Problem Set #3

1. Find $Cov(\hat{F}_n(x), \hat{F}_n(y))$

$$Cov(\hat{F}_n(x), \hat{F}_n(y)) = E[\hat{F}_n(x) * \hat{F}_n(y)] - E[\hat{F}_n(x)] * E[\hat{F}_n(y)]$$

We know that

$$E\left[\hat{F}_n(x)\right] = F(x)$$

So,

$$Cov(\hat{F}_n(x), \hat{F}_n(y)) = E[\hat{F}_n(x) * \hat{F}_n(y)] - F(x) * F(y)$$

$$E\left[\hat{F}_n(x) * \hat{F}_n(y)\right] = \frac{1}{n^2}\left[\sum_i I(X_i \le x) \sum_j I\left(X_j \le y\right)\right]$$

We divide this into two cases, where $i = j$ and $i \neq j$:

When $i \neq j$, which happens in $n(n-1)$ cases, these are independent, so we just have $F(x)F(y)$ since the expectation of the product of two independent variables is the product of the expectations.

When $i \neq j$, which happens in n cases, we have

$$E[I(X_i \le x) * I(X_i \le y)]$$

This will result in either $F(x)$ if $x < y$ or $F(y)$ if $y < x$. Therefore, we have two cases for our answer. If $x < y$:

$$Cov(\hat{F}_n(x), \hat{F}_n(y)) = \frac{1}{n^2}[n(n-1)F(x)F(y) + nF(x)] - F(x)F(y)$$

$$= \frac{n-1}{n}F(x)F(y) + \frac{F(x)}{n} - F(x)F(y)$$

$$= \frac{F(x) - F(x)F(y)}{n}$$

If $y < x$:

$$Cov(\hat{F}_n(x), \hat{F}_n(y)) = \frac{F(y) - F(x)F(y)}{n}$$

2. Find the plug-in estimate of k_F:

$$k_F = \frac{\int (x-\mu_F)^3 dF(x)}{\left(\int (x-\mu_F)^2 dF(x)\right)^{\frac{3}{2}}}$$

Using the method given in our lecture notes:

$$\widehat{k_F} = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X_n})^3}{(\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X_n})^2)^{\frac{3}{2}}}$$

$$\widehat{k_F} = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X_n})^3}{(\hat{\sigma}_n^2)^{\frac{3}{2}}}$$

$$\widehat{k_F} = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X_n})^3}{\hat{\sigma}_n^3}$$

3. Code is attached

4a. The representation of θ suggest that we would need the maximum value, since $F(x) = 1$ will occur when we have our max sample.

$$\hat{\theta}_n = X_{(n)} = \max\{X_1, \dots, X_n\}$$

4b. To find the bias, we use

$$B[\hat{\theta}_n] = E[\hat{\theta}_n] - \theta$$

Using the hint:

$$= \frac{n\theta}{n+1} - \theta$$

$$B[\hat{\theta}_n] = \frac{-\theta}{n+1}$$

4c. To find $\hat{\theta}_n^J$, we use the jackknife method:

$$\hat{\theta}_n^J = n\hat{\theta}_n - (n-1)\bar{\theta}_n^J$$

We know that

$$\bar{\theta}_n^J = \frac{1}{n}\sum_{i=1}^{n} \hat{\theta}_n^{-i}$$

So, we create some jackknife replications. For the first sample, we take out the first data point, so the estimate $\hat{\theta}_n^{-1}$ will be $X_{(n)}$, since that will still be the maximum. This will be the case until the last jackknife sample, where $X_{(n-1)}$ will be the new maximum. Therefore,

$$\bar{\theta}_n^J = \frac{(n-1)X_{(n)} + X_{(n-1)}}{n}$$

$$\hat{\theta}_n^J = nX_{(n)} - (n-1)\frac{(n-1)X_{(n)} + X_{(n-1)}}{n}$$

$$= \frac{n^2X_{(n)} - n^2X_{(n)} + 2nX_{(n)} - X_{(n)} - nX_{(n-1)} + X_{(n-1)}}{n}$$

$$\hat{\theta}_n^J = \frac{2nX_{(n)} - X_{(n)} - nX_{(n-1)} + X_{(n-1)}}{n}$$

4d. We have that

$$B\left[\hat{\theta}_n^J\right] = E\left[\hat{\theta}_n\right] - E\left[\hat{B}_J\left[\hat{\theta}_n\right]\right] - \theta$$

$$B\left[\hat{\theta}_n^J\right] = \frac{n\theta}{n+1} - E\left[(n-1)(\bar{\theta}_n^J - \hat{\theta}_n)\right] - \theta$$

$$= \frac{n\theta}{n+1} - (n-1)\left(\frac{(n-1)\left(\frac{n\theta}{n+1}\right) + \left(\frac{(n-1)\theta}{n+1}\right)}{n} - \frac{n\theta}{n+1}\right) - \theta$$

$$B\left[\hat{\theta}_n^J\right] = -\frac{\theta}{n(n+1)}$$

5a. To find the bias, we need to find the expected value of the estimator.

$$E\left[e^{\bar{X}_n}\right] = E[e^{\frac{1}{n}\sum_{i=1}^{n} X_i}]$$

$$= E[e^{\frac{1}{n}X_1} * e^{\frac{1}{n}X_2} * \ldots * e^{\frac{1}{n}X_n}]$$

$$= e^{\left(\frac{1}{n}\mu + \frac{\sigma^2}{2n^2}\right)n}$$

And σ^2 = 1:

$$E\left[e^{\bar{X}_n}\right] = e^{\mu + \frac{1}{2n}}$$

Therefore,

$$B\left[\hat{\theta}_n\right] = e^{\mu + \frac{1}{2n}} - e^{\mu}$$

$$= e^{\mu}(e^{\frac{1}{2n}} - 1)$$

Using Taylor expansions:

$$= e^{\mu}\left(1 + \frac{1}{2n} + \frac{1}{8n^2} + O\left(\frac{1}{n^3}\right) - 1\right)$$

$$= \frac{e^{\mu}}{2n} + \frac{e^{\mu}}{8n^2} + O\left(\frac{1}{n^3}\right)\ as\ n \to \infty$$

Therefore, the assumption holds with

$$a = \frac{e^{\mu}}{2}$$

$$b = \frac{e^{\mu}}{8}$$

Rest of 5 is in code.