Review of Vector Calculus

- Tensor index notation
  - We will limit our discussion only to 3-dimension.
  - We will assume a fixed orthonormal basis and a Euclidean metric

    ➔ no distinction between the contravariant tensor and the covariant tensor ➔ only subscript index will be used
  - Instead of vector notation $\boldsymbol{v}$, vector will be represented by its components with index v_i : $i = 1,2,3$
  - Instead of matrix notation $\mathbb{A}$, matrix will be represented by its components with indices A_{ij} : $i = 1,2,3$
  - References (not an ideal one, but better than nothing)
    - https://www.brown.edu/Departments/Engineering/Courses/En221/Notes/Index_notation/Index_notation.htm
    - https://www.continuummechanics.org/tensornotationbasic.html

Einstein Summation Convention

- Einstein summation convention
  - Inner product of two vectors

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = \sum_{i=1}^{3} v_i w_i \equiv v_i w_i$$

  - If we have a repeated indices, summation from 1 to 3 is implied ➔ called dummy indices
    - $\boldsymbol{x} = \mathbb{A}\boldsymbol{v} = \sum_{i=1}^{3} \mathbb{A}_{ji}\boldsymbol{v}_i \equiv A_{ji} v_i = x_j$
    - Index j is not repeated ➔ called a free index
    - Summation is also called as **contraction**
  - Each index can appear at most twice in any term
  - Each term must contain identical non-repeated indices
  - Example
    - $A_{ij} w_j + B_{ijk} y_k z_j + x_i$ ➔ valid
    - $A_{ij} w_j x_j$, $B_{jk} y_k + x_i$, $B_{ik} + x_i$ ➔ invalid
    - $\mathrm{tr}(\mathbb{A}) = A_{jj}$
  - The number of free indices determines the order (or rank) of tensor.
    - We can tell us whether it is a scalar, vector, matrix, 3rd-order (or rank-3) tensor, … by counting the number of the free indices

Tensor Index Notation

- Kronecker delta
  - $\delta_{ij} = \begin{cases} 0, i \neq j \\ 1, i = j \end{cases}$
  - $\delta_{ij} x_i = x_j$
  - Useful to represent inner product: $a_k b_k = \delta_{jk} a_j b_k$
- Levi-Civita symbol
  - $\epsilon_{ijk} = \begin{cases} +1, \text{if } (i,j,k) \text{ are even permutation of } (1,2,3) \\ -1, \text{if } (i,j,k) \text{ are odd permutation of } (1,2,3) \\ 0, \text{if } i = j, \text{or } j = k, \text{or } k = i \end{cases}$
    - Even permutation of $(1,2,3)$: $(1,2,3), (2,3,1)$ or $(3,1,2)$ or rotation of $(1,2,3)$
    - Odd permutation of $(1,2,3)$: $(3,2,1), (1,3,2)$ or $(2,1,3)$ or rotation of $(3,2,1)$
  - Cross product
    - $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\hat{x} + (a_z b_x - a_x b_z)\hat{y} + (a_x b_y - a_y b_x)\hat{z}$
    - $c_i = \epsilon_{ijk} a_j b_k$ or $\boldsymbol{c} = \epsilon_{ijk} a_j b_k \hat{\imath}$

Example

- Prove that $\boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = 0$
  - $\boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \delta_{ij} a_i (\boldsymbol{a} \times \boldsymbol{b})_j = \delta_{ij} a_i \epsilon_{jkl} a_k b_l$
  - Note that the products in tensor index notation are commutative and associative, but the order of indices can be changed only by a rule.
  - $= \delta_{ij} \epsilon_{jkl} a_i a_k b_l = \epsilon_{ikl} a_i a_k b_l$
  - Dummy indices can be replaced by different letters
  - $= \epsilon_{mnl} a_m a_n b_l = -\epsilon_{nml} a_m a_n b_l = -\epsilon_{ikl} a_k a_i b_l$
  - $\epsilon_{ikl} a_i a_k b_l = -\epsilon_{ikl} a_k a_i b_l$
  - ➔ $\epsilon_{ikl} a_i a_k b_l + \epsilon_{ikl} a_k a_i b_l = 0 = 2\epsilon_{ikl} a_i a_k b_l$

Product of Two ϵ_{ijk}

- δ_{ii} =?
- The only relation you should memorize!
  - $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$
  - Check!
    - When does the left-hand side becomes non-zero?
    - When does the left-hand side becomes zero?
  - $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$, but $\epsilon_{ijk} \neq \epsilon_{ikj}$ for example
  - Example: proof of BAC-CAB rule
    - $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$
    - $\big(\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})\big)_i = \epsilon_{ijk} A_j (\boldsymbol{B} \times \boldsymbol{C})_k = \epsilon_{ijk} A_j \epsilon_{klm} B_l C_m$
    - Note that the products in tensor index notation are commutative and associative, but the order of indices cannot be changed easily.
    - $= \epsilon_{ijk}\epsilon_{klm} A_j B_l C_m = \big(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\big) A_j B_l C_m$
  - Prove that $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c})$

Diadic Product

- In tensor index notation, simply $v_i w_j$.
- Takes in two vectors and returns a second order tensor
- Corresponds to outer product such as $|v\rangle\langle w|$
- Works as a linear transformation (or operator) that maps a given vector to another vector.
- Example
  - $|v\rangle = \boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, |w\rangle = \boldsymbol{w} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$
  - $|v\rangle\langle w| = \boldsymbol{v}\boldsymbol{w}^{\dagger} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1*2 & 1*3 & 1*4 \\ 2*2 & 2*3 & 2*4 \\ 3*2 & 3*3 & 3*4 \end{bmatrix} \Leftrightarrow v_i w_j$
  - In the language of matrix, it is called a rank-1 matrix.

Cross Product vs. Outer Product

- Cross Product takes in two vectors and returns a vector (first-order tensor)
- Diadic (outer) Product takes in two vectors and returns a second-order tensor (like matrix)
- Part of cross product can be considered as linear transform (operator)
  - $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\hat{x} + (a_z b_x - a_x b_z)\hat{y} + (a_x b_y - a_y b_x)\hat{z}$
  - $\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$
  - We might consider $(\boldsymbol{a} \times)$ as a linear operator that transform the input vector $\boldsymbol{b}$ to another vector $(a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x)$
  - This linear operator (or matrix) has rank-2, so it can be considered as the sum of two outer products.

Notations

- All the following discussions and chapter numbers will be based on "Introduction to Electrodynamics", David. J. Griffiths, 3rd ed. (1999), unless otherwise specified.
- Notations
  - $\hat{x}$: unit vector along x-axis, and similarly $\hat{y}$ for y-axis and $\hat{z}$ for z-axis
  - $\boldsymbol{r} = x\hat{x} + y\hat{y} + z\hat{z}$: position vector in 3-dimensional space
  - $r = (x^2 + y^2 + z^2)^{1/2}$
  - $\boldsymbol{s} = x\hat{x} + y\hat{y}$: position vector in xy-plane
  - $s = (x^2 + y^2)^{1/2}$
  - $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z$: del (or nabla) operator

1.2 Differential Calculus

- Vector Derivatives in Cartesian Coordinates
- $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z$
- Assumption
  - $f(x, y, z)$ is a scalar function of position
  - $\boldsymbol{V}(x, y, z)$ is a vector function of position
    - $\boldsymbol{V}(x, y, z)$ has 3 components $V_x(x, y, z), V_y(x, y, z), V_z(x, y, z)$
    - $\boldsymbol{V}(x, y, z) = V_x(x, y, z)\hat{x} + V_y(x, y, z)\hat{y} + V_z(x, y, z)\hat{z}$
- Gradient
  - $\nabla f(x, y, z) = \frac{\partial f(x,y,z)}{\partial x}\hat{x} + \frac{\partial f(x,y,z)}{\partial y}\hat{y} + \frac{\partial f(x,y,z)}{\partial z}\hat{z} \Rightarrow \partial_i f$
- Divergence
  - $\nabla \cdot \boldsymbol{V}(x, y, z) = \frac{\partial}{\partial x}V_x(x, y, z) + \frac{\partial}{\partial y}V_y(x, y, z) + \frac{\partial}{\partial z}V_z(x, y, z) \Rightarrow \partial_i V_i$
- Curl
  - $\nabla \times \boldsymbol{V}(x, y, z) = \left(\frac{\partial}{\partial y}V_z - \frac{\partial}{\partial z}V_y\right)\hat{x} + \left(\frac{\partial}{\partial z}V_x - \frac{\partial}{\partial x}V_z\right)\hat{y} + \left(\frac{\partial}{\partial x}V_y - \frac{\partial}{\partial y}V_x\right)\hat{z} \Rightarrow \epsilon_{ijk}\partial_j V_k$
- Laplacian
  - $\nabla^2 f(x, y, z) = \frac{\partial^2}{\partial x^2}f + \frac{\partial^2}{\partial y^2}f + \frac{\partial^2}{\partial z^2}f \Rightarrow \partial_i\partial_i f$
- $\boldsymbol{r} = x\hat{x} + y\hat{y} + z\hat{z}$ ➔ occasionally I will use $x_i \equiv r_i$ ➔ $\partial_i r_j = \partial_i x_j = ?$
- If $\boldsymbol{V}(x, y, z) = \boldsymbol{r} = x\hat{x} + y\hat{y} + z\hat{z}$, $V_i = r_i \equiv x_i$ ➔ $\nabla \cdot \boldsymbol{V}(x, y, z) = \partial_i V_i = ?$

1.2.6 Product Rules

- Assumption: $\boldsymbol{A}(\boldsymbol{r}), \boldsymbol{B}(\boldsymbol{r})$ are vector functions of positions
- $\boldsymbol{\nabla} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B})$
- $\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B} + \boldsymbol{A}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{\nabla} \cdot \boldsymbol{A})$
- $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{0}$

1.3 Integral Calculus

- 1.3.3 The fundamental theorem for Gradients

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}} (\nabla f) \cdot d\boldsymbol{l} = f(\boldsymbol{b}) - f(\boldsymbol{a})$$

- 1.3.4 The fundamental theorem for divergences

$$\int_{\mathcal{V}} (\nabla \cdot \boldsymbol{V}) d\tau = \oint_{\mathcal{S}} \boldsymbol{V} \cdot d\boldsymbol{a}$$

- 1.3.5 The fundamental theorem for curls

$$\int_{\mathcal{S}} (\nabla \times \boldsymbol{V}) \cdot d\boldsymbol{a} = \oint_{\mathcal{P}} \boldsymbol{V} \cdot d\boldsymbol{l}$$