8.1 The Continuity Equation

- Conservation of charge
  - Total charge in volume $\mathcal{V}$: $Q(t) = \int_{\mathcal{V}} \rho(\boldsymbol{r}, t) d\tau$
  - Current flowing out through the boundary $\mathcal{S}$: $\int_{\mathcal{S}} \boldsymbol{J} \cdot d\boldsymbol{a}$
  - Change in $Q(t)$ ➔ the same amount of charge must have passed in or out through surface
  - $\frac{dQ(t)}{dt} = -\int_{\mathcal{S}} \boldsymbol{J} \cdot d\boldsymbol{a}$ ➔ $\frac{d}{dt}\int_{\mathcal{V}} \rho(\boldsymbol{r}, t) d\tau = -\int_{\mathcal{S}} \boldsymbol{J} \cdot d\boldsymbol{a}$

  ➔ $\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\tau = -\int_{\mathcal{V}} (\nabla \cdot \boldsymbol{J}) d\tau$
  - $\frac{\partial \rho}{\partial t} = -\nabla \cdot \boldsymbol{J}$: typical form of local conservation law
    - Vector inside divergence ($\boldsymbol{J}$) corresponds to flow of the interested quantity (charge)

Energy Stored in the Electric Field

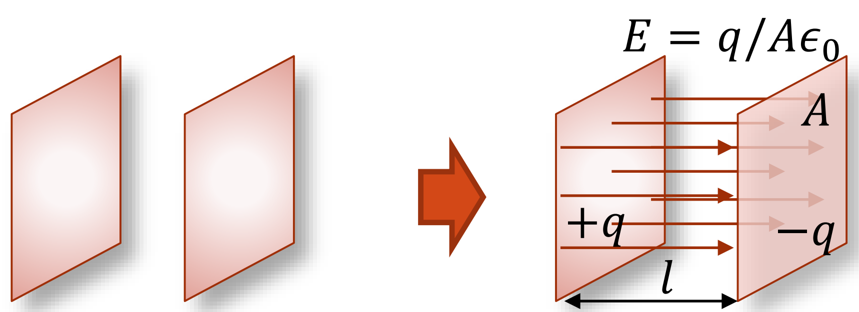
- Consider a capacitor composed of two electrodes facing each other and the area is A and the distance is l .
- Find out the energy stored in the capacitor.
  - When, there is q charge on the left electrode and $-q$ charge on the right electrode, the uniform electric field E is $E = q/A\epsilon_0$.
  - To move infinitesimal amount of charge dq from the right to the left, the amount of work necessary is $dW = Edq \cdot l$.
  - When there is $+Q$ and $-Q$ charge on the electrodes, respectively, the total amount of work stored in the capacitor is

$$\int dW = \int_0^Q (E \cdot l)dq = l \cdot \int_0^Q \frac{q}{A\epsilon_0} dq = \frac{l}{A\epsilon_0}\frac{Q^2}{2}$$

- Re-write in terms of E-field:

$$\frac{l}{A\epsilon_0}\frac{Q^2}{2} = \frac{l}{A\epsilon_0}\frac{(A\epsilon_0 E)^2}{2} = \frac{\epsilon_0}{2}(lA)E^2 = \frac{\epsilon_0}{2}E^2 \cdot (\text{volume})$$

- Therefore the energy stored in E-field is $W_E = \int \frac{\epsilon_0}{2} E^2 dv$



8.1.2 Poynting's Theorem

- Goal: calculate the amount of energy carried by fields
- Energy stored inside electric fields
  - Section 2.4.3
  - $W_e = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 d\tau$
- Energy stored inside magnetic fields
  - Section 7.2.4
  - $W_m = \frac{1}{2\mu_0}\int_{\text{all space}} B^2 d\tau$
- Total energy stored inside electromagnetic fields

$$U_{\text{em}} = \frac{1}{2}\int_{\text{all space}} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau$$

8.1.2 Poynting's Theorem

- Assumption
  - $\boldsymbol{E}$ and $\boldsymbol{B}$ are produced by some charge and current configuration at a time t
  - After dt, the charge moved by $d\boldsymbol{l} = \boldsymbol{v}dt$
- The amount of work dW, done by electromagnetic forces
  - Work done on charge q: $\boldsymbol{F} \cdot d\boldsymbol{l} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v}dt = q\boldsymbol{E} \cdot \boldsymbol{v}dt$
  - $q = \rho d\tau$, $\rho\boldsymbol{v} = \boldsymbol{J}$ ➔ $\boldsymbol{F} \cdot d\boldsymbol{l} = \boldsymbol{E} \cdot \boldsymbol{J}dtd\tau$
  - Work done on all the charges in a volume $\mathcal{V}$:

$$dW = \int_{\mathcal{V}} (\boldsymbol{E} \cdot \boldsymbol{J}dt)d\tau$$

- Rate of work done on all the charges

$$\frac{dW}{dt} = \int_{\mathcal{V}} (\boldsymbol{E} \cdot \boldsymbol{J})d\tau$$

  - $\boldsymbol{E} \cdot \boldsymbol{J}$: the work done per unit time, per unit volume ➔ power delivered per unit volume

8.1.2 Poynting's Theorem

- Goal: write $\boldsymbol{E}\cdot\boldsymbol{J}$ in terms of fields only
- From Ampere's law $\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}$,

$$\boldsymbol{E}\cdot\boldsymbol{J} = \frac{\mathbf{1}}{\mu_0}\boldsymbol{E}\cdot(\nabla\times\boldsymbol{B}) - \epsilon_0\boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t}$$

- From $\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{E}) - \boldsymbol{E}\cdot(\nabla\times\boldsymbol{B})$,

$$\boldsymbol{E}\cdot(\nabla\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{E}) - \nabla\cdot(\boldsymbol{E}\times\boldsymbol{B})$$

- Using $\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$,

$$\boldsymbol{E}\cdot(\nabla\times\boldsymbol{B}) = -\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t} - \nabla\cdot(\boldsymbol{E}\times\boldsymbol{B})$$

$$\Rightarrow \boldsymbol{E}\cdot\boldsymbol{J} = \frac{1}{\mu_0}\left\{-\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t} - \nabla\cdot(\boldsymbol{E}\times\boldsymbol{B})\right\} - \epsilon_0\boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t}$$

- Using $\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t} = \frac{1}{2}\frac{\partial\boldsymbol{B}^2}{\partial t}, \boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t} = \frac{1}{2}\frac{\partial\boldsymbol{E}^2}{\partial t}$,

$$\boldsymbol{E}\cdot\boldsymbol{J} = -\frac{1}{2\mu_0}\frac{\partial\boldsymbol{B}^2}{\partial t} - \frac{1}{\mu_0}\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}) - \frac{\epsilon_0}{2}\frac{\partial\boldsymbol{E}^2}{\partial t} = -\frac{d}{dt}\left(\frac{1}{2\mu_0}\boldsymbol{B}^2 + \frac{\epsilon_0}{2}\boldsymbol{E}^2\right) - \frac{1}{\mu_0}\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B})$$

- With divergence theorem for the 2nd term,

$$\frac{dW}{dt} = \int_{\mathcal{V}}(\boldsymbol{E}\cdot\boldsymbol{J})d\tau = -\frac{d}{dt}\int_{\mathcal{V}}\frac{1}{2}\left(\frac{1}{\mu_0}\boldsymbol{B}^2 + \epsilon_0\boldsymbol{E}^2\right)d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}}(\boldsymbol{E}\times\boldsymbol{B})\cdot d\boldsymbol{a}$$

where $\mathcal{S}$ is the surface bounding $\mathcal{V}$

8.1.2 Poynting's Theorem

- Interpretation of Poynting's theorem
  - $\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\frac{1}{\mu_0}\boldsymbol{B}^2 + \epsilon_0 \boldsymbol{E}^2\right) d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\boldsymbol{E}\times\boldsymbol{B})\cdot d\boldsymbol{a}$
  - LHS: the work done on the charges by the electromagnetic force
  - 1st term: decrease rate of total energy stored in fields, U_{em}
  - 2nd term: amount of energy flowing in through the surface $\mathcal{S}$
- Poynting vector
  - $\boldsymbol{S} \equiv \frac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B})$
  - The energy per unit time, per unit area, transported by the fields
  - Poynting's theorem: $\frac{dW}{dt} = -\frac{d}{dt}U_{\mathrm{em}} - \oint_{\mathcal{S}} \boldsymbol{S}\cdot d\boldsymbol{a}$

8.1.2 Poynting's Theorem

- Conservation of energy
  - The work W done on the charges ➔ increase of their mechanical energy (kinetic, potential, etc)
  - u_{mech}: mechanical energy density

$$W = \int_{\mathcal{V}} u_{\mathrm{mech}} d\tau \Rightarrow \frac{dW}{dt} = \frac{d}{dt}\int_{\mathcal{V}} u_{\mathrm{mech}} d\tau$$

  - Using energy density of the field $u_{\mathrm{em}} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$ and $U_{\mathrm{em}} = \int_{\mathcal{V}} u_{\mathrm{em}} d\tau$,

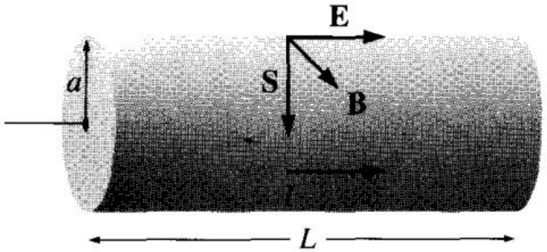
$$\frac{dW}{dt} = -\frac{d}{dt}U_{\mathrm{em}} - \oint_{\mathcal{S}} \boldsymbol{S}\cdot d\boldsymbol{a}$$

$$\Rightarrow \frac{d}{dt}\int_{\mathcal{V}} u_{\mathrm{mech}} d\tau = -\frac{d}{dt}\int_{\mathcal{V}} u_{\mathrm{em}} d\tau - \int_{\mathcal{V}} \nabla\cdot\boldsymbol{S} d\tau$$

$$\Rightarrow \int_{\mathcal{V}} \frac{\partial}{\partial t}(u_{\mathrm{mech}} + u_{\mathrm{em}}) d\tau = -\int_{\mathcal{V}} \nabla\cdot\boldsymbol{S} d\tau \Rightarrow \frac{\partial}{\partial t}(u_{\mathrm{mech}} + u_{\mathrm{em}}) = -\nabla\cdot\boldsymbol{S}$$

  - Compare with conservation of charge $\frac{\partial\rho}{\partial t} = -\nabla\cdot\boldsymbol{J}$

8.1.2 Poynting's Theorem



- Example 8.1
  - Wire: circular shape with radius a, length L
  - Current I flows through a wire with voltage difference V
  - Verify the energy conservation
    - Electric field is parallel to the wire: $E = V/L$
    - Magnetic field is circumferential at the surface of the wire:
      - Ampere's law: $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$
      - Integral form: $\int_C \boldsymbol{B} \cdot d\boldsymbol{l} = \int_\mathcal{V} (\mu_0 \boldsymbol{J}) d\tau + \int_\mathcal{V} \left(\mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right) d\tau$
      - $\frac{\partial \boldsymbol{E}}{\partial t} = 0$ for static field ➔ $B \cdot 2\pi a = \mu_0 I$ ➔ $B = \frac{\mu_0 I}{2\pi a}$
    - Poynting vector: $\boldsymbol{S} = \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B} = \frac{1}{\mu_0} \frac{V}{L} \frac{\mu_0 I}{2\pi a} = \frac{VI}{2\pi a L}$
    - The energy per unit time passing in through the surface of the wire: $\int_\mathcal{S} \boldsymbol{S} \cdot d\boldsymbol{a} = \frac{VI}{2\pi a L} \cdot 2\pi a L = VI$ ➔ Joule heating!

8.2.2 Maxwell's Stress Tensor

- Total electromagnetic force on the charges in volume $\mathcal{V}$

$$F = \int_{\mathcal{V}} (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \rho d\tau = \int_{\mathcal{V}} (\rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}) d\tau$$

- The force per unit volume
  - $\boldsymbol{f} = \rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}$
  - Goal: write $\boldsymbol{f}$ in terms of fields only by eliminating ρ and $\boldsymbol{J}$
  - From Gauss's law $\nabla \cdot \boldsymbol{E} = \rho / \epsilon_0$ ➔ $\rho = \epsilon_0 \nabla \cdot \boldsymbol{E}$
  - From Ampere's law $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$ ➔ $\boldsymbol{J} = \frac{1}{\mu_0} \nabla \times \boldsymbol{B} - \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$
  - $\boldsymbol{f} = \rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} = \boldsymbol{E}(\epsilon_0 \nabla \cdot \boldsymbol{E}) + \left(\frac{1}{\mu_0} \nabla \times \boldsymbol{B} - \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) \times \boldsymbol{B}$
  - Want to replace $\frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B}$ with different expression

8.2.2 Maxwell's Stress Tensor

- The force per unit volume (cont'd)
  - Want to replace $\frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B}$ with different expression
    - $\frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B}) = \left(\frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B}\right) + \left(\boldsymbol{E} \times \frac{\partial \boldsymbol{B}}{\partial t}\right)$
    - From Faraday's law $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$,
    - $\frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B} = \frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B}) - \boldsymbol{E} \times (-\nabla \times \boldsymbol{E})$
  - Recall $\boldsymbol{f} = \boldsymbol{E}(\epsilon_0 \nabla \cdot \boldsymbol{E}) + \left(\frac{1}{\mu_0} \nabla \times \boldsymbol{B}\right) \times \boldsymbol{B} - \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B}$

$$= \epsilon_0 \boldsymbol{E}(\nabla \cdot \boldsymbol{E}) + \left(\frac{1}{\mu_0} \nabla \times \boldsymbol{B}\right) \times \boldsymbol{B} - \epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B}) - \epsilon_0 \boldsymbol{E} \times (\nabla \times \boldsymbol{E})$$

$$= \epsilon_0 [\boldsymbol{E}(\nabla \cdot \boldsymbol{E}) - \boldsymbol{E} \times (\nabla \times \boldsymbol{E})] - \frac{1}{\mu_0} \boldsymbol{B} \times (\nabla \times \boldsymbol{B}) - \epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B})$$

    - I want to make the above result look more symmetric
      ➔ what is missing?

8.2.2 Maxwell's Stress Tensor

- The force per unit volume (cont'd)
  - $\boldsymbol{f} = \epsilon_0[\boldsymbol{E}(\nabla \cdot \boldsymbol{E}) - \boldsymbol{E} \times (\nabla \times \boldsymbol{E})] + \frac{1}{\mu_0}[\boldsymbol{B}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B} \times (\nabla \times \boldsymbol{B})] - \epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B})$
    - Want to simplify $\boldsymbol{E}(\nabla \cdot \boldsymbol{E}) - \boldsymbol{E} \times (\nabla \times \boldsymbol{E})$ pattern
    - Using $\boldsymbol{E} \times (\nabla \times \boldsymbol{E}) = \frac{1}{2}\nabla \boldsymbol{E}^2 - (\boldsymbol{E} \cdot \nabla)\boldsymbol{E}$,
    - $\boldsymbol{E}(\nabla \cdot \boldsymbol{E}) - \boldsymbol{E} \times (\nabla \times \boldsymbol{E}) = \boldsymbol{E}(\nabla \cdot \boldsymbol{E}) - \frac{1}{2}\nabla \boldsymbol{E}^2 + (\boldsymbol{E} \cdot \nabla)\boldsymbol{E}$
    - $\boldsymbol{f} = \epsilon_0\left[\boldsymbol{E}(\nabla \cdot \boldsymbol{E}) + (\boldsymbol{E} \cdot \nabla)\boldsymbol{E} - \frac{1}{2}\nabla \boldsymbol{E}^2\right] + \frac{1}{\mu_0}\left[\boldsymbol{B}(\nabla \cdot \boldsymbol{B}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \frac{1}{2}\nabla \boldsymbol{B}^2\right] - \epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B})$

$$= \epsilon_0[\boldsymbol{E}(\nabla \cdot \boldsymbol{E}) + (\boldsymbol{E} \cdot \nabla)\boldsymbol{E}] + \frac{1}{\mu_0}[\boldsymbol{B}(\nabla \cdot \boldsymbol{B}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{B}] - \frac{1}{2}\epsilon_0 \nabla \boldsymbol{E}^2 - \frac{1}{2}\frac{1}{\mu_0}\nabla \boldsymbol{B}^2 - \epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B})$$

$$= \epsilon_0[\boldsymbol{E}(\nabla \cdot \boldsymbol{E}) + (\boldsymbol{E} \cdot \nabla)\boldsymbol{E}] + \frac{1}{\mu_0}[\boldsymbol{B}(\nabla \cdot \boldsymbol{B}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{B}] - \nabla \frac{1}{2}\left(\epsilon_0 \boldsymbol{E}^2 + \frac{1}{\mu_0}\boldsymbol{B}^2\right) - \epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B})$$

8.2.2 Maxwell's Stress Tensor

- The force per unit volume (cont'd)
  - $\boldsymbol{E}(\nabla \cdot \boldsymbol{E}) + (\boldsymbol{E} \cdot \nabla)\boldsymbol{E}$ ➔ $E_j \partial_i E_i + E_i \partial_i E_j$
  - $\epsilon_0[\boldsymbol{E}(\nabla \cdot \boldsymbol{E}) + (\boldsymbol{E} \cdot \nabla)\boldsymbol{E}] + \frac{1}{\mu_0}[\boldsymbol{B}(\nabla \cdot \boldsymbol{B}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{B}] - \nabla \frac{1}{2}\left(\epsilon_0 \boldsymbol{E}^2 + \frac{1}{\mu_0}\boldsymbol{B}^2\right)$

$$= \epsilon_0\left(E_j \partial_i E_i + E_i \partial_i E_j\right) + \frac{1}{\mu_0}\left(B_j \partial_i B_i + B_i \partial_i B_j\right) - \frac{1}{2}\partial_j\left(\epsilon_0 E_k E_k + \frac{1}{\mu_0} B_k B_k\right)$$

- $E_j \partial_i E_i + E_i \partial_i E_j = \partial_i\left(E_i E_j\right)$
- $\frac{1}{2}\partial_j(\epsilon_0 E_k E_k) = \partial_i \delta_{ij}\left(\epsilon_0 \frac{1}{2} E_k E_k\right)$

$$= \partial_i\left(\epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j\right) - \partial_i \delta_{ij}\left(\epsilon_0 \frac{1}{2} E_k E_k + \frac{1}{\mu_0} B_k B_k\right)$$

$$= \partial_i\left[\epsilon_0\left(E_i E_j - \delta_{ij}\frac{1}{2} E_k E_k\right) + \frac{1}{\mu_0}\left(B_i B_j - \delta_{ij}\frac{1}{2} B_k B_k\right)\right] = \partial_i T_{ij}$$

where $T_{ij} \equiv \epsilon_0\left(E_i E_j - \delta_{ij}\frac{1}{2} E_k E_k\right) + \frac{1}{\mu_0}\left(B_i B_j - \delta_{ij}\frac{1}{2} B_k B_k\right)$: stress tensor

8.2.2 Maxwell's Stress Tensor

- The force per unit volume (cont'd)
  - Notation for Maxwell's stress Tensor: $\overleftrightarrow{\boldsymbol{T}}$
  - $\boldsymbol{f} = \epsilon_0[\boldsymbol{E}(\nabla \cdot \boldsymbol{E}) + (\boldsymbol{E} \cdot \nabla)\boldsymbol{E}] + \frac{1}{\mu_0}[\boldsymbol{B}(\nabla \cdot \boldsymbol{B}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{B}] - \nabla \frac{1}{2}\left(\epsilon_0 \boldsymbol{E}^2 + \frac{1}{\mu_0}\boldsymbol{B}^2\right) - \epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B})$
  - $f_j = \partial_i T_{ij} - \epsilon_0 \mu_0 \frac{\partial}{\partial t} S_j \Leftrightarrow \boldsymbol{f} = \nabla \cdot \overleftrightarrow{\boldsymbol{T}} - \epsilon_0 \mu_0 \frac{\partial \boldsymbol{S}}{\partial t}$
- Total electromagnetic force on the charges in volume $\mathcal{V}$

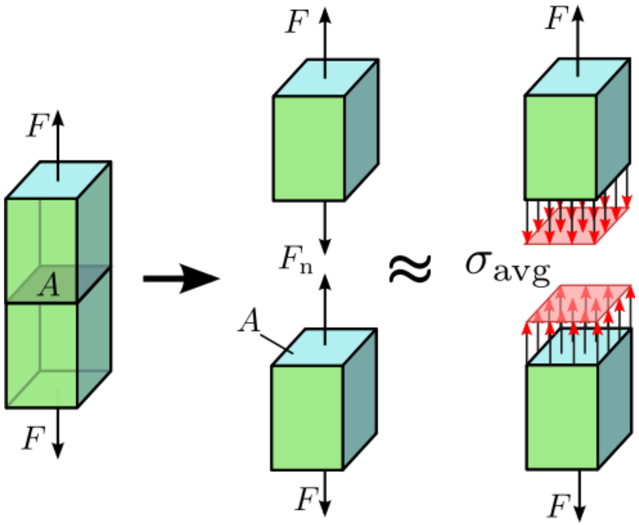
$$\boldsymbol{F} = \int_{\mathcal{V}} (\rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}) d\tau = \int_{\mathcal{V}} \boldsymbol{f} d\tau = \int_{\mathcal{V}} \left(\nabla \cdot \overleftrightarrow{\boldsymbol{T}} - \epsilon_0 \mu_0 \frac{\partial \boldsymbol{S}}{\partial t}\right) d\tau$$

$$= \oint_{\mathcal{S}} \overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a} - \epsilon_0 \mu_0 \frac{d}{dt} \int_{\mathcal{V}} \boldsymbol{S} d\tau$$

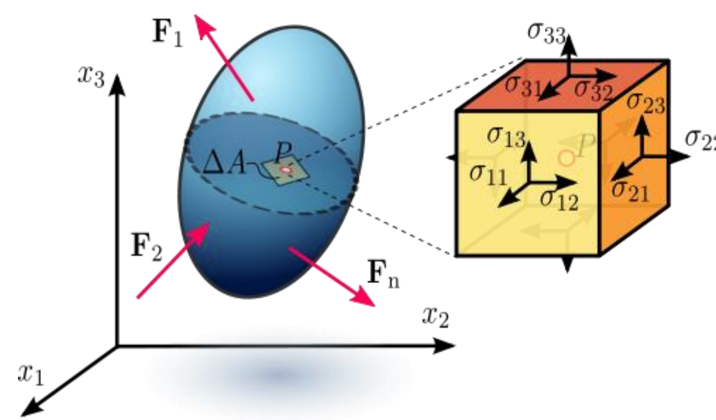
where divergence theorem is applied to the tensor

8.2.2 Maxwell's Stress Tensor

- Physical meaning of stress
  - Normal stress
    - Similar to pressure
  - Shear stress
- Generally, the force on the surface (called stress) is not necessarily parallel to the normal vector of the surface
  - Stress tensor can map the normal vector of the surface to a vector along any arbitrary direction



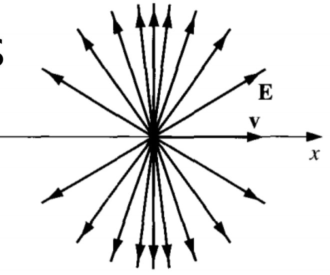
https://en.wikipedia.org/wiki/File:Axial_stress.svg

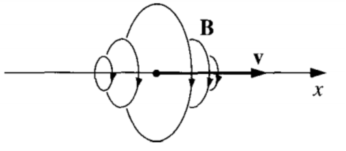


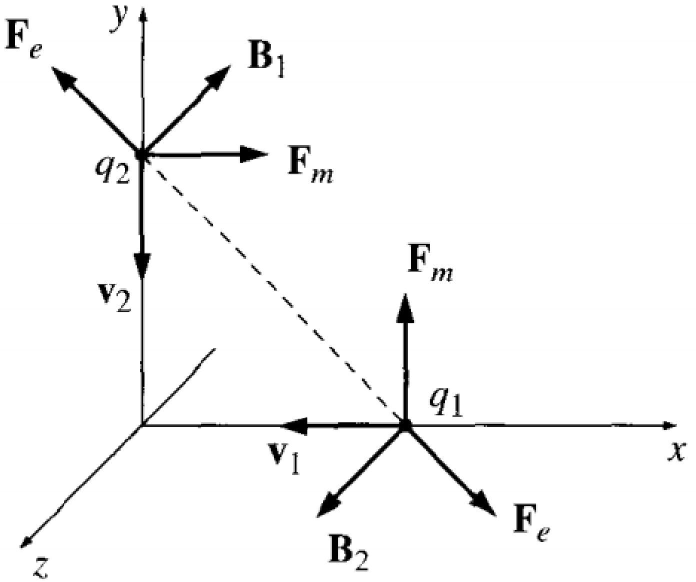
https://commons.wikimedia.org/wiki/File:Stress_in_a_continuum.svg

8.2.1 Newton's Third Law in Electrodynamics

- Why are we interested in the stress tensor?
  - Imagine two moving point charges
  - What do fields look like?
  - What is the forces on each other?







  - Is the third law satisfied?
    - If not, momentum conservation will be violated.
  - How can we save the momentum conservation?

8.2.3 Conservation of Momentum

- Newton's 2nd law
  - The force on an object is equal to the changing rate of its momentum

$$\boldsymbol{F} = \frac{d\boldsymbol{p}_{\text{mech}}}{dt}$$

$$\boldsymbol{F} = -\epsilon_0\mu_0 \frac{\partial}{\partial t}\int_{\mathcal{V}} \boldsymbol{S}\, d\tau + \oint_{\mathcal{S}} \overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a} = \frac{d\boldsymbol{p}_{\text{mech}}}{dt}$$

  - $\boldsymbol{p}_{\text{mech}}$: total (mechanical) momentum of the particles inside the volume $\mathcal{V}$
  - $\oint_{\mathcal{S}} \overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a}$: total force on the volume $\mathcal{V}$

$$\oint_{\mathcal{S}} \overleftrightarrow{\boldsymbol{T}} \cdot d\boldsymbol{a} = \frac{d}{dt}\boldsymbol{p}_{\text{mech}} + \frac{d}{dt}\int_{\mathcal{V}} (\epsilon_0\mu_0\boldsymbol{S})\, d\tau = \frac{d}{dt}\left[\boldsymbol{p}_{\text{mech}} + \int_{\mathcal{V}} (\epsilon_0\mu_0\boldsymbol{S})\, d\tau\right]$$

  - $\int_{\mathcal{V}} (\epsilon_0\mu_0\boldsymbol{S})\, d\tau$: can be interpreted as a momentum carried by the fields inside the volume $\mathcal{V}$

8.2.3 Conservation of Momentum

- Differential form

$$\oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} = \frac{d}{dt}\left[\mathbf{p}_{\text{mech}} + \int_{\mathcal{V}} (\epsilon_0\mu_0\mathbf{S})d\tau\right]$$

$$\int_{\mathcal{V}} (\nabla \cdot \overleftrightarrow{\mathbf{T}})d\tau = \frac{d}{dt}\left[\int_{\mathcal{V}} \boldsymbol{\wp}_{\text{mech}}d\tau + \int_{\mathcal{V}} (\epsilon_0\mu_0\mathbf{S})d\tau\right]$$

$$\frac{\partial}{\partial t}(\boldsymbol{\wp}_{\text{mech}} + \epsilon_0\mu_0\mathbf{S}) = \nabla \cdot \overleftrightarrow{\mathbf{T}}$$

  - $\boldsymbol{\wp}_{\text{mech}}$: density of mechanical momentum
  - $\boldsymbol{\wp}_{\text{em}} \equiv \epsilon_0\mu_0\mathbf{S}$: density of momentum in the fields