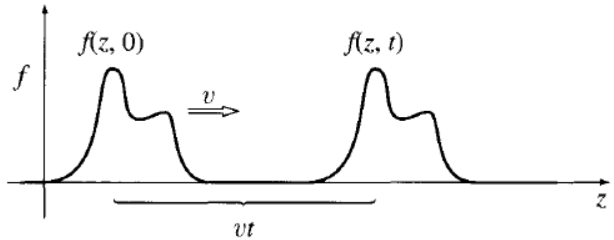
9.1.1 The Wave Equation



- General form of wave function
  - $f(z,t) = f(z - vt, 0) = g(z - vt)$ describes a wave propagating along z-axis with speed v.
  - Typical wave equation having the above function as solution:

$$\frac{\partial}{\partial z} f(z,t) + \frac{1}{v}\frac{\partial}{\partial t} f(z,t) = 0$$

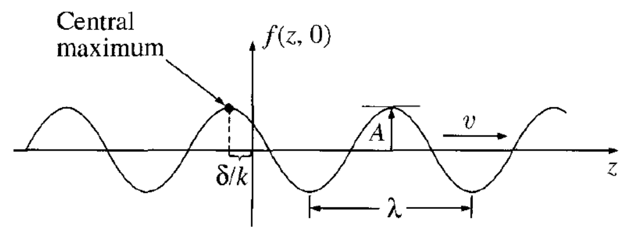
  - What is the solution of $\frac{\partial}{\partial z} f(z,t) - \frac{1}{v}\frac{\partial}{\partial t} f(z,t) = 0$?
  - What is the solution of $\frac{\partial^2}{\partial z^2} f(z,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2} f(z,t) = 0$?

$$\left(\frac{\partial}{\partial z} - \frac{1}{v}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t}\right) f(z,t) = 0$$

    - Allows waves along both directions. General form of solution:

$$f(z,t) = g(z - vt) + h(z + vt)$$

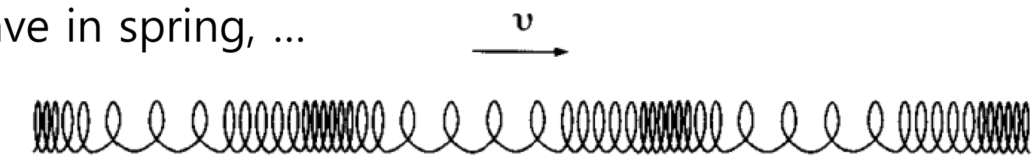
9.1.2 Sinusoidal waves



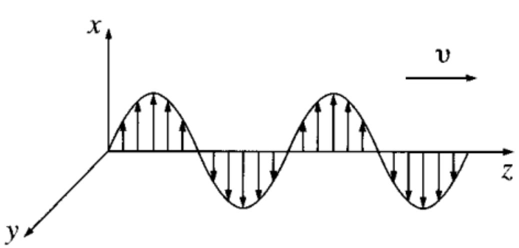
- Terminology
  - $f(z,t) = A\cos[k(z - vt) + \delta]$
  - A: amplitude
  - $k(z - vt) + \delta$: phase
  - δ: phase constant
  - The point whose phase is zero moves to $z = vt - \delta/k$ at time t
  - k: wave number. When the wavelength is λ, $k = 2\pi/\lambda$.
  - At fixed position z_0, $f(z_0,t)$ is oscillating with the period $T = \frac{2\pi}{kv}$
  - The frequency $f = 1/T = kv/2\pi = v/\lambda$
  - Angular frequency $\omega = 2\pi f = kv$
  - $f(z,t) = A\cos(kz - \omega t + \delta)$
  - Speed: $v = \omega/k$
- Complex notation
  - Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$
  - $f(z,t) = \mathrm{Re}\left[Ae^{i(kz-\omega t+\delta)}\right]$
  - Define **complex wave function** $\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz-\omega t)}$ with complex amplitude $\tilde{A} \equiv Ae^{i\delta}$.
  - Then actual wave function is $f(z,t) = \mathrm{Re}\left[\tilde{f}(z,t)\right]$
- Linear combinations of sinusoidal waves
  - Any arbitrary wave can be expressed as a linear combination of sinusoidal waves

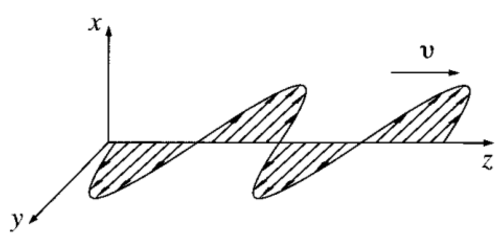
9.1.4 Polarization

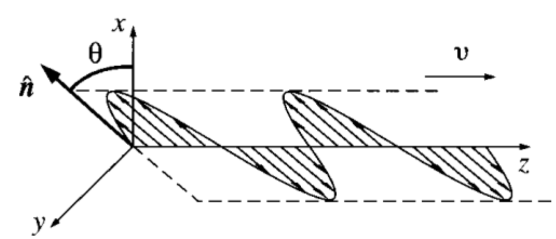
- Longitudinal wave
  - Displacement along the propagation direction
  - Sound wave, wave in spring, …



- Transverse wave
  - Displacement from equilibrium ⊥ propagation direction
  - Vibration of string, electromagnetic fields, …
  - There are 2-dimensions ⊥ propagation direction
  - ➔ Two independent states of polarization
- Propagation along z-axis
  - Vertical polarization: $\tilde{\boldsymbol{f}}_v(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{x}$
  - Horizontal polarization: $\tilde{\boldsymbol{f}}_h(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{y}$
  - Any direction $\hat{n}$ in the xy plane: $\tilde{\boldsymbol{f}}_v(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{n}$
    - $\hat{n}\cdot\hat{z} = 0$ and $\hat{n} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$
    - $\tilde{\boldsymbol{f}}_v(z,t) = \left(\tilde{A}\cos\theta\right)e^{i(kz-\omega t)}\hat{x} + \left(\tilde{A}\sin\theta\right)e^{i(kz-\omega t)}\hat{y}$: superposition of $\tilde{\boldsymbol{f}}_v$&$\tilde{\boldsymbol{f}}_h$







9.2 Electromagnetic Waves in Vacuum

- Vacuum: no charge & no current
- Maxwell's equation

(i) $\nabla \cdot \boldsymbol{E} = 0$ (ii) $\nabla \cdot \boldsymbol{B} = 0$

(iii) $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ (iv) $\nabla \times \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$

- Find a decoupled equation for $\boldsymbol{E}$
  - $\nabla \times (\nabla \times \boldsymbol{E}) = \nabla \times \left(-\frac{\partial \boldsymbol{B}}{\partial t}\right)$
  - Left-hand side: $\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\nabla^2 \boldsymbol{E}$
  - Right-hand side: $\nabla \times \left(-\frac{\partial \boldsymbol{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B}) = -\frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right)$
  - $-\nabla^2 \boldsymbol{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \Rightarrow \nabla^2 \boldsymbol{E} - \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \boldsymbol{0}$

9.2 Electromagnetic Waves in Vacuum

- Vacuum: no charge & no current
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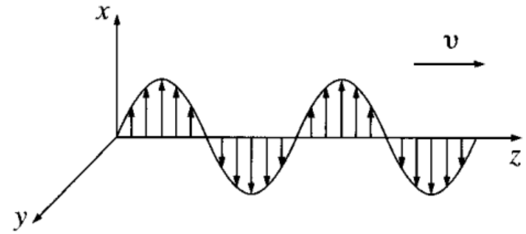
- Find a decoupled equation for $\boldsymbol{B}$
  - $\nabla \times (\nabla \times \boldsymbol{B}) = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right)$
  - Left-hand side: $\nabla \times (\nabla \times \boldsymbol{B}) = \nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B} = -\nabla^2 \boldsymbol{B}$
  - Right-hand side: $\nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right) = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \boldsymbol{E}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \boldsymbol{B}}{\partial t}\right)$
  - $-\nabla^2 \boldsymbol{B} = -\mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} \rightarrow \nabla^2 \boldsymbol{B} - \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} = \boldsymbol{0}$

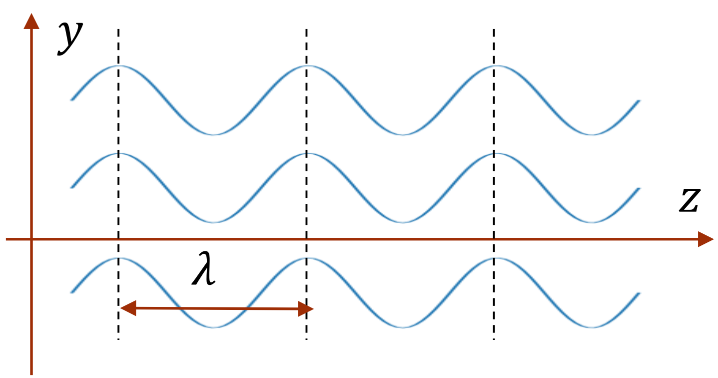
9.2 Electromagnetic Waves in Vacuum

- Two decoupled 2nd-order equations
  - $\nabla^2 \boldsymbol{E} - \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \boldsymbol{0}$, $\nabla^2 \boldsymbol{B} - \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} = \boldsymbol{0}$
  - How many equations do we have?
    - Recall $\boldsymbol{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$
    - $\nabla^2 E_x - \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0$, $\nabla^2 E_y - \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2} = 0$, …, $\nabla^2 B_x - \mu_0 \epsilon_0 \frac{\partial^2 B_x}{\partial t^2} = 0$, …
  - How many terms in each equation?
    - $\frac{\partial^2}{\partial x^2} E_x + \frac{\partial^2}{\partial y^2} E_x + \frac{\partial^2}{\partial z^2} E_x - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E_x = 0$ or $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) E_x = \vec{0}$
  - Recall one of the simplest solution of $\frac{\partial^2}{\partial z^2} f(z,t) - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} f(z,t) = 0$
    - $f(z,t) = A \cos[k(z - vt) + \delta]$
  - We will first consider sinusoidal solution for the wave equation

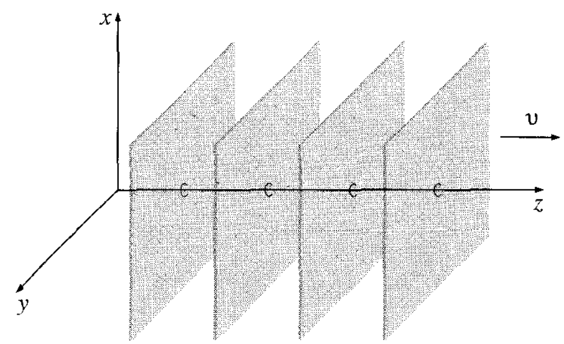
9.2.2 Monochromatic Plane Wave

- Simplest solution for $\nabla^2 E_x - \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0$
  - $E_x = E_0 \cos(kz - \omega t + \delta)$
  - Assume that $E_y = E_z = 0$
  - $\boldsymbol{E}(x, y, z, t) = E_x \hat{x} = E_0 \cos(kz - \omega t + \delta)\, \hat{x}$
  - Single frequency: called monochromatic
  - Plane wave
    - Waves are travelling in a single direction (here, z direction)
    - No x or y dependency



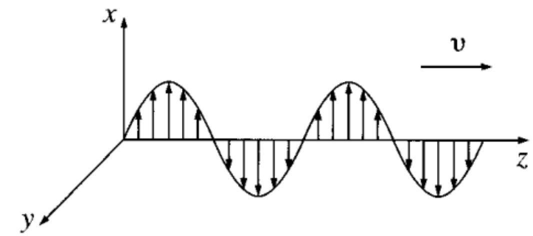


Plot of sinusoidal wave in yz-plane



Plot of position where the phase of the wave is the same in 3-D space

9.2.2 Monochromatic Plane Wave



- Complex field notation
  - $\boldsymbol{E}(x, y, z, t) = E_x \hat{x} = E_0 \cos(kz - \omega t + \delta)\, \hat{x}$
  - $= \mathrm{Re}\left[E_0 e^{i(kz-\omega t+\delta)}\right]\hat{x} = \mathrm{Re}\left[E_0 e^{i\delta}\hat{x} e^{i(kz-\omega t)}\right]$
  - $\widetilde{\boldsymbol{E}}(\boldsymbol{r}, t) = \left(E_0 e^{i\delta}\hat{x}\right)e^{i(kz-\omega t)} \equiv \widetilde{\boldsymbol{E}}_0 e^{i(kz-\omega t)}$
  - Generally $\widetilde{\boldsymbol{E}}_0$ can be any complex vector in 3-d space
- Solution for $\nabla^2\boldsymbol{B} - \mu_0\epsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} = \boldsymbol{0}$
  - We can also find a simplest solution (monochromatic plane wave) similar to $\widetilde{\boldsymbol{E}}(\boldsymbol{r}, t) = \widetilde{\boldsymbol{E}}_0 e^{i(kz-\omega t)}$.
  - $\widetilde{\boldsymbol{B}}(\boldsymbol{r}, t) = \widetilde{\boldsymbol{B}}_0 e^{i(kz-\omega t)}$
  - In principle, $\widetilde{\boldsymbol{B}}_0$ also can be any complex vector in 3-d space, BUT there are extra constraints on $\widetilde{\boldsymbol{E}}_0$ and $\widetilde{\boldsymbol{B}}_0$.

9.2.2 Monochromatic Plane Wave

- Constraints on $\tilde{\boldsymbol{E}}_0$ and $\tilde{\boldsymbol{B}}_0$
  - Gauss law $\nabla \cdot \boldsymbol{E} = 0$
    - $\nabla \cdot \hat{\boldsymbol{E}} = \nabla \cdot \left(\tilde{\boldsymbol{E}}_0 e^{i(kz-\omega t)}\right)$
    - $= \nabla \cdot \left[\left(\tilde{E}_{0,x} e^{i(kz-\omega t)}\right)\hat{x} + \left(\tilde{E}_{0,y} e^{i(kz-\omega t)}\right)\hat{y} + \left(\tilde{E}_{0,z} e^{i(kz-\omega t)}\right)\hat{z}\right]$
    - $= \partial_x\left(\tilde{E}_{0,x} e^{i(kz-\omega t)}\right) + \partial_y\left(\tilde{E}_{0,y} e^{i(kz-\omega t)}\right) + \partial_z\left(\tilde{E}_{0,z} e^{i(kz-\omega t)}\right)$
    - $= \tilde{E}_{0,z}(ik)e^{i(kz-\omega t)} = 0$ ➔ $\tilde{E}_{0,z} = 0$
  - Similar for $\nabla \cdot \boldsymbol{B} = 0$ ➔ $\tilde{B}_{0,z} = 0$
  - The electric field and the magnetic field should be perpendicular to the direction of propagation

➔ Electromagnetic waves should be transverse

9.2.2 Monochromatic Plane Wave

- Constraints on $\tilde{\boldsymbol{E}}_0$ and $\tilde{\boldsymbol{B}}_0$ (cont'd)
  - Faraday's law $\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B}$
    - $\nabla \times \hat{\boldsymbol{E}} = \nabla \times \left(\tilde{\boldsymbol{E}}_0 e^{i(kz-\omega t)}\right)$

$$= \nabla \times \left[\left(\tilde{E}_{0,x} e^{i(kz-\omega t)}\right)\hat{x} + \left(\tilde{E}_{0,y} e^{i(kz-\omega t)}\right)\hat{y}\right]$$

$$= \hat{x}\left[\partial_y(0) - \partial_z\left(\tilde{E}_{0,y} e^{i(kz-\omega t)}\right)\right] + \hat{y}\left[\partial_z\left(\tilde{E}_{0,x} e^{i(kz-\omega t)}\right) - \partial_x(0)\right]$$
$$+ \hat{z}\left[\partial_x\left(\tilde{E}_{0,y} e^{i(kz-\omega t)}\right) - \partial_y\left(\tilde{E}_{0,x} e^{i(kz-\omega t)}\right)\right]$$

$$= \hat{x}\left[-\partial_z\left(\tilde{E}_{0,y} e^{i(kz-\omega t)}\right)\right] + \hat{y}\left[\partial_z\left(\tilde{E}_{0,x} e^{i(kz-\omega t)}\right)\right]$$

$$= \hat{x}\left(-\tilde{E}_{0,y}(ik)e^{i(kz-\omega t)}\right) + \hat{y}\left(\tilde{E}_{0,x}(ik)e^{i(kz-\omega t)}\right) = e^{i(kz-\omega t)}(ik)\left[\left(-\tilde{E}_{0,y}\right)\hat{x} + \tilde{E}_{0,x}\hat{y}\right]$$

    - $-\frac{\partial}{\partial t}\boldsymbol{B} = -\frac{\partial}{\partial t}\left(\tilde{\boldsymbol{B}}_0 e^{i(kz-\omega t)}\right)$

$$= -\frac{\partial}{\partial t}\left(\left(\tilde{B}_{0,x} e^{i(kz-\omega t)}\right)\hat{x} + \left(\tilde{B}_{0,y} e^{i(kz-\omega t)}\right)\hat{y}\right)$$

$$= \left(\tilde{B}_{0,x}(i\omega)e^{i(kz-\omega t)}\right)\hat{x} + \left(\tilde{B}_{0,y}(i\omega)e^{i(kz-\omega t)}\right)\hat{y} = e^{i(kz-\omega t)}(i\omega)\left[\tilde{B}_{0,x}\hat{x} + \tilde{B}_{0,y}\hat{y}\right]$$

  - $-k\tilde{E}_{0,y} = \omega\tilde{B}_{0,x}$, $k\tilde{E}_{0,x} = \omega\tilde{B}_{0,y}$

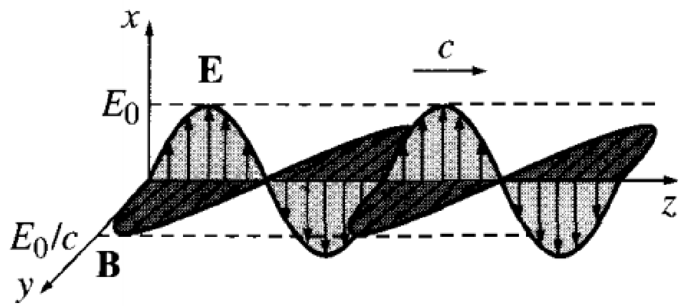
$$\omega\tilde{\boldsymbol{B}}_0 = \hat{z} \times \left(k\tilde{\boldsymbol{E}}_0\right) \Rightarrow \tilde{\boldsymbol{B}}_0 = \frac{k}{\omega}\left(\hat{z} \times \tilde{\boldsymbol{E}}_0\right)$$

9.2.2 Monochromatic Plane Wave

- Summary of constraints on $\tilde{\boldsymbol{E}}_0$ and $\tilde{\boldsymbol{B}}_0$
  - $\tilde{E}_{0,z} = 0$, $\tilde{B}_{0,z} = 0$
    - The electric field and the magnetic field should be perpendicular to the direction of propagation
  - $\tilde{\boldsymbol{B}}_0 = \frac{k}{\omega}\left(\hat{z} \times \tilde{\boldsymbol{E}}_0\right)$
    - $\boldsymbol{E}$ and $\boldsymbol{B}$ are in phase and mutually perpendicular

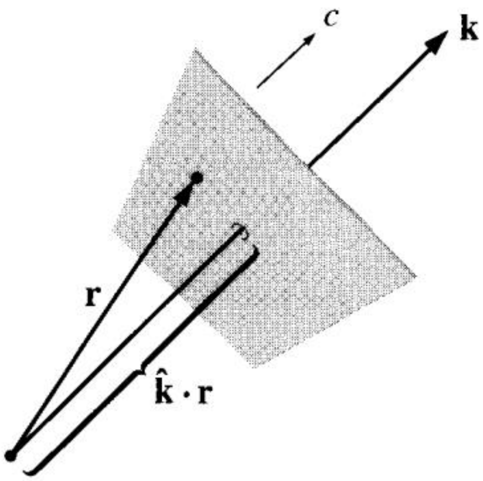
$$B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0$$

- Example 9.2
  - $\tilde{\boldsymbol{E}}(\boldsymbol{r}, t) = \tilde{\boldsymbol{E}}_0 e^{i(kz-\omega t)} = \tilde{E}_0 e^{i(kz-\omega t)} \hat{x} = E_0 e^{i\delta} e^{i(kz-\omega t)} \hat{x}$
  - $\tilde{\boldsymbol{B}}_0 = \frac{k}{\omega}\left(\hat{z} \times \tilde{\boldsymbol{E}}_0\right) = \frac{k}{\omega}\left(\hat{z} \times \left(E_0 e^{i\delta} \hat{x}\right)\right) = \frac{1}{c} E_0 e^{i\delta} \hat{y}$
  - $\boldsymbol{E}(\boldsymbol{r}, t) = E_0 \cos(kz - \omega t + \delta)\, \hat{x}$, $\boldsymbol{B}(\boldsymbol{r}, t) = \frac{1}{c} E_0 \cos(kz - \omega t + \delta)\, \hat{y}$



9.2.2 Monochromatic Plane Wave

- Generalization to an arbitrary propagation direction
  - z–direction is not special
  - Wave vector
    - Recall wave number k appearing $e^{i(kz-\omega t)}$
    - $\boldsymbol{k} \equiv k\hat{z}$ ➔ $kz = \boldsymbol{k} \cdot \boldsymbol{r}$
- Monochromatic plane wave propagating along $\hat{\boldsymbol{k}}$ direction
  - $\tilde{\boldsymbol{E}}(\boldsymbol{r}, t) = \tilde{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\hat{\boldsymbol{n}}$
  - $\tilde{\boldsymbol{B}}(\boldsymbol{r}, t) = \frac{1}{c}\tilde{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}}) = \frac{1}{c}\hat{\boldsymbol{k}} \times \tilde{\boldsymbol{E}}(\boldsymbol{r}, t)$
  - Derivation using tensor index notation
  - $\hat{\boldsymbol{n}}$: polarization vector and $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}} = \boldsymbol{0}$
- Actual (real) components
  - $\boldsymbol{E}(\boldsymbol{r}, t) = E_0 \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \delta)\, \hat{\boldsymbol{n}}$
  - $\boldsymbol{B}(\boldsymbol{r}, t) = \frac{1}{c} E_0 \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \delta)\, (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}})$



9.2.3 Energy and Momentum in EM Waves

- Energy density stored in EM fields (from ch. 8)
  - $u_{\text{em}} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$ and $\frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{em}}) = -\nabla \cdot \boldsymbol{S}$
  - A monochromatic plane wave:

$$\boldsymbol{E}(\boldsymbol{r}, t) = E_0 \cos(kz - \omega t + \delta)\,\hat{x}\ ,\ \boldsymbol{B}(\boldsymbol{r}, t) = \frac{1}{c} E_0 \cos(kz - \omega t + \delta)\,\hat{y}$$

  - $u_{\text{em}} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$

$$= \frac{1}{2}\left(\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) + \frac{1}{\mu_0}\frac{1}{c^2} E_0^2 \cos^2(kz - \omega t + \delta)\right)$$
$$= \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$$

  - Average energy density over many cycles:
    - Using $\cos 2\theta = 2\cos^2\theta - 1$,

$$\int_0^T \cos^2(kz - \omega t + \delta)\, dt = \frac{1}{2}\int_0^T \cos[2(kz - \omega t + \delta)] + 1 dt = \frac{T}{2}$$

$$\langle u_{em} \rangle = \frac{1}{T}\int_0^T u_{em} dt = \frac{1}{2}\epsilon_0 E_0^2$$

    - Note that it is the sum of the energy from both the electric field and the magnetic field for monochromatic plane wave

9.2.3 Energy and Momentum in EM Waves

- Energy flux density transported by the EM fields
  - Can you guess flow of energy per unit area, per unit time?
    - What about the direction?
  - Recall Poynting vector $\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E} \times \boldsymbol{B})$
  - A monochromatic plane wave:

$$\boldsymbol{E}(\boldsymbol{r}, t) = E_0 \cos(kz - \omega t + \delta)\,\hat{x}\ ,\ \boldsymbol{B}(\boldsymbol{r}, t) = \frac{1}{c} E_0 \cos(kz - \omega t + \delta)\,\hat{y}$$

  - $$\begin{aligned} \boldsymbol{S} &= \frac{1}{\mu_0}(\boldsymbol{E} \times \boldsymbol{B}) \\ &= \frac{1}{\mu_0}[E_0 \cos(kz - \omega t + \delta)\,\hat{x}] \times \left[\frac{1}{c} E_0 \cos(kz - \omega t + \delta)\,\hat{y}\right] \\ &= \frac{E_0^2}{\mu_0 c} \cos^2(kz - \omega t + \delta)\,\hat{z} = c\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{z} = c u_{em} \hat{z} \end{aligned}$$
  - Average energy flux density over many cycles:

$$\langle \boldsymbol{S} \rangle = \frac{1}{T}\int_0^T \boldsymbol{S} dt = \frac{1}{T}\frac{E_0^2}{\mu_0 c}\frac{T}{2}\hat{z} = \frac{1}{2} c\epsilon_0 E_0^2 \hat{z} = c\langle u_{em} \rangle \hat{z}$$

    - Intensity: the average power per unit area delivered by light

$$I \equiv \langle S \rangle = \frac{1}{2} c\epsilon_0 E_0^2$$

9.2.3 Energy and Momentum in EM Waves

- Momentum density carried by the EM fields
  - From ch. 8, $\boldsymbol{\wp}_{\mathrm{em}} \equiv \epsilon_0 \mu_0 \boldsymbol{S}$
  - A monochromatic plane wave:

$\boldsymbol{E}(\boldsymbol{r}, t) = E_0 \cos(kz - \omega t + \delta)\,\hat{x}$, $\boldsymbol{B}(\boldsymbol{r}, t) = \frac{1}{c} E_0 \cos(kz - \omega t + \delta)\,\hat{y}$
and $\boldsymbol{S} = c u_{em} \hat{z}$

  - $\boldsymbol{\wp}_{\mathrm{em}} \equiv \epsilon_0 \mu_0 \boldsymbol{S} = \frac{1}{c^2}(c u_{em} \hat{z}) = \frac{1}{c} u_{em} \hat{z}$
  - Average momentum density over many cycles:

$$\langle \boldsymbol{\wp}_{\mathrm{em}} \rangle = \frac{1}{T} \int_0^T \boldsymbol{\wp}_{\mathrm{em}}\, dt = \frac{1}{2c} \epsilon_0 E_0^2 \hat{z} = \frac{1}{c} \langle u_{em} \rangle \hat{z}$$

  - Radiation pressure (when absorbed)
    - In Δt, the momentum transfer is $\Delta \boldsymbol{p} = \langle \boldsymbol{\wp}_{\mathrm{em}} \rangle A (c \Delta t) = A \Delta t \langle u_{em} \rangle \hat{z}$.
    - Pressure (average force per unit area): $P = \frac{1}{A} \frac{\Delta p}{\Delta \mathrm{t}} = \langle u_{em} \rangle = \frac{I}{c}$
    - For a perfect reflector, pressure?

9.2.3 Energy and Momentum in EM Waves

- Summary
  - A monochromatic plane wave propagating along $\hat{\boldsymbol{k}}$:
    - $\boldsymbol{E}(\boldsymbol{r},t) = E_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta)\,\hat{\boldsymbol{n}}$
    - $\boldsymbol{B}(\boldsymbol{r},t) = \frac{1}{c} E_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta)\,(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}})$
    - $\tilde{\boldsymbol{E}}(\boldsymbol{r},t) = \tilde{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}\hat{\boldsymbol{n}}$, $\tilde{\boldsymbol{B}}(\boldsymbol{r},t) = \frac{1}{c}\hat{\boldsymbol{k}} \times \tilde{\boldsymbol{E}}(\boldsymbol{r},t)$
  - $u_{em} = \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$ and $\langle u_{em}\rangle = \frac{1}{2}\epsilon_0 E_0^2$
  - $\boldsymbol{\wp}_{\mathrm{em}} = \frac{1}{c} u_{em}\hat{z}$ and $\langle \boldsymbol{\wp}_{\mathrm{em}}\rangle = \frac{1}{c}\langle u_{em}\rangle \hat{z}$
- In QM, the momentum of a single photon is $\hbar\boldsymbol{k}$.
  - Can you derive this using the fact that the energy of a single photon is $\hbar\omega$, which was experimentally found by photoelectric effect and blackbody radiation combined with Planck's calculation?
- If you need to define operators that will work on a complex wave function, what will be the best guess for the energy and the momentum?