Problem Set #5

1a. If we have

$$R = \{X: X_n > c\}$$

So,

$$P(Type\ I\ error) = P\left(X_{(n)} > c \,\middle|\, \theta = \frac{1}{2}\right)$$
$$P(Type\ II\ error) = 1 - P\left(X_{(n)} > c \,\middle|\, \theta > \frac{1}{2}\right)$$

Then consider each case for c:

$$\beta(\theta) = P\big(X_{(n)} > c \mid \theta\big) = \begin{cases} 1, \ \ if\ c < 0 \\ 1 - \left(\frac{c}{\theta}\right)^n, \ \ if\ 0 \le c < \theta \\ 0, \ \ if\ \theta \le c \end{cases}$$

1b. Using equation 14 from Lecture 11:

$$\alpha = \sup_{\theta \in \Theta_0} \beta(\theta)$$

Which is the largest probability of type I error:

$$\left(1 - \left(\frac{c}{\frac{1}{2}}\right)^n\right) = \alpha$$

$$1 - (2c)^n = \alpha$$

$$c = \frac{(1-\alpha)^{\frac{1}{n}}}{2}$$

1c. Plugging in to b:

$$0.48 = \frac{(1-\alpha)^{\frac{1}{20}}}{2}$$

$$\alpha = 0.558$$

2a. Following the notes with $\sigma^2 = 1$:

$$R_\alpha = \left\{X : \bar{X}_n > \frac{\sigma z_{1-\alpha}}{\sqrt{n}}\right\}$$

$$\beta(\mu) = P\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} > \frac{\sqrt{n}(c-\mu)}{\sigma}\right)$$

$$\beta(\mu) = 1 - \phi\big(\sqrt{n}(c-\mu)\big)$$

We then take $\mu = 0$ as shown in Lecture 11:

$$\alpha = 1 - \phi\big(\sqrt{n}(c)\big)$$

$$c = \frac{\phi^{-1}(1-\alpha)}{\sqrt{n}}$$

2b. Finding the power under H_1:

$$\beta(\mu = 1) = P(\bar{X}_n > c \mid \mu = 1)$$

$$\beta(\mu = 1) = 1 - \phi\big(\sqrt{n}(c-1)\big)$$

Plug in c from part a:

$$= 1 - \phi\left(\sqrt{n}\left(\frac{\phi^{-1}(1-\alpha)}{\sqrt{n}} - 1\right)\right)$$

2c. Taking the limit for H_1:

$$lim_{n\to\infty}\left(1 - \phi\left(\sqrt{n}\left(\frac{\phi^{-1}(1-\alpha)}{\sqrt{n}} - 1\right)\right)\right)$$

$$lim_{n\to\infty}\left(1 - \phi\big(\phi^{-1}(1-\alpha) - \sqrt{n}\big)\right)$$

$$= 1 - 0 = 1$$

For H_0:

$$\lim_{n\to\infty}(1-\phi\big(\phi^{-1}(1-\alpha)\big)=\alpha$$

3a. Normal Wald Test for λ:

$$\left|\frac{\hat{\lambda}-\lambda}{\widehat{se}}\right|>z_{1-\frac{\alpha}{2}}$$

We can find the estimator of λ:

$$L(\lambda: X_1,\dots,X_n)=\prod_{i=1}^{n} e^{-\lambda}\ \frac{\lambda^{X_i}}{X_i!}$$

$$\ln L(\lambda: X_1,\dots,X_n)=-n\lambda-\sum_{i=1}^{n}\ln(X_i!)+\ln(\lambda)\sum_{i=1}^{n}X_i$$

$$\frac{d}{d\lambda}(\ln L)=-n+\frac{1}{\lambda}\sum_{i=1}^{n}X_i=0$$

$$\hat{\lambda}=\lambda_{MLE}=\frac{1}{n}\sum_{i=1}^{n}X_i$$

This is a maximum because the second derivative is always negative:

$$\frac{d^2}{d\lambda^2}=-\frac{1}{\lambda^2}\sum_{i=1}^{n}X_i$$

Next, we find the standard error using asymptotic normality from Lecture 10:

$$\widehat{se}=\frac{1}{\sqrt{nI(\hat{\lambda})}}=\frac{1}{\sqrt{\frac{n}{\hat{\lambda}}}}=\sqrt{\frac{\hat{\lambda}}{n}}$$

Therefore, we can construct the test. We reject the null hypothesis when

$$\left|\frac{\frac{1}{n}\sum_{i=1}^{n}X_i-\lambda_0}{\sqrt{\frac{\frac{1}{n}\sum_{i=1}^{n}X_i}{n}}}\right|>z_{1-\frac{\alpha}{2}}$$