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1. Consider the following case. This is the same discussion as Fig. 15 in Griffiths (pp. 408-411). Because the polarization of magnetic field is transverse to the plane of incidence (plane of incidence is defined as the incident wavevector **k** and the interface normal vector **z**), it is called TM wave (p-polarization, German parallel). Note the other case, i.e. Griffiths prof. 9.17, is that the polarization of electric field is transverse to the plane of incidence, it is called TE wave (s-polarization, German "senkrecht" for perpendicular). Of course, discussion on Fig. 15 (pp. 408-411) is for the TM-wave, p-polarization. In this problem, you need to start from **H**-field (not from **E**-field).

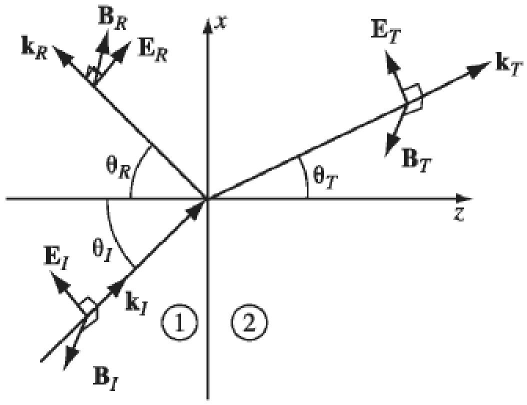

FIGURE 9.15

Assume a plane-wave solution $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$. The complex amplitudes of incident, reflected, and transmitted H-field are $\tilde{H}_i$, $\tilde{H}_r$, and $\tilde{H}_t$. The refractive index is n_1 and n_2 for the medium and medium 2, respectively. Explicitly derive the law of reflection and the law of refraction [use the parallel component of H, i.e. y-component, is continuous].

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$$\bar{H}(\bar{r},t) = \begin{cases} \{\tilde{H}_i e^{i(\bar{k}_i\cdot\bar{r}-\omega t)} + \tilde{H}_r e^{i(\bar{k}_r\cdot\bar{r}-\omega t)}\}\hat{y} & (z<0) \\ \tilde{H}_t e^{i(\bar{k}_t\cdot\bar{r}-\omega t)}\hat{y} & (z\geq 0) \end{cases}$$

B.C. $H_{\parallel} = H_y$ is continuous at $z=0$

$$\tilde{H}_i e^{i(\bar{k}_i\cdot\bar{r}-\cancel{\omega t})} + \tilde{H}_r e^{i(\bar{k}_r\cdot\bar{r}-\cancel{\omega t})} = \tilde{H}_t e^{i(\bar{k}_t\cdot\bar{r}-\cancel{\omega t})}$$

① $\bar{r}=0$: $\tilde{H}_i + \tilde{H}_r = \tilde{H}_t$

② $\bar{r} = (x,y,0)$, $\bar{k}_i = k_i(\sin\theta_i, 0, \cos\theta_i)$

$$\bar{k}_r = k_r(\sin\theta_r, 0, -\cos\theta_r)$$

$$\bar{k}_t = k_t(\sin\theta_t, 0, \cos\theta_t)$$

$\Rightarrow k_i \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t$

$\therefore \theta_i = \theta_r$ ($\because k_i = k_r$) $\rightarrow$ Law of reflection

$\frac{\omega}{c} n_1 \sin\theta_i = \frac{\omega}{c} n_2 \sin\theta_t \Rightarrow n_1 \sin\theta_i = n_2 \sin\theta_t \rightarrow$ Law of refraction

2. **Griffiths Problem 9.12**

Problem 9.12 In the complex notation there is a clever device for finding the time average of a product. Suppose $f(\mathbf{r}, t) = A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta_a)$ and $g(\mathbf{r}, t) = B\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta_b)$. Show that $\langle fg\rangle = (1/2)\mathrm{Re}(\tilde{f}\tilde{g}^*)$, where the star denotes complex conjugation. [Note that this only works if the two waves have the same $\mathbf{k}$ and ω, but they need not have the same amplitude or phase.] For example,

$$\langle u\rangle = \frac{1}{4}\mathrm{Re}\left(\epsilon_0\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^* + \frac{1}{\mu_0}\tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*\right) \quad \text{and} \quad \langle\mathbf{S}\rangle = \frac{1}{2\mu_0}\mathrm{Re}\left(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*\right).$$

Time average of fg

$$\langle fg\rangle = \frac{1}{T}\int_0^T A\cos(k\cdot r - \omega t + \delta_a)\,B\cos(k\cdot r - \omega t + \delta_b)\,dt$$

$$= \frac{AB}{2T}\int_0^T \left[\cos(2k\cdot r - 2\omega t + \delta_a + \delta_b) + \cos(\delta_a - \delta_b)\right]dt$$

$$= \frac{AB}{2T}\left[-\frac{1}{2\omega}\sin(2k\cdot r - 2\omega t + \delta_a + \delta_b) + \cos(\delta_a - \delta_b)t\right]\Big|_0^T$$

$$= \frac{AB}{2T}\cos(\delta_a - \delta_b)T$$

$$= \frac{1}{2}AB\cos(\delta_a - \delta_b)$$

In the complex notation: $\tilde{f} = \tilde{A}e^{i(k\cdot r - \omega t)}$, $\tilde{g} = \tilde{B}e^{i(k\cdot r - \omega t)}$

where $\tilde{A} = Ae^{i\delta_a}$, $\tilde{B} = Be^{i\delta_b}$.

$$\tilde{f}\tilde{g}^* = \tilde{A}e^{i(k\cdot r - \omega t)}\cdot\tilde{B}^*e^{-i(k\cdot r - \omega t)}$$

$$= AB\,e^{i(\delta_a - \delta_b)}$$

$$\Rightarrow \frac{1}{2}\mathrm{Re}(\tilde{f}\tilde{g}^*) = \frac{1}{2}AB\cos(\delta_a - \delta_b) = \langle fg\rangle.$$

3. **Griffiths Problem 9.17 [Fresnel equation까지만 유도하시오]**

Problem 9.17 Analyze the case of polarization *perpendicular* to the plane of incidence (i.e. electric fields in the y direction, in Fig. 9.15). Impose the boundary conditions (Eq. 9.101), and obtain the Fresnel equations for $\tilde{E}_{0_R}$ and $\tilde{E}_{0_T}$. Sketch $(\tilde{E}_{0_R}/\tilde{E}_{0_I})$ and $(\tilde{E}_{0_T}/\tilde{E}_{0_I})$ as functions of θ_I, for the case $\beta = n_2/n_1 = 1.5$. (Note that for this β the reflected wave is *always* 180° out of phase.) Show that there is no Brewster's angle for *any* n_1 and n_2: $\tilde{E}_{0_R}$ is *never* zero (unless, of course, $n_1 = n_2$ and $\mu_1 = \mu_2$, in which case the two media are optically indistinguishable). Confirm that your Fresnel equations reduce to the proper forms at normal incidence. Compute the reflection and transmission coefficients, and check that they add up to 1.

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$$\tilde{E}_I = \tilde{E}_{0I}\, e^{i(k_I\cdot r-\omega t)}\,\hat{y} \ , \quad \tilde{E}_R = \tilde{E}_{0R}\, e^{i(k_R\cdot r-\omega t)}\,\hat{y} \ , \quad \tilde{E}_T = \tilde{E}_{0T}\, e^{i(k_T\cdot r-\omega t)}\,\hat{y}$$

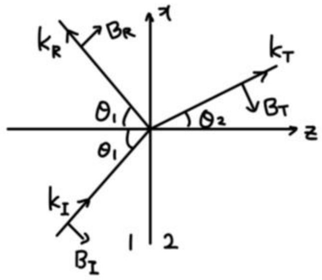
$$\bar{k}_I = k_I(\sin\theta_1, 0, \cos\theta_1) \ , \quad \bar{k}_R = k_R(\sin\theta_1, 0, -\cos\theta_1)$$

$$\bar{k}_T = k_T(\sin\theta_2, 0, \cos\theta_2)$$

$$\tilde{B}_I = \frac{1}{v_1}\tilde{E}_{0I}\, e^{i(k_I\cdot r-\omega t)}(-\cos\theta_1\hat{x} + \sin\theta_1\hat{z})$$

$$\tilde{B}_R = \frac{1}{v_1}\tilde{E}_{0R}\, e^{i(k_R\cdot r-\omega t)}(\cos\theta_1\hat{x} + \sin\theta_1\hat{z})$$

$$\tilde{B}_T = \frac{1}{v_2}\tilde{E}_{0T}\, e^{i(k_T\cdot r-\omega t)}(-\cos\theta_2\hat{x} + \sin\theta_2\hat{z})$$



$$\text{B.C.}: \left\{ \begin{array}{ll} \text{(i) } \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} & \text{(iii) } E_1^{\parallel} = E_2^{\parallel} \\ \text{(ii) } B_1^{\perp} = B_2^{\perp} & \text{(iv) } \frac{1}{\mu_1}B_1^{\parallel} = \frac{1}{\mu_2}B_2^{\parallel} \end{array} \right\}$$

$$\text{B.C. (iii)} \rightarrow \tilde{E}_{0I}\, e^{i(k_I\cdot r-\omega t)} + \tilde{E}_{0R}\, e^{i(k_R\cdot r-\omega t)} = \tilde{E}_{0T}\, e^{i(k_T\cdot r-\omega t)}$$

$$\Rightarrow \tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \quad (\because \text{law of reflection \& refraction}) \cdots (*)$$

$$\text{B.C. (ii)} \rightarrow \frac{1}{v_1}\tilde{E}_{0I}\sin\theta_1 + \frac{1}{v_1}\tilde{E}_{0R}\sin\theta_1 = \frac{1}{v_2}\tilde{E}_{0T}\sin\theta_2$$

$$\Rightarrow \tilde{E}_{0I} + \tilde{E}_{0R} = \left(\frac{v_1\sin\theta_2}{v_2\sin\theta_1}\right)\tilde{E}_{0T} = \tilde{E}_{0T} \quad (\text{law of refraction, same as B.C. (iii)})$$

$$\text{B.C. (iv)} \rightarrow \frac{1}{\mu_1}\left[-\frac{1}{v_1}\tilde{E}_{0I}\cos\theta_1 + \frac{1}{v_1}\tilde{E}_{0R}\cos\theta_1\right] = -\frac{1}{\mu_2}\frac{1}{v_2}\tilde{E}_{0T}\cos\theta_2$$

$$\Rightarrow \tilde{E}_{0I} - \tilde{E}_{0R} = \left(\frac{\mu_1 v_1\cos\theta_2}{\mu_2 v_2\cos\theta_1}\right)\tilde{E}_{0T} \equiv \alpha\beta\,\tilde{E}_{0T} \cdots (**)$$

with $\alpha \equiv \frac{\cos\theta_2}{\cos\theta_1}$, $\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}$

From the eq. (*) & (**),

$$\tilde{E}_{0T} = \left(\frac{2}{1+\alpha\beta}\right)E_{0I} \ , \quad \tilde{E}_{0R} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{0I} \rightarrow \overset{\text{real amplitude}}{E_{0R} = \left|\frac{1-\alpha\beta}{1+\alpha\beta}\right| E_{0I}}$$

$\Rightarrow$ Fresnel equations

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4. **Griffiths Problem 9.20**

Problem 9.20

(a) Show that the skin depth in a poor conductor ($\sigma \ll \omega\epsilon$) is $(2/\sigma)\sqrt{\epsilon/\mu}$ (independent of frequency). Find the skin depth (in meters) for (pure) water. (Use the static values of ϵ, μ, and σ; your answers will be valid, then, only at relatively low frequencies.)

(b) Show that the skin depth in a good conductor ($\sigma \gg \omega\epsilon$) is $\lambda/2\pi$ (where λ is the wavelength *in the conductor*). Find the skin depth (in nanometers) for a typical metal ($\sigma \approx 10^7 (\Omega\,\mathrm{m})^{-1}$) in the visible range ($\omega \approx 10^{15}$/s), assuming $\epsilon \approx \epsilon_0$ and $\mu \approx \mu_0$. Why are metals opaque?

(c) Show that in a good conductor the magnetic field lags the electric field by 45°, and find the ratio of their amplitudes. For a numerical example, use the "typical metal" in part (b).

(a) $\tilde{k} = k + i\kappa$, from eq. 126, $\kappa \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1\right]^{1/2}$

For a poor conductor $\sigma \ll \epsilon\omega$,

$$\kappa \simeq \omega\sqrt{\frac{\epsilon\mu}{2}}\left[1 + \frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2 - 1\right]^{1/2} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$$

$$\rightarrow d = \frac{1}{\kappa} = \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}} \;//$$

pure water $\begin{cases} \epsilon = 80.1\epsilon_0 \\ \mu = \mu_0(1+\chi_m) \simeq \mu_0 \quad (\chi_m = -9.0\times10^{-6}) \\ \sigma = 1/(2.5\times10^5) \end{cases}$

$\Rightarrow d = 1.19\times10^4\,\mathrm{m}$

(b) For a good conductor $\sigma \gg \epsilon\omega$,

$$\kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1\right]^{1/2} \simeq \omega\sqrt{\frac{\epsilon\mu}{2}}\sqrt{\frac{\sigma}{\epsilon\omega}} \simeq k$$

$$k = \frac{2\pi}{\lambda} \simeq \kappa \rightarrow d = \frac{1}{\kappa} = \frac{\lambda}{2\pi} \;//$$

Given constants : $\sigma \approx 10^7 (\Omega m)^{-1}$, $\omega \approx 10^{15}/s$, $\epsilon \simeq \epsilon_0$, $\mu \approx \mu_0$

$\Rightarrow \kappa \simeq 8\times10^7 \rightarrow d = \frac{1}{\kappa} \simeq 13\,\mathrm{nm}$ <u>opaque</u>

(c) $\phi \equiv \tan^{-1}(\kappa/k) \simeq 45° \;//$ $(\because k \simeq \kappa)$

From eq. 137,

$$\frac{B_0}{E_0} = \frac{K}{\omega} \equiv \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}} \overset{\sigma \gg \omega\epsilon}{\approx} \sqrt{\epsilon\mu\left(\frac{\sigma}{\epsilon\omega}\right)} = \sqrt{\frac{\mu\sigma}{\omega}}$$

$$\overset{(b)}{\Rightarrow} \frac{B_0}{E_0} \simeq 10^{-7}\,\mathrm{s/m}$$