전자기 hw 1 sol.

#1

$$\vec{A} = \hat{x} - 2\hat{y} + 3\hat{z}$$

$$\vec{B} = \hat{x} + \hat{y} - 2\hat{z}$$

want

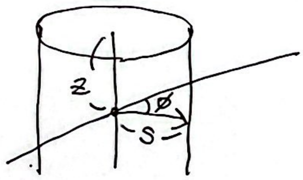
$$\vec{C} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} = \frac{\hat{x} + 5\hat{y} + 3\hat{z}}{\sqrt{1+25+9}}$$

$$= \frac{1}{\sqrt{35}}\left(\hat{x} + 5\hat{y} + 3\hat{z}\right) \quad \square$$

#2.

$$\nabla^2\vec{V} = \nabla(\nabla\cdot\vec{V}) - \nabla\times(\nabla\times\vec{V})$$

A. cylindrical coordinates



$$\vec{V} = V_s\hat{s} + V_\phi\hat{\phi} + V_z\hat{z}$$

① $\nabla(\nabla\cdot\vec{V})$

$$\nabla\cdot\vec{V} = \underbrace{\frac{1}{s}\frac{\partial}{\partial s}(sV_s) + \frac{1}{s}\frac{\partial V_\phi}{\partial\phi} + \frac{\partial V_z}{\partial z}}_{(*)}$$

$$\Rightarrow \nabla(\nabla\cdot\vec{V})$$

$$= \frac{\partial}{\partial s}\left[(*)\right]\hat{s}$$

$$+ \frac{1}{s}\frac{\partial}{\partial\phi}\left[(*)\right]\hat{\phi}$$

$$+ \frac{\partial}{\partial z}\left[(*)\right]\hat{z}$$

② $\nabla\times(\nabla\times\vec{V})$

$$\nabla\times\vec{V} = \overbrace{\left(\frac{1}{s}\frac{\partial V_z}{\partial\phi} - \frac{\partial V_\phi}{\partial z}\right)}^{①}\hat{s}$$

$$+ \underbrace{\left(\frac{\partial V_s}{\partial z} - \frac{\partial V_z}{\partial s}\right)}_{②}\hat{\phi}$$

$$+ \underbrace{\frac{1}{s}\left[\frac{\partial}{\partial s}(sV_\phi) - \frac{\partial V_s}{\partial\phi}\right]}_{③}\hat{z}$$

$$\Rightarrow \nabla\times(\nabla\times\vec{V})$$

$$= \left[\frac{1}{s}\frac{\partial}{\partial\phi}(③) - \frac{\partial}{\partial z}(②)\right]\hat{s}$$

$$+ \left[\frac{\partial}{\partial z}(①) - \frac{\partial}{\partial s}(③)\right]\hat{\phi}$$

$$+ \frac{1}{s}\left[\frac{\partial}{\partial s}(s\times②) - \frac{\partial}{\partial\phi}(①)\right]\hat{z}$$

약간의 노가다 계산을 거치면..

$$\nabla^2\vec{V} = \nabla(\nabla\cdot\vec{V}) - \nabla\times(\nabla\times\vec{V})$$

$$= \left(\frac{\partial^2 V_s}{\partial s^2} + \frac{1}{s^2}\frac{\partial^2 V_s}{\partial\phi^2} + \frac{\partial^2 V_s}{\partial z^2} + \frac{1}{s}\frac{\partial V_s}{\partial s} - \frac{2}{s^2}\frac{\partial V_\phi}{\partial\phi} - \frac{V_s}{s^2}\right)\hat{s}$$

$$+ \left(\frac{\partial^2 V_\phi}{\partial s^2} + \frac{1}{s^2}\frac{\partial^2 V_\phi}{\partial\phi^2} + \frac{\partial^2 V_\phi}{\partial z^2} + \frac{1}{s}\frac{\partial V_\phi}{\partial s} - \frac{V_\phi}{s^2} + \frac{2}{s^2}\frac{\partial V_s}{\partial\phi}\right)\hat{\phi}$$

$$+ \left(\frac{\partial^2 V_z}{\partial z^2} + \frac{\partial^2 V_z}{\partial s^2} + \frac{1}{s}\frac{\partial V_z}{\partial s} + \frac{1}{s^2}\frac{\partial^2 V_z}{\partial\phi^2}\right)\hat{z}$$

hw1 sol.

B. Spherical coordinates

비슷한 방법으로 계산, 답은 링크 참고.

(계산 과정 없으면 감점)

#3

A.

$$\int_{-\infty}^{\infty} f(x)\,\delta(ax)\,dx$$

set $ax = t \rightarrow a\,dx = dt$

$$\int_{-\infty}^{\infty} \frac{1}{a} f\left(\frac{t}{a}\right) \delta(t)\,dt$$

$$= \frac{1}{a} f(0)$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx$$

$$\therefore\ \delta(ax) = \frac{1}{a} \delta(x) \qquad \square$$

B.

$x - x_0 = t \rightarrow dx = dt$

$$\int_{-\infty}^{\infty} \delta(t) f(t + x_0)\,dt = f(x_0)$$

C.

$$\int_{-\infty}^{\infty} f(x)\,\delta(g(x))\,dx$$

$g(x) = t \rightarrow g'(x)\,dx = dt$

$$\int_{-\infty}^{\infty} f(g^{-1}(t)) \cdot \delta(t) \frac{dt}{|g'(g^{-1}(t))|}$$

$$= \sum_i \frac{f(a_i)}{|g'(a_i)|}$$

$$= \int_{-\infty}^{\infty} f(x) \sum_i \frac{\delta(x - a_i)}{|g'(a_i)|}\,dx$$

$$\therefore\ \delta(g(x)) = \sum_i \frac{\delta(x - a_i)}{|g'(a_i)|}$$

D.

$$\int_{-\infty}^{\infty} f(x)\,\delta'(x - x_0)\,dx$$

$$= \Big[f(x)\,\delta(x - x_0) \Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,\delta(x - x_0)\,dx$$

$$= -f'(x_0)$$

#4

Faraday E-field

$$\begin{cases} \nabla \cdot \vec{E} = 0 \\ \nabla \times \vec{E} = -\dfrac{\partial B}{\partial t} \end{cases}$$

Magnetic field

$$\begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{B} = \mu_0 \vec{J} \end{cases}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{\mathscr{r}^2} \times \hat{\mathscr{r}}\,d\tau'$$

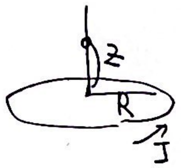
$$\Rightarrow \vec{E} = -\frac{1}{4\pi} \int \frac{\frac{\partial \vec{B}(\vec{r}')}{\partial t}}{\mathscr{r}^2} \times \hat{\mathscr{r}}\,d\tau'$$

$$\vec{B} = \begin{cases} \dfrac{\mu_0 N I}{2\pi s} \hat{\phi} & \text{inside a toroid} \\ 0 & \text{otherwise} \end{cases}$$

$$\Phi_B = \int \vec{B} \cdot d\vec{a} = \frac{\mu_0 N I}{2\pi} \int_a^{a+w} \frac{h}{s}\,ds$$

$$= \frac{\mu_0 N I}{2\pi} h \ln\left(1 + \frac{w}{a}\right) \simeq \frac{\mu_0 N I h w}{2\pi a} \quad (\because w \ll a)$$

$$\frac{d\Phi_B}{dt} = \frac{\mu_0 N k h w}{2\pi a}$$



$$\Rightarrow B_z = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}$$

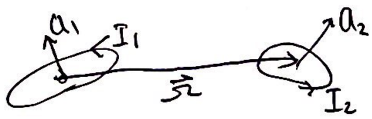
$R \to a$, $\mu_0 I \to -\frac{d\Phi_B}{dt}$

$$\therefore E(z) = \frac{\mu_0}{2}\frac{(-d\Phi_B/dt)\,a^2}{(a^2+z^2)^{3/2}}\hat{z}$$

$$= -\frac{\mu_0 N k h w a}{4\pi(a^2+z^2)^{3/2}} \quad \square$$

#5

(a)



$m_1 = a_1 I_1$

m_1이 만드는 자기장

$$\vec{B}_1(\vec{r}) = \frac{\mu_0}{4\pi r^3}\left[3(\vec{m}_1\cdot\hat{r})\hat{r}-\vec{m}_1\right]$$

$$\Phi_2 = \vec{B}_1 \cdot a_2$$

$$= \frac{\mu_0 I_1}{4\pi r^3}\left[3(a_1\cdot\hat{r})(a_2\cdot\hat{r}) - (a_1\cdot a_2)\right]$$

$$M_{21} = \frac{\Phi_2}{I_1} = \frac{\mu_0}{4\pi r^3}\left[3(a_1\cdot\hat{r})(a_2\cdot\hat{r}) - (a_1\cdot a_2)\right]$$

$$M_{12} = \cdots = M_{21}$$

(b)

고리 1 에 생기는 기전력 $\mathcal{E}_1$

$$\mathcal{E}_1 = -\frac{d\Phi_1}{dt} = -M\frac{dI_2}{dt}$$

$$P = I_1(-\mathcal{E}_1) = M I_1 \frac{dI_2}{dt}$$

$$W = \int P\,dt = M I_1 I_2$$

$$= \frac{\mu_0 I_1 I_2}{4\pi r^3}\left[3(a_1\cdot\hat{r})(a_2\cdot\hat{r}) - a_1\cdot a_2\right]$$

$$= \frac{\mu_0}{4\pi r^3}\left[3(m_1\cdot\hat{r})(m_2\cdot\hat{r}) - m_1\cdot m_2\right]$$

#6

Total Energy $E = E_E + E_B$

① E_E

$$E = \begin{cases} \dfrac{e}{4\pi\varepsilon_0 r^2}\hat{r} & (r>R) \\ 0 & (r<R) \text{ shell} \end{cases}$$

$$U_E = \int \frac{1}{2}\varepsilon_0 E^2 d\tau = \frac{e^2}{8\pi\varepsilon_0 R}$$

~~From eq~~

$$\vec{A} = \begin{cases} \dfrac{\mu_0\sigma\omega R}{3} r\sin\theta\,\hat{\phi} & (r<R) \\ \dfrac{\mu_0\sigma\omega R^4}{3}\dfrac{\sin\theta}{r^2}\hat{\phi} & (r>R) \end{cases}$$

$$\Rightarrow B = \nabla\times A$$

$$= \begin{cases} \dfrac{2}{3}\mu_0\sigma\omega R\,\hat{z} & (r<R) \\ \dfrac{\mu_0\sigma\omega R^4}{3r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) & (r>R) \end{cases}$$

$$E_B = \int \frac{B^2}{2\mu_0}d\tau = \frac{\mu_0 e^2\omega^2 R}{36\pi}$$

$$\therefore E = E_E + E_B = \frac{e^2}{8\pi\varepsilon_0 R} + \frac{\mu_0 e^2\omega^2 R}{36\pi}$$