Recall:

- Maxwell's Equation (from 7.3.3. Maxwell's Equations)
  - $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$: Gauss's law for electric field (ρ is charge density)
  - $\nabla \cdot \boldsymbol{B} = 0$: Gauss's law for magnetic field
  - $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$: Faraday's law
  - $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$: Ampere's law ($\boldsymbol{J}$ is current density)

- Constitutive relation
  - $\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}$: $\boldsymbol{P}$ is polarization density (from 4.3 The Electric Displacement)
  - $\boldsymbol{H} = \left(\frac{1}{\mu_0}\right) \boldsymbol{B} - \boldsymbol{M}$: $\boldsymbol{M}$ is magnetization density (from 6.3 The Auxiliary Field H)

- Maxwell's Equation (from 7.3.5 Maxwell's Equations in Matter)
  - $\nabla \cdot \boldsymbol{D} = \rho_f$: Gauss's law for electric field (ρ_f is (free) charge density)
  - $\nabla \cdot \boldsymbol{B} = 0$: Gauss's law for magnetic field
  - $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$: Faraday's law
  - $\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}_f$: Ampere's law ($\boldsymbol{J}_f$ is (free) current density)

Linear and Isotropic Material

- Constitutive relation in linear and isotropic material
  - Polarization (from 4.4 Linear Dielectrics)
    - $\boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E}$: χ_e is called electric susceptibility
      ➔ $\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} = \epsilon_0 (1 + \chi_e) \boldsymbol{E}$
    - By defining permittivity of the medium $\epsilon \equiv \epsilon_0 (1 + \chi_e)$
      ➔ $\boldsymbol{D} \equiv \epsilon \boldsymbol{E}$
    - $\frac{\epsilon}{\epsilon_0} = 1 + \chi_e = \epsilon_r$ is also called as relative permittivity or more frequently dielectric constant

  - Magnetization (from 6.4 Linear and Nonlinear Media)
    - $\boldsymbol{M} = \chi_m \boldsymbol{H}$: χ_m is called magnetic susceptibility
      ➔ $\boldsymbol{B} = \mu_0 (\boldsymbol{H} + \boldsymbol{M}) = \mu_0 (1 + \chi_m) \boldsymbol{H}$
    - By defining permeability of the medium $\mu \equiv \mu_0 (1 + \chi_m)$
      ➔ $\boldsymbol{B} \equiv \mu \boldsymbol{H}$ or more frequently $\boldsymbol{H} = \frac{1}{\mu} \boldsymbol{B}$

9.3.1 Propagation in Linear Media

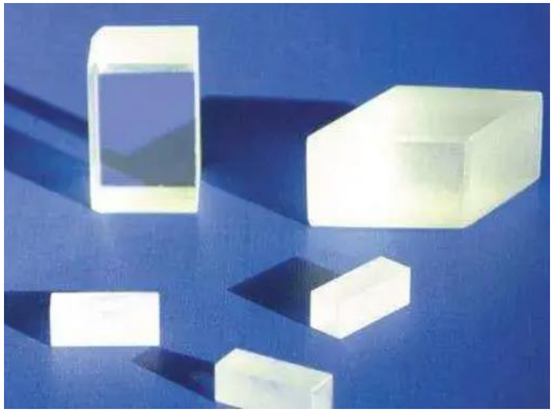
- Gauss's laws for linear and isotropic medium
  - $\nabla \cdot \boldsymbol{D} = \rho_f$
  - $\nabla \cdot \boldsymbol{B} = 0$
  - $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$
  - $\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}_f$
- No charge & no current: $\rho = \boldsymbol{J} = 0$

  (i) $\nabla \cdot \boldsymbol{E} = 0$ (ii) $\nabla \cdot \boldsymbol{B} = 0$

  (iii) $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ (iv) $\nabla \times \boldsymbol{B} = \mu\epsilon \frac{\partial \boldsymbol{E}}{\partial t}$
- What is the difference from the last lecture?

  (i) $\nabla \cdot \boldsymbol{E} = 0$ (ii) $\nabla \cdot \boldsymbol{B} = 0$

  (iii) $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ (iv) $\nabla \times \boldsymbol{B} = \mu_0\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$
- What is the speed of the light in the linear medium?



Picture of KTP

From https://hncrystal.en.made-in-china.com/productimage/ovBJzghUaPYu-2f1jooenEQPWtRRCrG/China-Potassium-Titanyl-Phosphate-KTiOPO4-KTP-160-.html

9.3.1 Propagation in Linear Media

- Speed of light in the linear media
  - $v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}$
  - $n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$: refractive index
  - For most of materials, $n \approx \sqrt{\epsilon_r}$
  - ϵ_r is almost always greater than 1 ➔ light travels more slowly in matter
- Vacuum → Linear media: $\epsilon_0 \to \epsilon, \mu_0 \to \mu$ and $c \to v$
  - Energy density: $u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu}B^2\right) = \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}\cdot\boldsymbol{H})$
  - Poynting vector: $\boldsymbol{S} = \frac{1}{\mu}(\boldsymbol{E}\times\boldsymbol{B}) = (\boldsymbol{E}\times\boldsymbol{H})$
  - Intensity: $I = \frac{1}{2}\epsilon v E_0^2$
  - Proof: Prob. 8.15

Reflection and Transmission at the Boundary

- Behavior of field at the interface between two materials
  - Transmission
    - Familiar with refraction of the light by the water
    - Equilateral prisms (or dispersion prism)
  - Partial reflection by glass
    - Modeling of partial mirror in quantum optics (Hong-Ou-Mandel interference)
    - Polarization dependence
      - Sunglass
      - Brewster's angle
    - Total internal reflection
      - Dove prism
      - Fresnel rhoms
      - Multi-mode fiber
  - Mirrors
    - Dielectric mirrors vs metal mirrors

Transition of Notation

- From now on, I will switch from Griffiths book to Haus and Cheng books.
  - Griffiths: "Introduction to Electrodynamics", David. J. Griffiths, 3rd ed. (1999)
  - Haus: "Waves and Fields in Optoelectronics", Hermann A. Haus (1984)
  - Cheng: "Field and Wave Electromagnetics", David K. Cheng 2nd ed. (1989)
- Major changes of notations
  - Griffiths book
    - $\boldsymbol{E}, \boldsymbol{B}$ are the fundamental fields, and $\boldsymbol{D}, \boldsymbol{H}$ will be calculated, if necessary.
    - Exponential part of the primary field: $e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t+i\phi}$
    - Complex field: $\widetilde{\boldsymbol{E}}(\boldsymbol{r},t) = \left(E_0 e^{i\delta}\hat{x}\right)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \equiv \widetilde{\boldsymbol{E}}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$
  - Haus and Cheng
    - $\boldsymbol{E}, \boldsymbol{H}$ are the fundamental fields, and $\boldsymbol{D}, \boldsymbol{B}$ will be calculated, if necessary.
    - Exponential part of the primary field: $e^{j\omega t-j\boldsymbol{k}\cdot\boldsymbol{r}+j\phi}$
    - Complex field: $\widetilde{\boldsymbol{E}}(\boldsymbol{r},t) = \left(E_0 e^{i\delta}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\hat{x}\right)e^{i\omega t} \equiv \boldsymbol{E}(\boldsymbol{r})e^{i\omega t}$
      - We will mainly use $\boldsymbol{E}(\boldsymbol{r})$ only to represent electric field assuming the frequency is fixed ➔ phasor (impedance calculation) used in electric circuits

(Haus) 1.2 Complex Vectors

- Maxwell's equations in linear media ➔ Linear

  (i) $\nabla \cdot \boldsymbol{E} = \rho_f / \epsilon$ (ii) $\nabla \cdot \boldsymbol{H} = 0$

  (iii) $\nabla \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t}$ (iv) $\nabla \times \boldsymbol{H} = \epsilon \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{J}_f$

  - Sinusoidal excitation (e.g. a sinusoidal ρ_f or $\boldsymbol{J}_f$) at frequency ω

    ➔ sinusoidal response ($\boldsymbol{E}$ and $\boldsymbol{B}$ change with frequency ω)

- Physical interpretation of complex vectors (phasors)
  - Example: complex scalar $V = |V| e^{j\phi}$

    $$\Rightarrow v(t) = \mathrm{Re}\left[V e^{j\omega t}\right] = |V| \cos(\omega t + \phi)$$

  - Complex vector $\boldsymbol{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$

    $$\boldsymbol{A} = |A_x| e^{j\phi_x} \hat{x} + \left|A_y\right| e^{j\phi_y} \hat{y} + |A_z| e^{j\phi_z} \hat{z} = \mathrm{Re}[\boldsymbol{A}] + j\mathrm{Im}[\boldsymbol{A}]$$

    where $\mathrm{Re}[\boldsymbol{A}] = \mathrm{Re}[A_x]\hat{x} + \mathrm{Re}\left[A_y\right]\hat{y} + \mathrm{Re}[A_z]\hat{z}$ & $\mathrm{Im}[\boldsymbol{A}]$ are real vectors

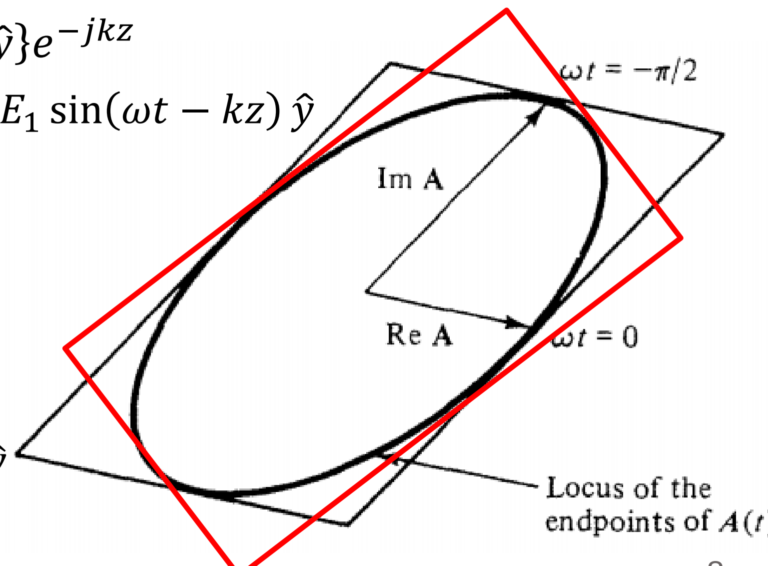
    ➔ $\boldsymbol{A}(t) = \mathrm{Re}\left[\boldsymbol{A} e^{j\omega t}\right]$

    $$= |A_x| \cos(\omega t + \phi_x)\, \hat{x} + \left|A_y\right| \cos\left(\omega t + \phi_y\right) \hat{y} + |A_z| \cos(\omega t + \phi_z)\, \hat{z}$$

    $$= \mathrm{Re}[\boldsymbol{A}] \cos \omega t - \mathrm{Im}[\boldsymbol{A}] \sin \omega t$$

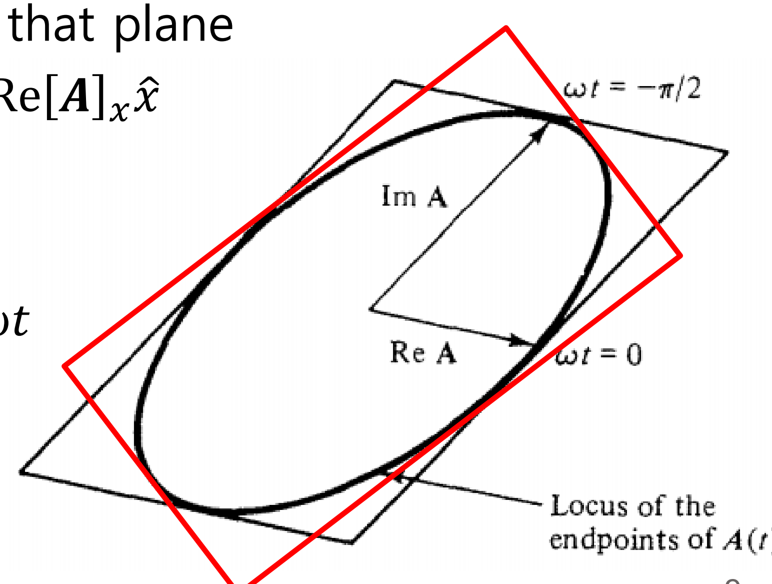
Representation of Polarization

- Recall
  - $\boldsymbol{E}(x,y,z,t) = E_0 \cos(\omega t - kz + \delta)\,\hat{x} = \mathrm{Re}\left[\{E_0 e^{j(\delta - kz)}\hat{x}\}e^{j\omega t}\right] = \mathrm{Re}\left[\widetilde{\boldsymbol{E}}(x,y,z)e^{j\omega t}\right]$
- $\widetilde{\boldsymbol{E}}(x,y,z) = \widetilde{\boldsymbol{E}}(\boldsymbol{r}) = E_0 e^{j\left(-\frac{\pi}{2} - kz\right)}\hat{y}$
  - $\boldsymbol{E}(\boldsymbol{r},t) = \mathrm{Re}\left[\widetilde{\boldsymbol{E}}(\boldsymbol{r})e^{j\omega t}\right] = E_0 \sin(\omega t - kz)\,\hat{y}$
  - Linear polarization
- $\widetilde{\boldsymbol{E}}(\boldsymbol{r}) = E_0 e^{j(-kz)}\hat{x} + E_0 e^{j\left(-\frac{\pi}{2} - kz\right)}\hat{y} = E_0\{\hat{x} - j\hat{y}\}e^{-jkz}$
  - $\boldsymbol{E}(\boldsymbol{r},t) = \mathrm{Re}\left[\widetilde{\boldsymbol{E}}(\boldsymbol{r})e^{j\omega t}\right] = E_0\{\cos(\omega t - kz)\,\hat{x} + \sin(\omega t - kz)\,\hat{y}\}$
  - Circular polarization
- $\widetilde{\boldsymbol{E}}(\boldsymbol{r}) = E_0 e^{j(-kz)}\hat{x} + E_1 e^{j\left(\frac{\pi}{2} - kz\right)}\hat{y} = \{E_0\hat{x} + jE_1\hat{y}\}e^{-jkz}$
  - $\boldsymbol{E}(\boldsymbol{r},t) = \mathrm{Re}\left[\widetilde{\boldsymbol{E}}(\boldsymbol{r})e^{j\omega t}\right] = E_0 \cos(\omega t - kz)\,\hat{x} - E_1 \sin(\omega t - kz)\,\hat{y}$
  - Elliptical polarization
  - Are $\hat{x}$ and $\hat{y}$ special direction?
- Arbitrary polarization
  - $\boldsymbol{A} = |A_x|e^{j\phi_x}\hat{x} + |A_y|e^{j\phi_y}\hat{y} + |A_z|e^{j\phi_z}\hat{z}$
  - Non-orthogonal axis: $\boldsymbol{A} = \mathrm{Re}[\boldsymbol{A}] + j\mathrm{Im}[\boldsymbol{A}]$
  - Non- $\frac{\pi}{2}$ phase difference: $\boldsymbol{A} = |\alpha|\hat{x} + |\beta|e^{j\phi}\hat{y}$
  - What about both? $\boldsymbol{A} = |\alpha|\hat{n} + |\beta|e^{j\phi}\hat{m}$



(Haus) 1.2 Complex Vectors

- Physical interpretation of complex vectors (phasors)
  - The locus of $\boldsymbol{A}(t)$ for all t forms an ellipse
  - $\boldsymbol{A}(t) = \mathrm{Re}\left[\boldsymbol{A}e^{j\omega t}\right] = \mathrm{Re}[\boldsymbol{A}]\cos\omega t - \mathrm{Im}[\boldsymbol{A}]\sin\omega t$
  - Both $\mathrm{Re}[\boldsymbol{A}]$ and $\mathrm{Im}[\boldsymbol{A}]$ are real vectors, but they are not necessarily orthogonal to each other
  - Vector $\boldsymbol{A}(t)$ must lie in the plane formed by $\mathrm{Re}[\boldsymbol{A}]$ and $\mathrm{Im}[\boldsymbol{A}]$.
    - 2D Cartesian coordinate system in that plane
    - Choose $\hat{x}$ along $\mathrm{Re}[\boldsymbol{A}]$ ➔ $\mathrm{Re}[\boldsymbol{A}] = \mathrm{Re}[\boldsymbol{A}]_x\hat{x}$
    - $\mathrm{Im}[\boldsymbol{A}] = \mathrm{Im}[\boldsymbol{A}]_x\hat{x} + \mathrm{Im}[\boldsymbol{A}]_y\hat{y}$
    - $\boldsymbol{A}(t) \equiv A_x(t)\hat{x} + A_y(t)\hat{y}$
    - $A_x(t) = \mathrm{Re}[\boldsymbol{A}]_x\cos\omega t - \mathrm{Im}[\boldsymbol{A}]_x\sin\omega t$
    - $A_y(t) = -\mathrm{Im}[\boldsymbol{A}]_y\sin\omega t$



(Haus) 1.2 Complex Vectors

- (cont'd) Physical interpretation of complex vectors (phasors)
  - What is the equation for ellipse?
    - $\alpha x^2 + \beta y^2 = \gamma$ whose major axis and minor axis are aligned along x- and y-axis and $\alpha, \beta, \gamma > 0$
    - $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \gamma$
  - From $A_y(t) = -\mathrm{Im}[\boldsymbol{A}]_y \sin \omega t$, $\sin \omega t = -\frac{A_y(t)}{\mathrm{Im}[\boldsymbol{A}]_y}$
  - From $A_x(t) = \mathrm{Re}[\boldsymbol{A}]_x \cos \omega t - \mathrm{Im}[\boldsymbol{A}]_x \sin \omega t$, $\cos \omega t = \frac{1}{\mathrm{Re}[\boldsymbol{A}]_x} \left\{ A_x(t) - \frac{\mathrm{Im}[\boldsymbol{A}]_x}{\mathrm{Im}[\boldsymbol{A}]_y} A_y(t) \right\}$
  - $\sin^2 \omega t + \cos^2 \omega t = 1$ ➔ $\left\{ \frac{A_y(t)}{\mathrm{Im}[\boldsymbol{A}]_y} \right\}^2 + \frac{1}{\mathrm{Re}[\boldsymbol{A}]_x^2} \left\{ A_x(t) - \frac{\mathrm{Im}[\boldsymbol{A}]_x}{\mathrm{Im}[\boldsymbol{A}]_y} A_y(t) \right\}^2 = 1$

➔ $\mathrm{Re}[\boldsymbol{A}]_x^2 A_y(t)^2 + \mathrm{Im}[\boldsymbol{A}]_y^2 A_x(t)^2 - 2\mathrm{Im}[\boldsymbol{A}]_y \mathrm{Im}[\boldsymbol{A}]_x A_x(t) A_y(t) + \mathrm{Im}[\boldsymbol{A}]_x^2 A_y(t)^2 = \mathrm{Im}[\boldsymbol{A}]_y^2 \mathrm{Re}[\boldsymbol{A}]_x^2$

➔ $\{\mathrm{Re}[\boldsymbol{A}]_x^2 + \mathrm{Im}[\boldsymbol{A}]_x^2\} A_y(t)^2 - 2\mathrm{Im}[\boldsymbol{A}]_y \mathrm{Im}[\boldsymbol{A}]_x A_x(t) A_y(t) + \mathrm{Im}[\boldsymbol{A}]_y^2 A_x(t)^2 = \mathrm{Im}[\boldsymbol{A}]_y^2 \mathrm{Re}[\boldsymbol{A}]_x^2$

  - $\begin{bmatrix} A_y(t) & A_x(t) \end{bmatrix} \begin{bmatrix} \mathrm{Re}[\boldsymbol{A}]_x^2 + \mathrm{Im}[\boldsymbol{A}]_x^2 & -\mathrm{Im}[\boldsymbol{A}]_y \mathrm{Im}[\boldsymbol{A}]_x \\ -\mathrm{Im}[\boldsymbol{A}]_y \mathrm{Im}[\boldsymbol{A}]_x & \mathrm{Im}[\boldsymbol{A}]_y^2 \end{bmatrix} \begin{bmatrix} A_y(t) \\ A_x(t) \end{bmatrix} = \mathrm{Im}[\boldsymbol{A}]_y^2 \mathrm{Re}[\boldsymbol{A}]_x^2$
  - Symmetric matrix ➔ diagonalizable with orthogonal transform
  - $\det \begin{bmatrix} a - \lambda & -b \\ -b & c - \lambda \end{bmatrix} = (\lambda - a)(\lambda - c) - b^2 = \lambda^2 - (a + c)\lambda + ac - b^2 = (\lambda - \lambda_1)(\lambda - \lambda_2)$
  - If positive (semi-)definite, $\lambda_1, \lambda_2 \geq 0 \Rightarrow (a + c) \geq 0, ac - b^2 \geq 0$
  - $ac - b^2 = \{\mathrm{Re}[\boldsymbol{A}]_x^2 + \mathrm{Im}[\boldsymbol{A}]_x^2\} \mathrm{Im}[\boldsymbol{A}]_y^2 - \mathrm{Im}[\boldsymbol{A}]_y^2 \mathrm{Im}[\boldsymbol{A}]_x^2 = \mathrm{Im}[\boldsymbol{A}]_y^2 \{\mathrm{Re}[\boldsymbol{A}]_x^2 + \mathrm{Im}[\boldsymbol{A}]_x^2 - \mathrm{Im}[\boldsymbol{A}]_x^2\} \geq 0$
  - $\begin{bmatrix} A_y(t) & A_x(t) \end{bmatrix} \mathbb{O} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \mathbb{O}^T \begin{bmatrix} A_y(t) \\ A_x(t) \end{bmatrix} = \gamma$

(Haus) 1.2 Complex Vectors

- Time average
  - Time average of products of sinusoidally time-dependent scalars
    - Example: V $= |V|e^{j\phi_V}, I = |I|e^{j\phi_I}$

$\rightarrow$ $v(t) = \mathrm{Re}\left[Ve^{j\omega t}\right] = |V|\cos(\omega t + \phi_V)\,, i(t) = \mathrm{Re}\left[Ie^{j\omega t}\right] = |I|\cos(\omega t + \phi_I)$

$$\langle v(t)i(t)\rangle = \frac{1}{T}\int_0^T v(t)i(t)dt$$

$$= \frac{1}{T}\int_0^T \frac{1}{2}\left(Ve^{j\omega t} + V^*e^{-j\omega t}\right)\frac{1}{2}\left(Ie^{j\omega t} + I^*e^{-j\omega t}\right)dt$$

$$= \frac{1}{4T}\int_0^T \left\{VIe^{j2\omega t} + VI^* + V^*I + (VI)^*e^{-j2\omega t}\right\}dt$$

$$= \frac{1}{4T}T(VI^* + V^*I) = \frac{1}{2}\frac{VI^* + V^*I}{2} = \frac{1}{2}\mathrm{Re}[VI^*]$$

(Haus) 1.2 Complex Vectors

- (cont'd) Time average
  - Time average of products of sinusoidally time-dependent vectors

$$\langle \boldsymbol{A}(t) \times \boldsymbol{B}(t) \rangle = \frac{1}{T}\int_0^T \boldsymbol{A}(t) \times \boldsymbol{B}(t) dt$$

$$= \frac{1}{T}\int_0^T \frac{1}{2}\left(\boldsymbol{A}e^{j\omega t} + \boldsymbol{A}^* e^{-j\omega t}\right) \times \frac{1}{2}\left(\boldsymbol{B}e^{j\omega t} + \boldsymbol{B}^* e^{-j\omega t}\right) dt$$

$$= \frac{1}{4T}\int_0^T \left\{(\boldsymbol{A} \times \boldsymbol{B})e^{j2\omega t} + \boldsymbol{A} \times \boldsymbol{B}^* + \boldsymbol{A}^* \times \boldsymbol{B} + (\boldsymbol{A}^* \times \boldsymbol{B}^*)e^{-j2\omega t}\right\} dt$$

$$= \frac{1}{4T} T(\boldsymbol{A} \times \boldsymbol{B}^* + \boldsymbol{A}^* \times \boldsymbol{B}) = \frac{1}{2}\frac{\boldsymbol{A} \times \boldsymbol{B}^* + \boldsymbol{A}^* \times \boldsymbol{B}}{2} = \frac{1}{2}\mathrm{Re}[\boldsymbol{A} \times \boldsymbol{B}^*]$$

1.4 The Complex Form of Maxwell's Eqs

- $\boldsymbol{E}(\boldsymbol{r},t), \boldsymbol{B}(\boldsymbol{r},t), \boldsymbol{J}(\boldsymbol{r},t), \rho(\boldsymbol{r},t) \Rightarrow \boldsymbol{E}(\boldsymbol{r})e^{j\omega t}, \boldsymbol{B}(\boldsymbol{r})e^{j\omega t}, \boldsymbol{J}(\boldsymbol{r})e^{j\omega t}, \rho(\boldsymbol{r})e^{j\omega t}$
  - $\nabla \cdot \left(\epsilon \boldsymbol{E}(\boldsymbol{r},t)\right) = \rho \Rightarrow \nabla \cdot \left(\epsilon \boldsymbol{E}(\boldsymbol{r})\right) = \rho$
  - $\nabla \cdot \left(\mu \boldsymbol{H}(\boldsymbol{r},t)\right) = 0 \Rightarrow \nabla \cdot \left(\mu \boldsymbol{H}(\boldsymbol{r})\right) = 0$
  - $\nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\mu \frac{\partial \boldsymbol{H}(\boldsymbol{r},t)}{\partial t} \Rightarrow \nabla \times \boldsymbol{E}(\boldsymbol{r}) = -j\omega\mu \boldsymbol{H}(\boldsymbol{r})$
  - $\nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \epsilon \frac{\partial \boldsymbol{E}(\boldsymbol{r},t)}{\partial t} + \boldsymbol{J}(\boldsymbol{r},t) \Rightarrow \nabla \times \boldsymbol{H}(\boldsymbol{r}) = j\omega\epsilon \boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{J}(\boldsymbol{r})$

- No charge & no current ($\rho = \boldsymbol{J} = 0$) & spatially uniform ϵ, μ
  - $\nabla \times \left(\nabla \times \boldsymbol{E}(\boldsymbol{r})\right) = -j\omega\mu\left(\nabla \times \boldsymbol{H}(\boldsymbol{r})\right) = \omega^2 \mu\epsilon \boldsymbol{E}(\boldsymbol{r})$
  - $\nabla \times \left(\nabla \times \boldsymbol{E}(\boldsymbol{r})\right) = \nabla\left(\nabla \cdot \boldsymbol{E}(\boldsymbol{r})\right) - \nabla^2 \boldsymbol{E}(\boldsymbol{r})$
  - $\nabla^2 \boldsymbol{E}(\boldsymbol{r}) + \omega^2 \mu\epsilon \boldsymbol{E}(\boldsymbol{r}) = 0$

- For plane wave,
  - $\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_+ e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}$ & $\boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{H}_+ e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}$

1.4 The Complex Form of Maxwell's Eqs

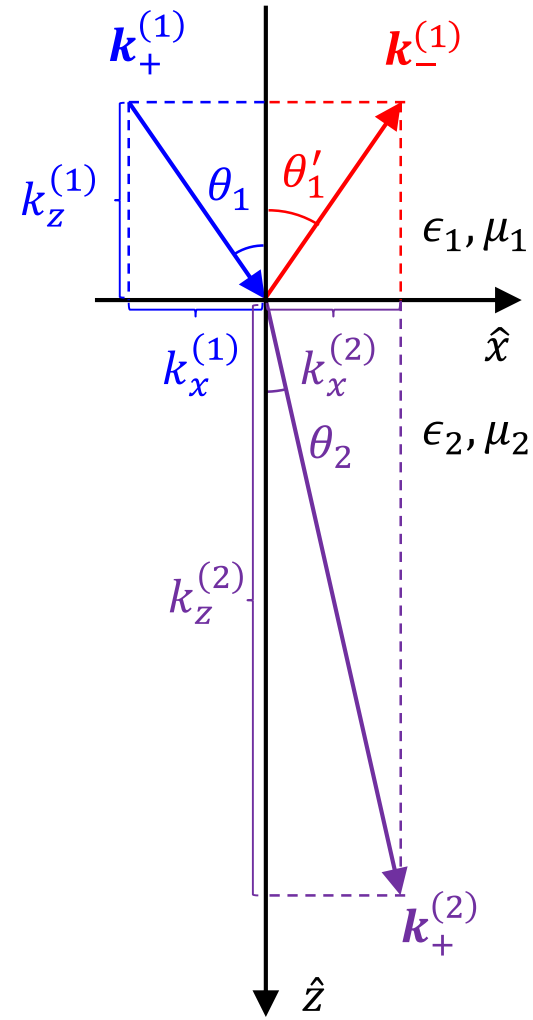
- Plane wave in source-free medium
  - $\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_+ e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}$ & $\boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{H}_+ e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}$ ➔ ∇ is replaced by $-j\boldsymbol{k}$
  - $\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = 0$ ➔ $-j\boldsymbol{k} \cdot \boldsymbol{E}_+ = 0$
  - $\nabla \cdot \boldsymbol{H}(\boldsymbol{r}) = 0$ ➔ $-j\boldsymbol{k} \cdot \boldsymbol{H}_+ = 0$
  - $\nabla \times \boldsymbol{E}(\boldsymbol{r}) = -j\omega\mu\boldsymbol{H}(\boldsymbol{r})$ ➔ $-j\boldsymbol{k} \times \boldsymbol{E}_+ = -j\omega\mu\boldsymbol{H}_+$
  - $\nabla \times \boldsymbol{H}(\boldsymbol{r}) = j\omega\epsilon\boldsymbol{E}(\boldsymbol{r})$ ➔ $-j\boldsymbol{k} \times \boldsymbol{H}_+ = j\omega\epsilon\boldsymbol{E}_+$
  - $\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_+) = \boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{E}_+) - \boldsymbol{E}_+(\boldsymbol{k} \cdot \boldsymbol{k}) = \boldsymbol{k} \times (\omega\mu\boldsymbol{H}_+) = -\omega^2\mu\epsilon\boldsymbol{E}_+$
    - $\boldsymbol{E}_+ k^2 = \omega^2\mu\epsilon\boldsymbol{E}_+$ ➔ $k^2 = \omega^2\mu\epsilon$ called dispersion relation
  - $\boldsymbol{H}_+ = \dfrac{1}{\omega\mu}\boldsymbol{k} \times \boldsymbol{E}_+ = \sqrt{\dfrac{\epsilon}{\mu}}\,\widehat{\boldsymbol{k}} \times \boldsymbol{E}_+$
  - (Intrinsic) impedance of medium: $\eta = \dfrac{|E|}{|H|} = \sqrt{\dfrac{\mu}{\epsilon}}$

Reflection and Transmission

- Boundary conditions (no charge, no current)
  - $D_1^{\perp} - D_2^{\perp} = 0$, $\boldsymbol{E}_1^{\parallel} - \boldsymbol{E}_2^{\parallel} = 0$
  - $B_1^{\perp} - B_2^{\perp} = 0$, $\boldsymbol{H}_1^{\parallel} - \boldsymbol{H}_2^{\parallel} = 0$
- Monochromatic, plane wave
  - Plane of incidence
  - Incident wave: $E_+^{(1)} e^{j\left(\omega_+^{(1)} t - \boldsymbol{k}_+^{(1)} \cdot \boldsymbol{r}\right)}$
  - Reflected wave: $E_-^{(1)} e^{j\left(\omega_-^{(1)} t - \boldsymbol{k}_-^{(1)} \cdot \boldsymbol{r}\right)}$
  - Transmitted wave: $E_+^{(2)} e^{j\left(\omega_+^{(2)} t - \boldsymbol{k}_+^{(2)} \cdot \boldsymbol{r}\right)}$
  - $\omega_+^{(1)} = \omega_-^{(1)} = \omega_+^{(2)} = \omega$
  - Tangential E & H continuous at all points on the boundary

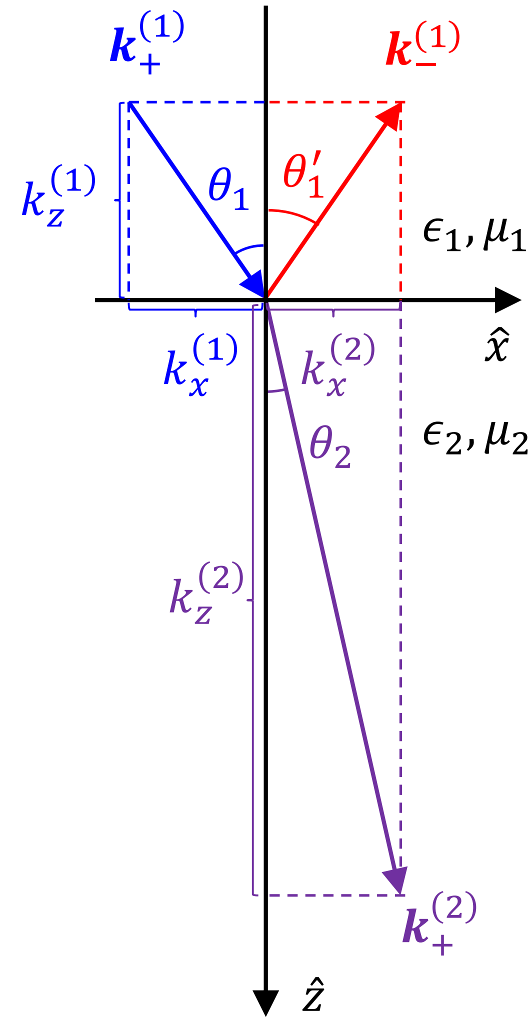
  ➔ $k_{x+}^{(1)} = k_{x-}^{(1)} = k_{x+}^{(2)} = k_x$

  ➔ $k_{y+}^{(1)} = k_{y-}^{(1)} = k_{y+}^{(2)} = k_y$
  - What determines $k_z^{(2)}$?



Reflection and Transmission

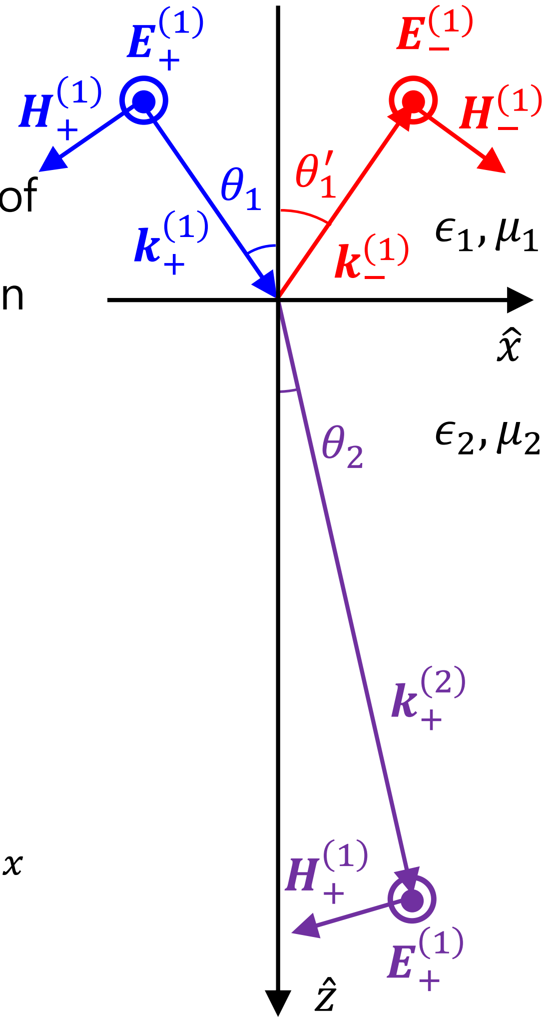
- Consequences
  1) All waves have the same ω
  2) All wave vectors are in one plane ➔ plane of incidence
  3) The angle of reflection θ_1' is the same as angle of incidence θ_1
     - $k_+^{(1)} \sin\theta_1 = k_-^{(1)} \sin\theta_1'$ ➔ $\theta_1' = \theta_1$
  4) The angle of transmission θ_2 is related to the angle of incidence θ_1 by $k_+^{(1)} \sin\theta_1 = k_+^{(2)} \sin\theta_2$
     - $k_+^{(1)} = k_-^{(1)} = k^{(1)} = \omega\sqrt{\mu_1\epsilon_1} = \omega\frac{n_1}{c}$
     - $k_+^{(2)} = k^{(2)} = \omega\sqrt{\mu_2\epsilon_2} = \omega\frac{n_2}{c}$
     - $n_1 \sin\theta_1 = n_2 \sin\theta_2$ or $\sqrt{\mu_1\epsilon_1} \sin\theta_1 = \sqrt{\mu_2\epsilon_2} \sin\theta_2$ ➔ Snell's law



(Haus) 2.1 TE Wave Reflected from Boundary

- Transverse electric wave reflected from boundary
  - Consider a plane wave with E-field polarized parallel to the surface of an interface between two media
    - What should be the electric polarization direction of the reflected and transmitted fields?
    - What should be the magnetic polarization direction of the reflected and transmitted fields?

  ➔ determined by boundary conditions
    - $k_{x+}^{(1)} = k_{x-}^{(1)} = k_{x+}^{(2)} = k_x$
    - $k_{z+}^{(1)} = k_{z-}^{(1)} = k_z^{(1)}$
  - Incident: $\hat{y}E_+^{(1)} e^{-j\boldsymbol{k}_+^{(1)}\cdot\boldsymbol{r}} = \hat{y}E_+^{(1)} e^{-jk_z^{(1)}z} e^{-jk_x x}$
  - Reflected: $\hat{y}E_-^{(1)} e^{-j\boldsymbol{k}_-^{(1)}\cdot\boldsymbol{r}} = \hat{y}E_-^{(1)} e^{jk_z^{(1)}z} e^{-jk_x x}$
  - Transmitted: $\hat{y}E_+^{(2)} e^{-j\boldsymbol{k}_+^{(2)}\cdot\boldsymbol{r}} = \hat{y}E_+^{(2)} e^{-jk_z^{(2)}z} e^{-jk_x x}$
  - Region 1: $E_y^{(1)} = \left(E_+^{(1)} e^{-jk_z^{(1)}z} + E_-^{(1)} e^{jk_z^{(1)}z}\right) e^{-jk_x x}$
  - Region 2: $E_y^{(2)} = E_+^{(2)} e^{-jk_z^{(2)}z} e^{-jk_x x}$

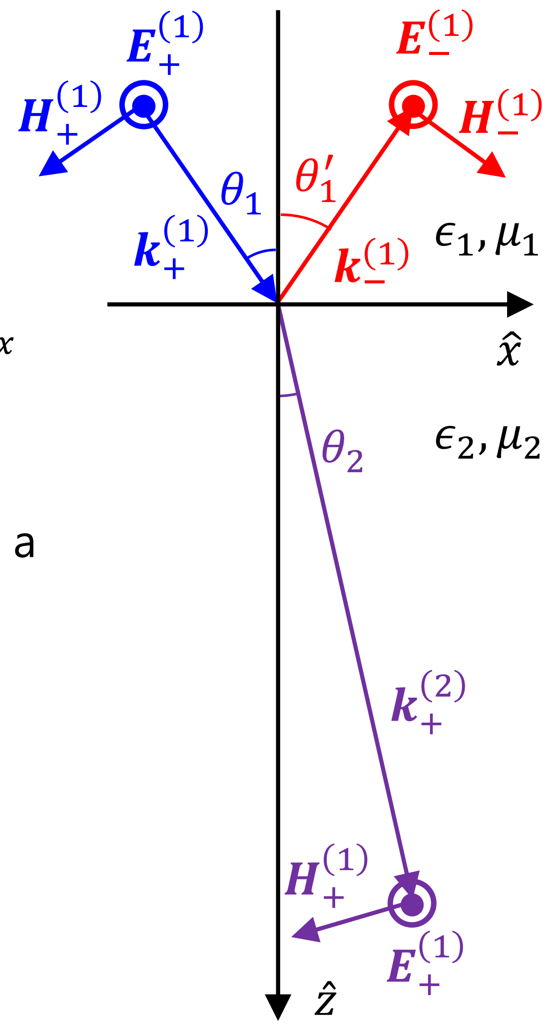


(Haus) 2.1 TE Wave Reflected from Boundary

- Region 1: $E_y^{(1)} = \left(E_+^{(1)} e^{-jk_z^{(1)}z} + E_-^{(1)} e^{jk_z^{(1)}z}\right) e^{-jk_x x}$
- Region 2: $E_y^{(2)} = E_+^{(2)} e^{-jk_z^{(2)}z} e^{-jk_x x}$
- In general, from $\nabla \times \boldsymbol{E} = -j\omega\mu\boldsymbol{H}$,

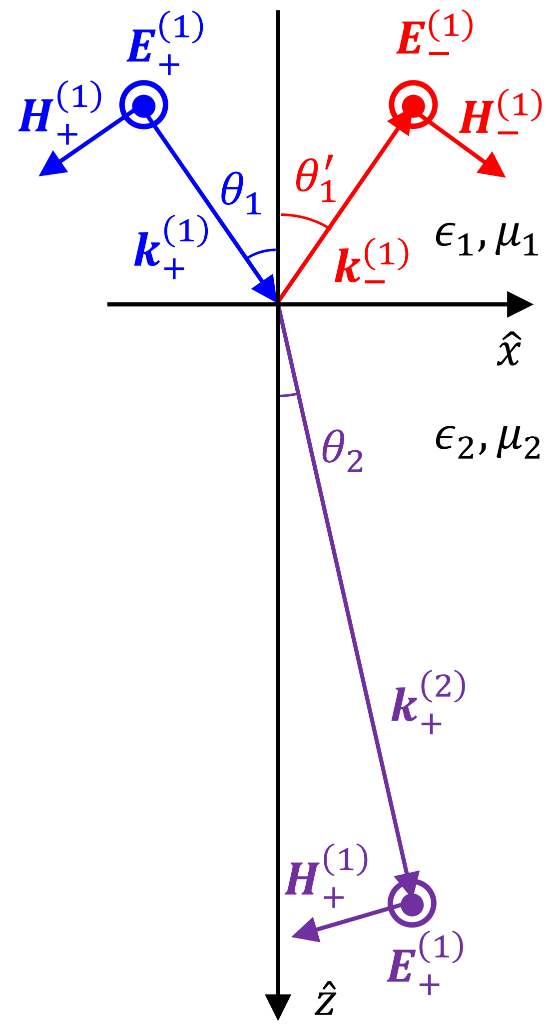
$$H_x = -\frac{1}{j\omega\mu}\left[\partial_y E_z - \partial_z E_y\right] = \frac{(jk_z)}{j\omega\mu} E_y = \frac{(k_z)}{\omega\mu} E_y$$

- Region 1: $H_x^{(1)} = \frac{k_z^{(1)}}{\omega\mu}\left(-E_+^{(1)} e^{-jk_z^{(1)}z} + E_-^{(1)} e^{jk_z^{(1)}z}\right) e^{-jk_x x}$
- Region 2: $H_x^{(2)} = -\frac{k_z^{(2)}}{\omega\mu} E_+^{(2)} e^{-jk_z^{(2)}z} e^{-jk_x x}$
- Characteristic **impedance** presented by medium 1 to a TE wave at inclination θ_1 with respect to the z axis:
  - For TE, $Z_0^{(1)} \equiv \frac{\omega\mu_1}{k_z^{(1)}} = \frac{\omega\mu_1}{\omega\sqrt{\epsilon_1\mu_1}\cos\theta_1} = \sqrt{\frac{\mu_1}{\epsilon_1}}\frac{1}{\cos\theta_1} \sim \frac{E_y}{H_x}$
  - cf) (intrinsic) **impedance** of medium: $\eta = \frac{|E|}{|H|} = \sqrt{\frac{\mu}{\epsilon}}$
- Wave **impedance** of TE: $Z(z) = -\frac{\text{Total } E_y(z)}{\text{Total } H_x(z)}$



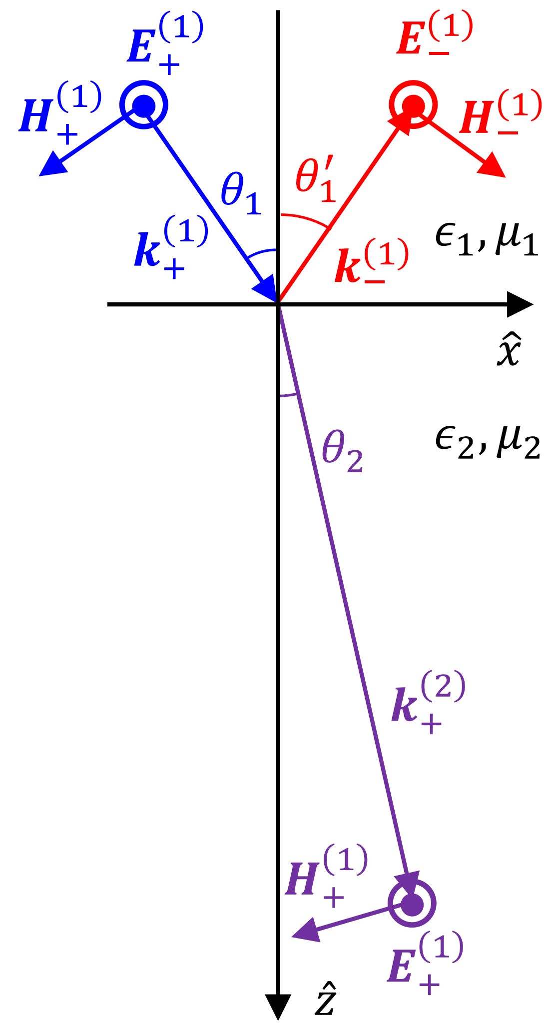
(Haus) 2.1 TE Wave Reflected from Boundary

- Region 1
  - $E_y^{(1)} = \left(E_+^{(1)} e^{-jk_z^{(1)} z} + E_-^{(1)} e^{jk_z^{(1)} z}\right) e^{-jk_x x}$
  - $H_x^{(1)} = -\frac{k_z^{(1)}}{\omega\mu}\left(E_+^{(1)} e^{-jk_z^{(1)} z} - E_-^{(1)} e^{jk_z^{(1)} z}\right) e^{-jk_x x}$
- Region 2
  - $E_y^{(2)} = E_+^{(2)} e^{-jk_z^{(2)} z} e^{-jk_x x}$
  - $H_x^{(2)} = -\frac{k_z^{(2)}}{\omega\mu} E_+^{(2)} e^{-jk_z^{(2)} z} e^{-jk_x x}$
- Characteristic impedance of TE: $Z_0^{(1)} \equiv \frac{\omega\mu_1}{k_z^{(1)}} = \sqrt{\frac{\mu_1}{\epsilon_1}} \frac{1}{\cos\theta_1}$
- At $z = 0$ for all x values, $E_y^{(1)} = E_y^{(2)}, H_x^{(1)} = H_x^{(2)}$
  - $E_+^{(1)} + E_-^{(1)} = E_+^{(2)}$
  - $\frac{1}{Z_0^{(1)}}\left(E_+^{(1)} - E_-^{(1)}\right) = \frac{1}{Z_0^{(2)}} E_+^{(2)}$
- Reflection coefficient: $\Gamma \equiv E_-^{(1)}/E_+^{(1)}$
- Transmission coefficient: $T \equiv E_+^{(2)}/E_+^{(1)}$
  - $1 + \Gamma = T$
  - $\frac{1}{Z_0^{(1)}}(1 - \Gamma) = \frac{1}{Z_0^{(2)}} T$
  - ➔ $T = \frac{2Z_0^{(2)}}{Z_0^{(2)} + Z_0^{(1)}}, \Gamma = \frac{Z_0^{(2)} - Z_0^{(1)}}{Z_0^{(2)} + Z_0^{(1)}}$



(Haus) 2.1 TE Wave Reflected from Boundary

- $Z_0^{(1)} \equiv \frac{\omega\mu_1}{k_z^{(1)}} = \sqrt{\frac{\mu_1}{\epsilon_1}}\frac{1}{\cos\theta_1} \rightarrow \frac{Z_0^{(1)}}{Z_0^{(2)}} = \frac{\omega\mu_1/k_z^{(1)}}{\omega\mu_2/k_z^{(2)}} = \frac{\mu_1 k_z^{(2)}}{\mu_2 k_z^{(1)}}$
- $\Gamma = \frac{Z_0^{(2)}-Z_0^{(1)}}{Z_0^{(2)}+Z_0^{(1)}} = \frac{1-Z_0^{(1)}/Z_0^{(2)}}{1+Z_0^{(1)}/Z_0^{(2)}} = \frac{1-\left(\mu_1 k_z^{(2)}\right)/\left(\mu_2 k_z^{(1)}\right)}{1+\left(\mu_1 k_z^{(2)}\right)/\left(\mu_2 k_z^{(1)}\right)}$
- $T = \frac{2Z_0^{(2)}}{Z_0^{(2)}+Z_0^{(1)}} = \frac{2}{1+Z_0^{(1)}/Z_0^{(2)}} = \frac{2}{1+\left(\mu_1 k_z^{(2)}\right)/\left(\mu_2 k_z^{(1)}\right)}$
- $\Gamma = \frac{1/Z_0^{(1)}-1/Z_0^{(2)}}{1/Z_0^{(1)}+1/Z_0^{(2)}} = \frac{\sqrt{\epsilon_1/\mu_1}\cos\theta_1-\sqrt{\epsilon_2/\mu_2}\cos\theta_2}{\sqrt{\epsilon_1/\mu_1}\cos\theta_1+\sqrt{\epsilon_2/\mu_2}\cos\theta_2}$
- Using $n = \sqrt{\epsilon/\epsilon_0}$ assuing $\mu_1 \approx \mu_2 \approx \mu_0$,
- $\Gamma = \frac{n_1\cos\theta_1-n_2\cos\theta_2}{n_1\cos\theta_1+n_2\cos\theta_2}$
- From Snell's law $n_1\sin\theta_1 = n_2\sin\theta_2$,
  - $\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1$ or $\cos\theta_2 = \sqrt{1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_1}$
  - $n_2\cos\theta_2 = \sqrt{n_2^2-n_1^2\sin^2\theta_1}$
- $\Gamma = \frac{n_1\cos\theta_1-\sqrt{n_2^2-n_1^2\sin^2\theta_1}}{n_1\cos\theta_1+\sqrt{n_2^2-n_1^2\sin^2\theta_1}}, T = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1+\sqrt{n_2^2-n_1^2\sin^2\theta_1}}$



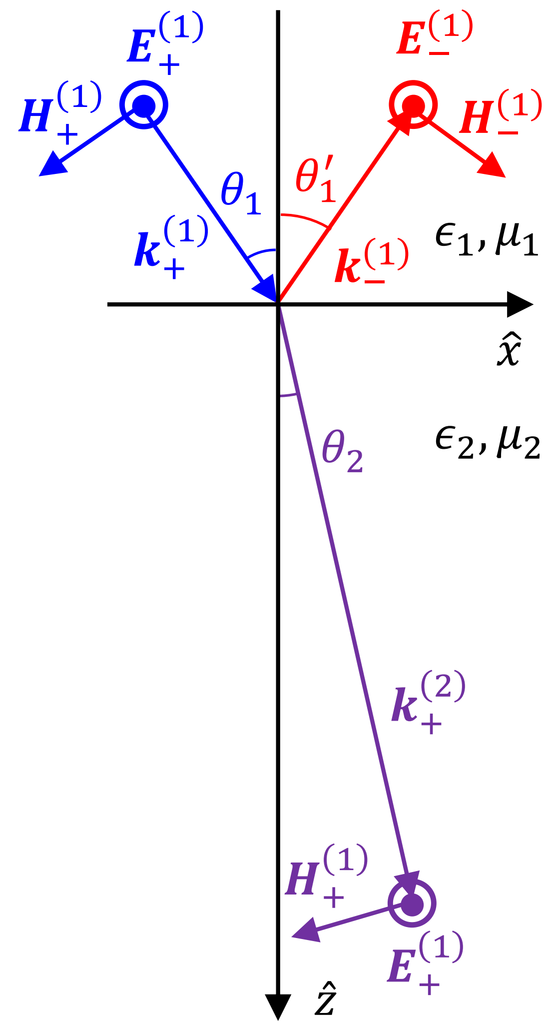
Power Flow

- $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}^*$
- Density of power flow along z-axis: $S_z = E_x H_y^* - E_y H_x^*$
- Reflectance: ratio of the reflected to incident power flow
  - $r = -\hat{z} \cdot \boldsymbol{S}_-^{(1)} / \hat{z} \cdot \boldsymbol{S}_+^{(1)}$
  - $r = E_{y-}^{(1)} H_{x-}^{(1)*} / \left(-E_{y+}^{(1)} H_{x+}^{(1)*}\right)$
  - $r = \left(\Gamma E_{y+}^{(1)}\right)\left(\frac{1}{Z_0^{(1)}} \Gamma E_{y+}^{(1)}\right)^* / \left(-E_{y+}^{(1)} \cdot \left(\frac{-1}{Z_0^{(1)}} E_{y+}^{(1)}\right)^*\right) = |\Gamma|^2$
- Transmittance: ratio of the transmitted to incident power flow
  - $t = \hat{z} \cdot \boldsymbol{S}_+^{(2)} / \hat{z} \cdot \boldsymbol{S}_+^{(1)}$
  - $t = \left(-E_{y+}^{(2)} H_{x+}^{(2)*}\right) / \left(-E_{y+}^{(1)} H_{x+}^{(1)*}\right)$
  - $t = \left(T E_{y+}^{(1)}\right)\left(\frac{-1}{Z_0^{(2)}} T E_{y+}^{(1)}\right)^* / \left(E_{y+}^{(1)} \cdot \left(\frac{-1}{Z_0^{(1)}} E_{y+}^{(1)}\right)^*\right) = |T|^2 \frac{Z_0^{(1)}}{Z_0^{(2)}}$

Signs of Γ and T

- For TE waves,
  - $$\Gamma = \frac{n_1 \cos\theta_1 - \sqrt{n_2^2 - n_1^2 \sin^2\theta_1}}{n_1 \cos\theta_1 + \sqrt{n_2^2 - n_1^2 \sin^2\theta_1}}, T = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + \sqrt{n_2^2 - n_1^2 \sin^2\theta_1}}$$
- Sign of Γ
  - Consider $n_2^2 - n_1^2 \sin^2\theta_1 > 0$ case
  - $(n_1 \cos\theta_1)^2 - (n_2^2 - n_1^2 \sin^2\theta_1) = n_1^2 - n_2^2$
  - If $n_1 < n_2$, medium 2 is called more dense
    - Reflection coefficient $\Gamma < 0$ ➔ $\boldsymbol{E}_-^{(1)} = \Gamma \boldsymbol{E}_+^{(1)}$

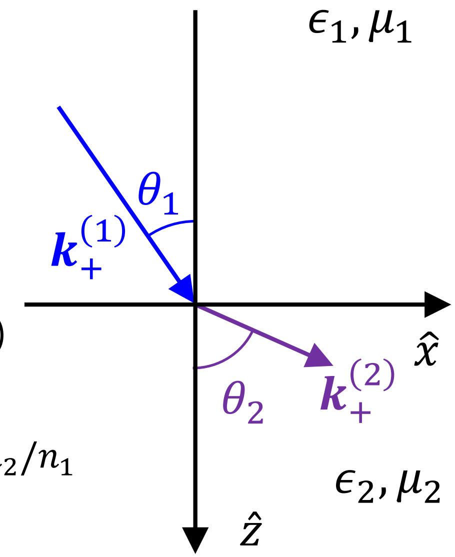
      ➔ the phase of reflected beam is shifted by 180°
    - Propagation from 2 to 1 should be in-phase
- Sign of T
  - $\boldsymbol{E}_+^{(2)} = T\boldsymbol{E}_+^{(1)}$ ➔ transmission is always in-phase



(Haus) 2.3 Total Internal Reflection

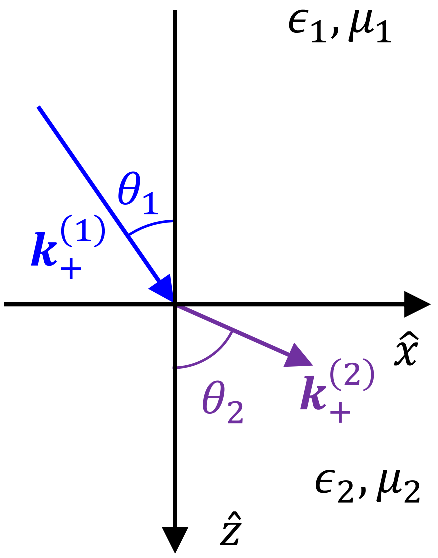
- For TE waves,
  - $\Gamma = \dfrac{n_1 \cos\theta_1 - \sqrt{n_2^2 - n_1^2 \sin^2\theta_1}}{n_1 \cos\theta_1 + \sqrt{n_2^2 - n_1^2 \sin^2\theta_1}}, T = \dfrac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + \sqrt{n_2^2 - n_1^2 \sin^2\theta_1}}$
- $n_2^2 - n_1^2 \sin^2\theta_1 < 0$?
  - It can happen only when $n_1 > n_2$.
  - Ex) seen from inside of water ➔ total internal reflection (TIR)
  - Snell's law $n_1 \sin\theta_1 = n_2 \sin\theta_2$ ➔ $\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1$

    ➔ problem when θ_1 becomes larger than the critical angle $\sin\theta_c = n_2/n_1$
  - Interpretation with wave vectors
    - $k^2 = \omega^2\mu\epsilon = \omega^2\frac{1}{v^2}$ ➔ $k = \frac{\omega}{v} = \frac{\omega}{c/n} = k_0 n$
    - $n_1 > n_2$ ➔ $k^{(1)} = k_0 n_1 > k^{(2)} = k_0 n_2$
  - From boundary condition: $k_{x+}^{(1)} = k_{x+}^{(2)}$
    - When θ_1 becomes larger than $\sin\theta_c = n_2/n_1$, $k_{x+}^{(2)} > k^{(2)}$
    - From $k^{(2)} = \sqrt{\left(k_{x+}^{(2)}\right)^2 + \left(k_{z+}^{(2)}\right)^2}$,
      - $k_{z+}^{(2)}$ should become imaginary $k_{z+}^{(2)} = \pm\sqrt{(k^{(2)})^2 - \left(k_{x+}^{(2)}\right)^2} = \pm j\alpha_z^{(2)}$



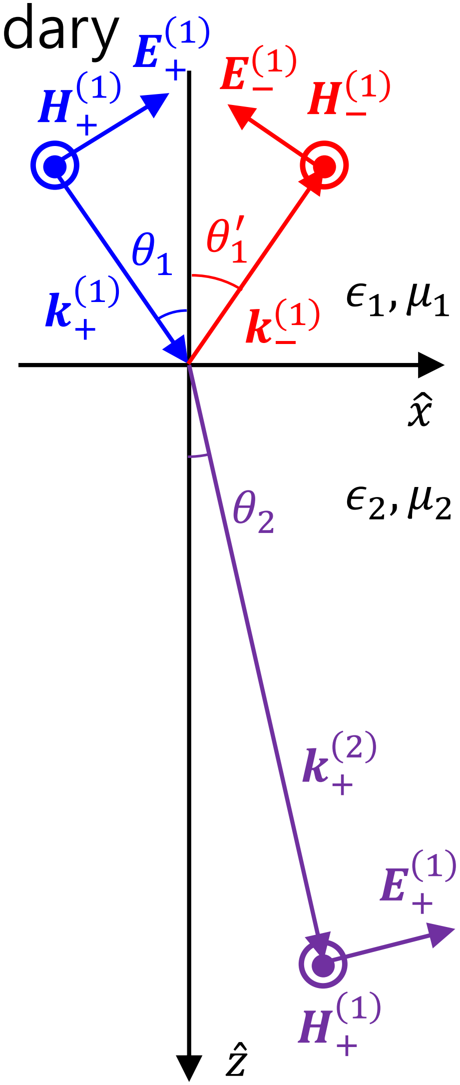
Phases of Γ and T

- For TE waves,
  - $\Gamma = \dfrac{n_1\cos\theta_1 - \sqrt{n_2^2 - n_1^2\sin^2\theta_1}}{n_1\cos\theta_1 + \sqrt{n_2^2 - n_1^2\sin^2\theta_1}} = \dfrac{1/Z_0^{(1)} - 1/Z_0^{(2)}}{1/Z_0^{(1)} + 1/Z_0^{(2)}} = \dfrac{Y_0^{(1)} - Y_0^{(2)}}{Y_0^{(1)} + Y_0^{(2)}}$
  - $T = \dfrac{2n_1\cos\theta_1}{n_1\cos\theta_1 + \sqrt{n_2^2 - n_1^2\sin^2\theta_1}} = \dfrac{2/Z_0^{(1)}}{1/Z_0^{(1)} + 1/Z_0^{(2)}} = \dfrac{2Y_0^{(1)}}{Y_0^{(1)} + Y_0^{(2)}}$
- Reflection coefficient
  - $\Gamma = \dfrac{Y_0^{(1)} - jX_0^{(2)}}{Y_0^{(1)} + jX_0^{(2)}} = \dfrac{|A|e^{j\phi}}{|A|e^{-j\phi}} = e^{j2\phi}$ ➔ $|\Gamma| = 1$, $\arg\Gamma = 2\phi(\theta_1, n_1, n_2)$
  - $\boldsymbol{E}_-^{(1)} = \Gamma\boldsymbol{E}_+^{(1)}$ ➔ the phase shift of reflected beam is determined by θ_1
    ➔ homework about Fresnel rohm phase retarder
- Transmitted wave
  - $T = \dfrac{2Y_0^{(1)}}{Y_0^{(1)} + jX_0^{(2)}} = |T|e^{j\phi}$
  - With $k_{x+}^{(1)} = k_{x+}^{(2)} = k_x$ and $k_{z+}^{(2)} = \sqrt{(k^{(2)})^2 - (k_x)^2} = -j\alpha_z^{(2)}$
  - $E_y^{(2)} = E_+^{(2)} e^{-jk_z^{(2)}z} e^{-jk_x x} = TE_+^{(1)} e^{-j\left(-j\alpha_z^{(2)}\right)z} e^{-jk_x x} = |T|e^{j\phi} E_+^{(1)} e^{-\alpha_z^{(2)}z} e^{-jk_x x}$
  - $H_x^{(2)} = -\dfrac{k_z^{(2)}}{\omega\mu} E_+^{(2)} e^{-jk_z^{(2)}z} e^{-jk_x x} = \dfrac{\alpha_z^{(2)}}{\omega\mu} |T|e^{j\phi} E_+^{(1)} e^{-\alpha_z^{(2)}z} e^{-jk_x x}$



(Haus) 2.2 TM Wave Reflected from Boundary

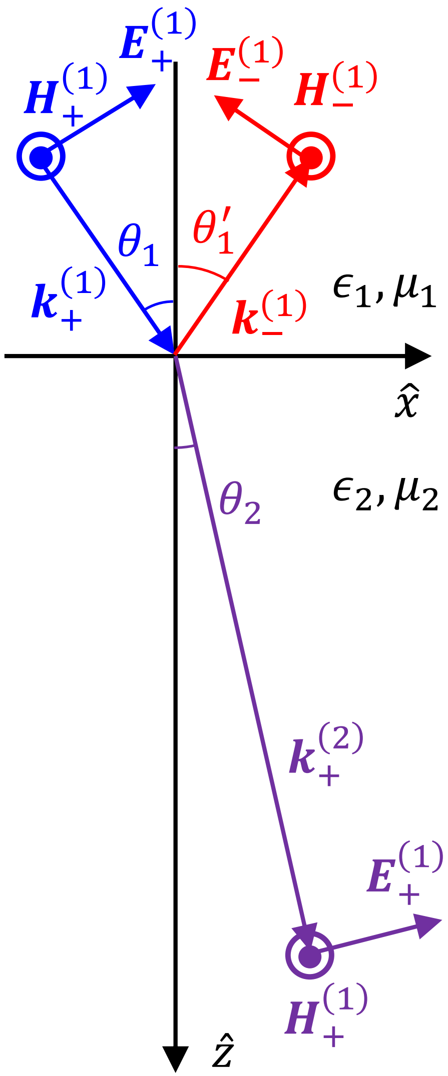
- Transverse magnetic wave reflected from boundary
  - Consider a plane wave with H-field polarized parallel to the surface of an interface between two media
    - $\boldsymbol{H}_+^{(1)} \parallel \boldsymbol{H}_-^{(1)} \parallel \boldsymbol{H}_+^{(2)}$
    - $k_{x+}^{(1)} = k_{x-}^{(1)} = k_{x+}^{(2)} = k_x$, $k_{z+}^{(1)} = k_{z-}^{(1)} = k_z^{(1)}$
  - Region 1
    - $H_y^{(1)} = \left(H_+^{(1)} e^{-jk_z^{(1)}z} + H_-^{(1)} e^{jk_z^{(1)}z}\right) e^{-jk_x x}$
    - Ampere's law $\nabla \times \boldsymbol{H} = j\omega\epsilon \boldsymbol{E}$ ➔ $E_x = \frac{1}{j\omega\epsilon}\left(-\partial_z H_y\right)$
    - $E_x^{(1)} = \frac{k_z^{(1)}}{\omega\epsilon}\left(H_+^{(1)} e^{-jk_z^{(1)}z} - H_-^{(1)} e^{jk_z^{(1)}z}\right) e^{-jk_x x}$
  - Region 2
    - $H_y^{(2)} = H_+^{(2)} e^{-jk_z^{(2)}z} e^{-jk_x x}$
    - $E_x^{(2)} = \frac{k_z^{(2)}}{\omega\epsilon} H_+^{(2)} e^{-jk_z^{(2)}z} e^{-jk_x x}$
  - Characteristic **impedance** presented by medium 1 to a TM wave at inclination θ_1 with respect to the z axis:
    - For TM, $Z_0^{(1)} \equiv \frac{k_z^{(1)}}{\omega\epsilon} = \frac{\omega\sqrt{\epsilon\mu}\cos\theta_1}{\omega\epsilon} = \sqrt{\frac{\mu}{\epsilon}}\cos\theta_1 \sim \frac{E_x}{H_y}$



(Haus) 2.2 TM Wave Reflected from Boundary

- Region 1
  - $H_y^{(1)} = \left(H_+^{(1)} e^{-jk_z^{(1)} z} + H_-^{(1)} e^{jk_z^{(1)} z}\right) e^{-jk_x x}$
  - $E_x^{(1)} = \frac{k_z^{(1)}}{\omega\epsilon}\left(H_+^{(1)} e^{-jk_z^{(1)} z} - H_-^{(1)} e^{jk_z^{(1)} z}\right) e^{-jk_x x}$
- Region 2
  - $H_y^{(2)} = H_+^{(2)} e^{-jk_z^{(2)} z} e^{-jk_x x}$
  - $E_x^{(2)} = \frac{k_z^{(2)}}{\omega\epsilon} H_+^{(2)} e^{-jk_z^{(2)} z} e^{-jk_x x}$
- Characteristic impedance of TM: $Z_0^{(1)} \equiv \frac{k_z^{(1)}}{\omega\epsilon_1} = \sqrt{\frac{\mu_1}{\epsilon_1}} \cos\theta_1$
- At $z = 0$ for all x values, $H_y^{(1)} = H_y^{(2)}, E_x^{(1)} = E_x^{(2)}$
  - $H_+^{(1)} + H_-^{(1)} = H_+^{(2)}$
  - $Z_0^{(1)}\left(H_+^{(1)} - H_-^{(1)}\right) = Z_0^{(2)} H_+^{(2)}$
- Reflection coefficient of TM: $\Gamma^{\mathrm{TM}} \equiv -H_-^{(1)}/H_+^{(1)}$
- Transmission coefficient: $T^{\mathrm{TM}} \equiv H_+^{(2)}/H_+^{(1)}$
  - $1 - \Gamma = T$
  - $Z_0^{(1)}(1 + \Gamma) = Z_0^{(2)} T$

➔ $T = \frac{2Z_0^{(1)}}{Z_0^{(2)}+Z_0^{(1)}}, \Gamma = \frac{Z_0^{(2)}-Z_0^{(1)}}{Z_0^{(2)}+Z_0^{(1)}}$



Duality

- Duality

  (i) $\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = 0$ (ii) $\nabla \times \boldsymbol{E}(\boldsymbol{r}) = -j\omega\mu\boldsymbol{H}(\boldsymbol{r})$

  (iii) $\nabla \cdot \boldsymbol{H}(\boldsymbol{r}) = 0$ (iv) $\nabla \times \boldsymbol{H}(\boldsymbol{r}) = j\omega\epsilon\boldsymbol{E}(\boldsymbol{r})$

  - $\boldsymbol{E} \rightarrow \boldsymbol{H},\ \boldsymbol{H} \rightarrow -\boldsymbol{E},\ \mu \Leftrightarrow \epsilon$

- Transverse Electric wave

  - $Z_0^{(1)} \equiv \frac{\omega\mu_1}{k_z^{(1)}} = \sqrt{\frac{\mu_1}{\epsilon_1}}\frac{1}{\cos\theta_1}$

  $\Rightarrow \frac{Z_0^{(1)}}{Z_0^{(2)}} = \frac{\omega\mu_1/k_z^{(1)}}{\omega\mu_2/k_z^{(2)}} = \frac{\mu_1 k_z^{(2)}}{\mu_2 k_z^{(1)}}$

  - $\Gamma^{\mathrm{TE}} = \frac{Z_0^{(2)} - Z_0^{(1)}}{Z_0^{(2)} + Z_0^{(1)}} = \frac{1 - Z_0^{(1)}/Z_0^{(2)}}{1 + Z_0^{(1)}/Z_0^{(2)}}$

  $= \frac{1 - \left(\mu_1 k_z^{(2)}\right)/\left(\mu_2 k_z^{(1)}\right)}{1 + \left(\mu_1 k_z^{(2)}\right)/\left(\mu_2 k_z^{(1)}\right)}$

  - $T^{\mathrm{TE}} = \frac{2Z_0^{(2)}}{Z_0^{(2)} + Z_0^{(1)}} = \frac{2}{1 + Z_0^{(1)}/Z_0^{(2)}}$

  $= \frac{2}{1 + \left(\mu_1 k_z^{(2)}\right)/\left(\mu_2 k_z^{(1)}\right)}$

- Transverse Magnetic wave

  - $Z_0^{(1)} \equiv \frac{k_z^{(1)}}{\omega\epsilon_1} = \sqrt{\frac{\mu_1}{\epsilon_1}}\cos\theta_1$

  $\Rightarrow \frac{Z_0^{(2)}}{Z_0^{(1)}} = \frac{k_z^{(2)}/\omega\epsilon_2}{k_z^{(1)}/\omega\epsilon_1} = \frac{\epsilon_1 k_z^{(2)}}{\epsilon_2 k_z^{(1)}}$

  - $-\Gamma^{\mathrm{TM}} = \frac{Z_0^{(1)} - Z_0^{(2)}}{Z_0^{(1)} + Z_0^{(2)}} = \frac{1 - Z_0^{(2)}/Z_0^{(1)}}{1 + Z_0^{(2)}/Z_0^{(1)}}$

  $= \frac{1 - \left(\epsilon_1 k_z^{(2)}\right)/\left(\epsilon_2 k_z^{(1)}\right)}{1 + \left(\epsilon_1 k_z^{(2)}\right)/\left(\epsilon_2 k_z^{(1)}\right)}$

  - $T^{\mathrm{TM}} = \frac{2Z_0^{(1)}}{Z_0^{(2)} + Z_0^{(1)}} = \frac{2}{1 + Z_0^{(2)}/Z_0^{(1)}}$

  $= \frac{2}{1 + \left(\epsilon_1 k_z^{(2)}\right)/\left(\epsilon_2 k_z^{(1)}\right)}$

Brewster's Angle

- Transverse Electric field
  - Define $\beta \equiv \frac{n_2}{n_1}$
  - For $\Gamma^{\mathrm{TE}} = \frac{Z_0^{(2)} - Z_0^{(1)}}{Z_0^{(2)} + Z_0^{(1)}}$ to become zero, $Z_0^{(2)} = Z_0^{(1)}$

$\rightarrow \sqrt{\frac{\mu_1}{\epsilon_1}} \frac{1}{\cos\theta_1} = \sqrt{\frac{\mu_2}{\epsilon_2}} \frac{1}{\cos\theta_2}$ $\rightarrow$ $\sqrt{\epsilon_1}\cos\theta_1 = \sqrt{\epsilon_2}\cos\theta_2$

$\rightarrow \cos\theta_2 = \frac{\sqrt{\epsilon_1}}{\sqrt{\epsilon_2}}\cos\theta_1 = \left(\frac{n_1}{n_2}\right)\cos\theta_1 = \frac{1}{\beta}\cos\theta_1$

  - Snell's law $\sin\theta_2 = \left(\frac{n_1}{n_2}\right)\sin\theta_1 = \frac{1}{\beta}\sin\theta_1$
  - $1 = \cos^2\theta_2 + \sin^2\theta_2 = \frac{1}{\beta^2}\cos^2\theta_2 + \frac{1}{\beta^2}\sin^2\theta_2 = \frac{1}{\beta^2}$
  - $\beta = 1$ $\rightarrow$ impossible with two different materials
- Transverse Magnetic field
  - For $\Gamma^{\mathrm{TM}} = \frac{Z_0^{(2)} - Z_0^{(1)}}{Z_0^{(2)} + Z_0^{(1)}}$ to become zero, $Z_0^{(2)} = Z_0^{(1)}$

$\rightarrow \sqrt{\frac{\mu_1}{\epsilon_1}}\cos\theta_1 = \sqrt{\frac{\mu_2}{\epsilon_2}}\cos\theta_2$ $\rightarrow$ $\frac{1}{\sqrt{\epsilon_1}}\cos\theta_1 = \frac{1}{\sqrt{\epsilon_2}}\cos\theta_2$

$\rightarrow \cos\theta_2 = \left(\frac{n_2}{n_1}\right)\cos\theta_1 = \beta\cos\theta_1$

$\rightarrow 1 = \cos^2\theta_2 + \sin^2\theta_2 = \beta^2\cos^2\theta_1 + \frac{1}{\beta^2}\sin^2\theta_1$

$\rightarrow \beta^2 = \beta^4\cos^2\theta_1 + (1 - \cos^2\theta_1)$

$\rightarrow \beta^2 - 1 = (\beta^4 - 1)\cos^2\theta_1$

$\rightarrow \cos^2\theta_1 = \frac{1}{\beta^2+1}$ $\rightarrow$ $1 + \tan^2\theta_1 = \frac{1}{\cos^2\theta_1} = 1 + \beta^2$

$\rightarrow \tan\theta_B = \frac{n_2}{n_1}$

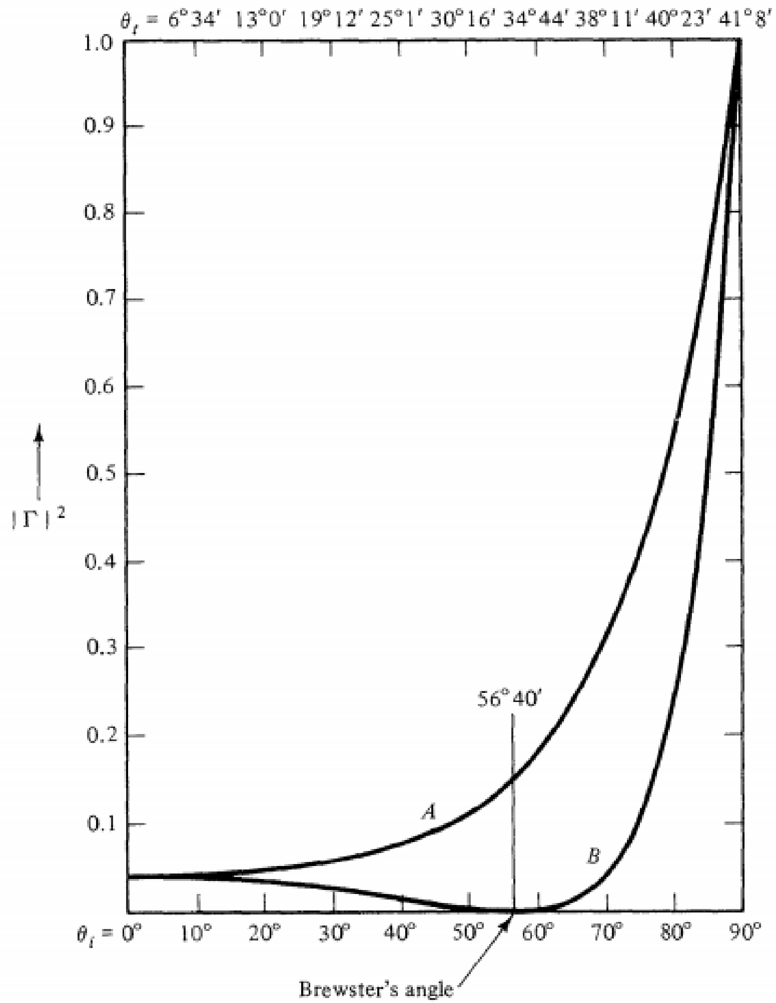


Figure 2.2 Square of reflection coefficient as a function of the angle of incidence: (a) curve A, TE; (b) curve B, TM. $n = 1.52$ index of glass. (After O. D. Chwolson, *Lehrbuch der Physik*, Vol. II, No. 2, 2nd ed., Braunschweig, Vieweg, 1922, p. 716.)