01 Solving Differential Equation using Power Series

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Solving Differential Equation

0.1 Series Solutions for Differential Equations

Example 1. Solve for

$$y' - y = 0$$

Let's assume that the following solution exists

$$y = f(x) = C_o + C_1 x + C_2 x^2 + C_3 x^3 + ...$$

$$y = \sum_{n=0}^{\infty} C_n x^n$$

Now get the derivative of the above power series,

$$y' = \sum_{n=1}^{\infty} n C_n x^{n-1}$$

Now substitute these values to the original equation for the values of y' and y.

$$\sum_{n=1}^{\infty} n C_n x^{n-1} - \sum_{n=0}^{\infty} C_n x^n = 0$$

The main goal of this process is to determine the values of the coefficients. To make the terms similar, we need to make changes to the indices of the summation to make them both n=0.

$$\sum_{n=0}^{\infty} (n+1) C_{n+1} x^{n-1+1} - \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) C_{n+1} x^{n} - \sum_{n=0}^{\infty} C_n x^n = 0$$

Combining the summations, we get

$$\sum_{n=0}^{\infty} [(n+1) C_{n+1} x^{n} - C_n x^n] = 0$$

Now factoring out the x^n,

$$\sum_{n=0}^{\infty}[(n+1)C_{n+1} - C_n]x^n = 0$$

Basically to satisfy the equation,

$$(n+1)C_{n+1} - C_n = 0$$

This will now give us a recursive relationship,

$$C_{n+1} = \frac{C_n}{n+1}$$

Now we plugin some values to n starting from zero,

$$n=0 \qquad C_1 = \frac{C_0}{0+1} \qquad C_1 = C_0 \tag{1}$$
$$n=1 \qquad C_2 = \frac{C_1}{1+1} \qquad C_2 = \frac{C_1}{2} \qquad C_2 = \frac{C_0}{2} \tag{2}$$
$$n=2 \qquad C_3 = \frac{C_2}{2+1} \qquad C_3 = \frac{\frac{C_0}{2}}{3} \qquad C_3 = \frac{1}{3}\frac{C_0}{2} \qquad C_3 = \frac{C_0}{3!} \tag{3}$$
$$n=3 \qquad C_4 = \frac{C_3}{3+1} \qquad C_4 = \frac{1}{4}\frac{C_0}{3!} \qquad C_4 = \frac{C_0}{4!} \tag{4}$$

To summarize the above series, we could write,

$$C_n = \frac{C_0}{n!} \tag{5}$$

We then plug this to our assumed solution of the differential equation,

$$y = \sum_{n=0}^{\infty} C_n x^n \tag{6}$$
$$y = \sum_{n=0}^{\infty} \frac{C_0}{n!} x^n \tag{7}$$
$$y = C_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{8}$$
$$\tag{9}$$

$$y = C_0 e^x \tag{10}$$