# ON A CERTAIN SUM OF THE DERIVATIVES OF DIRICHLET L-FUNCTIONS

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ABSTRACT. We consider a sum of the derivatives of Dirichlet L-functions over the zeros of Dirichlet L-functions. We give an asymptotic formula for the sum.

### 1. Introduction

Let  $s = \sigma + it$  denote a complex variable. The Dirichlet *L*-function attached to  $\chi$  is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (\sigma > 1),$$

where  $\chi(n)$  is a Dirichlet character modulo q. For  $\chi$  (mod 1) we get the Riemann  $\zeta$ -function  $L(s,\chi)=\zeta(s)$ . The Generalized Riemann Hypothesis (GRH) states that all zeros of every Dirichlet L-function in the strip  $0 < \sigma < 1$  lie on the line  $\sigma = 1/2$ . We denote the zeros in the strip  $0 < \sigma < 1$  by  $\rho_{\chi} = \beta_{\chi} + i\gamma_{\chi}$ . A Dirichlet character is said to be primitive when it is not induced by any other character of modulus strictly less than q. The unique principal character modulo q is denoted by  $\chi_0$ . When  $\chi = \chi_0$ , we have  $L(s,\chi_0) = \zeta(s) \prod_{p|q} (1-p^{-s})$ , where, and in what follows, p denotes a prime number. For a Dirichlet character  $\chi$  (mod q) the Gauss sum is defined by

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) \exp\left(2\pi i \frac{a}{q}\right).$$

For a primitive character  $\chi \pmod{q}$  we have  $|\tau(\chi)| = \sqrt{q}$ . In this paper, T is a positive number which always tends to  $+\infty$  and  $\varepsilon > 0$ . Our main theorem is

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**Theorem 1.1.** Let  $c_1$  be a positive constant. Let  $\chi \pmod{q}$  be a primitive character. Then, uniformly for  $q \leq \exp(c_1 \sqrt{\log T})$ , we have

$$\sum_{0 < \gamma_{\chi} \le T} L'(\rho_{\chi}, \chi) = \frac{1}{4\pi} T \left( \log \frac{qT}{2\pi} \right)^2 + a_1 \frac{T}{2\pi} \log \frac{qT}{2\pi} + a_2 \frac{T}{2\pi} + a_3 + O\left( T \exp\left( -c\sqrt{\log T} \right) \right),$$

where the implicit constant is absolute, c is a positive absolute constant depends on  $c_1$  and

$$a_1 = \sum_{p|q} \frac{\log p}{p-1} + \gamma_0 - 1,$$

$$a_2 = \frac{1}{2} \left( \sum_{p|q} \frac{\log p}{p-1} \right)^2 + (\gamma_0 - 1) \sum_{p|q} \frac{\log p}{p-1}$$
$$-\frac{3}{2} \sum_{p|q} p \left( \frac{\log p}{p-1} \right)^2 + 1 - \gamma_0 - \gamma_0^2 + 3\gamma_1$$

with the stieltjes constants  $\gamma_0, \gamma_1$  and

$$a_3 = \frac{\omega \chi(-1)\tau(\overline{\chi})\tau(\overline{\omega}\chi)}{q\varphi(q)} \frac{L'(\beta,\omega)}{\beta} \left(\frac{qT}{2\pi}\right)^{\beta}$$

when  $L(s, \omega)$  with a quadratic character  $\omega \pmod{q}$  has an exceptional zero  $\beta$ , otherwise  $a_3 = 0$ .

Assuming the GRH, we can replace the error term by  $(qT)^{\frac{1}{2}+\varepsilon}$  uniformly for  $q \ll T^{1-\varepsilon}$ .

Remark 1. Let q be a prime power. If we could obtain the estimate

(1) 
$$\sum_{\gamma_{\chi} \le T} |L'(\rho_{\chi}, \chi)|^2 \ll T(\log qT)^4,$$

where the implicit constant is absolute, we could replace the error term by  $\sqrt{qT}(\log qT)^{\frac{7}{2}}$  under the GRH. We will give the details at the last section. In view of Gonek's formula (12), the above estimate (1) may be plausible.

When q = 1, the above theorem implies Fujii's Theorem 1 in [2]. Our proof is a generalization of his method. However, it is not easy to obtain his Theorem 2 in [2] and we give a weaker statement. Kaptan, Karabulut and Yıldırım [6] consider more general cases and give the

asymptotic formula, that is for  $\mu \geq 1$  and  $q \leq (\log T)^A$  with any fixed

$$\sum_{0 < \gamma_{\chi} < T} L^{(\mu)}(\rho_{\chi}, \chi) = \frac{(-1)^{\mu}}{\mu + 1} \frac{T}{2\pi} \left( \log \frac{qT}{2\pi} \right)^{\mu + 1} + O(T(\log T)^{\mu + \varepsilon})$$

for any fixed  $\varepsilon > 0$ . Our result is the case  $\mu = 1$  in their paper and gives a more sophisticated formula. Jakhlouti and Mazhouda [5] consider the sum

$$\sum_{\substack{\rho_{a,\chi}\\0<\gamma_{a,\chi}\leq T}}L'(\rho_{a,\chi},\chi)X^{\rho_{a,\chi}},$$

where  $\rho_{a,\chi} = \beta_{a,\chi} + i\gamma_{a,\chi}$  are the zeros of  $L(s,\chi) - a$  for any fixed complex number a and X is a fixed positive number. They also fix  $\chi$ throughout their paper. Hence our main theorem treats a special case of their sum, but our result gives a more precise form because we do not fix  $\chi$ .

### 2. Preliminaries

The Dirichlet L-function attached to a primitive character  $\chi \pmod{q}$ satisfies the functional equation

(2) 
$$L(s,\chi) = \Delta(s,\chi)L(1-s,\overline{\chi}),$$

where

$$\Delta(s,\chi) = \varepsilon(\chi) 2^s \pi^{s-1} q^{\frac{1}{2}-s} \Gamma(1-s) \sin \frac{\pi}{2} (s+\kappa)$$

when we put

$$\kappa = \frac{1 - \chi(-1)}{2}$$

and

$$\varepsilon(\chi) = \frac{\tau(\chi)}{i^{\kappa} \sqrt{q}}.$$

We note that  $\Delta(s,\chi)$  is a meromorphic function with only real zeros and poles satisfying the functional equation

$$\Delta(s, \chi)\Delta(1 - s, \overline{\chi}) = 1.$$

By Stirling's formula, we can show that

**Lemma 2.1.** For  $-1 \le \sigma \le 2$  and  $t \ge 1$ , we have

(3) 
$$\Delta(1-s,\chi) = \frac{\tau(\chi)}{\sqrt{q}} e^{-\frac{\pi i}{4}} \left(\frac{qt}{2\pi}\right)^{\sigma-\frac{1}{2}} \exp\left(it \log \frac{qt}{2\pi e}\right) \left(1 + O\left(\frac{1}{t}\right)\right)$$

and

(4) 
$$\frac{\Delta'}{\Delta}(s,\chi) = -\log\frac{qt}{2\pi} + O\left(\frac{1}{t}\right).$$

A theorem from [4] and an application of the Phragmen-Lindelöf principle yields the estimate

(5) 
$$L(s,\chi) \ll (q(|t|+2))^{\frac{3}{16}+\varepsilon} \text{ for } \frac{1}{2} \le \sigma \le 1 + \frac{1}{\log qT},$$

(6) 
$$L(s,\chi) \ll (q(|t|+2))^{\frac{1}{2}} \log q(|t|+2)$$
 for  $-\frac{1}{\log qT} \le \sigma < \frac{1}{2}$ 

uniformly in  $|t| \ll T$  for any non-principal Dirichlet character  $\chi \pmod{q}$ . When we assume the GRH, the bound of (5) can be replaced by  $(q(|t|+2))^{\varepsilon}$ . For the principal character, we need the restriction  $|s-1| \gg 1$  in (5). For the logarithmic derivative it is known that for  $q \geq 1$  and  $\chi \pmod{q}$ 

$$\frac{L'}{L}(s,\chi) = \sum_{|t-\gamma_{\chi}| \le 1} \frac{1}{s - \rho_{\chi}} + O(\log q(|t| + 2)) \quad \text{for} \quad -1 \le \sigma \le 2, \ |t| \ge 1$$

(see [9, p. 225]). For  $q \ge 1$ ,  $\chi \pmod{q}$  and  $t \ge 0$  we have (see [9, p. 220])

(8) 
$$N(t+1,\chi) - N(t,\chi) := \#\{\rho_{\chi} = \beta_{\chi} + i\gamma_{\chi} : t < \gamma_{\chi} \le t+1\} \\ \ll \log q(t+2).$$

Hence for any  $T_0 \ge 0$ , there exists a  $t = t(\chi), t \in (T_0, T_0 + 1]$ , such that

(9) 
$$\min_{\gamma_{\chi}} |t - \gamma_{\chi}| \gg \frac{1}{\log q(t+2)}.$$

By the expression (7), it follows that for  $q \ge 1$ ,  $\chi \pmod{q}$  and t satisfying (9)

(10) 
$$\frac{L'}{L}(\sigma + it, \chi) \ll (\log q(|t| + 2))^2 \quad \text{for} \quad -1 \le \sigma \le 2$$

uniformly. This estimate is valid for  $|s - \rho_{\chi}| \gg (\log(q(|t| + 2)))^{-1}$ though t is not satisfying (9).

We will apply the following approximate functional equation for  $L(s,\chi)$ .

**Lemma 2.2** (A. F. Lavrik [7]). We let  $0 \le \sigma \le 1$ ,  $2\pi xy = t$ ,  $x \ge 1$ and  $y \ge 1$ . Then for t > 0, we get

$$L(s,\chi) = \sum_{n \le x} \frac{\chi(n)}{n^s} + \Delta(s,\chi) \sum_{n \le y} \frac{\overline{\chi}(n)}{n^{1-s}} + O\left(\sqrt{q} \left(y^{-\sigma} + x^{\sigma-1} (qt)^{\frac{1}{2}-\sigma}\right) \log 2t\right).$$

On the other hand, for  $t > t_0 > 0$  and  $\sigma > 1$ , using partial summation, we get

(11) 
$$L(s,\chi) = \sum_{n \le qt} \frac{\chi(n)}{n^s} + O\left(\frac{q|s|}{\sigma}(qt)^{-\sigma}\right).$$

We will use the following modified Gonek's lemma ([3, Lemma 5]).

**Lemma 2.3.** Let  $\{b_n\}_{n=1}^{\infty}$  be a sequence of complex numbers such that  $b_n \ll n^{\varepsilon}$  for any  $\varepsilon > 0$ . Let a > 1 and let m be a non-negative integer. Then for any sufficiently large T,

$$\frac{1}{2\pi} \int_{1}^{T} \left( \sum_{n=1}^{\infty} \frac{b_n}{n^{a+it}} \right) \Delta (1 - a - it, \chi) \left( \log \frac{qt}{2\pi} \right)^{m} dt$$

$$= \frac{\tau(\chi)}{q} \sum_{1 \le n \le qT/2\pi} b_n e\left( -\frac{n}{q} \right) (\log n)^m + O\left( \left| \sum_{n=1}^{\infty} \frac{b_n}{n^a} \right| (qT)^{a-1/2} (\log qT)^m \right).$$

This is provided implicitly by Steuding in [10].

# 3. Proof in the unconditional case

In this section we prove the claim of the unconditional part of Theorem 1.1. Let  $(\log 2q)^{-1} \ll b \le 1$  and  $T \ge 2$  be such that

$$\min_{\gamma_{\chi}} |b - \gamma_{\chi}| \gg \frac{1}{\log 2q} \quad \text{and} \quad \min_{\gamma_{\chi}} |T - \gamma_{\chi}| \gg \frac{1}{\log qT}.$$

We prove the theorem under this situation. At the end of the proof, we remove this restriction. Let  $a = 1 + (\log qT)^{-1}$  and define the contour C as the positively oriented rectangular path with vertices a+ib, a+iT, 1-a+iT and 1-a+ib. By the residue theorem, our sum can be written as a contour integral

$$\sum_{0 < \gamma_{\chi} \le T} L'(\rho_{\chi}, \chi) = \frac{1}{2\pi i} \int_{C} \frac{L'}{L}(s, \chi) L'(s, \chi) ds + E,$$

where E consists of the terms  $L'(\rho_{\chi}, \chi)$  with  $0 < \gamma_{\chi} < b$ .

For zeros  $\rho_{\chi} = \beta_{\chi} + i\gamma_{\chi}$  with  $0 < \gamma_{\chi} < b$  we have

$$L'(\rho_{\chi}, \chi) \ll q^{\frac{1}{2}} (\log 2q)^2$$

by (5), (6) and the Cauchy's integral formula applied to the circle with centre  $\rho_{\chi}$  and radius  $(\log 2q)^{-1}$ . Therefore, by (8), we have

$$E = \sum_{0 < \gamma_{\chi} < b} L'(\rho_{\chi}, \chi) \ll q^{\frac{1}{2}} (\log 2q)^2 \sum_{0 < \gamma_{\chi} < b} 1 \ll q^{\frac{1}{2}} (\log 2q)^3.$$

Next we consider the contour integral

$$\frac{1}{2\pi i} \int_C \frac{L'}{L}(s,\chi) L'(s,\chi) ds$$

$$= \frac{1}{2\pi i} \left\{ \int_{a+ib}^{a+iT} + \int_{1-a+iT}^{1-a+ib} + \int_{a+iT}^{1-a+iT} + \int_{1-a+ib}^{a+ib} \right\} \frac{L'}{L}(s,\chi) L'(s,\chi) ds$$

$$= I_1 + I_2 + I_3 + I_4,$$

say.

By the Laurent expansion of the Riemann  $\zeta$ -function, it is easily seen that

$$I_{1} = \frac{1}{2\pi} \int_{b}^{T} \frac{L'}{L}(a+it,\chi)L'(a+it,\chi)dt$$

$$= \frac{1}{2\pi} \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{\chi(m)\Lambda(m)\chi(n)\log n}{(mn)^{a}} \int_{b}^{T} \frac{dt}{(mn)^{it}}$$

$$\ll \left| \frac{\zeta'}{\zeta}(a) \right| |\zeta'(a)| \ll (\log qT)^{3},$$

where  $\Lambda(m)$  is the von-Mangoldt function. To estimate the integral on the horizontal line, we will show the following lemma.

**Lemma 3.1.** Let  $\chi$  be a primitive character, then

$$\int_{1-a}^{a} L'(\sigma + iT, \chi) d\sigma \ll \sqrt{qT} \log qT.$$

*Proof.* Let

$$\delta = \frac{1}{\log qT}.$$

Then  $L(w,\chi)$  is analytic on the disk  $|s-w| \leq \delta$ , for  $s = \sigma + iT$  with  $1-a \leq \sigma \leq a$ . Therefore, by Cauchy's integral formula,

$$L'(s,\chi) = \frac{1}{2\pi i} \int_{|s-w|=\delta} \frac{L(w,\chi)}{(s-w)^2} dw$$

$$\ll \log qT \int_0^{2\pi} |L(s+\delta e^{i\theta},\chi)| d\theta.$$

Thus it suffices to prove that

$$\int_{1-a}^a \int_0^{2\pi} |L(s+\delta e^{i\theta},\chi)| d\theta d\sigma = \int_0^{2\pi} \int_{1-a}^a |L(s+\delta e^{i\theta},\chi)| d\sigma d\theta \ll \sqrt{qT}.$$

From the functional equation and, for  $1 - a \le \sigma \le 1/2$ , we have

$$\begin{split} &\int_{1-a}^{\frac{1}{2}} |L(s+\delta e^{i\theta},\chi)| d\sigma \\ &= \int_{1-a}^{\frac{1}{2}} |\Delta(s+\delta e^{i\theta},\chi) L(1-s-\delta e^{i\theta},\overline{\chi})| d\sigma \\ &= \int_{\frac{1}{2}}^{a} \left| \Delta(1-\sigma+iT+\delta e^{i\theta},\chi) L(\sigma-iT-\delta e^{i\theta},\overline{\chi}) \right| d\sigma. \end{split}$$

On the second equality, we change the variable  $\sigma$  to  $1 - \sigma$ . Since

$$\Delta(1 - (\sigma - iT - \delta e^{i\theta}), \chi)$$

$$= \overline{\Delta(1 - (\sigma + iT - \delta e^{-i\theta}), \overline{\chi})}$$

$$= \overline{\frac{\tau(\overline{\chi})}{\sqrt{q}} e^{\frac{\pi i}{4}} \left(\frac{qT}{2\pi}\right)^{\sigma - \delta \cos \theta - \frac{1}{2}} \exp\left(iT \log \frac{qT}{2\pi e}\right) \left(1 + O\left(\frac{1}{T}\right)\right)}$$

by Lemma 2.1, the integral can be bounded by

$$\int_{\frac{1}{2}}^{a} (qT)^{\sigma-\delta\cos\theta-\frac{1}{2}} \left| L(\sigma+iT-\delta e^{-i\theta},\chi) \right| d\sigma.$$

Therefore we obtain

$$\begin{split} &\int_{1-a}^{a} |L(s+\delta e^{i\theta},\chi)| d\sigma \\ &\ll \int_{\frac{1}{2}}^{a} |L(\sigma+iT+\delta e^{i\theta},\chi)| d\sigma \\ &+ \int_{\frac{1}{2}}^{a} (qT)^{\sigma-\delta\cos\theta-\frac{1}{2}} \left| L(\sigma+iT-\delta e^{-i\theta},\chi) \right| d\sigma \\ &\ll \int_{\frac{1}{2}}^{a} (qT)^{\sigma-\frac{1}{2}} \left| L(\sigma+iT\pm\delta e^{\pm i\theta},\chi) \right| d\sigma. \end{split}$$

On the last inequality, we use the facts that

$$(qT)^{\delta} = e$$

with  $\delta = (\log qT)^{-1}$ . This integral is

$$= \left\{ \int_{\frac{1}{2}}^{1} + \int_{1}^{a} \right\} (qT)^{\sigma - \frac{1}{2}} \left| L(\sigma + iT \pm \delta e^{\pm i\theta}, \chi) \right| d\sigma$$
$$= S_{1} + S_{2},$$

say. Using Lemma 2.2, we have

$$S_1 \ll (qT)^{-\frac{1}{2}} \sum_{n \ll \sqrt{qT}} n^{\delta} \int_{\frac{1}{2}}^1 \left(\frac{qT}{n}\right)^{\sigma} d\sigma + \sum_{n \ll \sqrt{qT}} n^{\delta - 1} \int_{\frac{1}{2}}^1 n^{\sigma} d\sigma + \sqrt{q} \log 2T \int_{\frac{1}{2}}^1 (qT)^{\frac{\sigma + \delta - 1}{2}} d\sigma \ll \sqrt{qT}.$$

On the other hand, by (11), we get

$$S_2 \ll (qT)^{-\frac{1}{2}} \sum_{n \le \frac{qT}{2}} n^{\delta} \int_1^a \left(\frac{qT}{n}\right)^{\sigma} d\sigma + \sqrt{qT} \int_1^a \frac{d\sigma}{\sigma}$$

$$\ll \sqrt{qT}.$$

Hence we complete the proof.

By (10) and the above lemma, we get

$$I_3 + I_4 \ll (\log qT)^2 \int_{1-a}^a |L'(\sigma + iT, \chi)| d\sigma$$
$$\ll \sqrt{qT} (\log qT)^3.$$

Now we consider  $I_2$ . By the functional equation, we have

$$\frac{L'}{L}(1-a+it,\chi)L'(1-a+it,\chi)$$

$$= \left(\frac{\Delta'}{\Delta}(1-a+it,\chi) - \frac{L'}{L}(a-it,\overline{\chi})\right)$$

$$\times (\Delta'(1-a+it,\chi)L(a-it,\overline{\chi}) - \Delta(1-a+it,\chi)L'(a-it,\overline{\chi}))$$

$$= \frac{\Delta'}{\Delta}(1-a+it,\chi)\Delta'(1-a+it,\chi)L(a-it,\overline{\chi})$$

$$-2\Delta'(1-a+it,\chi)L'(a-it,\overline{\chi})$$

$$+\Delta(1-a+it,\chi)\frac{L'}{L}(a-it,\overline{\chi})L'(a-it,\overline{\chi}).$$

Thus we can divide  $I_2$  into the following three integrals:

$$I_{2} = \frac{1}{2\pi} \int_{T}^{b} \frac{L'}{L} (1 - a + it, \chi) L' (1 - a + it, \chi) dt$$

$$= \frac{1}{\pi} \int_{b}^{T} \Delta' (1 - a + it, \chi) L' (a - it, \overline{\chi}) dt$$

$$- \frac{1}{2\pi} \int_{b}^{T} \frac{\Delta'}{\Delta} (1 - a + it, \chi) \Delta' (1 - a + it, \chi) L (a - it, \overline{\chi}) dt$$

$$- \frac{1}{2\pi} \int_{b}^{T} \Delta (1 - a + it, \chi) \frac{L'}{L} (a - it, \overline{\chi}) L' (a - it, \overline{\chi}) dt$$

$$= J_{1} + J_{2} + J_{3},$$

say. We take complex conjugates of  $J_i$  (i = 1, 2, 3) to apply Lemma 2.3. Then we have

$$\overline{J_1} = \frac{1}{\pi} \int_b^T \Delta'(1 - a + it, \chi) L'(a - it, \overline{\chi}) dt$$

$$= \frac{1}{\pi} \int_b^T \Delta'(1 - a - it, \overline{\chi}) L'(a + it, \chi) dt$$

$$= -\frac{1}{\pi} \int_b^T L'(a + it, \chi) \Delta(1 - a - it, \overline{\chi}) \log \frac{qt}{2\pi} dt$$

$$+ O\left(\sum_{n=1}^\infty \frac{\log n}{n^a} \int_b^T \frac{(qt)^{a - \frac{1}{2}}}{t} dt\right)$$

$$= \frac{1}{\pi} \int_b^T \sum_{n=1}^\infty \frac{\chi(n) \log n}{n^{a+it}} \Delta(1 - a - it, \overline{\chi}) \log \frac{qt}{2\pi} dt + O\left((qT)^{a - \frac{1}{2}} (\log qT)^2\right)$$

$$= 2 \frac{\tau(\overline{\chi})}{q} \sum_{1 \le n \le qT/2\pi} \chi(n) e^{\left(-\frac{n}{q}\right)} (\log n)^2 + O\left((qT)^{a-\frac{1}{2}} (\log qT)^3\right).$$

On the third equality, we use the approximation (4). For convenience, we put  $x = qT/2\pi$ . By partial summation, the above sum can be calculated as

$$\begin{split} &\sum_{1 \leq n \leq x} \chi(n) e\left(-\frac{n}{q}\right) (\log n)^2 \\ &= (\log x)^2 \sum_{m=1}^q \chi(m) e\left(-\frac{m}{q}\right) \sum_{\substack{n \leq x \\ n \equiv m \bmod q}} 1 \\ &- 2 \int_1^x \left(\sum_{m=1}^q \chi(m) e\left(-\frac{m}{q}\right) \sum_{\substack{n \leq y \\ n \equiv m \bmod q}} 1\right) \frac{\log y}{y} dy \\ &= \left(\frac{x}{q} \chi(-1) \tau(\chi) + O(\sqrt{q})\right) (\log x)^2 - 2 \int_1^x \left(\frac{y}{q} \chi(-1) \tau(\chi) + O(\sqrt{q})\right) \frac{\log y}{y} dy \\ &= \frac{\chi(-1) \tau(\chi)}{q} \left(x (\log x)^2 - 2 \int_1^x \log y dy\right) + O\left(\sqrt{q} (\log x)^2 + \sqrt{q} \int_1^x \frac{\log y}{y} dy\right) \\ &= \frac{\chi(-1) \tau(\chi)}{q} \left(x (\log x)^2 - 2x \log x + 2x\right) + O\left(\sqrt{q} (\log x)^2\right), \end{split}$$

and we can see that

$$\frac{\chi(-1)\tau(\chi)\tau(\overline{\chi})}{q^2} = \frac{\overline{\tau(\chi)}\tau(\chi)}{q^2} = \frac{q}{q^2} = \frac{1}{q}.$$

Therefore we obtain

$$J_1 = 2\left(\frac{T}{2\pi} \left(\log \frac{qT}{2\pi}\right)^2 - \frac{T}{\pi} \log \frac{qT}{2\pi} + \frac{T}{\pi}\right) + O\left((qT)^{a-\frac{1}{2}} (\log qT)^3\right).$$

Next we consider  $J_2$ . We have, by (4) again,

$$\overline{J_2} = -\frac{1}{2\pi} \int_b^T \frac{\Delta'}{\Delta} (1 - a + it, \chi) \Delta' (1 - a + it, \chi) L(a - it, \overline{\chi}) dt$$

$$= -\frac{1}{2\pi} \int_b^T L(a + it, \chi) \frac{\Delta'}{\Delta} (1 - a - it, \overline{\chi}) \Delta' (1 - a - it, \overline{\chi}) dt$$

$$= -\frac{1}{2\pi} \int_b^T \sum_{n=1}^\infty \frac{\chi(n)}{n^{a+it}} \Delta (1 - a - it, \overline{\chi}) \left(\log \frac{qt}{2\pi}\right)^2 dt$$

$$+ O\left(\sum_{n=1}^\infty \frac{1}{n^a} \int_b^T (qt)^{a - \frac{1}{2}} \frac{\log qt}{t} dt\right)$$

$$= -\frac{\tau(\overline{\chi})}{q} \sum_{1 \le n \le qT/2\pi} \chi(n) e\left(-\frac{n}{q}\right) (\log n)^2 + O\left((qT)^{a - \frac{1}{2}} (\log qT)^3\right).$$

This sum is the same as the previous one. Hence we get

$$J_2 = -\left(\frac{T}{2\pi} \left(\log \frac{qT}{2\pi}\right)^2 - \frac{T}{\pi} \log \frac{qT}{2\pi} + \frac{T}{\pi}\right) + O\left((qT)^{a-\frac{1}{2}} (\log qT)^3\right).$$

Finally, we calculate  $J_3$ . We have

$$\begin{split} \overline{J_3} &= \overline{-\frac{1}{2\pi}} \int_b^T \Delta(1-a+it,\chi) \frac{L'}{L} (a-it,\overline{\chi}) L'(a-it,\overline{\chi}) dt \\ &= -\frac{1}{2\pi} \int_b^T \frac{L'}{L} (a+it,\chi) L'(a+it,\chi) \Delta(1-a-it,\overline{\chi}) dt \\ &= -\frac{1}{2\pi} \int_b^T \left( \sum_{n=1}^\infty \frac{\chi(n) \Lambda(n)}{n^{a+it}} \right) \left( \sum_{n=1}^\infty \frac{\chi(n) \log n}{n^{a+it}} \right) \Delta(1-a-it,\overline{\chi}) dt \\ &= -\frac{\tau(\overline{\chi})}{q} \sum_{1 < mn < qT/2\pi} \chi(mn) e\left( -\frac{mn}{q} \right) \Lambda(m) \log n + O\left( (qT)^{a-\frac{1}{2}} (\log qT)^3 \right). \end{split}$$

By the orthogonality of Dirichlet characters, we see that

$$\sum_{mn \le x} \chi(m) \chi(n) e\left(-\frac{mn}{q}\right) \Lambda(m) \log n$$

$$= \sum_{a=1}^{q} \chi(a) \sum_{b=1}^{q} \chi(b) e\left(-\frac{ab}{q}\right) \sum_{\substack{mn \le x \\ m \equiv a \bmod q \\ n \equiv b \bmod q}} \Lambda(m) \log n$$

$$= \frac{1}{\varphi(q)^2} \sum_{\substack{\psi \bmod q \\ \psi' \bmod q}} \sum_{a=1}^q \overline{\psi}(a) \chi(a) \sum_{b=1}^q \overline{\psi'}(b) \chi(b) e\left(-\frac{ab}{q}\right)$$
$$\times \sum_{mn \le r} \psi(m) \psi'(n) \Lambda(m) \log n.$$

We will divide the sum into four parts, according to the following conditions:

- (i)  $\psi = \psi_0$ ,  $\psi' = \psi'_0$ , (ii)  $\psi = \psi_0$ ,  $\psi' \neq \psi'_0$ , (iii)  $\psi \neq \psi_0$ ,  $\psi' = \psi'_0$ ,
- (iv)  $\psi \neq \psi_0, \ \psi' \neq \psi'_0$

where  $\psi_0 = \psi'_0$  is the principal character modulo q. Before discussing further, we will remind some facts on the sum of Dirichlet characters (see [1, Sec. 8]). We define  $G(n,\chi)$  as

$$G(n,\chi) := \sum_{q=1}^{q} \chi(a) e\left(\frac{an}{q}\right).$$

If a Dirichlet character  $\chi \pmod{q}$  is primitive, then we have

$$G(a,\chi) = \overline{\chi}(a)\tau(\chi).$$

Now we consider the above four parts.

(i) In this case, we have

$$\begin{split} &\frac{1}{\varphi(q)^2} \sum_{a=1}^q \chi(a) \sum_{b=1}^q \chi(b) e\left(-\frac{ab}{q}\right) \sum_{mn \leq x} \psi_0(m) \psi_0(n) \Lambda(m) \log n \\ &= \frac{1}{\varphi(q)^2} \sum_{a=1}^q \chi(a) G(-a, \chi) \sum_{mn \leq x} \psi_0(m) \psi_0(n) \Lambda(m) \log n \\ &= \frac{\chi(-1) \tau(\chi)}{\varphi(q)} \sum_{mn \leq x} \psi_0(m) \psi_0(n) \Lambda(m) \log n. \end{split}$$

By Perron's formula we get

$$\sum_{mn \le x} \psi_0(m) \Lambda(m) \psi_0(n) \log n$$

$$= \frac{1}{2\pi i} \int_{a-iU}^{a+iU} \frac{L'}{L}(s, \psi_0) L'(s, \psi_0) \frac{x^s}{s} ds + R,$$

where R is the error term appearing in Perron's formula (see [8, p.140]) and satisfies that

$$R \ll \sum_{\substack{\frac{x}{2} < mn < 2x \\ mn \neq x}} |\Lambda(m) \log n| \min\left(1, \frac{x}{U|x - mn|}\right) + \frac{(4x)^a}{U} \sum_{mn=1}^{\infty} \frac{|\Lambda(m) \log n|}{(mn)^a}.$$

We will choose an appropriate U later. The first term of the error term R can be estimated as follows;

$$\frac{x}{U} \sum_{\frac{x}{2} < mn < x - 1} \frac{\Lambda(m) \log n}{x - mn} + \sum_{x - 1 \le mn \le x + 1} \Lambda(m) \log n$$

$$+ \frac{x}{U} \sum_{x + 1 < mn < 2x} \frac{\Lambda(m) \log n}{mn - x}$$

$$\ll \frac{x}{U} \log x \sum_{m < x - 1} \frac{\Lambda(m)}{m} \sum_{\frac{x}{2m} < n < \frac{x - 1}{m}} \frac{1}{\frac{x}{m} - n}$$

$$+ (\log x)^2 \sum_{x - 1 \le l \le x + 1} \sum_{l = mn} 1$$

$$+ \frac{x}{U} \log x \sum_{m < 2x} \frac{\Lambda(m)}{m} \sum_{\frac{x + 1}{m} < n < \frac{2x}{m}} \frac{1}{n - \frac{x}{m}}$$

$$\ll \frac{x}{U} (\log x)^2 \sum_{m < 2x} \frac{\Lambda(m)}{m} + (\log x)^2 \sum_{x - 1 \le l \le x + 1} d(l)$$

$$\ll \frac{x}{U} (\log x)^3 + x^{\varepsilon},$$

where d(l) is the divisor function. On the last estimates, we use

$$\sum_{m \le x} \frac{\Lambda(m)}{m} = \log x + O(1)$$

and

$$d(x) \ll x^{\varepsilon}.$$

The second is

$$\ll \frac{(4x)^a}{U} \sum_{mn=1}^{\infty} \frac{|\Lambda(m)\log n|}{(mn)^a} \ll \frac{x^a}{U} (\log qT)^3.$$

Therefore

$$R \ll \frac{x}{U}(\log x)^3 + x^{\varepsilon}.$$

Since  $L(s, \psi_0) = \zeta(s) \prod_{p|q} (1 - p^{-s})$ , there is an absolute constant C > 0 such that

$$L(s, \psi_0) \neq 0$$
 for  $\sigma \ge 1 - \frac{C}{\log(|t| + 2)}$ 

(see [8, p.172]). With regard to this zero-free region for  $L(s, \psi_0)$ , let  $a' = 1 - C/\log U$  and  $U = \exp\left(4c_1\sqrt{\log qT}\right)$ . By the residue theorem, the integral is

$$\frac{1}{2\pi i} \int_{a-iU}^{a+iU} \frac{L'}{L}(s,\psi_0) L'(s,\psi_0) \frac{x^s}{s} ds$$

$$= \operatorname{Res}_{s=1} \frac{L'}{L}(s,\psi_0) L'(s,\psi_0) \frac{x^s}{s}$$

$$+ \frac{1}{2\pi i} \left\{ \int_{a+iU}^{a'+iU} + \int_{a'+iU}^{a'-iU} + \int_{a'-iU}^{a-iU} \right\} \frac{L'}{L}(s,\psi_0) L'(s,\psi_0) \frac{x^s}{s} ds.$$

By an argument similar to the proof of Lemma 3.1, we can see that the integral on the horizontal line can be estimated as

$$\int_{a\pm iU}^{a'\pm iU} \frac{L'}{L}(s,\psi_0) L'(s,\psi_0) \frac{x^s}{s} ds \ll \frac{(\log qU)^3}{U} x^a (qU)^{\frac{3}{16} + \varepsilon} (a - a') \ll xU^{-\frac{1}{2}} = x \exp\left(-2c_1\sqrt{\log x}\right),$$

noting the condition  $q \leq \exp\left(c_1\sqrt{\log T}\right) \leq \exp\left(4c_1\sqrt{\log qT}\right) = U$  and (10). Since  $L'/L(s,\psi_0) \ll |s-1|^{-1}$  and  $L'(s,\psi_0) \ll |s-1|^{-2}$  in the neighbourhood around s=1, the integral on the vertical line can be bounded by

$$\ll x^{a'} (qU)^{\frac{3}{16} + \varepsilon} (\log qU)^3 \int_{-U}^{U} \frac{dt}{1 + |t|} + x^{a'} (\log qU)^3 \int_{-1}^{1} \frac{dt}{|a' + it|}$$

$$\ll x^{a'} (qU)^{\frac{3}{16} + \varepsilon} (\log U)^4$$

$$\ll x^{a'} U^{\frac{1}{2}} = x \exp\left(\left(2c_1 - \frac{C}{4c_1}\right)\sqrt{\log x}\right).$$

When we put  $c_1 = \sqrt{C}/4$ , we obtain that

$$\frac{1}{2\pi i} \int_{a-iU}^{a+iU} \frac{L'}{L}(s,\psi_0) L'(s,\psi_0) \frac{x^s}{s} ds$$

$$= \operatorname{Res}_{s=1} \frac{L'}{L}(s,\psi_0) L'(s,\psi_0) \frac{x^s}{s} + O\left(x \exp\left(-\frac{\sqrt{C}}{2}\sqrt{\log x}\right)\right).$$

Note that

$$\operatorname{Res}_{s=1}^{\underline{L'}} (s, \psi_0) L'(s, \psi_0) \frac{x^s}{s}$$

$$= \frac{1}{2!} \lim_{s \to 1} \frac{d^2}{ds^2} (s-1)^3 \frac{L'}{L} (s, \psi_0) L'(s, \psi_0) \frac{x^s}{s}.$$

To calculate this residue, we observe that

$$L'(s, \psi_0) = \zeta'(s) \prod_{p|q} (1 - p^{-s}) + \zeta(s) \left( \prod_{p|q} (1 - p^{-s}) \right)'$$

$$= \left( -\frac{1}{(s-1)^2} + \sum_{k=1}^{\infty} \gamma_k k(s-1)^{k-1} \right) \prod_{p|q} (1 - p^{-s})$$

$$+ \left( \frac{1}{s-1} + \sum_{k=0}^{\infty} \gamma_k (s-1)^k \right) \left( \prod_{p|q} (1 - p^{-s}) \sum_{p|q} \frac{\log p}{p^s - 1} \right)$$

and

$$\frac{L'}{L}(s, \psi_0) = \frac{\zeta'}{\zeta}(s) + \sum_{p|q} \frac{\log p}{p^s - 1}$$

$$= -\frac{1}{s - 1} + \sum_{k=0}^{\infty} \eta_k (s - 1)^k + \sum_{p|q} \frac{\log p}{p^s - 1},$$

where  $\gamma_k$  is the k-th Stieltjes constant and can be defined by the limit

$$\gamma_k = \lim_{n \to \infty} \left\{ \left( \sum_{m=1}^n \frac{(\log m)^k}{m} \right) - \frac{(\log n)^{k+1}}{k+1} \right\},\,$$

and  $\eta_k$  can be represented by the sum

$$\eta_k = (-1)^k \left\{ \frac{k+1}{k!} \gamma_k + \sum_{n=0}^{k-1} \frac{(-1)^{n-1}}{(k-n-1)!} \eta_n \gamma_{k-n-1} \right\}.$$

Hence we get

$$\begin{aligned} & \operatorname{Res} \frac{L'}{L}(s, \psi_0) L'(s, \psi_0) \frac{x^s}{s} \\ &= \frac{1}{2!} \lim_{s \to 1} \frac{d^2}{ds^2} \prod_{p|q} \left( 1 - \frac{1}{p^s} \right) \frac{x^s}{s} \\ &- \frac{2}{2!} \lim_{s \to 1} \frac{d}{ds} \prod_{p|q} (1 - p^{-s}) \left( \sum_{p|q} \frac{\log p}{p^s - 1} + \eta_0 + \sum_{p|q} \frac{\log p}{p^s - 1} \right) \frac{x^s}{s} \\ &- \frac{2}{2!} \lim_{s \to 1} \prod_{p|q} \left( 1 - \frac{1}{p^s} \right) \left\{ \gamma_1 + \gamma_0 \sum_{p|q} \frac{\log p}{p^s - 1} + \eta_1 \right. \\ &- \sum_{p|q} \frac{\log p}{p^s - 1} \left( \eta_0 + \sum_{p|q} \frac{\log p}{p^s - 1} \right) \right\} \frac{x^s}{s} \\ &= \frac{\varphi(q)}{q} x \left\{ \frac{1}{2} (\log x)^2 - \left( \sum_{p|q} \frac{\log p}{p - 1} + \gamma_0 + 1 \right) \log x - \frac{1}{2} \left( \sum_{p|q} \frac{\log p}{p - 1} \right)^2 \right. \\ &+ \frac{3}{2} \sum_{p|q} p \left( \frac{\log p}{p - 1} \right)^2 + (1 - \gamma_0) \sum_{p|q} \frac{\log p}{p - 1} + \gamma_0^2 + \gamma_0 - 3\gamma_1 + 1 \right\}. \end{aligned}$$

Therefore we can see that

$$\frac{\chi(-1)\tau(\chi)}{\varphi(q)} \sum_{mn \le x} \psi_0(m)\psi_0(n)\Lambda(m) \log n$$

$$= \frac{\tau(\overline{\chi})}{q} x \left\{ \frac{1}{2} (\log x)^2 - \left( \sum_{p|q} \frac{\log p}{p-1} + \gamma_0 + 1 \right) \log x - \frac{1}{2} \left( \sum_{p|q} \frac{\log p}{p-1} \right)^2 + \left( 1 - \gamma_0 \right) \sum_{p|q} \frac{\log p}{p-1} + \gamma_0^2 + \gamma_0 - 3\gamma_1 + 1 \right\}$$

$$+ O\left( x \exp\left( -\frac{\sqrt{C}}{2} \sqrt{\log x} \right) \right).$$

Here we note that  $\tau(\chi)/\varphi(q) \ll 1$ .

(ii) In the same way, we obtain

$$\frac{1}{\varphi(q)^2} \sum_{\substack{\psi' \bmod q \\ \psi' \neq \psi'_0}} \sum_{b=1}^q \overline{\psi'}(b) \chi(b) \sum_{a=1}^q \chi(a) e\left(-\frac{ab}{q}\right) 
\times \sum_{mn \le x} \psi_0(m) \psi'(n) \Lambda(m) \log n 
= \frac{\chi(-1)\tau(\chi)}{\varphi(q)^2} \sum_{\substack{\psi' \neq \psi'_0}} \sum_{b=1}^q \overline{\psi'}(b) \sum_{mn \le x} \psi_0(m) \psi'(n) \Lambda(m) \log n.$$

The sum of  $\overline{\psi}$  is 0. Hence we see that the sum in this case vanishes.

(iii) This case is the same as the case (ii).

(iv)

$$\frac{1}{\varphi(q)^2} \sum_{\substack{\psi \neq \psi_0 \\ \psi' \neq \psi'_0}} \sum_{a=1}^q \overline{\psi}(a) \chi(a) \sum_{b=1}^q \overline{\psi'}(b) \chi(b) e\left(-\frac{ab}{q}\right)$$

$$\times \sum_{mn \leq x} \psi(m) \psi'(n) \Lambda(m) \log n$$

$$= \frac{1}{\varphi(q)^2} \sum_{\substack{\psi \neq \psi_0 \\ \psi' \neq \psi'_0}} \sum_{a=1}^q \overline{\psi}(a) \chi(a) \psi'(-a) \overline{\chi}(-a) \tau(\overline{\psi'}\chi)$$

$$\times \sum_{mn \leq x} \psi(m) \psi'(n) \Lambda(m) \log n$$

$$= \frac{\chi(-1)}{\varphi(q)^2} \sum_{\substack{\psi \neq \psi_0 \\ \psi' \neq \psi'_0}} \sum_{a=1}^q \overline{\psi}(a) \psi'(-a) \tau(\overline{\psi'}\chi) \sum_{mn \leq x} \psi(m) \psi'(n) \Lambda(m) \log n$$

$$= \frac{\chi(-1)}{\varphi(q)^2} \sum_{\substack{\psi \neq \psi_0 \\ \psi' \neq \psi'_0}} \psi'(-1) \tau(\overline{\psi'}\chi) \sum_{a=1}^q \overline{\psi}(a) \psi'(a) \sum_{mn \leq x} \psi(m) \psi'(n) \Lambda(m) \log n$$

$$= \frac{\chi(-1)}{\varphi(q)} \sum_{\substack{\psi \neq \psi_0 \\ \psi' \neq \psi'_0}} \psi(-1) \tau(\overline{\psi}\chi) \sum_{mn \leq x} \psi(m) \psi(n) \Lambda(m) \log n.$$

To show the last equality, we use the fact that the sum over a does not equal to 0 if and only if  $\psi = \psi'$ .

In this case, we know the fact that there is an absolute constant C'>0 such that

$$L(s, \chi) \neq 0$$
 for  $\sigma > 1 - \frac{C'}{\log q(|t| + 2)}$ 

unless  $\chi$  is a quadratic character, in which case  $L(s,\chi)$  has at most one, necessarily real, zero  $\beta < 1$  (see [8, p. 360]). By the same argument as in the case (i), when we put  $c_1 = \sqrt{C'}/4$  we have

$$\sum_{mn \le x} \psi(m) \Lambda(m) \psi(n) \log n$$

$$= -L'(\beta, \psi) \frac{x^{\beta}}{\beta} + O\left(x \exp\left(-\frac{\sqrt{C'}}{2} \sqrt{\log x}\right)\right)$$

when  $L(s, \psi)$  with a quadratic character  $\omega$  has an exceptional zero  $\beta$ . If there is no exceptional zero, then the first term vanishes. Hence when  $L(s, \omega)$  has an exceptional zero  $\beta$  we have

$$\frac{\chi(-1)}{\varphi(q)} \sum_{\psi \neq \psi_0} \psi(-1)\tau(\overline{\psi}\chi) \sum_{mn \leq x} \psi(m)\psi(n)\Lambda(m) \log n$$

$$= -\frac{\chi(-1)}{\varphi(q)} \omega(-1)\tau(\overline{\omega}\chi) L'(\beta, \omega) \frac{x^{\beta}}{\beta} + O\left(\sqrt{q}x \exp\left(-\frac{\sqrt{C'}}{2}\sqrt{\log x}\right)\right),$$

otherwise the main term does not appear.

From the above, when we put  $c_1 = \min\{\sqrt{C}/4, \sqrt{C'}/4\}$  and  $c = c_1/2$ , we have

$$J_{3} = -\frac{T}{2\pi} \left\{ \frac{1}{2} \left( \log \frac{qT}{2\pi} \right)^{2} - \left( \sum_{p|q} \frac{\log p}{p-1} + \gamma_{0} + 1 \right) \log \frac{qT}{2\pi} - \frac{1}{2} \left( \sum_{p|q} \frac{\log p}{p-1} \right)^{2} + \frac{3}{2} \sum_{p|q} p \left( \frac{\log p}{p-1} \right) + (1-\gamma_{0}) \sum_{p|q} \frac{\log p}{p-1} + \gamma_{0}^{2} + \gamma_{0} + \gamma_{1} + 1 \right\} + \frac{\omega \chi(-1)\tau(\overline{\chi})\tau(\overline{\omega}\chi)}{q\varphi(q)} \frac{L'(\beta,\omega)}{\beta} \left( \frac{qT}{2\pi} \right)^{\beta} + O\left(T \exp\left(-c\sqrt{\log T}\right)\right).$$

We note that  $\tau(\chi)\sqrt{q}/q \ll 1$ .

To complete the proof, we take away the condition on T. When T increases continuously in  $|T - \gamma_{\chi}| \ll (\log qT)^{-1}$ , the number of relevant  $L'(\rho_{\chi}, \chi)$  is at most  $O(\log qT)$  and the order of each term is  $O((qT)^{\frac{3}{16}+\varepsilon})$ . Thus the contribution of these terms is smaller than the

error in our main theorem. Therefore the proof in the unconditional case is completed.

#### 4. The conditional estimate

In this section, we assume the GRH. We choose  $a' = 1/2 + (\log qT)^{-1}$  and U = qT. In the case (i), by Cauchy's theorem,

$$\frac{1}{2\pi i} \int_{a-iU}^{a+iU} \frac{L'}{L}(s,\psi_0) L'(s,\psi_0) \frac{x^s}{s} ds$$

$$= \operatorname{Res}_{s=1} \frac{L'}{L}(s,\psi_0) L'(s,\psi_0) \frac{x^s}{s}$$

$$+ \frac{1}{2\pi i} \left\{ \int_{a+iU}^{a'+iU} + \int_{a'+iU}^{a'-iU} + \int_{a'-iU}^{a-iU} \right\} \frac{L'}{L}(s,\psi_0) L'(s,\psi_0) \frac{x^s}{s} ds,$$

The integral on the horizontal line is

$$\int_{a'+iU}^{a\pm iU} \frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}ds \ll \frac{x^a}{U}(qU)^{\varepsilon}(\log qU)^3 \ll (qT)^{\varepsilon}.$$

As for the vertical line, we note that

$$\frac{L'}{L}(s,\psi_0) = \frac{\zeta'}{\zeta}(s) + \sum_{p|q} \frac{\log p}{p^s - 1} \ll \log 2q$$

for s = a' + it and  $0 \le |t| \le 1$ . Thus we have

$$\int_{a'-iU}^{a'+iU} \frac{L'}{L}(s,\psi_0) L'(s,\psi_0) \frac{x^s}{s} ds$$

$$= i \int_{-U}^{U} \frac{L'}{L} (a'+it,\psi_0) L'(a'+it,\psi_0) \frac{x^{a'+it}}{a'+it} dt$$

$$\ll x^{a'} (\log qU)^3 \int_{1}^{U} \frac{(qt)^{\varepsilon}}{t} dt + x^{a'} (\log 2q)^2 \int_{-1}^{1} \frac{q^{\varepsilon}}{a'} dt$$

$$\ll (qT)^{\frac{1}{2}+\varepsilon}.$$

Concerning the case (iv), we can see that

$$\sum_{nm \le x} \psi(m) \Lambda(m) \psi(n) \log n \ll (qT)^{\frac{1}{2} + \varepsilon}$$

by the similar argument. Therefore we can replace the error term in our theorem by  $(qT)^{\frac{1}{2}+\varepsilon}$ .

# 5. The Details of Remark 1

We consider the case when q is a prime power. Let  $q=p^{\alpha}$ ,  $a'=-(\log qT)^{-1}=1-a$  and U=qT. In the case (i), by the residue theorem

$$\frac{1}{2\pi i} \int_{a-iU}^{a+iU} \frac{L'}{L}(s,\psi_0) L'(s,\psi_0) \frac{x^s}{s} ds$$

$$= \operatorname{Res}_{s=1} \frac{L'}{L}(s,\psi_0) L'(s,\psi_0) \frac{x^s}{s} + \operatorname{Res}_{s=0} \frac{L'}{L}(s,\psi_0) L'(s,\psi_0) \frac{x^s}{s}$$

$$+ \sum_{\substack{\rho \neq 0 \\ |\Im \rho| \leq U}} L'(\rho,\psi_0) \frac{x^{\rho}}{\rho}$$

$$+ \frac{1}{2\pi i} \left\{ \int_{a+iU}^{a'+iU} + \int_{a'+iU}^{a'-iU} + \int_{a'-iU}^{a-iU} \right\} \frac{L'}{L}(s,\psi_0) L'(s,\psi_0) \frac{x^s}{s} ds,$$

where  $\rho$  runs over the zeros of  $L(s, \psi_0)$ . With regard to the residue at s = 0, we can see that

$$\frac{L'}{L}(s,\psi_0) = \frac{\zeta'}{\zeta}(s) + \frac{\log p}{p^s - 1} \quad (q = p^{\alpha})$$

and

$$\frac{\log p}{p^s - 1} = \frac{1}{s} \cdot \frac{s \log p}{e^{s \log p} - 1} = \frac{1}{s} \sum_{n=0}^{\infty} \frac{B_n}{n!} (s \log p)^n,$$

where  $B_n$  is the *n*-th Bernoulli number, and hence we have

$$\operatorname{Res}_{s=0}^{L'} \frac{L'}{L}(s, \psi_0) L'(s, \psi_0) \frac{x^s}{s}$$

$$= \lim_{s \to 0} \frac{d}{ds} s \frac{L'}{L}(s, \psi_0) L'(s, \psi_0) x^s$$

$$= \lim_{s \to 0} \frac{d}{ds} s \left( \frac{\zeta'}{\zeta}(s) + \frac{1}{s} \sum_{n=0}^{\infty} \frac{B_n}{n!} (s \log p)^n \right) L'(s, \psi_0) x^s$$

$$= L''(0, \psi_0) + \left( \frac{\zeta'}{\zeta}(0) + B_1 \log p + \log x \right) L'(0, \psi_0)$$

$$= 3\zeta'(0) \log p - \frac{3}{2}\zeta(0) (\log p)^2 + \zeta(0) \log x \ll (\log qT)^2.$$

The integral on the horizontal line is

$$\int_{1-a\pm iU}^{a\pm iU} \frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}ds$$

$$\ll \left\{\int_{\frac{1}{2}\pm iU}^{a\pm iU} + \int_{1-a\pm iU}^{\frac{1}{2}\pm iU} \right\} \frac{L'}{L}(s,\psi_0)L'(s,\psi_0)\frac{x^s}{s}ds$$

$$\ll \frac{x^a}{U}(qU)^{\varepsilon}(\log qU)^3 + \frac{x^{\frac{1}{2}}}{U}(qU)^{\frac{1}{2}}(\log qU)^4$$

$$\ll (qU)^{\varepsilon}(\log qU)^3 + \sqrt{q}(\log qU)^4.$$

On the integral along the vertical line, since  $|s - \rho_{\psi_0}| \gg 1$ , by (7), we can see that

$$\frac{L'}{L}(s,\psi_0) \ll \log q(|t|+2).$$

Therefore we have

$$\int_{1-a-iU}^{1-a+iU} \frac{L'}{L}(s,\psi_0) L'(s,\psi_0) \frac{x^s}{s} ds$$

$$= i \int_{-U}^{U} \frac{L'}{L} (1-a+it,\psi_0) L'(1-a+it,\psi_0) \frac{x^{1-a+it}}{1-a+it} dt$$

$$\ll (\log qU)^2 \left| \int_{-U}^{U} \zeta (1-a+it) \frac{dt}{1-a+it} \right|$$

$$\ll (\log qU)^2 \left( \log U \int_{1}^{U} t^{-\frac{1}{2}} dt + \int_{-1}^{1} \frac{dt}{|1-a+it|} \right)$$

$$\ll \sqrt{U} (\log qU)^3.$$

Here we use the well-known estimate

$$\zeta(s) \ll (|t|+2)^{\frac{1}{2}} \log(|t|+2)$$
 for  $-\frac{1}{\log T} \le \sigma < \frac{1}{2}$ .

The sum over  $\rho$  consists of two sums as

$$\sum_{\substack{\rho \neq 0 \\ |\Im \rho| \leq U}} L'(\rho, \psi_0) \frac{x^{\rho}}{\rho} = \sum_{|\gamma| \leq U} L' \left( \frac{1}{2} + i\gamma, \psi_0 \right) \frac{x^{\frac{1}{2} + i\gamma}}{\frac{1}{2} + i\gamma}$$

$$+ \sum_{\substack{\left|\frac{2\pi k}{\log p}\right| \leq U \\ k \neq 0}} L' \left( \frac{2\pi i k}{\log p}, \psi_0 \right) \frac{x^{\frac{2\pi k}{\log p} i} \log p}{2\pi i k}$$

$$= S_1 + S_2,$$

say. Since

$$L'\left(\frac{1}{2}+i\gamma,\psi_0\right) = \zeta'\left(\frac{1}{2}+i\gamma\right)(1-p^{-\frac{1}{2}-i\gamma}),$$

we have

$$S_1 \ll x^{\frac{1}{2}} \sum_{\gamma \leq U} \frac{\left| L'\left(\frac{1}{2} + i\gamma, \psi_0\right) \right|}{\gamma}$$

$$\ll x^{\frac{1}{2}} \left( \sum_{\gamma \leq U} \frac{\left| \zeta'\left(\frac{1}{2} + i\gamma\right) \right|^2}{\gamma} \right)^{\frac{1}{2}} \left( \sum_{\gamma \leq U} \frac{1}{\gamma} \right)^{\frac{1}{2}}$$

$$\ll x^{\frac{1}{2}} (\log U)^{\frac{7}{2}}$$

by partial summation and the fact that

(12) 
$$\sum_{0 < \gamma < T} \left| \zeta' \left( \frac{1}{2} + i\gamma \right) \right|^2 \approx T(\log T)^4$$

proved by Gonek [3].

On the other hand, since

$$L'\left(\frac{2\pi ik}{\log p}, \psi_0\right) = \zeta\left(\frac{2\pi ik}{\log p}\right)\log p,$$

we see that

$$S_2 \ll (\log p)^2 \sum_{\frac{2\pi k}{\log p} \le U} \frac{\left| \zeta\left(\frac{2\pi i k}{\log p}\right) \right|}{2\pi k} \ll \sqrt{U} \log U (\log q)^2$$

by the estimate

$$\zeta(s) \ll (|t|+2)^{\frac{1}{2}} \log(|t|+2)$$
 for  $-\frac{1}{\log T} \le \sigma < \frac{1}{2}$ 

again. Therefore we can see that

$$\frac{\chi(-1)\tau(\chi)}{\varphi(q)} \sum_{mn \le x} \psi_0(m)\psi_0(n)\Lambda(m) \log n$$

$$= \frac{\tau(\overline{\chi})}{q} x \left\{ \frac{1}{2} (\log x)^2 - \left( \sum_{p|q} \frac{\log p}{p-1} + \gamma_0 + 1 \right) \log x - \frac{1}{2} \left( \sum_{p|q} \frac{\log p}{p-1} \right)^2 + \left( 1 - \gamma_0 \right) \sum_{p|q} \frac{\log p}{p-1} + \gamma_0^2 + \gamma_0 - 3\gamma_1 + 1 \right\}$$

$$+ O\left( x^{\frac{1}{2}} (\log U)^{\frac{7}{2}} \right).$$

As for the case (iv), we need to deal with the Dirichlet L-functions with primitive and also imprimitive characters. However, it is sufficient to consider these with only primitive characters, for we put  $q = p^{\alpha}$ . For primitive characters, the integral on the vertical line can be estimated as

$$\begin{split} & \int_{1-a-iU}^{1-a+iU} \frac{L'}{L}(s,\psi) L'(s,\psi) \frac{x^s}{s} ds \\ & = \int_{1-a-iU}^{1-a+iU} \Delta(s,\psi) \left\{ \left( \frac{\Delta'}{\Delta}(s,\psi) \right)^2 L(1-s,\overline{\psi}) \\ & - 2 \frac{\Delta'}{\Delta}(s,\psi) L'(1-s,\overline{\psi}) + \frac{L'}{L}(1-s,\overline{\psi}) L'(1-s,\overline{\psi}) \right\} \frac{x^s}{s} ds \\ & \ll q^{a-\frac{1}{2}} \left| \int_{-U}^{U} \left( t^{a-\frac{1}{2}} \exp\left( it \log \frac{2\pi e}{qt} \right) + O(t^{a-\frac{3}{2}}) \right) \right. \\ & \times \left( (\log qU)^2 L(a-it,\psi) + \frac{L'}{L}(a-it,\psi) L'(a-it,\psi) \right) \frac{x^{1-a+it}}{1-a+it} dt \right| \\ & \ll x^{1-a} q^{a-\frac{1}{2}} \left( (\log U)^2 \sum_{n=1}^{\infty} \frac{1}{n^a} \left| \int_{1}^{U} \left( t^{a-\frac{3}{2}} \exp\left( it \log \frac{2\pi exn}{qt} \right) + O(t^{a-\frac{5}{2}}) \right) dt \right| \right. \\ & + \sum_{m=2}^{\infty} \frac{\Lambda(m)}{m^a} \sum_{n=1}^{\infty} \frac{\log n}{n^a} \left| \int_{1}^{U} \left( t^{a-\frac{3}{2}} \exp\left( it \log \frac{2\pi exmn}{qt} \right) + O(t^{a-\frac{5}{2}}) \right) dt \right| \right. \\ & + O(q^{a-\frac{1}{2}} (\log U)^3). \end{split}$$

Since

$$\frac{d^2}{dt^2} \left( t \log \frac{2\pi exn}{qt} \right) = -t^{-1},$$

by the second derivative test,

$$\int_{1}^{U} t^{a-\frac{3}{2}} \exp\left(it \log \frac{2\pi exn}{qt}\right) dt$$

$$\ll \sum_{l \leq \lceil \log U \rceil + 1} \int_{\frac{U}{2^{l}}}^{\frac{U}{2^{l-1}}} t^{a-\frac{3}{2}} \exp\left(it \log \frac{2\pi exn}{qt}\right) dt$$

$$\ll \sum_{l \leq \lceil \log U \rceil + 1} 1 \ll \log U.$$

Therefore we obtain

$$\int_{1-a-iU}^{1-a+iU} \frac{L'}{L}(s,\psi)L'(s,\psi)\frac{x^s}{s} ds \ll q^{a-\frac{1}{2}}(\log U)^4.$$

On the sum  $S_1$ , we assume the estimate (1). By partial summation and this assumption, we have

$$S_{1} \ll x^{\frac{1}{2}} \sum_{0 < \gamma_{\psi} \leq U} \frac{\left| L'\left(\frac{1}{2} + i\gamma_{\psi}, \psi\right) \right|}{\gamma_{\psi}}$$

$$\ll x^{\frac{1}{2}} \left( \sum_{0 < \gamma_{\psi} \leq U} \frac{\left| L'\left(\frac{1}{2} + i\gamma_{\psi}, \psi\right) \right|^{2}}{\gamma_{\psi}} \right)^{\frac{1}{2}} \left( \sum_{0 < \gamma_{\psi} \leq U} \frac{1}{\gamma_{\psi}} \right)^{\frac{1}{2}}$$

$$\ll x^{\frac{1}{2}} (\log U)^{\frac{7}{2}}.$$

On the other hand, the counterpart of the sum  $S_2$  does not appear. When  $\psi \pmod{q}$  is induced by  $\psi^* \pmod{d}$  with  $d \mid q$ , we see that

$$L(s,\psi) = L(s,\psi^*) \prod_{\substack{p \mid q \\ p \nmid d}} \left( 1 - \frac{\psi^*(p)}{p^s} \right).$$

However we assume that  $q = p^{\alpha}$ . Thus the products on the right-hand side is 1. Hence there is no zeros on the imaginary axis.

Therefore we can replace the estimate of the error term by

$$\sqrt{qT}(\log qT)^{\frac{7}{2}}$$
.

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