Electric Fields and Magnetic Fields

- Maxwell's equations in real, time-dependent form
  - All the fields in Maxwell's equation represent vector fields depending on the positions.
  - At each point $\boldsymbol{r} = (x, y, z)$ in space, electric field $\boldsymbol{E}$ is composed of three components E_x, E_y, E_z which are functions of position.
  - E.g. $E_x(x, y, z) = E_x(\boldsymbol{r})$, $\boldsymbol{E} = E_x(\boldsymbol{r})\hat{\boldsymbol{x}} + E_y(\boldsymbol{r})\hat{\boldsymbol{y}} + E_z(\boldsymbol{r})\hat{\boldsymbol{z}} = E_i(\boldsymbol{r})\hat{\boldsymbol{r}}_i$
  - $\boldsymbol{E}\ (V/m)$: electric field
  - $\boldsymbol{D}\ (C/m^2)$: displacement
  - $\boldsymbol{B}\ (W/m^2)$: magnetic field
  - $\boldsymbol{H}(A/m)$: magnetic field (intensity)

Maxwell's Equation in Integral Form

- Gauss' law for electricity

$$\oint_{\mathcal{S}} \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{q_{enc}}{\epsilon_0} \Leftrightarrow \int_{\mathcal{V}} (\nabla \cdot \boldsymbol{E}) d\tau = \int_{\mathcal{V}} \frac{\rho}{\epsilon_0} d\tau$$

  - $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$
- Gauss' law for magnetism

$$\oint_{\mathcal{S}} \boldsymbol{B} \cdot d\boldsymbol{a} = 0 \Leftrightarrow \int_{\mathcal{V}} (\nabla \cdot \boldsymbol{B}) d\tau = 0$$

  - $\nabla \cdot \boldsymbol{B} = 0$
- Faraday's law of induction

$$\oint_{\mathcal{P}} \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{\partial}{\partial t} \Phi_B \Leftrightarrow \int_{\mathcal{S}} (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{a} = -\int_{\mathcal{S}} \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{a}$$

  - $\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t} \boldsymbol{B}$
- Ampere's law

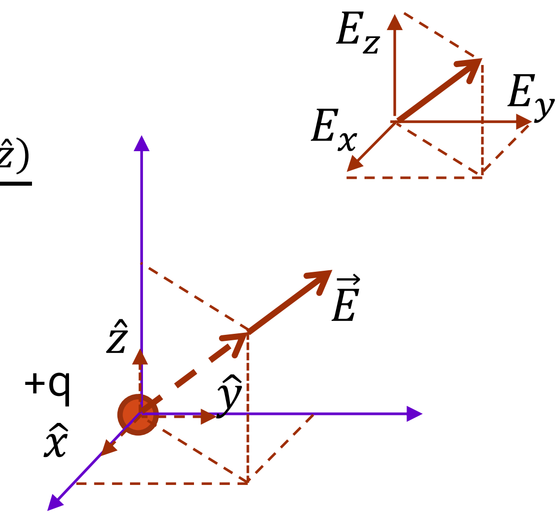
$$\oint_{\mathcal{P}} \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I_{enc} + \mu_0 \epsilon_0 \frac{\partial \Phi_E}{\partial t} \Leftrightarrow \int_{\mathcal{S}} (\nabla \times \boldsymbol{B}) \cdot d\boldsymbol{a} = \int_{\mathcal{S}} \mu_0 \left(\boldsymbol{J} + \frac{\epsilon_0 \partial \boldsymbol{E}}{\partial t} \right) \cdot d\boldsymbol{a}$$

  - $\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \frac{\epsilon_0 \partial}{\partial t} \boldsymbol{E} \right)$

7.3.3. Maxwell's Equations

- Maxwell's Equation
  - $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$: Gauss's law for electric field (ρ is charge density)
  - $\nabla \cdot \boldsymbol{B} = 0$: Gauss's law for magnetic field
  - $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$: Faraday's law
  - $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$: Ampere's law ($\boldsymbol{J}$ is current density)
- Force law
  - $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$
- Continuity eq. $\nabla \cdot J = -\frac{\partial \rho}{\partial t}$ can be derived from Ampere's law.
- Physical constants
  - Permittivity of vacuum: $\epsilon_0 = 8.85 \times 10^{-12} (\mathrm{C^2/Nm^2})$
  - Permeability of vacuum: $\mu_0 = 4\pi \times 10^{-7} (\mathrm{N/A^2})$
  - $1/c^2 \equiv \epsilon_0 \mu_0$ or $c = 1/\sqrt{\epsilon_0 \mu_0} = 299\,792\,458\ (\mathrm{m/s})$

Overview of Gauss' law for electricity



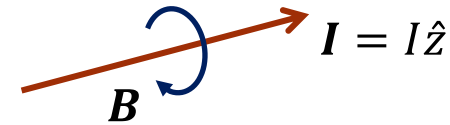
- $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$ where ρ is charge density
- From Coulomb's law, $\boldsymbol{E} = \frac{q\hat{r}}{4\pi\epsilon_0 r^2} = \frac{q(x\hat{x}+y\hat{y}+z\hat{z})}{4\pi\epsilon_0 r^3}$

where $r = (x^2 + y^2 + z^2)^{1/2}$

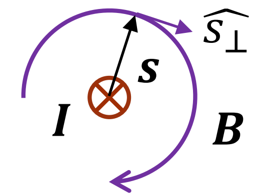
- $\nabla \cdot \boldsymbol{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$
- $E_x = \frac{qx}{4\pi\epsilon_0 r^3} = \frac{q}{4\pi\epsilon_0}\frac{x}{(x^2+y^2+z^2)^{3/2}}$
- $\frac{\partial E_x}{\partial x} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}\frac{x\cdot 2x}{(x^2+y^2+z^2)^{5/2}}\right) = \frac{q}{4\pi\epsilon_0}\frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}$
- Similarly, $\frac{\partial E_y}{\partial y} = \frac{q}{4\pi\epsilon_0}\frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}}$ and $\frac{\partial E_z}{\partial z} = \frac{q}{4\pi\epsilon_0}\frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}$
- Therefore, $\nabla \cdot \boldsymbol{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$ except where the charge is located.

Overview of Ampere's law

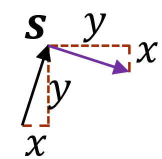
- $\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \frac{\epsilon_0 \partial}{\partial t} \boldsymbol{E} \right)$ where $\boldsymbol{J}$ is current density



- Consider a simple case where the magnetic field is generated by a current $\boldsymbol{I}$ in a wire.



- $\boldsymbol{B} = \frac{\mu_0 I}{2\pi s} \widehat{s_\perp} = \frac{\mu_0 I}{2\pi s} \left(\frac{\boldsymbol{s}_\perp}{s} \right) = \frac{\mu_0 I}{2\pi} \frac{y\hat{x} - x\hat{y}}{x^2 + y^2}$ where $s = (x^2 + y^2)^{1/2}$



$\boldsymbol{s}_\perp = \hat{z} \times \boldsymbol{s}$
$= y\hat{x} - x\hat{y}$

- $B_x = \frac{\mu_0 I}{2\pi} \frac{y}{x^2 + y^2}, \; B_y = \frac{\mu_0 I}{2\pi} \frac{-x}{x^2 + y^2}, \; B_z = 0$

- $\nabla \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ B_x & B_y & B_z \end{vmatrix} = \left(\partial_y B_z - \partial_z B_y \right) \hat{x} - \left(\partial_x B_z - \partial_z B_x \right) \hat{y} + \left(\partial_x B_y - \partial_y B_x \right) \hat{z}$

- Because $B_z = 0$ and B_x and B_y do not depend on z, both x and y components vanish.

- $\partial_x B_y = \frac{\mu_0 I}{2\pi} \frac{-(x^2 + y^2) - x \cdot (-1) 2x}{(x^2 + y^2)^2} = \frac{\mu_0 I}{2\pi} \frac{x^2 - y^2}{(x^2 + y^2)^2}$

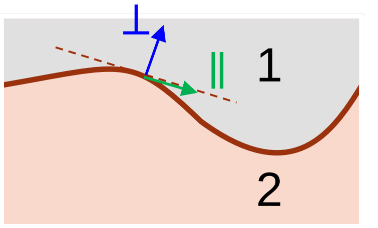
- $\partial_y B_x = \frac{\mu_0 I}{2\pi} \frac{(x^2 + y^2) + y \cdot (-1) 2y}{(x^2 + y^2)^2} = \frac{\mu_0 I}{2\pi} \frac{x^2 - y^2}{(x^2 + y^2)^2}$

- Therefore, $\nabla \times \vec{B} = 0$ when it is calculated not along the wire. $\frac{\epsilon_0 \partial \vec{E}}{\partial t}$ is a correction made by Maxwell for consistency and called as displacement current.

7.3.5 Maxwell's Equations in Matter

- Constitutive relation
  - $\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}$: $\boldsymbol{P}$ is polarization density
  - $\boldsymbol{H} = \left(\frac{1}{\mu_0}\right)\boldsymbol{B} - \boldsymbol{M}$: $\boldsymbol{M}$ is magnetization density
- Maxwell's Equation
  - $\nabla \cdot \boldsymbol{D} = \rho_f$: Gauss's law for electric field (ρ_f is (free) charge density)
  - $\nabla \cdot \boldsymbol{B} = 0$: Gauss's law for magnetic field
  - $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$: Faraday's law
  - $\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}_f$: Ampere's law ($\boldsymbol{J}_f$ is (free) current density)

7.6 Boundary Conditions



- Perpendicular components of displacement field at the interface are discontinuous if there is non-zero surface charge density
  - $D_1^{\perp} - D_2^{\perp} = \sigma_f$
- Parallel components of electric field at the interface are continuous independent of any sources
  - $\boldsymbol{E}_1^{\parallel} - \boldsymbol{E}_2^{\parallel} = 0$
- Perpendicular components of magnetic field at the interface are continuous independent of any sources
  - $B_1^{\perp} - B_2^{\perp} = 0$
- Parallel components of $\boldsymbol{H}$ field at the interface are discontinuous if there is non-zero surface current density
  - $\boldsymbol{H}_1^{\parallel} - \boldsymbol{H}_2^{\parallel} = \boldsymbol{K}_f \times \hat{n}$