

Review of Vector Calculus

- Tensor index notation
 - We will limit our discussion only to 3-dimension.
 - We will assume a fixed orthonormal basis and a Euclidean metric
 - ➔ no distinction between the contravariant tensor and the covariant tensor ➔ only subscript index will be used
 - Instead of vector notation \mathbf{v} , vector will be represented by its components with index $v_i : i = 1, 2, 3$
 - Instead of matrix notation \mathbb{A} , matrix will be represented by its components with indices $A_{ij} : i = 1, 2, 3$
 - References (not an ideal one, but better than nothing)
 - https://www.brown.edu/Departments/Engineering/Courses/En221/Notes/Index_notation/Index_notation.htm
 - <https://www.continuummechanics.org/tensornotationbasic.html>

Einstein Summation Convention

- Einstein summation convention

- Inner product of two vectors

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = \sum_{i=1}^3 v_i w_i \equiv v_i w_i$$

- If we have a repeated indices, summation from 1 to 3 is implied → called dummy indices
 - $\mathbf{x} = \mathbb{A}\mathbf{v} = \sum_{i=1}^3 \mathbb{A}_{ji} \mathbf{v}_i \equiv A_{ji} v_i = x_j$
 - Index j is not repeated → called a free index
 - Summation is also called as **contraction**
- Each index can appear at most twice in any term
- Each term must contain identical non-repeated indices
- Example
 - $A_{ij} w_j + B_{ijk} y_k z_j + x_i$ → valid
 - $A_{ij} w_j x_j, B_{jk} y_k + x_i, B_{ik} + x_i$ → invalid
 - $\text{tr}(\mathbb{A}) = A_{jj}$
- The number of free indices determines the order (or rank) of tensor.
 - We can tell us whether it is a scalar, vector, matrix, 3rd-order (or rank-3) tensor, ... by counting the number of the free indices

Tensor Index Notation

- Kronecker delta

- $\delta_{ij} = \begin{cases} 0, i \neq j \\ 1, i = j \end{cases}$

- $\delta_{ij}x_i = x_j$

- Useful to represent inner product: $a_k b_k = \delta_{jk} a_j b_k$

- Levi-Civita symbol

- $\epsilon_{ijk} = \begin{cases} +1, \text{if } (i, j, k) \text{ are even permutation of } (1, 2, 3) \\ -1, \text{if } (i, j, k) \text{ are odd permutation of } (1, 2, 3) \\ 0, \text{if } i = j, \text{ or } j = k, \text{ or } k = i \end{cases}$

- Even permutation of (1,2,3) : (1,2,3), (2,3,1) or (3,1,2) or rotation of (1,2,3)

- Odd permutation of (1,2,3) : (3,2,1), (1,3,2) or (2,1,3) or rotation of (3,2,1)

- Cross product

- $\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\hat{x} + (a_z b_x - a_x b_z)\hat{y} + (a_x b_y - a_y b_x)\hat{z}$

- $c_i = \epsilon_{ijk} a_j b_k$ or $\mathbf{c} = \epsilon_{ijk} a_j b_k \hat{i}$



Example

- Prove that $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$
 - $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \delta_{ij} a_i (\mathbf{a} \times \mathbf{b})_j = \delta_{ij} a_i \epsilon_{jkl} a_k b_l$
 - Note that the products in tensor index notation are commutative and associative, but the order of indices can be changed only by a rule.
 - $= \delta_{ij} \epsilon_{jkl} a_i a_k b_l = \epsilon_{ikl} a_i a_k b_l$
 - Dummy indices can be replaced by different letters
 - $= \epsilon_{mnl} a_m a_n b_l = -\epsilon_{nml} a_m a_n b_l = -\epsilon_{ikl} a_k a_i b_l$
 - $\epsilon_{ikl} a_i a_k b_l = -\epsilon_{ikl} a_k a_i b_l$
 - $\Rightarrow \epsilon_{ikl} a_i a_k b_l + \epsilon_{ikl} a_k a_i b_l = 0 = 2\epsilon_{ikl} a_i a_k b_l$



Product of Two ϵ_{ijk}

- $\delta_{ii} = ?$
- The only relation you should memorize!
 - $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$
 - Check!
 - When does the left-hand side becomes non-zero?
 - When does the left-hand side becomes zero?
 - $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$, but $\epsilon_{ijk} \neq \epsilon_{ikj}$ for example
 - Example: proof of BAC-CAB rule
 - $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$
 - $(\mathbf{A} \times (\mathbf{B} \times \mathbf{C}))_i = \epsilon_{ijk}A_j(\mathbf{B} \times \mathbf{C})_k = \epsilon_{ijk}A_j\epsilon_{klm}B_lC_m$
 - Note that the products in tensor index notation are commutative and associative, but the order of indices cannot be changed easily.
 - $= \epsilon_{ijk}\epsilon_{klm}A_jB_lC_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})A_jB_lC_m$
 - Prove that $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$

Diadic Product

- In tensor index notation, simply $v_i w_j$.
- Takes in two vectors and returns a second order tensor
- Corresponds to outer product such as $|v\rangle\langle w|$
- Works as a linear transformation (or operator) that maps a given vector to another vector.
- Example

$$\square \quad |v\rangle = \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, |w\rangle = \mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\square \quad |v\rangle\langle w| = \mathbf{v}\mathbf{w}^\dagger = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 * 2 & 1 * 3 & 1 * 4 \\ 2 * 2 & 2 * 3 & 2 * 4 \\ 3 * 2 & 3 * 3 & 3 * 4 \end{bmatrix} \Leftrightarrow v_i w_j$$

- In the language of matrix, it is called a rank-1 matrix.

Cross Product vs. Outer Product

- Cross Product takes in two vectors and returns a vector (first-order tensor)
- Diadic (outer) Product takes in two vectors and returns a second-order tensor (like matrix)
- Part of cross product can be considered as linear transform (operator)

$$\square \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y) \hat{x} + (a_z b_x - a_x b_z) \hat{y} + (a_x b_y - a_y b_x) \hat{z}$$

$$\square \quad \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

- We might consider $(\mathbf{a} \times)$ as a linear operator that transform the input vector \mathbf{b} to another vector $(a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x)$
- This linear operator (or matrix) has rank-2, so it can be considered as the sum of two outer products.

Notations

- All the following discussions and chapter numbers will be based on "Introduction to Electrodynamics", David. J. Griffiths, 3rd ed. (1999), unless otherwise specified.
- Notations
 - \hat{x} : unit vector along x-axis, and similarly \hat{y} for y-axis and \hat{z} for z-axis
 - $\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$: position vector in 3-dimensional space
 - $r = (x^2 + y^2 + z^2)^{1/2}$
 - $\mathbf{s} = x\hat{x} + y\hat{y}$: position vector in xy-plane
 - $s = (x^2 + y^2)^{1/2}$
 - $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z$: del (or nabla) operator

1.2 Differential Calculus

- Vector Derivatives in Cartesian Coordinates

- $\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} = \hat{x} \partial_x + \hat{y} \partial_y + \hat{z} \partial_z$

- Assumption

- $f(x, y, z)$ is a scalar function of position
- $\mathbf{V}(x, y, z)$ is a vector function of position
 - $\mathbf{V}(x, y, z)$ has 3 components $V_x(x, y, z), V_y(x, y, z), V_z(x, y, z)$
 - $\mathbf{V}(x, y, z) = V_x(x, y, z)\hat{x} + V_y(x, y, z)\hat{y} + V_z(x, y, z)\hat{z}$

- Gradient

- $\nabla f(x, y, z) = \frac{\partial f(x, y, z)}{\partial x} \hat{x} + \frac{\partial f(x, y, z)}{\partial y} \hat{y} + \frac{\partial f(x, y, z)}{\partial z} \hat{z} \Rightarrow \partial_i f$

- Divergence

- $\nabla \cdot \mathbf{V}(x, y, z) = \frac{\partial}{\partial x} V_x(x, y, z) + \frac{\partial}{\partial y} V_y(x, y, z) + \frac{\partial}{\partial z} V_z(x, y, z) \Rightarrow \partial_i V_i$

- Curl

- $\nabla \times \mathbf{V}(x, y, z) = \left(\frac{\partial}{\partial y} V_z - \frac{\partial}{\partial z} V_y \right) \hat{x} + \left(\frac{\partial}{\partial z} V_x - \frac{\partial}{\partial x} V_z \right) \hat{y} + \left(\frac{\partial}{\partial x} V_y - \frac{\partial}{\partial y} V_x \right) \hat{z} \Rightarrow \epsilon_{ijk} \partial_j V_k$

- Laplacian

- $\nabla^2 f(x, y, z) = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f \Rightarrow \partial_i \partial_i f$

- $\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z} \rightarrow$ occasionally I will use $x_i \equiv r_i \rightarrow \partial_i r_j = \partial_i x_j = ?$

- If $\mathbf{V}(x, y, z) = \mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$, $V_i = r_i \equiv x_i \rightarrow \nabla \cdot \mathbf{V}(x, y, z) = \partial_i V_i = ?$

1.2.6 Product Rules

- Assumption: $\mathbf{A}(\mathbf{r}), \mathbf{B}(\mathbf{r})$ are vector functions of positions
- $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
- $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$
- $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

1.3 Integral Calculus

- 1.3.3 The fundamental theorem for Gradients

$$\int_a^b (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$$

- 1.3.4 The fundamental theorem for divergences

$$\int_V (\nabla \cdot \mathbf{V}) d\tau = \oint_S \mathbf{V} \cdot d\mathbf{a}$$

- 1.3.5 The fundamental theorem for curls

$$\int_S (\nabla \times \mathbf{V}) \cdot d\mathbf{a} = \oint_P \mathbf{V} \cdot d\mathbf{l}$$