ { for single mode → polarization can be any direction ⇒ only one is utilized
  for LASER → well defined atom interact with cavity (?)

(well defined)

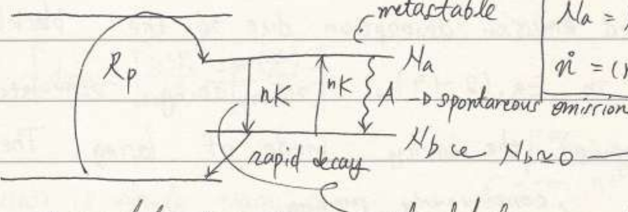
The corresponding rate equation for photons

$$\dot{n} = (n+1) K \cdot N_a - nK N_b - \gamma n \qquad (2\text{-}22)$$

($(n+1)K N_a$: stimulated, spontaneous; nKN_b: absorption)

where γ is the cavity decay rate. Eqs (2-20) and (2-21) are basic rate equations for lasering.

Below is the 4-level laser model, where the lower level decays so rapidly that the population $N_b \simeq 0$. Then the rate equations are simplified as



$$\boxed{\begin{aligned} \dot{N}_a &= R_p - nKN_a - AN_a \\ \dot{n} &= (n+1)KN_a - \gamma n \end{aligned}}$$ → photon

$N_b \simeq 0$ → simplify

⟨4-level⟩

No.
Date . .

N_a, n unknown, 2equation → quadratic equation.

$$\beta = \frac{3^*}{P} = \frac{1}{P'}$$

→ if $\beta = 1$

→ thresholdless laser

No. HW2

Date . .

$$\cancel{\rho(E)}\ \cancel{\rho(p,x)}\ \frac{dx^3dp^3}{h^3} = V\frac{dp^3}{dh^3}$$

$$dp^3 = 4\pi p^2 dp = \frac{\hbar^2\omega^2}{\pi c^2}\, d\left(\frac{h\omega}{2\pi c}\right)$$

$$\hbar\omega = pc \qquad \frac{\hbar\omega}{c} = \frac{h\omega}{2\pi c} = p$$

$$= \frac{\omega^2}{2\pi^2 c^3}\, d\omega$$

$\Rightarrow$ 광자의 편광 → ×2

$$Z = 2\times\int e^{-\beta E}\frac{dx^3 d^3p}{h^3}$$

$$= \int e^{-\beta E}\frac{\omega^2}{\pi^2c^3}\, d\omega\ V$$

→ density of mode $= \dfrac{\omega^2}{\pi^2 c^3}$

i) $r \gg 1$

$$n_{ss} \simeq \frac{p'}{2}\left[(r-1) + r + \left(\frac{4}{p'}-2\right)\frac{1}{2}\right]$$

$$\simeq \frac{p'}{2}(r-2)$$

$$= p'(r-1)$$

$$n_{ss} = \frac{p'}{2}\left[(r-1) + \sqrt{(r-1)^2 + 4r/p'}\right]$$

i) $r<1$

$$n_{ss} = \frac{p'}{2}\left[(r-1) + \sqrt{1+\left(\frac{4}{p'}-2\right)r + r^2}\right]$$

$$\simeq \frac{p'}{2}\left[(r-1) + 1 + \frac{1}{2}\left(\frac{4}{p'}-2\right)r\right]$$

$$= \frac{p'}{2}\left[\frac{2}{p'} + \frac{2}{p'}r\right]$$

$$\frac{p'}{2}\left((x-1) + \sqrt{(x-1)^2 + \frac{4x}{p'}}\right) + A = 0$$

$$\left((x-1)+2\right)^2 \quad \frac{4x}{p'}$$

$$= (x-1)^2 + 4(x-1) + 4$$

$$f'(x) = 1 + \frac{x-1+\frac{2}{p'}}{\sqrt{(x-1)^2+\frac{4x}{p'}}} = 0$$

$$\left(\frac{1}{p'}-1\right)x = 0$$

$$\left(x-1+\frac{2}{p'}\right)^2 = (x-1)^2 + \frac{4}{p'}(x-1) + \left(\frac{2}{p'}\right)^2$$

$$(x-1)^2 + \frac{4x}{p'} = \left[(x-1) + \frac{2A}{p'}\right]^2 = 1 + r$$

$$(x-1)^2 + \frac{4x}{p'} = (x-1)^2 + \frac{4A}{p'}(x-1) + \left(\frac{2A}{p'}\right)^2$$

$$\frac{4}{p'} = \frac{4}{p'^2} \qquad p'^2 = p'$$

$$-\frac{4A}{p'} + \left(\frac{2A}{p'}\right)^2 = 0 \qquad \frac{A}{p'} = 1$$

$$(x-1)^2 + \frac{4x}{p'}$$

$$= \left[(x-1) + \frac{2A}{p'}\right]^2$$

$$= (x-1)^2 + \frac{4A}{p'}(x-1) + \left(\frac{2A}{p'}\right)^2$$

$$\frac{1}{p'}(1-A)x = -\frac{A}{p'} + \left(\frac{A}{p'}\right)^2 \qquad A = 1$$

$$x = \frac{1}{1-A}\left[-A + \frac{A^2}{p'}\right]$$

threshold ; $\Delta N \geq \Delta N_{th} = \dfrac{\pi \Delta \omega_a}{\lambda^2 \gamma_{rad} L_m}$ No. Date

$\dfrac{dN_a}{dt} = R_p - nKN_a - AN_a$ $\quad A \quad \ln\left(\dfrac{1}{R_1 R_2}\right)$

$\dot{n} = (n+1)KN_a - \gamma n$

$\Rightarrow N_a = \dfrac{R_p}{(nK+A)}$ $\quad K = \dfrac{A}{p'}$

$0 = (n+1)\dfrac{R_p K}{(nK+A)} - \gamma n$ $\quad r = \dfrac{R_p}{\gamma p'}$

$0 = (n+1)R_p K - \gamma n(nK+A)$ $\quad f(x) = (x-1) + \sqrt{(x-1)^2 + \dfrac{4x}{p'}}$

$f'(x) = 1 + \dfrac{2x-2+\frac{4}{p'}}{2\sqrt{(x-1)^2+\frac{4x}{p'}}}$

$\cancel{\gamma n^2}$

$\gamma K n^2 + (\gamma A - K R_p)n - R_p K = 0$ $\quad \sqrt{g+1} - 1 = f(0)$

$\Rightarrow \cancel{\dfrac{\gamma}{p'}n^2 + \dfrac{1}{p'}(\gamma -}$

$\cancel{f(x) = (x-1) + \sqrt{(x-1)^2 + g}}$

$\cancel{f'(x) = 1 + \dfrac{(x-1)}{\sqrt{(x-1)^2+g}} = 0}$

$N_a = \dfrac{R_p}{nK+A}$

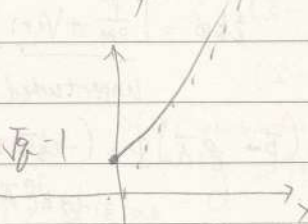
$(n+1)\cdot K \dfrac{R_p}{nK+A} - \gamma n = 0$ $\quad \cancel{(x-1)^2 = (x-1)^2 + g}$

$(n+1)KR_p - \gamma n(nK+A) = 0$

$\gamma n(nK+A) - (n+1)KR_p = 0$ $\quad K = \dfrac{A}{p'}$

$\gamma \cdot Kn^2 + (\gamma A - KR_p)n - KR_p = 0$

$\dfrac{\gamma}{p'}n^2 + \left(\gamma - \dfrac{R_p}{p'}\right)n - \dfrac{R_p}{p'} = 0$



$n^2 + \left(p' - \dfrac{R_p}{\gamma}\right)n - \dfrac{R_p}{\gamma} = 0$

$\Rightarrow n^2 + p'\cdot(1-r)n - rp' = 0$

$\therefore n = \dfrac{1}{2}\left((r-1) + \sqrt{(r-1)^2 + \dfrac{4r}{p'}}\right)p'$ $\quad \dfrac{A}{}$

$\Rightarrow N_a = \dfrac{R_p = r\cdot \gamma p'}{A\frac{1}{2}\left((r-1) + \sqrt{(r-1)^2 + \frac{4r}{p'}}\right) + A}$ $\quad x = 0$

R_p

No.

Date

3. Semiclassical theory of atom-field interaction.

In the semiclassical theory, the atom is treated quantum-mechanically while the field is treated classically.

3.1 Electric dipole interaction

$\left\{\begin{array}{l}\text{- E1 approximation} \\ \text{- Equation for } C_a \text{ and } C_b \\ \text{- Equation for density matrix} \\ \text{- Inclusion of damping}\end{array}\right.$

Consider a hydrogen-like atom subject to an EM field. Schrödinger equation is

$$i\hbar\,\partial_t\Psi(\vec{x},t) = \left[\frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{x},t)\right)^2 + V(\vec{x},t)\right]\Psi(\vec{x},t) \qquad \Rightarrow e=-|e|<0 \quad (3-1)$$

where $\vec{A}(\vec{x},t)$ is a vector potential associated with the applied EM field.

Suppose the wavelength of the EM field is much larger than the size of atom. (i.e. dipole approximation). Then we can neglect the variation of $\vec{A}(\vec{x},t)$ over the atom and replace it with $\vec{A}(\vec{R},t)$ with $\vec{R}$ the position vector of the nucleus.

Define $\phi(\vec{x},t)$ such that

$$\Psi(\vec{x},t) = \exp\left(\frac{ie}{\hbar c}\vec{A}(\vec{R},t)\cdot\vec{x}\right)\phi(\vec{x},t) \qquad (3-2)$$

Then eq (3-1) becomes (using $\vec{p} = -i\hbar\vec{\nabla}$)

$$i\hbar\left[\frac{ie}{\hbar c}\dot{\vec{A}}\cdot\vec{x}\,\phi + \dot{\phi}\right] = \left[\frac{p^2}{2m} + V(r)\right]\phi \qquad (3-3)$$

Since $\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ (Coulomb gauge), we obtain a new form of Schrödinger equation

$$i\hbar\dot{\phi} = \Big[\underbrace{\frac{p^2}{2m} + V(r)}_{\text{unperturbed Hamiltonian}} \underbrace{-e\overbrace{\vec{x}}^{\text{dipole moment}}\cdot\vec{E}}\Big]\phi = (H_0 + \underbrace{H_I}_{\text{interaction Hamiltonian}})\phi \qquad (3-4)$$

※. $\left(\vec{p} - \frac{e}{c}\vec{A}\right)\Psi = \left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)e^{\frac{ie}{\hbar c}\vec{A}\cdot\vec{x}}\phi$

$= e^{\frac{ie}{\hbar c}\vec{A}\cdot\vec{x}}\left[\frac{e}{c}\vec{A}\phi - i\hbar\vec{\nabla}\phi - \frac{e}{c}\vec{A}\phi\right] = e^{\frac{ie}{\hbar c}\vec{A}\cdot\vec{x}}(-i\hbar\vec{\nabla})\phi$

⇒ 대입하여 정리하면 (3-2) → (3-4) 된다.

Hence, the atom-field interaction, represented by the interaction Hamiltonian H_I, is in the form of electric dipole interaction:

$$H_I = -\vec{\mu}\cdot\vec{E} \qquad (3-5)$$

where $\vec{\mu}$ is the dipole moment operator of the atom. Since it is customary to use ψ for wavefunction, we will use Eq (3-4) with ϕ replaced with ψ from now on call it our Schrödinger equation.

3.2. Equation of motion

Consider an atom is excited by an EM field of frequency ω, nearly resonant to atomic transition from the ground state to one of the excited states.

state a ———— $\hbar\omega_a$

$\hbar\omega_0 = \hbar(\omega_a - \omega_b)$

state b ———— $\hbar\omega_b$ $\Rightarrow$ two-level system

If other excited levels are way off resonance, the atom can be considered as a two-level system, So, we write

$$\psi(\vec{x},t) = C_a(t)\, e^{-i\omega_a t} \underbrace{u_a(\vec{x})}_{\text{eigen fuction for state a}} + C_b(t)\, e^{-i\omega_b(t)} \underbrace{u_b(\vec{x})}_{\text{eigen fuction for state b}} \qquad (3-6)$$

or equivalently using Dirac's notation

$$|\psi\rangle = C_a e^{-i\omega_a t}|a\rangle + C_b e^{-i\omega_b t}|b\rangle \qquad (3-7)$$

$$i\hbar\frac{d}{dt}|\psi\rangle = (H_0 + H_I)|\psi\rangle \qquad (3-8)$$

if DC electric field $\Rightarrow$ H_0, or H_I 포함

with $(H_0)_{aa} = \langle a|H_0|a\rangle = \hbar\omega_a$ $\cancel{= \hbar\omega_a + \hbar\delta_a}$ $\quad (H_0)_{bb} = \hbar\omega_b$ $\cancel{= \hbar\omega_b + \hbar\delta_b}$, $(H_0)_{ab} = 0$

$(H_I)_{ab} = -\mu E_0 \cos\omega t$ $\cancel{= -\mu_0 E_0 \cos\omega t - \mu\mathcal{E}}$ $= (H_I)^*_{ba}$, $(H_I)_{aa} = (H_I)_{bb} = 0$

where μ is the induced dipole moment along the electric field direction (of unit vector $\hat{e}$)

fix phase $\mathcal{E}$ $\quad \mu = \langle a|e\vec{x}\cdot\hat{e}|b\rangle \qquad (3-10)$

~~tt~~ Without loss of generality we can assume μ to be real so is E_0.

Substitute~~ing~~ Eq (3-7) into (3-8)

No.

Date

$$|\Psi\rangle = C_a e^{-i\omega_a t}|a\rangle + C_b e^{-i\omega_b t}|b\rangle$$

$$\downarrow \quad \rightarrow i\hbar\frac{d}{dt}|\Psi\rangle = (H_0+H_I)|\Psi\rangle$$

$$i\hbar(\dot{C}_a - i\omega_a C_a)e^{-i\omega_a t}|a\rangle + i\hbar(\dot{C}_b - i\omega_b)e^{-i\omega_b t}|b\rangle$$

$$= C_a e^{-i\omega_a t}(H_0+H_I)|a\rangle + C_b e^{-i\omega_b t}(H_0+H_I)|b\rangle \quad (3-11)$$

Applying $\langle a|$ to Eq. (3-11) and using (3-9)

$$\dot{C}_a = -\frac{i}{\hbar}C_b e^{i\omega_0 t}(-\mu E_0\cos\omega t) = i\frac{\Omega}{2}\left[e^{i(\omega_0+\omega)t} + e^{i(\omega_0-\omega)t}\right]C_b \quad (3-12)$$

expand by $\cos x = \frac{e^{ix}+e^{-ix}}{2}$

$$(i\hbar\dot{C}_a + \hbar\omega_a C_a)e^{-i\omega_a t} = C_a e^{-i\omega_a t}(H_0)_{aa} + C_b e^{-i\omega_b t}(H_I)_{ab}$$

Similarly

resonant term $\propto \frac{1}{\omega_0-\omega}$ $\quad$ anti- 〃 $\propto \frac{1}{\omega_0+\omega}$

$$\dot{C}_b = i\frac{\Omega}{2}\left[e^{-i(\omega_0-\omega)t} + e^{-i(\omega_0+\omega)t}\right]C_a \quad (3-13)$$

where Ω is the Rabi frequency as defined in Eq (2-9)

The sum frequency terms (or anti-resonant terms) in Eqs (3-13) and (3-14) contribute to the solutions negligibly in the optical domain, and thus can be neglected. This approximation called the "rotating wave approximation" (RWA).

Under the rotating wave approximation, we get the equations of motion ($\Delta = \omega - \omega_0$) (Eq 2-8)

$$\dot{C}_a = i\frac{\Omega}{2}e^{-i\Delta t}C_b, \quad \dot{C}_b = i\frac{\Omega}{2}e^{i\Delta t}C_a \quad (3-14)$$

3.3 Density matrix

For an observable $\hat{O}$, the expectation value is

$$\langle\hat{O}\rangle = \langle\Psi|\hat{O}|\Psi\rangle$$

$$= C_a^* C_a O_{aa} + C_b^* C_b O_{bb} + C_a^* C_b e^{i\omega_0 t}O_{ab} + C_b^* C_a e^{-i\omega_0 t}O_{ba} \quad (3-15)$$

~~con~~

Containing bilinear combinations of the amplitudes. Thus, it is convenient to define a density matrix whose matrix elements are these bilinear products and to have an equation of motion for the density matrix directly. The density matrix for our two-level system is defined as

$$\rho = |\psi\rangle\langle\psi| \quad (3-16)$$
$$= C_a C_a^* |a\rangle\langle a| + C_a C_b^* e^{-i\omega_0 t} |a\rangle\langle b| + C_b C_a^* e^{i\omega_0 t} |b\rangle\langle a| + C_b C_b^* |b\rangle\langle b|$$

so by definition

$$\rho_{aa} = C_a C_a^*,\ \underbrace{\rho_{ab} = C_a C_b^* e^{-i\omega_0 t}},\ \underbrace{\rho_{ba} = C_b C_a^* e^{i\omega_0 t}}_{\text{explicit time dependence}},\ \rho_{bb} = C_b C_b^* \quad (3-17)$$

state a
$\hbar\omega_0$
state b

$C_a^* C_a(t),\ \overset{C_b^*(t)}{C_b(t)} \Rightarrow$ Slowly varying part (no explicit time dependence)

In terms of the density matrix, the expectation value is given by

$$\langle \hat{O} \rangle = \rho_{aa} O_{aa} + \rho_{bb} O_{bb} + \rho_{ba} O_{ab} + \rho_{ab} O_{ba} = \mathrm{Tr}(\rho \hat{O}) \quad (3-18)$$

where $\mathrm{Tr}()$ is the trace operator.

The equation of motion for the density matrix is (using $H^\dagger = H$)

$$i\hbar \dot{\rho} = \left(i\hbar \frac{d}{dt}|\psi\rangle\right)\langle\psi| + |\psi\rangle\left(i\hbar\frac{d}{dt}\langle\psi|\right) = H|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi|H^\dagger = [H, \rho] \quad (3-19)$$

or

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] \quad (3-20)$$

The density matrix above is defined (can define single wave function.) for a system where wavefunction exists. Such system is called "pure". One can still define a density matrix for a statistical ensemble of pure systems, referred to as being "mixed". {can't define w/ single wave function, but can define density matrix.}

No.

Date

For the mixed case, the density matrix is defined as

$$\rho = \sum_{\Psi} P_{\Psi} |\Psi\rangle\langle\Psi| \qquad (3\text{-}21)$$

where P_{Ψ} is the probability that the system has the state vector $|\Psi\rangle$.

Explicitly, the equations of motion are

$$\dot{\rho}_{aa} = -\frac{i}{\hbar}[H,\rho]_{aa} = -\frac{i}{\hbar}\left[(H\rho)_{aa} - (\rho H)_{aa}\right] \qquad (3\text{-}22)$$

$$= -\frac{i}{\hbar}\left(H_{aa}\rho_{aa} + H_{ab}\rho_{ba} - \rho_{aa}H_{aa} - \rho_{ab}H_{ba}\right)$$

$$= -\frac{i}{\hbar}H_{ab}\rho_{ba} + c.c.$$

$$= -\frac{i}{\hbar}(-\mu E_0 \cos\omega t)(e^{i\omega_0 t} c_a^* c_b) + c.c.$$

$$= \frac{i}{2}\Omega(e^{i\omega t} + e^{-i\omega t}) e^{i\omega_0 t} c_a c_a^* + c.c.$$

$$\cong \frac{i}{2}\Omega e^{-i\omega t} e^{i\omega_0 t} c_a^* c_b + c.c. \Rightarrow \text{only near resonant term}$$

$$= \frac{i}{2}\Omega \underbrace{e^{-i\omega t}\rho_{ba}}_{\text{slowly varying}} + c.c. \quad (\because \rho_{ba} \propto e^{i\omega_0 t})$$

→ near resonance : slowly varying

In the last step RWA is used. Practically, RWA is consistent with

$$H_{ab} \cong -\frac{1}{2}\mu E_0 e^{-i\omega t} \qquad (3\text{-}23)$$

Similarly, one can show $\rho_{ab} \propto e^{-i\omega_0 t}$

$$\dot{\rho}_{ab} = -i\omega_0 \rho_{ab} - \frac{i}{2}\Omega e^{-i\omega t}(\rho_{aa} - \rho_{bb}) \qquad (3\text{-}24)$$

$$\dot{\rho}_{bb} = -\frac{i}{2}\Omega e^{i\omega t}\rho_{ba} + c.c. = -\dot{\rho}_{aa}$$

$\because \rho_{aa} + \rho_{bb} = \text{constant}$

Eqs. (3-22) and (3-24) are the equations of motion for the density matrix without damping.

No.

Date . .

3.4 ~~Inclusing~~ Inclusion of decay

The decay of population can ~~occur~~ occur by spontaneous emissions, inelastic scattering and /or energy loss to non-radiative channels. Denoting the total decay rate of the upper (lower) state as Γ_a (Γ_b),

diagonal elements
$$\begin{cases} \dot{\rho}_{aa} = \underbrace{-\Gamma_a \rho_{aa}} + \frac{i}{2}\Omega e^{-i\omega t}\rho_{ba} + c.c \\ \dot{\rho}_{bb} = \underbrace{-\Gamma_b \rho_{bb}} + \frac{i}{2}\Omega e^{i\omega t}\rho_{ab} + c.c \end{cases} \qquad (3\text{-}25)$$

where the decay of the upper state into the lower state is neglected.

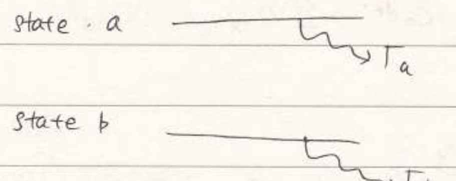


Fig 7.

$|C_a|^2 \propto e^{-\Gamma_a t}$

$\Rightarrow C_a \propto e^{-\frac{\Gamma_a}{2}t}$

$C_b \propto e^{-\frac{\Gamma_b}{2}t}$

In terms of probability amplitude these are equivalent to

$$\dot{C}_a = -(\Gamma_a/2)C_a + \frac{i}{2}\Omega e^{-i\Delta t}C_b \qquad (3\text{-}26)$$
$$\dot{C}_b = -(\Gamma_b/2)C_b + \frac{i}{2}\Omega e^{i\Delta t}C_a$$

$$\Downarrow$$
$$\rho_{ab} = C_a C_b^* \propto e^{-\frac{(\Gamma_a+\Gamma_b)t}{2}}$$

Hence, decay of the off-diagonal element is given by

off diagonal element
$$\left\{ \dot{\rho}_{ab} = -\gamma_{ab}\rho_{ab} - i\omega_0\rho_{ab} - \frac{i}{2}\Omega e^{-i\omega t}(\rho_{aa}-\rho_{bb}) \right. \qquad (3\text{-}27)$$

where

$$\underbrace{\gamma_{ab} = \frac{\Gamma_a+\Gamma_b}{2}} \qquad (3\text{-}28)$$

If the lower state does not decay (i.e. the ground state) and if the decay of the upper state is purely radiative

$$\dot{\rho}_{aa} = \underbrace{-\Gamma_0\rho_{aa}} + \frac{i}{2}\Omega e^{-i\omega t}\rho_{ba} + c.c \quad \text{does not decay}$$
$$\dot{\rho}_{bb} = -\dot{\rho}_{aa} = \Gamma_0\rho_{aa} - \frac{i}{2}\Omega e^{-i\omega t}\rho_{ba} + c.c \qquad (3\text{-}29)$$

Only C_a decays exponentially at the rate of $\Gamma_0/2$ whereas C_b does not, and thus

$$\dot{\rho}_{ab} = \underbrace{-\frac{\Gamma_0}{2}}\rho_{ab} - i\omega_0\rho_{ab} - \frac{i}{2}\Omega e^{-i\omega t}(\rho_{aa}-\rho_{bb}) \qquad (3\text{-}30)$$

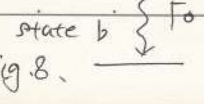


Fig. 8.

No.

Date . .

Problem4

11.11

$$C_a(t) = C_a^{(0)}(t) + \Omega C_a^{(1)}(t) + \Omega^2 C_a^{(2)}(t) + \cdots$$

$$C_b(t) = C_b^{(0)}(t) + \Omega C_b^{(1)}(t) + \Omega^2 C_b^{(2)}(t) + \cdots$$

$$\frac{dC_a(t)}{dt} = i\frac{\Omega}{2}\left[e^{i(\omega+\omega_0)t} + e^{i(\omega_0-\omega)t}\right]C_b$$

$$\frac{dC_b(t)}{dt} = i\frac{\Omega}{2}\left[e^{-i(\omega_0-\omega)t} + e^{-i(\omega_0+\omega)t}\right]C_a$$

$C_a(0) = 0$

$$C_a(t) = \int_0^t i\frac{\Omega}{2}\left[e^{i(\omega+\omega_0)t'} + e^{i(\omega_0-\omega)t'}\right]C_b\,dt'$$

$C_b(0) = 1$

$$C_b(t) = \int_0^t i\frac{\Omega}{2}\left[e^{-i(\omega_0-\omega)t'} + e^{-i(\omega_0+\omega)t'}\right]C_a\,dt' + 1$$

$$\Rightarrow C_a(t) = \cancel{\int_0^t\int_0^{t'}}$$

$$= \int_0^t i\frac{\Omega}{2}\left[e^{i(\omega_0+\omega)t'} + e^{i(\omega_0-\omega)t'}\right]\left\{\int_0^{t'} i\frac{\Omega}{2}\left[e^{-i(\omega_0-\omega)t''} + e^{-i(\omega_0+\omega)t''}\right]\overset{C_a(t)}{}\,dt'' + 1\right\}dt'$$

$$= C_a^{(0)}(t) + \Omega C_a^{(1)}(t) + \Omega^2 C_a^{(2)}(t) + \cdots$$

$$= \frac{\Omega}{2(\omega_0+\omega)}\left(e^{i(\omega_0+\omega)t} - 1\right) + \frac{\Omega}{2(\omega_0-\omega)}\left(e^{i(\omega_0-\omega)t} - 1\right)$$

$$+ \int_0^t i\frac{\Omega}{2}\left[e^{i(\omega_0+\omega)t'} + e^{i(\omega_0-\omega)t'}\right]\cdot\int_0^{t'} i\frac{\Omega}{2}\left[e^{-i(\omega_0-\omega)t''} + e^{-i(\omega_0+\omega)t''}\right]C_a(t'')\,dt''dt'$$

$$\Rightarrow C_a^{(1)}(t) = \frac{1}{2(\omega_0+\omega)}\left(e^{i(\omega_0+\omega)t} - 1\right) + \frac{1}{2(\omega_0-\omega)}\left(e^{i(\omega_0-\omega)t} - 1\right)$$

$$\simeq \frac{1}{2(\omega_0-\omega)}\left(e^{i(\omega_0+\omega)t} - 1\right) \quad (\text{we assume } \omega_0 \simeq \omega)$$

$$C_b(t) \simeq 1 + \frac{i\Omega^2}{4(\omega_0-\omega)}\int_0^t\left[e^{2i\omega t'} + 1 - e^{-i(\omega_0-\omega)t'} - e^{-i(\omega_0+\omega)t'}\right]dt'$$

$$= 1 + \frac{i\Omega^2}{4(\omega_0-\omega)}\left\{\frac{1}{2i\omega}\left(e^{2i\omega t} - 1\right) + t + \frac{1}{i\underset{(\omega_0-\omega)}{\wedge}}\left(e^{-i(\omega_0-\omega)t} - 1\right) + \frac{1}{i(\omega_0+\omega)}\left(e^{-i(\omega_0+\omega)t} - 1\right)\right\}$$

$$\simeq 1 + \frac{i\Omega^2}{4(\omega_0-\omega)}\left\{t + \frac{1}{i(\omega_0-\omega)}\left(e^{-i(\omega_0-\omega)t} - 1\right)\right\}$$

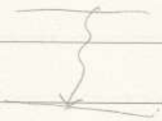
$\Rightarrow$ all from $\exp(-i(\omega_0-\omega)t)$ terms

$$\Rightarrow C_b^{(0)}(t) = 1,\ C_b^{(1)}(t) = 0,\ C_b^{(2)}(t) =$$

$$C_a^{(0)}(t) = 0,\ C_a^{(1)}(t) = \frac{1}{2(\omega_0-\omega)}\left(e^{i(\omega_0+\omega)t} - 1\right),\ C_a^{(2)}(t) = 0$$

Problem5.

I) $|\psi\rangle = C_a(t)|a\rangle + C_b(t)e^{i\varphi}|b\rangle$



$\rho_1 = |\psi\rangle\langle\psi| = \begin{bmatrix} |C_a(t)|^2 & C_a(t)C_b^*(t)e^{-i\varphi} \\ C_a^*(t)C_b(t)e^{i\varphi} & |C_b(t)|^2 \end{bmatrix}$ — ~~identity matrix~~

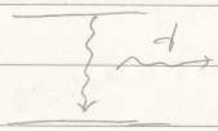
$|\psi(\vec{x})|^2 = |C_a(t)\psi_a(\vec{x}) + C_b(t)\psi_b(\vec{x})|^2$ 아마 Coherent 하지 않는~~도~~??

$= |C_a(t)|^2|\psi_a(\vec{x})|^2 + |C_b(t)|^2|\psi_b(\vec{x})|^2$

$+ 2\mathrm{Re}\left[C_a(t)C_b^*(t)e^{-i\varphi}\psi_a(\vec{x})\psi_b^*(\vec{x})\right]$

$= \rho_{DC}(\vec{x}) + \rho_{AC}(\vec{x})$

I)



$C_b(t)e^{i\varphi}$ $\langle\mu\rangle^2$

No.
Date

$$\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}\rho_{aa}&\rho_{ab}\\\rho_{ba}&\rho_{bb}\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix} \quad \sigma_+\rho=\begin{bmatrix}\rho_{ba}&\rho_{bb}\\0&0\end{bmatrix}$$

$$\begin{bmatrix}0&0\\1&0\end{bmatrix} \quad \begin{bmatrix}0&0\\1&0\end{bmatrix} \quad \sigma_-\rho=\begin{bmatrix}0&0\\\rho_{aa}&\rho_{ab}\end{bmatrix}$$

Problem6

$$\rho\sigma_+=\begin{bmatrix}0&\rho_{aa}\\0&\rho_{ba}\end{bmatrix}$$

$$\rho\sigma_-=\begin{bmatrix}\rho_{ab}&0\\\rho_{bb}&0\end{bmatrix}$$

~~$\dot\rho_{aa}=$~~

$\sigma_-\rho\sigma_+$

$$=\begin{bmatrix}0&0\\\rho_{aa}&\rho_{ab}\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}$$

$$\rho_{ba}=\begin{bmatrix}0&0\\0&\rho_{aa}\end{bmatrix}$$

$$\dot C_a=-(\Gamma_a/2)C_a+\frac{i}{2}\Omega e^{-i\Delta t}C_b$$
$$\dot C_b=-(\Gamma_b/2)C_b+\frac{i}{2}\Omega e^{i\Delta t}C_a$$

$$C_a^*\dot C_a+C_a\dot C_a^*=-(\Gamma_a/2)|C_a|^2+\frac{i}{2}\Omega e^{-i\Delta t}C_bC_a^*$$
$$-(\Gamma_a/2)|C_a|^2-\frac{i}{2}\Omega e^{+i\Delta t}C_b^*C_a$$

ρ_{ab}

$$=-\Gamma_a\rho_{aa}+\Omega\cos(\Delta t)C_bC_a^*$$
$$=-\Gamma_a\rho_{aa}+\Omega\cos(\Delta t)\rho_{ba}$$
$$=\frac{d}{dt}(C_a^*C_a)$$
$$=\dot\rho_{aa}$$

$\Delta=\omega-\omega_0$

$\Rightarrow\Delta+\omega_0=\omega$

$e^{i\omega t}\cdot e^{-i\omega_0 t}$

$$\dot\rho_{bb}=C_b^*\dot C_b+\dot C_b^*C_b$$
$$=-\Gamma_b\rho_{bb}+\frac{i}{2}\Omega e^{-i\Delta t}C_b^*C_a-\frac{i}{2}\Omega e^{+i\Delta t}C_bC_a^*$$
$$=-\Gamma_b\rho_{bb}+\frac{i}{2}\Omega e^{-i\Delta t}\rho_{ab}-\frac{i}{2}\Omega e^{+i\Delta t}\rho_{ab}$$

$$\dot\rho_{ab}=(\dot{C_aC_b^*e^{-i\omega_0 t}})$$
$$=\dot C_aC_b^*e^{-i\omega_0 t}+C_a\dot C_b^*e^{-i\omega_0 t}$$
$$-i\omega\underbrace{C_aC_b^*e^{-i\omega_0 t}}_{\rho_{ab}}$$

$$\dot\rho_{ab}=\cancel{\frac{d}{dt}(C_a^*C_b)}$$
$$=\frac{d}{dt}(C_a(t)C_b^*(t))$$
$$=\dot C_a(t)C_b^*(t)+C_a(t)\dot C_b^*(t)$$
$$=-(\Gamma_a/2)C_aC_b^*+\frac{i}{2}\Omega e^{-i\Delta t}|C_b|^2$$
$$-(\Gamma_b/2)C_aC_b^*-\frac{i}{2}\Omega e^{i\Delta t}|C_a|^2$$
$$=-[(\Gamma_a+\Gamma_b)/2]\rho_{ab}+\frac{i}{2}\Omega e^{-i\Delta t}(\rho_{bb}-\rho_{aa})$$
$$=-\gamma_{ab}\rho_{ab}+\frac{i}{2}\Omega e^{-i\Delta t}(\rho_{bb}-\rho_{aa})$$

☆ CJ Foot Bloch eq 풀이 $\sigma_+=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $\sigma_-=\begin{pmatrix}0&0\\1&0\end{pmatrix}$

No.
Date . .

Problem 7. $H=\frac{1}{2}\hbar\omega_0\sigma_z-\frac{1}{2}\hbar\Omega(\sigma_+e^{-i\omega t}+\sigma_-e^{+i\omega t})$

$$\dot\rho=-\frac{i}{\hbar}[H,\rho]+\frac{\Gamma_0}{2}(2\sigma_-\rho\sigma_+-\sigma_+\sigma_-\rho-\rho\sigma_+\sigma_-)$$

$$[H,\rho]=H\rho-\rho H$$

$$=\frac{1}{2}\hbar\omega_0\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}\rho_{aa}&\rho_{ab}\\\rho_{ba}&\rho_{bb}\end{bmatrix}-\frac{1}{2}\hbar\omega_0\begin{bmatrix}\rho_{aa}&\rho_{ab}\\\rho_{ba}&\rho_{bb}\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

$$-\frac{1}{2}\hbar\Omega e^{-i\omega t}\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}\rho_{aa}&\rho_{ab}\\\rho_{ba}&\rho_{bb}\end{bmatrix}-\frac{1}{2}\hbar\Omega e^{+i\omega t}\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}\rho_{aa}&\rho_{ab}\\\rho_{ba}&\rho_{bb}\end{bmatrix}$$

$$+\frac{1}{2}\hbar\Omega e^{-i\omega t}\begin{bmatrix}\cancel{0}&\cancel{1}\\\cancel{0}&\cancel{0}\end{bmatrix}\begin{bmatrix}\rho_{aa}&\rho_{ab}\\\rho_{ba}&\rho_{bb}\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}+\frac{1}{2}\hbar\Omega e^{i\omega t}\begin{bmatrix}\cancel{0}&\cancel{0}\\\cancel{1}&\cancel{0}\end{bmatrix}\begin{bmatrix}\rho_{aa}&\rho_{ab}\\\rho_{ba}&\rho_{bb}\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}$$

$$=\frac{1}{2}\hbar\omega_0\begin{bmatrix}\cancel{\rho_{aa}}&\rho_{ab}\\-\rho_{ba}&\cancel{-\rho_{bb}}\end{bmatrix}-\frac{1}{2}\hbar\omega_0\begin{bmatrix}\cancel{\rho_{aa}}&-\rho_{ab}\\\rho_{ba}&\cancel{-\rho_{bb}}\end{bmatrix}$$

$$-\frac{1}{2}\hbar\Omega e^{-i\omega t}\begin{bmatrix}\rho_{ba}&\rho_{bb}\\0&0\end{bmatrix}-\frac{1}{2}\hbar\Omega e^{+i\omega t}\begin{bmatrix}0&0\\\rho_{aa}&\rho_{ab}\end{bmatrix}$$

$$+\frac{1}{2}\hbar\Omega e^{-i\omega t}\begin{bmatrix}0&\rho_{aa}\\0&\rho_{ba}\end{bmatrix}+\frac{1}{2}\hbar\Omega e^{i\omega t}\begin{bmatrix}\rho_{ab}&0\\\rho_{bb}&0\end{bmatrix}$$

$$=\frac{1}{2}\hbar\omega_0\begin{bmatrix}0&2\rho_{ab}\\-2\rho_{ba}&0\end{bmatrix}+\frac{1}{2}\hbar\Omega e^{-i\omega t}\begin{bmatrix}-\rho_{ba}&\rho_{aa}-\rho_{bb}\\0&\rho_{ba}\end{bmatrix}+\frac{1}{2}\hbar\Omega e^{i\omega t}\begin{bmatrix}\rho_{ab}&0\\(\rho_{bb}-\rho_{aa})&-\rho_{ab}\end{bmatrix}$$

$$=[H,\rho]$$

$$\sigma_-\sigma_+=\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

$$\sigma_+\sigma_-=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix}$$

$$\frac{\Gamma_0}{2}(2\sigma_-\rho\sigma_+-\sigma_+\sigma_-\rho-\rho\sigma_+\sigma_-)$$

$$=\frac{\Gamma_0}{2}\left(2\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}\rho_{aa}&\rho_{ab}\\\rho_{ba}&\rho_{bb}\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}-\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}\rho_{aa}&\rho_{ab}\\\rho_{ba}&\rho_{bb}\end{bmatrix}-\begin{bmatrix}\rho_{aa}&\rho_{ab}\\\rho_{ba}&\rho_{bb}\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)$$

$$=\frac{\Gamma_0}{2}\left(\begin{bmatrix}0&0\\0&2\rho_{aa}\end{bmatrix}-\begin{bmatrix}\rho_{aa}&\rho_{ab}\\0&0\end{bmatrix}-\begin{bmatrix}\rho_{aa}&0\\-\rho_{ba}&0\end{bmatrix}\right)=\frac{\Gamma_0}{2}\begin{bmatrix}-2\rho_{aa}&-\rho_{ab}\\-\rho_{ba}&2\rho_{aa}\end{bmatrix}$$

$$\dot\rho_{aa}=+\frac{i}{2}\Omega e^{-i\omega t}\rho_{ba}-\frac{i}{2}\Omega e^{i\omega t}\rho_{ab}-\Gamma_0\rho_{aa}$$

$$\dot\rho_{bb}=\frac{i}{2}\Omega(\rho_{ab}e^{i\omega t}-\rho_{ba}e^{-i\omega t})+\Gamma_0(\rho_{aa})$$

$$\dot\rho_{ab}=-i\omega_0\rho_{ab}-\frac{i}{2}\Omega e^{-i\omega t}(\rho_{aa}-\rho_{bb})-\frac{\Gamma_0}{2}\rho_{ab}$$

$$=\begin{bmatrix}0&0\\\rho_{aa}&\rho_{ab}\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&\rho_{aa}\end{bmatrix}$$

$$\sigma_-\rho\sigma_+=\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}\rho_{aa}&\rho_{ab}\\\rho_{ba}&\rho_{bb}\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}$$

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4. Spectral Line Broadening

- 3 Maser broadening
- power bradening
- collisional broadening
- Doppler broadening.

4.1 Power Broadening.

Power broadening comes from the saturability of atomic oscillator.

Consider a two-level system described by Eqs (3-29) and (3-30), which has the non-decaying ground state. An example is $^1S_0 - ^1P_1$ transition of atomic barium at 533 nm.

$\frac{i}{2}\Omega(\sigma_1 - i\sigma_2) + c.c. = \Omega\sigma_2$

state a

Γ_0

state b

$$\dot{\rho}_{aa} = -\Gamma_0\rho_{aa} + \frac{i}{2}\Omega e^{-i\omega t}\rho_{ba} + c.c.$$

$$\dot{\rho}_{ab} = -\gamma_{ab}\rho_{ab} - i\omega_0\rho_{ab} - \frac{i}{2}\Omega e^{-i\omega t}(\rho_{aa} - \rho_{bb}) \qquad (4-1)$$

$$1 = \rho_{aa} + \rho_{bb}$$

$\dot{\rho}_{bb} \neq +\Gamma_0\rho_{bb} - \Omega\sigma_2$ ← ρ_{bb}가 아니라 ρ_{aa} ☆

where $\Gamma_0/2 = \gamma_{ab}$ for purely radiative cases. Suppose the atom is continuously driven by a near resonant laser of frequency ω and intensity I_0. In the steady state.

$$\dot{\rho}_{ab} = -i\omega\rho_{ab}, \quad \dot{\rho}_{aa} = \dot{\rho}_{bb} = 0$$

donnot become zero (∵ extra field) → oscillate with external field frequency!

Define

☆
$$\rho_{ab} = (\sigma_1 + i\sigma_2)e^{-i\omega t} \qquad (4-3)$$
$$\rho_{aa} - \rho_{bb} = \sigma_3$$

→ $\sigma_1, \sigma_2, \sigma_3$: slowly varying variable ⇒ in steady state, there becomes constant

In terms of there new variables σ_1

$$0 = -\Gamma_0(\sigma_3 + 1)/2 + \Omega\sigma_2 \qquad (4-4)$$

$$0 = -(\Delta\sigma_2 + \gamma_{ab}\sigma_1) + i(-\gamma_{ab}\sigma_2 + \Delta\sigma_1 - \frac{1}{2}\Omega\sigma_3)$$

$\dot{\rho}_{aa} = -\Gamma_0\rho_{aa} + \Omega\sigma_2$

$\dot{\rho}_{bb} = +\Gamma_0\rho_{aa} - \Omega\sigma_2$

$\Gamma_0 = \frac{1}{T_1} \Rightarrow \rho$의 diagonal 성분의 decay.

$\gamma_{ab} = \frac{1}{T_2} \Rightarrow$ dephasing

(ρ의 off-diagonal의 decoherence.)

Or

$$\Gamma_0 = 2\Omega\sigma_2 - \Gamma_0\sigma_3$$

$$0 = \gamma_{ab}\sigma_1 + \Delta\sigma_2 \qquad (4\text{-}5)$$

$$0 = 2\Delta\sigma_1 - 2\gamma_{ab}\sigma_2 - \Omega\sigma_2$$

Solutions are

$$\sigma_1 = \frac{-\Omega\Delta/2}{(\omega-\omega_0)^2 + \gamma_{ab}^2 + \Omega^2\gamma_{ab}/\Gamma_0}$$

$$\sigma_2 = -\frac{\gamma_{ab}}{\Delta}\sigma_1 \qquad (4\text{-}6)$$

$$\sigma_3 = \frac{\Omega^2\gamma_{ab}/\Gamma_0}{(\omega-\omega_0)^2 + \gamma_{ab}^2 + \Omega^2\gamma_{ab}/\Gamma_0} - 1$$

and corresponding density matrix elements are

$I_{sat} = \frac{\hbar\omega_0\gamma_{ab}}{\sigma_{rad}}$

$= \frac{\hbar\omega_0\gamma_{ab}}{6\pi\left(\frac{c}{\omega_0}\right)^2}$

$$\rho_{ab} = \frac{\Omega(-\Delta + i\gamma_{ab})/2}{(\omega-\omega_0)^2 + \gamma_{ab}^2 + \Omega^2\gamma_{ab}/\Gamma_0} e^{-i\omega t}$$

$$\rho_{aa} = \frac{\Omega^2\gamma_{ab}/(2\Gamma_0)}{(\omega-\omega_0)^2 + \gamma_{ab}^2 + \Omega^2\gamma_{ab}/\Gamma_0}. \qquad (4\text{-}7)$$

$= \frac{\hbar}{6\pi}\cdot\frac{1}{c^2}\left(\frac{2\pi c}{\lambda}\right)^3\gamma_{ab}$

$= \frac{2\pi h}{3\lambda^3}\cdot c\,\gamma_{ab}$

$= \frac{\pi}{3}\frac{hc}{\lambda^3\tau}$

$\left(\gamma_{ab} = \frac{\Gamma_0}{2} = \frac{1}{2}\cdot\frac{1}{\tau}\right)$

2승 μ가 아니냐 μ^{*2}에 해당 ≠ polarization 크기 조절 → $\neq \frac{2e^2\omega_0^2}{3mc^3}$

ε 편광조절 보정 크기에 맞춤.

$(8^*:1\sim3)$

Note

formal intensity

$$\Omega^2 = \frac{\mu^2 E_0^2}{\hbar^2} = \left(\frac{8\pi\mu^2}{\hbar^2 c}\right)\frac{cE_0^2}{8\pi} = \left(\frac{6\pi c^2}{\hbar\omega_0^3}\cdot\frac{4\mu^2\omega_0^3}{3\hbar c^3}\right)\cdot I_0$$

$$= \frac{6\pi c^2}{\hbar\omega_0^3}\cdot\Gamma_0\cdot I_0 = \frac{6\pi(c/\omega_0)^2\Gamma_0\cdot I_0}{\hbar\omega_0} = \frac{\sigma_{rad}\Gamma_0\cdot I_0}{\hbar\omega_0} \qquad (4\text{-}8)$$

Where σ_{rad} is the radiative scattering cross section as in Eq (1-13). Define the saturation intensity I_{sat} s.t.

$$\Omega^2\gamma_{ab}/\Gamma_0 = \frac{\sigma_{rad} I_0 \gamma_{ab}}{\hbar\omega_0} \equiv \gamma_{ab}^2\frac{I_0}{I_{sat}} \qquad (4\text{-}9)$$

or

$$I_{sat} \equiv \frac{\hbar\omega_0\gamma_{ab}}{\sigma_{rad}} = \frac{\hbar\omega_0}{\left(\frac{2\sigma_{rad}}{2\gamma_{ab}}\right)} = \frac{(\text{single photon energy})}{2(\text{cross section})\times(\text{dephasing time})}$$

No.

Date . .

Then

$$\rho_{ab} = \frac{\Omega(-\Delta + i\gamma_{ab})/2}{(\omega-\omega_0)^2 + \gamma_{ab}^2(1+I_0/I_{sat})} e^{-i\omega t}$$

$$\rho_{aa} = \frac{\gamma_{ab}^2 (I_0/I_{sat})/2}{(\omega-\omega_0)^2 + \gamma_{ab}^2(1+I_0/I_{sat})} \qquad (4\text{-}11)$$

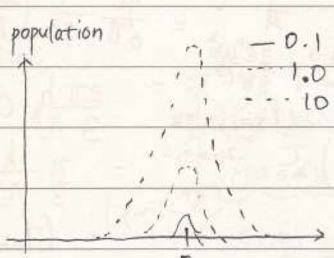
First, the linewidth is given by the dephasing rate γ_{ab}, not by the population decay rate Γ_0. Second, it is broadened

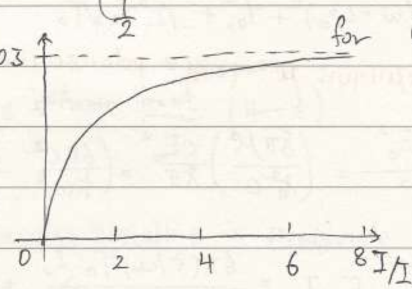
$$\Delta\omega(\text{FWHM}) = 2\gamma_{ab}\sqrt{1+I_0/I_{sat}} \qquad (4\text{-}12)$$

Such broadening is called "power broadening". Since all of the atoms experiencing the same broadening, it is called "homogeneous" broadening.

Third, the excited state population is saturated. Particularly, on resonance $(\omega=\omega_0)$

$$\rho_{aa} = \frac{1}{2}\frac{(I_0/I_{sat})}{(1+I_0/I_{sat})} \simeq \begin{cases} \frac{1}{2}(I_0/I_{sat}), & \text{for } (I_0/I_{sat}) \ll 1 \\ \frac{1}{2} & \text{for } (I_0/I_{sat}) \gg 1 \end{cases}$$





frequency (normalized)

I_0/I_{sat} : 100 times bigger, but height is not bigger 100 times.

but FWHM is bigger.

4.2 Elastic Collisions – Collisional Broadening

Elastic collisions between atoms in a gas or between atoms and phonons in a solid can cause ρ_{ab} to decay. During a collision the energy level of atom is slightly shifted. Since individual atoms experience different and independent, hence random phase shifts, the sum of all atomic dipoles, proportional to ρ_{ab}, decays exponentially (so called dephasing of ρ_{ab})

Consider Eq (3-27) with $\Omega = 0$, ☆ but with $\omega_0 \to \omega_0 + \delta\omega(t)$, where $\delta\omega(t)$ represents the energy level shift during a collision process.

$$\dot{\rho}_{ab} = -[\gamma_{ab} + i\omega_0 + i\delta\omega(t)]\rho_{ab} \qquad (4-14)$$

$$\dot{\rho}_{ab} = -\gamma_{ab}\rho_{ab} - i\omega_0\rho_{ab} - \tfrac{i}{2}\Omega e^{-i\omega t}(\rho_{aa} - \rho_{bb})$$ we don't consider external field

A formal solution

$$\rho_{ab}(t) = \rho_{ab}(0)\, \exp\left[-(i\omega_0 + \gamma_{ab})t - i\int_0^t \delta\omega(t')\,dt'\right] \qquad (4-15)$$

Average over an ensemble of atoms

$$\langle \rho_{ab}(t) \rangle_{\text{ensemble average}} = \rho_{ab}(0)\, e^{-(i\omega_0 + \gamma_{ab})t} \left\langle \exp\left[-i\int_0^t \delta\omega(t')\,dt'\right] \right\rangle \qquad (4-16)$$

Expanding the exponential

$$\left\langle \exp\left[-i\int_0^t \delta\omega(t')\,dt'\right] \right\rangle$$

Why not t'??

$$= 1 - i\int_0^t \langle \delta\omega(t') \rangle dt' - \frac{1}{2}\int_0^t dt' \int_0^{t} dt'' \langle \delta\omega(t')\delta\omega(t'') \rangle + \cdots \qquad (4-17)$$

All odd terms should vanish since $\delta\omega(t)$ randomly fluctuates from atom to atom. Under Markoff approximation.

$$\langle \delta\omega(t')\delta\omega(t'') \rangle = 2\gamma_{coll}\, \delta(t' - t'') \qquad (4-18)$$

each one fluctuate $+, - \Rightarrow$ if $t' = t''$, have nonzero contribution.

dimension : $\left[\frac{1}{T^2}\right]$ $\qquad \left[\frac{1}{T}\right]$ $\qquad \left[\frac{1}{T}\right]$

The 2n-th correlation is given by the sum of all distinguishable products of pairs like Eq. (4-18). (Prove it in Problem 8)

$$\langle \delta\omega(t_1)\delta\omega(t_2)\delta\omega(t_3)\cdots\delta\omega(t_{2n-1})\delta\omega(t_{2n})\rangle$$
$$= \underbrace{\frac{(2n)!}{2^n n!}}_{\text{we have to consider all possible pairing}} \langle \delta\omega(t_1)\delta\omega(t_2)\rangle\langle\delta\omega(t_3)\delta\omega(t_4)\rangle\cdots\langle\delta\omega(t_{2n-1})\delta\omega(t_{2n})\rangle \quad (4\text{-}19)$$

Therefore,

$$\left\langle \exp\left[-i\int_0^t \delta\omega(t')dt'\right]\right\rangle$$
$$= \sum_{n=0}^{\infty} \frac{(-i)^{2n}(2n)!}{(2n)!\,2^n n!}\underbrace{(2\gamma_{coll}t)^n}_{\delta(t'-t'')\text{로 적분후 다시 적분} \to t} = \sum_{n=0}^{\infty}\frac{(-\gamma_{coll}t)^n}{n!} = \exp(-\gamma_{coll}t) \quad (4\text{-}20)$$

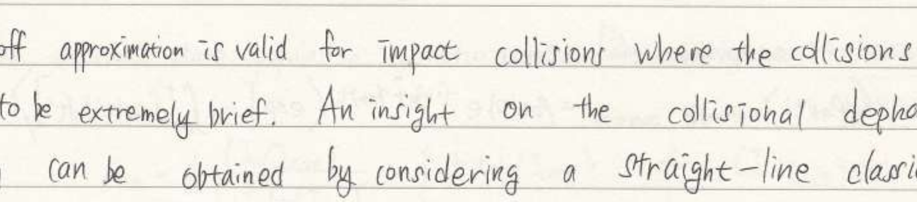
The total decay rate of ρ_{ab} is then

$$\gamma = \gamma_{ab} + \gamma_{coll} \quad (4\text{-}21)$$

The Markoff approximation is valid for impact collisions where the collisions are assumed to be extremely brief. An insight on the collisional dephasing rate γ_{coll} can be obtained by considering a straight-line classical trajectory along with van der Waals type potential ($\sim 1/r^6$) between particles. For example,

$$\delta\omega_s(t) \simeq \frac{C_6}{\left[b^2 + v_r^2(t-t_c)^2\right]^3} \quad (4\text{-}22)$$

where b is the impact parameter, v_r the relative velocity between collision partners and t_c the time when the collision occurs.



$$e^{-\gamma_{coll}\cdot t} = \left\langle \exp\left[-i\int_0^t \delta\omega(t')dt'\right]\right\rangle = \left\langle \exp\left[-i\sum_s \int_{-\infty}^{\infty}\delta\omega_s(t)dt\right]\right\rangle$$

$$= \left\langle \prod_s \exp\left[-i\int_{-\infty}^{\infty}\delta\omega_s(t)dt\right]\right\rangle \cong \prod_s \left\langle \exp\left[-i\int_{-\infty}^{+\infty}\delta\omega_s(t)dt\right]\right\rangle \quad (4\text{-}23)$$

Where $\delta\omega_s$ is the level shift associated with a single collision process whereas $\delta\omega$ is composed of such single-collision level shifts. Let N be the number of collisions during time interval $(0 \sim t)$. These collisions are all statistically independent. → each single-collision is equivalent

$$e^{-\gamma_{coll}t} = \left\langle \exp\left[-i\int_{-\infty}^{+\infty}\delta\omega_s(t)dt\right]\right\rangle^N \quad (4\text{-}24)$$

Define

$$\Theta_s = \Theta_s(b) \equiv -\int_{-\infty}^{+\infty}\delta\omega_s(t;b)\,dt \quad (4\text{-}25)$$

i) b가 크면 $\Theta_s(b) \to 0$ ⇒ $\langle e^{i\Theta_s(b)}\rangle \to 1$

ii) b가 작으면 $\Theta_s(b)$: random variable ⇒ $\langle e^{i\Theta_s(b)}\rangle \to 0$

$$e^{-\gamma_{coll}t} = \langle e^{i\Theta_s}\rangle^N = \langle 1-(1-e^{i\Theta_s})\rangle^N \quad (e^{i\Theta_s} \simeq 1)$$

$$\simeq 1 - N\left(1-\langle e^{i\Theta_s}\rangle\right)$$

$$\simeq \exp\left[-N\left(1-\langle e^{i\Theta_s}\rangle\right)\right] \quad (4\text{-}26)$$

very small quantity

density of perturber

$$\gamma_{coll}t \simeq \underbrace{n_p v_r \cdot \int_0^t dt' \cdot \underbrace{\int_0^{\infty} 2\pi b \cdot db\left[1-\langle e^{i\Theta_s(b)}\rangle\right]}_{=\sigma_{coll}\text{ (collisional cross section)}}}_{\text{Number of perturbers colliding with the atom during }(0,\dots,t)}$$

relative velocity of flux, perturbers coming in

The quantity in [ ] is negligible unless the impact parameter is far less than a critical value b_0 for which Θ_s becomes the order of unity. Typically, $b_0 \sim 10\,\text{Å}$.

Why?!

$$\gamma_{coll} = n_p v_r \underbrace{\sigma_{coll}}_{\text{collision cross section}} \simeq n_p v_r \int_0^{b_0} 2\pi b\,db = \underbrace{n_p}_{\propto\text{ Pressure}} v_r(\pi b_0^2)$$

for $b < b_0$, phaseshift become important

e.g. Ar, Ne → buffer gas

→ Other perturber can be different atom ⇒ number ∝ pressure

⇒ collisional broadening $\underset{\text{is also called}}{=}$ pressure broadening.

for van der walls interaction

$$\delta\omega_s \simeq \frac{C_6}{\left[b^2+v_r^2(t-t_c)^2\right]^3}$$

Where the collisional cross section is given by

$$\sigma_{coll} = \int_0^{\infty} 2\pi b\, db \left\{ 1 - \left\langle \exp\left[-i\int_{-\infty}^{+\infty} \delta\omega_s(t;b)\, dt\right]\right\rangle \right\}$$

with $\delta\omega_s(t)$ given by Eq. (4-22)

The collision-induced decay of ρ_{ab} results in a line broadening of $2\gamma_{coll}$ (FWHM).

This is called "collisional broadening" or "pressure broadening"

(of course, being homogeneous broadening).

ε function of intensity

4.3 Doppler Broadening.

Atoms in thermal equilibrium at temperature T have velocities according to the Maxwell-Boltzmann velocity distribution function.

$$f(\vec{v})\, d^3v = C\exp\left(-\frac{M\vec{v}^2}{2k_BT}\right) d^3v \qquad (4\text{-}29)$$

Where k_B is the Boltzmann constant, M is the mass of the atom and C is a normalization constant such that

$$\int f(\vec{v})\, d^3v = 1 \qquad (4\text{-}30)$$

In 1-D, a normalized form of the M-B distribution function is

$$f(v) = \frac{1}{u\sqrt{\pi}} e^{-(v/u)^2} \qquad (4\text{-}31)$$

Where u is the mean thermal velocity given by

$$u = \sqrt{\frac{2k_BT}{M}} \qquad (4\text{-}32)$$

Suppose an atom of resonance frequency of ω_0, moving at $\vec{v}$, interact with a plane of EM wave of wave vector $\vec{k}$ and frequency ω. Due to the Doppler shift, the frequency of the EM wave seen by the atom is

$$\omega' = \omega - \vec{k}\cdot\vec{v}$$

or equivalently, the atomic resonance frequency appears to be shifted in the laboratory frame as

$$\omega_0' = \omega_0 + \vec{k}\cdot\vec{v}$$

from $\sigma_{sc} \simeq 6\pi \left(\frac{c}{\omega_0}\right)^2 \cdot \frac{(\Gamma_0/2)^2}{(\omega-\omega_0)^2+(\Gamma_0/2)^2}$ (first Lecture)

The fluorescence (absorption) lineshape of such collection of atoms measured by the EM wave in $\vec{k}$ direction is proportional to fluorescence (absorption) cross section averaged over the velocity distribution

$$\bar{\sigma}(\omega) = \frac{1}{u\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(v/u)^2}\, \sigma(\omega;\omega_0+kv)\, dv \qquad (4-33)$$

on resonance, $I_0 \ll 1$, ⇒ lineshape factor → 1

where $\sigma(\omega;\omega_0+kv)$ is given by

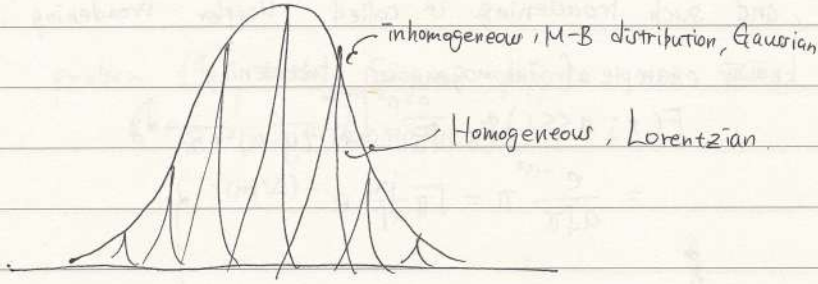
∵ γ_{ab}가 있는 이유

$$\sigma(\omega;\omega_0+kv) = \sigma^0(\omega_0)\frac{\gamma_{ab}^2}{(\omega-\omega_0-kv)^2+\gamma_{ab}^2(1+I_0/I_{sat})} \qquad (4-34)$$

Where $\sigma^0(\omega_0)$ is the unsaturated on-resonance cross section as given by Eq. (A4.1-2).
부록에 해당

Eq. (4-33) is a convultion of Gaussian M-B distribution function and a Lorentzian homogeneous lineshape of individual atoms. Such an integral is called the Voigt integral

$$\bar{\sigma}(\omega) = \underbrace{\frac{1}{u\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(v/u)^2}\, \sigma(\omega;\omega_0+kv)\, dv}_{\text{convolution}}$$



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Simplifying Eq. (4-33)

$$\bar{\sigma}(\omega) = \sigma^{\circ}(\omega_0)\left(\frac{\gamma_{ab}}{ku}\right)^2 \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-y^2}\frac{1}{(y-x)^2+a^2}dy \qquad (4\text{-}35)$$

$$\left\{ \underbrace{x \equiv \frac{\omega-\omega_0}{ku}}_{\text{normalized detuning}}, \ y \equiv \frac{v}{u}, \ a \equiv \frac{\gamma}{ku} \ \text{ with } \ \gamma \equiv \gamma_{ab}\sqrt{1+I_0/I_{sat}} \right. \qquad (4\text{-}36)$$

normalized with "u"

→ we can do it only numerically

Consider the normalized Voigt integral

$$F(x;a) \equiv \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-y^2}\frac{1}{(y-x)^2+a^2}dy \qquad (4\text{-}37)$$

i) If the homogeneous broadening is much larger than the inhomogeneous broadening, $ku \ll \gamma$ $(a \gg 1)$

$$F(x;a\gg1) \simeq \frac{1/\sqrt{\pi}}{x^2+a^2}\int_{-\infty}^{+\infty} e^{-y^2}dy$$
$$= \frac{1}{x^2+a^2} = \frac{(ku)^2}{(\omega-\omega_0)^2+\gamma^2} \qquad (4\text{-}38)$$

and thus

$$\bar{\sigma}(\omega) \simeq \frac{\sigma^{\circ}(\omega_0)}{1+I_0/I_{sat}} \cdot \frac{\gamma^2}{(\omega-\omega_0)^2+\gamma^2} \qquad (4\text{-}39)$$

which is the same as Eq. (A4.1-1) and is consistent with Fig 1.

ii) When $ku \gg \gamma$, i.e. $a \ll 1$, the lineshape is much broader than the homogeneous lineshape, and such broadening is called "Doppler broadening", which is the well known example of inhomogeneous broadening.

$$F(x;a\ll1) \simeq \frac{e^{-x^2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{1}{(y-x)^2+a^2}dy$$
$$= \frac{e^{-x^2}}{a\sqrt{\pi}}\cdot\pi = \sqrt{\pi}\frac{ku}{\gamma}e^{-(\Delta/ku)^2} \qquad (4\text{-}40)$$

Lorentzian is sharply peaked

→ integral = π/a

$\gamma \neq \gamma_{ab}$ $\left(\gamma = \gamma_{ab}\sqrt{1+I_0/I_{sat}}\right)$

square root

⇒ slower than homogeneous case ⇒ all atoms has different ω_0 (∵ different velocity) → in order to saturate all atoms, we need more bigger power

Thus

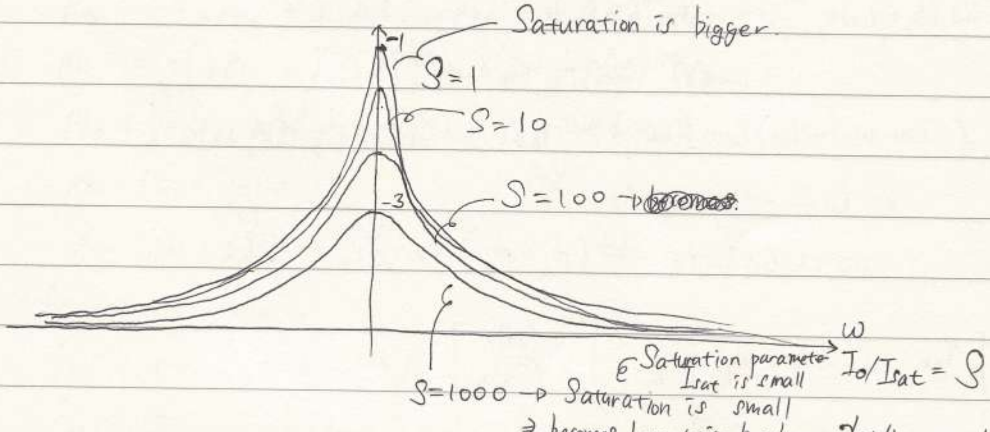
$$\bar{\sigma}(\omega) = \frac{\sigma^0(\omega_0)}{\sqrt{1+I_0/I_{sat}}}\left(\frac{\gamma_{ab}}{ku/\sqrt{\pi}}\right)\exp\left[-\left(\frac{\omega-\omega_0}{ku}\right)^2\right] \qquad (4-41)$$

e.g. we set laser at $\omega=\omega_0$ we are not touching other

First, the lineshape is a Gaussian with a width of ku. Second, on resonance cross section has additional reduction factor (γ_{ab}/ku) from σ^0_{ab}. Third, its saturation behavior is different from the homogeneous broadening case.

$ku \ll \gamma_{ab}$ ⇒ large reduction.

The saturation progresses more slowly since more atoms get involved as the intensity increases. However, as the intensity increases further, γ eventually becomes comparable or even larger than ku, and thus the homogeneous broadening, i.e., power broadening, becomes dominant and the fluorescence or absorption becomes fully saturated.



<Fig3. Voigt integral>

Solve problem 11 with Computer program, not 손으로
(eg. mathematica)

No.

Date Problem 8

$$\dot\rho_{aa} = -\Gamma_a\rho_{aa} + \frac{i}{2}\Omega e^{-i\omega t}\rho_{ba} - \frac{i}{2}\Omega e^{i\omega t}\rho_{ab} + R_p$$

$$\dot\rho_{bb} = -\Gamma_b\rho_{bb} + \frac{i}{2}\Omega e^{i\omega t}\rho_{ab} - \frac{i}{2}\Omega e^{-i\omega t}\rho_{ba}$$

$$\dot\rho_{ab} = -\gamma_{ab}\rho_{ab} - i\omega_0\rho_{ab} - \frac{i}{2}\Omega e^{-i\omega t}(\rho_{aa}-\rho_{bb})$$

for steady state

$$\dot\rho_{aa} = 0 = -\Gamma_a\rho_{aa} + \frac{i}{2}\Omega e^{-i\omega t}e^{+i\omega t}(\sigma_1 - i\sigma_2) - \frac{i}{2}\Omega e^{i\omega t}e^{-i\omega t}(\sigma_1+i\sigma_2) + R_p$$

$$= -\Gamma_a\rho_{aa} + \Omega\sigma_2 + R_p$$

$$\dot\rho_{bb} = 0 = -\Gamma_b\rho_{bb} - \Omega\sigma_2$$

$$\Rightarrow \rho_{aa} = \frac{\Omega}{\Gamma_a}\sigma_2 + \frac{R_p}{\Gamma_a}$$

$$\rho_{bb} = -\frac{\Omega}{\Gamma_b}\sigma_2$$

$$\Rightarrow \sigma_3 = \rho_{aa}-\rho_{bb} = \Omega\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right)\sigma_2 + \frac{R_p}{\Gamma_a}$$

$$\dot\rho_{ab} = -i\omega(\sigma_1+i\sigma_2)e^{-i\omega t} = -\gamma_{ab}(\sigma_1+i\sigma_2)e^{-i\omega t} - i\omega_0(\sigma_1+i\sigma_2)e^{-i\omega t}$$

$$-1 \qquad - \frac{i}{2}\Omega\cdot\left(\Omega\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right)\sigma_2 + \frac{R_p}{\Gamma_a}\right)e^{-i\omega t}$$

$$\Rightarrow \left(-i(\omega-\omega_0)+\gamma_{ab}\right)\sigma_1 = \left((\omega_0-\omega) - i\gamma_{ab} - \frac{i}{2}\Omega^2\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right)\right)\sigma_2 - \frac{\Omega R_p}{2\Gamma_a}i$$

$$\Rightarrow \gamma_{ab}\sigma_1 = (\omega_0-\omega)\sigma_2 - \cancel{\frac{\Omega R_p}{2\Gamma_a}}$$

$$-(\omega-\omega_0)\sigma_1 = \left(-\gamma_{ab} - \frac{\Omega^2}{2}\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right)\right)\sigma_2 - \cancel{\frac{\Omega R_p}{2\Gamma_a}} - \frac{\Omega R_p}{2\Gamma_a}$$

$$\begin{bmatrix} \gamma_{ab} & (\omega-\omega_0) \\ -(\omega-\omega_0) & \left(\gamma_{ab}+\frac{\Omega^2}{2}\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right)\right)\end{bmatrix}\vec\sigma_{12} = \begin{bmatrix} \cancel{-\frac{\Omega R_p}{2\Gamma_a}}\ 0 \\ \cancel{0}\ -\frac{\Omega R_p}{2\Gamma_a}\end{bmatrix}$$

$$\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix} = \frac{1}{\gamma_{ab}^2 + \frac{\Omega^2}{2}\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right)\gamma_{ab} + (\omega-\omega_0)^2}\begin{bmatrix}\gamma_{ab}+\frac{\Omega^2}{2}\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right) & -(\omega-\omega_0) \\ (\omega-\omega_0) & \gamma_{ab}\end{bmatrix}\begin{bmatrix}0\\ -\frac{\Omega R_p}{2\Gamma_a}\end{bmatrix}$$

$$= \frac{1}{\gamma_{ab}^2 + \frac{\Omega^2}{2}\gamma_{ab}\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right) + (\omega-\omega_0)^2}\begin{bmatrix}(\omega-\omega_0)\frac{\Omega R_p}{2\Gamma_a} \\ -\frac{\gamma_{ab}\Omega R_p}{2\Gamma_a}\end{bmatrix}$$

No.

Date

☆ $\rho_{aa} = \frac{\Omega}{\Gamma_a}\sigma_2 + \frac{R_p}{\Gamma_a} = \frac{R_p}{\Gamma_a}\left(1 + \frac{-\frac{\gamma_{ab}\Omega^2}{2\Gamma_a}}{\gamma_{ab}^2 + \frac{\Omega^2}{2}\gamma_{ab}\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right) + (\omega-\omega_0)^2}\right)(\rho_{aa} + \dot{\rho}_{bb}) =$

$\rho_{bb} = -\frac{\Omega}{\Gamma_b}\sigma_2 = \frac{\frac{\gamma_{ab}\Omega^2}{2\Gamma_a}R_b}{\gamma_{ab}^2 + \frac{\Omega^2}{2}\gamma_{ab}\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right) + (\omega-\omega_0)^2}$

$\dot{\rho}_{aa} = -\Gamma_0\rho_{aa} + \frac{i}{2}\Omega e^{-i\omega t}\rho_{ba} - \frac{i}{2}\Omega e^{i\omega t}\rho_{ab}$ $\qquad \gamma_{ab} = (\Gamma_a + \Gamma_b)/2 = \Gamma$

$\dot{\rho}_{ab} = -\gamma_{ab}\rho_{ab} - i\omega_0\rho_{ab} - \frac{i}{2}\Omega e^{-i\omega t}(\rho_{aa} - \rho_{bb})$

$1 = \rho_{aa} + \rho_{bb}$

$\rho_{aa} = -\Gamma\rho_{aa} + \frac{i}{2}\Omega e^{-i\omega t}\rho_{ba} - \frac{i}{2}\Omega e^{i\omega t}\rho_{ab}$

$\rho_{ab} = -\Gamma\rho_{ab} - i\omega_0\rho_{ab} - \frac{i}{2}\Omega e^{-i\omega t}(\rho_{aa} - \rho_{bb})$

$\dot{\rho}_{ba}$

$i(x+iy) - i(x-iy)$

$\cancel{\dot{\rho}_{ab} = -\Gamma_a}$ $\qquad \rho_{ab} = (\sigma_1 + i\sigma_2)e^{-i\omega t}$

$\cancel{\dot{\rho}_{aa} = -\Gamma_a\rho_{aa} + \frac{i}{2}}$ $\qquad \frac{i}{2}\Omega e^{-i\omega t}(\sigma_1 + i\sigma_2) - \frac{i}{2}\Omega e^{i\omega t}(\sigma_1 + i\sigma_2)e^{-i\omega t}$

$\sigma_2 \qquad = 0 \quad 2\Omega\sigma_2/2 = \Omega\sigma_2$

$\dot{\rho}_{aa} = -\Gamma_0\rho_{aa} + \frac{i}{2}\Omega e^{-i\omega t}\rho_{ba} - \frac{i}{2}\Omega e^{+i\omega t}\rho_{ab} + R_p$

$\dot{\rho}_{bb} = -\Gamma_b\rho_{bb} + \frac{i}{2}\Omega e^{+i\omega t}\rho_{ab} - \frac{i}{2}\Omega e^{-i\omega t}\rho_{ba}$

$\dot{\rho}_{ab} = -\gamma_{ab}\rho_{ab} - i\omega_0\rho_{ab} - \frac{i}{2}\Omega e^{-i\omega t}(\rho_{aa} - \rho_{bb})$

$\rho_{ab} = (\sigma_1 + i\sigma_2)e^{-i\omega t}$

$\rho_{aa} - \rho_{bb} = \sigma_3$

⇒ for steady state

$\dot{\rho}_{aa} = 0 = -\Gamma_a\rho_{aa} + \Omega\sigma_2 + R_p$ $\qquad \{ \} = \Omega\left(\frac{1}{\Gamma_a} + \frac{1}{\Gamma_b}\right)\sigma_2 + \frac{R_p}{\Gamma_a}$

$\dot{\rho}_{bb} = 0 = -\Gamma_b\rho_{bb} - \Omega\sigma_2$ $\qquad \sigma_3 = \rho_{aa} - \rho_{bb}$

⇒ $\qquad \rho_{bb} = -\frac{\Omega}{\Gamma_b}\sigma_2 \qquad \rho_{aa} = \frac{\Omega}{\Gamma_a}\sigma_2 + \frac{R_p}{\Gamma_a}$

☆

⇒ $\rho_{ab} = \frac{+(\omega-\omega_0) - i\gamma_{ab}}{\gamma_{ab}^2 + \frac{\Omega^2}{2}\gamma_{ab}\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right) + (\omega-\omega_0)^2} \times \boxed{\frac{\Omega R_p}{2\Gamma_a}}$

$\dot{\rho}_{ab} = (\sigma_1 + i\sigma_2)e^{-i\omega t}(-i\omega)$

$= -\gamma_{ab}(\sigma_1 + i\sigma_2)e^{-i\omega t} - i\omega_0(\sigma_1 + i\sigma_2)e^{-i\omega t}$

☆

$\sigma_3 = \Omega\left(\frac{1}{\Gamma_a} + \frac{1}{\Gamma_b}\right)\sigma_2 + \frac{R_p}{\Gamma_a}$ $\qquad - \frac{i}{2}\Omega e^{-i\omega t}\sigma_3$

$= \frac{R_p}{\Gamma_a}\left(\frac{-\frac{R_p}{\Gamma_a}\cdot\frac{\gamma_{ab}\Omega^2}{2}\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right)}{\gamma_{ab}^2 + \frac{\Omega^2}{2}\gamma_{ab}\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right) + (\omega-\omega_0)^2}\right)$ $\quad -i(\omega-\omega$

$(i(\omega-\omega_0) - \gamma_{ab})(\sigma_1 + i\sigma_2) = \frac{i}{2}\Omega\left[\Omega\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right)\sigma_2 + \frac{R_p}{\Gamma_a}\right]$

$= \frac{\gamma_{ab}^2 + (\omega-\omega_0)^2}{\gamma_{ab}^2 + \frac{\Omega^2}{2}\gamma_{ab}\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right) + (\omega-\omega_0)^2}$ //

No.
Date

$\frac{1}{2}\varepsilon_0 E^2 = \frac{E^2}{8\pi}$

$\Rightarrow I_{sat} = ?$

$6\pi\left(\frac{c}{\omega_0}\right)^2 = \sigma_{abs}$

$\frac{\Omega^2}{2}\gamma_{ab}\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right) = \frac{I}{I_{sat}}$ $\qquad \frac{4\mu^2\omega_0^3}{3\hbar c^3} = \Gamma_0 \quad \frac{4\mu_0^2\omega_0^3}{3\hbar c^3} =$

$\Omega = \frac{\mu E}{\hbar}$

$$\Omega^2 = \frac{\mu^2 E^2}{\hbar^2} = \frac{\mu^2}{\hbar^2}\frac{8\pi}{c}\left(\frac{c}{8\pi}E^2\right) = \frac{\mu^2 8\pi}{\hbar^2 c} I = \frac{4\mu^2\omega_0^3}{3\hbar c^3}\times\frac{8\pi c^2}{\omega_0^3\hbar}\times 3\times I$$

$$= \Gamma_0 \times 6\pi\left(\frac{c}{\omega_0}\right)^2\frac{1}{\hbar\omega_0} I$$

$$= \underline{\Gamma_0 \times \sigma_{rad} \times I/\hbar\omega_0}$$

$$\gamma_{ab}^2 + \underbrace{\frac{\Omega^2}{2}\gamma_{ab}\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right)}_{} + (\omega-\omega_0)^2$$

$$\gamma_{ab}^2\left(1+\frac{I}{I_{sat}}\right)$$

$$\Rightarrow \gamma_{ab}\frac{\cancel{I}}{I_{sat}} = \frac{\Omega^2}{2}\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right)$$

$$= \frac{\Gamma_0}{2}\sigma_{rad}\cdot\cancel{I}\frac{1}{\hbar\omega_0}\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right)$$

$$\Rightarrow \cancel{2\gamma_{ab}\cdot I}$$

$$I_{sat} = \frac{2\gamma_{ab}\hbar\omega_0}{\Gamma_0\sigma_{rad}\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right)}$$

$$= \frac{(\Gamma_a+\Gamma_b)\hbar\omega_0}{\Gamma_0\sigma_{rad}\left(\frac{\Gamma_a+\Gamma_b}{\Gamma_a\Gamma_b}\right)}$$

$$= \frac{\Gamma_a\Gamma_b\hbar\omega_0}{\Gamma_0\sigma_{rad}}$$

$\cancel{\frac{e^2}{r} = mr\omega_0^2 = \frac{\hbar^2}{r^3}} = \frac{(\hbar\omega_0)^2}{r^3} \Rightarrow m = \frac{\hbar^2}{r^4}$ $\quad e^2 = \frac{(\hbar\omega_0)^2}{r^2}$ $\quad e = \frac{\hbar\omega_0}{r}$ $\quad \Gamma_0 = \frac{2e^2\omega_0^2}{3mc^3} = \frac{1}{(3mc^2)} 2\left(\frac{e\omega_0}{c}\right)^2$ $\quad \frac{4\mu^2\omega_0^3}{3\hbar c^3}$

No.
Date . .

Problem 9

증명 $\delta\omega(t_1), \delta\omega(t_2) \to$ 독립 $\Rightarrow \mathrm{Cov}(\delta\omega(t_1), \delta\omega(t_2)) = 0$

$\frac{1}{2}a_0^2 e^2 = \mu^2$

$= \frac{\mu^2\omega_0^2}{3mc^4 a_0^2}$

$\Gamma_0 = \frac{2e^2\omega_0^2}{3mc^3}$

$= \frac{4\mu^2\omega_0^3}{3\hbar c}$

$\cancel{a_0^2 mc\,\omega_0 = \hbar}$

$\cancel{a_0^2 mc\,\omega_0^2 = \hbar\omega_0}$

$2ma_B^2\omega_0^2 = \hbar\omega_0 \Rightarrow \frac{1}{ma_B^2} = \frac{\omega_0}{\hbar}$

$\frac{\hbar^2}{r^3} = mr\omega_0^2$

$\frac{1}{2}a_B^2 e^2 = \mu^2 \Rightarrow e^2 = \frac{2\mu^2}{a_B^2}$

$\omega_0 = \frac{me^4}{\hbar^3}$

$\frac{2e^2\omega_0^2}{3mc^3} = \frac{4\mu^2\omega_0^2}{3mc^3 a_B^2} = \underline{\frac{4\mu^2\omega_0^3}{3\hbar c^3}}$ //

$\Gamma_0 = \frac{2}{3}\frac{e^2\omega_0^2}{mc^3}$

$= \frac{2e^2\omega_0^2}{3mc^3}$

$\Rightarrow \frac{1}{2}a_B^2 e^2 = \mu^2$

$\frac{4\mu^2\omega_0^3}{3\hbar c^3}$

$= \underline{\frac{4\omega_0^2}{3ma_B^2 c^3}\mu^2}$ //

$\frac{e^2}{a_B^2} = \frac{\hbar^2}{a_B^3 m} = ma_B\omega_0^2$

$ma_B^2\omega_0^2 = \hbar\omega_0$

$\Rightarrow \omega_0 = \frac{\hbar}{ma_B^2} = \frac{me^4}{\hbar^3}$

$a_B = \frac{\hbar^2}{me^2}$

$= \hbar^3$

$mc^2 = \frac{\hbar^2}{ma_B^2}$

$\cancel{\Rightarrow mc = \frac{\hbar}{a_B}}$

$\cancel{\Rightarrow m = \frac{\hbar}{a_B c}}$

$\cancel{\frac{1}{m\omega_0}} = \frac{\hbar^3}{m^2 e^4}$

$= \frac{a_B^2 c^2}{e^4}$

$\frac{4\mu^2\omega_0^{\cancel{3}}}{3\hbar \cancel{c^3}} = \frac{4\mu^2\omega_0^{\cancel{2}}}{3ma_B^2\cancel{c^3}}$

$mc^2 = \frac{\hbar^2}{ma_B^2}$

$\frac{\omega_0}{\hbar} = \frac{1}{ma_B^2}$

$m = \frac{\hbar}{a_B c}$

$\frac{L^2}{mr^3} = F$

$\Rightarrow \underline{ma_B^2\omega_0^2 = \hbar\omega_0}$ //

$\frac{L^2}{mr^2} = V$

$m = \frac{\hbar}{a_B c}$

$\omega_0 \quad \frac{1}{m} = \frac{e^4}{\omega_0\hbar^3}$

$+ i\frac{\Omega}{2} e^{-i\omega t}\rho_{ba}$

No.

Date . .

$\dot{\rho}_{ab} = -\gamma_{ab}\rho_{ab} - i\omega_0\rho_{ab} - \frac{i}{2}\Omega e^{-i\omega t}(\rho_{aa}-\rho_{bb})$

$\dot{\rho}_{aa} = -\Gamma_a\rho_{aa} + \frac{i}{2}\Omega e^{-i\omega t}\rho_{ba} + c.c.$

$\dot{\rho}_{bb} = -\Gamma_b\rho_{bb} - \frac{i}{2}\Omega e^{-i\omega t}\rho_{ba} + c.c.$

$\Omega = 0$

for $\dot{\rho}_{ab} = -\gamma_{ab}\rho_{ab} - i\omega_0\rho_{ab}$

$\Rightarrow \dot{\rho}_{ab} = -(\gamma_{ab} + i(\omega_0 + \delta\omega(t)))\rho_{ab}$

$\Rightarrow \rho_{ab}(t) = \rho_{ab}(0)\cdot\exp\left(-\int_0^t (\gamma_{ab} + i(\omega_0 + \delta\omega(t')))\,dt'\right)$

$= \rho_{ab}(0)\exp(-(\gamma_{ab}+i\omega_0)t)\cdot\exp\left(-\int_0^t i\delta\omega(t')dt'\right)$

$\left\langle \exp\left(-\int_0^t i\delta\omega(t')dt'\right)\right\rangle$

$= \left\langle 1 - i\int_0^t \delta\omega(t_1)dt_1 + \frac{(-i)^2}{2!}\int_0^t \delta\omega(t_1)\int_0^t \delta\omega(t_2)dt_1dt_2 + \cdots \right\rangle$

odd $\rightarrow$ 0

even $\rightarrow \left\langle \frac{(-i)^{2m}}{(2m)!}\int_0^t \delta\omega(t_1)\int_0^t \delta\omega(t_2)\cdots\int_0^t \delta\omega(t_{2m})dt_1dt_2\cdots dt_{2m}\right\rangle$

~~for~~ markoff approximation

$\langle \delta\omega(t_i)\delta\omega(t_j)\rangle = 2\gamma_{coll}\,\cancel{\delta\omega(t)}\,\delta(t_i - t_j)$

$\Rightarrow \frac{(-i)^{2m}}{(2m)!}\int_0^t\int_0^t\cdots\int_0^t \langle \delta\omega(t_1)\delta\omega(t_2)\cdots\delta\omega(t_{2m})\rangle dt_1dt_2\cdots dt_{2m}$

$t_i = t_j$ 인 2각들인 경우만 0이 안된다

~~generally $t_1 = t_2$~~ 각각~~의~~이 $\omega(t_i)$에 대해 2각이 이든

~~$\langle\delta\omega(t_1)$~~

있어야 0 이 되지 않는다

6

11 $\Rightarrow$ $\delta\omega(t_1)\cdots\delta\omega(t_{2m})$ 2각 짓는 방법은 $\frac{(2m)!}{2!2!\cdots2!}$

$\frac{24}{4}$

$\frac{(4!)}{2^2}$ 2라고 ~~$\langle\delta\omega(t_1)\delta\omega(t_1)\delta\omega(t_2)\delta\omega(t_2)\rangle$~~

0

$\langle \delta\omega(t_1)\delta\omega(t_2)\rangle = \cancel{\langle\delta\omega(t_1)\rangle\langle\delta\omega_2\rangle}$

$+ 2Cor\langle \delta\omega_1(t_1)\delta\omega(t_2)\rangle$

Problem 9

Set $\delta\omega(t_1)\delta\omega(t_2)$

$\delta\omega(t_{2n-1})\delta\omega(t_{2n})$

$\exp(-(t_1-t_2)^2)$

No. $\cancel{\exp(-(t_1-t_3)^2)}$

Date $\cdots \exp((t_{2n-1}-t_{2n})^2)$

$$\langle \delta\omega(t_1)\delta\omega(t_2)\cdots\delta\omega(t_n)\rangle$$

$$= \sum_{i=1}^{M} \frac{\delta\omega_i(t_1)\delta\omega_i(t_2)\cdots\delta\omega_i(t_n)}{M}$$

$t_1 = t_2 \neq t_3 = t_4 \cdots t$

$t_1 = t_2$

$t_3 = t_4 = \cdots = t_{2n}$

$\delta\omega(t_1)\delta\omega$

$\delta\omega(t_1)\delta\omega(t_1)$

$t_1 = t_2 = t_3 = t_4$

$t_1 = t_2,\ t_3 = t_4$ $\qquad \sum \frac{\delta\omega_i(t_1)\delta\omega_i(t)}{M} = 2\gamma_{coll}\delta(t_1-t_1)$

$$\langle \delta\omega(t_1)\delta\omega(t_2)\delta\omega(t_3)\delta\omega(t_4)\rangle$$

$$= \frac{1}{M}\sum_{i=1}^{M} \delta\omega(t_1)\delta\omega(t_2)\delta\omega(t_3)\delta\omega(t_4)$$

$$\langle XY\rangle = \sum_{i=1}^{N}\frac{(X-\bar{X})(Y-\bar{Y})}{N}$$

$$= \langle \delta\omega(t_1)\delta\omega(t_1)\rangle\langle\delta\omega(t_3)\delta\omega(t_3)\rangle$$

$$= N\langle X\rangle\langle Y\rangle - \mathrm{Cov}(X,Y)$$

$\iiint \delta\omega(t_1)^4\, dt_1\, dt_2\, dt_3\, dt_4$

$\leq \iint \delta\omega(t_1)\delta\omega(t_2)\, dt_1\, dt_2 \iint \delta\omega(t_3)\delta\omega(t_4)\, dt_3\, dt_4$

$$\langle XY\rangle = \sum_{i=1}^{N}\frac{(XY-\bar{X}\bar{Y})}{N}$$

$$= \frac{1}{M}\sum_{i=1}^{M}\delta\omega(t_1)\delta\omega(t_2)\delta\omega(t_3)\delta\omega(t_4)$$

$$= \sum_{i=1}^{N}\frac{(X-X_i)}{N}\sum_{j=1}^{N}\frac{(Y_j-\bar{Y}_j)}{N}$$

$t_1=t_2=t_3=t_4$

$$\langle \delta\omega(t_1)\delta(t_1)\rangle$$

$$-\sum_{i\neq j}^{N}(X_i-\bar{X})(Y_j-\ldots)$$

t_2, t_1, t_3

$t_1 = t_2$ $\quad \langle\delta\omega(t_1)\delta\omega(t_1)\rangle\langle\delta\omega(t_2)\cdots\delta\omega(t_n)\rangle$

$\text{짝} \Rightarrow 0$

$$\frac{1}{M}\sum_{i=1}^{M}(\delta\omega(t_2))^{n-2}$$

$n \equiv 0 \pmod 2$

$t_3 = t_4$

$2n$

$$\langle\delta\omega(t_1)\delta\omega(t_2)\rangle\langle\delta\omega(t_3)\cdots\delta\omega(t_n)\rangle$$

$$+\langle\delta\omega(t_1)\delta\omega(t_3)\rangle\langle\delta\omega(t_2)\delta\omega(t_4)\cdots\delta\omega(t_n)\rangle$$

$$+\langle\delta\omega(t_1)\delta\omega(t_n)\rangle\langle\delta\omega(t_2)\delta\omega(t_3)\cdots\rangle$$

$\varepsilon = 0$

No.

Date . .

$\frac{1}{4\pi}\langle v\rangle$ $4\pi \times \frac{1}{4\pi}\langle v\rangle$

Problem 10

$\nu_{coll} = n_p v_r \sigma_{coll}$

$pV = nRT$

$\Rightarrow n_p = \frac{P}{RT}$

$\Rightarrow v_r = \cancel{\frac{3}{4}\langle v\rangle}\ \left(\langle v\rangle \times \frac{1}{4\pi}\right) \times 4\pi$

$= \langle v\rangle$

$\Rightarrow \nu_{coll} = \frac{P}{RT} \times \sigma_{coll} \times \langle v\rangle \times N_A$ $\quad \langle v\rangle = \sqrt{\frac{8RT}{\pi M}}$

$= P \sigma_{coll} \sqrt{\frac{8}{\pi M R T}} = 7.07 \times 10^{-18} \times N_A = 4.26 \times 10^{6}\ Hz$

1atm $5.37 \times 10^{-15} \times N_A$

$= 3.23 \times 10^{9}\ Hz$

natural line width: $\nu_{ab} = \frac{\Gamma_0}{2}$?

$M = 137.327$ $\quad \Gamma_0 = 20\,MHz \times 2\pi$

$\lambda = 553\,nm$

$\frac{hc}{\lambda}\ \Delta E\ \Delta t = \frac{\hbar}{2}$

$\Delta t = \frac{\lambda}{2c} \times \frac{1}{2\pi}$

$\Gamma_0 \Rightarrow \nu = \frac{2c}{\lambda} \times 2\pi$

$\left(\langle v\rangle \cdot \frac{1}{4\pi}\right) \cdot 4\pi \times \sigma_{coll}$ $R(\theta,\varphi) = \frac{1}{4}\langle v\rangle$ ~~$\frac{3}{4}\langle v\rangle \sigma_{coll}$~~

$\frac{1}{4}\langle v\rangle$ $R = \frac{1}{4}\langle v\rangle \Delta A$

σ_{coll}

No.

Date

Problem 11.

$$\tilde{\Gamma}_0 = \frac{4\mu^2\omega_0^3}{3\hbar c^3} \qquad \omega_0 = \frac{2\pi c}{\lambda}$$

$$= \frac{2e^2\omega_0^2}{3mc^3}$$

$$= \frac{e^2\omega_0^2}{6\pi\epsilon_0 mc^3}$$

$$= \frac{2\pi e^2 c^2}{3\epsilon_0 m\lambda^2 c^3}$$

$$= \frac{2\pi e^2}{3\epsilon_0 m\lambda^2 c} = 7.26\times 10^7$$

$$I_{sat} = \frac{\pi}{3}\frac{hc}{\lambda^3\cdot\tau}$$

$$= \underbrace{\frac{\pi}{3}\frac{hc\Gamma_0}{\lambda^3}}$$

$$= 1.55\times 10^2\ W/m^2$$

$$\frac{\tilde{\Gamma}_0}{2\pi} = 11.56\ MHZ$$

~~calcul~~ Calculated by

$$\tilde{\Gamma}_0 = \frac{4\mu^2\omega_0^3}{3\hbar c^3}$$

$$\omega_0 = \frac{2\pi c h}{\lambda h}$$

$$= \frac{2\pi c}{\lambda}$$

20MHz

$$\gamma_{ab} = \frac{\Gamma_0'}{2},$$

$$\dot{\rho}_{aa} = -\Gamma_0\rho_{aa} + \frac{1}{2}\Omega e^{-i\omega t}\rho_{ba} + c.c \qquad \gamma_{ab} = \frac{1}{2}\Gamma_0$$

$\Rightarrow \rho_{aa}$

$$\gamma_{ab}^2 + \Omega^2\gamma_{ab}/\Gamma_0 \qquad \sigma_{rad} = 6\pi\left(\frac{c}{\omega_0}\right)^2$$

$$= \gamma_{ab}^2 + \frac{1}{2}\Omega^2$$

$$= \gamma_{ab}^2\left(1 + \underbrace{\frac{\Omega^2}{2\gamma_{ab}^2}}\right) \qquad \Omega^2 = \left(\frac{\mu_0 E}{\hbar}\right)^2$$

$$= \frac{I}{I_{sat}} = \frac{2\Omega^2}{\Gamma_0^2 \cdot I_{sat}} = \frac{\sigma_{rad}\tilde{\Gamma}_0\cdot I}{\Gamma_0^2\cdot\hbar\omega_0}$$

$$\Rightarrow I_{sat} = \frac{\Gamma_0^2\hbar\omega_0}{\tilde{\Gamma}_0\cdot\sigma_{rad}}$$

$$= \frac{\Gamma_0^2\hbar\omega_0^3}{\tilde{\Gamma}_0 6\pi c^2} = \underbrace{\frac{4\pi^2\Gamma_0^2\hbar c}{3\tilde{\Gamma}_0\lambda^3}}$$

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No.

Date

$$\bar{\sigma}(\omega) = \frac{1}{u\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(v/u)^2} \sigma(\omega; \omega_0 + kv)\, dv$$

$$\sigma(\omega; \omega_0 + kv) = \mathring{\sigma}(\omega_0) \frac{\gamma_{ab}^2}{(\omega - \omega_0 - kv)^2 + \gamma_{ab}^2 (1 + \mathring{I}_0 / I_{sat})}$$

$$\gamma_{ab} = \Gamma_0 / 2$$

$\cancel{\gamma_{ab}=}$

$$\gamma = \gamma_{ab} \sqrt{1 + I/I_{sat}}, \quad \frac{v}{u} = y \Rightarrow v = u \cdot y, \quad \frac{\omega - \omega_0}{ku} = x, \quad \frac{\gamma}{ku} = a$$

$$\Rightarrow \sigma(\omega; \omega_0 + kv) = \sigma^{\circ}(\omega_0) \frac{\gamma_{ab}^2}{(x-y)^2 + a^2} \cdot \frac{1}{(ku)^2}$$

$$\Rightarrow \bar{\sigma}(\omega) = \frac{1}{u\sqrt{\pi}} \left(\frac{\gamma_{ab}}{ku}\right)^2 \int_{-\infty}^{+\infty} e^{-y^2} \cdot \frac{1}{(x-y)^2 + a^2} u \cdot dy$$

$$= \frac{1}{\sqrt{\pi}} \left(\frac{\gamma_{ab}}{ku}\right)^2 \int_{-\infty}^{+\infty} e^{-y^2} \frac{1}{(x-y)^2 + a^2} dy$$

$$k = \frac{2\pi}{\lambda}$$

$$u = \sqrt{\frac{2k_B T}{M}} = \sqrt{\frac{2RT}{M}}$$

$$a = \frac{\gamma}{ku} = \frac{\gamma_{ab}}{ku} \sqrt{1 + I/I_{sat}} = \frac{\Gamma_0}{2ku} \sqrt{1 + I/I_{sat}}$$

$$\omega = \underset{\frac{2\pi c}{\lambda}}{\omega_0} + ku x$$

No.

Date

Spectral hole burning
Lamp dip
Crossover resonance

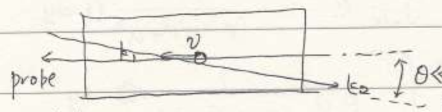
5. Lamb-dip Spectroscopy

5.1. Spectral hole burning

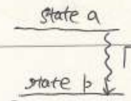
Consider two counte-propagating laser beams through an atomic vapor cell as shown in Fig.1. Define the following variables

$$\begin{cases} \omega_{1,2} = \text{frequency of laser } 1,2 \\ \omega_0 = \text{resonance frequency of atom} \\ I_{1,2} = \text{intensity of laser } 1,2 \ (\text{assuming } \underset{\text{induced saturation}}{I_2 \gg I_1}) \end{cases}$$

Our atom is the two-level system described by Eq (4-1). Assume that the medium is inhomogeneously broadened, i.e. $ku \gg \gamma_{ab}$



$\theta \ll 1 \Rightarrow$ well overlaped over the cell $\to$ good saturation



ground state does not decay

$$\dot{\rho}_{aa} = -\Gamma_0 \rho_{aa} + \frac{i}{2}\Omega e^{-i\omega t}\rho_{ba} + c.c$$

$$\dot{\rho}_{ab} = -\gamma_{ab}\rho_{ab} - i\omega_0\rho_{ab} - \frac{i}{2}\Omega e^{-i\omega t}(\rho_{aa} - \rho_{bb})$$

$$1 = \rho_{aa} + \rho_{bb}$$

<Fig. 1>

i) With laser 2 off, absorption signal of laser 1 is determined by the absorption cross section given by Eqs. (4-33) and (4-34) with $\omega = \omega_1$ and $I_0 = I_1$.

(probe laser on, frequency $= \omega_1$)

$$\bar{\sigma}(\omega) = \sigma^0(\omega_0) \frac{1}{u\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(v/u)^2} \frac{\gamma_{ab}^2}{(\omega_1 - \omega_0 - kv)^2 + \gamma_{ab}^2(1 + I/I_{sat})} dv \qquad (5-1)$$

In Eq. (5-1) the probability of finding the atoms with their velocities in the interval $(v, v+dv)$ is given by the Maxwell-Boltzman velocity distribution function.

ii) With laser 2 on, the # of these atoms in the ground state is reduced due to the saturation effect induced by laser 2. The reduction factor is just the negative of the population inversion under saturation.

$$\underbrace{-\sigma_3(v; \omega_2, I_2)}_{\text{correct factor to be inserted in the integral}} = 1 - \frac{\gamma_{ab}^2 (I_2/I_{sat})}{(\omega_2 - \underbrace{\omega_0 + kv}_{})^2 + \gamma_{ab}^2(1 + I_2/I_{sat})} \qquad (5-2)$$

$\omega_0 - kv$: resonance frequency

$\Rightarrow \sigma_3 = \rho_{aa} - \rho_{bb} = 2\rho_{aa} - 1$

note that in Lecture4, $\rho_{aa} = \dfrac{\gamma_{ab}^2 (I_0/I_{sat})/2}{(\omega-\omega_0)^2 + \gamma_{ab}^2 (1 + I_0/I_{sat})}$

excited, ground state population of that velocity group will be equal.

if Laser 2 does not induce saturation, Eq (5-1) : perfectly okay.

if saturation is very large, particular velocity of atom will be saturated,

→ cross section will decrease for that particular velocity → proportional to negative of population inversion

⇒ particular velocity group of atom undergoing saturation by Laser 2 makes detuning factor disappearing → $kv = \omega_0 - \omega_2$

pump laser.

Since Laser 2 is propagating in the opposite direction to laser 1, the sign in front of kv is reversed. The saturation occurs for a group of atoms with their velocities close to $(\omega_0 - \omega_2)/k$. Such selective reduction of absorption in an homogeneously broaden lineshape is called "spectral hole burning". As ω_1 is scanned, one would observe an absorption lineshape proportional to

$$\sigma(\omega_1;\omega_2) = \frac{\sigma^0(\omega_0)}{1+I_1/I_{sat}} \frac{1}{u\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(v/u)^2} \Big[1 - \underbrace{\frac{\gamma_{ab}^2 (I_2/I_{sat})}{(\omega_2-\omega_0+kv)^2+\gamma_2^2}}_{I_2 \text{가 만드는 } N_2-N_1} \Big] \underbrace{\frac{\gamma_1^2}{(\omega_1-\omega_0-kv)^2+\gamma_1^2}}_{I_1 \text{에 의한 } \sigma_{abs.}} dv \quad (5\text{-}3)$$

where

$$\gamma_1 = \gamma_{ab}\sqrt{1 + I_1/I_{sat}}, \quad \gamma_2 = \gamma_{ab}\sqrt{1 + I_2/I_{sat}} \quad (5\text{-}4)$$

we assume Laser 1 doesn't induce saturation.

∵ $\gamma_{ab} \ll ku$

We further assume $\gamma_1, \gamma_2 \ll ku$, and $I_1/I_{sat} \ll 1$ so that $\gamma_1 \simeq \gamma_{ab}$

assume medium is Doppler broadened ⇒ inhomogeneous broadening is dominant

sharply peaked at $\Delta_1 = kv$ (∵ $\gamma_{ab} \ll ku$)

$$\Rightarrow \frac{\sigma(\omega_1;\omega_0)}{\sigma^0(\omega_0)} \simeq \frac{1}{u\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(v/u)^2} \frac{\gamma_{ab}^2}{(\Delta_1 - kv)^2+\gamma_{ab}^2} dv - \frac{1}{u\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(v/u)^2} \underbrace{\frac{\gamma_{ab}^2 (I_2/I_{sat})}{(\Delta_2+kv)^2+\gamma_2^2}}_{\text{slowly varying}} \underbrace{\frac{\gamma_{ab}^2}{(\Delta_1-kv)^2+\gamma_{ab}^2}}_{\text{sharply peaked at } kv_0=\Delta_1} dv$$

$$\simeq \left(\frac{\sqrt{\pi}\gamma_{ab}}{ku}\right) e^{-(\Delta_1/ku)^2} - \left(\frac{I_2}{I_{sat}}\right)\left(\frac{\gamma_{ab}}{ku\sqrt{\pi}}\right) e^{-(\Delta_1/ku)^2} \int_{-\infty}^{+\infty} \frac{1}{(\delta_2+x)^2+\gamma^2} \frac{1}{(\delta_1-x)^2+1} dx \quad (5\text{-}5)$$

where $\Delta_{1,2} = \omega_{1,2} - \omega_0$, $\delta_{1,2} = \Delta_{1,2}/\gamma_{ab}$, $\gamma = \gamma_2/\gamma_{ab}$ (5-6)

Using the identity (Problem 12)

$$\int_{-\infty}^{+\infty} \frac{1}{(\delta_2+x)^2+\gamma^2} \frac{1}{(\delta-x)^2+1} dx = \frac{\pi(\gamma^{-1}+1)}{(\delta_1+\delta_2)^2+(\gamma+1)^2} \quad (5\text{-}7)$$

No.

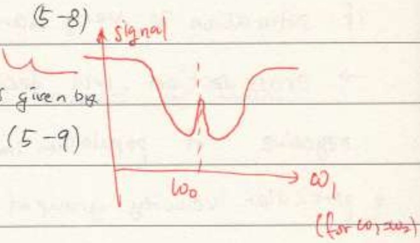
Date . .

we get unsaturated cross section (reduction due to inhomogeneous broadening)

$$\frac{\sigma(\omega_1;\omega_2)}{\sigma^0(\omega_0)} = \underbrace{\frac{\sqrt{\pi}\,\gamma_{ab}}{ku}e^{-(\Delta_1(ku))^2}}_{\text{we get if pump laser is not exist.}}\left[1-\frac{\gamma_{ab}^2(\gamma^2-1)(\gamma^{-1}+1)}{(\Delta_1+\Delta_2)^2+\gamma_{ab}^2(1+\gamma)^2}\right] \quad (5\text{-}8)$$

function of Δ_1, Δ_2

$\gamma = \frac{\gamma_2}{\gamma_{ab}} = \sqrt{1+\frac{I_2}{I_{sat}}}$, $\therefore \frac{I_2}{I_{sat}} = \gamma^2 - 1$



A dip occurs when $\Delta_1 = -\Delta_2$, i.e. $\omega_1 = 2\omega_0 - \omega_2$ and its width is given by
= hole

$$\Delta\omega_{\text{hole-burning}} = 2\gamma_{ab}\left(1+\sqrt{1+I_2/I_{sat}}\right) \quad (5\text{-}9)$$

$$= 2\gamma_{ab}(1+\gamma)$$

~ power broadened lineWidth by Laser 2.

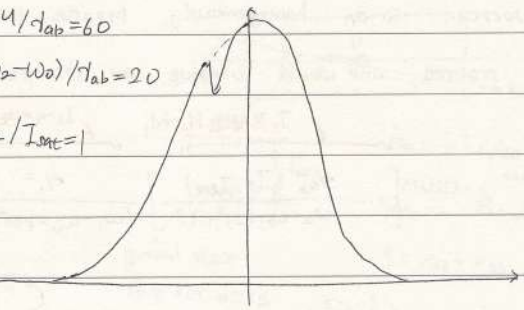
$\Delta_1 + \Delta_2 = 0$, $\frac{(\gamma^2-1)(1+\gamma)}{(1+\gamma)^2\gamma} \to 1$ as $\gamma \to \infty$, complete hole burning?

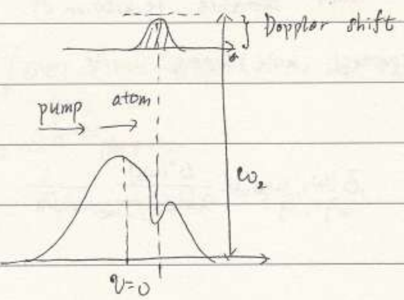
→ It's not actually the case.

$ku/\gamma_{ab} = 60$

$(\omega_2 - \omega_0)/\gamma_{ab} = 20$

$I_2/I_{sat} = 1$





Although our model gives an almost complete hole burning for $I_2/I_{sat} \gg 1$, in the actual lineshape measured by Laser 1 never shows such hole burnings. This is due to the coherent interaction between lasers 1 and 2 through the medium. The strong pump (Laser 2) induces so-called "dressed states", which has four levels instead 2, and Laser 1 can then be amplified or absorbed depending on its detuning, resulting in a less pronounced and broader hole burning. Therefore, our model is valid for I_2/I_{sat} not so much larger than 1.

depth of hole → lessened

→ at most 50% hole burning

No.
Date

5.2 Lamb dip

Suppose $\omega_1 = \omega_2 = \omega$. Then Eq. (5-8) becomes

from $\frac{\bar{\sigma}(\omega_1,\omega_2)}{\sigma^0(\omega_0)} = \left(\frac{\sqrt{\pi}\gamma_{ab}}{ku}\right)e^{-(\Delta/ku)^2}\left[1 - \frac{\gamma_{ab}^2(\gamma^2-1)(\gamma^{-1}+1)}{(\Delta_1+\Delta_2)^2 + \gamma_{ab}^2(1+\gamma)^2}\right]$

$$\frac{\bar{\sigma}(\omega)}{\sigma^0(\omega_0)} = \left(\frac{\sqrt{\pi}\gamma_{ab}}{ku}\right) e^{-(\Delta/ku)^2}\left[1 - \frac{\gamma_{ab}^2\cdot(\gamma^2-1)(\gamma^{-1}+1)/4}{\Delta^2 + \gamma_{ab}^2(1+\gamma)^2/4}\right] \qquad (5\text{-}10)$$

A dip occurs ~~with~~ when $\omega = \omega_0$ ($\Delta = 0$) and its width is given by

$$\Delta\omega_{Lamb\text{-}dip} = \gamma_{ab}\left(1 + \sqrt{1 + I_2/I_{sat}}\right) \qquad (5\text{-}11)$$

i) if $I_2/I_{sat} \ll 1$ i.e. $\gamma \simeq 1$ by $I_2/I_{sat} = \gamma^2 - 1$

$$\frac{\bar{\sigma}(\omega)}{\sigma^0(\omega_0)} \simeq \left(\frac{\sqrt{\pi}\gamma_{ab}}{ku}\right) e^{-(\Delta/ku)^2}\left[1 - \frac{I_2}{2I_{sat}}\frac{\gamma_{ab}^2}{(\omega-\omega_0)^2 + \gamma_{ab}^2}\right] \qquad (5\text{-}12)$$

For weak pumping by Laser 2, one can see a dip at $\omega = \omega_0$ where linewidth is just homogeneous line width on the much broader inhomogeneous lineshape. This is called "Lamb dip".
→ we can get information of γ_{ab} with inhomogeneous broadening (although inhomogeneous is strong) (useful technique)

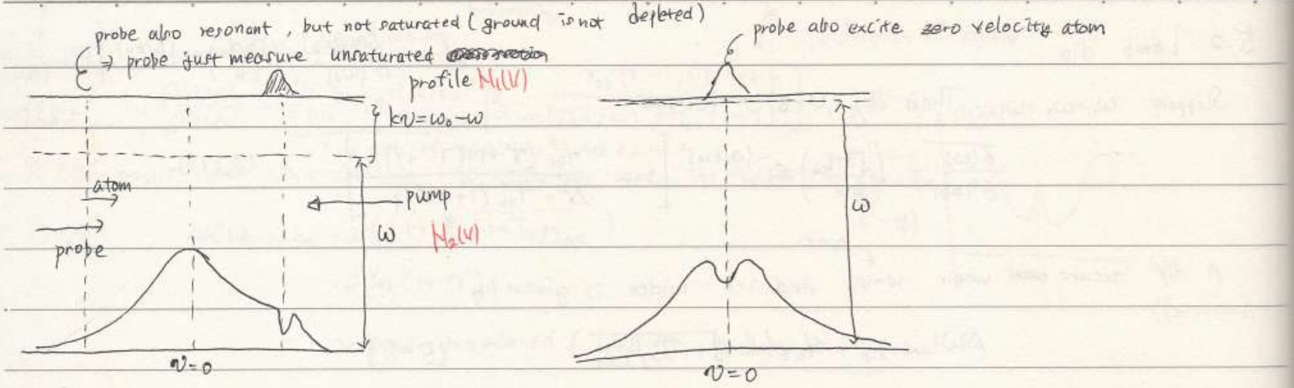
ii) if $I_2/I_{sat} \gg 1$, i.e., $\gamma \gg 1$ $\gamma^2 \simeq I_2/I_{sat}$ $(\gamma^2 = 1 + I_2/I_{sat})$

$$\frac{\bar{\sigma}(\omega)}{\sigma^0(\omega_0)} = \left(\frac{\gamma_{ab}}{ku/\sqrt{\pi}}\right) e^{-(\Delta/ku)^2}\left[1 - \frac{(\gamma_{ab}/2)^2\cdot(I_2/I_{sat})}{(\omega-\omega_0)^2 + (\gamma_{ab}/2)^2\cdot(I_2/I_{sat})}\right] \qquad (5\text{-}13)$$

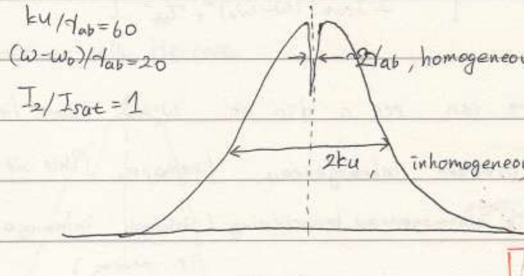
Once again our simple model predicts a complete Lamb dip, which never happens in real experiments where the coherent interaction of pump and probe in the dressed state picture introduces the detuning-dependent amplification or absorption of probe laser, and thus a less pronounced dip.

Originally, the Lamb dip was studied in the gas laser (by setting 2 mirrors), where the gain medium is inhomogeneous broaden and the intracavity laser field itself serves as a set of two counter-propagating laser beams. As the cavity is tuned, a dip in the output power appears when the cavity frequency matches the atomic resonance frequency ω_0. In this case, $I_1 = I_2 \gg 1$, and thus our analysis is not quite applicable. In modern laboratories, the Lamb dip is often used ☆ for frequency locking of a laser to an atomic transition
← fix laser frequency or stabilize frequency (reduce fluctuation)

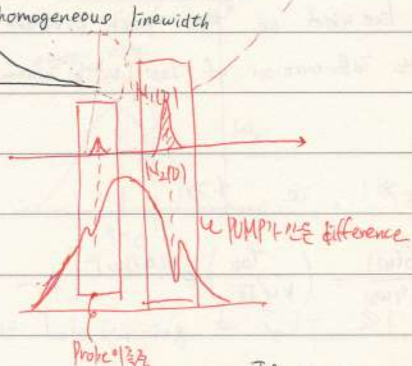
No.
Date . .



No saturation in the probe absorption. Saturation of the zero velocity group measured



⟨Fig.3⟩

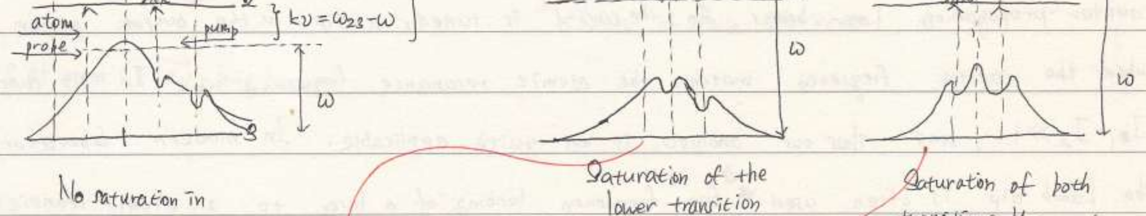


5.3 Cross over resonance

So, far, we have considered Lamb dips in a two-level system. If there are more than one transition within the inhomogeneous lineshape, variety of anomalous Lamb dips can occur. We will consider two cases associated with hyperfine (of F quantum number) levels.

i) Common ground state

Lamb dips at $\omega = \omega_{13}, \omega_{23}$. Crossover "dip" at $\omega = (\omega_{13}+\omega_{23})/2$



No saturation in probe absorption

Saturation of the lower transition measured

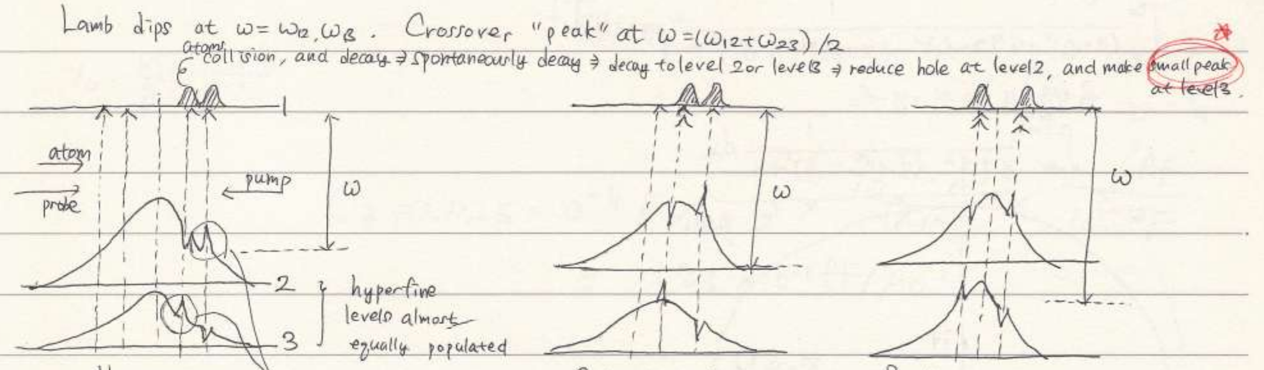
Saturation of both transitions Measured

⟨Fig.4⟩

ii) Common excited state

Lamb dips at $\omega = \omega_{12}, \omega_{13}$. Crossover "peak" at $\omega = (\omega_{12} + \omega_{23})/2$

atom collision, and decay ⇒ spontaneously decay ⇒ decay to level 2 or level 3 ⇒ reduce hole at level 2, and make small peak at level 3.



No saturation in probe absorption

Saturation of the upper transition measured

Population increase due to optical pumping measured

Peaks due to spontaneous decay → can give info on velocity changing collisions (Sensitive to collision) — if velocity change, they get broaden.

(= VCC)

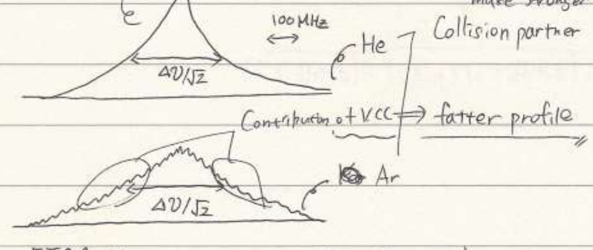
⟨Fig. 5⟩

~~Intensity of probe beam detector.~~

Example

Velocity changing collisions of Kr^* in He or Ar buffer gas : PRA 17(5), 1609 (1978).

we do not see VCC, atom: Kr^* ⇒ Ar is heavier than He ⇒ Ar can make stronger VCC.



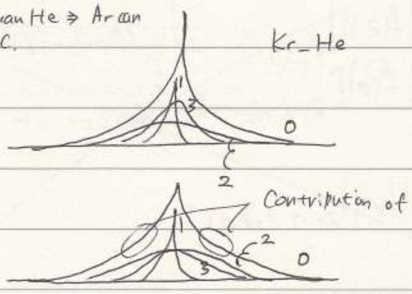


FIG1. Typical recording of the 557-nm line of Kr I in the presence of He and Ar perturbers. Upper trace : P_{Kr}, 8 mTorr; P_{He}, 260 mTorr. Lower trace: P_{Kr}; 8 mTorr; P_{Ar} 220 mTorr. The horizontal line represents the Doppler width $/\sqrt{2}$. In this experiment the frequency scale is deteremined by Fabry-Perot fringes spaced by 83 MHz.

FIG2. Typical fits for He and Ar perturbers. Upper trace: P_{Kr}, 8 mTorr; P_{He}, 110 mTorr. Lower trace: P_{Kr}, 8 mTorr; P_{Ar}, 120 mTorr. The solid line corresponds to the recorded profile and large dots represent the calculated profiles corresponding to the best fit. The three dotted lines represent the contributions from the three terms of Eq(3) : Curve 1 corresponds to the narrow resonance (first term), curve 2 corresponds to the Gaussian background (second term) arising from Kr^*-Kr collisions, and curve 3 corresponds to the contribution of VCC (third term) arising from Kr^*-He and Kr^*-Ar collisions.

velocity changing collision rate $\sim 10^4$/s·mTorr

phase changing collision rate $\sim 10^6$/s·mTorr

No.

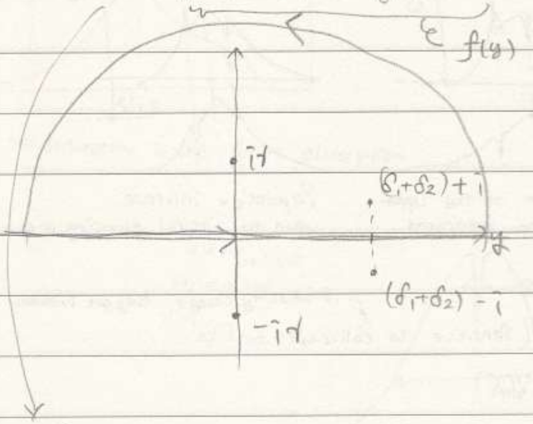
Date

$$\int_{-\infty}^{+\infty} \frac{1}{(\delta_2+x)^2+\gamma^2}\,\frac{1}{(\delta_1-x)^2+1}\,dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{(x+\delta_2)^2+\gamma^2(x-\delta_1)^2+(x+\delta_2)^2(x-\delta_1)^2+\gamma^2}\,dx$$

$\delta_2 + x = y \Rightarrow x = y - \delta_2$

$$\Rightarrow \int_{-\infty}^{+\infty} \underbrace{\frac{1}{y^2+\gamma^2}\,\frac{1}{(\delta_1+\delta_2-y)^2+1}}_{f(y)}\,dy$$



$$\oint f(z)\,dz = 2\pi i \;\mathrm{Res}\{f(z)\}\Big|_{z=i\gamma,\ \delta_1+\delta_2+i}$$

$$f(z) = \frac{1}{(z+i\gamma)(z-i\gamma)(\delta_1+\delta_2+i+z)(\delta_1+\delta_2-i+z)} \Rightarrow \frac{1}{2\gamma i}\,\frac{1}{(\delta_1+\delta_2+i(1-\gamma))(\delta_1+\delta_2-i(\gamma+1))}$$

$$\mathrm{Res}\{f(z)\}\Big|_{z=i\gamma} = \frac{1}{2\gamma i}\,\frac{1}{(\delta_1+\delta_2-i\gamma)^2+1}$$

$$\mathrm{Res}\{f(z)\}\Big|_{z=\delta_1+\delta_2+i} = \frac{1}{(\delta_1+\delta_2+i(1+\gamma))(\delta_1+\delta_2+i(1-\gamma))\,2(\delta_1+\delta_2+i)}$$

$$\Rightarrow \mathrm{Res}\{f(z)\}\Big|_{z=i\gamma,\ \delta_1+\delta_2+i} =$$

$$f(z) = \frac{1}{(z+i\gamma)(z-i\gamma)(z-(\delta_1+\delta_2)+i)(z-(\delta_1+\delta_2)-i)}$$

$$\mathrm{Res}\{f(z)\}\Big|_{z=i\gamma} = \frac{1}{2\gamma i}\,\frac{1}{(-(\delta_1+\delta_2)+i(\gamma+1))(-(\delta_1+\delta_2)+i(\gamma-1))}$$

$$\mathrm{Res}\{f(z)\}\Big|_{z=\delta_1+\delta_2+i} = \frac{1}{(\delta_1+\delta_2+i(\gamma+1))(\delta_1+\delta_2+i(-\gamma+1))(+2i)}$$

$= i(\gamma+1) - i(\gamma^2+\gamma)$

$= i(-\gamma+1) - i(\gamma^2-\gamma)$

$$\Rightarrow \mathrm{Res}\{f(z)\}\Big|_{z=i\gamma,\ \delta_1+\delta_2+i} = \frac{+1}{((\delta_1+\delta_2)^2+(\gamma^2+1))^2\,2\gamma i}\left[\frac{\delta_1+\delta_2+i(\gamma+1)+(\delta_1+\delta_2)-i(\gamma+1)\gamma}{\delta_1+\delta_2+i(1-\gamma)}\right]$$

$$= \frac{+1}{((\delta_1+\delta_2)^2+(\gamma^2+1)^2)\,2\gamma i}\left[\frac{\delta_1+\delta_2+i(-\gamma+1)}{\delta_1+\delta_2+i(1-\gamma)}\right](1+\gamma)$$

$$= \frac{(1+\gamma)}{(\delta_1+\delta_2)^2+(\gamma+1)^2}\cdot\frac{1}{2\gamma}$$

No.

Date . .

$$\therefore \int_{-\infty}^{+\infty} \frac{1}{(\delta_2+x)^2+\gamma^2}\,\frac{1}{(\delta_1-x)^2+1}\,dx = \frac{\pi\left(\frac{1}{\gamma}+1\right)}{(\delta_1+\delta_2)^2+(\gamma+1)^2}$$

$fF/\mu m^2$

$$V_0 = \frac{kT}{q}\ln\frac{N_A N_D}{n_i^2}$$

f 10^{-15}

n 10^{-9}

p 10^{-12}

$$C_2 = 1.725\times10^{-4}\,F/m^2 \times \frac{10^{-12}\,m^2}{1\,\mu m^2}\times\frac{1\,fF}{10^{-15}F}$$

$$= 1.725\times10^{-1}\,fF/\mu m^2$$

~~$1 = 10\exp\left(\frac{V_1}{V_T}\right)$~~

$$1\exp(V_1/V_T) = 10\exp(V_f/V_T)$$

$$\Rightarrow \frac{1}{10} = \exp(\Delta V/V_T)$$

$$\therefore \Delta V = -V_T\ln 10$$

No.

Date . .

6. Optical Bloch equation ⇒ originally developed for describe nuclear spin with extra magnetic field

6.1 Derivation

state a, state b, Γ_0

$\dot{\rho}_{aa} = -\Gamma_0 \rho_{aa} + \frac{i}{2}\Omega e^{-i\omega t}\rho_{ba} + c.c$

$\dot{\rho}_{ab} = -\gamma_{ab}\rho_{ab} - i\omega_0 \rho_{ab} - \frac{i}{2}\Omega e^{-i\omega t}(\rho_{aa} - \rho_{bb})$

$1 = \rho_{aa} + \rho_{bb}$

Recall the transformation

$$\rho_{ab} = (\sigma_1 + i\sigma_2) e^{-i\omega t}$$

$$\rho_{aa} - \rho_{bb} = \sigma_3 \quad \text{with} \quad \rho_{aa} + \rho_{bb} = 1 \qquad (6-1)$$

Rewrite the density matrix equation, Eq(4-1) → if $\vec{E} \to \vec{0}$

$$\dot{\sigma}_1 = -\gamma_{ab}\sigma_1 - \Delta\sigma_2$$

$$\dot{\sigma}_2 = -\gamma_{ab}\sigma_2 + \Delta\cancel{\sigma}_1 - \frac{1}{2}\Omega\sigma_3 \qquad (6-2)$$

$$\dot{\sigma}_3 = -\Gamma_0(1+\sigma_3) + 2\Omega\sigma_2$$

$\dot{\sigma}_1 = -\gamma_{ab}\sigma_1 + \omega_0\sigma_2$

$\dot{\sigma}_2 = -\gamma_{ab}\sigma_2 - \omega_0\sigma_1$

$\dot{\sigma}_3 = -\Gamma_0(1+\sigma_3)$

⇒ σ_i : real value

Defining $\vec{S}$ vector as (bloch vector)

$$S_1 = 2\sigma_1, \quad S_2 = -2\sigma_2, \quad S_3 = \sigma_3 \qquad (6-3)$$

The equation of motion becomes

$$\dot{S}_1 = -\gamma_{ab} S_1 + \underbrace{\Delta}_{K_3 = \Delta} S_2$$

$$\dot{S}_2 = -\gamma_{ab} S_2 - \Delta S_1 + \underbrace{\Omega}_{K_1 = \Omega} S_3 \qquad (6-4)$$

$$\dot{S}_3 = -\Gamma_0(1+S_3) - \Omega S_2 \quad K_2 = 0$$

→ $\dot{S}_1 = -\gamma_{ab} S_1 - \omega_0 S_2$

$\dot{S}_2 = -\gamma_{ab} S_2 + \omega_0 S_1$

or

$\vec{K} = (\Omega, 0, \Delta)$

$$\dot{\vec{S}} = -\Gamma(\vec{S} + \vec{e}_3) + \vec{S} \times \vec{K} \qquad (6-5)$$

optical bloch equation ⇒ $\vec{S}$ will precess around $\vec{K}$, decay (decreasing of magnitude $|\vec{S}|$), after long time be $\vec{e}_3$

where

$$\underbrace{\Gamma = \begin{bmatrix} \gamma_{ab} & 0 & 0 \\ 0 & \gamma_{ab} & 0 \\ 0 & 0 & \Gamma_0 \end{bmatrix}}_{\text{decay matrix}}, \quad \underbrace{\vec{K} = \begin{bmatrix} \Omega \\ 0 \\ \omega - \omega_0 \end{bmatrix}}_{\text{torque vector}}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (6-6)$$

which is nothing but a classical spin vector $\vec{S}$ precessing under torque vector $\vec{K}$ with damping.

$= \frac{6 rad\,\Gamma_0 \cdot I_0}{\hbar\omega_0}$ //

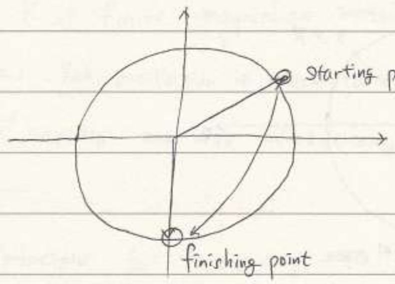
Notice that $\Omega^2 = \frac{\mu^2 E_0^2}{\hbar^2} = \frac{6\pi (C/\omega_0)^2 \cdot \Gamma_0 I_0}{\hbar\omega_0}$

No.

Date . .

6.2. Evolution of the Bloch vector

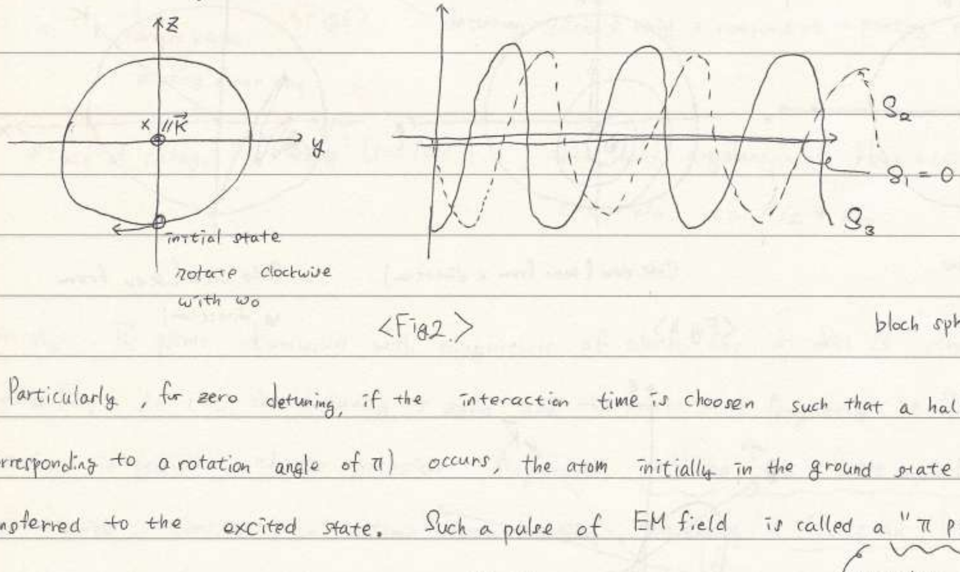
i) Suppose $\vec{K} = 0$. Then $\vec{S}$ will decay to $\vec{S} = -\vec{e}_3$, pointing to the $-z$ direction or downward in Fig. 1.



<Fig 1>

ii) Suppose $\Gamma = 0$ ($\Gamma_0 = 0$, $\omega = \omega_0$) (negligible damping). Let $\vec{S}(0) = -\vec{e}_3$ (initially ground state) i.e., $\rho_{aa}(0) = 0$, $\rho_{bb}(0) = 1$ (initially in the ground state). With zero detuning the torque vector $\vec{K}$ is in the $+x$ direction, and thus we get full Rabi oscillations at the angular frequency of Ω.

With finite detuning the torque vector is on the x-z plane (upward), and we get partial Rabi oscillations at the angular frequency of $\sqrt{\Omega^2 + (\omega - \omega_0)^2}$.



<Fig 2>

bloch sphere : boundary of bloch vector

iii) Particularly, for zero detuning, if the interaction time is choosen such that a half Rabi oscillation (corresponding to a rotation angle of π) occurs, the atom initially in the ground state will be perfectly transferred to the excited state. Such a pulse of EM field is called a "π pulse" (See Prob 13)

2π pulse : come back to ground state

$\pi/2$ pulse : wait 1/4 times of period ⇒ point half of equator of bloch sphere.

iv) With damping, the amplitude of S decreases and spirals toward a steady state, which is obtained by letting all derivatives equal to zero in Eq (6-4), which is equivalent to Eq (4-6)

$$S_1 = -\frac{\Omega\Delta}{\Delta^2 + \gamma_{ab}^2 + \Omega^2\gamma_{ab}/\Gamma_0}$$

$$S_2 = -\frac{\Omega\gamma_{ab}}{\Delta^2 + \gamma_{ab}^2 + \Omega^2\gamma_{ab}/\Gamma_0} \qquad (6\text{-}7)$$

$$S_3 = -\frac{\Delta^2 + \gamma_{ab}^2}{\Delta^2 + \gamma_{ab}^2 + \Omega^2\gamma_{ab}/\Gamma_0}$$

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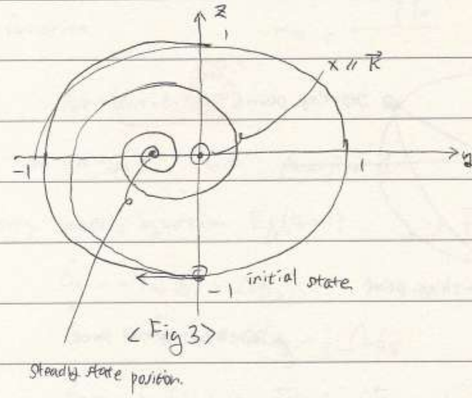
For example, if $\Delta=0$, $\Gamma_0/\gamma_{ab}=2$, $\Omega/\gamma_{ab}=10$, we get $S_1=0$, $S_2=-10/51$, $S_3=1/51$,

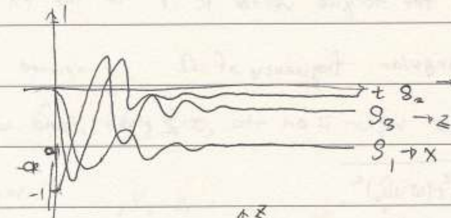
as shown in the numerical simulation below

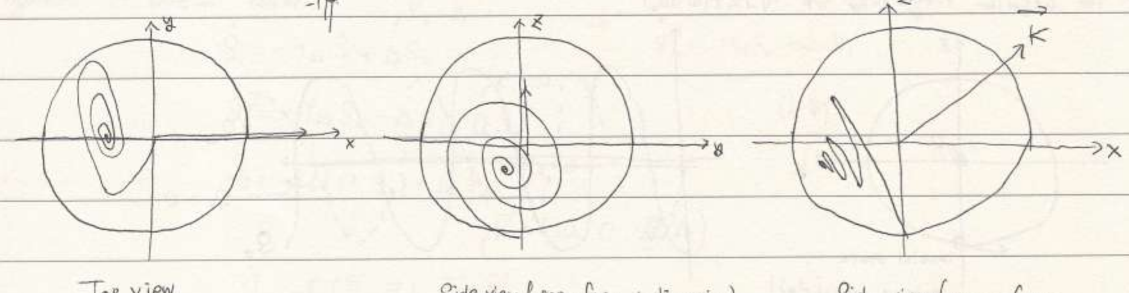


<Fig 3>

• Another Example.

$\Delta=5$, $\Gamma_0/\gamma_{ab}=2$, $\Omega/\gamma_{ab}=10$



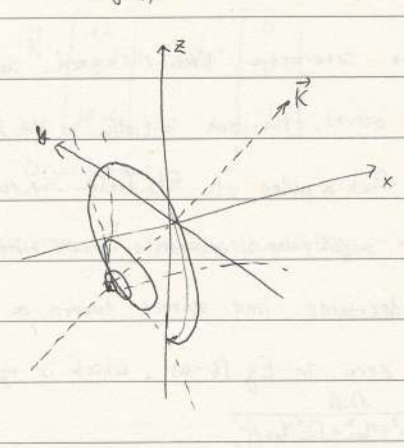


Top view

Side view (seen from x direction)

Side view (seen from y direction)

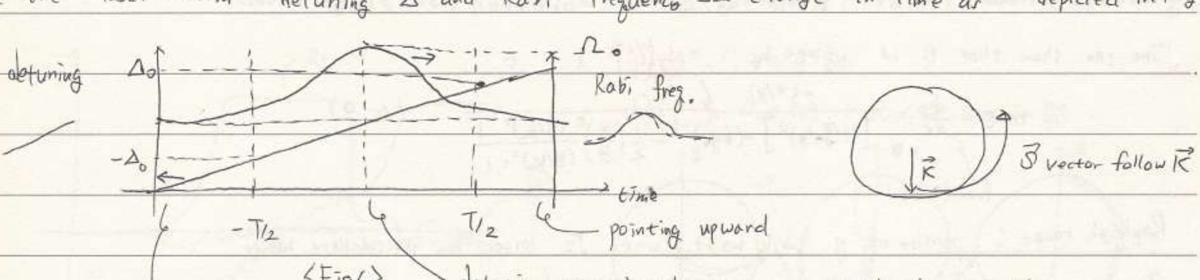
<Fig.4>



6.3 Adiabatic Following

Suppose the two level atom is initially in the ground state (i.e. bloch vector $\vec{S}$ pointing downward). Suppose the torque vector $\vec{K}$ of finite magnitude initially points downward and rotates slowly upward. If the instaneous Rabi oscillation is much faster than the rate at which $\vec{K}$ swings, $\vec{S}$ will spin around $\vec{K}$ rapidly and thus adiabatically follow it. This is called "adiabatic following".

One can use this principle for inverting the two level atom very efficiently. This is known as "adiabatic inversion". Let us neglect the damping for the time being. Assume that the laser-atom detuning Δ and Rabi frequency Ω change in time as depicted in Fig 6.

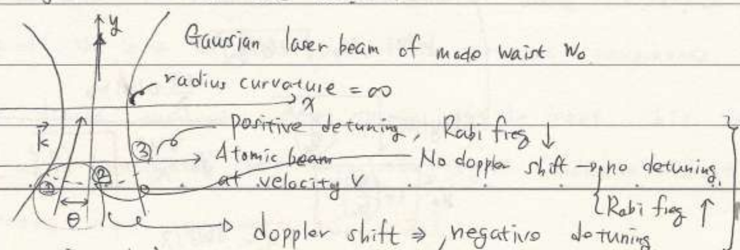


<Fig6>

→ rate of change = π/T [rad/sec] , rotational angularspin of bloch vector around torgue vector

$t=0 \to \omega_0, \quad t=-T/2 \to \Delta_0$

Initially, $\vec{K}$ points downward with magnitude of about Δ_0, at which $\vec{S}$ starts to precess around $\vec{K}$. At $t=0$, the detuning is zero and the precession frequency is Ω_0. The adiabatic inversion is possible if the precession frequencies, both Δ_0, and Ω_0 are much larger than π/T. As an example consider an atomic beam ~~traveling~~ traversing a Gaussian laser beam just below the focal plane as shown in Fig 7. The laser is on resonance with the atoms at rest. One can show that both the Rabi frequency and the frequency detuning behave in the same way as depicted in Fig 6. For details see Sec 6.4.



<Fig 7>

No.

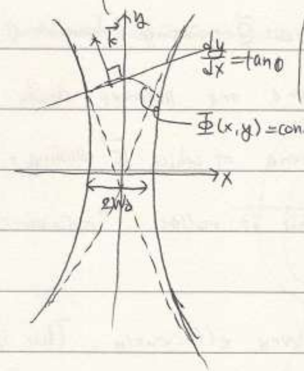
Date

$\vec{k}$ is orthogonal to wave front

constant phase surface = wave front

6.4. Adiabatic inversion using a Gaussian beam.

⇒ we don't consider z component. since atom does'nt have z component (going along x)



For a Gaussian beam propagating in y direction, the phase factor of the electric field is described by (on the $z=0$ plane) "= Phase"

$$\underbrace{\Phi(x,y) = -ky + \tan^{-1}(y/y_0) - \frac{kx^2}{2R(y)}} \qquad (6-8)$$

where $y_0 = \pi w_0^2/\lambda$, the confocal parameter or the Rayleigh range, and $R(y) = y(1+y_0^2/y^2)$, the radius of curvature.

The wave vector is normal to the surface of constant ~~plan~~ phase. The angle θ between the wave vector and the axis of the Gaussian beam is given by $\theta = \tan^{-1}(dy/dx)$, where dy/dx is obtained from $\Phi(x,y) = \text{const}$.

One can show that θ is given by

$$\tan\theta = \frac{dy}{dx} = \frac{-(x/y)}{[1+(y_0/y)^2] - (ky_0)^{-1} - \frac{1}{2}\left(\frac{x}{y}\right)^2 \frac{(y/y_0)^2-1}{(y/y_0)^2+1}} \qquad (6-9)$$

$\frac{1}{ky_0}\left(\frac{y_0}{y}\right)^2$?

from (6-8)

Rayleigh range : position of y while waist become $\sqrt{2}$ larger than the smallest waist

$R(y)$: radius curvatue of constant phase wave front

e.g. $R(y=y_0) = \sqrt{2}\, y_0$

For typical experimental conditions the expression is further simplified. Suppose $w_0 = 30\,\mu m$, $\lambda = 791\,nm$ ($^1S_0 - {}^3P_1$ transition of atomic barium), and then $y_0 = 3.6\,mm$ and $ky_0 = 2.9\times 10^4$. Choose $y \gtrsim y_0$. Since $x \ll y_0$ for the points within the laser beam profile, the second and the third term in the denominator are negligible, and thus

$$\theta \simeq \tan^{-1}\frac{-xy}{y^2+y_0^2} \simeq -\left(\frac{y}{y_0^2+y^2}\right)x \qquad (6-10)$$

which is linear in x, and thus the laser-atom frequency detuning Δ due to the Doppler shift is also linear in x. The value of Δ near the beam boundary, Δ_0, is

$$\Delta_0 \simeq kv\theta\Big|_{x=w(y)} = \frac{kv|y|\cdot w(y)}{y^2+y_0^2} = \left(\frac{v}{w_0}\right)^{\cancel{2}} \frac{2(|y|/y_0)}{\sqrt{1+(y/y_0)^2}} \qquad (6-11)$$

where we used the relation for the Gaussian beam

$$w(y) = w_0\sqrt{1+(y/y_0)^2}$$

$$\frac{\frac{2\pi}{\lambda}v|y|\,w_0\sqrt{1+\left(\frac{y}{y_0}\right)^2}}{y_0^2\left[1+\left(\frac{y}{y_0}\right)^2\right]} = \frac{\frac{2\pi}{\lambda}v|y|\,w_0}{y_0\frac{\pi w_0^2}{\lambda}\sqrt{1+\left(\frac{y}{y_0}\right)^2}}$$

$$y_0 = \pi w_0^2/\lambda$$

No.

Date . .

whereas the rate at which $\vec{K}$ swings is approximately π/T with T the transit time across the pump beam at height y. The transit time is just $\sqrt{\pi}\,w(y)/v$, and thus the swing rate R is

$$R = \frac{\sqrt{\pi}\,v}{w_0\sqrt{1+(y/y_0)^2}} \quad (6\text{-}13)$$

$\neq R(y)$

The ratio Δ_0/R is

using $\Delta_0 = \frac{v}{w_0}\cdot\frac{2\frac{|y|}{y_0}}{\sqrt{1+(y/y_0)^2}}$

$$\Delta_0/R = \frac{2}{\sqrt{\pi}}\frac{y}{y_0} \simeq \frac{y}{y_0} \quad (6\text{-}14)$$

we want this value is much bigger than 1.

Therefore, for adiabatic following to occur, $y \gg y_0$.

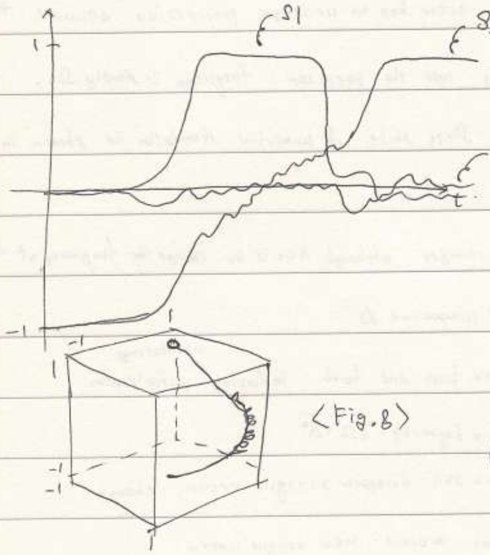
if we use focus Gaussian beam, below the ~~below the~~ smallest spot $y=y_0$, if we choose this y much bigger the confocal parameter, we can satisfy this.

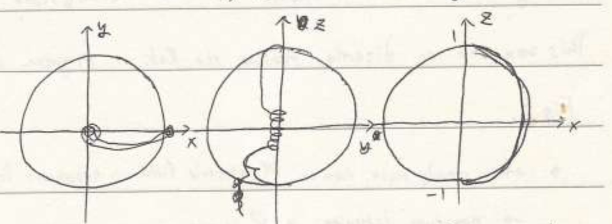
If we choose $y=3y_0$, and assume $v=350\,m/s$, for example, we get $\Delta_0 = 2.2\times10^7\,sec^{-1}$, $R=6.5\times10^6\,sec^{-1}$.

In Fig.8, the numerical solution of Eq (6-4) is shown for the parameters assumed above.

$\gamma_{ab}=0,\ \Gamma_0=0,\ y=3y_0$

$\Omega_0=10,\ \Delta_0=1.9,\ R=0.56$ time in $\frac{w_0}{v}$



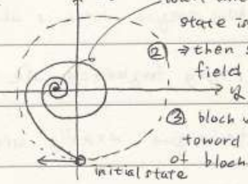


Top view | Side view (seen from x direction) | Side view (seen from y direction)

⟨Fig.8⟩

→ x, y : off diagonal density matrix
→ determine induced dipole moment
→ have polarization → can radiate
→ transmission doesnot disappear immediatley, but exponentially
: "FID"

decay is exponential
x, y decays with γ_{ab}
z decays with Γ_0



① wait until steady state is reached
② → then shut off input field → $\Omega=0$
③ bloch vector decay toword the bottom of bloch sphere

6.5 Free induction decay and optical nutation

✓ Free-induction decay

Suppose a steady state has been reached as in Fig.3 with a resonant driving EM field. Wile monitoring the transmission of the driving laser, one turns off the driving field suddenly. In this case, the transmitted signal does not disappear at once. Instead, it decays exponentially. Since the signal exist without an input, it is called "free-induction decay" (FID). In the steady state before turning off the laser the Bloch vector has nonzero components S_1 and S_2, which determine the induced dipole moment. After turning off the input, this induced dipole moment undergoes relaxation while generating the transmission signal (radiation) without the input laser beam (recall that an oscillating dipole generates radiation).

No.

Date . .

✓ Optical nutation

Suppose one introduces a sudden Stark shift to the sample above instead of turning off the input laser. For this resonant excitation before applying the Stark field, the Bloch vector is on y-z plane in the steady state, almost parallel to y axis. The sudden Stark shift makes the atoms no longer resonant with the EM field. This is like introducing a large (negative) detuning suddenly.

The resulting torque vector is a vector sum of previous one and new one in the z-direction. The Bloch vector then precesses around the new torque vector, resulting in a ringing in the transmission signal. This phenomenon is called "optical nutation". The precession frequency is $\sqrt{\Omega^2+\Delta^2}$.

Optical nutation can occur again when the DC electric field is suddenly turned off. This is because the torque vector changes its direction again, and thus Bloch vector has to undergo precession around the new torque vector until it reaches a new steady state. In this case the precession frequency is simply Ω. This way one can directly measure the Rabi frequency and the Stark shift. A numerical simulation is shown in Fig. 9.

⇒ after steady state, turn on DC electric field → resonant frequency changes, although there is no change in frequency of AC field

→ negative detuning ⇒ $\vec{R}$ is not along x, tilted due to z component Δ

→ steady state Bloch vector spiral to new $\vec{R}$ vector ⇒ oscillate back and forth, inducing oscillating polarization

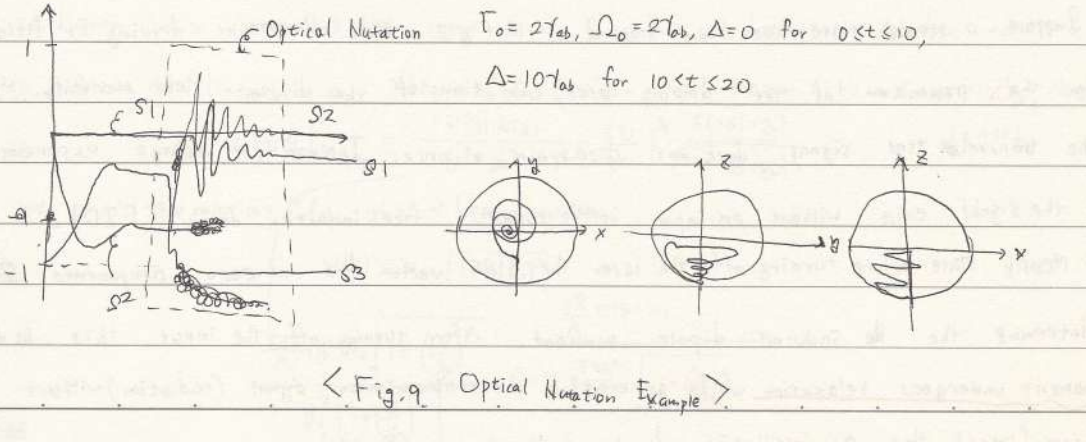
⇒ generate radiation → ringing signal, "optical nutation" → frequency = $\sqrt{\Omega^2+\Delta^2}$

⇒ wait for new steady state, and turn off electric field → Stark shift disappear ⇒ torque vector change

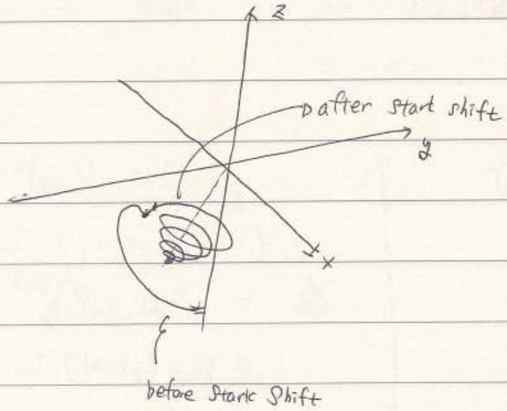
⇒ now torque vector is along the x axis → bloch vector spiral around new torque vector

⇒ ringing frequency = Ω → measure two different frequency, ⇒ determine stark shift value

⇒ technique usually used to measure Stark shift & Rabi frequency in optical medium



< Fig. 9 Optical Nutation Example >


z
after Start shift
y
x
before Stark Shift

No. Problem13

Date . .

ω

$\cancel{t = \omega_0^2 = \omega}$

$$\int e^{-\gamma t} \cos\omega t \, dt$$

$\Rightarrow$

$$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2} \qquad \frac{\cancel{\omega_0}}{(a+j\omega)^2+\omega_0^2}$$

$$e^{-at}\cos\omega_0 t \cdot u(t) \qquad e^{-at}\sin\omega_0 t \, u(t)$$

$$\frac{a+j\omega}{a^2+\omega_0^2+2aj\omega+\omega_0^2-\omega^2}$$

$$\omega^4 - 2(a^2+\omega_0^2)$$

$$\frac{a^2+\omega^2}{(a^2+\omega_0^2-\omega^2)^2+(2a\omega)^2} = (\omega_0^2-a^2) \pm \sqrt{2a^2}$$

$$\omega = \sqrt{\omega_0^2-a^2} \quad \frac{|A|^2(a^2+\omega^2)+\omega_0^2|B|^2}{(a^2+\omega_0^2-\omega^2)^2+(2a\omega)^2}$$

$$(a^2+\omega_0^2)^2 - 2(a^2+\omega_0^2)\omega^2 + \omega^4$$

$$\omega^2 = \omega_0^2 - a^2$$

$$(a^2+\omega_0^2)^2 - 2(a^2+\omega_0^2)\omega^2 + \omega^4 + 4a^2\omega^2$$

$$\omega^4 - 2(\omega_0^2-a^2)\omega^2$$

$$\cancel{\omega^2 = \frac{}{2(4a^2+1)}}$$

$$\cancel{\omega^2 = \frac{1}{2}\left(2(-a^2+\omega_0^2) \pm 2\sqrt{(a^2-\omega_0^2)^2-(a^2+\omega_0^2)^2}\right)}$$

set $\omega_0 = 10$

No.

Date . .

$$\Omega^2 = \frac{6\pi(c/\omega_0)^2\Gamma_0 I_0}{\hbar\omega_0} = \frac{6\pi c^2\Gamma_0 I_0}{\hbar\omega_0^3} \qquad \Omega = \sqrt{\frac{6\pi c^2\Gamma_0 I_0}{\hbar\omega_0^3}} \qquad \Delta t = \frac{2W}{v} = \frac{40\mu m}{400 m/s} = 0.1\times10^{-6}\ sec$$

$= 10^{-7}\ sec$

$\Rightarrow \Omega = \frac{\pi}{10^{-7}}$

$= \pi\times10^{7}\ Hz$

$= \pi\times10^{4}\ kHz$

$\gg \Gamma_0/2\pi$

$\Gamma_0 \simeq 300\times10^3 = 3\times10^5$

$$\dot{S}_2 = -\gamma_{ab}S_2 + \Omega S_3 \qquad \gamma_{ab} = \Gamma_0/2$$

$$\dot{S}_3 = -\Gamma_0(1+S_3) - \Omega S_2$$

$$\dot{S}_2 = -\frac{\cancel{\Gamma_0}}{2}S_2 + \Omega S_3 \Rightarrow \cancel{\ddot{S}_2}$$

$$\dot{S}_3 = -\Gamma_0(1+S_3) - \Omega S_2$$

$$\& \quad \ddot{S}_3 = \cancel{-\Gamma_0(1+\dot{S}_3)}$$

$$= -\Gamma_0\dot{S}_3 - \Omega\dot{S}_2$$

$$= -\Gamma_0\dot{S}_3$$

$$\left(e^{\frac{\Gamma_0}{2}t}S_2\right)^{\cdot} = \Omega S_3 e^{\frac{\Gamma_0}{2}t} = \dot{\phi}_1$$

$$\cancel{\left((S_3+1)e^{\Gamma_0 t}\right)^{\cdot} = -\Omega S_2 e^{\Gamma_0 t} = \dot{\phi}_2}$$

$$\left(S_3 e^{\Gamma_0 t}\right)^{\cdot} = -\Gamma_0 e^{\Gamma_0 t} - \Omega e^{\Gamma_0 t}S_2 = \dot{\phi}_2$$

$$\int^{u(t)} e^{-\gamma t} e^{i\omega t} e^{i(\omega_0 t+\varphi)}\, dt$$

$$= \frac{-\gamma\ e^{i\varphi}}{\gamma - i\omega - i\omega_0 - i\varphi}$$

$$\dot{S}_3 e^{\frac{\Gamma_0}{2}t} = -\Gamma_0(1+S_3)e^{\frac{\Gamma_0}{2}t} - \Omega S_2 e^{\frac{\Gamma_0}{2}t}$$

$$\ddot{S}_3 e^{\frac{\Gamma_0}{2}t} + \frac{\Gamma_0}{2}\dot{S}_3 e^{\frac{\Gamma_0}{2}t} = -\frac{\Gamma_0^2}{2}(1+S_3)e^{\frac{\Gamma_0}{2}t} - \Gamma_0\dot{S}_3 e^{\frac{\Gamma_0}{2}t} - \Omega^2 S_3 e^{\frac{\Gamma_0}{2}t}$$

(γ_{ab}, $\Gamma_0\gamma_{ab}$ written above $\frac{\Gamma_0}{2}$)

$$\cancel{\Rightarrow \ddot{S}_3 + \frac{\Gamma_0}{2}\dot{S}_3 = -\left(\frac{\Gamma_0^2}{2}+\Omega^2\right)S_3 - \Gamma_0\dot{S}_3 - \frac{\Gamma_0^2}{2}}$$

$$\cancel{\Rightarrow \ddot{S}_3 + \frac{3\Gamma_0}{2}\dot{S}_3 + \left(\frac{\Gamma_0^2}{2}+\Omega^2\right)\left(S_3 + \frac{\Gamma_0^2/2}{\frac{\Gamma_0^2}{2}+\Omega^2}\right) = 0}$$

$$((a^2+\omega_0^2-\omega^2)^2 + (2a\omega)^2)$$

$$2\omega(a^2+\omega_0^2-\omega^2)$$

$$-(a^2+\omega^2)\left(4\omega(a^2+\omega_0^2-\omega^2) + 8a^2\omega\right)$$

$$\ddot{S}_3 + \gamma_{ab}\dot{S}_3 = -\Gamma_0\gamma_{ab}(1+S_3) - \Gamma_0\dot{S}_3 - \Omega^2 S_3$$

$$\Rightarrow \ddot{S}_3 + \underset{(\Gamma_0+}{\gamma_{ab})}\dot{S}_3 + (\Gamma_0\gamma_{ab}+\Omega^2)S_3 = -\Gamma_0\gamma_{ab}$$

$$(a^2+\omega_0^2-\omega^2)^2 + 4a^2\omega^2$$

$$-(a^2+\omega^2)\left(-2(a^2+\omega_0^2-\omega^2) + 4a^2\omega^2\right)$$

$$\ddot{S}_3 + \overbrace{(\Gamma_0+\gamma_{ab})}^{=2\Gamma}\dot{S}_3 + \overbrace{(\Gamma_0\gamma_{ab}+\Omega^2)}^{\omega^2}\left(S_3 + \frac{\Gamma_0\gamma_{ab}}{\Gamma_0\gamma_{ab}+\Omega^2}\right) = 0$$

$$\Rightarrow S_3 = e^{-\Gamma t}(A\cos\omega t + B\sin\omega t) - \frac{\Gamma_0\gamma_{ab}}{\Gamma_0\gamma_{ab}+\Omega^2}$$

$$A = \frac{\Gamma_0\gamma_{ab}}{\Gamma_0\gamma_{ab}+\Omega^2} - 1$$

$$\dot{S}_3 = -\Gamma e^{-\Gamma t}(A\cos\omega t + \cancel{B\sin\omega t}) + \omega e^{-\Gamma t}(-\cancel{A\sin\omega t} + B\cos\omega t)$$

$$= 0 \Rightarrow B = \left(-1 + \frac{\Gamma_0\gamma_{ab}}{\Gamma_0\gamma_{ab}+\Omega^2}\right)\frac{\Gamma}{\omega}$$

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No.

Date . .

5 0 0 2 we set $w_0 = 50$

10 0 0 2

5 0 0 4

10 0 0 4

5 0 3 2

10 0 3 2

5 0 3 4

10 0 3 4

$$e^{-\Gamma t}\left(A\cos\omega t - A\frac{\Gamma}{\omega}\sin\omega t\right)$$

$$-\Gamma e^{-\Gamma t}\left(\cancel{A\cos\omega t} + A\frac{\Gamma}{\omega}\sin\omega t\right)$$

$$+e^{-\Gamma t}\left(-\omega A\sin\omega t + \cancel{A\Gamma\cos\omega t}\right)$$

$$P3 = [\Omega \quad 0 \quad \Delta \quad \underset{\downarrow \atop \Gamma_0/2}{\gamma_{ab}} \quad \Gamma_0]$$

No.

Date

$$2\omega\left\{(a^2+\omega_0^2-\omega^2)^2+(2a\omega)^2\right\}-\omega^2\left\{-4\omega(a^2+\omega_0^2-\omega^2)+8a^2\omega\right\}$$

$$\Rightarrow (a^2+\omega_0^2-\omega^2)^2+(2a\omega)^2-\omega^2\left(-2(a^2+\omega_0^2-\omega^2)+4a^2\right)$$

$$=(a^2+\omega_0^2-\omega^2)^2+2\omega^2(a^2+\omega_0^2-\omega^2)$$

$$=(a^2+\omega_0^2-\omega^2)(a^2+\omega_0^2+\omega^2) \quad \Rightarrow \omega^2=\omega_0^2+a^2.$$

after π/ku, summation of density matrix very small ⇒ make coherence disappear

No. Ch7 More Applications of Bloch Equation

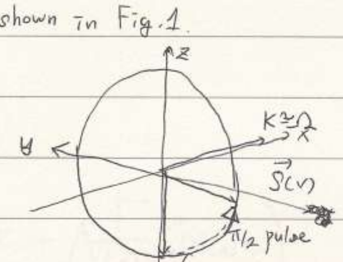
Date . .

7.1 Photon echo. ← Technique

atoms see ~~see~~ different laser frequency → doppler shifted by kv
⇒ Create off diagonal density matrix element of individual atoms
⇒ oscillation frequency of off-diagonal density matrix element will be frequency that atoms see. → $\omega - kv$ ⇒ although they are in phase at $t=0$, after time, their sum become zero. large

Consider a collection of two-level atoms in a form of gas. This medium is inhomogeneously broadened due to independent Doppler shifts of individual atoms. Suppose these atoms are excited by short laser pulse. whose center frequency is resonant with the atoms at rest. We assume that the Rabi frequency Ω is much larger than the amount of inhomogeneous broadening so that for all members of the ensemble $\Omega \gg \Delta$ with Δ = Doppler shift.

Thus, the torque vector $\vec{K}$ is approximately along the x direction for all atoms. With the pulse area of $\pi/2$, we can rotate the Bloch vectors of individual atoms to the $-y$ direction as shown in Fig.1.



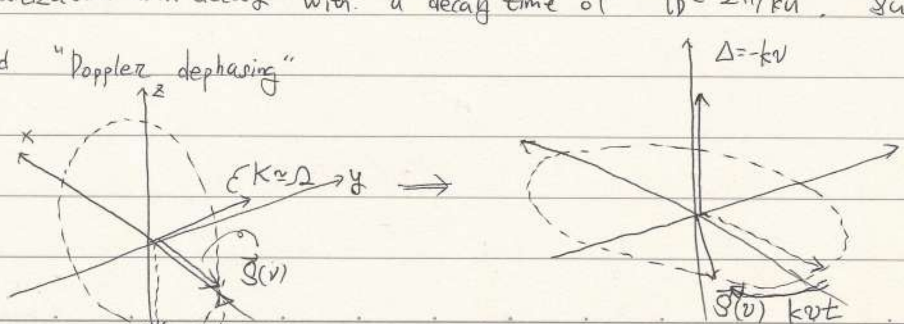
detuning effect is negligible ⇒ effectively all atoms can be put into this state.
($\because \Omega \gg \Delta$)

Now individual atoms experience dephasing of ρ_{ab} or decay of $\vec{S}$ at the rate of γ_{ab}.

However, there exist an additional dephasing process due to Doppler broadening.

The induced dipole of the atom with velocity v in the direction of laser beam oscillates at $\omega - kv$
(after $\pi/2$ pulse)
although laser is off, due to detuning, in this bloch vector representation, it rotates

So in the rotating frame of Bloch picture, $\vec{S}$ associated with this atom will rotate in the $-\phi$ direction at a rate of $\Delta = -kv$. For the mean thermal velocity u, the rotation angle becomes 2π in a time $2\pi/ku$. Therefore, the total dipole moments or the polarization will decay with a decay time of $T_D = 2\pi/ku$. Such dephasing is called "Doppler dephasing"

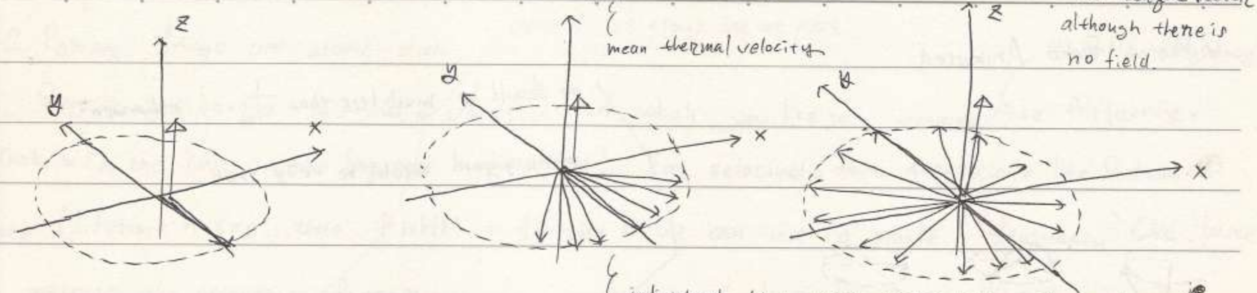


<Fig.1>

No. Date

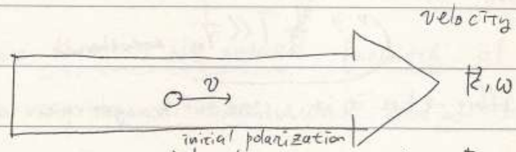
No field after $t=0$ ⇒ after this we consider individual (phase of) velocity component ⇒ For convinient we introduce this torque vector although there is no field.

$t=0$ $t \ll 2\pi/ku$ (mean thermal velocity)



individual bloch vector will rotate, due torque associated with detuning for different speed depending on their velocity ⇒ tend to spread ⇒ after long time they would be in all place in equitorial plane

⟨Fig.2⟩



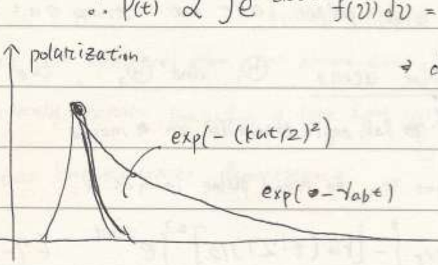
⇒ almost constant, since Ω was very large. (does not depend on velocity too much)

$\rho_{ab}(t) \simeq |\rho_{ab}| e^{-i\omega_0 t}$, in atom's reference

$\tilde{\rho}_{ab}(t) \simeq |\rho_{ab}| e^{-i(\omega_0 + kv)t}$, in lab's reference frame

$-\left(\frac{v}{u}\right)^2 - ikvt = -\left(\frac{v}{u} + \frac{i}{2}kut\right)^2 - \left(\frac{kut}{2}\right)^2$

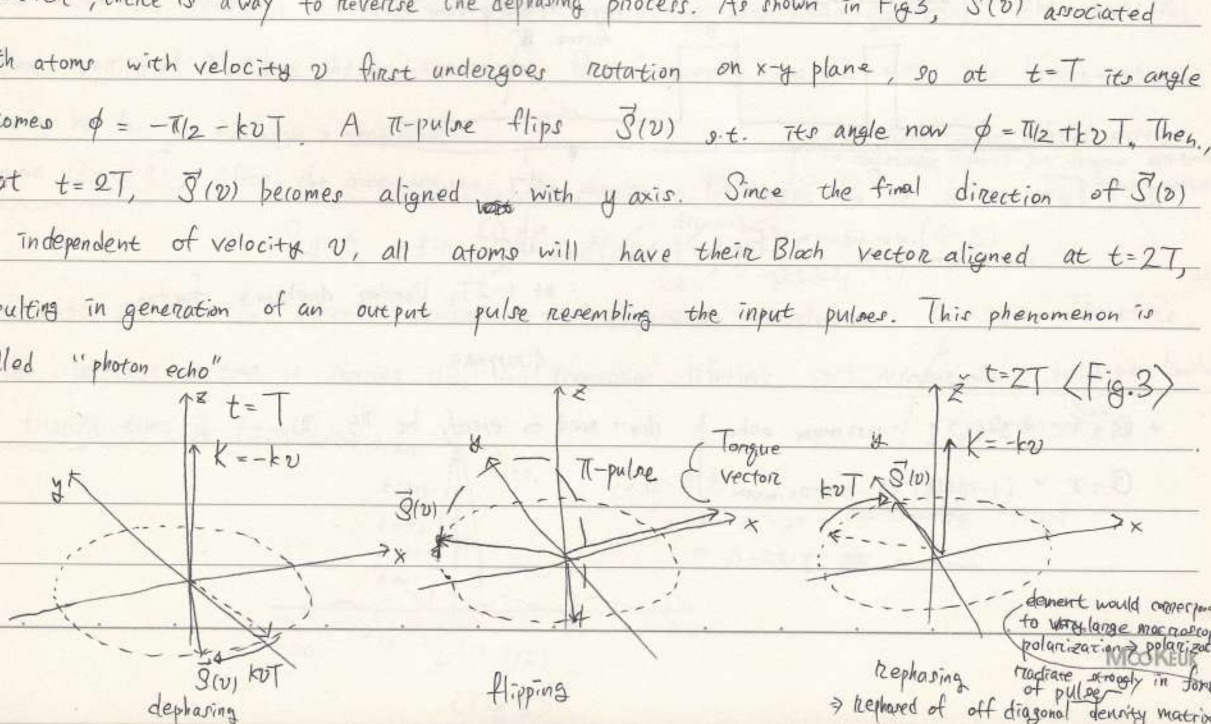
$$\therefore P(t) \propto \int e^{-ikvt} f(v)\,dv = \int e^{-\left(\frac{v}{u}\right)^2 - ikvt}\,dv \propto e^{-\left(\frac{kut}{2}\right)^2}$$



⇒ doppler dephasing much faster than homogeneous dephasing.

However, there is a way to reverse the dephasing process. As shown in Fig 3, $\vec{S}(v)$ associated with atoms with velocity v first undergoes rotation on x-y plane, so at $t=T$ its angle becomes $\phi = -\pi/2 - kvT$. A π-pulse flips $\vec{S}(v)$ s.t. its angle now $\phi = \pi/2 + kvT$. Then, at $t=2T$, $\vec{S}(v)$ becomes aligned with y axis. Since the final direction of $\vec{S}(v)$ is independent of velocity v, all atoms will have their Bloch vector aligned at $t=2T$, resulting in generation of an output pulse resembling the input pulses. This phenomenon is called "photon echo"

⟨Fig.3⟩



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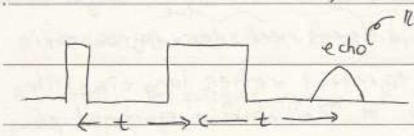
✓ Photon Echo Animated

T should be much less than $\frac{1}{\gamma_{ab}}$, otherwise, echo signal would be very small



$\Rightarrow T \ll \frac{1}{\gamma_{ab}}$, ~~otherwise~~

⇒ photon echo can overcome doppler dephasing effect → but it is subject to homogeneous dephasing rate

The time sequence of photon echo is shown in Fig. 4. Photon echo is possible even for the time interval between two input pulses T much greater than dephasing time $T_D = 2\pi/ku$. The echo signal is still subject to the homogeneous dephasing rate γ_{ab}, so it scales as $Exp(-2\gamma_{ab}T)$. The condition of $\pi/2$ and π pulses for the input pulses are not stringent in practice. For arbitrary input pulses with pulse areas Θ_1, and Θ_2, the echo polarization is given by

Rabi angle that pulse can induce

→ we can still get echo sygnal even when the first pulse is not $\pi/2$, or second pulse is not π.

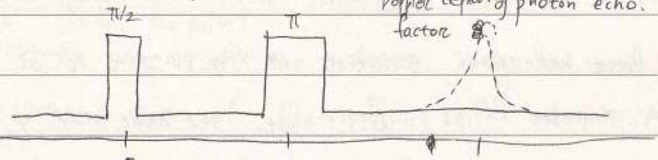
$$P(t) \cong -\frac{N\mu}{2} \sin\Theta_1 (1-\cos\Theta_2) e^{-\gamma_{ab}t} \exp\left\{-\left[ku(t-2T)/2\right]^2\right\} e^{-i\omega t} \qquad (7\text{-}1)$$

where N is the atomic number density. Θ_1, Θ doesn't need to be $\pi/2$, π respectively

# of atoms in ensemble

Doppler dephasing factor

photon echo.



⟨Fig. 4⟩

at $t=2T$, Doppler dephasing factor disappear

⇒ $\Theta_1 = \pi/2 \Rightarrow \sin\Theta_1 = 1$: maximum echo
$\Theta_2 = \pi \Rightarrow (1-\cos\Theta_2) = 2$: maximum

don't need to exactly be $\pi/2$, π

Technique

7.2 Ramsey fringe and atomic clock

Suppose you have a two-level transition of which you like to measure the frequency. — tunable frequency
— even if all atoms are at rest

What will be the limit in the frequency measurement? For relatively fast transitions like Sodium D lines (lifetime ~16ns, thus FWHM = 10MHz), we can use a single frequency CW laser to measure the resonance fluorescence as a function of laser frequency. We will get a Lorentzian lineshape with a full width equal to the inverse of the lifetime of the transition. So, in this case the limit is the natural linewidth of the transition. ⇒ we can not get the frequency better than FWHM

Transitions with very slow lifetimes can be used as frequency standards (e.g. atomic clock) since we can measure the transition frequency more accurately. Examples are microwave transitions in hyperfine structure of atoms and molecules. However, these transitions are so narrow, the accuracy is rather limited by the "transition-time" broadening of moving atoms.

hydrogen, $1\frac{5}{2}$?, Cs? --→ Hz ? level

measurement time = atom travel, at certain speed, and we measure in a certain region of interaction ⇒ measurement time = transit time of the interaction reason ⇒ we call broadens due to this transit-time broadening

Suppose this two-level atom has almost zero linewidth
→ extremely narrow linewidth ⇒ tune light source to that transition line ⇒ measure fluorescence — also extremely narrow

For example, consider a two-level atom interacting with near-resonant EM field for a time period of τ. Let the on-resonance Rabi frequency be Ω and the laser-atom detuning be Δ.

Assume $\Omega\tau \ll 1$. — Rabi angle is very small. After the interaction, the population inversion S_3 is given by — ⇒ sinc function reflect effect of detuning → width of sinc function determine frequency resolution

$$S_3 \simeq -1 + \frac{(\Omega\tau)^2}{2}\,\text{sinc}^2(\Delta\tau/2)$$

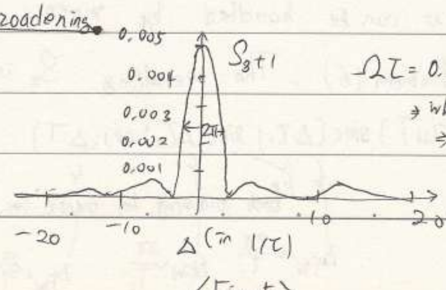
detuning; $\sim -\{\cos\Omega t\}$ ~ Rabi oscillation (7-2) starting from ground state

The width of the sinc function determines the frequency resolution: $\Delta \sim 2\pi/\tau$. Therefore, if the interaction time is shorter than the transition lifetime, our measurement will be limited by transit time broadening.

※ $\text{sinc}(x) = \frac{\sin x}{x}$

$\Omega\tau = 0.1$

approximately FWHM
⇒ when $\Delta\tau \sim 2\pi$ ⇒ $\Delta \sim 2\pi/\tau$: resolution



<Fig.5>

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Date

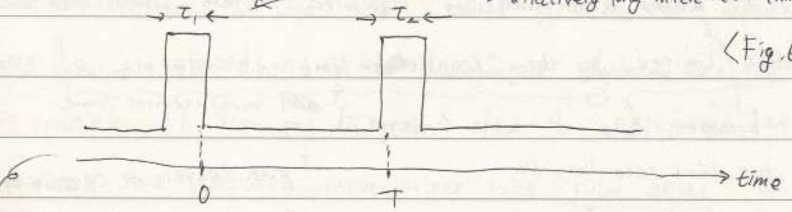
The transit time broadening is determined by the interaction time. The longer interaction time, the better resolution. This is why slowing and trapping of atoms are so important for precision measurements. For a beam of atoms, we need a large laser beam for ensuring a long interaction time. For very slow transitions, this size becomes impractically long.

However, there is a smart way of overcoming this difficulty, thanks to H. Ramsey.† This technique is known as "Ramsey fringe". ← = "oscillatory field" technique

Consider two separate fields of the same frequency interact with a two-level atom. These fields are in phase. The in-phase condition is modeled with a single EM field (with detuning Δ) with its amplitude nonvanishing

inphase → measurement time effectively become T.

→ effectively long interaction time → very long spectrum resolution



⟨Fig.6⟩

PT.3.1856)

† Senge Haroche et al., Physics Today, page 27, January 2013 (http://dx.doi.org/10.1063/

→ individual pulse will be limited by transit time broadening.

two fields are connected in phase

→ It is convenient introduce the detuning even we don't have any field

→ we are considering two connected interaction in phase

for $-\tau_1 < t < 0$ (the first field) and $T < t < T+\tau_2$ (the second field). Both the Rabi frequency of the first and the second fields are Ω. Let $\Theta_1 = \Omega\tau_1$, $\Theta_2 = \Omega\tau_2$ (on-resonance pulse area) and assume $\Theta_1 \Theta_2 \ll 1$. For the first and second fields the torque vector is $\vec{R} = (\Omega, 0, \Delta)$. For the time evolution between the fields $\vec{R}_d = (0, 0, \Delta)$ since the two fields are in phase. We can assume an extremely small Rabi frequency during that time in order to impose the in phase condition for the two pulses.

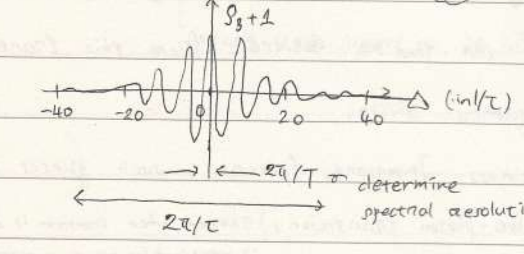
Initially, the Bloch vector is $\vec{S} = (0, 0, -1)$. So $\vec{S}$ precesses around $\vec{R}$ for $-\tau_1 < t < 0$, then around $\vec{R}_d$ for $0 < t < \tau_1$, and finally around $\vec{R}$ again for $T < t < T+\tau_2$. The whole ~~proce~~ process can be handled by matrix multiplication. We quote only the answer here (Problem 16). The resulting S_z is given by

$$S_3 \simeq -1 + \Theta_1\Theta_2 \exp(-\gamma_{ab}T)\,\mathrm{sinc}(\Delta\tau_1)\,\mathrm{sinc}(\Delta\tau_2)\cos(\Delta T) \qquad (9\text{-}3)$$

both accounting for transit-time broadening

$\Delta_{FW} \sim \frac{2\pi}{\tau_1}$ $\quad \Delta_{FW} \sim \frac{2\pi}{\tau_2}$ $\quad \Delta_{FW} \sim \frac{2\pi}{T}$ $\Rightarrow T \gg \tau_1, \tau_2 \Rightarrow$ improve spectral resolution

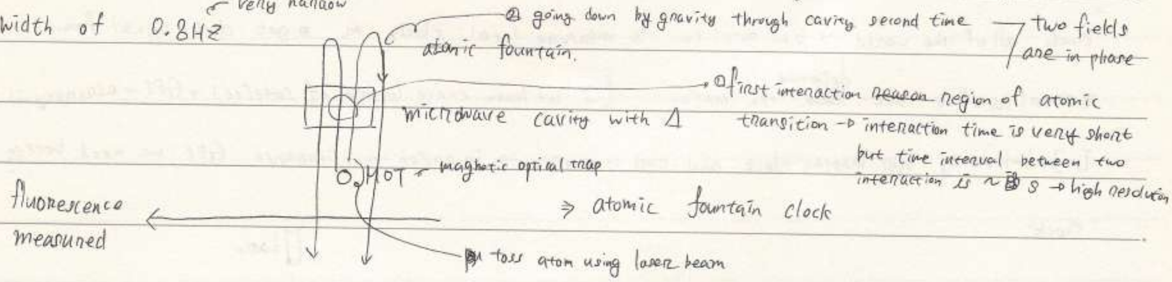
Note that the frequency resolution, the width of the central peak, is given by the $2\pi/T$, the inverse of the time difference between the two fields. Since the two fields are in phase, effectively it is like a single very long pulse of duration of T! Practically, we want to choose T comparable to the dephasing time, $(\gamma_{ab})^{-1}$. maximum time we can achieve.



$\Theta_1 = 0.1$
$\Theta_2 = 0.1$ two fields are in phase
$\gamma_{ab} T = 1$
$\tau/T = 0.1$

⟨Fig.7⟩

In atomic clocks the frequency of a very slow transition is continuously measured and updated using the Ramsey fringe. The slower transition will give the better resolution as long as one can chose the time difference between two separate fields comparable to the dephasing time. With an atomic beam (e.g. $v \sim 500 m/s$) the spacial distance needed for the necessary time difference becomes impractically long. One can overcome this difficulty by using an atomic fountain or very slow atomic beam source. A current state-of-art atomic clock employs an atomic fountain with a hyperfine transition of cesium (^{133}Cs $F=3, m=0 \leftrightarrow F=4, m=0$) whose transition frequency is 9,162,631,770Hz and linewidth of 0.8HZ ← very narrow



⟨Fig.8⟩

No.

Date . .

can achieve $a10^{-15}$

Repeated measurements with statistical averaging one can obtain frequency accuracy down to $4\times10^{-14}\tau^{-1/2}$ with τ the averaging time (not the interaction time) in the repeated measurements.

The use of atomic fountain offers tremendous advantages over the use of thermal atomic beam in two aspects: slower atomic velocity (thus longer interaction time) and higher atomic density (thus larger signal to noise ratio). "1 second" can then be derived from this transition frequency by successive application of frequency division.

The quest for a better atomic clock continues. Important factors in such quests are

i) much narrower transition (e.g. two-photon transitions) ⇒ happens when transition is almost forbidden → people look for two-photon transition

ii) higher transition frequency (e.g. optical frequency instead of microwave. Even X-ray or gamma ray clocks are investigated.) ∵ accuracy = $\frac{\text{Linewidth}}{\text{Center frequency}}$ ← better clock

∗ hyperfine transition ~ GHz → optical frequency ⇒ can be achieved by

iii) high density and slow atomic source (e.g. fountain, MOT, clocks in micro-gravity environment in space) ← instead of using thermal beam ⇒ just drop atom and move very slowly ⇒ achieve long interaction time

neutral atoms trapped in optical lattice ⇒ use two photon transition to measure transition → better clock

Chain of electronic circuit → divide frequency until we get 1 sec (?)

linked phase locked ⇒ ~~division~~ division give precise half

signal to noise ratio better noise ratio

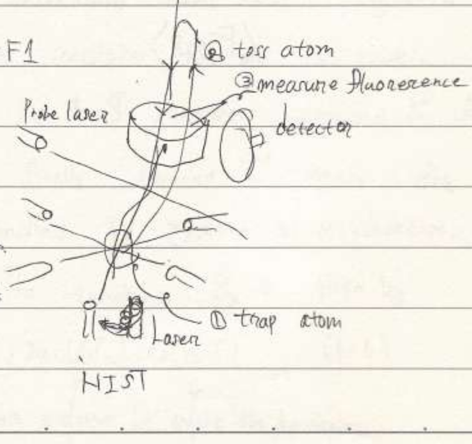
longer interaction time

e.g. NIST, in USA ⇒ small atomic clock in satelite → ~~stored~~ synced with master clock → offer accurate clock all of the world → use satelite to measure local clock or get clock signal from several satelite and determine the location. (∵ we know exact location of satelite) → GPS → accuracy is about to [m] ⇒ limited by our master clock and clock on satelite → in order to improve GPS, we need better clock

∨ Cs Atomic Fountain Clock at NIST: NIST-F1

- Based on Cs atomic fountain
- Accuracy better than 1×10^{-15}
- In service since 1999

⇒ people want to replace fountain clock to optical lattice clock ← better resolution



No.

Date . .

✓ Atomic Clock Ensemble in Space (ACES)

- A cold atomic beam of Cs in micrognavity.

- $v = 0.05 m/s$, $T = 5 sec$, resulting an accuracy $\sim 1 \times 10^{-16}$ → much better than fountain clock

- Launch planned in 2020 (not sure, keep delayed)

initially it was supposed to be in 2013, but delayed.

No.

Date . .

$$\begin{bmatrix} \sin\beta & \cos\beta \\ -\cos\beta & -\sin\beta \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix}$$

$$S_1'(-\tau_1) = A\cos\beta - B\sin\beta$$

$$S_2'(-\tau_1) = \cancel{\cos\beta + B\sin\beta} = A\sin\beta + B\cos\beta$$

$$\begin{bmatrix} S_1'(-\tau_1) \\ S_2'(-\tau_1) \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} S_1'(-\tau_1) \\ S_2'(-\tau_1) \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\Rightarrow A = \cos\beta \cdot S_1'(-\tau_1) + \sin\beta\, S_2'(-\tau_1)$$

$$\begin{bmatrix} S_2'(-\tau_1) \\ -S_1'(-\tau_1) \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} S_2'(-\tau_1) \\ -S_1'(-\tau_1) \end{bmatrix}$$

$$\Rightarrow C = \cos\beta\, S_2'(-\tau_1) - \sin\beta\, S_1'(-\tau_1)$$

$$\Rightarrow \vec{S}_1[\vec{S}(0)]' = \begin{bmatrix} \cos\beta \cdot S_1'(-\tau_1) + \sin\beta\, S_2'(-\tau_1) \\ \cos\beta \cdot S_2'(-\tau_1) - \sin\beta\, S_1'(-\tau_1) \\ S_3'(-\tau_1) \end{bmatrix}\vec{K}$$

$$[\vec{S}(0)] = \begin{bmatrix} -\cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}[S_3(0)]$$

$$[\vec{S}(-\tau_1)]' = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & -\cos\alpha \end{bmatrix}[S(-\tau_1)]$$

$$= \begin{bmatrix} \cos\alpha\, S_1(-\tau_1) + \sin\alpha\, S_3(-\tau_1) \\ S_2(-\tau_1) \\ -\sin\alpha\, S_1(-\tau_1) + \cos\alpha\, S_3(-\tau_1) \end{bmatrix}$$

$$= \begin{bmatrix} +\cos\alpha\cos\beta\, S_1'(-\tau_1) + \cos\alpha\sin\beta\, S_2'(-\tau_1) - \sin\alpha\, S_3'(-\tau_1) \\ \cos\beta\, S_2'(-\tau_1) - \sin\beta\, S_1'(-\tau_1) \\ \sin\alpha \cdot \cos\beta\, S_1'(-\tau_1) + \sin\alpha\sin\beta\, S_2'(-\tau_1) + \cos\alpha\, S_3'(-\tau_1) \end{bmatrix}$$

$\dot S_1 = -\gamma_{ab} S_1 + \Delta S_2$

$\dot S_2 = -\gamma_{ab} S_2 - \Delta S_1 + \Omega S_3$

$\dot S_3 = -T_0(1+S_3) - \Omega S_2$

1 – 7)

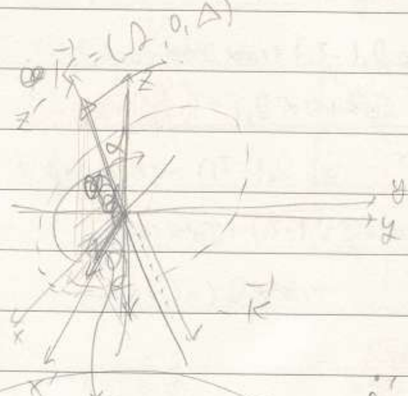
$-\tau_1 < t < 0$

$\dot S_1 \simeq \Delta S_2$

$\dot S_2 \simeq -\Delta S_1 + \Omega S_3$

$\dot S_3 \simeq -\Omega S_2$

$\dot{\vec S} = -\vec K \times \vec S$



$[\sin\alpha, 0, 0]_K$

$\dot S_1' = K S_2'$

$\dot S_2' = -K S_1'$

$S_1'(t) = A\cos Kt + B\sin Kt$

$S_2'(t) = C\cos Kt + D\sin Kt$

$\Rightarrow \dot{\vec S} = -\frac{\beta}{\tau_1}\hat e_{z'} \times \vec S$

$$\cancel{\Rightarrow \begin{bmatrix} S_1'(t) \\ S_2'(t) \\ S_3'(t) \end{bmatrix} = \begin{bmatrix} -\sin\alpha \\ \\ 0 \quad -\cos\alpha \end{bmatrix}}$$

$\cancel{S_1'(0) = A}$

$S_2'(0) = C$

$S_1'(-\tau_1) = A\cos\beta - B\sin\beta$

$S_2'(-\tau_1) = C\cos\beta - D\sin\beta$

$\dot S_1'(-\tau_1) = K(A\sin\beta + B\cos\beta)$

$= K(C\cos\beta + D\sin\beta) = KS_2'(-\tau_1)$

$\dot S_2'(-\tau_1) = K(C\sin\beta + D\cos\beta)$

$= -K(A\cos\beta + B\sin\beta)$

$= -KS_1'(-\tau_1)$

$-[0, 0, K] \times \begin{bmatrix} S_1' \\ S_2' \\ S_3' \end{bmatrix}$

$(-\hat e_3' K) \times (S_1'\hat e_1' + S_2'\hat e_2' + S_3'\hat e_3')$

$\dot S_1' = -\hat e_2' K S_1' + \hat e_1' K S_2'$

No.

Date . .

$$\mathcal{S}(0)=\begin{bmatrix} +\cos\alpha\cos\beta\, S_1'(-\tau_1)+\cos\alpha\sin\beta\, S_2'(-\tau_1)+\sin\alpha\, S_3'(-\tau_1) \\ \cos\beta\, S_2'(-\tau_1)-\sin\beta\, S_1'(-\tau_1) \\ \sin\alpha\cos\beta\, S_1'(-\tau_1)+\sin\alpha\sin\beta\, S_2'(-\tau_1)+\cos\alpha\, S_3'(-\tau_1) \end{bmatrix}$$

$$= \cancel{-\sin^2\alpha\cos\beta\cdot S_1(-\tau_1)-\sin\alpha\cos\alpha\cos\beta\, S_2(-\tau_1)-\sin\alpha\sin\beta\, S_2(-\tau_1)}$$

$$=\begin{bmatrix} \cos^2\alpha\cos\beta\, S_1(-\tau_1)+\cos\alpha\sin\alpha\cos\beta\, S_3(-\tau_1)+\cos\alpha\sin\beta\, S_2(-\tau_1)+\sin^2\alpha\, S_1(-\tau_1) \\ -\sin\alpha\cos\alpha\, S_3(-\tau_1) \\ \cos\beta\, S_2(-\tau_1)-\cos\alpha\sin\beta\, S_1(-\tau_1)-\sin\alpha\sin\beta\, S_3(-\tau_1) \\ \sin\alpha\cos\alpha\cos\beta\, S_1(-\tau_1)+\sin^2\alpha\cos\beta\, S_3(-\tau_1)+\sin\alpha\sin\beta\, S_2(-\tau_1)-\sin\alpha\cos\alpha\, S_1(-\tau_1) \\ +\cos^2\alpha\, S_3(-\tau_1) \end{bmatrix}$$

$$=\begin{bmatrix} \cos^2\alpha\cos\beta+\sin^2\alpha & \cos\alpha\sin\beta & -\cos\alpha\sin\alpha(1-\cos\beta) \\ -\cos\alpha\sin\beta & \cos\beta & -\sin\alpha\sin\beta \\ \sin\alpha\cos\alpha(\cos\beta-1) & \sin\alpha\sin\beta & \sin^2\alpha\cos\beta+\cos^2\alpha \end{bmatrix}$$

$\mathcal{S}$

$$=\begin{bmatrix} \cos\beta+\sin^2\alpha(1-\cos\beta) & \cos\alpha\sin\beta & -\cos\alpha\sin\alpha(1-\cos\beta) \\ -\cos\alpha\sin\beta & \cos\beta & -\sin\alpha\sin\beta \\ -\cos\alpha\sin\alpha(1-\cos\beta) & \sin\alpha\sin\beta & 1-\sin^2\alpha(1-\cos\beta) \end{bmatrix} = A_1$$

$\mathcal{S}$

$\boxed{(1)}$

No.

Date

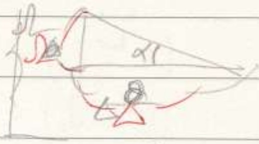
ii) $$\frac{\sin(\varepsilon\sqrt{1+x^2})}{\sqrt{1+x^2}} \approx \frac{\varepsilon\sqrt{1+x^2} - \frac{\varepsilon^3}{6}(1+x^2)\sqrt{1+x^2}}{\sqrt{1+x^2}} + O(\varepsilon^4)$$

$$= \varepsilon - \frac{\varepsilon^3}{6}(1+x^2) + O(\varepsilon^4)$$

$$= \frac{\varepsilon x - \frac{\varepsilon^3}{6}x^3}{x} + O(\varepsilon^3)$$

$$= \frac{\sin(\varepsilon x)}{x} + O(\varepsilon^3) \quad \boxed{//}$$

iii) $$S_z(0) = \cancel{\cos\alpha\sin\alpha(\ldots)} + \qquad 1-2\sin^2\left(\frac{\beta}{2}\right)$$



$$-1+\sin^2\alpha(1-\cos\beta)$$

$$= -1 + \frac{\Omega^2}{\Omega^2+\Delta^2}\frac{\sin^2\beta}{1+\cos\beta}$$

$$\sin\alpha = \frac{\Delta\Omega}{\sqrt{\Omega^2+\Delta^2}}$$

$$= -1 + \frac{\Omega^2}{\Omega^2+\Delta^2}\frac{\sin^2(\sqrt{\Omega^2+\Delta^2}\cdot\tau_1)}{(1+\cos\beta)}$$

$$\sin\alpha = \frac{\Omega}{\sqrt{\Omega^2+\Delta^2}} \quad //$$

$$\cancel{= -1 + \left\{\frac{\sin\left(\sqrt{1+\frac{\Omega^2}{\Delta^2}}\cdot\Delta\tau_1\right)}{\sqrt{\frac{\Omega^2}{\Delta^2}+1}}\right\}^2 \frac{1}{1+\cos(\sqrt{\Omega^2+\Delta^2}\,\tau_1)}}$$

$$\cancel{\simeq -1 + \left(\frac{\sin(\Delta\tau_1)}{\frac{\Omega}{\Delta}}\right)^2 \frac{1}{1+\cos(\sqrt{\Omega^2+\Delta^2}\,\tau_1)}}$$

$$\frac{\sin\left(\Omega\tau_1\sqrt{1+\left(\frac{\Delta}{\Omega}\right)^2}\right)}{\sqrt{1+\left(\frac{\Delta}{\Omega}\right)^2}}$$

if $\Omega\tau_1 \ll 1$

$$= -1 + 2\sin^2\alpha\sin^2\frac{\beta}{2}$$

$$= -1 + \cancel{\frac{\Delta^2}{\Delta^2}} \times \left[\frac{1}{1+\left(\frac{\Delta}{\Omega}\right)^2}\sin^2\left(\sqrt{1+\left(\frac{\Delta}{\Omega}\right)^2}\,\Omega\tau_1\right)\right]$$

$$= -1 + 2\frac{\Omega^2}{\Omega^2+\Delta^2}\sin^2\left(\sqrt{\Omega^2+\Delta^2}\,\frac{\tau_1}{2}\right)$$

$$= -1 + 2\frac{1}{\left(1+\left(\frac{\Delta}{\Omega}\right)^2\right)\cdot\left(\frac{\Omega\tau_1}{2}\right)^2}\sin^2\left(\sqrt{1+\left(\frac{\Delta}{\Omega}\right)^2}\cdot\Omega\tau_1/2\right)\cdot\left(\frac{\Omega\tau_1}{2}\right)^2$$

$$\times \frac{1}{1+\cos(\sqrt{\Omega^2+\Delta^2}\,\tau_1)}$$

$$\simeq -1 + 2\frac{\sin^2\left(\frac{\Delta}{\Omega}\Omega\tau_1/2\right)}{\left(\frac{\Delta}{\Omega}\right)^2\cdot\left(\frac{\Omega\tau_1}{2}\right)^2}\cdot\left(\frac{\Omega\tau_1}{2}\right)^2$$

$$1 + \cancel{\left(\frac{\Delta}{\Omega}\right)^2} + \left(\frac{\sin(\Delta\tau_1)}{\frac{\Delta}{\Omega}}\right)^2\cdot\frac{1}{2}$$

$$= -1 + \left(\frac{\sin(\Delta\tau_1)}{\Delta\tau_1}\right)^2\frac{(\Omega\tau_1)^2}{2} = -1 + \frac{1}{2}(\Omega\tau_1)^2\,\mathrm{sinc}^2($$

$$= -1 + \frac{(\Omega\tau_1)^2}{2}\left(\frac{\sin\left(\frac{\Delta\tau_1}{2}\right)}{\frac{\Delta\tau_1}{2}}\right)^2 = -1 + \frac{(\Omega\tau_1)^2}{2}\,\mathrm{sinc}^2(\Delta\tau_1/2) \quad //$$

$\boxed{//}$

No.

Date . .

show that $S(\tau_2+T)$ $A_2 = \sim$?

$\dot{S}_1 = -\gamma_{ab} S_1 + \Delta S_2$

$\dot{S}_2 = -\gamma_{ab} S_2 - \Delta S_1$

$\dot{S}_3 = -\Gamma_0 (1+S_3) \simeq \cancel{\Gamma_0 t}$ 0

$\Rightarrow$ $(e^{\gamma_{ab}t} S_1)^{\cdot} = \Delta S_2 e^{\gamma_{ab}t} = \dot{Z}_1 = \Delta Z_2$

$(e^{\gamma_{ab}t} S_2)^{\cdot} = -\Delta S_1 e^{\gamma_{ab}t} = \dot{Z}_2 = -\Delta Z_1$

$\Rightarrow Z_1 = A\cos(\Delta t) + B\sin(\Delta t)$

$Z_2 = C\cos(\Delta t) + D\sin(\Delta t)$

$\cancel{Z(T)}$

$Z_1(0) = A = S_1(0)$

$Z_2(0) = C = S_2(0)$

$\dot{Z}_1(0) = \Delta \cdot B = \Delta \cdot S_2(0)$ $\Rightarrow$ $B = S_2(0)$

$\dot{Z}_2(0) = +\Delta \cdot D = -\Delta S_1(0)$ $D = -S_1(0)$

$\Rightarrow$ $S_1(t) = e^{-\gamma_{ab}t} S_1(0)\cos(\Delta t) + e^{-\gamma_{ab}t} S_2(0)\sin(\Delta t)$

$S_2(t) = e^{-\gamma_{ab}t} S_2(0)\cos(\Delta t) - e^{-\gamma_{ab}t} S_1(0)\sin(\Delta t)$

$\Rightarrow$ $S_3(t) = S_3(0)$

$\Rightarrow$ $\cancel{S(t)}$

$$\vec{S}(T) = \begin{bmatrix} e^{-\gamma_{ab}T}\cos(\gamma) & e^{-\gamma_{ab}T}\sin(\gamma) & 0 \\ -e^{-\gamma_{ab}T}\sin(\gamma) \cancel{\cos(\gamma)} & +e^{-\gamma_{ab}T}\cos(\gamma) \cancel{\sin(\gamma)} & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{S}(0)$$

$A_3 = A_1|_{\cancel{\alpha} \to \beta}$ (동일한 상황)

No.

Date . .

IV) ~~$S_3(0) = A_1 S$~~ $\doteq \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

$$\vec{S}(0) = A_1 \vec{S}(-\tau_1) = \begin{bmatrix} \cos\alpha \sin\alpha (1-\cos\beta) \\ \sin\alpha \sin\beta \\ -1 + \sin^2\alpha (1-\cos\beta) \end{bmatrix}$$

$$\vec{S}(T) = A_2 \vec{S}(0)$$

$$= \begin{bmatrix} e^{-\gamma_{ab}T} \cos(\gamma) \cos\alpha \sin\alpha (1-\cos\beta) + e^{-\gamma_{ab}T} \sin(\gamma) \sin\alpha \sin\beta \\ -e^{-\gamma_{ab}T} \sin(\gamma) \cos\alpha \sin\alpha (1-\cos\beta) + e^{-\gamma_{ab}T} \sin\alpha \sin\beta \cos(\gamma) \\ -1 + \sin^2\alpha (1-\cos\beta) \end{bmatrix}$$

$$\Rightarrow \vec{S}(T+\tau_2) = A_3 \vec{S}(T)$$

$$A_3 = \begin{bmatrix} \cos\delta + \sin^2\alpha(1-\cos\delta) & \cos\alpha \sin\delta & -\cos\alpha \cdot \sin\alpha (1-\cos\delta) \\ -\cos\alpha \sin\delta & \cos\delta & -\sin\alpha \sin\delta \\ -\cos\alpha \sin\alpha (1-\cos\delta) & \sin\alpha \sin\delta & 1-\sin^2\alpha(1-\cos\delta) \end{bmatrix}$$

$$\Rightarrow S_3(T+\tau_2)$$

$$= -\cos\alpha \sin\alpha (1-\cos\delta) \cdot \Big(\underbrace{e^{-\gamma_{ab}T} \cos(\gamma) \cos\alpha \sin\alpha (1-\cos\beta)}_{(\Delta x)^2} + \underbrace{e^{-\gamma_{ab}T} \sin(\gamma) \sin\alpha \sin\beta}_{\Delta x}\Big)$$

$$+ \sin\alpha \sin\delta \Big(-e^{-\gamma_{ab}T} \sin(\gamma) \cos\alpha \sin\alpha (1-\cos\beta) + e^{-\gamma_{ab}T} \sin\alpha \sin\beta \cancel{\sin\alpha}\ \cos\gamma\Big)$$

$$+ \Big(-(1-\sin^2\alpha(1-\cos\delta))(1-\sin^2\alpha(1-\cos\beta))\Big)$$

Δx $\quad \Delta x^2 \quad \Delta x^2 \quad \Delta x^2 \quad \Delta x$

$$\simeq \cancel{-2\cos\alpha \sin\alpha \sin^2\left(\frac{\beta}{2}\right) e^{-\gamma_{ab}T} \sin(\gamma)}$$

$$\simeq + \sin^2\alpha \sin\beta \sin\delta\, e^{-\gamma_{ab}T} \cancel{\sin\gamma}\ \cos\gamma - 1 + 2\sin^2\alpha \sin^2\frac{\delta}{2} + 2\sin^2\alpha \sin^2\left(\frac{\beta}{2}\right)$$

교차항.

제곱항

그래서 제곱할 듯?

or

T에 대한 의존성이 없어서?

8. Rate Equation Approximation

Bloch equation = Semiclassical equations

The density matrix equation or the Bloch equation in the previous sections are exact semiclassical equations except for the rotating wave approximation. The population ρ_{aa} and ρ_{bb} depends on ρ_{ab} and vice versa, as in Eq.(4-1). However, the rate equations ~~for laser, Eqs.(2-20)~~ ~~an~~ discussed in Sec. 2 only deal with populations as in the rate equations for laser, Eqs.(2-20) and (2-22). Are these two approaches consistent? When they are consistent and when they are not? These are the questions to be answered in this section.

8.1 From the density matrix.

due to significant damping (dephasing)

Suppose ρ_{ab} rapidly approaches the steady state value, which is given by Eq.(4-7).

$$\underbrace{\rho_{ab}^{ss}}_{\text{steady state solution}} = -\frac{\Omega(\Delta - i\gamma_{ab})/2}{\Delta^2+\gamma_{ab}^2+\Omega^2\gamma_{ab}/\Gamma_0} e^{-i\omega t} \qquad (8-1)$$

※ Density matrix equation (Ground state does not decay)

$$\dot{\rho}_{aa} = -\Gamma_0 \rho_{aa} + \frac{i}{2}\Omega e^{-i\omega t}\rho_{ba} + c.c.$$

$$\dot{\rho}_{ab} = -\gamma_{ab}\rho_{ab} - i\omega_0\rho_{ab} - \frac{i}{2}\Omega e^{-i\omega t}(\rho_{aa} - \rho_{bb})$$

$$1 = \rho_{aa} + \rho_{bb}$$

※ Laser rate equation

$$\dot{N}_a = R_p - nK(N_a - N_b) - AN_a \qquad N_a : \text{population of excited state}$$

$$\dot{n} = (n+1)KN_a - nKN_b - \gamma n \qquad n : \text{photon number}$$

→ No equation corresponding off-diagonal density matrix equation

⇒ are they consistent?

⇒ Laser rate equation can be derived by Density matrix equation under certain approximation.

From Eq.(4-6)

substitute ρ_{ab}^{ss}

$$\rho_{aa}^{ss} - \rho_{bb}^{ss} = \frac{-\Delta^2-\gamma_{ab}^2}{\Delta^2+\gamma_{ab}^2+\Omega^2\gamma_{ab}/\Gamma_0} = \frac{2}{\Omega}(\Delta+i\gamma_{ab})\rho_{ab}^{ss} e^{+i\omega t} \qquad (8-2)$$

Plugging this into ρ_{aa} equation and using Eq.(4-8) $\rho_{ab}^{ss} = \frac{\Omega(\rho_{aa}^{ss} - \rho_{bb}^{ss})}{2(\Delta + i\gamma_{ab})} e^{-i\omega t}$, $-\frac{i}{2}\Omega e^{i\omega t} \rho_{ab}^{ss} \rightarrow -\frac{i}{2}\Omega \frac{\Omega(\rho_{aa}^{ss} - \rho_{bb}^{ss})}{2(\Delta + i\gamma_{ab})}$

(density matrix equation에서 ρ_{aa} 항에서)

$$\dot{\rho}_{aa} \simeq -\Gamma_0 \rho_{aa} + \frac{1}{4}\Omega^2 \cdot \frac{\rho_{aa}^{ss} - \rho_{bb}^{ss}}{i\Delta - \gamma_{ab}} + c.c.$$

Lorentzian lineshape

$$= -\Gamma_0 \rho_{aa} - \frac{\sigma_{rad}\Gamma_0 \cdot I}{2\hbar\omega_0\gamma_{ab}} (\rho_{aa}^{ss} - \rho_{bb}^{ss}) \cdot \frac{\gamma_{ab}^2}{\Delta^2 + \gamma_{ab}^2}$$

$\Rightarrow$ homogeneous cross section $\sigma_h^0 = \frac{\Gamma_0}{2\gamma_{ab}} \cdot \frac{\gamma_{ab}^2}{\Delta^2 + \gamma_{ab}^2}$

$$= -\Gamma_0 \rho_{aa} - \frac{\sigma_h^0 \cdot I}{\hbar\omega_0} \cdot (\rho_{aa}^{ss} - \rho_{bb}^{ss}) \qquad (8-3)$$

$\frac{\Gamma_0}{2\gamma_{ab}}$: reduction factor of homogeneous broadening

$$\Omega^2 = \frac{\mu^2 E_0^2}{\hbar^2} = \left(\frac{8\pi\mu^2}{\hbar^2 c}\right)\frac{cE_0^2}{8\pi} = \left(\frac{6\pi c^2}{\hbar\omega_0^3}\frac{4\mu^2\omega_0^3}{3\hbar c^3}\right) I_0 = \frac{6\pi c^2}{\hbar\omega_0^3}\Gamma_0 \cdot I_0 = \frac{6\pi(c/\omega_0)^2\Gamma_0 I_0}{\hbar\omega_0} = \frac{\sigma_{rad}\Gamma_0 \cdot I_0}{\hbar\omega_0}$$

Where $\sigma_h^0(\omega)$ is the unsaturated homogeneous (absorption/emission) cross section given by

$$\sigma_h^0(\omega) = \sigma_{rad}\left(\frac{\Gamma_0}{2\gamma_{ab}}\right)\frac{\gamma_{ab}^2}{(\omega - \omega_0)^2 + \gamma_{ab}^2} \qquad (8-4)$$

Dropping 'ss' superscript with substitution $\rho_{aa} \rightarrow N_a$, $\rho_{bb} \rightarrow N_b$, we get the rate equation.

number of atoms in excited state

number of atoms in ground state

$$\dot{N}_a = -\Gamma_0 N_a - \frac{\sigma_h^0 I}{\hbar\omega_0} N_a + \frac{\sigma_h^0 I}{\hbar\omega_0} \cdot N_b \qquad (8-5)$$

$$1 = N_a + N_b$$

spontaneous emission; stimulated emission; (stimulated) absorption

$\Rightarrow \frac{\sigma_h^0 I}{\hbar\omega_0}$ = number of photons absorbed/emitted per second per atom (intensity)

$\Rightarrow$ we replaced ρ_{ab}^0 using population difference $\Rightarrow$ rate equation is only in terms of population

$I/\hbar\omega_0$ $\qquad$ σ_h^0 : cross section

Note:

but rate equation cannot describe coherent transient phenomena (e.g. Rabi oscillation, optical nutation, free induction decay ...) 11:16

$\Rightarrow$ only give us exponential decay

1. Since the steady state value of ρ_{ab}, ρ_{aa} and ρ_{bb} are used in deriving Eq (8-5), the rate equation should give the same steady state solution as the density matrix equation. Due to the same reason, the transient feature of the rate equation solutions should not be trusted. No ringing, only exponential decay.

2. The unsaturated homogeneous cross section σ_h^0 can be replaced with σ_e^0, unsaturated emission cross section, and σ_a^0, unsaturated absorption cross section if they different from each other.

In rate equation, we use same cross section for stimulated emission and absorption. But molecule can have different crosssection $\Rightarrow$ replace same cross section by σ_e^0, σ_a^0

No.

Date

8.2 From Einstein's rate equation.

In Einstein's rate equation, $Bu(\omega)$ is the ~~transm~~ transition rate from the lower state to the upper state. For a two-level system initially in ground state, the excited state population at time t, when a classical field is turned on at $t=0$, is given by Eg. (2-11)

$$|C_a(t)|^2 = \Omega^2 \left[\frac{\sin(\Delta t/2)}{\Delta}\right]^2 \quad (8-6)$$

which becomes delta-function-like in Δ as $t\to\infty$ ← assume no decay but in reality, there is decay ⇒ we need correction to this equation

Averaging over the atomic natural line shape. ← normalized Lorentz lineshape

$$P_a = \int_{-\infty}^{+\infty} \Omega^2 \left[\frac{\sin(\Delta t/2)}{\Delta}\right]^2 \frac{1}{\pi}\frac{\Gamma_0/2}{\Delta^2+(\Gamma_0/2)^2}\, d\Delta$$

← it has finite lineshape, but sinc fct → delta fct

$$\simeq \frac{\Omega^2}{\pi\Gamma_0}\int_{-\infty}^{+\infty}\left[\frac{\sin(\Delta t/2)}{\Delta}\right]^2 d\Delta = \frac{\Omega^2}{\Gamma_0}t \quad (8-7)$$

← using $\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^2 = \pi$

The transition rate is just P_a/t. Therefore, our rate equation becomes

$$\dot{N}_a = -\Gamma_0 N_a - \frac{\Omega^2}{\Gamma_0}(N_a - N_b) \quad (8-8)$$

↓ using $\Omega^2 = \frac{\sigma_{rad}\Gamma_0 I_0}{\hbar\omega_0}$

From the relation, Eg (4-8),

$$\dot{N}_a = \underbrace{-\Gamma_0 N_a}_{\text{spontaneous emis.}} - \underbrace{\frac{\sigma_{rad} I}{\hbar\omega}(N_a - N_b)}_{\text{stimulated absorption, emission}} \quad (8-9)$$

⇒ only cross section is different ⇒ σ_{rad} v.s. $\sigma_h^0(\Delta)$

The discrepancy between Egs. (8-5) and (8-9) comes from the fact that in Einstein's rate equation the atom is ① assumed to be resonant with the field and ② only the radiative decay is considered.
← No other dephasing is considered

8.3. Limitation of the rate equation. ← ⇒ off-diagonal element approach as steady state extremely fast → damping dominant

In deriving the rate equation we used the relation between $(\rho_{aa}-\rho_{bb})$ and ρ_{ab} that is valid only in the steady state. Therefore, the rate equation is bound to give incorrect description for coherent transient effects like optical nutation, Rabi oscillation and adiabatic following.

Furthermore there must be serious damping present in the system for the rate equation to be applicable. This fact is also reflected in the transition rate Ω^2/Γ_0, inversely proportional to Γ_0 (so, if $\Gamma_0 = 0$, the transition rate becomes infinite!)
← ⇒ damping have to be significant

For example, consider the Rabi oscillation with negligible damping ($\Omega \gg \Gamma_0$). Suppose at time t a reasonant EM field is turned on. In Bloch picture, the state vector $\vec{S}$ rotates around the x axis while it decays to its steady state position. From Eq.(4-7), the steady state value of N_a on resonance is

this will show ringing effect

$$N_a^{ss} = \frac{R}{1+2R} \qquad (8-10)$$

where

$$R \equiv \Omega^2/\Gamma_0^2 \qquad (8-11)$$

The rate equation solution from Eq.(8-8) is

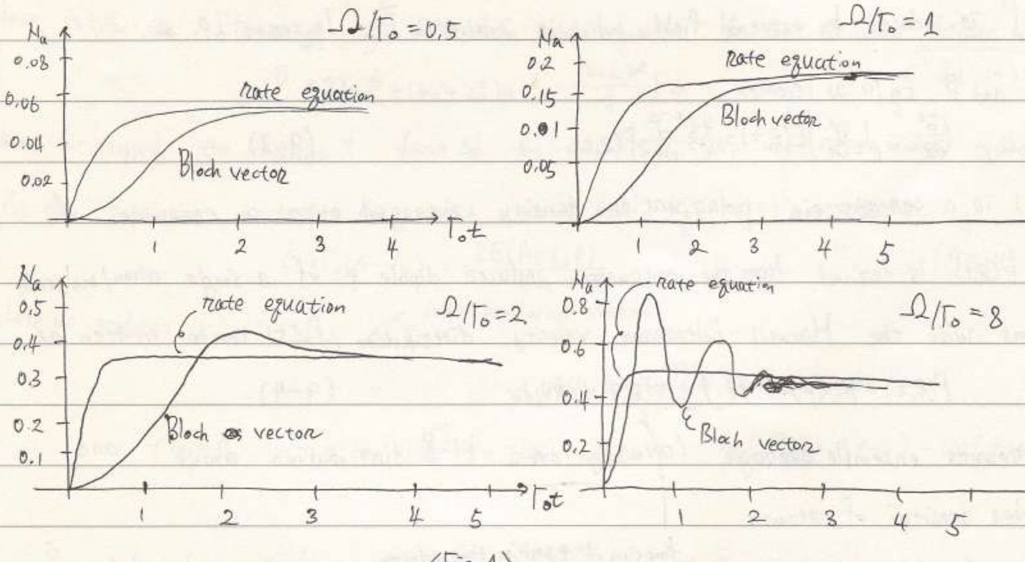
$$N_a(t) = N_a^{ss}\{1-\exp(-\Gamma_0(1+2R)t)\} \qquad (8-13)$$

solve rate equation analytically

In Fig1 the rate equation solution is compared with the exact Bloch vector solution.

$$\rho_{ab} = \frac{\Omega(-\Delta+i\gamma_{ab})/2}{(\omega-\omega_0)^2+\gamma_{ab}^2+\Omega^2\gamma_{ab}/\Gamma_0}e^{-i\omega t} \qquad (\gamma_{ab} = \Gamma_0/2\ ?)$$

$$\rho_{aa} = \frac{\Omega^2\gamma_{ab}/(2\Gamma_0)}{(\omega-\omega_0)^2+\gamma_{ab}^2+\Omega^2\gamma_{ab}/\Gamma_0}$$



<Fig.1>

The discrepancy becomes serious particularly when $\Omega \gg \Gamma_0$. However, if we only interested in the steady state populations or if the system is dephasing-dominant, the rate equation is useful tool.

advantage of rate equation

No. Lec 9.

Date Part 1.

Ch 9. Coherent pulse propagation

Equations for E, ρ, σ

9.1 Maxwell-Schrödinger equation

when we apply electric field, polarization can be induced ⇒ like dielectric material

Consider a nonmagnetic source-free ($\rho = \vec{J} = 0$) polarizable (i.e. dielectric) medium. The Maxwell equations are

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B},\quad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{E}+4\pi\vec{P}\right) \qquad (9-1)$$

$$\vec{\nabla}\cdot\left(\vec{E}+4\pi\vec{P}\right) = 0$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -4\pi\vec{\nabla}(\vec{\nabla}\cdot\vec{P}) - \nabla^2\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\times\vec{B} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\vec{E}+4\pi\vec{P})$$

Applying a curl on the curl-E equation, we obtain a wave equation for $\vec{E}(\vec{x},t)$.

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = \frac{4\pi}{c^2}\frac{\partial^2\vec{P}}{\partial t^2} - 4\pi\vec{\nabla}(\vec{\nabla}\cdot\vec{P}) \qquad (9-2)$$

We restrict ourselves to 1-D problems, where $\vec{E}$ and $\vec{P}$ are aligned along the x-direction and the variation of $\vec{E}$ and $\vec{P}$ occurs in the z-direction, and thus the divergence of $\vec{P}$ vanishes.

$\vec{E}, \vec{P}$ only have one component (x-component), spatial dependent on z

The polarization $\vec{P}$ is induced by external field, which is induced in $\vec{E}$. In terms of the amplitudes of $\vec{E}$ and $\vec{P}$, Eq.(9-2) becomes

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E(z,t) = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}P(z,t) \qquad (9-3)$$

Note that $P(z,t)$ is a macroscopic polarization density averaged over an ensemble of atoms/molecules. $P(z,t)$ is derived from the microscopic induced dipole p of a single atom/molecule. Considering atoms with the Maxwell-Boltzmann velocity distribution, $P(z,t)$ can be written as

$$P(z,t) = N\langle p\rangle = N\int_{-\infty}^{+\infty} p(z,t;v)\,f(v)\,dv \qquad (9-4)$$

function of position, time, velocity

where $\langle\dots\rangle$ represents ensemble average (average over M-B distribution above) and N is the number density of atoms.

The equation for the induced dipole p (real) and its quadrature counterpart q (also real) can be obtained from the density matrix equation. Consider an atom with $v=0$

$$p = \mu\rho_{ab} + c.c. = 2\mathrm{Re}[\mu\rho_{ab}] \qquad (9-5)$$

$$q = i\mu\rho_{ab} + c.c. = -2\mathrm{Im}[\mu\rho_{ab}]$$

No.

Date

such that

$$p = 2\mu|\rho_{ab}|\cos(\omega t-\phi), \quad q = 2\mu|\rho_{ab}|\sin(\omega t-\phi)$$

quadrature conjugate

cosine quadrature — sine quadrature

with

$$\rho_{ab} = (\sigma_1 + i\sigma_2)e^{-i\omega t}, \quad \tan\phi = \frac{\sigma_2}{\sigma_1} \qquad (9\text{-}6)$$

We use the density matrix equation without RWA for deriving the equations for p and q.

$$\dot{\rho}_{aa} = -\Gamma_0\rho_{aa} + i\frac{\mu E}{\hbar}\rho_{ba} + c.c.$$

E is real (function of z, t, $Re(\mathcal{E}(z,t)e^{i(kz-i\omega t)})$) — slowly varying electric field envelope

$$\dot{\rho}_{ab} = -\gamma_{ab}\rho_{ab} - i\omega_0\rho_{ab} - i\frac{\mu E}{\hbar}(\rho_{aa}-\rho_{bb}) \qquad (9\text{-}7)$$

$$1 = \rho_{aa} + \rho_{bb}$$

where

$$E = E(z,t) = Re\left[\mathcal{E}(z,t)e^{i(kz-\omega t)}\right] \qquad (9\text{-}8)$$

with $\mathcal{E}$ a slowly varying envelope (see sec 9.2.) and $k=\omega/c$. The equation for p and q are then

$$\dot{p} = \mu\dot{\rho}_{ab} + c.c. = -\gamma_{ab}p - \omega_0 q$$

do the same thing

$$\dot{q} = i\mu\dot{\rho}_{ab} + c.c. = -\gamma_{ab}q + \omega_0 q + \frac{2\mu^2 E}{\hbar}\sigma_3 \qquad (9\text{-}9)$$

Where $\sigma_3 = \rho_{aa} - \rho_{bb}$. Eliminating q, we obtain the equation for the induced dipole moment.

$$\ddot{p} + 2\gamma_{ab}\dot{p} + (\omega_0^2 + \gamma_{ab}^2)p = -\frac{2\omega_0\mu^2}{\hbar}E\sigma \qquad (9\text{-}10)$$

where we dropped the subscript 3 from σ_3 for simplicity. We need one more equation, the equation for the population inversion σ, which comes from Eqs. (9-7) and (9-9).

$$\dot{\sigma} + \Gamma_0(\sigma - \sigma_0) = \frac{2E(\dot{p} + \gamma_{ab}p)}{\hbar\omega_0} \qquad (9\text{-}11)$$

where we replaced "-1" with σ_0. initial population inversion

purely imaginary ⇒ + c.c. 하면 vanish

$$-\gamma_{ab}\mu\rho_{ab} - i\omega_0\mu\rho_{ab} - i\frac{\mu^2 E}{\hbar}(\rho_{aa}-\rho_{bb}) + c.c. = -\gamma_{ab}(\mu\rho_{ab} + c.c.) - \omega_0(i\mu\rho_{ab} + c.c.)$$

$$\ddot{p} = -\gamma_{ab}\dot{p} - \omega_0\dot{q} = -\gamma_{ab}\dot{p} + \underbrace{\omega_0\gamma_{ab}q}_{\omega_0 q = \sim} - \omega_0^2 p - \frac{2\omega_0\mu^2 E}{\hbar}\sigma_3 = -\gamma_{ab}\dot{p} + \gamma_{ab}(-\gamma_{ab}p - \dot{p}) - \omega_0^2 p - \frac{2\omega_0\mu^2 E}{\hbar}\sigma_3$$

$$\dot{\sigma} = 2\dot{\rho}_{aa} = -\Gamma_0(\sigma+1) - 2\frac{\mu E}{\hbar}(i\rho_{ab} + c.c.) = -\Gamma_0(\sigma+1) - 2\frac{E}{\hbar}q = -\Gamma_0(\sigma+1) + \frac{2E}{\hbar\omega_0}(\dot{p} + \gamma_{ab}p)$$

No.

Date

macroscopic eq. ⇒ from Maxwell eq.

E : total electric field = $E_{External} + E_{generated}$

source is polarization

So, we obtain the "Maxwell - Schrödinger equations" as follows

for single atom

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E(z,t) = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}P(z,t) \qquad (9\text{-}3)$$

$$\ddot{p} + 2\gamma_{ab}\dot{p} + (\omega_0^2+\gamma_{ab}^2)p = -\frac{2\omega_0\mu^2}{\hbar}E\sigma \qquad (9\text{-}10)$$

$$\dot{\sigma} + \Gamma_0(\sigma - \sigma_0) = \frac{2E(\dot{p}+\gamma_{ab}p)}{\hbar\omega_0} \qquad (9\text{-}11)$$

microscopic eq.
⇒ from schrödinger eq.

source of population inversion] driving force (σ is multiplied at)

$$P(z,t) = N\langle p\rangle = N\cdot\int_{-\infty}^{+\infty} p(z,t;v)f(v)\,dv \qquad (9\text{-}4)$$

⇒ total macroscopic polarization (Ensemble averaged)

9.2. Slowly-varying-envelope approximation ⇒ Equations for $E, p, \sigma \to \mathcal{E}, \mathcal{P}, R$

The Maxwell - Schrödinger equations can be cast in a more convenient form, i.e., in terms of slowly varying envelopes. Switch to the complex notation with slowly varying envelopes $\mathcal{E}$ and $\mathcal{P}$,

$$E = Re\left[\mathcal{E}e^{i(kz-\omega t)}\right], \quad Np = Re\left[\mathcal{P}e^{i(kz-\omega t)}\right]$$

$$\left|\frac{1}{c}\frac{\partial\mathcal{E}}{\partial t}\right|, \left|\frac{\partial\mathcal{E}}{\partial z}\right| \ll \left|\frac{\omega}{c}\mathcal{E}\right|, |k\mathcal{E}| \text{ and } \left|\frac{1}{c}\frac{\partial\mathcal{P}}{\partial t}\right|, \left|\frac{\partial\mathcal{P}}{\partial z}\right| \ll \left|\frac{\omega}{c}\mathcal{P}\right|, |k\mathcal{P}| \qquad (9\text{-}12)$$

Also define $R = N\sigma$, the population inversion density $R_0 = N\sigma_0$. Then, Eqs (9-3), (9-10) and (9-11) becomes

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\mathcal{E} \simeq i2\pi k\langle\mathcal{P}\rangle \qquad (9\text{-}13\ a)$$

slowly - varying - envelope approximation

slowly-varying-envelope ⇒ $\frac{\partial^2}{\partial z^2}\mathcal{E}, \frac{\partial^2}{\partial t^2}\mathcal{E} \to 0$

$k^2 = \frac{\omega^2}{c^2}$ ⇒ cancel

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E(z,t) = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}P(z,t)$$

$$\Rightarrow \frac{\partial^2}{\partial z^2}\left[\mathcal{E}e^{i(kz-\omega t)}\right] = \frac{\partial}{\partial z}\left[\frac{\partial\mathcal{E}}{\partial z} + ik\mathcal{E}\right]e^{i(kz-\omega t)} = \left[\frac{\partial^2\mathcal{E}}{\partial z^2} + 2ik\frac{\partial\mathcal{E}}{\partial z} - k^2\mathcal{E}\right]e^{i(kz-\omega t)}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\mathcal{E}e^{i(kz-\omega t)}\right] = \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{\partial\mathcal{E}}{\partial t} - i\omega\mathcal{E}\right]e^{i(kz-\omega t)} = \frac{1}{c^2}\left[\frac{\partial^2}{\partial t^2}\mathcal{E} - 2i\omega\frac{\partial\mathcal{E}}{\partial t} - \omega^2\mathcal{E}\right]e^{i(kz-\omega t)}$$

$$-\frac{4\pi}{c^2}\omega^2\mathcal{P}\frac{1}{2ik} = 2i\pi k\mathcal{P} \Rightarrow \text{take ensemble average}$$

largest contribution

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\mathcal{E} \simeq 2i\pi k\langle\mathcal{P}\rangle$$

⇒ When we evaluate electric field, we need ensemble average of that quantity

⇒ one group of velocity v.
atom see $\omega - kv$ frequency

$$\dot{\mathcal{P}} + (\gamma_{ab} - i\Delta')\mathcal{P} = -i\frac{\mu^2}{\hbar}\mathcal{E}R \qquad (9\text{-}13\ b)$$

$$\dot{R} + \Gamma_0(R - R_0) = Im\left(\frac{\mathcal{P}\mathcal{E}^*}{\hbar}\right) \qquad (9\text{-}13\ c)$$

where $\Delta' = \omega - \omega_0 - kv$ for atoms with velocity v along the z direction. Eq (9-13) is the Maxwell - Schrödinger equation in terms of slowly-varying envelopes.

Note that $\mathcal{P}$ and R equations are microscopic equations for a group of atoms with velocity v whereas $\mathcal{E}$ equation is macroscopic equation requiring an ensemble average on $\mathcal{P}$. Also

$\mathcal{P}$: script P = $\mathcal{P}$ No. 4

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note that $\mathcal{P}$ is the envelope of N_p (Eq. 9-4) and is the field seen by the atoms moving with v at $z = z_0 + vt$.

$$\ddot{P} + 2\gamma_{ab}\dot{P} + (\omega_0^2 + \gamma_{ab}^2)P = -\frac{2\omega_0 \mu^2}{\hbar} E\sigma$$

$$\dot{\sigma} + \Gamma_0(\sigma - \sigma_0) = \frac{2E(\dot{P} + \gamma_{ab}P)}{\hbar\omega_0}$$

real quantity, and oscillating ⇒ take time average

$(\omega_0^2 - \omega^2)\mathcal{P} = (\omega_0 - \omega)(\omega_0 + \omega)\mathcal{P} \simeq 2\omega(\omega_0 - \omega)\mathcal{P}$

$$\Rightarrow N_p = Re[\mathcal{P} e^{i(kz - \omega t)}] \Rightarrow \ddot{\mathcal{P}} - 2i\omega\dot{\mathcal{P}} - \omega^2\mathcal{P} + 2\gamma_{ab}(\dot{\mathcal{P}} - i\omega\mathcal{P}) + (\omega_0^2 + \gamma_{ab}^2)\mathcal{P} = -\frac{2\omega_0^2 \mu^2}{\hbar}\mathcal{E}R$$

$$2i\omega\dot{\mathcal{P}} + 2\omega(\omega - \omega_0)\mathcal{P} + i2\gamma_{ab}\omega\mathcal{P} = \frac{2\omega_0\mu^2}{\hbar}\mathcal{E}R$$

slowly varying envelope ⇒ $\ddot{\mathcal{P}} \to 0$ $\gamma_{ab}^2 \ll \omega_0^2 \Rightarrow \gamma_{ab}^2 \sim 0$

$\dot{P}$ is dominant, by $-i\omega\mathcal{P}$ ⇒ $\dot{P} \gg \gamma_{ab}P$ ⇒ $\dot{P} \sim Re[-i\omega\mathcal{P}e^{i(kz-\omega t)}]$

$$\left\langle \frac{2NE(\dot{P} + \gamma_{ab}P)}{\hbar\omega_0} \right\rangle_{time} \simeq \left\langle 2E\frac{Re[-i\omega\mathcal{P}e^{i(kz-\omega t)}]}{\hbar\omega_0} \right\rangle_{time}$$ use theory of time average

$$\simeq Re\left[\frac{\mathcal{E}(-i\mathcal{P})^*}{\hbar}\right] = Im\left[\frac{\mathcal{P}\mathcal{E}^*}{\hbar}\right]$$

$\omega \simeq \omega_0 \Rightarrow$ cancel

It is convenient to rewrite Eq. (9-13) in terms of new variables

$$Z = z, \quad T = t - z/c \quad \text{or} \quad z = Z, \ t = T + Z/c$$

external electric field form of pulse enter the medium at $t=0$, propagate to z-direction ⇒ most of time medium doesn't see electric field ⇒ when electric field is arrive, medium interact ⇒ (9-14) make electric field ⇒ describe respond as we follow the pulse along its propagation

Noting

$$\frac{\partial}{\partial Z} = \left(\frac{\partial z}{\partial Z}\right)_T\left(\frac{\partial}{\partial z}\right) + \left(\frac{\partial t}{\partial Z}\right)_T\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial T} = \left(\frac{\partial z}{\partial T}\right)_Z\frac{\partial}{\partial z} + \left(\frac{\partial t}{\partial T}\right)_Z\frac{\partial}{\partial t} = \frac{\partial}{\partial t}$$

The M-S equations are then

~~connecting~~ concentrating on the interaction around the pulse arrival ($T=0$) for the group of atoms $z \simeq ct$

$$\frac{\partial \mathcal{E}}{\partial Z} = 2\pi i k\langle\mathcal{P}\rangle \qquad (9\text{-}15a)$$

$$\frac{\partial \mathcal{P}}{\partial T} + (\gamma_{ab} - i\Delta')\mathcal{P} = -i\frac{\mu^2}{\hbar}\mathcal{E}R \qquad (9\text{-}15b)$$

$$\frac{\partial R}{\partial T} + \Gamma_0(R - R_0) = Im\left(\frac{\mathcal{P}\mathcal{E}^*}{\hbar}\right) \qquad (9\text{-}15c)$$

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\mathcal{E} \simeq i2\pi k\langle\mathcal{P}\rangle$$

$$\dot{\mathcal{P}} + (\gamma_{ab} - i\Delta')\mathcal{P} = -i\frac{\mu^2}{\hbar}\mathcal{E}R$$

$$\dot{R} + \Gamma_0(R - R_0) = Im\left(\frac{\mathcal{P}\mathcal{E}^*}{\hbar}\right)$$

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Date . .

Eq. (9-15 a) can be written in terms of the Intensity $I = \langle \frac{c}{4\pi} E^2 \rangle_{time} = \frac{c}{8\pi} \mathcal{E}\mathcal{E}^*$, where $\langle \dots \rangle$ indicates the time averaging over a few cycles of harmonic oscillations (See A 9)

$$\frac{\partial I}{\partial Z} = -\frac{\omega}{2} \mathrm{Im}\left[\langle \mathcal{P} \rangle \mathcal{E}^*\right] \qquad (9-15\acute{a})$$

Here $I = I(Z, T)$ changes as a function of Z, in the direction of propagation. T dependence comes indirectly, through $\mathcal{P}$ (and R) equations. The time dependence of the wave is ~~convertio~~ conveniently seen in the frame moving along the wave. In that frame the wave is described as a function of T.

$$\frac{\partial I}{\partial Z} = \frac{c}{8\pi}\left(\mathcal{E}^* \frac{\partial \mathcal{E}}{\partial Z} + \mathcal{E}\frac{\partial \mathcal{E}^*}{\partial Z}\right) = \frac{c}{8\pi} 2 \mathrm{Re}\left[\mathcal{E}^* \frac{\partial \mathcal{E}}{\partial Z}\right]$$
$$= \frac{c}{4\pi} \mathrm{Re}\left[\mathcal{E}^* 2\pi i k \langle \mathcal{P} \rangle\right] = -\frac{\omega}{2} \mathrm{Im}\left[\mathcal{E}^* \langle \mathcal{P} \rangle\right]$$

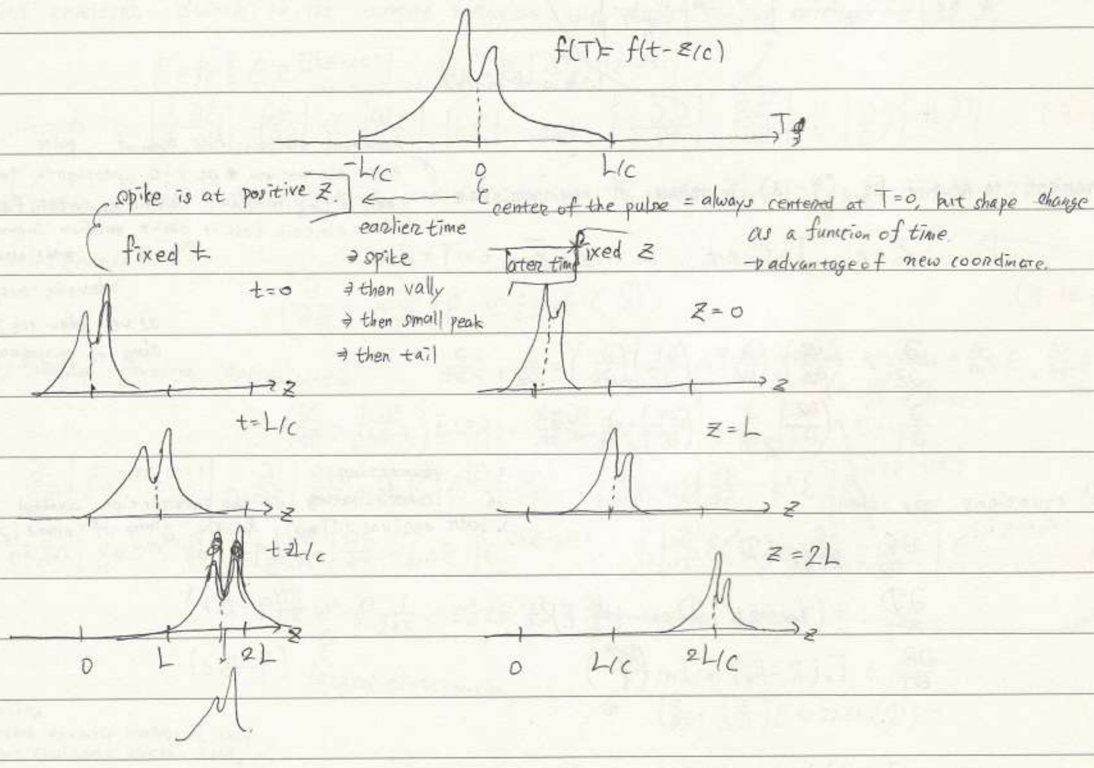

Fig. 1 physical picture of wave equation in the new coordinates ⇒ an EM pulse seen by an observer moving along with the pulse

9.3 Rabi oscillations

As an examples for the use of the Maxwell-Schrödinger equations, let us first consider the Rabi oscillations. Suppose the number of photons in the field is much greater than the number of molecules in the sample so that the field amplitude does not change much as it transverses the sample. Let us consider the case of exact resonance ($\Delta'=0$) and negligible damping. From Eqs. (9-13b) and (9-13c)

$$\dot{\mathcal{P}} = -i\frac{\mu^2}{\hbar}\mathcal{E}R, \quad \dot{R} = \mathrm{Im}\left(\frac{\mathcal{P}\mathcal{E}^*}{\hbar}\right) \qquad (9\text{-}16)$$

for simplicity, let neglect damping part and detuning

where $\mathcal{E}$ is a constant which can be taken to be real. Then

$$\ddot{R} = \frac{\mathcal{E}}{\hbar}\mathrm{Im}(\dot{\mathcal{P}}) = -\left(\frac{\mu\mathcal{E}}{\hbar}\right)^2 R = -\Omega^2 R \qquad (9\text{-}17)$$

With the initial condition, $R(0) = -N$, $\mathcal{P}(0) = 0$,

$$R(t) = -N\cos\Omega t, \quad \mathcal{P}(t) = i\mu N\sin\Omega t \qquad (9\text{-}18)$$

coefficient determined by $\dot{\mathcal{P}} = -i\mu\Omega(-N)\cos\Omega t$

$$\dot{\mathcal{P}} + (\gamma_{ab} - i\Delta')\mathcal{P} = -i\frac{\mu^2}{\hbar}\mathcal{E}R$$

$$\dot{R} + \Gamma_0(R - R_0) = \mathrm{Im}\left(\frac{\mathcal{P}\mathcal{E}^*}{\hbar}\right)$$

This is just the Rabi oscillation. The energy density in the field varies as

$$\delta U(t) = -N\hbar\omega(1-\cos\Omega t)/2 \qquad (9\text{-}19)$$

by $\delta U = -\hbar\omega N\rho_{aa} = -\hbar\omega\left(\frac{N+R}{2}\right)$

∵ increase in excited state population will decrease energy

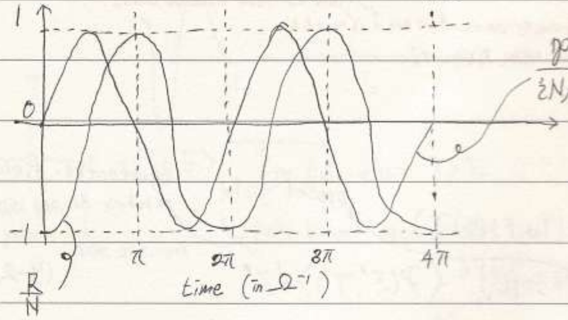

Fig.2

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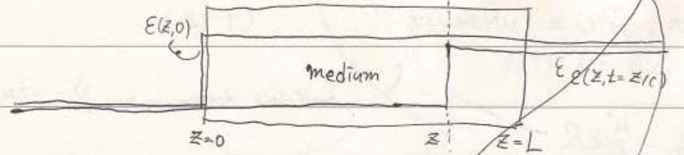
9.4 Free induction decay revisited

Suppose a traveling applied field with envelope $\mathcal{E}_0$ induces a steady state polarization with envelope $\mathcal{P}_0$. At $t=0$ the applied field is turned off in such a way that $\mathcal{E}_0 = 0$ for $t>0$ at $Z=0$ (see Fig.3).

Since the applied field propagates in the z direction, $\mathcal{E}_0 = 0$ for $t > z/c$ at z. So for $T>0$, the right-hand sides of Eqs. (9-15b) and (9-15c) become negligible. Let $\omega = \omega_0$ but assume velocity v for the molecules. Then for $T>0$ (assuming on-resonance condition)

$$\frac{\partial \mathcal{E}}{\partial Z} = 2\pi i k \langle \mathcal{P} \rangle$$
$$\frac{\partial \mathcal{P}}{\partial T} + (\gamma_{ab} + ikv)\mathcal{P} \simeq 0 \qquad (9\text{-}20)$$



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⟨Fig.3⟩

$$\frac{\partial \mathcal{E}}{\partial Z} = 2\pi i k \langle \mathcal{P} \rangle$$
$$\frac{\partial \mathcal{P}}{\partial T} + (\gamma_{ab} - i\Delta')\mathcal{P} = \cancel{-i\frac{\mu}{\hbar}\mathcal{E}R} \to 0$$
$$\frac{\partial R}{\partial T} + \Gamma_0 (R - R_0) = \cancel{Im\left(\frac{\mathcal{P}\mathcal{E}^*}{\hbar}\right)} \to 0$$

after $T>0$

$P = \chi E$
$\Rightarrow E \sim \lambda f \Rightarrow \lambda \ll 1$ 이라 $\Rightarrow$ λ 1차항만 고려

① $\omega \neq \omega_0$인 electric field 존재 → $\Delta' \approx -kv$ ← due to free induction decay.
② ω_0로 진동하는 파동과 고려?

Integrating $\mathcal{E}$ and $\mathcal{P}$ equations,

~~$\mathcal{P}(Z,T)$~~

$$\mathcal{P}(Z,T) = \mathcal{P}_0 \exp(-(\gamma_{ab} + ikv)T)$$
$$\mathcal{E}(Z,T) = \mathcal{E}(0,T) + 2\pi i k \int_0^Z \langle \mathcal{P}(Z',T) \rangle dZ' \qquad (9\text{-}21)$$

applied field — generated field by polarization which decay exponentially

with the initial condition $\mathcal{P}(Z, T=0) = \mathcal{P}_0$. The first term in $\mathcal{E}$ expression represents the applied field propagating the medium without modification (Fig 4). The second term represents a generated field by the medium. The resulting electric field is the sum of these two. Since $\mathcal{P}$ does not depend on Z explicitly and ☆☆ $\mathcal{E}(0, T>0) = 0$,

$$\mathcal{E}(L, T>0) = 2\pi i k L \langle \mathcal{P}_0 e^{-ikvT} \rangle e^{-\gamma_{ab} T} \qquad (9\text{-}22)$$

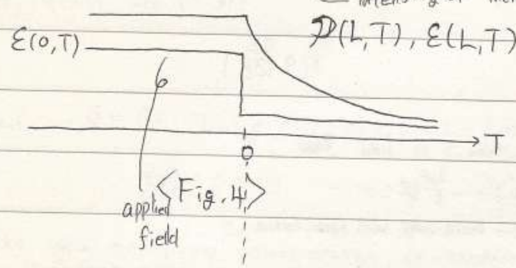
corresponding to a free-induction-decay signal. Expression for $\mathcal{P}_0$ can be readily obtained from Eq. (4-11).

$\because$ $\mathcal{P}(Z,T)$ does not depend on Z

dipole moment

$$\mathcal{P}_0 = \frac{N\mu\Omega(-\Delta' + i\gamma_{ab})}{\Delta'^2 + \gamma_{ab}^2(1 + I_o/I_{sat})} \qquad (9\text{-}23)$$

intensity of incident field



⟨Fig. 4⟩

9.5 Area Theorem ⇒ $\frac{\partial \Theta}{\partial Z} = g \sin\Theta$ with $g \equiv \frac{2\pi^{3/2} R_0 \mu^2}{\hbar u}$ (coupling constants)

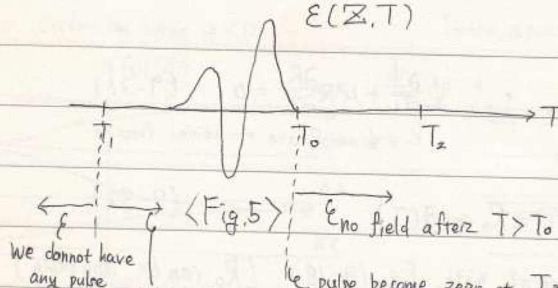
Consider an EM pulse propagating in an inhomogeneously broaden medium. Define the pulse area in terms of the slowly varying envelope $\mathcal{E}$. Assume $\mathcal{E}$ to be real. $= \frac{\mu}{\hbar}\int_{-\infty}^{+\infty} \mathcal{E}(Z,T)\,dT$

$$\Theta(Z) = \frac{\mu}{\hbar}\int_{T_1}^{T_2} \mathcal{E}(Z,T)\,dT \qquad (9\text{-}25)$$

T_1 and T_2 are chousen such that $\mathcal{E}$ is completely enclosed in the interval (T_1, T_2) as shown in Fig. 4. At $T=T_0$, $\mathcal{E}$ goes to zero. Integrating Eq. (9-15a)

$$\frac{\partial \Theta}{\partial Z} = \frac{2\pi i \mu k}{\hbar}\int_{T_1}^{T_2} \langle \mathcal{P} \rangle dT \qquad (9\text{-}26)$$

$$\frac{\partial \mathcal{E}}{\partial Z} = 2\pi i k \langle \mathcal{P} \rangle$$



⟨Fig. 5⟩

Z: real

Θ: real, imaginary

$$\frac{\partial \Theta}{\partial Z} = \frac{2\pi i \mu k}{\hbar}\int_{T_1}^{T_2}\langle \mathcal{P}\rangle dT$$

⇒ purely imaginary

$$\frac{\partial \mathcal{P}}{\partial T} + ikv\mathcal{P} = -\frac{i\mu^2}{\hbar}\mathcal{E}R$$

where $\langle \ldots \rangle$ means the averaging over the Maxwell-Boltzmann distribution. Let us assume the case of on resonance $(\Delta = 0)$ and negligible damping. From Eq. (9-15b)

R, v: real ⇒ purely real term ⇒ should vanish

$$\int_{T_1}^{T_2}\langle \mathcal{P}\rangle dT = \frac{i}{k}\int_{T_1}^{T_2}\left\langle \frac{1}{v}\cdot\frac{\partial \mathcal{P}}{\partial t}\right\rangle dT - \frac{\mu^2}{\hbar k}\int_{T_1}^{T_2}\left\langle \frac{\mathcal{E}R}{v}\right\rangle dT \qquad (9\text{-}27)$$

Since $\partial\Theta/\partial Z$ is real, the left-hand side of Eq (9-27) must be purely imaginary. The second term on the right-hand side is real, and thus it should vanish. ⇒ for simplicity let, assume resonance, no damping

No.

Date

at $T=T_1$, there is no electric field (not arrived at that time) $\Rightarrow \mathcal{P}(T_1)=0 \Rightarrow \langle \mathcal{P}(T_1)\rangle = 0$

$$\int_{T_1}^{T_2} \langle \mathcal{P} \rangle dT = \frac{i}{k}\int_{T_1}^{T_2}\left\langle \frac{1}{v}\frac{\partial \mathcal{P}}{\partial t}\right\rangle dT = \frac{i}{k}\left\langle \frac{1}{v}\int_{T_1}^{T_2}\frac{\partial \mathcal{P}}{\partial T}dT\right\rangle$$

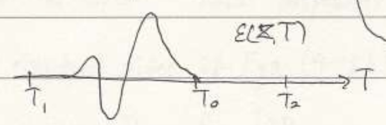
$$= \frac{i}{k}\left\langle \frac{\mathcal{P}(T_2)-\mathcal{P}(T_1)}{v}\right\rangle = \frac{i}{k}\left\langle \frac{\mathcal{P}(T_2)}{v}\right\rangle$$

$$= \frac{i}{k}\left\langle \frac{1}{v}\mathcal{P}(T_0;v)\, e^{-ikv(T_2-T_0)}\right\rangle \qquad (9\text{-}28)$$

$\mathcal{P}$: function of v $\Rightarrow$ explicitly put v after ';'

after T_0, there is no field $\Rightarrow \frac{\partial \mathcal{P}}{\partial T} + ikv\mathcal{P} = -\frac{i\mu^2}{\hbar}\mathcal{E}R \simeq 0$

$\Rightarrow$ polarization should vary with phase factor



Now the velocity integral $\langle \dots \rangle$ (M-B average) can be done in the complex plane by noting that $k(T_2-T_0)>0$. why?

$$\int_{T_1}^{T_2}\langle \mathcal{P}\rangle dT = \frac{i}{k}\int_{-\infty}^{+\infty} f(v)\mathcal{P}(T_0;v)\frac{e^{-ikv(T_2-T_0)}}{v}dv$$

$$= \frac{i}{k}(-\pi i)\times[\text{residue}\,(v=0)] = \frac{\pi}{k}f(0)\mathcal{P}(T_0;0) \qquad (9\text{-}29)$$

$f(0) = \frac{1}{\sqrt{\pi}u}$ — mean thermal velocity

where $f(v)$ is given by Eq. (4-31). Therefore, $\frac{\partial \Theta}{\partial Z} = \frac{2\pi i \mu k}{\hbar}\int_{T_1}^{T_2}\langle \mathcal{P}\rangle dT$

$$\frac{\partial \Theta}{\partial Z} = i\frac{2\pi^{3/2}\mu}{\hbar u}\mathcal{P}(T_0;0) \qquad (9\text{-}30)$$

Obviously, $\mathcal{P}(T_0;0)$ is purely imaginary for our choice of $\mathcal{E}$ to be real. Let $\mathcal{P}(T_0;0) = i\Psi$ ($\Psi \in \mathbb{R}$). We can obtain Ψ (real) from Eqs. (9-15b) and (9-15c) with $v=0$ ($\Delta'=0$), R_0

$$\frac{\partial}{\partial T}\Psi = -\frac{\mu^2}{\hbar}\mathcal{E}R,\quad \frac{\partial R}{\partial T} = \frac{\Psi\cdot\mathcal{E}}{\hbar}\quad \therefore\ \Psi\frac{\partial \Psi}{\partial T} + \mu^2 R\frac{\partial R}{\partial T} = 0 \qquad (9\text{-}31)$$

$\Rightarrow \Psi$ and R are sinusoidal function

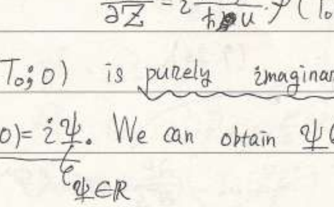
Therefore, the solutions are in the form of

$$\Psi(T) = -\mu R_0 \sin\Phi(T),\quad R(T) = R_0\cos\Phi(T) \qquad (9\text{-}32)$$

where the choice of sign in Ψ is consistent with Eq. (9-18) (R_0 can be anything)

we don't know what it is, but function of T.

$$\int_{-\infty}^{+\infty}[\dots]dv + \pi i\times(\text{Residue at } v=0) + (\text{Integral on arc at } |v|=\infty) = 0$$



$$e^{-ik(v_r+iv_i)(T_2-T_1)} \propto e^{kv_i(T_2-T_0)} \to 0 \quad \text{for } v_i \to -\infty$$

$v = v_r + iv_i$

$$\begin{cases} \frac{\partial \mathcal{P}}{\partial T} + (\gamma_{ab} - i\Delta')\mathcal{P} = -i\frac{\mu^2}{\hbar}\mathcal{E}R \Rightarrow \text{neglect damping } (\gamma_{ab}), \text{ and zero detuning, and } v=0 \Rightarrow \Delta'=0 \\ \frac{\partial R}{\partial T} + \Gamma_0(R-R_0) = \mathrm{Im}\left[\frac{\mathcal{P}\mathcal{E}^*}{\hbar}\right] \Rightarrow \text{neglect damping } (\Gamma_0) \end{cases}$$

No.

Date

at $T=T_1$, there is no electric field (not arrived at that time) $\Rightarrow \mathcal{P}(T_1)=0 \Rightarrow \langle \mathcal{P}(T_1)\rangle = 0$

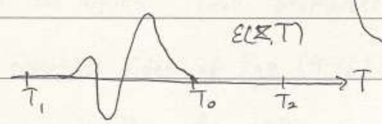
$$\int_{T_1}^{T_2}\langle \mathcal{P}\rangle dT = \frac{i}{k}\int_{T_1}^{T_2}\left\langle \frac{1}{v}\frac{\partial \mathcal{P}}{\partial t}\right\rangle dT = \frac{i}{k}\left\langle \frac{1}{v}\int_{T_1}^{T_2}\frac{\partial \mathcal{P}}{\partial T}dT\right\rangle$$

$$= \frac{i}{k}\left\langle \frac{\mathcal{P}(T_2)-\mathcal{P}(T_1)}{v}\right\rangle = \frac{i}{k}\left\langle \frac{\mathcal{P}(T_2)}{v}\right\rangle$$

$$= \frac{i}{k}\left\langle \frac{1}{v}\mathcal{P}(T_0;v)\, e^{-ikv(T_2-T_0)}\right\rangle \qquad (9\text{-}28)$$

$\mathcal{P}$: function of v $\Rightarrow$ explicitly put v after ";"

after T_0, there is no field $\Rightarrow \frac{\partial \mathcal{P}}{\partial T} + ikv\mathcal{P} = -\frac{i\mu^2}{\hbar}\mathcal{E}R \simeq 0$ $\Rightarrow$ polarization should vary with phase factor



Now the velocity integral $\underbrace{\langle \dots \rangle}_{\text{M-B average}}$ can be done in the complex plane by noting that $k(T_2-T_0)>0$. why?

$$\int_{T_1}^{T_2}\langle \mathcal{P}\rangle dT = \frac{i}{k}\int_{-\infty}^{+\infty} f(v)\mathcal{P}(T_0;v)\frac{e^{-ikv(T_2-T_0)}}{v}dv$$

$$= \frac{i}{k}(-\pi i)\times[\text{residue}(v=0)] = \frac{\pi}{k}f(0)\mathcal{P}(T_0;0) \qquad (9\text{-}29)$$

$f(0) = \frac{1}{\sqrt{\pi}u}$ — mean thermal velocity

where $f(v)$ is given by Eq. (4-31). Therefore,

$$\frac{\partial \Theta}{\partial Z} = \frac{2\pi i \mu k}{\hbar}\int_{T_1}^{T_2}\langle \mathcal{P}\rangle dT$$

$$\frac{\partial \Theta}{\partial Z} = i\frac{2\pi^{3/2}\mu}{\hbar u}\mathcal{P}(T_0;0) \qquad (9\text{-}30)$$

Obviously, $\mathcal{P}(T_0;0)$ is purely imaginary for our choice of $\mathcal{E}$ to be real. Let $\mathcal{P}(T_0;0) = i\Psi$ ($\Psi \in \mathbb{R}$). We can obtain Ψ (real) from Eqs. (9-15b) and (9-15c) with $v=0$ ($\Delta'=0$). R_0

$$\frac{\partial}{\partial T}\Psi = -\frac{\mu^2}{\hbar}\mathcal{E}R,\quad \frac{\partial R}{\partial T} = \frac{\Psi\cdot\mathcal{E}}{\hbar}. \quad \therefore \Psi\frac{\partial \Psi}{\partial T} + \mu^2 R\frac{\partial R}{\partial T} = 0 \qquad (9\text{-}31)$$

$\Rightarrow \Psi$ and R are sinusoidal function

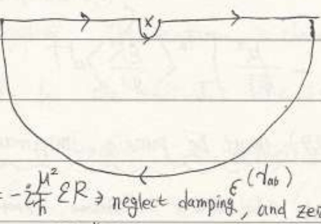
Therefore, the solutions are in the form of

$$\Psi(T) = -\mu R_0 \sin\Phi(T),\quad R(T) = R_0\cos\Phi(T) \qquad (9\text{-}32)$$

where the choice of sign in Ψ is consistent with Eq. (9-18) (R_0 can be anything).

we don't know what it is, but function of T.

$$\int_{-\infty}^{+\infty}[\dots]dv + \pi i\cdot\times(\text{Residue at } v=0) + (\text{integral on arc at } |v|=\infty) = 0$$



$$e^{-ik(v_r+iv_i)(T_2-T_1)} \propto e^{kv_i(T_2-T_0)} \to 0 \quad \text{for } v_i \to -\infty$$

$v = v_r + iv_i$

$$\begin{cases} \frac{\partial \mathcal{P}}{\partial T} + (\gamma_{ab} - i\Delta')\mathcal{P} = -i\frac{\mu^2}{\hbar}\mathcal{E}R \Rightarrow \text{neglect damping } (\gamma_{ab})\text{, and zero detuning, and } v=0 \Rightarrow \Delta'=0 \\ \frac{\partial R}{\partial T} + \Gamma_0(R-R_0) = \mathrm{Im}\left[\frac{\mathcal{P}\mathcal{E}^*}{\hbar}\right] \Rightarrow \text{neglect damping } (\Gamma_0) \end{cases}$$

$$\frac{\partial \Psi}{\partial T} = -\frac{\mu^2}{\hbar}\mathcal{E}R$$

$$\frac{\partial}{\partial T}\left(-\mu R_0 \sin\Phi\right) = -\mu R_0 \cos\Phi \frac{\partial \Phi}{\partial T} = -\frac{\mu^2}{\hbar}\mathcal{E}R_0\cos\Phi$$

result in Rabi oscillation

$R(t) = -N\cos\Omega t$, $\mathcal{P}(t) = i\mu N \sin\Omega t$ ⇒ coefficient is chosen to be consistent with this eq.

Plugging Eqs. (9-32) into (9-31)

$$\frac{\partial \Phi}{\partial T} = \frac{\mu \mathcal{E}}{\hbar} \qquad (9\text{-}33)$$

So we identify $\Phi = \Theta$. Thus $\mathcal{P}(T_0; 0) = i\Psi = -i\mu R_0 \sin\Theta(T_0)$, and therefore, we finally get

$$\frac{\partial \Theta}{\partial Z} = g\sin\Theta \quad \text{with} \quad g \equiv \frac{2\pi^{3/2} R_0 \mu^2}{\hbar u} \qquad (9\text{-}34)$$

where g is the gain coefficient (= coupling coefficient) (unsaturated, on-resonance). The statement in Eq. (9-34) is called the area theorem, first derived by S. McCall and E. Hahn (1969).

$$\frac{\partial \Theta}{\partial Z} = i\frac{2\pi^{3/2}\mu}{\hbar u}\mathcal{P}(T_0; 0)$$

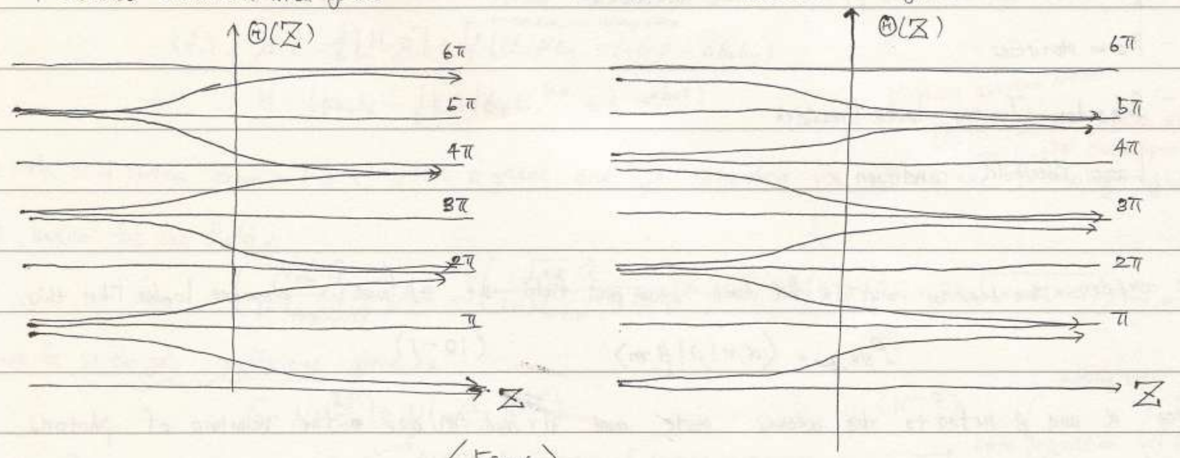
Note from Eq. (9-34) that Θ does not depend on Z when $\Theta = n\pi$ with $n = 0, 1, 2, \cdots$. If $g > 0$ (amplifying medium), Θ evolves toward $(2n+1)\pi$ whereas if $g < 0$ (absorbing medium) Θ evolves toward $2n\pi$ (Fig. 5). Explicitly,

$$\int_{\Theta_0}^{\Theta} \frac{d\Theta'}{\sin\Theta'} = \ln\left|\frac{\tan(\Theta/2)}{\tan(\Theta_0/2)}\right| = \int_0^{Z} g\, dZ' = gZ \qquad (9\text{-}35)$$

$$\frac{\partial \Theta}{\partial Z} = g\sin\Theta \quad \text{with} \quad g \equiv \frac{2\pi^{3/2} R_0 \mu^2}{\hbar u}$$

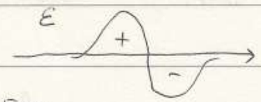
$$\therefore \tan(\Theta/2) = \tan(\Theta_0/2)\, e^{gZ}$$

Pulse area evolution when $g < 0$ Pulse area evolution when $g > 0$



⟨Fig. 6⟩

$\Theta = 0$ does not necessarily mean $\mathcal{E} = 0$



pulse area remain (unchange) in the medium

No. Lec10.

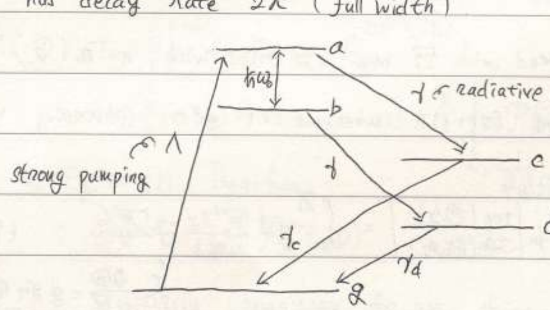
Date Quantum Theory of Laser : Photon statistics, Laser linewidth and threshold condition.

10. Quantum Theory of Laser

How coherent is the laser light? More specifically, how monochromatic can be the laser? To answer this question, we need a fully quantum mechanical ~~descript~~ description of the laser.

10.1. Equation of Motion

Consider a single atom in a cavity. The atom is described by the three-level system shown in Fig. 1. The atom is pumped from the ground level g to the upper level a. The lasing levels a and b decay to levels c and d, respectively, at γ, which in turn decays fast to the ground state at γ_c and γ_d, respectively. We assume $\gamma_c, \gamma_d \gg \gamma$ so that the atom is mostly in states a, b and g. The cavity has decay rate 2κ (full width)



<Fig. 1>

⇒ 5 levels, but effectively 3-level system

$\gamma_c, \gamma_d \gg \gamma$ ⇒ once atom is put in level c and d, quickly decay to ground state g.

⇒ basically we neglect level c, d.

($\because$ population in level c, d would be very small)

⇒ effectively, g, a, b level → 3-level

⇒ - Equation of motion for photon number distribution.
- Photon statistics
- Schawlow-Towner laser linewidth
- Laser threshold condition

We consider the density matrix for both atom and field s.t. its matrix element looks like this.

$$\rho_{\alpha n, \beta m} = \langle \alpha, n | \rho | \beta, m \rangle \qquad (10-1)$$

Where α and β refer to the atomic state and n and m are the number of photons in the cavity, referring to the field state.

For N atoms ρ is defined for all atoms plus the field. In this case the density matrix in Eq. (10-1) is called the population matrix, corresponding to $\rho_{\alpha\beta}$ of the semiclassical theory. Specifically,

$= \lim_{n,m\to\infty} \rho_{\alpha n,\beta m}$

consistent with full density matrix equation $0 = \sum_k \sum_\alpha (\mathrm{Tr}_{\{\alpha'\}} \rho_{\{\alpha'\},n;\{\alpha'\}m})_{\alpha,\alpha} = \sum_\alpha \rho_{\alpha,n;\alpha,m}$

$$\sum_\alpha \rho_{\alpha n,\alpha m} = \sum_k \sum_\alpha (\rho^k_{n,m})_{\alpha,\alpha}$$ change sigma, summation of all atomic state except k th atom → taking full trace, we are not touching photon number state

$\rho = \rho_{\alpha\beta} \otimes \rho_{nm}$

k refers to atom $\ni 1, 2, \cdots, N$

$$\rho_{\alpha n,\beta m} = \sum_k \rho^k_{\alpha n,\beta m} = \sum_k (\rho^k_{n,m})_{\alpha,\beta} \qquad (10\text{-}2)$$

where k summation is for all atoms, and the population matrix for k th atom is

$$\rho^k_{n,m} \equiv \sum_{\{\alpha'\}} \rho_{\{\alpha'\},n;\{\alpha'\},m} = \mathrm{Tr}[\rho] \qquad (10\text{-}3)$$

$\rightarrow (\rho^k_{n,m})_{\alpha\beta}$: k번째 atom의 α,β density matrix

with $\{\alpha'\} = \alpha_1, \alpha_2, \cdots, \alpha_{k-1}, \alpha_{k+1}, \cdots, \alpha_N$, meaning taking trace over the atomic states of all atoms except k th atoms. (k번째atom만 고려하기 위해서 α_k를 제외한 other atoms 을 trace) Nonetheless, the equation of motion, shown below, in terms of $\rho_{\alpha n,\beta m}$ is the same for both cases, the N atom case and the ~~orignal~~ single atom case. The equation of motion comes from the master equation $(\rho_{nm})_{\alpha\beta} = \rho_{\alpha n,\beta m}$

$$\dot\rho_{nm} = -\frac{i}{\hbar} \mathrm{Tr}_{atom}[H_I, \rho]_{nm} + (L\rho)_{nm} \qquad (10\text{-}4)$$

where ~~$\alpha\beta$ sum 1, 2, ..., k, ..., N~~ $\Rightarrow \mathrm{Tr}_{atom}(\rho) = \sum_{k=1}^{N} \sum_\alpha (\rho^k_{\alpha\alpha})_{n,m}$

$$H_I = i\hbar g(\sigma_+ a - \sigma_- a^\dagger) \qquad (10\text{-}5a)$$ Murray Sargent Ch 14참고

(σ_+: raising operator of atomic state; σ_-: lowering "; a: destruction "; $a^\dagger$: creation operator of photon number)

with g the coupling constant, between atom and field

$$g = \frac{\mu}{\hbar}\sqrt{\frac{2\pi\hbar\omega_0}{V}} \qquad (10\text{-}5b)$$

and the Liouville operator (dissipation super-operator) is given by

$$L\rho = \kappa(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \qquad (10\text{-}6)$$ Christopher quantum optics pg. 210 참고

(κ: half decay rate of cavity)

which is similar to Eq. (3-31). Component-wisely

$$(L\rho)_{nm} = -\kappa(n+m)\rho_{nm} + 2\kappa\sqrt{(n+1)(m+1)}\,\rho_{n+1,m+1} \qquad (10\text{-}7)$$

(cf.) $$\begin{cases} \dot\rho = -\frac{i}{\hbar}[H,\rho] + \frac{\Gamma_0}{2}(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-) \\ H = \frac{1}{2}\hbar\omega_0\sigma_z - \frac{1}{2}\hbar\Omega(\sigma_+ e^{-i\omega t} + \sigma_- e^{+i\omega t}) \end{cases}$$

(Γ_0: radiative decay rate)

Milburn quantum optics Ch 12 참고 + A short introduction to Lindblad Master equation 참고

For the level system shown in Fig. 1, after a great deal of algebra we obtain the following equation of motion for the field.

$$\dot\rho_{nm} = -\left(\frac{N'_{nm} G}{1+N_{nm}\mathcal{S}/G}\right)\rho_{nm} + \left(\frac{\sqrt{nm}\,G}{1+N_{n-1,m-1}\mathcal{S}/G}\right)\rho_{n-1,m-1} - \kappa(n+m)\rho_{nm} + 2\kappa\sqrt{(n+1)(m+1)}\,\rho_{n+1,m+1} \quad (10\text{-}8)$$

where G is the gain coefficient given by

$$G = N_a\left(\frac{2g^2}{\gamma}\right) = N\left(\frac{\Lambda}{\gamma+\Lambda}\right)\left(\frac{2g^2}{\gamma}\right) \qquad (10\text{-}9)$$

express N_a in terms of pumping parameter → rate equation of upper level neglecting the laser effect (pumping term, decaying term) $\Rightarrow \dot N_a = \Lambda N_g - \gamma N_a = \Lambda(N-N_a) - \gamma N_a = 0 \Rightarrow N_a = \frac{N\Lambda}{\gamma+\Lambda}$

and $\mathcal{S}$ is the self saturation coefficient

$$\mathcal{S} = \frac{4g^2}{\gamma^2} G \qquad (10\text{-}10)$$

and

$$N'_{nm} = \frac{1}{2}(n+1+m+1) + \frac{(n-m)^2\mathcal{S}}{8G} \qquad (10\text{-}11)$$
$$N_{nm} = \frac{1}{2}(n+1+m+1) + \frac{(n-m)^2\mathcal{S}}{16G}$$

note that denominator difference.

in 3 level system, when pumping is very large population is mostly at ground, upper level $\Rightarrow N \simeq N_a + N_g$

$$\frac{N'_{nm}G}{1+N_{nm}\mathcal{S}/G} = \left(\frac{\frac{1}{2}(n+1+m+1)G + \frac{1}{8}(n-m)^2\frac{4g^2}{\gamma^2}G}{1+\left(\frac{1}{2}(n+1+m+1)\mathcal{S} + \frac{(n-m)^2}{16}\frac{4g^2}{\gamma^2}\right)}\right)$$

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No.

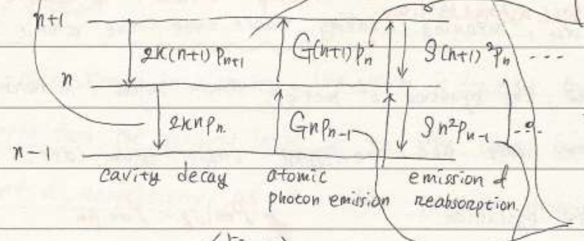
Date

population of having n photons in cavity — due to cavity decay — due to cavity decay from n+1 state to n state

Particularly, the diagonal element, $\rho_{nn} = p_n$, the photon number distribution function for the laser field, satisfies

$$\dot{p}_n = -\left[\frac{(n+1)G}{1+(n+1)S/G}\right]p_n + \left[\frac{nG}{1+nS/G}\right]p_{n-1} - 2\kappa n p_n + 2\kappa(n+1)p_{n+1} \qquad (10\text{-}12)$$

Eq.(10-12) describes probability flow among the field states as shown in Fig.2.



corresponds to $-\left[\frac{(n+1)G}{1+(n+1)S/G}\right]p_n$ (neglect denominator for a moment)

⇒ when we have n photons, atom emit photon by stimulated, spontaneous emission
→ n ⇒ stimulated emission
1 ⇒ spontaneous emission

photon emission rate from n-1 state to n state

⟨Fig.2⟩

comes from denominator

The atom emits a photon to the laser field at a rate of $G(n+1)$ with the familiar $(n+1)$ factor (spontaneous + stimulated emission). This increases the photon number by 1, so that the probability p_n decreases. Similarly, there is a probability inflown from $(n-1)$ state by the photon emission, corresponding to the second term in Eq.(10-12). The denominator factors account for the saturation effects. When we expand the denominator factor of the first term, we get $(n+1)G(n+1)S/G\cdot p_n = S(n+1)^2 \cdot p_n$, which corresponds to photon emission followed reabsorption. The cavity decay brings $(n+1)$ state to n state (inflow) and n state to $(n-1)$ state (outflow).

10.2. Laser photon statistics

Far above threshold, i.e., $G \gg 2\kappa$ (gain much larger than loss), $S\langle n\rangle/G \gg 1$, and we can neglect the factor "1" in the denominators. In the steady state,

$$0 = -\frac{G^2}{S}p_n + \frac{G^2}{S}p_{n-1} - 2\kappa n p_n + 2\kappa(n+1)p_{n+1} \qquad (10\text{-}14)$$

group second, and third term ⇒ difference is only n and (n+1) ⇒ if Group vanish, each group should vanish.

Due to the detailed balance between adjacent levels in the steady state,

$$2\kappa n p_n = \frac{G^2}{S}p_{n-1} \qquad (10\text{-}15)$$

recursion relation

The normalized solution is

$\sum p_n = 1$

$$p_n = \frac{\lambda^n}{n!}e^{-\lambda} \quad \text{with} \quad \lambda = \frac{G^2}{2\kappa S} \qquad (10\text{-}16)$$

The photon number distribution is Poissonian distribution well above threshold.

group second and third term

$$\dot{p}_n = -\left[\frac{(n+1)G}{1+(n+1)S/G}\right]p_n + \left[\frac{nG}{1+nS/G}\right]p_{n-1} - 2\kappa n p_n + 2\kappa(n+1)p_{n+1}$$

The exact solution of Eq. (10-12) is obtained from the detailed balance equality,

$$2\kappa p_n n = \left[\frac{nG}{1+nS/G}\right]p_{n-1} \qquad (10\text{-}19)$$

$$p_n = p_0 \prod_k^n \frac{(G/2\kappa)}{(1+kS/G)} \qquad (10\text{-}20)$$

kappa

$$= p_0 \prod_k^n \frac{(G/2\kappa)}{(1+k\,S/G)} \qquad k=1,2,3,\cdots \quad (10\text{-}21)$$

Considering n_p satisfying

~~then for $k<n_p$~~

$$\frac{G/2\kappa}{1+n_pS/G} = 1$$

다시보기

$$\frac{G/2\kappa}{1+n_pS/G} = 1 \Rightarrow n_p = \frac{G}{S}\left(\frac{G}{2\kappa}-1\right)$$

then for $k<n_p$ the multiplicative factor is larger than unity while it is less than unity for $k>n_p$.
$\Rightarrow n_p$ is ~~peaked~~ when P_n is peak. Hence, p_n increases for n up to n_p and goes monotonically to zero for $n>n_p$.

Thus the distribution peaks at

$$n_p = \frac{G}{2\kappa}\left(\frac{G-2\kappa}{S}\right) \qquad (10\text{-}22)$$

For the distribution given by Eq. (10-20), one can show (Problem 21) that the average number of photons is

$$\langle n\rangle = \frac{G}{2\kappa}\left(\frac{G-2\kappa}{S}\right) + \frac{G}{S}p_0 \underbrace{\simeq}_{\text{true when } G\gg 2\kappa} \frac{G}{2\kappa}\left(\frac{G-2\kappa}{S}\right) = n_p \qquad (10\text{-}23)$$

and its variance is

$$\langle n^2\rangle = \langle n\rangle^2 + \frac{G^2}{2\kappa S} \qquad (10\text{-}24)$$

classically we cannot have non-classical field

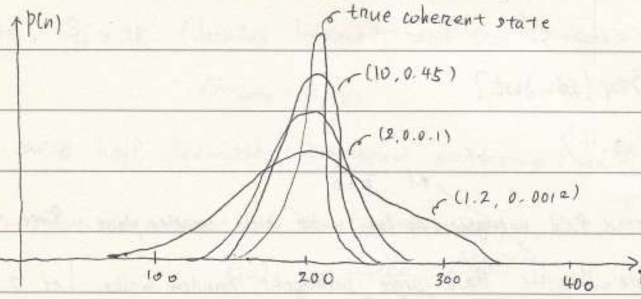
Measure of coherence of photon number

Mandel Q parameter for the field is defined as

coherent state $\Rightarrow Q=0$
super-Poissonian $\Rightarrow Q>0$
sub-Poissonian $\Rightarrow Q<0 \Rightarrow$ Possible in only quantum mechanics

$$Q \equiv \frac{(\text{variance})}{(\text{variance of coherent state})} - 1 = \frac{\langle n^2\rangle - \langle n\rangle^2}{\langle n\rangle} - 1 = \frac{\cdot}{G/(2\kappa-1)} \qquad (10\text{-}25)$$

Hence, the laser field is in general super-Poissonian $(Q>0)$. It becomes Poissonian $(Q=0)$ very far above threshold, i.e., $G\gg 2\kappa$. $\Rightarrow$ Not truly coherent state, $\rightarrow$ just approximation



Coherent state vs. laser. The narrowest one is p_n for the coherent state with $\langle n\rangle = 200$. The others are p_n for the ~~others~~ laser with $(G/2\kappa, S/2\kappa) = (10, 0.45), (2, 0.01), (1.2, 0.0012)$, all with the same $\langle n\rangle$, in order of increasing distribution width.

⟨Fig. 3⟩

No.

Date Part2.

10.3 Natural Linewidth

A finite laser linewidth comes from the amplitude and phase fluctuations of the laser field. When the laser is operating far above threshold, $\Delta n / \langle n \rangle \ll 1$ so that the amplitude fluctuation is negligible. However, the phase fluctuation is not negligible even far above threshold. The source of the phase fluctuation is the spontaneous emission of atoms. The electric field experience a random phase change whenever spontaneous emission occurs into the lasing mode. Such change occurs in a time scale much shorter than the overall evolution of the field.

The electric field can be ~~eq~~ represented by a vector in the complex plane (positive frequency part of field operator)

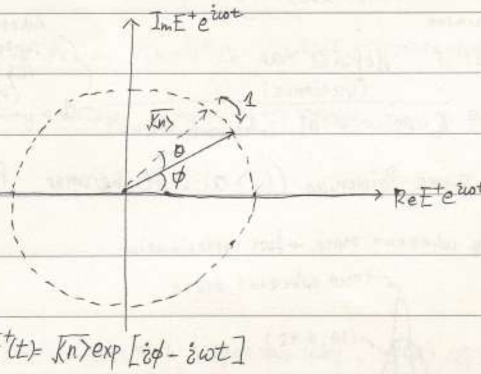
$$E^+(t) = \sqrt{\langle n \rangle}\, \exp(i\phi - i\omega t) \qquad (10-26)$$

where the field is normalized with respect to single photon amplitude E_0

$$\frac{E_0^2}{8\pi} = \frac{\hbar\omega}{\underbrace{V}_{\text{mode volume}}} \qquad (10-27)$$

and $E^{\pm}$ is the positive (negative) frequency part of the electric field operator

→ intensity proportional to photon number



$E^+(t) = \sqrt{\langle n \rangle} \exp[i\phi - i\omega t]$

⟨Fig. 4⟩

Due to the spontaneous emission events, the electric field undergoes random walks in the complex plane. Since $\langle n \rangle \gg 1$, we can neglect the change in the amplitude. But the phase angle undergoes random walks. Let s denote the displacement of each walk. Each walk has a unity step size (E_0 if not normalized) oriented in random direction so that the mean-square size $\overline{s^2} = 1/2$ (c.f. $\overline{s} = 0$). During the time period of τ, there will be $G_s \tau$ spontaneous emission events ($G_s = N_a^{sat} \Gamma_a$ is the saturated spontaneous emission rate into the laser field; N_a^{sat}: saturated upper state population). According to the standard one-dimensional random walk theory (see Reif for example), the probability that a distance l is traveled after $G_s\tau$ steps is given by

$$P(l) = \frac{1}{\sqrt{2\pi \overline{s^2} G_s \tau}} \exp\left[-l^2 / 2\overline{s^2} G_s\tau\right] = \frac{1}{\sqrt{\pi G_s \tau}} \exp\left[-l^2 / G_s\tau\right] \qquad (10-28)$$

electric field operator spreads → phase space in

as denominator ↑ → θ get broaden

No.

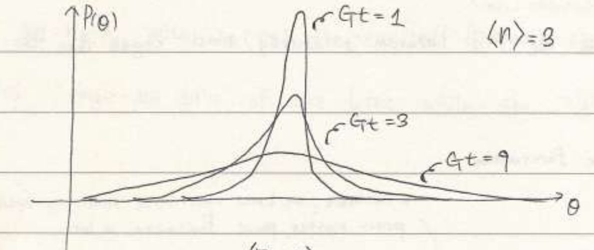
Date

In terms of the angular displacement $\theta = \frac{l}{\sqrt{\langle n \rangle}}$, noting $P(l)dl = P(\theta)d\theta$

$$P(\theta) = \sqrt{\frac{\langle n \rangle}{\pi G_s \tau}}\exp\left[-\theta^2 \langle n \rangle / G_s \tau\right] \qquad (10\text{-}29)$$

The probability distribution in Eq.(10-29) diffuses out as τ increases. In fact, it satisfies an one-dimensional diffusion equation with a diffusion constant $D = G_s/4\langle n \rangle$ (see Problem 22)

$$\frac{\partial P}{\partial \tau} = D\frac{\partial^2 P}{\partial \theta^2} \quad \text{with } D = \frac{G_s}{4\langle n \rangle} \qquad (10\text{-}30)$$



$\langle$Fig.5$\rangle$

$$\frac{\partial P}{\partial \tau} = D\frac{\partial^2 P}{\partial \theta^2} \quad \text{with } D = \frac{G_s}{4\langle n \rangle}$$

$$P(\theta) = \frac{\sqrt{\langle n \rangle}}{\sqrt{\pi G_s \tau}}\exp(-\theta^2\langle n \rangle / G_s \tau)$$

$$-\frac{\theta^2}{4D\tau} + i\theta = -\frac{1}{4D\tau}(\theta - 2iD\tau)^2 - D\tau$$

Recall the power spectral density (spectrum) is the Fourier transform of the two-time correlation function (Wiener-Khinchin theorem). Consider the two-time correlation function of the electric field

$$\left\langle E^-(t)E^+(t+\tau)\right\rangle = \left\langle E^-(0)E^+(\tau)\right\rangle = \langle n \rangle e^{-i\omega\tau}\left\langle e^{i\theta}\right\rangle \qquad (10\text{-}31)$$

$$\left\langle e^{i\theta}\right\rangle = \int P(\theta)e^{i\theta}d\theta = e^{-D\tau}$$

So that

$$\left\langle E^-(t)E^+(t+\tau)\right\rangle = \langle n \rangle e^{-i\omega\tau - D\tau} \quad \text{for } \tau > 0 \qquad (10\text{-}32)$$

The fourier transform of this (in steady state) is the power spectrum, which is a Lorentzian with a full width of $2D = G_s/2\langle n \rangle$. In steady state, $G_s \simeq 2\kappa$ (detailed balance), and thus the laser's natural linewidth well above threshold is

$$\Delta\omega_{laser} = \kappa/\langle n \rangle \qquad (10\text{-}33)$$

with κ the cavity half linewidth. Semiclassical consideration (see Siegman) of the laser operating just below threshold can also give an estimate of the laser linewidth,

$$\Delta\omega_{S-T} = \frac{N_u}{N_u - N_l}\frac{\hbar\omega(2\kappa)^2}{P_{out}} \qquad (10\text{-}34)$$

P_{out}: output power

where $N_u(N_l)$ is the population in the upper (lower) lasing level. This linewidth formula is called the Schawlow-Towns formula. Since $N_u \gg N_l$ and the laser output $P_{out} = 2\kappa\hbar\omega\langle n \rangle$, $\Delta\omega_{S-T}$ becomes

$$\Delta\omega_{S-T} \simeq \frac{2\kappa}{\langle n \rangle} \qquad (10\text{-}35)$$

which is twice larger than our quantum result.

No.

Date . .

This discrepancy is due to the fact that just below threshold the contribution from amplitude fuctuations is as important as that of phase fluctuations! The formula with 1/2 correction factor is called "modified" Schawlow-Townes formula.

In most lasers, actual linewidths are much broader than the S-T linewidth due to technical noises such as cavity vibrations, gain fluctuations caused by instability in current, voltage, pressure and/or temperature. In the diode laser, however, due to its small cavity length the S-T linewidth is usually much larger than the technical noises.

cavity vibration, gain fluctuation, instability of current and voltage, pressure, temperature

← linewidth ~ S-T formula

∴ if well above threshold ⇒ linewidth is mainly due to phase fluctuation

10.4 General form of laser threshold condition

$Q = \frac{1}{G/2\kappa - 1}$

in experience, we know when laser starts to operate, photon number, phase fluctuate a lot

Recall that in deriving the photon number distribution for laser far above threshold we assumed $G \gg 2\kappa$. Furthermore, when $G = 2\kappa$, the Mandel Q in Eq. (10-25) is not well defined. In fact, there exist dramatic changes in photon statistics when $G = 2\kappa$. The condition $G = 2\kappa$ thus specifies the laser threshold. In the strong pumping limit ($\Lambda \gg \gamma$)

← photon statistics become extremely chaotic

$$G \simeq \frac{2Ng^2}{\gamma} = 2\kappa \qquad (10\text{-}36)$$

where we replaced N_a with N. Note in the absence of lasing in the level system shown in Fig. 1, from the semiclassical density matrix equations

most of population is in the excited state

$$\dot{\rho}_{aa} = 0 = \Lambda \rho_{gg} - \gamma \rho_{aa} \longrightarrow \rho_{gg} = (\gamma/\Lambda)\rho_{aa} \ll \rho_{aa} \qquad (10\text{-}37)$$

$$\dot{\rho}_{cc} = 0 = \gamma \rho_{aa} - \gamma_c \rho_{cc} \longrightarrow \rho_{cc} = (\gamma/\gamma_c)\rho_{aa} \ll \rho_{aa}$$

So the population inversion, $\Delta N = N_a - N_b \simeq N$. In addition, $\gamma_{ab} = \gamma$. From Eqs. (2-16) and (10-5b)

∵ both level decay in γ

$A = \Gamma_0 = B\frac{\hbar\omega^3}{\pi^2 c^3} = \frac{4\mu^2\omega^3}{3\hbar c^3}$, $g = \frac{\mu}{\hbar}\sqrt{\frac{2\pi\hbar\omega_0}{V}}$

$$g^2 = \frac{2\pi\mu^2\omega}{\hbar V} = \left(\frac{4\mu^2\omega^3}{3\hbar c^3}\right)\frac{3\pi c^3}{2\omega^2 V} = \Gamma_{rad}\frac{\sigma_{rad}\cdot c}{4V} \qquad (10\text{-}38)$$

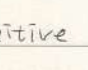
with $\sigma_{rad} = 3\lambda^2/2\pi$. Then, the above threshold condition can be rewritten in a more intuitive form

$$(\Delta N)\cdot\left(\frac{\sigma_{rad}\cdot c}{V}\right)\left(\frac{\Gamma_{rad}}{\Delta\omega_f}\right) = 2\kappa \qquad (10\text{-}39)$$

where $\Delta\omega_f = 2\gamma_{ab}$ is the fluorescence linewidth of the lasing transition. Substituting

$$\kappa = \frac{c}{2L}(1-R) \qquad (10\text{-}40)$$

half decay rate ⇒ multiply 1/2



Per time L/c, probability of exiting the cavity is $1-R$ ⇒ (Decay rate) $\frac{1-R}{L/c} = \frac{c}{L}(1-R)$

with R the cavity reflectivity and L the cavity length. Noting that the total population inversion ΔN can be written as $\Delta N = \Delta n \cdot V \cdot L'/L$ with L' the length of the gain medium in the cavity and Δn the inversion density, we finally get

inversion density

medium is filled in only L'

No.

Date . .

$$\text{emission} \rightarrow \sigma_{em} = \sigma_{rad} \underbrace{\frac{\Gamma_{rad}}{\Delta\omega_f}}_{\text{reduction factor}}$$

$$\underbrace{\Delta n \cdot \sigma_{em} \cdot L'}_{\text{single path gain}} = \underbrace{1 - R}_{\text{single path loss}} \qquad (10-41)$$

where $\sigma_{em} = \sigma_{rad}\Gamma_{rad}/\Delta\omega_f$, the emission cross section (Eq. (A4.1-2)). If σ_{em} is not the same as the absorption cross section, ~~~~ σ_{abs}, which is often the case ~~of~~ for dye lasers,

$$\underbrace{(n_a\sigma_{em} - n_b\sigma_{abs})L' = 1 - R}_{\text{new threshold condition}} \qquad (10-42)$$

where $\Delta n = n_a - n_b$ with $n_a (n_b)$ the excited (ground) state population density. The lefthand side of Eq. (10-42) is called "single-pass gain" of the laser while the righthand side is just the cavity loss (or single pass loss).

$$\rho_{aa} \rightarrow \frac{R_p}{2\gamma}$$

$$\rho_{bb} \rightarrow \frac{R_p}{2\gamma}$$

$$\sigma_2 \rightarrow \frac{-\gamma_{ab}\frac{\Omega R_p}{2\Gamma_a}}{\frac{1}{2}\Omega^2\left(\frac{1}{\Gamma_a}+\frac{1}{\Gamma_b}\right)\gamma_{ab}} = \frac{-\frac{1}{2}\Omega R_p}{\frac{1}{2}\Omega^2\frac{2}{\gamma}\gamma} = -\frac{R_p}{2\Omega} = \frac{R_p}{2\Omega}$$

$$\rho_{aa} \rightarrow \frac{R_p}{\Gamma_a}$$

σ_2

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11. Survey of Various Lasers

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- Charles H. Townes, *How the laser happened - Adventures of a Scientist* (Oxford, New York, 1999)
  - historical account of MASER development
- R. Feynman *et al.*, The Feynman Lectures on Physics, Vol. III (Addison Wesley, Reading, 1979)
  - maser principles

Web Resources

http://www.ece.ucdavis.edu/~cornett/research/tisapph.html - Ti:sapphire laser

http://www.lasalle.edu/academ/chem/laser_web/index.htm - general review of various lasers

http://www.repairfaq.org/sam/lasersam.htm - amateur laser building and useful links

http://sbfel3.ucsb.edu/www/vl_fel.html - information on free-electron lasers

11.1 The beginning: the ammonia MASER

C. Townes invented the MASER in 1954. The first MASER used a beam of ammonia molecules as a gain medium. In a rotating ammonia molecule, the nitrogen atom can flip across the plane formed by three hydrogen atoms. The stationary states $|I\rangle$ and $|II\rangle$ of the ammonia molecule are two superposition states of $|1\rangle$ and $|2\rangle$ and have split eigenenergies $E_0 \pm A$ with A the coupling rate between $|1\rangle$ and $|2\rangle$. In the DC electric field, the energy splitting further increases.

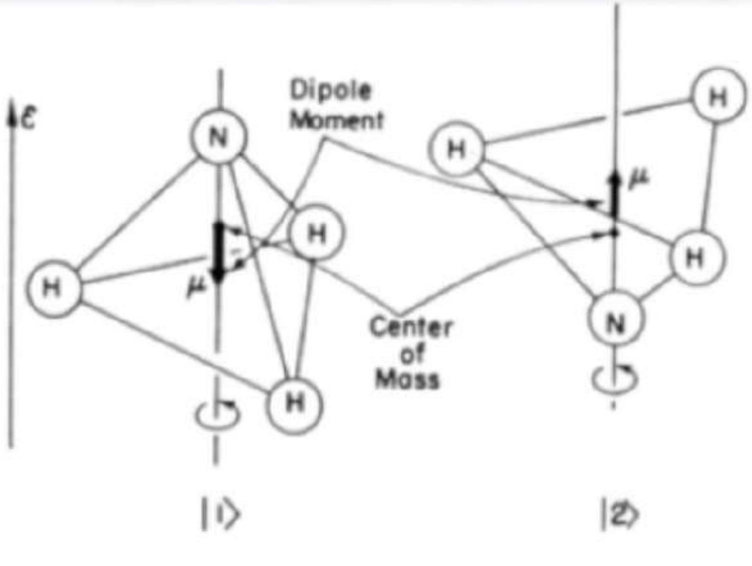


Fig. 1 A physical model of two base states for the ammonia molecule. These states have the electric dipole moments μ.

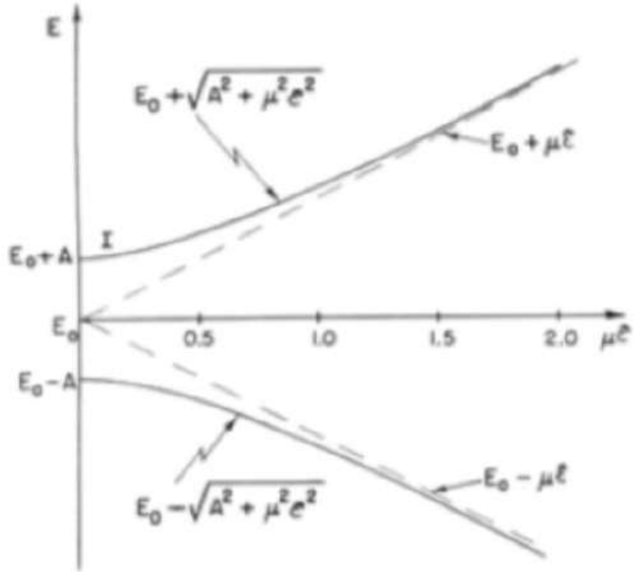


Fig. 2 Energy levels of the ammonia molecule in an electric field.

The stationary state $|I\rangle$ is a low-electric-field seeker whereas the state $|II\rangle$ is a high-electric-field seeker, so when an electric field *gradient* is applied transversely, these states can be separated in a Stern-Gerlach style. In the ammonia MASER, the ammonia molecules in the higher energy state $|I\rangle$ enters a microwave cavity which is resonant with the transition from $|I\rangle$ to $|II\rangle$ and emit photons into the cavity and exit. Since the photoemission is stimulated by the already present photons, the resulting microwave field becomes *coherent*.

MASER stands for <u>m</u>icrowave <u>a</u>mplification by <u>s</u>timulated <u>e</u>mission of <u>r</u>adiation.

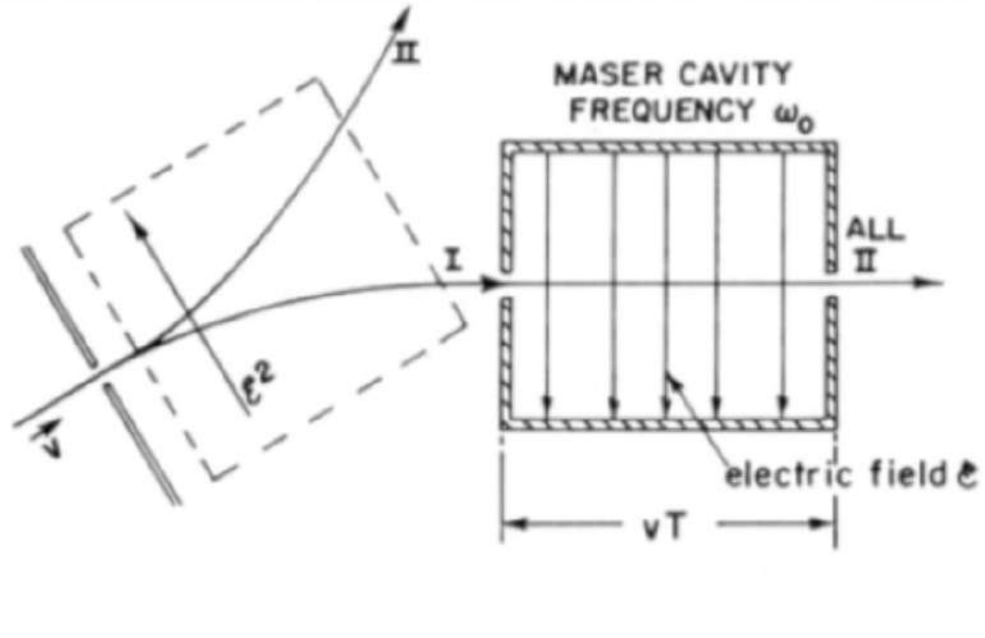
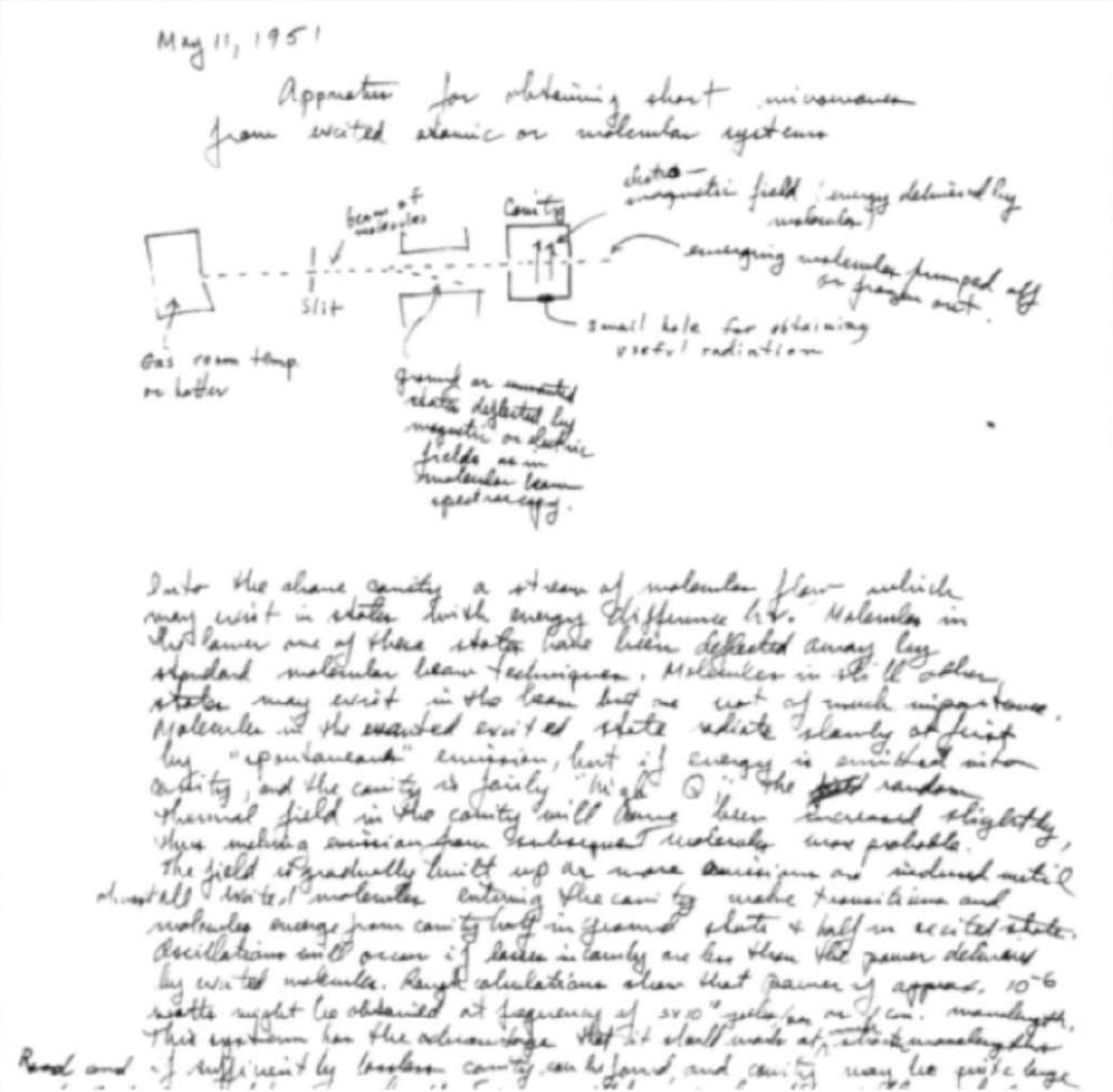


Fig. 3 Schematic diagram of the ammonia maser.

"Apparatus for obtaining short microwaves from excited atomic or molecular systems"



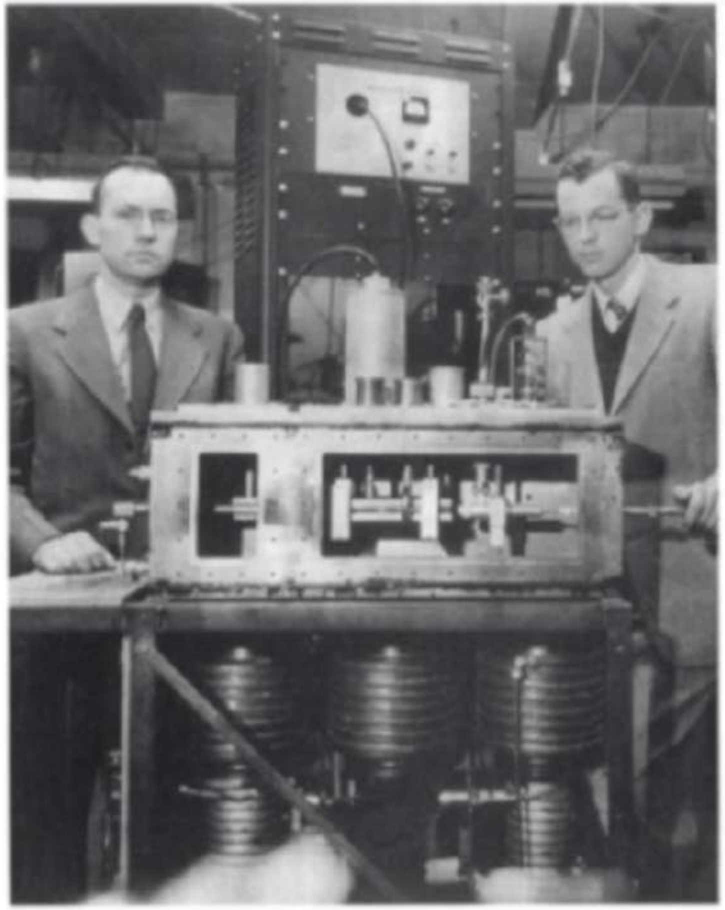
May 11, 1951

Apparatus for obtaining short microwaves from excited atomic or molecular systems

Into the above cavity a stream of molecules flow which may exist in states with energy difference hν. Molecules in the lower one of these states have been deflected away by standard molecular beam techniques. Molecules in still other states may exist in the beam but are not of much importance. Molecules in the wanted excited state radiate slowly at first by "spontaneous" emission, but if energy is emitted into cavity, and the cavity is fairly "high Q", the random thermal field in the cavity will have been increased slightly, thus making emission from subsequent molecules more probable. The field is gradually built up as more emissions are induced until almost all excited molecules entering the cavity make transitions and molecules emerge from cavity half in ground state & half in excited state. Oscillations will occur if losses in cavity are less than the power delivered by excited molecules. Rough calculations show that powers of approx. 10^{-6} watts might be obtained at frequency of 3×10^{10} cycles/sec or 1 cm. wavelength. This system has the advantage that it should work at still smaller wavelengths if sufficiently lossless cavity can be found, and cavity may be quite large

Read and

Fig. 4 C. Townes' lab note entry on the maser dated on May 11, 1951.



James Gordon (at right) and I were photographed with the second maser at Columbia University. The normally evacuated metal box where maser action occurred is opened up to show the four rods (quadrupole focuser) which sent excited molecules into a resonant cavity (the small cylinder to the right of the four rods). The microwaves that were generated emerged through the vertical copper waveguide near my hand. This second maser was essentially a duplicate of the first operating one, and it was built to examine the purity of maser signals, by allowing the two to beat together, thus producing a pure audio signal.

"It is perhaps an often-used device among dramatists to have a scientist scribble his thinking on the back of an envelope, but that is what I did on that morning of April 26, 1951, in Franklin Park. I took an envelope from my pocket to try to figure out how many molecules it would take to make an oscillator able to produce and amplify millimeter waves."

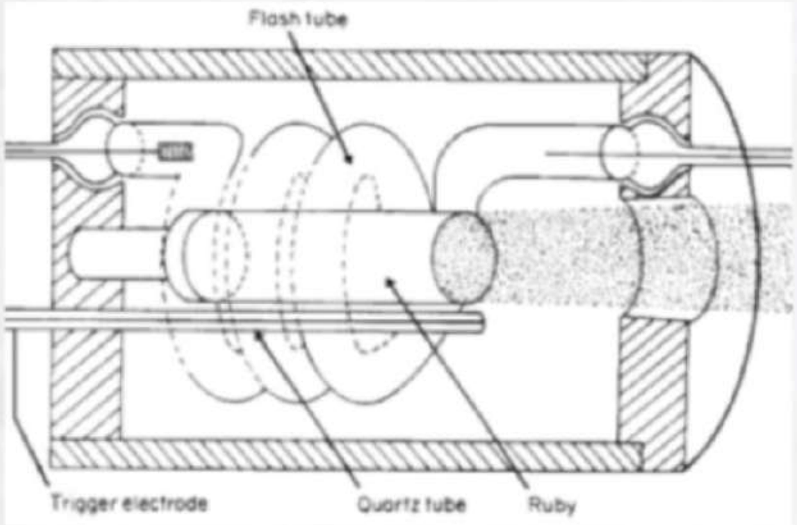
"One day, Rabi and Kusch - both of them Nobel laureates - came to my office and said, "Look, you should stop the work you are doing. It isn't going to work. You're wasting money. Just stop." ... I was lucky that I had come to Columbia with tenure."

"Three months later, during a seminar with most of the rest of my students in early April of 1954, Jim Gordon burst in and shouted, "It is working!" We stopped the seminar and went to the lab to see the evidence for oscillation and to celebrate."

Excerpts from "How the laser happened - Adventures of a Scientist" by Charles H. Townes (Oxford, New York, 1999).

11.2 The first "optical" MASER: ruby LASER

Charles Townes and Art Schawlow (Townes's former postdoc and brother-in-law) proposed an optical MASER (Phys. Rev., December 1958). Inspired by this, Ted Maiman developed the first LASER, a ruby laser, in May, 1960. Ironically, at one conference in 1959, Schawlow told Maiman that *pink* ruby might not be a good system because it is a three-level system requiring a lot of pumping. Schawlow recommended *black* ruby which is a four-level system but difficult to get. In the first ruby laser, evidence for lasing was indirect. Maiman only showed the ruby spectrum became narrow. He could not actually show the laser beam out of his device until his later experiments.



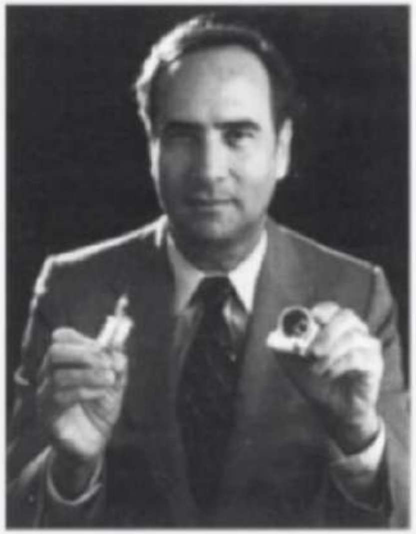


Fig. 5 Schematic of the original ruby laser and its inventor, Ted Maiman.

In pink ruby, Cr^{3+} ions present as impurities in Al_2O_3 crystal act as gain particles. Typical Cr^{3+} concentrations are ~0.05% by weight. An intense flash lamp pumping excites Cr^{3+} ions to F bands, which decay to ^2E lasing upper level in 50 nsec. The decay from ^2E (or E specifically) level is slow, taking 2 msec. Therefore, a reasonably fast pumping will result in all populations shared among $2A$, E and $^4A_2$ levels.

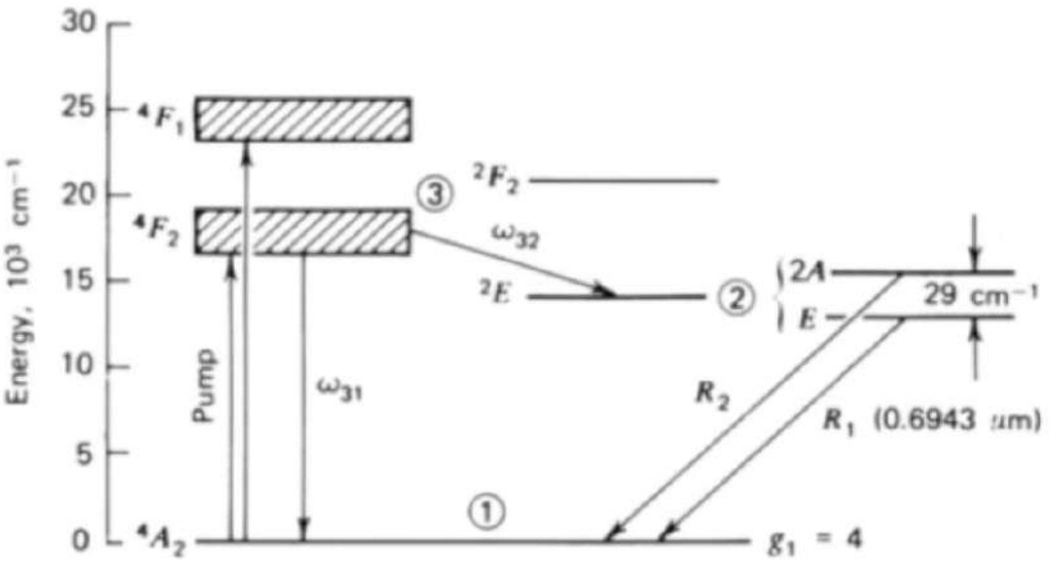


FIGURE 6a Energy levels pertinent to the operation of a ruby laser.

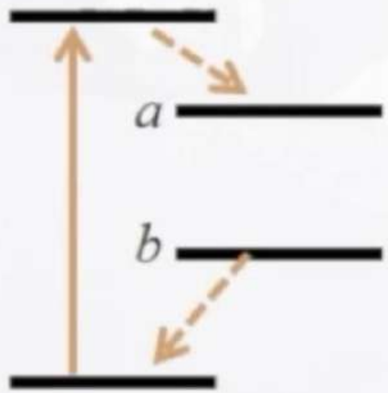


Fig. 6b 4 level system is easier than a 3-level system to achieve population inversion.

- One can write down a threshold condition for the ruby laser as

$$\Delta n \sigma_{em} l = 1 - R$$

where l=ruby rod length and Δn=population inversion density.

- Taking $\sigma_{em} \sim 10^{-20}\ \mathrm{cm}^2, l = 10\ \mathrm{cm}, R = 0.97,$ we obtain $\Delta n_{th} \approx 3\times 10^{17}\ \mathrm{cm}^{-3}$.
- Since there are two upper levels, one of which participates the laser oscillation,

$$n_1 + 2n_2 = n, n_2 - \frac{1}{2} n_1 = \Delta n_{th},$$

where 1/2 factor comes from the degeneracy of the ground state.

- With $n = 2\times 10^{19}\ \mathrm{cm}^{-3}$, we obtain $n_1 \sim 2n_2 \sim 10^{19}\ \mathrm{cm}^{-3}$ at threshold. In order to maintain necessary population density n_2, we need an optical pumping by which

$$\text{(total number atoms in level 2)} = 2n_2 A l$$

$$\approx \text{(total number photons absorbed in level 1)} = \frac{\Delta t A \Delta v I(v) n_1 \sigma_{abs} l}{\hbar \omega}$$

where Δt=pumping duration, A=ruby rod cross sectional area, Δv=absorption bandwidth, and $I(v)$=pumping intensity per frequency. From this we obtain a threshold pumping energy

$$\left[\Delta t \Delta v I(v)\right] = \frac{2n_2 \hbar \omega}{n_1 \sigma_{abs}} = \frac{10^{19}\mathrm{cm}^{-3} 3\times 10^{-12}\mathrm{ergs}}{10^{19}\mathrm{cm}^{-3} 10^{-19}\mathrm{cm}^2} = 3\ \mathrm{J/cm}^2.$$

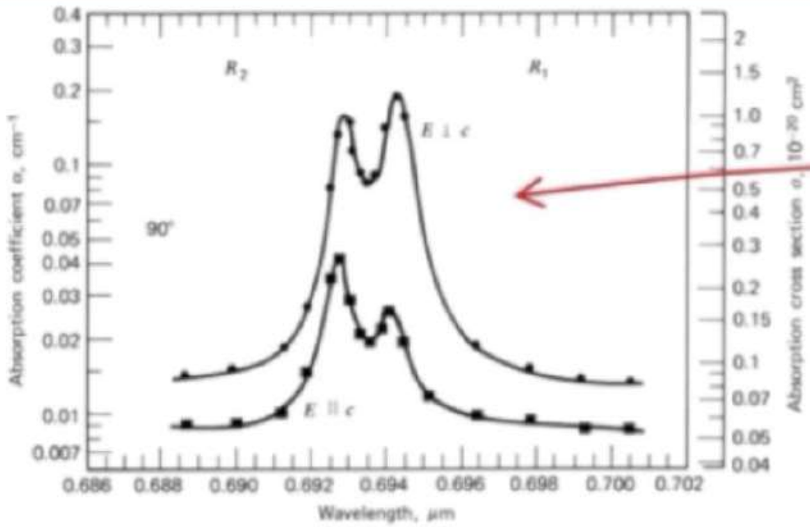


FIGURE 7 Absorption coefficient and absorption cross section as functions of wavelength for $E \parallel c$ and $E \perp c$. Sample was a pink ruby laser rod having a 90° c axis orientation with respect to the rod axis and a Cr^{3+} concentration of 1.58×10^{19} cm^{-3}.

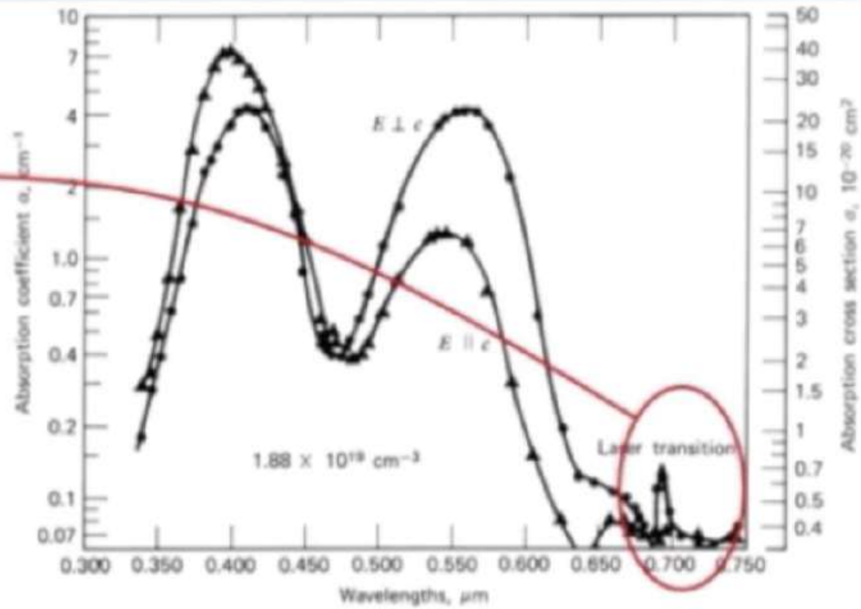


FIGURE 8 Absorption coefficient and absorption cross section as functions of wavelength for $E \parallel c$ and $E \perp c$. The 300°K data were derived from transmittance measurements on pink ruby with an average Cr ion concentration of 1.88×10^{19} cm^{-3}.

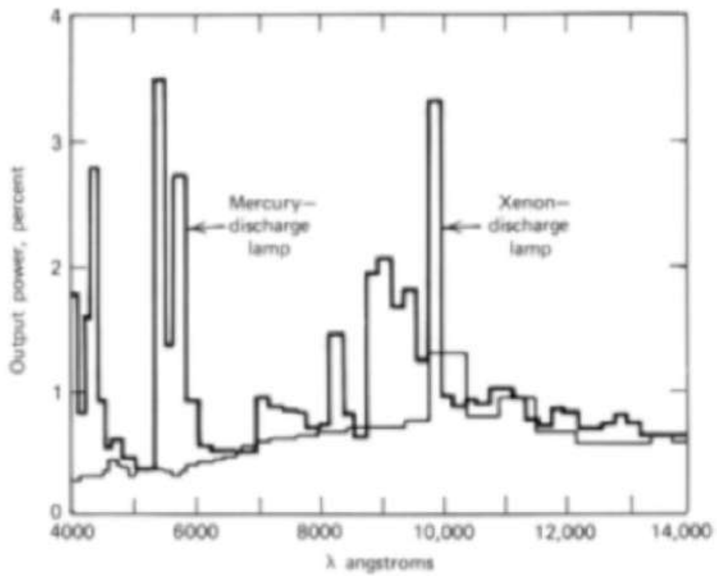


FIGURE 9 Spectral output characteristics of two commercial high-pressure lamps. Output is plotted as a fraction of electrical input to lamp over certain wavelength intervals (mostly 200 Å) between 0.4 and 1.4 μm.

Assuming about 10% of lamp output power in the ruby absorption band, about 20% of it actually absorbed by the crystal and about 50% electrical-optical-conversion efficiency, we obtain a threshold electric energy per pulse per unit area on the ruby rod

$$\frac{3\ \mathrm{J/cm^2}}{0.1\times0.2\times0.5} = 300\ \mathrm{J/cm^2}\ !$$

which means a not-so-efficient lasing.

11.3 The first CW (gas) laser: He-Ne LASER

- The first continuous laser and first gas laser invented by Ali Javan (Iranian) in 1961 at Bell Labs.
- 1.0 mmHg of He mixed with 0.1 mmHg of Ne, pumped by electric discharge.
- Populations in metastable states 2^1S and 2^3S of He are transferred to 3S and 2S states of Ne by resonant collisions.
- These upper levels 3S and 2S decay slowly (10^{-7} sec) whereas the lower level 2p decays fast (10^{-8} sec), enabling population inversion.
- The first laser operated at 1.15 μm, followed by 0.633 μm and 3.39 μm.
- The low lying 1S state is metastable. It should be emptied by wall collisions. (That is why we need a *long* laser tube)

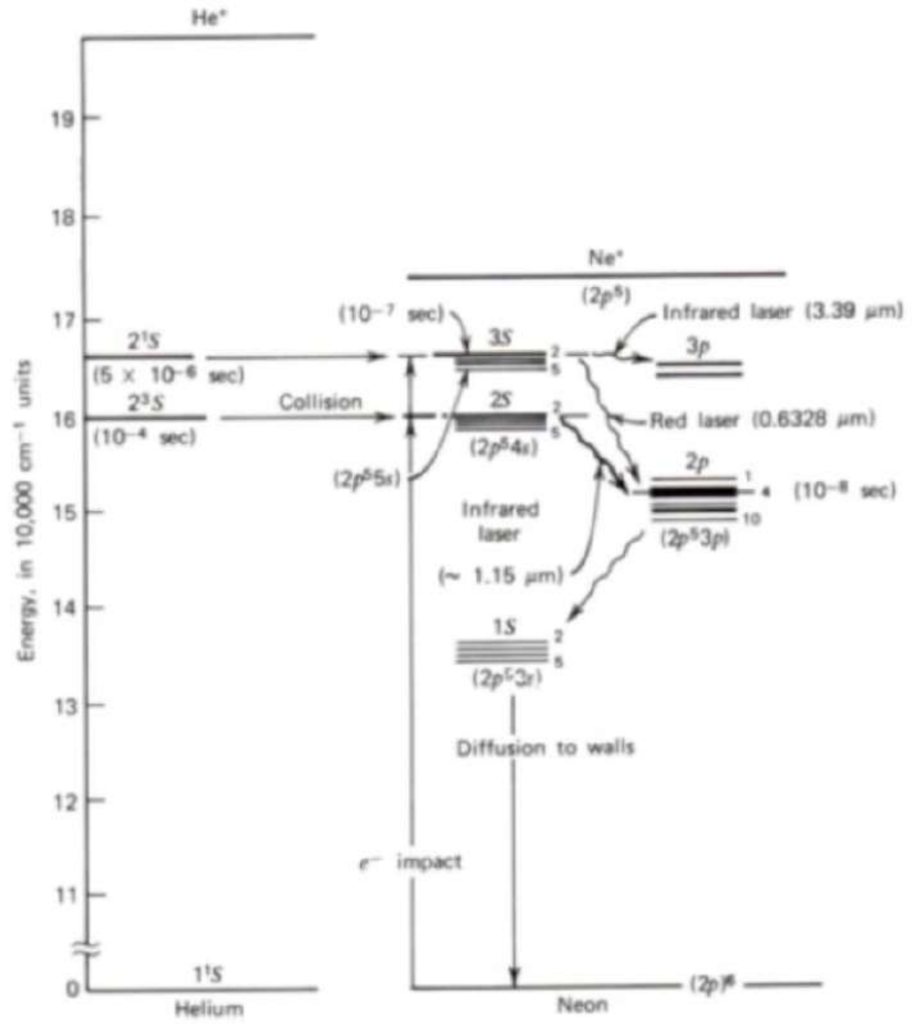


FIGURE 10 He-Ne energy levels. The dominant excitation paths for the red and infrared laser transitions are shov

Fig. 11 Left: Ali Javan, William Bennett, and Donald Herriott adjust the helium neon laser, the first laser to generate a continuous beam of light at 1.15 microns and the first of many electrical discharge pumped gas lasers. Right: Ali Javan and his invention (fondly called "Adam"), which is still kept at his MIT office.

Green, Yellow and Orange Cylindrical Helium Neon Lasers

- *543.5 nm (green), 594.1 nm (yellow) and 612.0 nm (orange)*
- *44.5 mm (1.75") diameter TEM_{00} cylindrical lasers*

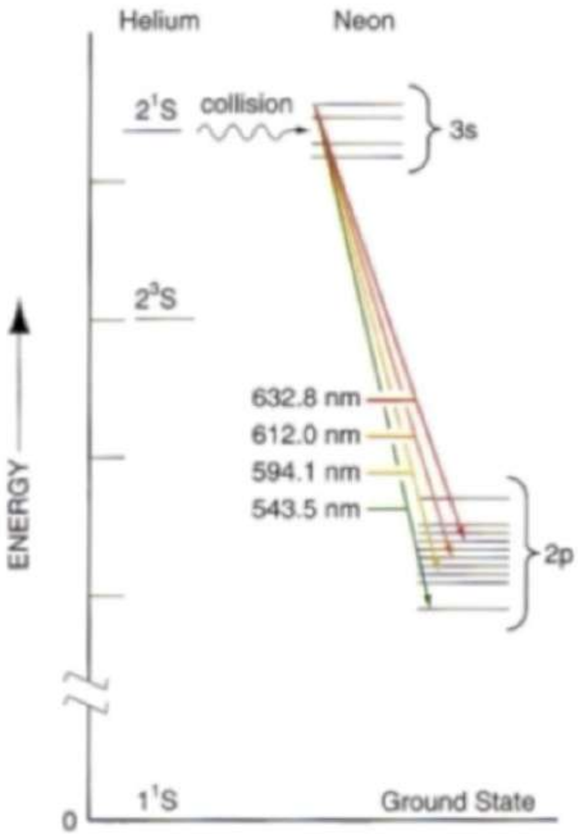


Fig. 12 By using a wavelength-selection element inside the cavity, other colors than red can be obtained.

11.4 CO_2 Laser

- Invented by Kumar N. Patel in 1964 at Bell Labs.
- Uses vibrational transitions.
- Three normal modes: symmetric stretch(1), bending(2), asymmetric stretch(3). CO_2 molecules vibronic state specified by (ν_1, ν_2, ν_3).

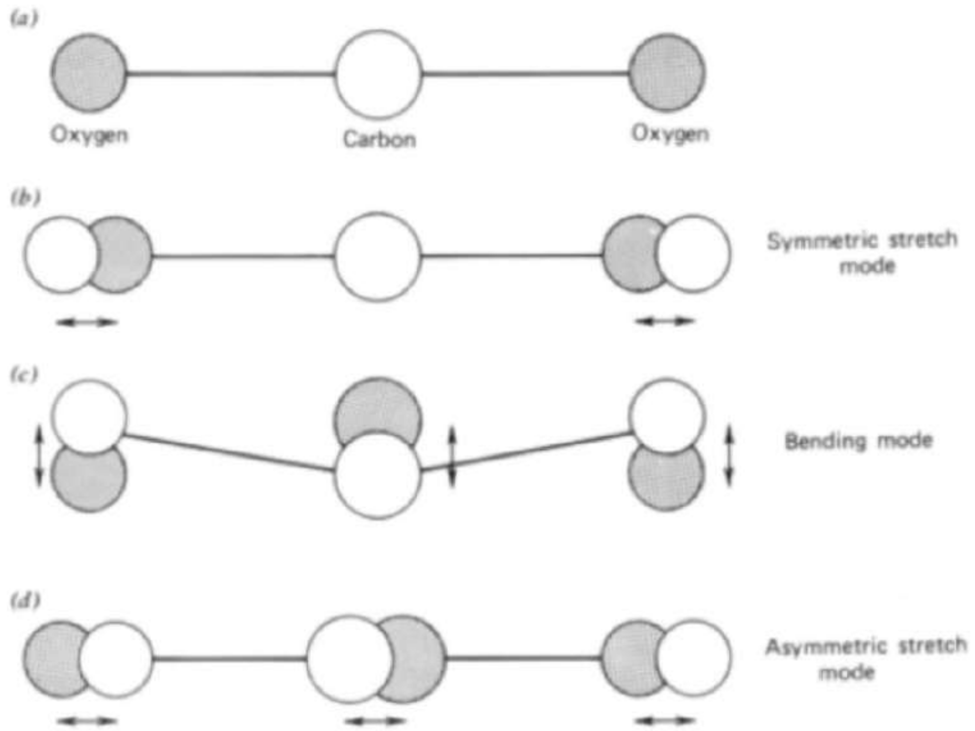


FIGURE 13 (*a*) Unexcited CO_2 molecule. (*b*), (*c*), and (*d*) The three normal modes of vibration of the CO_2 molecule.

- N_2 molecules excited by discharge tend to collect in ν=1 state, which resonantly collide with CO_2 molecules transferring energy to CO_2's (00^01) state.
- Lasing occurs between the upper metastable (00^01) state and the lower (10^00) at 10.6 μm. The lower (10^00) state decays to (01^00) and then to (00^00) rapidly, helped by collisions with added He molecules and others. (superscript 0 refers angular momentum)

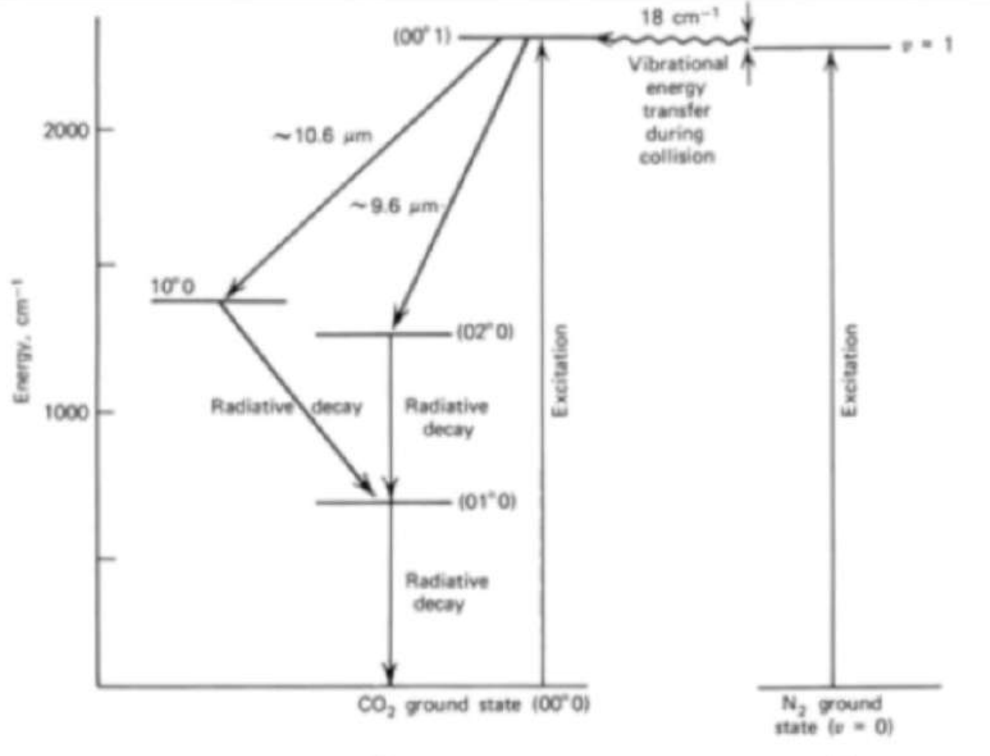


FIGURE 14 (a) Some of the low-lying vibrational levels of the carbon dioxide (CO_2) molecule, including the upper and lower levels for the 10.6 and 9.6μm laser transitions. (b) Ground state ($v = 0$) and first excited state ($v = 1$) of the nitrogen molecule, which plays an important role in the selective excitation of the (00^01)CO_2 level.

- High quantum efficiency (45%), easy population inversion, and rapid recycling result in a high overall efficiency of about 30%. A few kilowatt power is common for a one-meter tube.

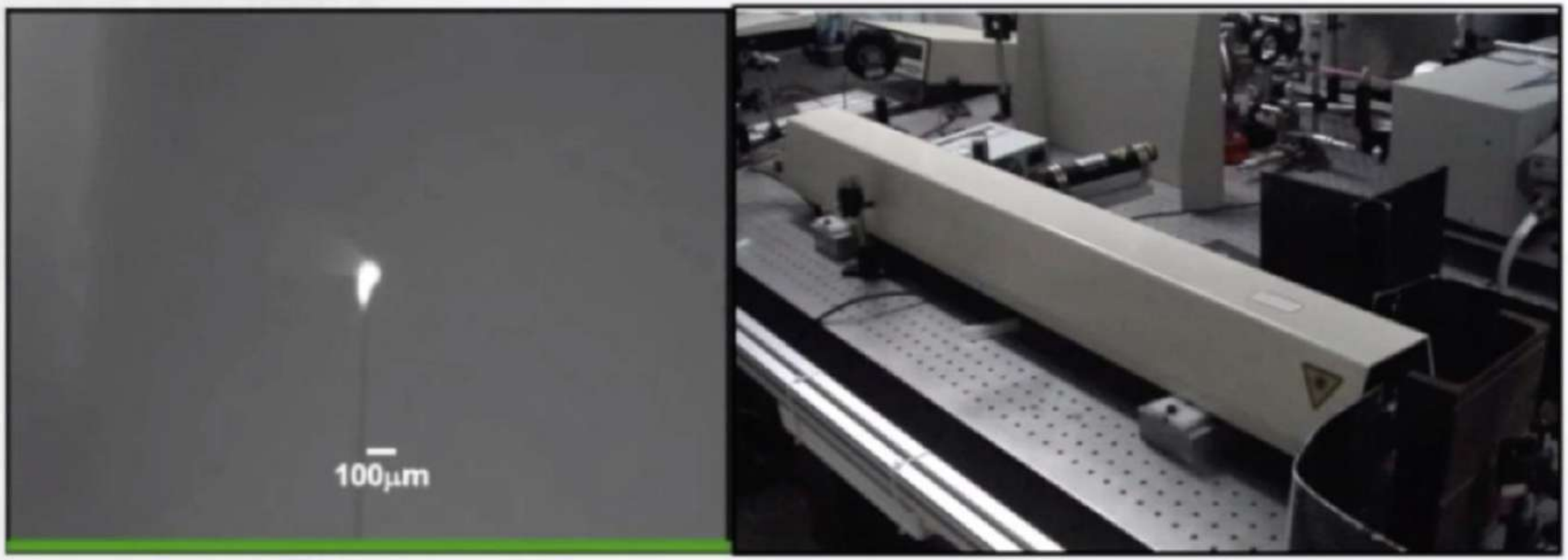
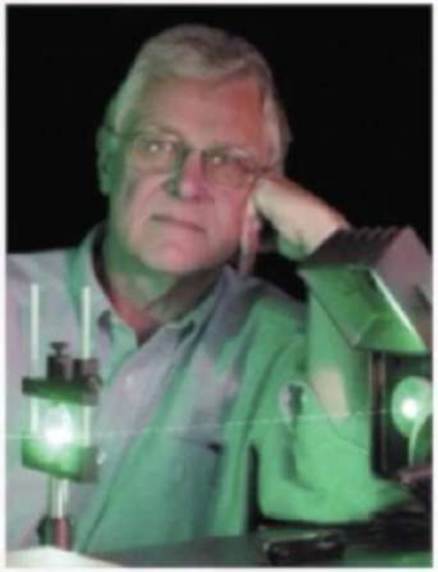


Fig. 15a A 10-W beam of CO_2 laser melting a tip of a fused silica fiber forming a microsphere.

Fig. 15b Laser machining based on CO_2 lasers

11.5 Noble Gas Lasers: Ar-Ion Laser

- Invented by William Bridges in 1964 at Hughes Aircraft.
- Capable of producing 10 wavelengths in UV and up to 25 in the visible region, ranging from 275 nm to 363.8 nm and 408.9 nm to 686.1 nm, respectively
- The gain bandwidth on each transition is on the order of 2.5 gigahertz at gas pressures of approximately 0.1 torr.



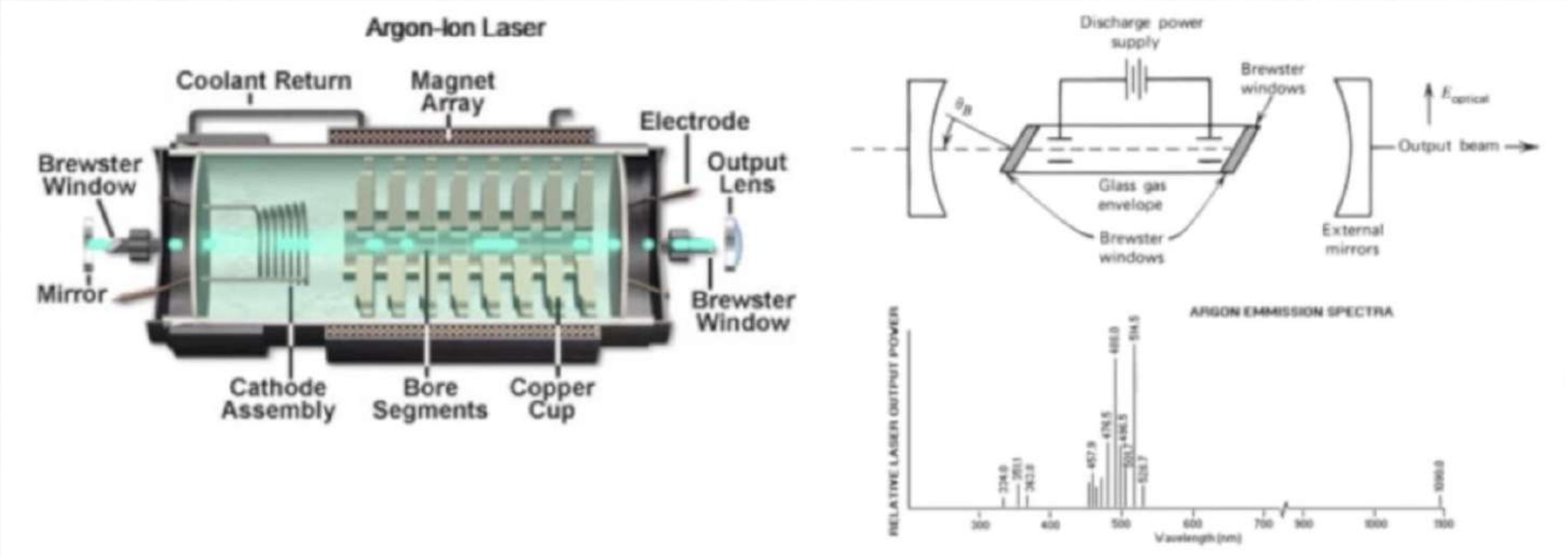


Fig. 16 Layout of a typical argon-ion laser and its emission lines.

- The neutral argon atom is pumped to the 4p state by two collisions with electrons. The first ionizes the atom and the second excites it from the ground ion state to the 4p metastable state.
- The ion decays rapidly from the lower lasing level 4s to the ground 3p state emitting an ultraviolet photon at about 72 nm.
- There are many competing transition lines. Multi-line or single-line operation can be selected.
- A magnetic field produced by a solenoid envelopes the plasma and enhances population inversion. It tends to force the free electrons toward the center of the tube, increasing the probability of a pumping collision.
- Proper pressure inside the cavity must be maintained to optimize the gain. The pressure-balanced design of the plasma tube brings stability.

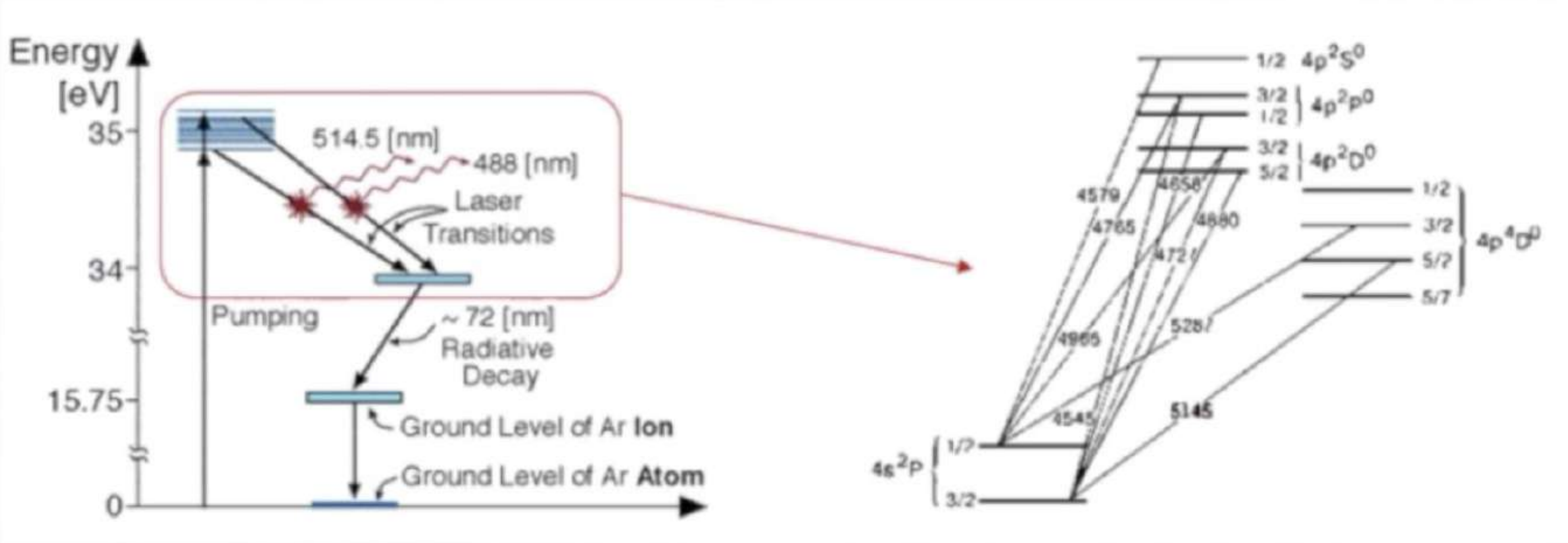


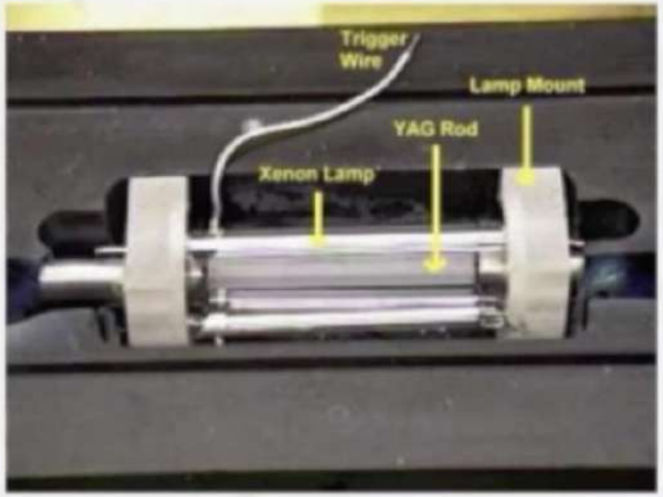
Fig. 17 Energy levels of Ar^{+}.

- The argon gas is kept in a sealed plasma tube with a pressure of about 1 torr.
- The ionization of the neutral argon gas atoms inside the plasma tube occurs by a voltage pulse of about 8 kilovolts. A high DC current (45 amps) and a high voltage (600 volts DC) are used throughout the tube to keep the gas ionized.
- Stimulated emission can occur for both the Ar^+ and Ar_2^+ ions. For Ar^+, the lasing emission occurs in the VIS region at 488.0 nm and 514.5 nm. The Ar_2^+ ion produces laser emission in the UV region at wavelengths of 334.0, 351.1, and 363.8 nm.

Fig. 18 Argon-ion laser in action.

11.6 Nd: YAG, Nd: Glass Lasers

- Invented by J. E. Geusic, H. M. Markos and L. G. Van Uiteit in 1964 at Bell Labs.
- Nd:YAG stands for Neodymium: Yttrium Aluminum Garnet. The active medium for the laser is triply ionized neodymium (1% doped) which is incorporated into a crystalline or glass structure. The most common host for neodymium is a synthetic crystal with a garnet-like structure, yttrium aluminum garnet. $Y_3Al_5O_{12}$, a hard brittle material, is commonly referred to as YAG.
- Originally flash-lamp pumpd. Recently diode-laser pumped.



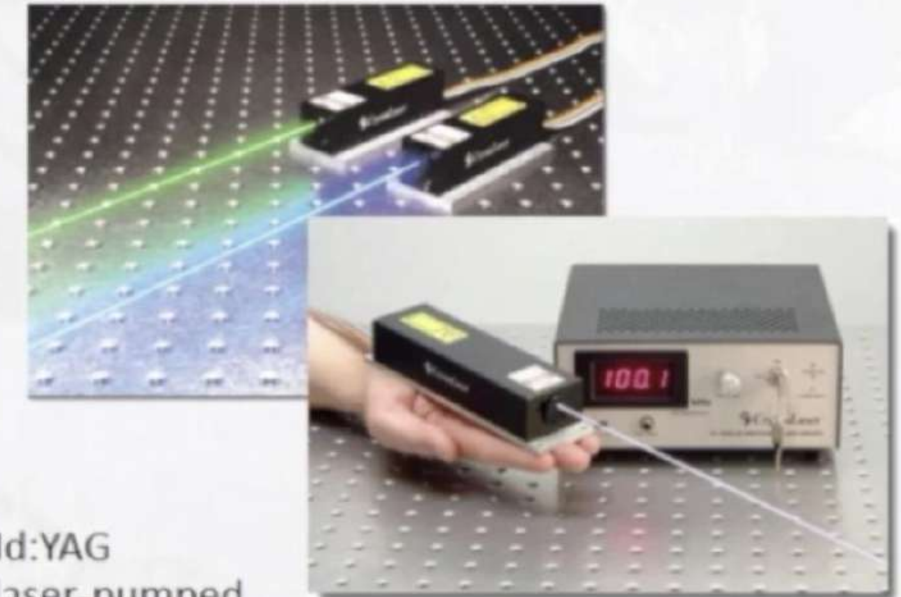


Fig. 19 A traditional flash-lamp pumped Nd:YAG compared with recently-developed diode-laser-pumped Q-switched Nd:YAG lasers - a.k.a. DPSS (diode-pumped solid-state) lasers.

- Nd^{3+} in YAG is a *four-level* system. Nd^{3+} ions are pumped at 720 nm and 830 nm from the ground state ($^4I_{9/2}$) by flashlamps, tungsten arc lamps, or GaAlAs diode laser to E4 states ($^2H_{9/2}$ and $^4F_{5/2}$, respectively), which rapidly decay to a metastable state E3 ($^4F_{3/2}$) with a lifetime of 0.55 ms.
- The ions return to the ground state E1 ($^4I_{9/2}$) rapidly from the lower laser level E2 ($^4I_{11/2}$) through vibrational relaxation. Linewidth of the laser transition is 6 cm^{-1} (180 GHz).
- Emission cross section $\sigma = 9\times10^{-19}$ cm^2, which is 75 times larger than that of ruby, and thus Nd:YAG lasers can be easily operated in CW mode.

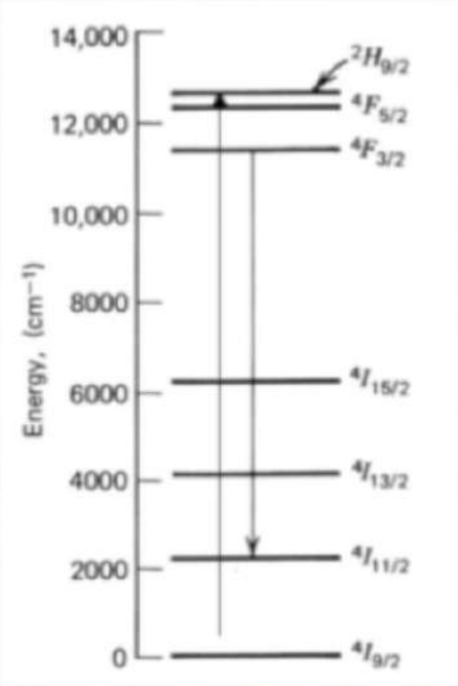


Fig. 20 Nd:YAG forms a four-level system.

- In Nd:Glass laser, Nd^{3+} ions are present as impurities in *glass*. It is a four-level system.
- Flourescence linewidth is much broader, being about 300 cm^{-1}, 50 times larger than that of Nd:YAG. This is mostly inhomogeneous broadening due to the amorphous structure of glass. The broader linewidth results in the higher lasing threshold. But it makes mode locking (picosecond) easy.
- Optical pumping efficiency was measured to be 40%.
- Excited state lifetime varies from 0.2 ms to 0.9 ms depending on host glass and Nd^{3+} concentration.

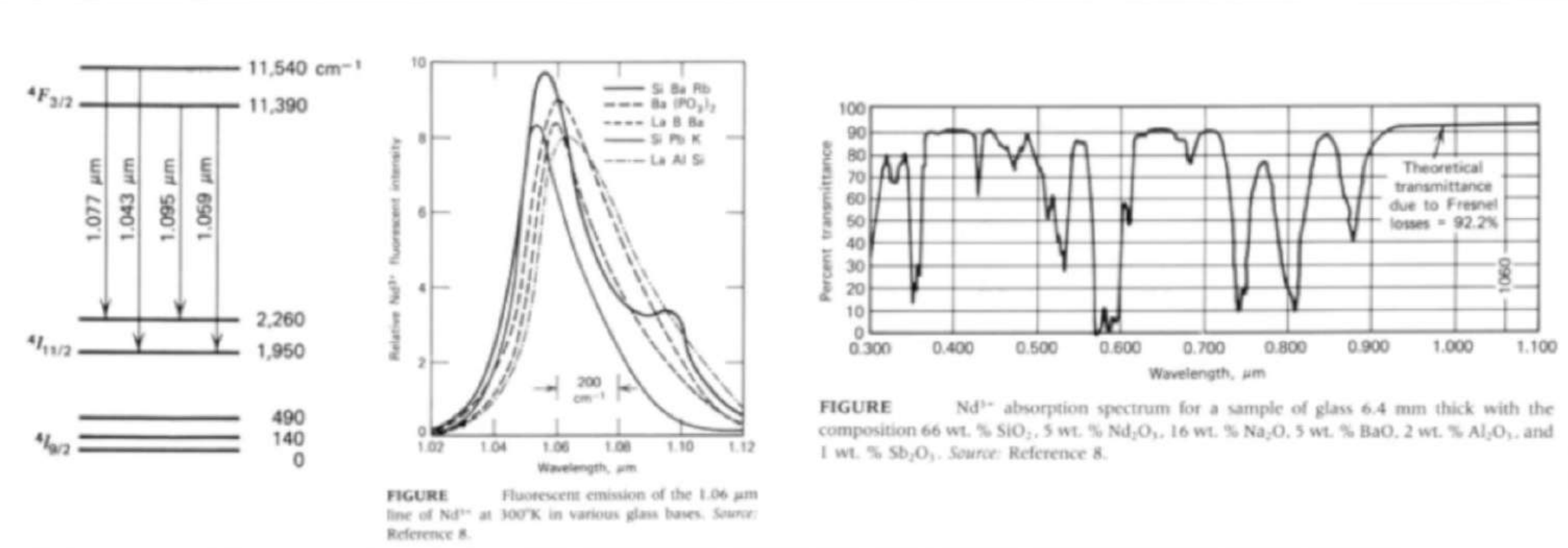


FIGURE Fluorescent emission of the 1.06 μm line of Nd^{3+} at 300°K in various glass bases. *Source:* Reference 8.

FIGURE Nd^{3+} absorption spectrum for a sample of glass 6.4 mm thick with the composition 66 wt. % SiO_2, 5 wt. % Nd_2O_3, 16 wt. % Na_2O, 5 wt. % BaO, 2 wt. % Al_2O_3, and 1 wt. % Sb_2O_3. *Source:* Reference 8.

Fig. 21 (a) Energy levels, (b) emission spectrum and (c) absorption spectrum of Nd:Glass.

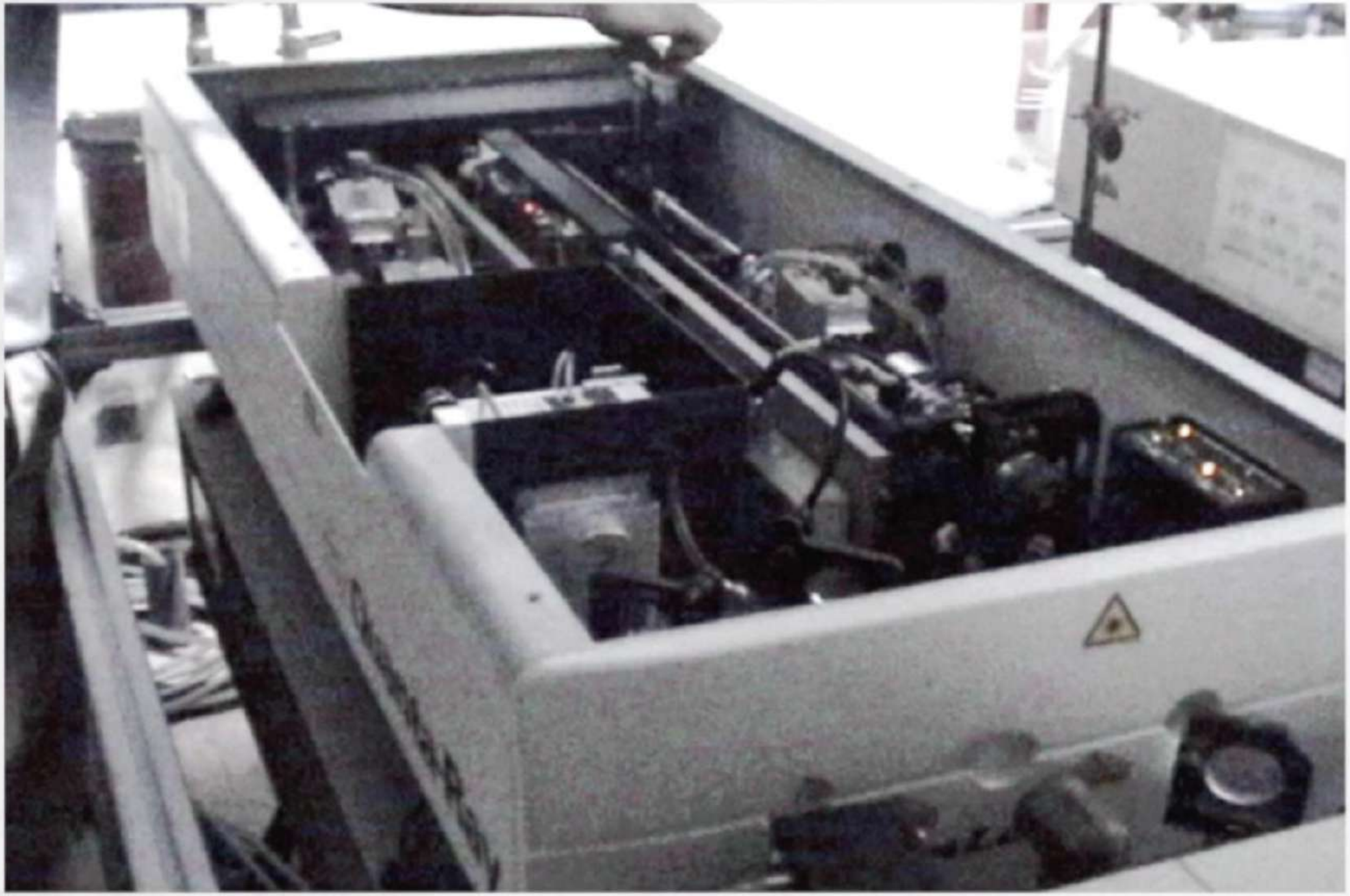


Fig. 22 Q-switched Nd:YAG laser with a second-harmonic crystal assembly.

11.7 Ti-Sapphire Laser

- Invented by Peter Moulton in 1982 at MIT Lincoln Labs.
- Broadly tunable solid state lasers.
- Titanium metal ions (Ti^{3+} ion) are doped into a transparent sapphire (Al_2O_3) host at 0.1 % by weight.
- Provides a very wide tuning range (660 ~ 1180 nm), the broadest tuning range for any single solid-state, gas, or liquid laser medium. Femtosecond lasing is possible when mode locked.

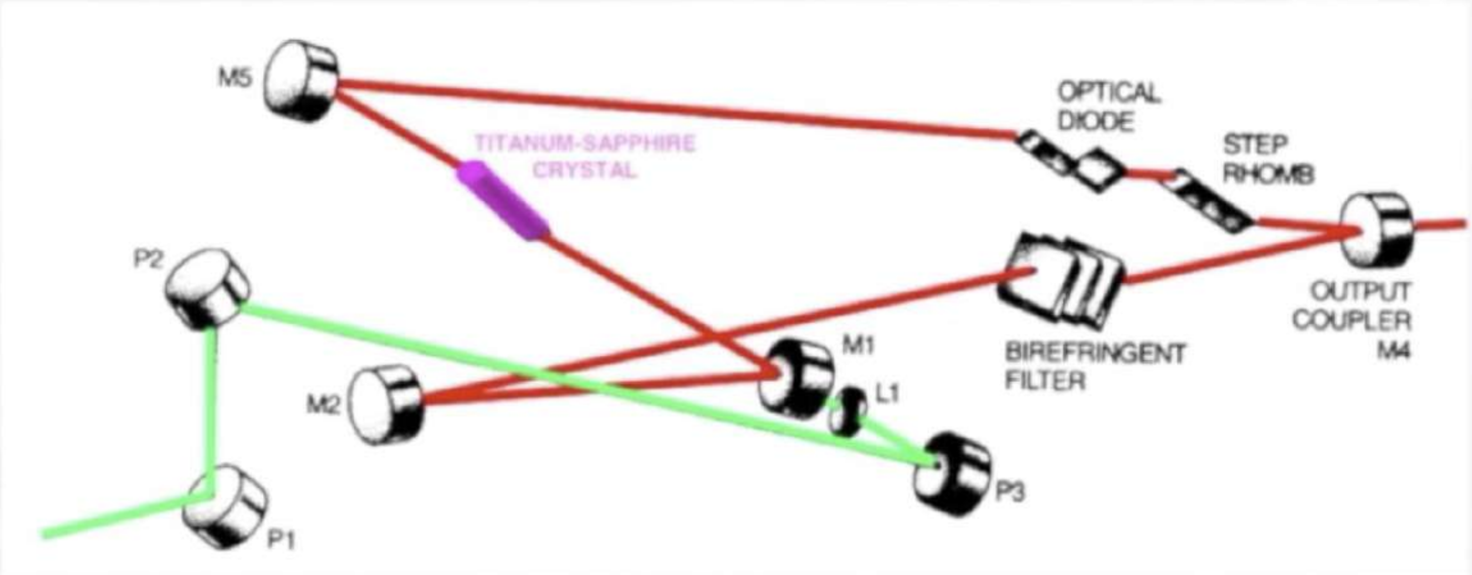
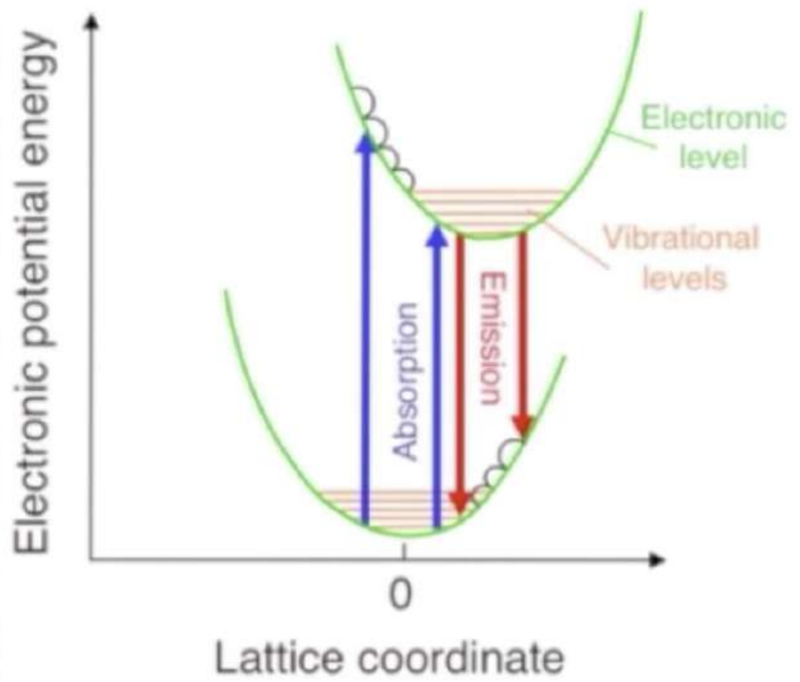


Fig. 23 Layout of a CW Ti:sapphire laser (Coherent 899-21)

- The Ti:sapphire laser is called a vibronic laser because of broad vibronic energy band on top of the electronic levels.
- Ti:sapphire laser is a four-level system. Ti^{3+} ions are optically pumped to the ^2E excited state, commonly by other lasers like argon ion lasers.
- Fast relaxation to the lowest vibrational levels in the upper electronic level ^2E, followed by emission to the highest vibrational levels in the lower electronic level $^2T_2$.
- Absorption and emission bands are widely separated.



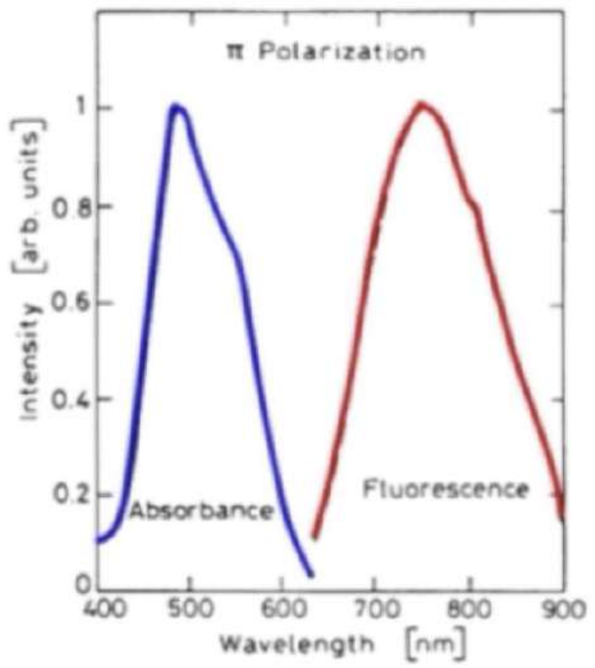
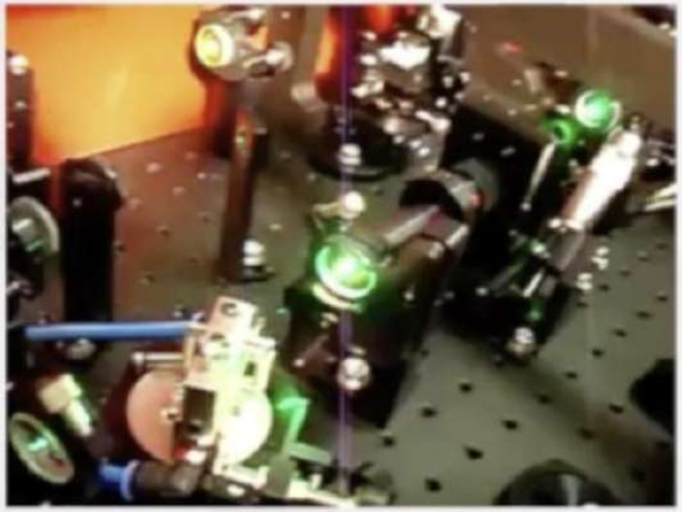
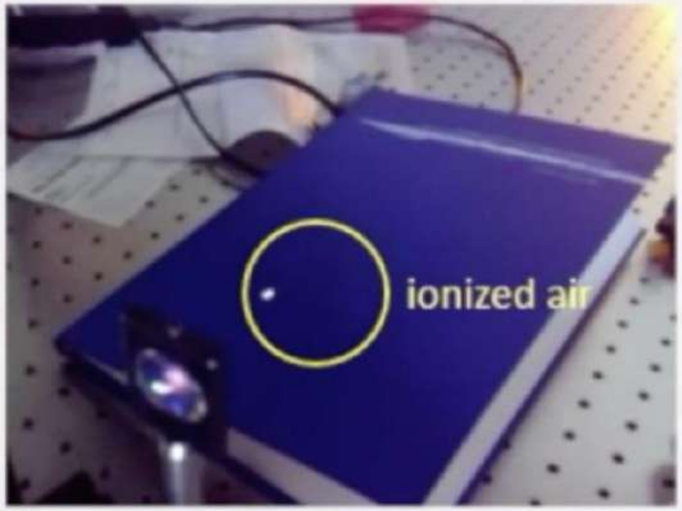


Fig. 24 Energy levels of Ti^{3+} in sapphire host and its absorption and emission spectra.

- Ti:sapphire lasers are available as both CW and pulsed. The typical CW power output at 800 nm is up to about 1 W. A typical commercial Ti:sapphire laser (mode locked) has a pulse length in picoseconds or femtoseconds.





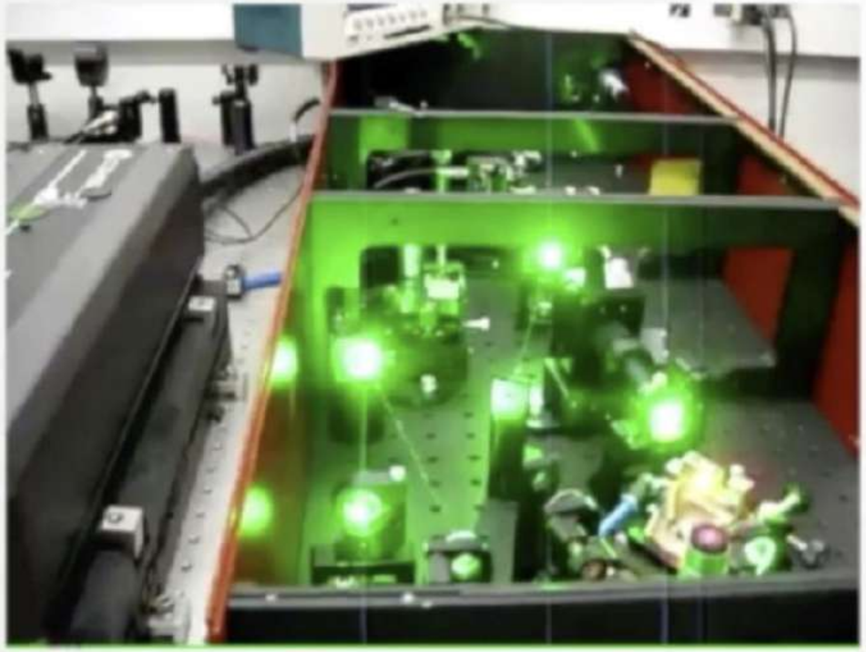


Fig. 25 Mode-locked femtosecond Ti:sapphire lasers. A 100 GW 35 fs 1 kHz laser is focused down to creates an electric field large enough to ionize the air.

- Ti:sapphire lasers are available as both CW and pulsed. The typical CW power output at 800 nm is up to about 1 W. A typical commercial Ti:sapphire laser (mode locked) has a pulse length in picoseconds or femtoseconds.

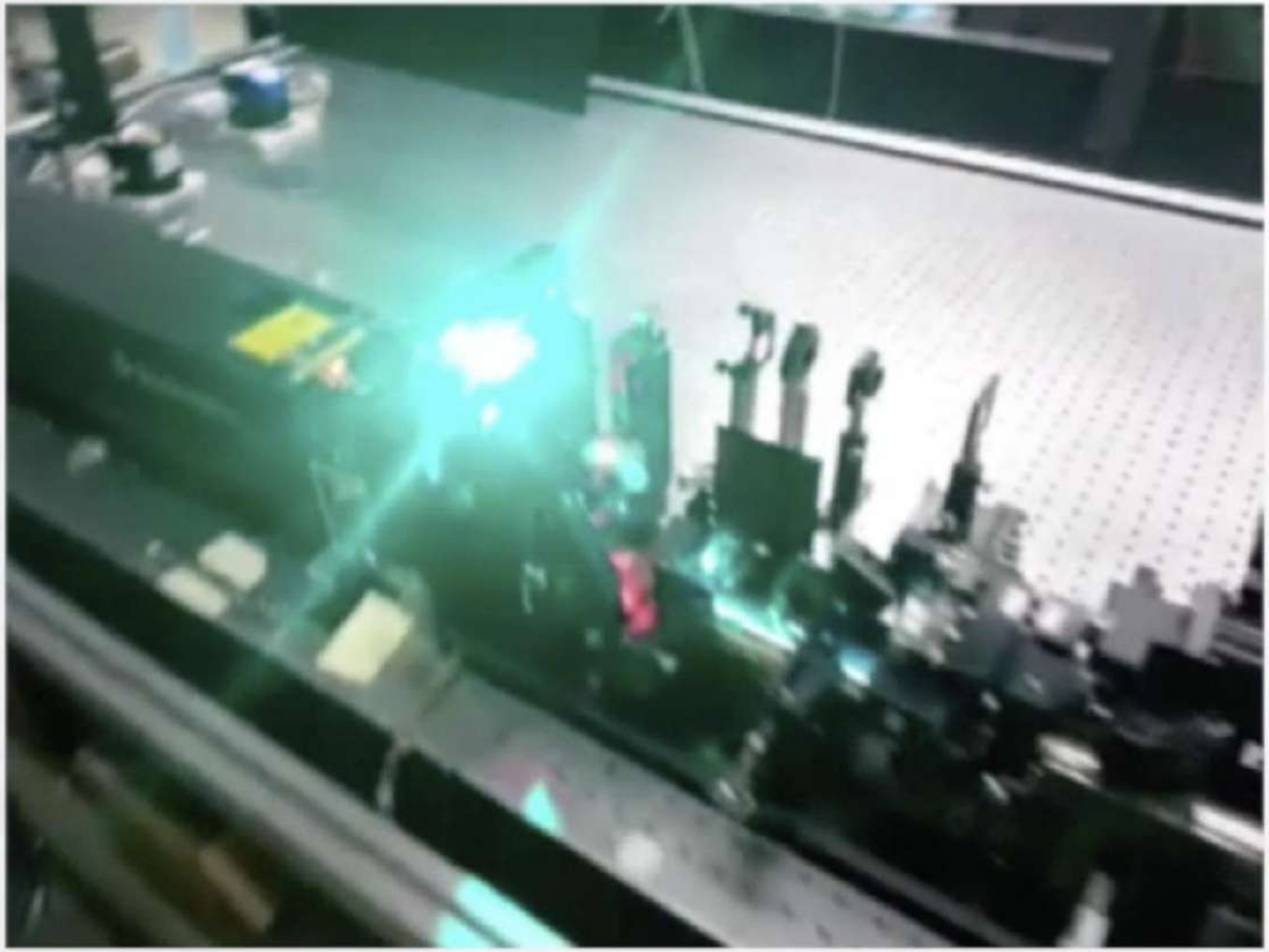
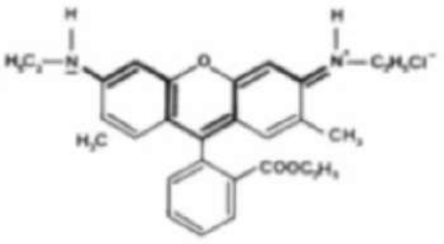


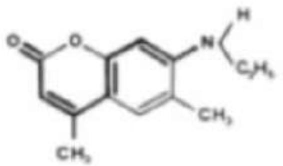
Fig. 26 A CW-ring Ti:sapphire laser (Coherent 899-21).

11.8 Dye Lasers

- Invented by P. Sorokin and J. Lankard in 1966 at IBM Labs.
- Liquid organic dye is used as a gain medium.
- The most useful feature of dye lasers is their tunability; that is, the lasing wavelength for a given dye may be varied over a wide range.
- Taking advantage of the broad fluorescent linewidths (50-100 nm) available in organic dyes, one uses a wavelength-dispersive optical element such as a diffraction grating, a prism or an etalon in the laser cavity to perform selective tuning.



Rhodamine 6G



Coumarin

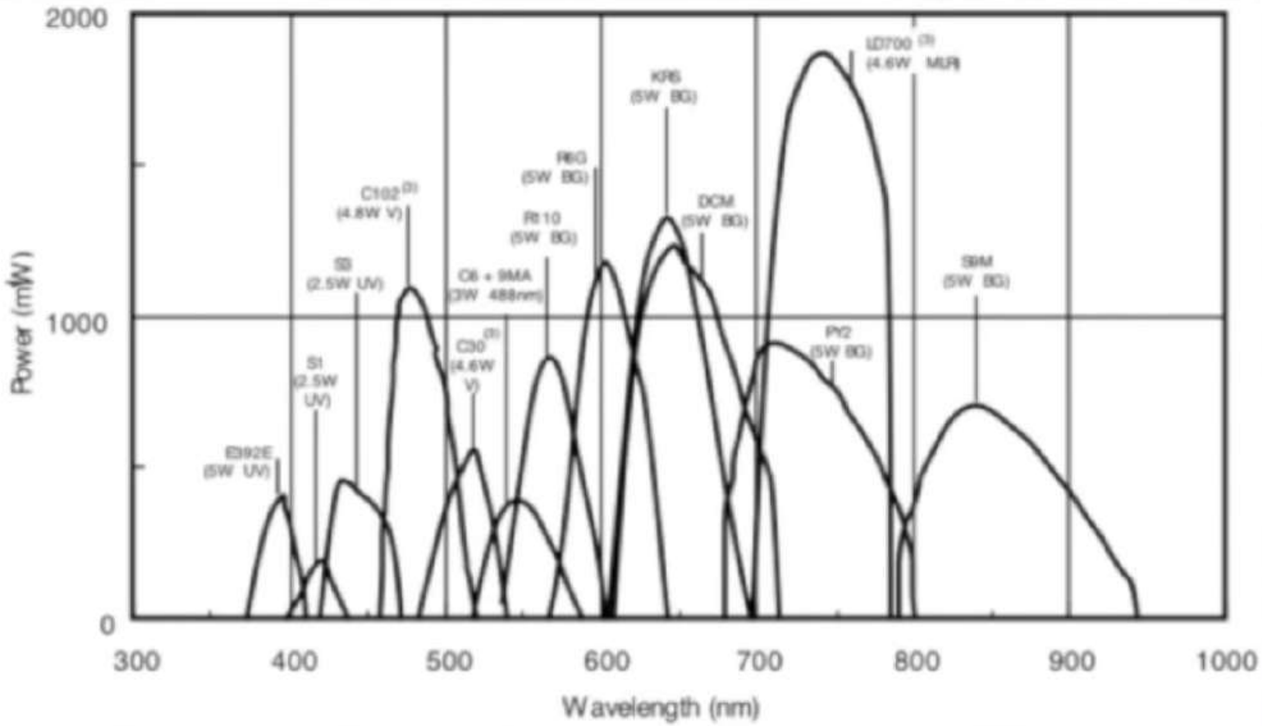
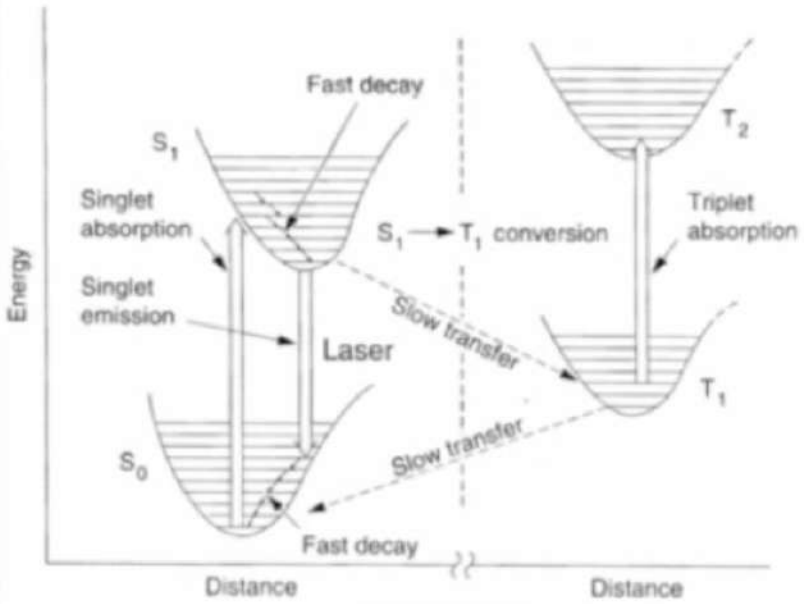


Fig. 27 Dye laser tuning curve covering from 380 nm to 930 nm.

- The dye laser is a four-level system. It is optically pumped by a flash lamp (pulsed) or a CW pump laser.
- In a singlet state, the spin of the excited electron is antiparallel to the spin of the remaining electrons. In a triplet state, the spins are parallel.
- There exists a small probability that an excited molecule wil decay to the triplet state T1, inducing quenching of laser oscillation.
- Therefore, dye lasers usually operate in a pulsed mode. In CW operation, the dye has to be circulated in order to prevent photobleaching (quenching).



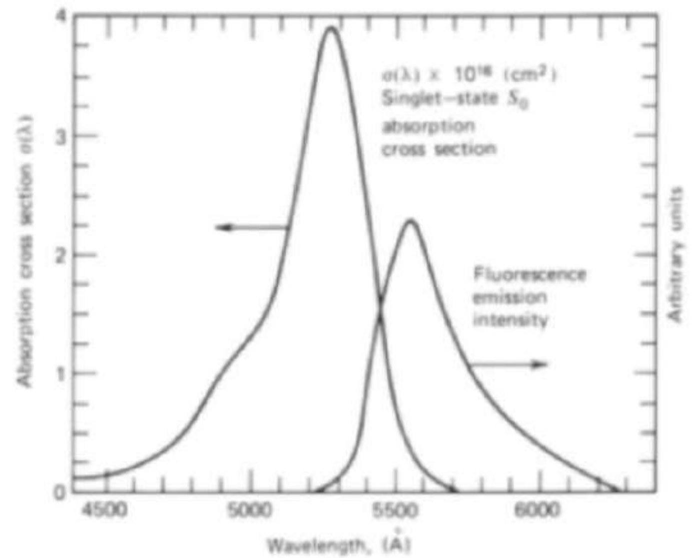


FIGURE Singlet-state absorption and fluorescence spectra of rhodamine 6G obtained from measurements with a 10^{-4} molar ethanol solution of the dye. *Source:* Reference 22.

Fig. 28 Energy levels of laser dye and its absorption and emission spectra.

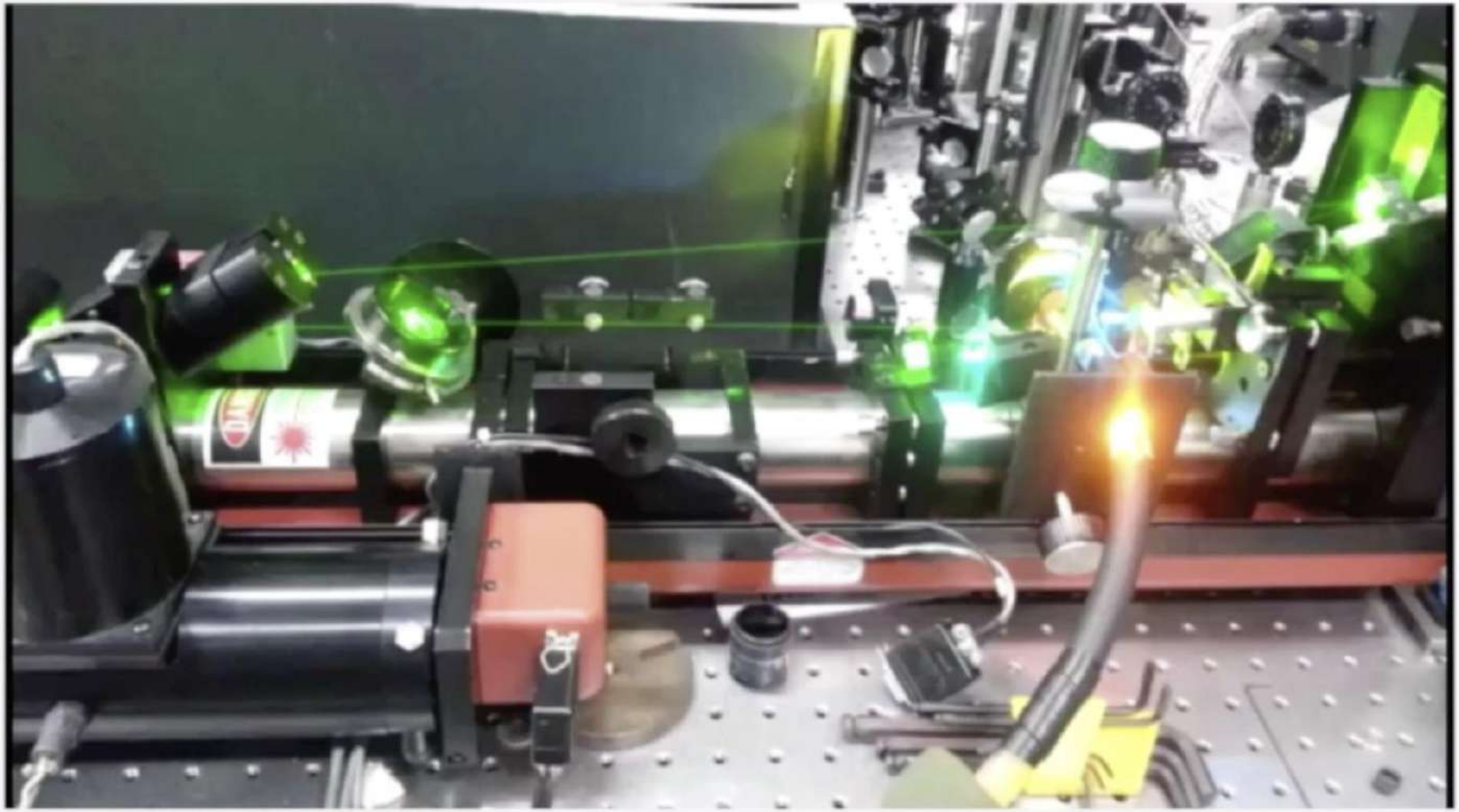


Fig. 29 CW ring dye laser (Coherent 699-21).

11.9 Chemical Lasers: Excimer Laser

- Invented by Nikolai Basov in 1970.
- The word *excimer* is a combination of *excited dimers*, a mixture of a rare-gas such as argon, krypton, and xenon; and a halide like fluorine, chlorine, and bromine.
- These dimer molecules such as argon fluoride, krypton fluoride, and xenon chloride can form only when the atoms are excited, and the molecule only exists as long as it is excited.
- When the molecule drops to the ground state, the molecule breaks.
- The fact that the molecule only exists while excited benefits the laser by keeping the ground state unpopulated, which means that every excited molecule contributes to the population inversion; also there is no absorption of laser light by molecules in the ground state as could occur in other types of lasers.
- Therefore, excimer lasers can produce a large power.

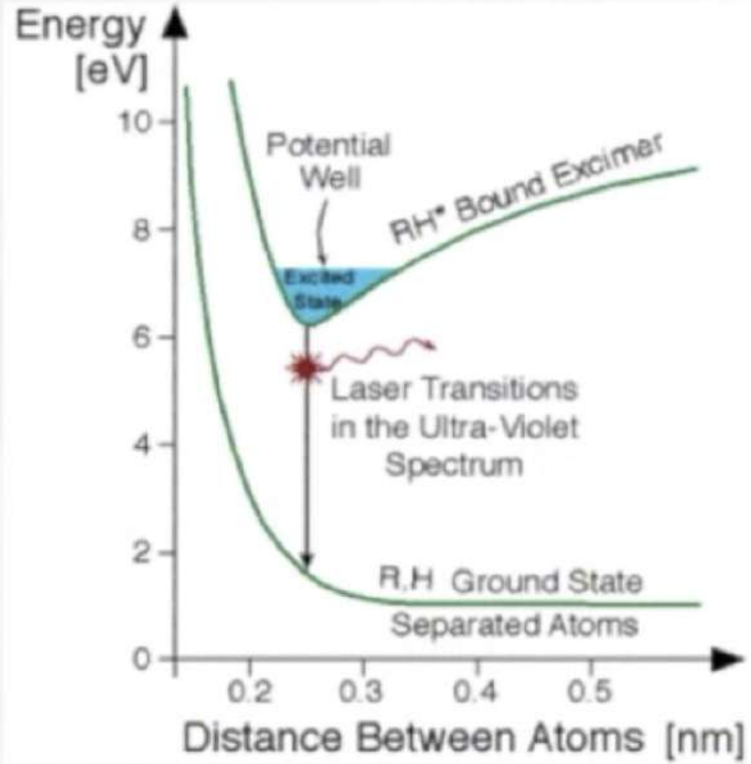


Fig. 30 Energy levels of excimer laser.

- In the tube of the excimer laser, a rare gas and a halide are mixed with an inert gas at high pressure. Either a transverse discharge or an electron beam can excite this gas mixture.
- Most commercial excimer lasers produce pulses ranging from 3 to 35 nanoseconds in duration. The output wavelength of most excimer lasers falls between 150 and 350. nanometers, for example: ArF at 193 nm, KrF at 249 nm, XeCl at 308 nm and XeF at 350 nm
- The laser cavity is made to be sealed and repeatedly refilled, which is necessary because the laser gas degrades during use. For this reason the laser cavity, optics, and electrodes must be designed to resist corrosion by the halogens present in the laser gas. Passive components typically are coated with Teflon, while the electrodes are made of halogen-resistant materials such as nickel.

Fig. 31 Left: Excimer lasers for machining. Right:Inner works of a commercial excimer laser.

11.10 Semiconductor Lasers: LED's and Laser Diodes

- First GaAs semiconductor diode laser was invented by Robert Hall in 1962 at General Electric Labs.
- When a device is called a 'laser diode', this generally refers to the combination of the semiconductor chip that does the actual lasing along with a monitor photodiode chip (for used for feedback control of power output) housed in a package (usually with 3 leads) that looks like a metal can transistor with a window on the top.
- Direct electric current pumping, fast modulation (>20 GHz), integratable in circuits.
- Important applications in optical fiber communication and optical data storage.

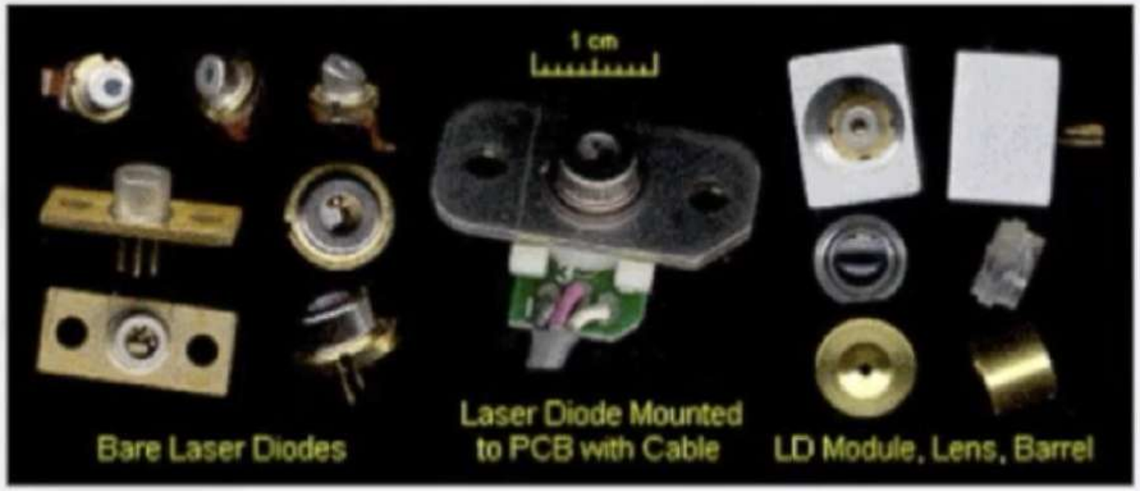


Fig. 32 A variety of small laser diodes.

- The energy level differences between the conduction and valence band electrons in these semiconductors are what provide the mechanism for laser action.
- For example, in GaAs/$Ga_{1-x}Al_xAs$ lasers ($0 \le x \le 1$), a thin (0.1-0.2 μm) region (active region) of GaAs is sandwiched between two regions of $Ga_{1-x}Al_xAs$, forming potential wells for electrons and holes. A forward bias achieves inversion for lasing.
- The sandwich structure also acts as a waveguide for modal confinement.

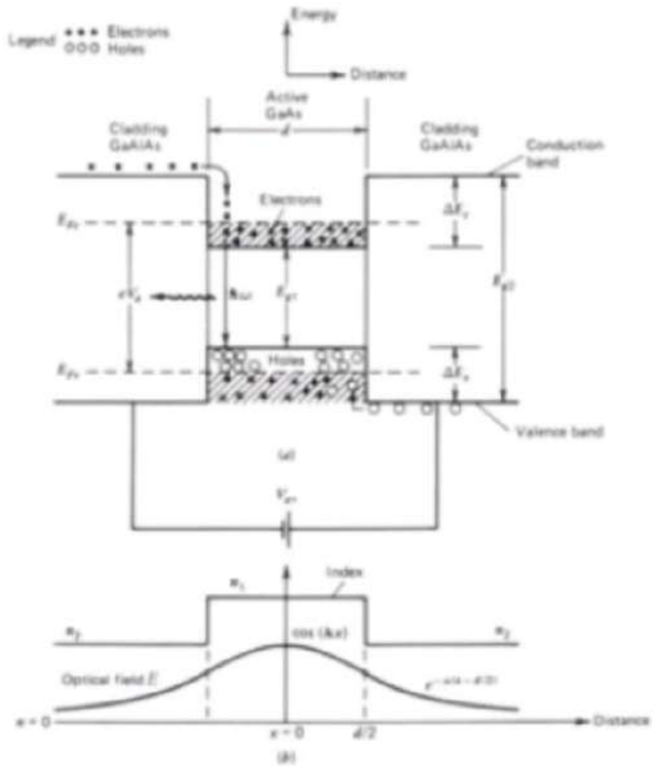


FIGURE (a) The conduction and valence band edges under large positive bias in a double heterojunction GaAlAs/GaAs/GaAlAs laser diode. (b) The index of refraction profile and the optical field (fundamental mode) profile.

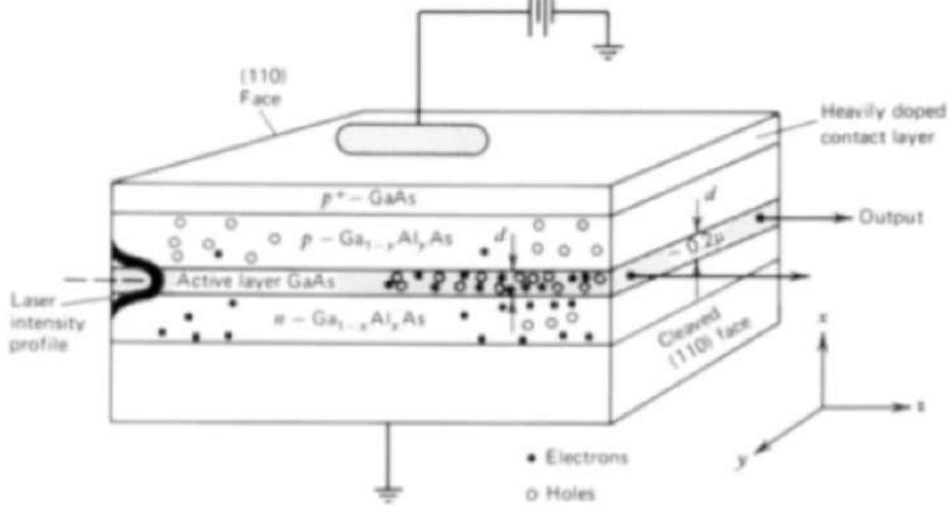


FIGURE A typical double heterostructure GaAs-GaAlAs regions, respectively. Frequencies near $\nu = E_g/\hbar$ are amplified by stimulating electron-hole recombination.

Fig. 33 Typical structure of a double heterojunction laser diode and its band edge level structure.

- The cavity is formed by two cleaved faces with coating. One end is about 95% reflective, becoming the back facet; the other is 70% transparent (determined by index of refraction of GaAs, n=3.5), allowing the light to escape.
- Epitaxial growth of $Ga_{1-x}Al_xAs$ on top of GaAs (and vice versa) is possible due to their almost same lattice constants.

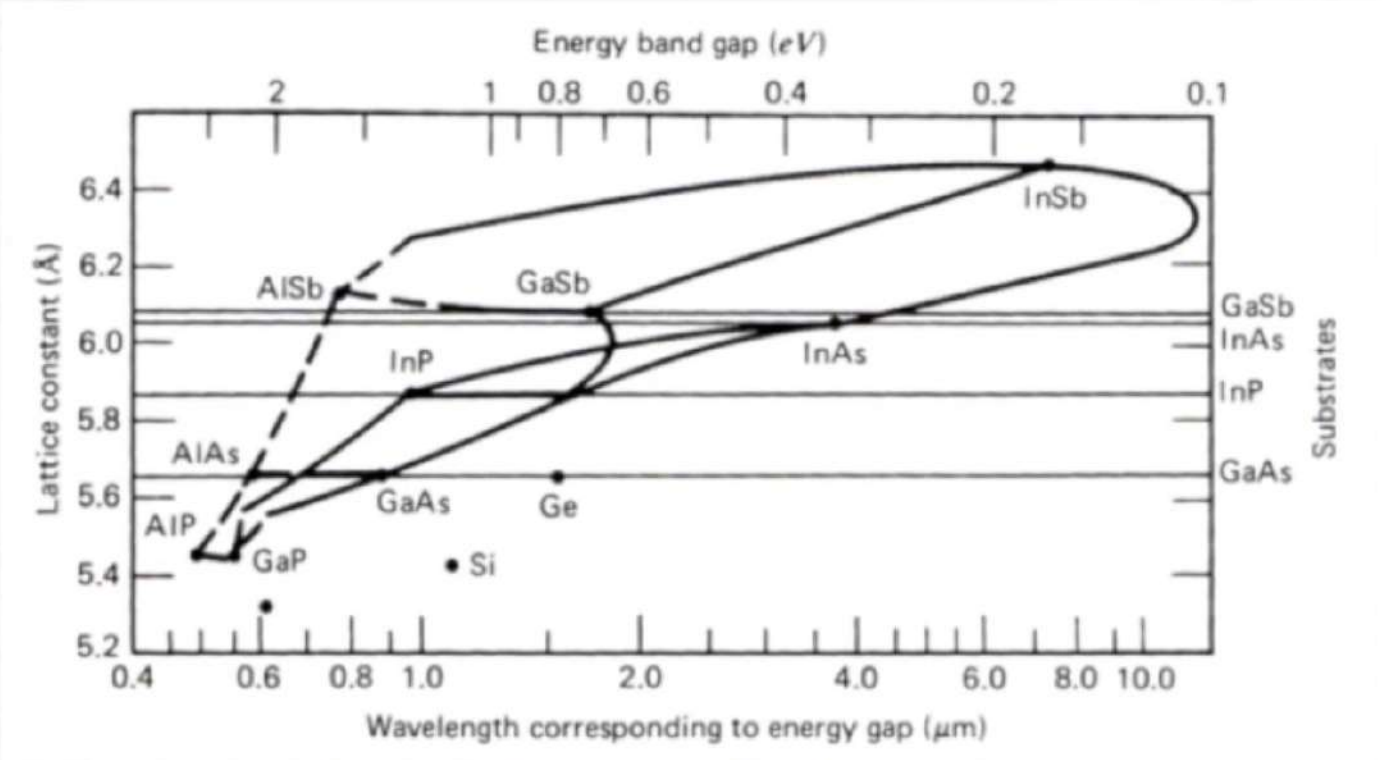


Fig. 34 Lattice constant of III-V compounds.

- In *quantum well* lasers, the thickness of active GaAs region is reduced to 100 Å or so, and as a result, a quantum potential well is achieved for electrons and holes.
- Since the gain is inversely proportional to the well thickness, the threshold carrier density (or transparency density) is reduced accordingly.

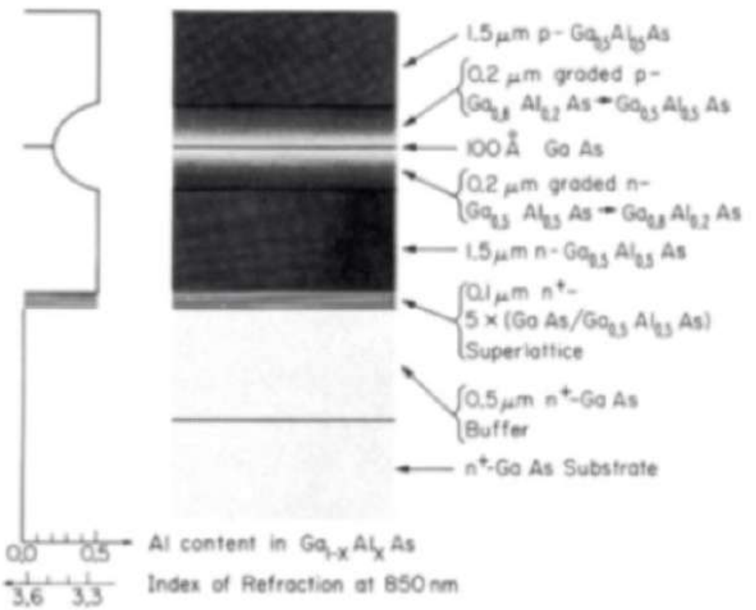


FIGURE 12.5 The actual layered structure grown by molecular beam epitaxy used to fabricate QW lasers. *Source:* Courtesy M. Mittelstein, L. Anders and P. Derry, California Institute of Technology., Pasadena, Calif.

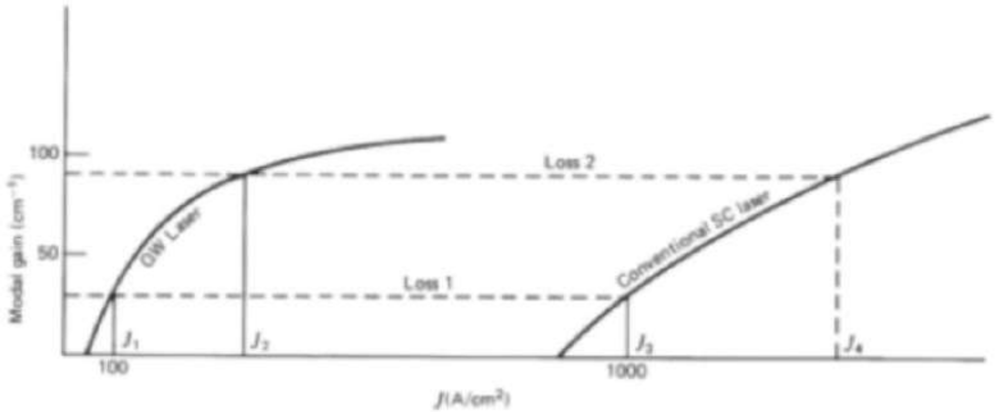


FIGURE 12.6 Qualitative plots of modal gain vs. pump current density for a DH laser and a QW well laser. The large ratio of threshold currents is due to large ratios of their active region thicknesses. The horizontal lines represent two loss values that must be compensated by the gain. The requisite current densities are J_1 and J_2 for the QW laser and J_3 and J_4 for the DH laser.

Fig. 35 Structure of an actual quantum-well laser. Threshold characteristics of the QW laser.

11.11 Free-Electron Laser

- A device that converts the kinetic energy of free electrons to electromagnetic radiation.
- Invented by J. Madey in 1977 in its present form.
- A Free electron laser generates *tunable, coherent, high power* radiation, currently spanning wavelengths from millimeter to visible and potentially ultraviolet to x-ray.
- Synchrotron radiation also relies on kinetic energy of electrons, but its spectrum is so broad that its power per unit frequency is extremely low.

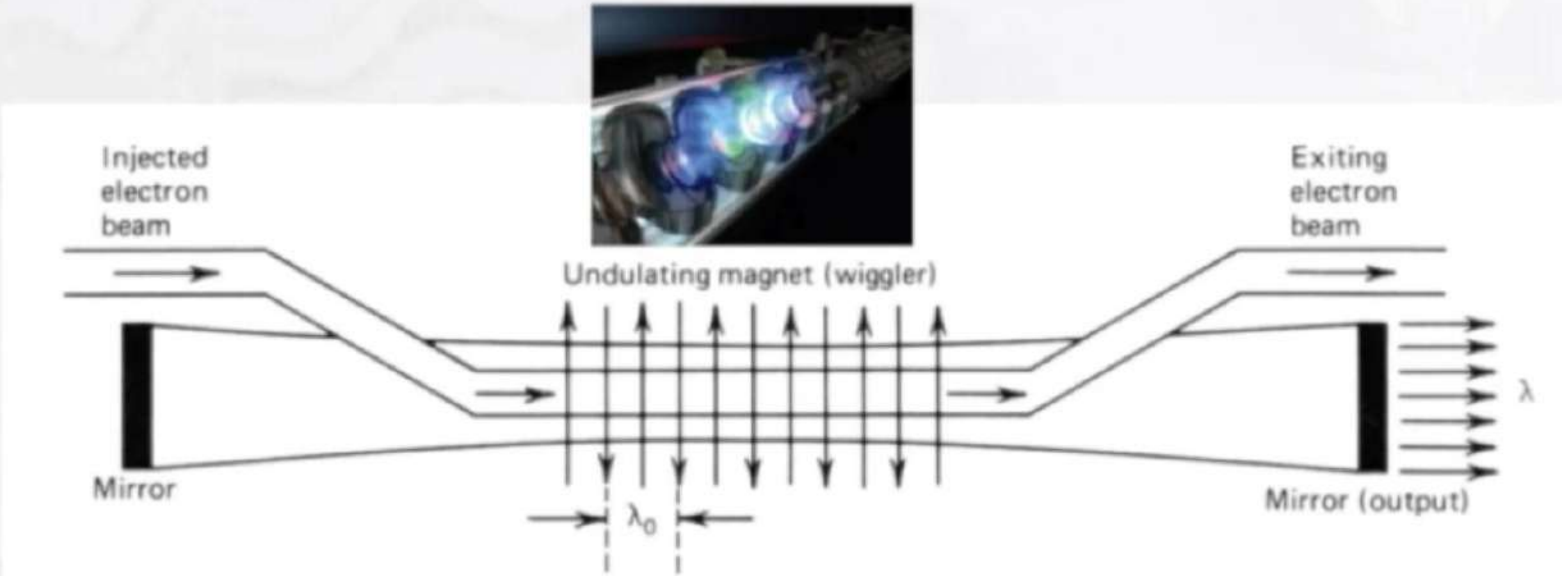


Fig. 36 An experimental arrangement of a typical free-electron laser.

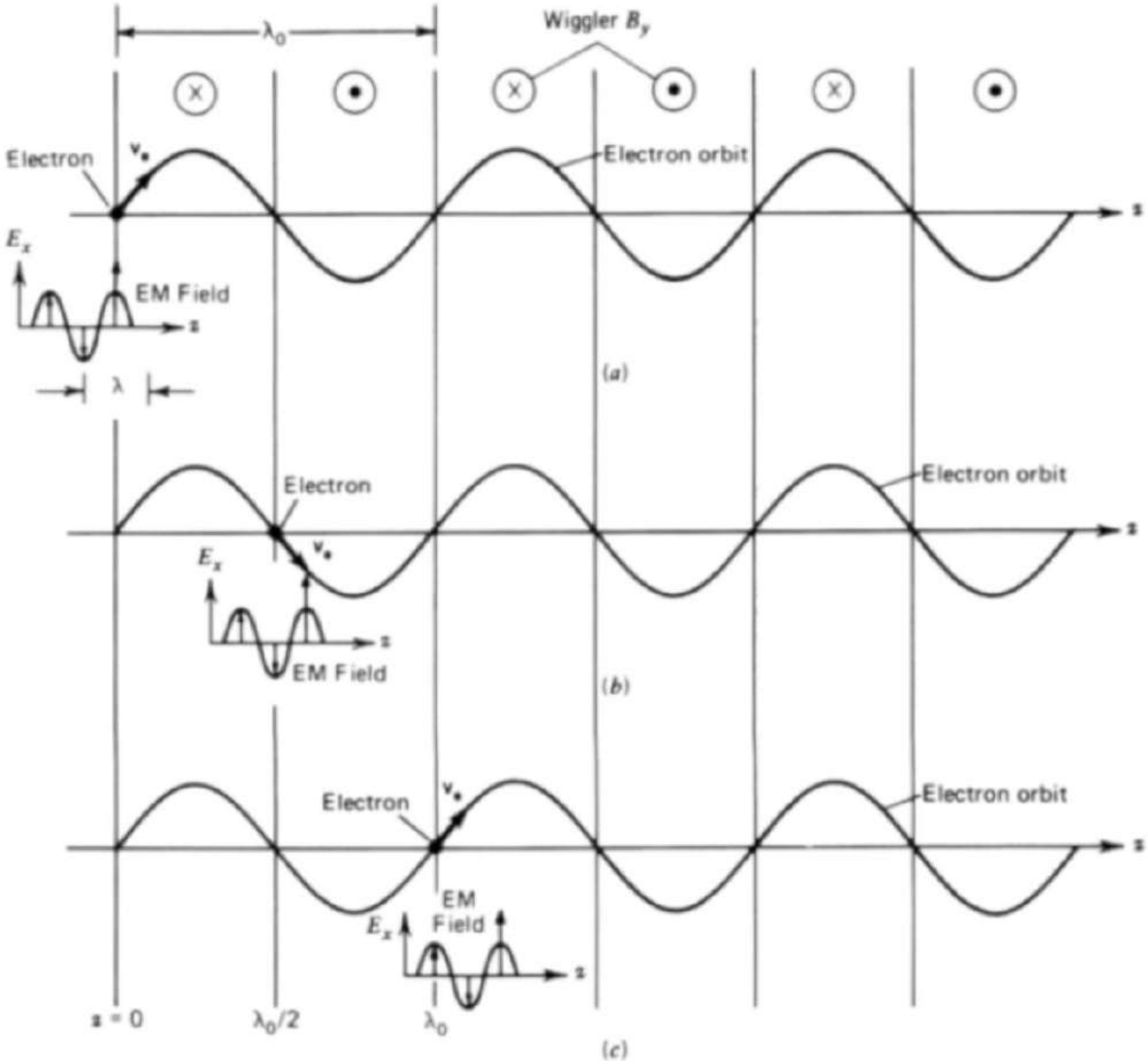


Fig. 37 The electron orbit in a periodic wiggler field. An electron is shown at (a) $z = 0$, (b) $z = \lambda_0/2$, and (c) $z = \lambda_0$. The traveling electromagnetic wave in the vicinity of the electron is also shown. The field first encountered by the electron at $z = 0$ is shown in (a), (b), and (c) with a bold tall arrow.

- Condition for in-phase energy transfer ($E_z v_z$) is obtained by considering a phase lag of electron oscillation *w. r. t.* optical oscillation.

$$-k\lambda_0 + \omega(\lambda_0 / v_z) = -\frac{2\pi}{\lambda}\lambda_0 + \frac{2\pi c}{\lambda}\frac{\lambda_0}{v_z} = \frac{2\pi\lambda_0}{\lambda}\left(\frac{c}{v_z} - 1\right) = 2\pi \times \text{integer}$$

- Maximum gain with 2π phase lag.

$$\lambda = \lambda_0\left(\frac{c}{v_z} - 1\right) = \lambda_0\left(\beta_z^{-1} - 1\right)$$

- For *highly relativistic electrons* (v_z~c),

$$\frac{1}{\gamma^2} = 1 - \left(\beta_z^2 + \beta_\perp^2\right) \rightarrow \frac{1}{\gamma^2} + \beta_\perp^2 = 1 - \beta_z^2 \approx 2(1 - \beta_z),$$

and thus

$$\lambda \cong \lambda_0(1 - \beta_z) \cong \frac{\lambda_0}{2}\left(\frac{1}{\gamma^2} + \beta_\perp^2\right)$$

- Therefore, for a given wiggler period l_0, by varying γ (energy of electrons) one can vary light wavelength l continuously.
- Typical value of l_0~several cm, undulator length~several meters, B~kG, E~several MeV to GeV.

- Electrons are bunched via the resonant interaction. Then collective motion of the bunches radiates powerful coherent synchrotron radiation.

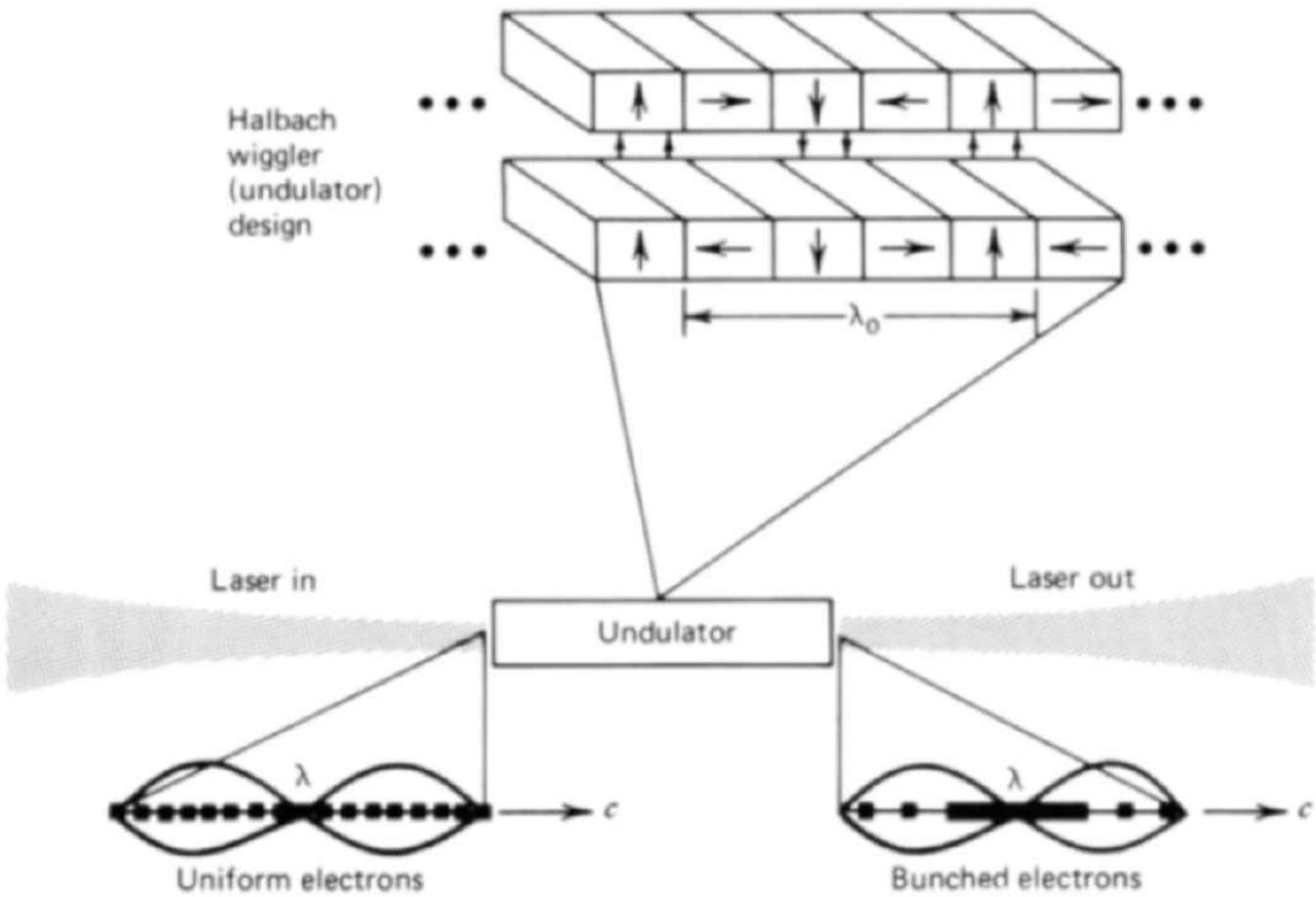
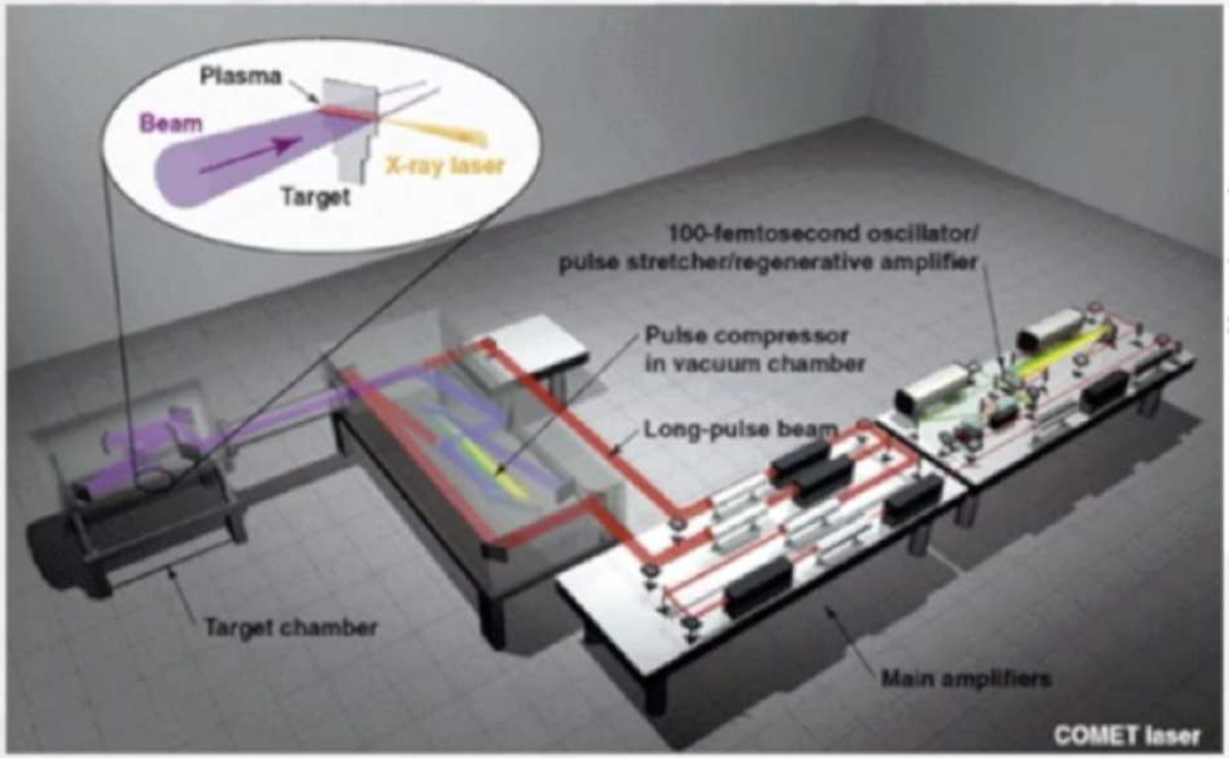


Fig. 38 A practical design (the Halbach magnet) for constructing the wiggler is shown. Eight permanent magnets are used in one wiggler period.

11.12 X-Ray Lasers

- Intense and coherent x-ray source are needed for material science, x-ray microscopy and basic research tool for structure of matter.
- X-ray lasing was first demonstrated at Livermore Labs. in 1984 using high power lasers (Nova).
- Soft X-ray (14.7 nm with palladium target) amplification has been observed in laser-induced plasma using a table-top terawatt laser in mid 1990's. Excited ions generated soft x-ray photons.



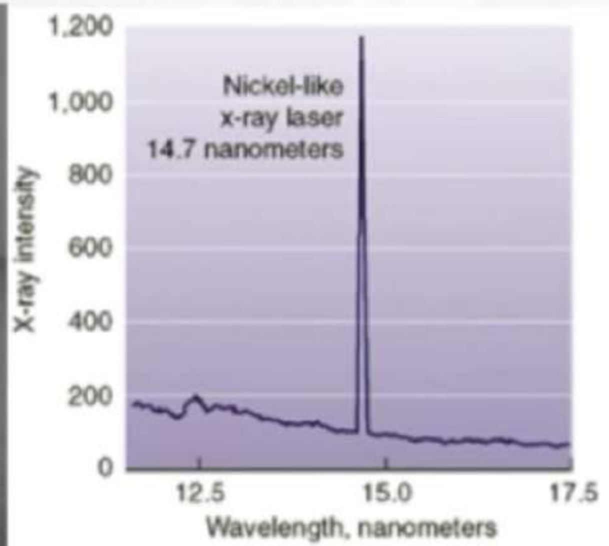


Fig. 39 Tabletop soft-x-ray laser developed at Livermore Labs.

11.13 Gamma-Ray Lasers

- Need for high power light source in the range of 10^{18} W ($=10^3$ PW$=10^6$ TW) in cancer therapy, nuclear fusion, advanced propulsion, and countermeasures against biological and chemical weapons, --> gamma-ray laser.
- Typical design of a gamma-ray laser uses isomers, nuclei in an isomeric energy level, or a practically non-decaying excited state of nucleus (~MeV in energy).
- Isomers are first excited to a gateway level by x-ray pumping (~tens of keV), which can quickly decay to an output level. The output level then decay to low neucleus state emitting high energy photons (several MeV).
- Alternatively, by scattering ultraviolet photons from 500-MeV electrons *inside a free-electron laser*, 12-MeV gamma-ray photons could be obtained. By collimating the gamma flux, one could achieve a nearly monoenergetic beam.

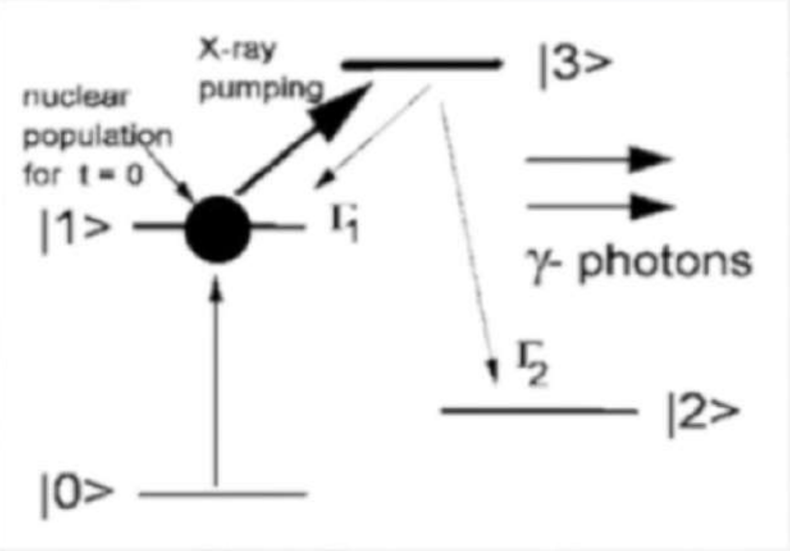


Fig. 40 Energy levels for a gamma-ray laser.

11.14 Home-Built Lasers (Amateur Laser Building)

- Scientific American, "Amateur Scientist" Column
  - Helium-Neon Laser, September, 1964, pg. 227.
  - More on the Helium-Neon Laser, December, 1965, pg. 106.
  - Argon Ion Laser, February, 1969, pg. 118.
  - Tunable Dye Laser, February, 1970, pg. 116.
  - Carbon Dioxide Laser, September, 1971, pg. 218.
  - Infrared Diode Laser, March, 1973, pg. 114.
  - Nitrogen Laser, June, 1974, pg. 122.
  - Mercury-Vapor Ion Laser, October, 1980, pg. 204.
  - Copper Chloride Laser, April, 1990, pg. 114.

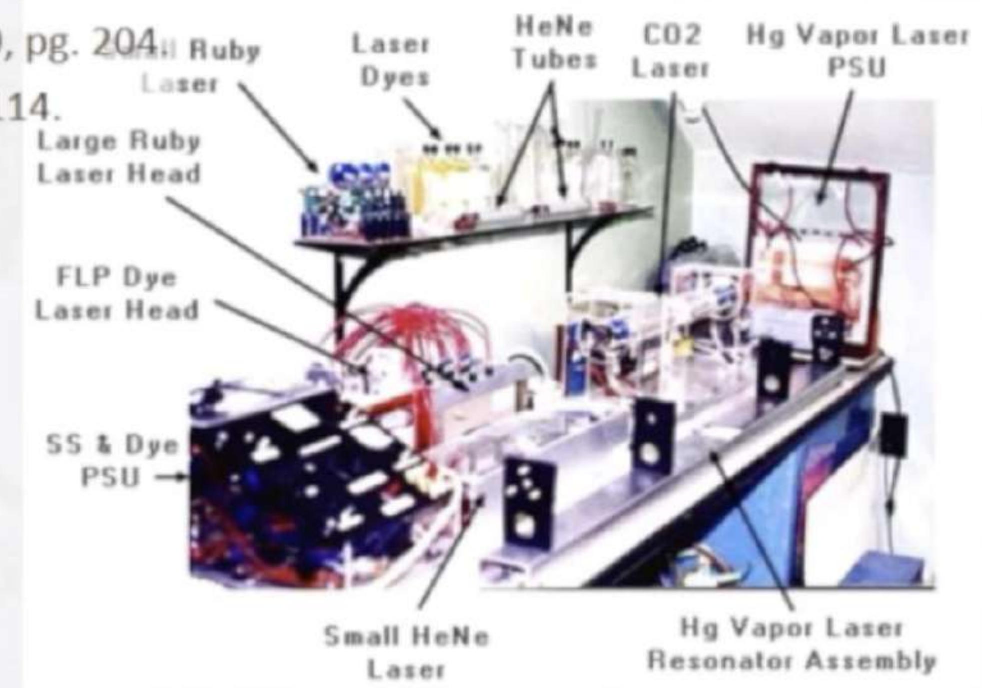


Fig. 41 Typical home laboratory hosting several home-built lasers.

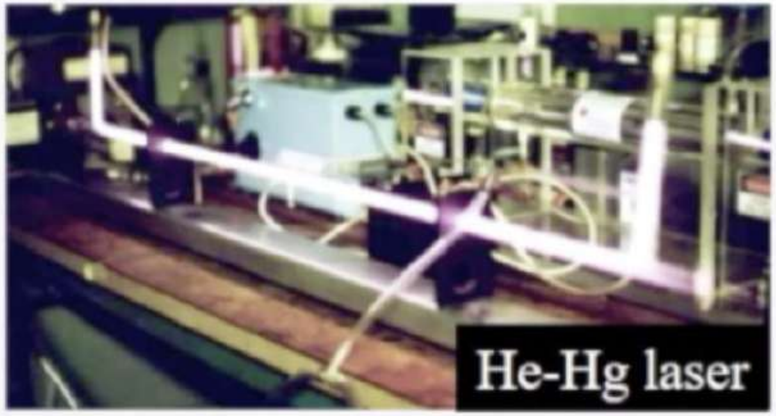


Fig. 42 Various home-built lasers. Source: Chris Chagaris (pyro@grolen.com)

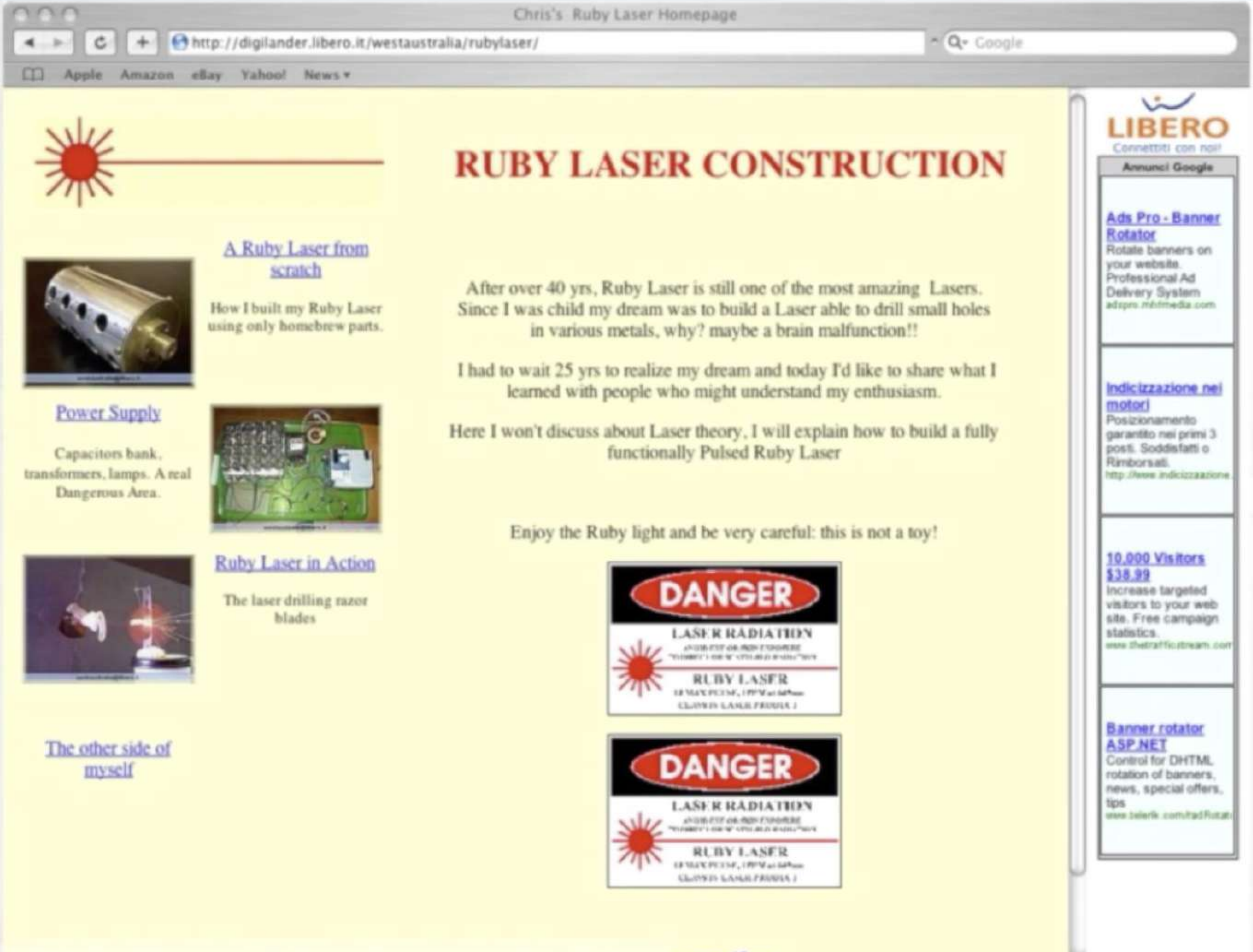


Fig. 43 Do-it-youself Ruby Laser on the Web.

11.15 Uncovered but interesting lasers

- One-atom maser (micromaser), single-atom laser (cavity-QED microlaser), Single-trapped-atom laser
- Quantum cascade laser
- Random laser (chaotic laser)

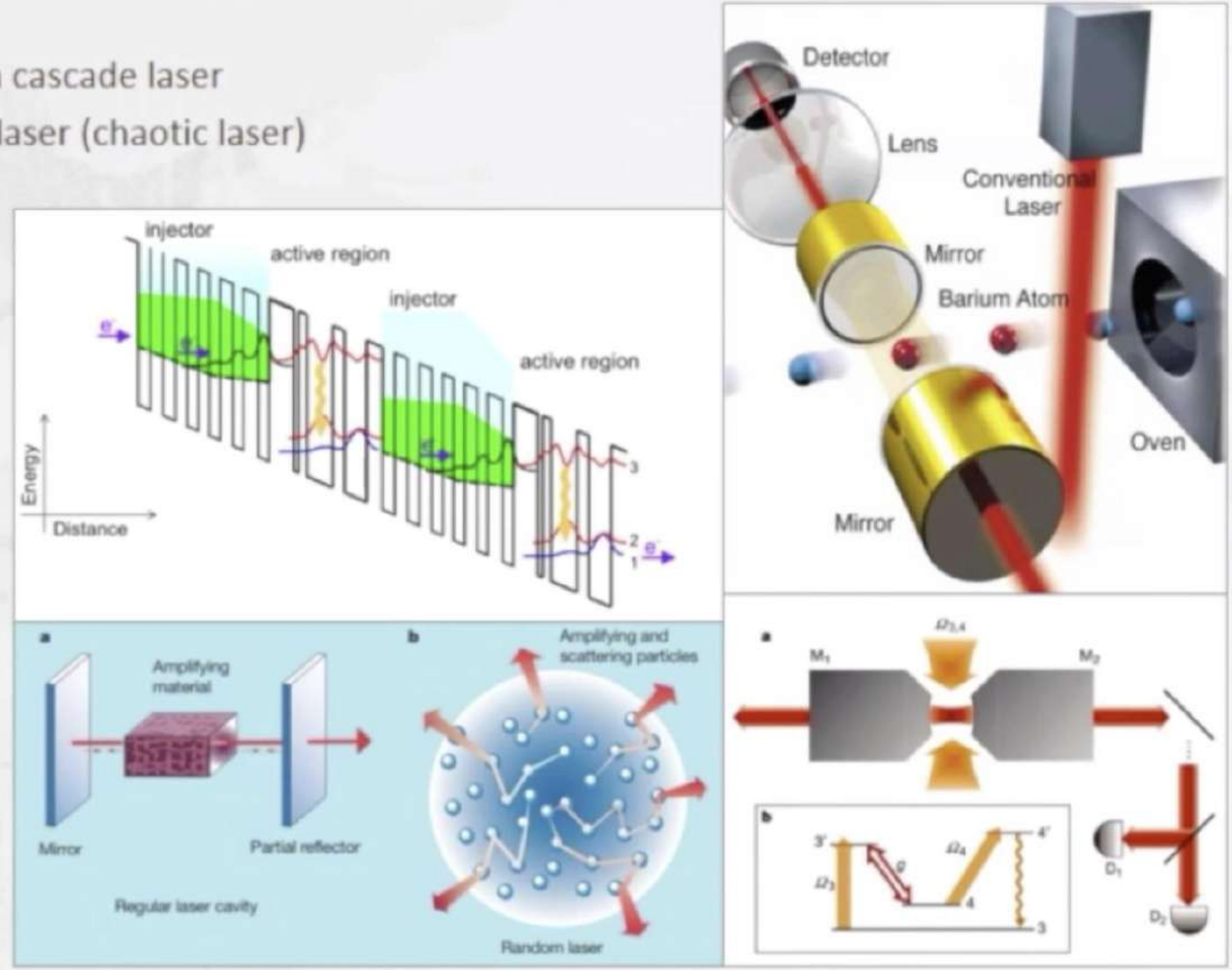
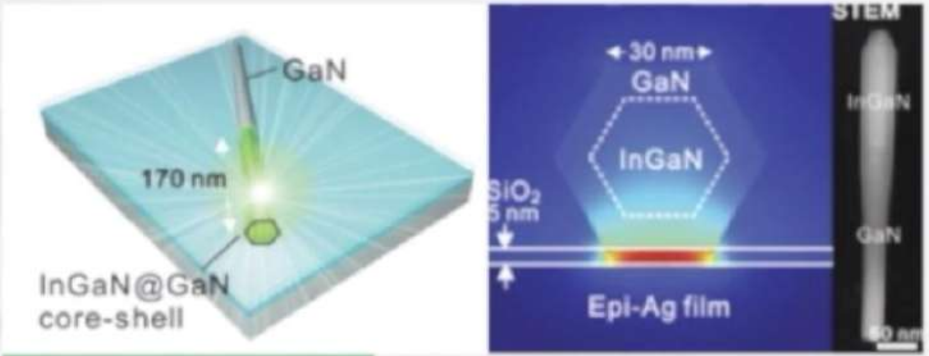
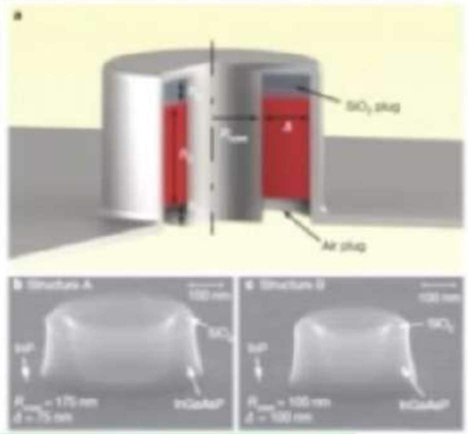


Fig. 44 Future of lasers?

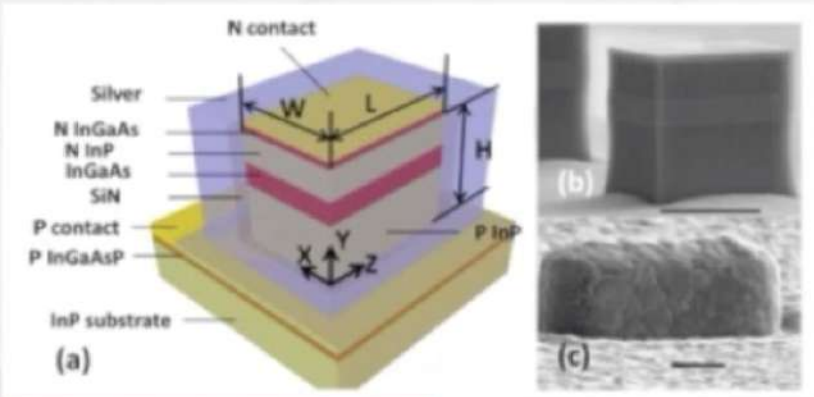
- Surface Plasmon Amplification by Stimulated Emission of Radiation (SPASER)
- Submicron-scale lasers (nano lasers)



Science **337**, 450 (2012)



Nature **482**, 204 (2012)



Opt. Express **21**, 4728 (2013)

Fig. 45 Surface-Plasmonic Nano lasers

11.16 Laser Weapon

- The US Navy has developed a laser which it says can shoot down small aircraft such as drones. Officials say the laser system will be deployed in 2014 aboard the USS Ponce.

Fig. 46 US Navy laser weapon to be deployed in 2014

12. Normal mode Splitting

12-1. Jaynes-Cumming model (quantum picture)

Electromagnetic radiation is described in terms of modes, which are defined by a set of boundary conditions. These boundary conditions are usually imposed by a cavity. The optical cavities are typically formed by set of planar or curved mirrors. These cavities are open cavities, meaning that atoms or molecules in the cavity can couple to the vacuum reservoir (i.e. infinitely many modes) via the substantial part of solid angle whereas the microwave cavities are examples of closed cavities.

When a two-level atom is made to couple to a single radiation mode (defined by a cavity) with negligible atomic and cavity damping (called the strong coupling limit), the atom and the cavity mode ~~form~~ form a new quantum entity with its own eigenstates and eigenvalues. The model describing such a system is known as the Jaynes-Cumming model.

The Hamiltonian is

+1 : excited state, −1 : ground state

$$H = \frac{1}{2}\hbar\omega_p(\sigma_3+1) + \hbar\omega_c a^\dagger a \pm \hbar g(\sigma_- a^\dagger + \sigma_+ a) \quad (12-1)$$

energy of two level atom; energy of field; atomic lowering operator; photon creation operator; atomic raising operator; photon anhililation operator; $a^\dagger a$: photon number operator; interaction hamiltonian → interaction of two level atom and radiation mode; frequency of cavity; frequency of atomic transition

where σ_+, σ_- and σ_3, the Pauli spin matrices satisfying $[\sigma_+, \sigma_-] = \sigma_3$ act as

$\sigma_- a^\dagger$

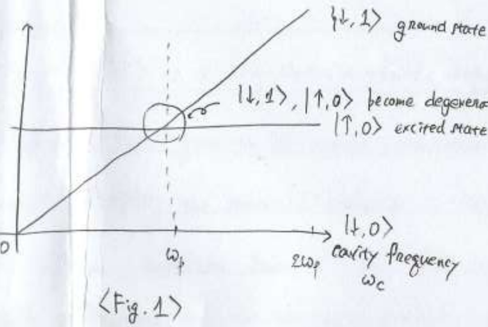
lower atomic state from excited state to ground state, emitting photon into cavity

$\sigma_+ a^-$

excite atom from lower level to upper level anhiliating photon

the atomic raising, lowering, and projection (σ_3) operators while $a^\dagger$ and a are the photon creation and annihilation operators. The atomic (cavity) resonance frequency is denoted by ω_p (ω_c). Without the atom-cavity interaction (the third term in H), the system is described by eigenstates $|\uparrow, n\rangle$, and $|\downarrow, n\rangle$ with $\uparrow(\downarrow)$ indicating the atom being in the excited (ground) state and $n = 0, 1, 2, \ldots$

Let us consider the lowest three states, $|\downarrow,0\rangle$, $|\downarrow,1\rangle$ and $|\uparrow,0\rangle$. As shown in Fig.1, the eigenenergies of $|\downarrow,1\rangle$ and $|\uparrow,0\rangle$ are degenerate when $\omega_p = \omega_c$.



⟨Fig.1⟩

↓: ground state, 1 : 1 photon in cavity

→ neglect atomic damping and cavity damping

→ assume coupling constant is much larger than atomic decay rate, cavity decay rate.

→ cavity for radiation mode

→ simplest way to having a radiation mode is to set up a cavity and define mode by cavity

→ excite that a cavity mode. creating photon 1, 2, 3, .. and so on.

→ convinient to assume cavity for radiation mode.

However with the atom-cavity interaction turned on, the system is described by new eigenstates $|+\rangle$ and $|-\rangle$ and new eigenvalues, which are obtained by diagonalizing the Hamiltonian

$$|\uparrow,0\rangle, |\downarrow,1\rangle \quad \hbar\begin{bmatrix} \omega_p & g \\ g & \omega_c \end{bmatrix}$$

expectation value of interaction hamiltonian (12-2) between $|\downarrow,1\rangle$ state

expectation value of interaction hamiltonian between $|\uparrow,0\rangle$ state

The eigenvalues are

$$E_\pm = \hbar\left[\omega_p + \frac{\Delta}{2} \pm \sqrt{\left(\frac{\Delta}{2}\right)^2 + g^2}\right] \quad (12-3)$$

$$(\omega_p - \lambda)(\omega_c - \lambda) - g^2 = 0$$

$$\lambda^2 - (\omega_p + \omega_c)\lambda + \omega_p\omega_c - g^2 = 0$$

$$\lambda = \frac{1}{2}(\omega_p + \omega_c) \pm \sqrt{\frac{(\omega_p+\omega_c)^2}{4} - \omega_p\omega_c + g^2}$$

$$= \omega_p + \Delta/2 \pm \sqrt{(\Delta/2)^2 + g^2}$$

with $\Delta = \omega_c - \omega_p$. The eigenstates $|+\rangle$ and $|-\rangle$ are given by the linear superpositions of $|\downarrow,1\rangle$ and $|\uparrow,0\rangle$ states.

$$|+\rangle = \cos\theta \cdot |\uparrow,0\rangle + \sin\theta\,|\downarrow,1\rangle$$

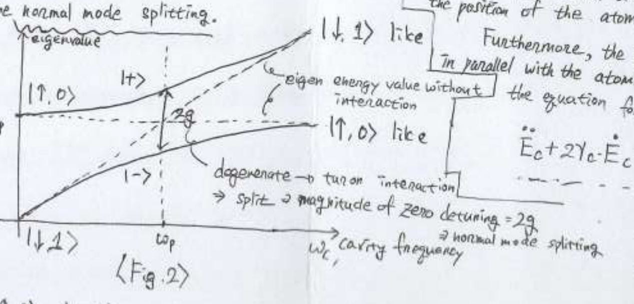
$$|-\rangle = -\sin\theta\,|\uparrow,0\rangle + \cos\theta\,|\downarrow,1\rangle$$

with $\theta = \tan^{-1}\left[(\Delta/2g) + \sqrt{1 + (\Delta/2g)^2}\right]$ (12-4)

Particularly, when $\Delta = 0$, $\theta = \pi/4$, and thus

$$E_\pm = \hbar(\omega_p \pm g), \quad |\pm\rangle = \frac{1}{\sqrt{2}}\left(\pm|\uparrow,0\rangle + |\downarrow,1\rangle\right) \quad (12-6)$$

The splitting of the energy eigenvalues by $2g$ is called the normal mode splitting.



⟨Fig.2⟩

→ $|\uparrow,0\rangle$, $|\downarrow,1\rangle$: no longer eigenstate

Suppose the system is in $|\uparrow,0\rangle$ state initially. It is not eigenstate, so the state changes in time as

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle e^{-i(\omega_p+g)t} - |-\rangle e^{-i(\omega_p-g)t}\right]$$

$$= \frac{1}{\sqrt{2}} e^{-i\omega_p t}\left[|+\rangle e^{-igt} - |-\rangle e^{+igt}\right]$$

$$= e^{-i\omega_p t}\left(|\uparrow,0\rangle \cos gt - i\,|\downarrow,1\rangle \sin gt\right)$$

cf. Rabi oscillation. → no external field in vacuum. Rabi oscillation → vacuum field drive oscillation

The state oscillates between $|\uparrow,0\rangle$ and $|\downarrow,1\rangle$ states. The oscillation is called the vacuum Rabi oscillation and the coupling constant g is also called the vacuum Rabi frequency. In the presence of the atomic and cavity damping effects, our consideration is approximately valid as long as the coupling constant g is much larger than atomic cavity decay rates. For correct inclusion of damping effects, one has to use master equation as seen in Eqs. (3-31) and (10-4).

10.2 Semiclassical description of the normal mode splitting

However, one can still get a good approximate description of the system using the semiclassical Maxwell-Schrödinger (M-S) equations. Particularly, M-S equations can give the correct lineshape of the normal mode splitting. However, since the field is treated classically, we can never get correct quantum statistical information on the system using this approach.

Let us consider a situation in which we drive an atom-cavity composite with a classical field (eg. CW laser). The cavity field, while continuously excited by the driving field, coherently interacts with the atoms inducing the atomic dipole moment. We begin with the following coupled equations for p and E.

First, the equation for the dipole moment comes from Eq. (9-10),

$\omega_0 \gg \gamma_{ab}$ → neglect γ_{ab}^2, replace ω_0 as ω_p

$$\ddot{p} + 2\gamma_p\dot{p} + \omega_p^2 p = \frac{2\omega_p\mu^2}{\hbar} N E_c \quad (10-6)$$

with γ_p (γ_c below) the atomic (cavity) decay rate (half width). E_c is the cavity field. We replaced the inversion with $-N$, with N the number of atoms ($N=1$).

The equation for the field is given by Eq. (9-3). But we are interested in a standing wave field evaluated at the position of the atom at rest.

Furthermore, the cavity damping has to be included in parallel with the atomic damping of Eq. (10-6). Therefore, the equation for the cavity field becomes

$$\ddot{E}_c + 2\gamma_c\dot{E}_c + \omega_c^2 E_c = \frac{4\pi\omega_p^2}{V} p + \xi E_L \quad (10-7)$$

$$\ddot{p} + 2\gamma_{ab}\dot{p} + (\omega_0^2 + \gamma_{ab}^2)p = -\frac{2\omega_0\mu^2}{\hbar}E\,\sigma$$

$\sigma = -N$ ← population inversion

→ consider two level atom, ground state donnot decay.

where V is the cavity mode volume and E_L is the driving laser field. Here, the macroscopic polarization is replaced with P/V since the dipole p is contained in the cavity mode. The parameter ξ represent the coupling of the driving field into the cavity field.

We solve Eqs. (10-6) and (10-7) using the slow varying envelope approximation.

$$P/V = \mathrm{Re}\left[\mathcal{P}e^{-i\omega t}\right], \quad E_c = \mathrm{Re}\left[\mathcal{E}e^{-i\omega t}\right],$$

slowly varying envelop

$$E_L = E_0\cos\omega t \quad (12-8)$$

probe laser

driving laser frequency; with $\Delta_{p,c} = \omega - \omega_{p,c}$,

$\Delta_p = \omega - \omega_p$, $\Delta_c = \omega - \omega_c$

$$\dot{\mathcal{P}} + (\gamma_p - i\Delta_p)\mathcal{P} = i\frac{\mu^2 N}{\hbar V}\mathcal{E}$$

$$\dot{\mathcal{E}} + (\gamma_c - i\Delta_c)\mathcal{E} = 2i\pi\omega_p\mathcal{P} + i\frac{\xi E_0}{2\omega} \quad (12-9)$$

decay term; extra source term

assume atom in cavity is stationary

In the steady state, $\dot{\mathcal{P}} = \dot{\mathcal{E}} = 0$, and we get (describe lineshape)

$$\mathcal{E} = i\frac{\xi E_0}{2\omega}\cdot\frac{(\gamma_p - i\Delta_p)}{Ng^2 + (\gamma_c - i\Delta_c)(\gamma_p - i\Delta_p)} \quad (12\text{-}10)$$

where we used the definition of the coupling constant g:

atom-field coupling constant

$$g = \frac{\mu}{\hbar}\sqrt{\frac{2\pi\hbar\omega_p}{V}} \quad (12-11)$$

Therefore, the cavity transmission proportional to $|\mathcal{E}|^2$ becomes

$$T \propto \left|\frac{(\gamma_p - i\Delta_p)}{Ng^2 + (\gamma_c - i\Delta_c)(\gamma_p - i\Delta_p)}\right|^2 \quad (12-12)$$

transform as Lorentzian

$$T \propto \frac{\Delta_p^2 + \gamma_p^2}{(\Delta_+^2 + \Gamma_+^2)(\Delta_-^2 + \Gamma_-^2)} \quad (12-13)$$

inverse of Lorentzian; two Lorentzian fct

where

$$\Delta_\pm = \omega - (\omega_+ \pm R), \quad \Gamma_\pm = \gamma_+ \pm I$$

$$\omega_\pm = \frac{1}{2}(\omega_c \pm \omega_p), \quad \gamma_\pm = \frac{1}{2}(\gamma_c \pm \gamma_p)$$

$$2R^2 = (Ng^2 + \omega_-^2 - \gamma_-^2) + \sqrt{(Ng^2 + \omega_-^2 - \gamma_-^2)^2 + 4\omega_-^2\gamma_-^2}$$

$$2I^2 = -(Ng^2 + \omega_-^2 - \gamma_-^2) + \sqrt{(Ng^2 + \omega_-^2 - \gamma_-^2)^2 + 4\omega_-^2\gamma_-^2}$$

$N \cdot g^2$

→ we have atom-cavity combined system → drive system with laser through cavity, measure transmission as a function of laser frequency.

→ Numerator makes two Lorentzian more pronounced separation of

∵ it becomes smallest when $\Delta_p = 0$

longest contribution : $-\frac{\omega^2}{c^2}$

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E(z,t) = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}P(z,t)$$

$$-\omega^2 E - \ddot{E} = -4\pi\omega_0^2 P$$

→ does not include cavity decay

→ phenomenologically include decay

→ replace P as dipole p divided by mode volume V

coupling between atom and field, and determines cavity transmission

with extra probe laser including

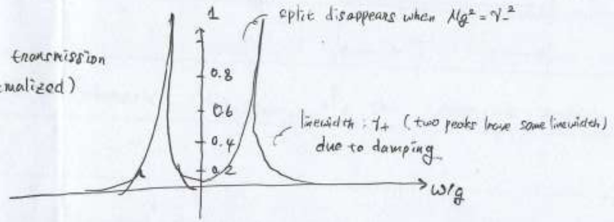
E_L: extra laser field amplitude → driving atom-cavity combined system

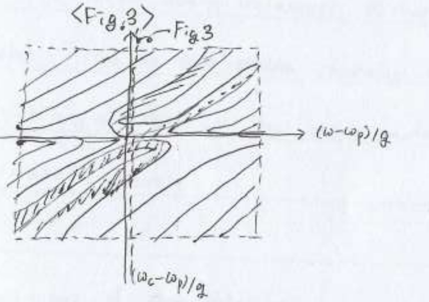
When $\omega_c = \omega_p$, we get (if $Ng^2 > \gamma^2$) ← product of two Lorentzian and inverted Lorentzian

$$T \propto \frac{(\omega-\omega_c)^2+\gamma_c^2}{\left[(\omega-\omega_p+\sqrt{Ng^2-\gamma_-^2})^2+\gamma_+^2\right]\left[(\omega-\omega_p-\sqrt{Ng^2-\gamma_-^2})^2+\gamma_+^2\right]} \quad (12\text{-}14)$$

we have peak at $\omega = \omega_p \pm \sqrt{Ng^2-\gamma_-^2}$
→ for $Ng^2 \gg \gamma_-^2$
→ $\omega \simeq \omega_p \pm \sqrt{N} g$

When ω is scanned across ω_p, the above expression clearly exhibits a two peak lineshape as shown in Fig.3. The two peaks are separated approximately by $2\sqrt{N}g$. For $N=1$ we get the normal mode splitting of Sec. 12.1.

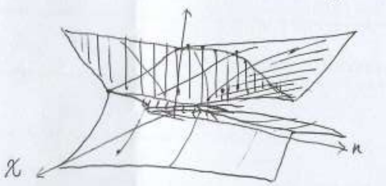




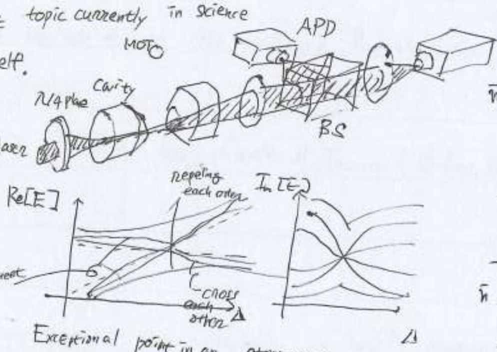
⟨Fig.4. Contour plot (log-scale) of Eq. (12-14). $\gamma_c = \gamma_p = 0.25g$.
Fig.3 corresponds to a scan along dotted line.⟩

Note that the two-peak structure disappears when $\sqrt{N}\cdot g = |\gamma_-| = \frac{1}{2}|\gamma_p - \gamma_c|$. In non-Hermitian physics, this condition corresponds to an exceptional point, at which two eigenstates ~~disappear~~ appear to coalesce to a single state. The coalescence in the parameter space made of two system parameters such as g and Δ, the eigenenergy surfaces exhibit Möbius-strip-like topology. The exceptional point is one of the hottest topic currently in science and technology. Try google search for yourself.

← energy loss does not occur in hermitian system → need to talk about width, separation of two peaks, converge to single peak



Exceptional point in a microcavity - K. An group

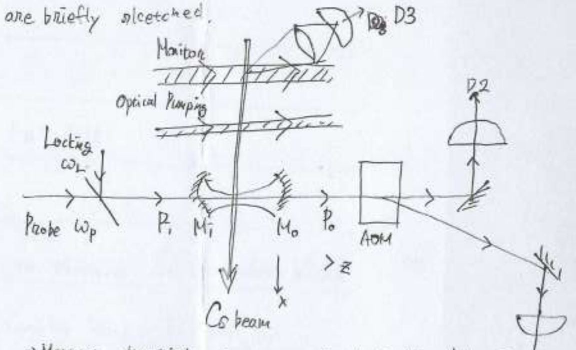


Exceptional point in an atom-cavity system,
Y. Choi et al. Phys. Rev. Lett. 104, 153601 (2010) - K. An group

→ Exceptional point phenomena
→ can be used to sense external perturbation.

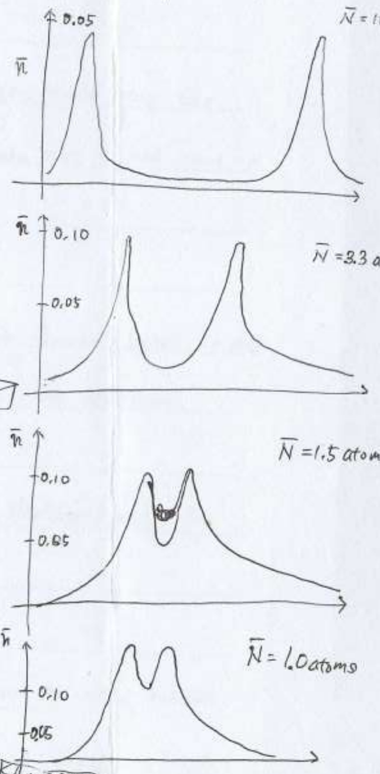
12.3 Normal mode splitting experiments

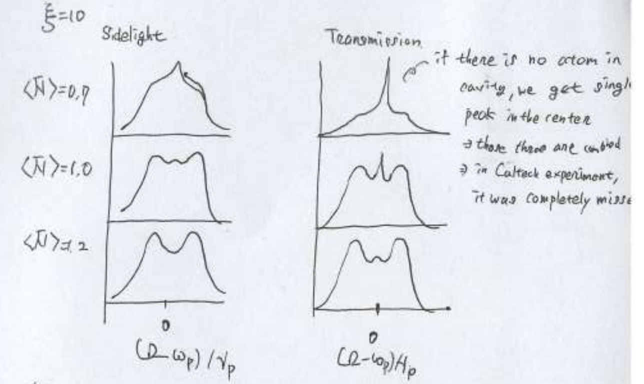
Experimental ~~exp~~ verification of the normal mode splitting for a single atom was done by Caltech group led by J. Kimble in 1992 (PRL 68, 1132), MIT group (K. An was involved), who initiated and advocated the experiment, lost the lead, but came back in 1996 with a more complete study (PRL 77, 2901). Below the results from these experiments are briefly sketched.



→ Measure transmission spectrum under the condition that the number of atom in cavity inside in this mode is about one

⟨Fig.5. Experimental setup for observing the normal mode splitting (Caltech experiment)⟩

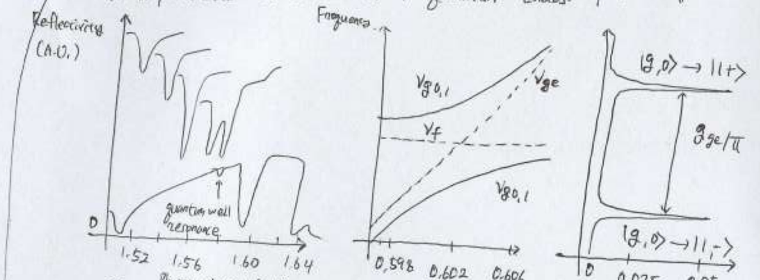




⟨Fig.6 Normal mode splitting. Caltech experiment (Left) and MIT experiment (Right). The central peak seen in MIT data is due to the strong atom number fluctuations when $\langle N \rangle \sim 1$ with appreciable probability of being no atom in the cavity.⟩

Normal mode splitting is often considered as a signature of quantum coupled systems. It has been demonstrated in various physical systems

i. semiconductors → exciton QED
ii. superconducting Josephson junctions → Circuit QED ← first qubit experiment in circuit QED
field toward qubit operation → coupling is large → split is huge and
iii. macroscopic oscillator at quantum limit (with ultra-small inertia and at low temperature) → optomechanics
make much easier to h quantum qubit
iv. asymmetric microcavities → quantum chaos



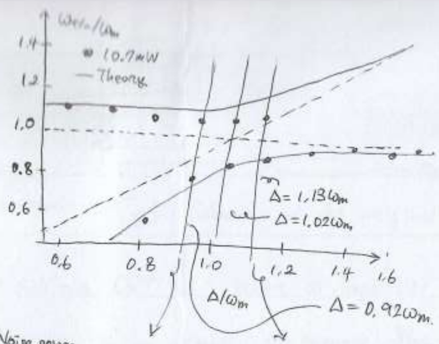
Semiconductor quantum well (excitation - photon) PRL 69, 3314 (1992)

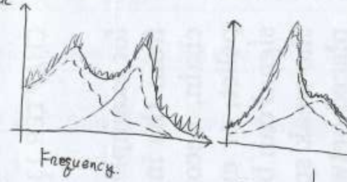
Circuit QED (antenna-photon), Nature 454, 315 (2008)

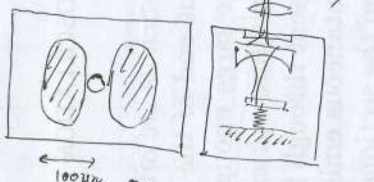
⟨Fig.7a⟩

→ when people want to build qubit → demonstrate Rabi ~~os~~ oscillation to observe normal splitting

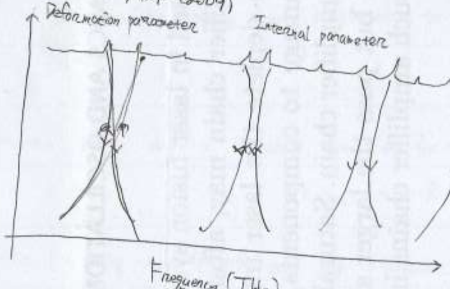
Mechanical system coupled with light → people want to build quantum mechanical circuit using optomechanical oscillator







Opto-mechanics (phonon-photon), Nature 460, 724 (2009)



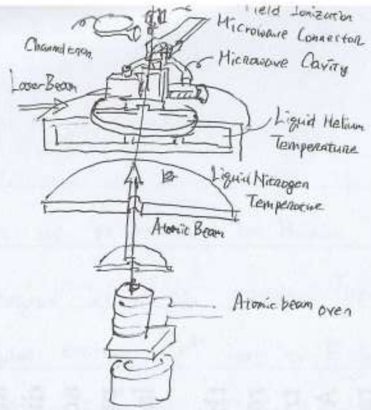
Asymmetric microcavity (photon-photon), PRL 103, 134101 (2009)

- one may orbit to another by dynamical tunneling. SNU group led by K. An

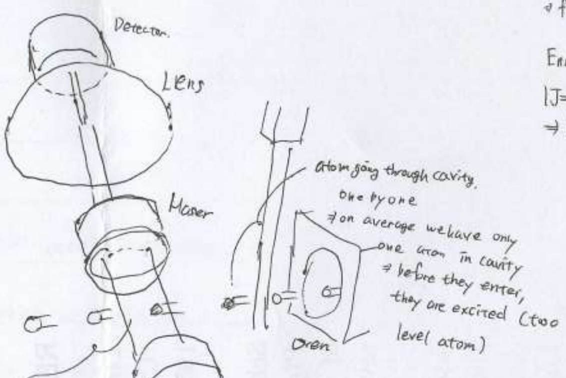
⟨Fig. 7b⟩

12.4 Single-atom maser & single-atom laser

A maser or a laser can be made with only one atom as a gain medium if the loss is made small enough to satisfy the well-above-threshold condition, $G \gg 2K$



The micromaser, D. Meschede and H. Walther, Phys. Rev. Lett. 54, 551 (1985)



The microlaser K. An et al., Phys. Rev. Lett. 73, 3375 (1994)

emit photon in cavity mode (due to strong atom field coupling constant)

→ if loss is small enough, its emission is enough to build a laser with only one atom in cavity

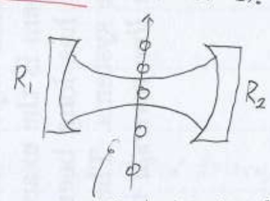
In the single-atom maser/laser, the atoms traverse the cavity one by one in such a way that on average only one atom is inside the cavity. In other words, when a preceding atom exits the cavity, a new one enters the cavity. Each atom is prepared in the excited state before it enters the cavity. In addition, the probability of emitting a photon during the transit is much smaller than unity. In the threshold condition $\Delta n\, 6_{em} L = 1 - R$, we thus have $\Delta n = \frac{1}{V}$ and $6_{em} = 6_{rad}\left(\frac{\Gamma_0}{2\gamma_{ab}}\right)$ with $2\gamma_{ab}$ replaced with the transit time broadening (See Problem 23).

V is mode volume

$2\gamma_{ab}$: flourescence bandwidth

(∵ during transit atom, population doesnot change much)

$R = \sqrt{R_1 R_2}$



atom is moving through cavity → transit time broadening → a spectrum is broadened → act as a dephasing rate (radiative decay rate is different)

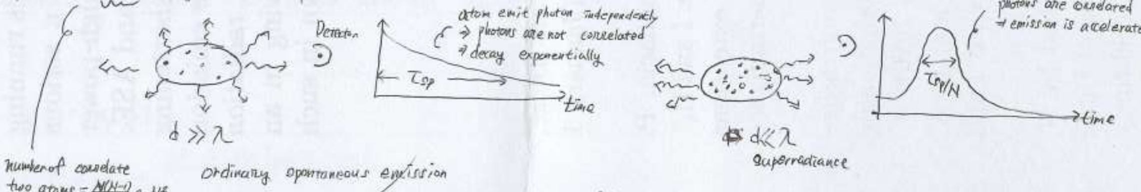
(∵ we are monitoring emission of each atom)

→ excited state decays slowly in this experiment, but transit time broadening is much larger than decay rate

→ emission cross section is reduced

12.5 Superradiance

In 1954, R. Dicke suggested enhanced emission of N excited atoms when they are correlated. Emission strength becomes proportional to N^2 in this case → superradiance.



number of correlate two atoms $= \frac{N(N-1)}{2} \sim N^2$

→ peak intensity is much higher

→ superradiance doesnot occur often. → condition for superradiance is extremely stringent

→ first experiment was conducted in 1970,

Ensemble of N two-level atoms (spin 1/2) in terms of eigenstates $|J=N/2, M\rangle$ of total angular momentum (↑: excited state, ↓: ground state)

→ Dicke ladder made of Dicke states

e.g. $N=2$, $\begin{cases} |1,1\rangle = |\uparrow\uparrow\rangle \\ |1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |1,-1\rangle = |\downarrow\downarrow\rangle \end{cases}$

all atoms are excited (all arrows are upward)

$|J=\frac{N}{2}, M=\frac{N}{2}\rangle$

$|J=\frac{N}{2}, M=\frac{N}{2}-1\rangle$

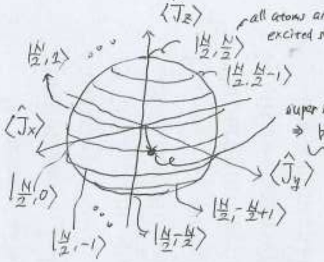
$|J=\frac{N}{2}, M=1\rangle$

$|J=\frac{N}{2}, M=0\rangle$

$|J=\frac{N}{2}, M=-1\rangle$

$|J=\frac{N}{2}, M=1-\frac{N}{2}\rangle$

$|J=\frac{N}{2}, M=-\frac{N}{2}\rangle$



super radiance signal is peak → bright state → largest emission rate

N-atom Bloch Sphere with total spin $\hat{J}_\mu = \sum_{i=1}^{N} \hat{\sigma}_i^\mu$

→ in order to see super radiance, all atoms are prepared to be excited state, and atoms are very close inter atomic distance is much smaller than transition wavelength → they are correlated (in phase) → when one atom flips emitting photon, state changes to $|J=\frac{N}{2}, M=\frac{N}{2}-1\rangle$. When they reach to $|J=\frac{N}{2}, M=0\rangle$, emission rate becomes extremely fast proportional to N^2.

→ driving force of superradiance

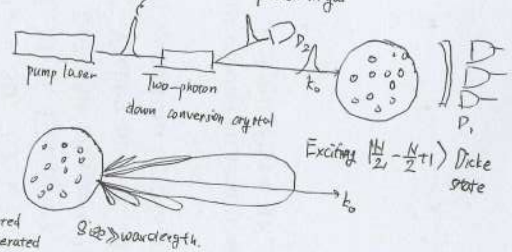
can be calculated

→ initially atoms can be in $|J=\frac{N}{2}, M=0\rangle$ (Dicke proposed)

→ quite recently people achieved this

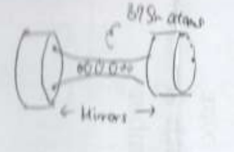
A dicke state or a superradiant state directly excited by a coherent laser pulse can induce superradiance immediately without initial delay for phase correlation to buildup.

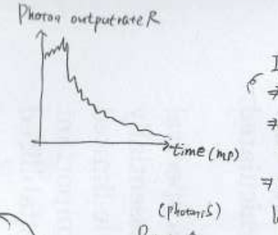
Trigger for presence of one photon in gas



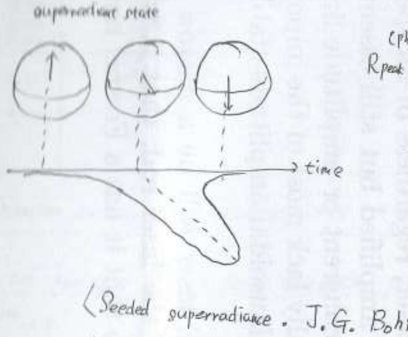
⟨Single-Photon superradiance, M.O. Scully and A.A. Svidzinsky, Science 325, 1510 (2009)⟩

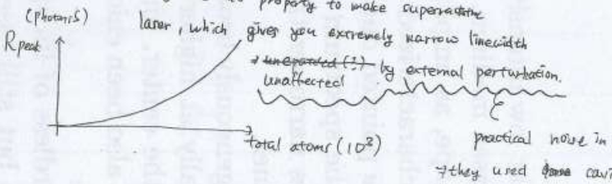
⟨Review by M. Gross and S. Haroche, Phys. Rep. 93, 301 (1982)⟩





Implemented by NIST Group
⇒ made $|J=\frac{N}{2}, M=0\rangle$
⇒ used pulse laser until all the atoms to be excited ⇒ they get superradiance
⇒ They use this property to make superradiance laser, which gives you extremely narrow linewidth
⇒ ~~unaffected (?)~~ unaffected by external perturbation.





practical noise in linewidth
⇒ they used ~~some~~ cavity with very broadline width, and atoms have narrow linewidth ⇒ cavity does not store many photon ⇒ just reinforce the fluorescence from atom ⇒ atoms undergo superradiance at transition frequency which is very narrow
⇒ even if cavity length frequency change, superradiance frequency does not change much generating very narrow linewidth ⇒ it can be used for optical clock. ∵ frequency is extremely well defined

⟨Seeded superradiance. J.G. Bohnet et al., Nature 484, 78 (2012);
M.A. Norcia et al., Sci. Adv. 2, e1601231 (2016)⟩

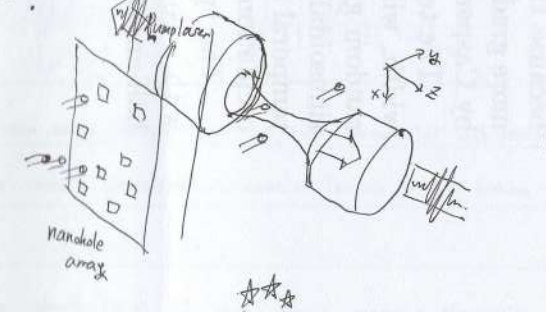
⇒ in order to see superradiance, we don't need to put all atoms up there
⇒ we can have superradiance by exciting its lowest state in ladder structure $(J=\frac{N}{2}, M=\frac{N}{2}-\frac{N}{2}+1)$
⇒ Just one atom is excited ⇒ we still have correlation between atom, we have superradiance
→ Single photon superradiance

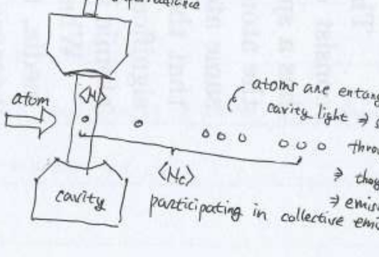
Superradiance can be generated by time-separated correlated atoms when individual atoms are prepared in the same superposition state $|\psi_{atom}\rangle = \frac{1}{\sqrt{2}}\left(|e\rangle + e^{i\varphi}|g\rangle\right)$ and then made to interact with a cavity for a fixed time τ. One can show that the mean photon number in the cavity is given by

$$\langle n\rangle \simeq \langle n\rangle_{NC} + \langle n\rangle_C = \frac{\frac{1}{2}\rho_{ee}\langle N_c\rangle (g\tau)^2}{1-(\rho_{ee}-1/2)\langle N_c\rangle (g\tau)^2} + \underbrace{\left(|\rho_{eg}|\langle N_c\rangle g\tau\right)^2}_{\text{correspond to superradiance}}$$

⇒ clearly showing $\langle N_c\rangle^2$ term
$g = \frac{\mu}{\hbar}\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}$
← can be solved analytically using master eq. ⇒ this is solution.

under the condition, $g\tau \ll 1$, where g is the atom cavity coupling constant and N_c is the number atoms going through the cavity during the cavity field decay time κ^{-1}.





atoms are entangled through cavity light ⇒ so that even atom had gone through cavity, they are all correlated
⇒ they affect emission process particular inside cavity
⇒ emission $\propto N^2$
participating in collective emission

☆☆☆
⟨J.Kim et al, Science 359, 662 (2018) - SNU group led by K.An⟩ ← 꼭 읽기!!

⇒ it is important to have the atom put in same shipment(?) before they enter to cavity ⇒ achieved by nano array ← array of nanoscopic hole
⇒ atoms go through there hole and separation between the holes is equal to transition wave length of the atom ⇒ pump laser un-resonance with atom can excite same spontaneous state(?) after station of atom to the cavity, go through ~~an~~ anti node

4 ⇒ ∵ separation ~~of~~ along horizontal direction is also transition wave length ⇒ have coherent interaction between atom and field

maintaining correlation along atoms