Linear Differential Equations

Notes by: ALEXIUS ACADEMIA

General Form:

$$b_0\,(x)\,\frac{d^2\,y}{dx^2} + b_1\,(x)\,\frac{d^{n-1}\,y}{dx^{n-1}} + \ldots + b_{n-1}\,(x)\,\frac{dy}{dx} + b_n(x)\,y = R(x)$$

If $R(x) = 0$, the differential equation is **homogeneous**

$R(x) \neq 0$, the differential equation is **non-homogeneous**

Solution of Homogeneous Differential Equations:

$$y = C_1\,y_1 + C_2\,y_2 + \ldots + C_n\,y_n$$

Solution of Non-Homogeneous Differential Equations:

$$y = y_c + y_p$$

where: y_c – Complimentary

y_p – Particular

Homogeneous Equation with Constant Coefficient:

$$a_0\,y^n + a_1\,y^{(n-1)} + \ldots + a_{n-1}\,y' + a_n\,y = 0$$

Introducing a notation:

D - Differential Operator

$D = \frac{d}{dx}$

$Dy = \frac{dy}{dx}$

$y^{(n)} = \frac{d^n\,y}{dx^n} = D^n\,y$

Rules of Auxillary Equation

Type of Root	**General Solution**
Distinct	$y = C_1\,e^{r_1\,x} + C_2\,e^{r_2\,x}$
Repeated Single Root	$y = C_1\,e^{rx} + C_2\,x\,e^{rx}$
Complex or Imaginary $\alpha \pm \beta i$	$y = e^{\alpha x}\,[C_1\cos(\beta x) + C_2\sin(\beta x)]$

Example 1:

Solve for **y'' + y' - 6y = 0**

Treat this as a quadratic equation:

$$r^2 + r - 6 = 0$$

By factoring we get,

$$(r + 3)(r - 2) = 0$$

We now have the roots.

$$r = -3$$

$$r = 2$$

By observation, these are distinct roots, so we apply the general solution:

$$y = C_1 e^{-3x} + C_2 e^{2x}$$

Example 2:

Solve for **y'' - 8y' + 16y = 0**

Treat this as a quadratic equation:

$$r^2 - 8r + 16 = 0$$

By factoring we get,

$$(r - 4)(r - 4) = 0$$

We now have the root.

$$r = 4$$

By observation, these is a repeated root, so we apply the general solution:

$$y = C_1 e^{4x} + C_2 x e^{4x}$$

Example 3:

Solve for **y'' - 6y' + 13y = 0**

Treat this as a quadratic equation:

$$r^2 - 6r + 13 = 0$$

Since we can't factor this, we use the quadratic formula,

$$r = \frac{6 \pm \sqrt{(-6)^2 - 4(13)}}{2(1)} = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i$$

We now have the root.

$$r = 3 \pm 2i$$

Base on this, the roots are imaginary, now

$$\alpha = 3$$

$$\beta = 2$$

$$y = e^{3x}[C_1 \cos(2x) + C_2 \sin(2x)]$$

Example 4:

Solve for **y'' + 3y' + 2y =** x^2

Now since this equation is a non-homogeneous equation, we are going to have two parts in the solution. The complimentary and the particular part.

For the complimentary we have,

$$r^2 + 3r + 2 = 0$$

By factoring, we get

$$(r + 1)(r + 2) = 0$$

We now have the roots,

$$r = -1 \text{ and } r = -2$$

Our complimentary solution is,

$$y_c = C_1 e^{-x} + C_2 e^{-2x}$$

For the particular solution, the method that will be used is the method of undetermined coefficients:

In this method, we treat the left equation as a quadratic again and try to determine the coefficient.

$$y_p = \mathrm{A}x^2 + \mathrm{B}x + C$$

Now if we get the derivative of this, we get

$$y_p' = 2\mathrm{A}x + B$$

Then if we get the derivative again, we obtain

$$y_p'' = 2A$$

Then we plug this in to the original D.E.

$$2A + 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2$$

Let's try to rearrange the equation and expand,

$$2Ax^2 + 2Bx + 2C + 2A + 6Ax + 3B = x^2$$

Factor out the x^2 and x, we get

$$2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = (1)x^2 + 0(x) + 0$$

By taking each corresponding order of term from each side of the equation,

$$2Ax^2 = x^2 \qquad , A = 1/2$$

$$(6A + 2B) = 0 \qquad , B = -3/2$$

$$(2A + 3B + 2C) = 0 \qquad , C = 7/4$$

Then plug these coefficient to our particular solution

$$y_p = (1/2)x^2 - (3/2)x + (7/4)$$

The general solution now is

$$\mathbf{y = (1/2)x^2 - (3/2)x + (7/4) + C_1 e^{-x} + C_2 e^{-2x}}$$