

# Chapter 1

## Abelian Varieties

These are notes for BUNTES Fall 2017, the topic is [Abelian varieties](#), they were last updated October 23, 2017. We are using Milne's [abelian varieties](#) notes primarily, for more details see [the webpage](#). These notes are by Alex, feel free to email me at [alex.j.best@gmail.com](mailto:alex.j.best@gmail.com) to report typos/suggest improvements, I'll be forever grateful.

### 1.1 Introduction (Angus)

#### 1.1.1 Definitions

**Definition 1.1.1** (Abelian varieties). An **abelian variety** is a [complete](#) connected [algebraic group](#).

**Definition 1.1.2** (Algebraic groups). An **algebraic group** is an algebraic variety  $G$  along with regular maps  $m: G \times G \rightarrow G$ ,  $e: * \rightarrow G$ ,  $\text{inv}: G \rightarrow G$  such that the following diagrams commute.

$$\begin{array}{ccccc}
 * \times G & \xrightarrow{e \times \text{id}} & G \times G & \xleftarrow{\text{id} \times e} & G \times * \text{ identity} \\
 & \searrow \sim & \downarrow m & \swarrow \sim & \\
 & & G & & 
 \end{array}$$

$$\begin{array}{ccccc}
 G & \xrightarrow{\text{inv}, \text{id}} & G \times G & \xleftarrow{\text{id}, \text{inv}} & G \text{ Inverse} \\
 \downarrow & & \downarrow m & & \downarrow \\
 * & \xrightarrow{e} & G & \xleftarrow{e} & *
 \end{array}$$

$$\begin{array}{ccc}
 G \times G \times G & \xrightarrow{\text{id} \times m} & G \times G \text{ Associativity} \\
 m \times \text{id} \downarrow & & \downarrow m \\
 G \times G & \xrightarrow{m} & G
 \end{array}$$

**Definition 1.1.3** (Complete varieties). A variety  $X$  is **complete** if every projection map

$$X \times Y \rightarrow Y$$

is closed.

**Example 1.1.4** (Abelian varieties).

- Elliptic curves.
- Weil restriction  $\text{Res}_{K/Q} E$  of an elliptic curve  $E$ .
- Jacobian varieties of curves.

Plan:

- Some motivation via elliptic curves.
- Gathering some material about “completeness”.
- Prove that [abelian varieties](#) are abelian.

### 1.1.2 Elliptic curves ( $\text{char}(k) \neq 2, 3$ )

**Theorem 1.1.5.** *TFAE for a projective curve  $E$  over  $k$ .*

1.  $E$  is given by  $Y^2Z = X^3 + aXZ^2 + bZ^3$ ,  $4a^3 + 27b^2 \neq 0$ .
2.  $E$  is nonsingular of genus 1 with a distinguished point  $P_0$ .
3.  $E$  is nonsingular with an [algebraic group](#) structure.
4. (if  $k \subseteq \mathbb{C}$ ) such that  $E(\mathbb{C}) = \mathbb{C}/\Lambda$  for some lattice  $\Lambda \subseteq \mathbb{C}$ .

*Proof.* Strategy: [Item 1](#)  $\iff$  [Item 2](#)  $\iff$  [Item 3](#) and [Item 2](#)  $\implies$  [Item 4](#)  $\implies$  [Item 1](#).

[Item 1](#)  $\implies$  [Item 2](#) is done.

[Item 2](#)  $\implies$  [Item 1](#): Riemann-Roch states that  $l(D) = l(K - D) + \deg(D) + 1 - g$  so here  $l(D) = l(K - D) + \deg(D)$  further is  $D > 0$  then  $l(K - D) = 0$  in which case  $l(D) = \deg(D)$ . Consider  $L(nP_0)$  for  $n > 0$  Riemann-Roch implies that  $l(nP_0) = n$  then it always contains the constants.

$$L(P_0) = k$$

$$L(2P_0) = k \oplus kx$$

$$L(3P_0) = k \oplus kx \oplus ky$$

$$\vdots$$

$$L(6P_0) = k \oplus kx \oplus ky \oplus kx^2 \oplus ky^2 \oplus kxy \oplus kx^3/\sim$$

so we must have a relation which after manipulation is of the desired form. We get an embedding

$$E \hookrightarrow \mathbf{P}^2$$

$$P \mapsto (x(P) : y(P) : 1) (P \neq P_0)$$

$$P_0 \mapsto (0 : 1 : 0)$$

and thus  $E$  is of the desired form.  $\square$

**Definition 1.1.6** (Elliptic curves). An **elliptic curve** over  $k$  is any/all of [that 5](#).

Which of the above characterisations generalise to abelian varieties?

1. No, in general we don't know that the equations look like.
2. One could possibly replace “genus” with a condition on the dimension of cohomology groups.
3. Yes, this is essentially the definition.
4. Yes, stay tuned!

### 1.1.3 Complete varieties

Idea: if  $X \times Y$  had product topology (instead of its Zariski topology) then **complete** is equivalent to compact.

We'd like to gather a few results about **complete varieties** we can use to access properties of **abelian varieties** (like abelianness).

**Proposition 1.1.7.** *Let  $V$  be a **complete variety**. Given any morphism  $\phi: V \rightarrow W$   $\phi(V)$  is closed.*

*Proof.* Let  $\Gamma_\phi = \{(v, \phi(v))\} \subseteq V \times W$  be the graph of  $\phi$ . It's a closed subvariety of  $V \times W$ . Under the projection  $V \times W \rightarrow W$ , the image of  $\Gamma_\phi$  is  $\phi(V)$  and thus closed.  $\square$

**Corollary 1.1.8.** *If  $V$  is **complete** and connected, any regular function on  $V$  is constant.*

*Proof.* A regular function is a morphism  $f: V \rightarrow \mathbf{A}^1$ . By the above  $f(V) \subseteq \mathbf{A}^1$  is closed, and this is a finite set of points. But connected implies we just have one point.  $\square$

**Corollary 1.1.9.** *Let  $V$  be a **complete** connected variety. Let  $W$  be an affine variety. Given  $\phi: V \rightarrow W$ , then  $\phi(V)$  is a point.*

*Proof.* We have an embedding  $W \hookrightarrow \mathbf{A}^n$ . On  $\mathbf{A}^n$  we have the coordinate functions  $\mathbf{A}^n \xrightarrow{x_i} \mathbf{A}^1$ . The composition

$$V \xrightarrow{\phi} W \hookrightarrow \mathbf{A}^n \rightarrow \mathbf{A}^1$$

be the above is constant. Thus the coordinates of  $\phi(V)$  are constant, so  $\phi(V) = \{\text{pt}\}$ .  $\square$

A final result of interest that I won't prove today:

**Theorem 1.1.10.** *Projective varieties are **complete**.*

The main goal of this section is to prove the following theorem:

**Theorem 1.1.11 (Rigidity).** *Let  $V, W$  be varieties such that  $V$  is **complete** and  $V \times W$  is geometrically irreducible. Let  $\alpha: V \times W \rightarrow U$  be a morphism such that  $\exists u_0 \in U(k), v_0 \in V(k), w_0 \in W(k)$  with  $\alpha(V \times \{w_0\}) = \alpha(\{v_0\} \times W) = \{u_0\}$ . Then  $\alpha(V \times W) = \{u_0\}$ .*

*Proof.* Since  $V \times W$  is geometrically irreducible,  $V$  must be connected. Denote the projection  $q: V \times W \rightarrow W$ . Let  $U_0 \ni x_0$  be an open neighborhood. We consider the set

$$Z = \{w \in W : \alpha((v, w)) \notin U_0 \text{ for some } v \in V\} = q(\alpha^{-1}(U \setminus U_0))$$

Since  $q$  is closed,  $Z \subseteq W$  is closed. Since  $w_0 \in W \setminus Z$ ,  $W \setminus Z$  is a nonempty open subset of  $W$ .

Consider  $w \in W \setminus Z$ . Since  $V \times \{w\} \cong V$  it is **complete** and connected. Thus

$$\alpha(V \times \{w\}) = \{\text{pt}\} = \alpha((v_0, w)) = \{u_0\}$$

which implies that

$$\alpha(V \times (W \setminus Z)) = \{u_0\}$$

Since  $V \times (W \setminus Z) \subseteq V \times W$  is open and  $V \times W$  is irreducible, it is dense. So  $\alpha(V \times W) = \{u_0\}$ .  $\square$

**Proposition 1.1.12.** *Let  $A, B$  be [abelian varieties](#). Every morphism  $\alpha: A \rightarrow B$  is the composition of a homomorphism and a translation.*

*Proof.* First compose by a translation on  $B$  such that  $\alpha(0) = 0$ . Consider the map

$$\begin{aligned}\phi: A \times A &\rightarrow B \\ (a, a') &\mapsto \alpha(a + a') - \alpha(a) - \alpha(a')\end{aligned}$$

Then

$$\begin{aligned}\phi(A \times \{0\}) &= \alpha(a + 0) - \alpha(a) - \alpha(0) = 0 \\ \phi(\{0\} \times A) &= \alpha(0 + a) - \alpha(0) - \alpha(a) = 0.\end{aligned}$$

By the [rigidity theorem 11](#)  $\phi(A \times A) = \{0\}$  hence  $\alpha(a + a') = \alpha(a) + \alpha(a')$ .  $\square$

**Corollary 1.1.13.** *Abelian varieties are abelian.*

*Proof.* The inversion map  $a \mapsto -a$  sends 0 to 0, thus is a homomorphism. Therefore

$$a + b - a - b = a + b - (a + b) = 0$$

and so

$$a + b = b + a. \quad \square$$

## 1.2 Abelian varieties over $\mathbf{C}$ (Alex)

The goal of this talk is to understand what [abelian varieties](#) look like over  $\mathbf{C}$ . The goal for me is to understand what a (principal) polarisation is and why it is important.

First immediate question: why study complex theory at all? The most classical field, algebraically closed, archimidean, characteristic 0.

Recall/rapidly learn the picture for elliptic curves, given  $E$  an elliptic curve we have for some  $\Lambda$  a rank 2 lattice in  $\mathbf{C}$

$$\begin{aligned}\mathbf{C}/\Lambda &\xrightarrow{\sim} E(\mathbf{C}) \subseteq \mathbf{P}^2(\mathbf{C}) \\ z &\mapsto (\wp(z) : \wp'(z) : 1) \\ 0 &\mapsto (0 : 1 : 0)\end{aligned}$$

where

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left( \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right).$$

This is a meromorphic function whose image lands in

$$y^2 = 4x^3 - g_2x - g_3.$$

So the  $\mathbf{C}$  points of an elliptic curve are topologically a torus.

Naturally one asks: does this generalise? Let  $A$  be an [abelian variety](#) over  $\mathbf{C}$ , what does  $A(\mathbf{C})$  look like? Another torus?

**Proposition 1.2.1.**  *$A(\mathbf{C})$  is a compact, connected, complex lie group.*

**Proposition 1.2.2.** *Let  $A$  be an [abelian variety](#) of dimension  $g$  over  $\mathbf{C}$ . Then we have*

$$A(\mathbf{C}) \cong V/\Lambda$$

where  $V$  is a  $g$  dimensional complex vector space and  $\Lambda$  is a full rank lattice of  $V$  (i.e  $\Lambda$  is a discrete subgroup of  $V$  s.t.  $\mathbf{R} \otimes \Lambda = V$ ).

*Proof.* Differential geometry gives us a map of complex manifolds

$$\exp: \mathrm{Tgt}_0(A(\mathbf{C})) \rightarrow A(\mathbf{C})$$

this is a holomorphism. And since  $A(\mathbf{C})$  is abelian, this is a homomorphism also. In general this is locally an isomorphism around 0.

Claim:  $\exp$  is injective. There exists a neighborhood  $U \ni 0$  s.t.  $\exp(U) \cong U$ . Consider the image  $\exp(\mathrm{Tgt}_0 A(\mathbf{C}))$ . For  $x \in \exp(\mathrm{Tgt}_0 A(\mathbf{C}))$ ,  $\{U + x\}$  are all open and give a cover. Thus  $\exp(\mathrm{Tgt}_0 A(\mathbf{C}))$  is open. Since  $A(\mathbf{C})$  is connected we are thus reduced to showing  $\exp(\mathrm{Tgt}_0 A(\mathbf{C}))$  is closed also. Since  $\exp$  is a homomorphism, the image is a subgroup. So its complement is the union of its non-trivial cosets, which is open. Thus  $\exp(\mathrm{Tgt}_0 A(\mathbf{C}))$  is closed. Giving  $\exp(\mathrm{Tgt}_0 A(\mathbf{C})) = A(\mathbf{C})$ , which proves the claim.

$\exp$  is a local isomorphism, which gives that  $\ker(\exp)$  is discrete, i.e. a lattice. We now have

$$A(\mathbf{C}) \cong \mathrm{Tgt}_0 A(\mathbf{C}) / \ker(\exp)$$

so as  $A(\mathbf{C})$  is compact we cannot have a kernel which is not full rank, as otherwise the quotient could not be compact.  $\square$

**Definition 1.2.3.** We call any such  $V/\Lambda$  a **complex torus**.

From the above isomorphism we can now read off properties of  $A(\mathbf{C})$  as a group.

**Proposition 1.2.4.**  $A(\mathbf{C})$  is divisible, and  $A(\mathbf{C})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{2g}$ .

*Proof.*

$$A(\mathbf{C}) \cong V/\Lambda \cong (\mathbf{R}/\mathbf{Z})^{2g}$$

isomorphisms as groups, thus  $A(\mathbf{C})$  is divisible. Further,  $(\mathbf{R}/\mathbf{Z})[n] = (\frac{1}{n}\mathbf{Z})/\mathbf{Z}$ .  $\square$

Question: Given a **complex torus**  $V/\Lambda$ , does there exist an **abelian variety**  $A$  such that  $A(\mathbf{C}) \cong V/\Lambda$ ?

**Example 1.2.5.**

- $\mathbf{C}/\Lambda \cong E(\mathbf{C})$  always in dim 1
- $\mathbf{C}^2/\Lambda^2 \cong (E \times E)(\mathbf{C})$  sometimes yes in higher dimension
- $\mathbf{C}^2 / \langle (i, 0), (i\sqrt{p}, i), (1, 0), (0, 1) \rangle_{\mathbf{Z}}$

for  $p$  prime??? (I guess not, see Mumford)

**Theorem 1.2.6** (Chow). *If  $X$  is an analytic submanifold of  $\mathbf{P}^n(\mathbf{C})$  then  $X$  is an algebraic subvariety.*

By this theorem it is enough to analytically imbed  $V/\Lambda \hookrightarrow \mathbf{P}^m$ . We can try and do this by mimicing the elliptic curve strategy, find enough functions  $\theta: V/\Lambda \rightarrow \mathbf{C}$ .

**Proposition 1.2.7.** *Let  $X = V/\Lambda$ . Then*

$$H^r(X, \mathbf{Z}) \cong \{\text{alternating } r\text{-forms } \Lambda \times \cdots \times \Lambda \rightarrow \mathbf{Z}\}.$$

*Proof.*  $\pi: V \rightarrow V/\Lambda$  is a universal covering map, so

$$\Lambda = \pi^{-1}(0) \cong \pi_1(X, 0).$$

Because all these spaces are nice

$$H^1(X, \mathbf{Z}) \cong \text{Hom}(\pi_1(X), \mathbf{Z}) \cong \text{Hom}(\Lambda, \mathbf{Z}).$$

To extend to  $r \neq 1$  use the Künneth formula:

$$\begin{array}{ccc} \wedge^r(H^1(X_1 \times X_2, \mathbf{Z})) & \xlongequal{\quad} & H^r(X_1 \times X_2, \mathbf{Z}) \\ \parallel \text{Künneth} & & \parallel \text{Künneth} \\ \wedge^r(H^1(X_1, \mathbf{Z}) \otimes H^1(X_2, \mathbf{Z})) & & \\ \parallel & & \parallel \\ \bigoplus_{p+q=r} (\wedge^p(H^1(X_1, \mathbf{Z})) \otimes \wedge^q(H^1(X_2, \mathbf{Z}))) & \xlongequal{\quad} & \bigoplus_{p+q=r} (H^p(X_1, \mathbf{Z}) \otimes H^q(X_2, \mathbf{Z})) \end{array}$$

Since we know the proposition for  $S^1 = \mathbf{R}/\mathbf{Z}$  by taking products and applying the above we get it for all complex tori  $V/\Lambda$ .  $\square$

**Proposition 1.2.8.** *There is a correspondence*

$$\begin{aligned} \{\text{Hermitian forms } H \text{ on } V\} &\leftrightarrow \{\text{Alternating forms } E: V \times V \rightarrow \mathbf{R}, E(iu, iv) = E(u, v)\} \\ H &\mapsto \text{im } H \\ E(iu, v) + iE(u, v) &\leftarrow E. \end{aligned}$$

Now we will consider **line bundles** on  $X = V/\Lambda$ , that is

$$L \xrightarrow{\pi} X$$

such that for any  $x \in X$  there exists  $U \ni x$  with  $\pi^{-1}(U) \cong \mathbf{C} \times U$ . We can obtain these from hermitian forms and some auxilliary data as follows.

**Definition 1.2.9.** If  $H$  is a hermitian form on  $V$  such that  $E(\Lambda \times \Lambda) \subseteq \mathbf{Z}$  there exists a map

$$\alpha: \Lambda \rightarrow \mathbf{C}^* = \{z \in \mathbf{C}^* : |z| = 1\}$$

such that

$$\alpha(u + v) = e^{i\pi E(u, v)} \alpha(u) \alpha(v).$$

Further, there is a **line bundle**  $L(H, \alpha)$  on  $X$  which is defined by quotienting  $\mathbf{C} \times V$  by  $\Lambda$  which acts via

$$\phi_u(\lambda, v) = (\alpha(u) e^{\pi H(v, u) + \frac{1}{2} \pi H(u, u)} \lambda, v + u) \text{ for } u \in \Lambda,$$

we'll denote by  $e_u$  the factor  $\alpha(u) e^{\pi H(v, u) + \frac{1}{2} \pi H(u, u)}$  for brevity.

**Theorem 1.2.10** (Appell-Humbert). *Any **line bundle** on  $X$  is of the form  $L(H, \alpha)$  for some  $H, \alpha$  as above. Further*

$$L(H_1, \alpha_1) \otimes L(H_2, \alpha_2) = L(H_1 + H_2, \alpha_1 \alpha_2).$$

*In fact we have the following diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(\Lambda, \mathbf{C}) & \longrightarrow & \{\text{data } (H, \alpha)\} & \longrightarrow & \{\text{gp. of Herm. } H \text{ w/ } E(\Lambda \times \Lambda) \subseteq \mathbf{Z}\} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Pic}^0(X) & \longrightarrow & \text{Pic}(X) & \longrightarrow & \ker(H^2(X, \mathbf{Z}) \rightarrow H^2(X, \mathcal{O}_X)) \longrightarrow 0 \end{array}$$

where  $\text{Pic}(X)$  is the group of all **line bundles** on  $X$  and  $\text{Pic}^0$  is the subgroup of those which are topologically trivial.

We wanted functions  $X \rightarrow \mathbf{C}$ . Now we can instead consider sections  $s$  of  $L(H, \alpha) \xrightarrow{\pi} X$  i.e. maps  $s: X \rightarrow L(H, \alpha)$  with  $\pi \circ s = \text{id}$ . Denote the space of such sections  $H^0(X, L(H, \alpha))$ .

**Definition 1.2.11** (Theta functions). The sections of  $L(H, \alpha)$  correspond to holomorphic functions

$$\theta: V \rightarrow \mathbf{C}$$

such that  $\theta(z + u) = e_u \theta(z)$ , we will call such a  $\theta$  a **theta function** for  $(H, \alpha)$ .

If  $H$  is not positive definite the space of such functions is 0!

**Proposition 1.2.12.** *If  $H$  is positive definite, then the dimension of  $H^0(X, L(H, \alpha))$  is  $\sqrt{\det E}$  where we really mean the determinant of a matrix for  $E$  with respect to an integral basis.*

**Theorem 1.2.13** (Lefschetz). *Given a positive definite  $H$ , there exists an imbedding  $X \hookrightarrow \mathbf{P}^m$ .*

*Proof.* Sketch: Let  $L = L(H, \alpha)$ , consider  $L(H, \alpha)^{\otimes 3} = L(3H, \alpha^3)$ , take a basis of  $\theta_0, \dots, \theta_d$  of  $H^0(X, L^{\otimes 3})$ .

Claim:  $\Theta: z \mapsto (\theta_0(z) : \dots : \theta_d(z)) \subseteq \mathbf{P}^d$  is an embedding.

To see that this is well defined, we must give a section of  $L^{\otimes 3}$  not vanishing at  $z$  for all  $z \in X$ . Let  $\theta \in H^0(X, L) \setminus \{0\}$ . Then pick  $a, b$  such that the section of  $L^{\otimes 3}$  given by

$$\theta(z - a)\theta(z - b)\theta(z + a + b)$$

does not vanish. This is possible and thus we have a nonvanishing section of  $L^{\otimes 3}$ .

For injectivity, show that if the above section has the same values on  $z_1, z_2$  then it is a **theta function** for some sublattice. Almost all sections aren't **theta functions** for a sublattice (this uses [Proposition 12](#)).

Something similar must be done for tangent vectors.  $\square$

**Definition 1.2.14** (Riemann forms). A **Riemann form** is  $E: \Lambda \times \Lambda \rightarrow \mathbf{Z}$  alternating such that

$$E_{\mathbf{R}}: V \times V \rightarrow \mathbf{R}$$

has the property that  $E(iu, iv) = E(u, v)$  and the corresponding Hermitian form is positive definite.

**Definition 1.2.15** (Polarizable tori). A **complex torus**  $X = V/\Lambda$  is **polarizable** if there exists a **Riemann form**  $E$  on  $\Lambda$ .

**Example 1.2.16** (Proposition). Every  $\mathbf{C}/\Lambda$  where  $\Lambda = \langle 1, \tau \rangle_{\mathbf{Z}}$  is **polarizable**.

To see this take

$$E(u, v) = \frac{uv}{\text{im } \tau}$$

as a **Riemann form**.

Putting everything together we have obtained an equivalence of categories

$$\{\text{abelian varieties over } \mathbf{C}\} \leftrightarrow \{\text{polarizable complex tori}\}.$$

**Definition 1.2.17** (Isogenies of complex tori). An **isogeny** of complex tori is a homomorphism  $V/\Lambda \rightarrow V'/\Lambda'$  with finite kernel.

**Definition 1.2.18** (Dual vector spaces). Given  $V$  a complex vector space, let

$$V^* = \{f: V \rightarrow \mathbf{C} : f(u+v) = f(u) + f(v), f(\alpha v) = \bar{\alpha}f(v)\}$$

and given  $\Lambda \subset V$  a lattice, let

$$\Lambda^* = \{f \in V^* : f(\lambda) \in \mathbf{Z} \forall \lambda \in \Lambda\}.$$

**Definition 1.2.19** (Dual tori). If  $X = V/\Lambda$ ,  $X^\vee = V^*/\Lambda^*$  is the **dual torus**.

**Proposition 1.2.20** (Existence of Weil pairing).

$$X \times X^\vee \rightarrow \mathbf{C}$$

so

$$X[n] \times X^\vee[n] \rightarrow \left( \frac{1}{n^2} / \frac{1}{n} \mathbf{Z} \right) \cong \mathbf{Z}/n\mathbf{Z}$$

this is called the **Weil pairing**.

Can a **complex torus** be isogenous to its own dual? If  $X$  is **polarizable** then

$$\begin{aligned} X &\rightarrow X^\vee \\ v &\mapsto H(v, -) \end{aligned}$$

is an **isogeny**.

**Definition 1.2.21.** A **polarization** is an **isogeny**  $X \rightarrow X^\vee$ .

## 1.3 Rational Maps into Abelian Varieties (Maria)

Note all varieties are irreducible today.

### 1.3.1 Rational maps

$V, W$  varieties  $/K$ . Consider pairs  $(U, \phi_U)$ , where  $\emptyset \neq U \subset V$  an open subset so  $U$  is dense, and  $\phi_U: U \rightarrow W$  is a regular map.

**Definition 1.3.1.**  $(U, \phi_U), (U', \phi_{U'})$  are equivalent if  $\phi_U$  and  $\phi_{U'}$  agree on  $U \cap U'$ . An equivalence class  $\phi$  of  $\{(U, \phi_U)\}$  is a **rational map**  $\phi: V \dashrightarrow W$ . If  $\phi: V \dashrightarrow W$  is defined at  $v \in V$  if  $v \in U$  for some  $(U, \phi_U) \in \phi$ .

**Note 1.3.2.** The set  $U_1 = \bigcup U$  where  $\phi$  is defined is open and  $(U_1, \phi_1) \in \phi$  where  $\phi_1: U_1 \rightarrow W$  restricts to  $\phi_U$  on  $U$ .

**Example 1.3.3.**

1. Let  $\emptyset \neq W \subseteq V$  be open. Then the **rational map**  $V \dashrightarrow W$  induced by  $\text{id}: W \rightarrow W$  will not extend to  $V$ . To avoid this, assume  $W$  is **complete** (so  $W = V$ ).
2.  $C: y^2 = x^3$ , then  $\alpha: \mathbf{A}^1 \rightarrow C, a \mapsto (a^2, a^3)$  is a regular map, restricting to an isomorphism  $\mathbf{A}^1 \setminus \{0\} \rightarrow C \setminus \{0\}$ . The inverse of  $\alpha|_{\mathbf{A}^1 \setminus \{0\}}$  represents  $\beta: C \dashrightarrow \mathbf{A}^1$  which does not extend to  $C$ . This corresponds on function fields to

$$\begin{aligned} K(t) &\rightarrow K(x, y) \\ t &\mapsto y/x \end{aligned}$$

which does not send  $K[y]_{(t)}$  to  $K[x, y]_{(x, y)}$ .



3. Given a nonsingular surface  $V$ ,  $P \in V$  then  $\exists \alpha: W \rightarrow V$  regular that induces an isomorphism  $\alpha: W \setminus \alpha^{-1}(P) \rightarrow V \setminus P$ , but  $\alpha^{-1}(P)$  is a projective line. The **rational map** represented by  $\alpha^{-1}$  is not regular on  $V$  (where to send  $P$ ?).

**Theorem 1.3.4** (Milne 3.1). A **rational map**  $\phi: V \dashrightarrow W$  from a nonsingular variety  $V$  to a **complete variety**  $W$  is defined on an open subset  $U \subseteq V$  whose complement has codimension  $\geq 2$ .

*Proof.* ( $V$  a curve)  $V$  nonsingular curve,  $\emptyset \neq U \subseteq V$  open,  $\phi: U \rightarrow W$  a regular map.

$$\begin{array}{c}
 & & V \\
 & \nearrow & \uparrow p \\
 U & \longrightarrow & U' \subseteq Z \subseteq V \times W \ni (v, w) \\
 & \searrow & \downarrow q \\
 & & W \ni w
 \end{array}$$

$U'$  is the image of  $U$ ,  $Z = \overline{U'}$ .  $W$  is **complete**,  $Z$  closed implies  $p(Z) \subseteq V$  is closed. Also,  $U \subseteq p(Z) \implies p(Z) = V$ .

$$U \xrightarrow{\sim} U' \rightarrow U$$

so

$$U' \xrightarrow{\sim} U$$

$$Z \twoheadrightarrow V$$

this implies  $Z \xrightarrow{\sim} V$ . Then  $q|_Z: Z \rightarrow W$  is the extension of  $\phi$  to  $V$ .  $\square$

**Theorem 1.3.5** (Milne 3.2). A **rational map**  $\phi: V \dashrightarrow A$  from a nonsingular variety  $V$  to an **abelian variety**  $W$ , extends to all of  $V$ .

*Proof.* **Theorem 4 Lemma 6**  $\square$

**Lemma 1.3.6.** Let  $\phi: V \dashrightarrow G$  be a map from a nonsingular variety to a group variety. Then either  $\phi$  is defined on all of  $V$  or the set where  $\phi$  is not defined is closed of pure codimension 1.

*Proof.* Fix  $(U, \phi_U) \in \phi$  and consider

$$\Phi: V \times V \dashrightarrow G$$

represented by

$$\begin{aligned}
 U \times U &\xrightarrow{\phi_U \times \phi_U} G \times G \xrightarrow{\text{id} \times \text{inv}} G \times G \xrightarrow{m} G \\
 (x, y) &\mapsto \phi_U(x)\phi_U(y)^{-1}
 \end{aligned}$$

Check  $\phi$  is defined at  $x$  iff  $\Phi$  is defined at  $(x, x)$  (and in this case  $\Phi(x, x) = e$ ). This is equivalent to the map  $\Phi^*: \mathcal{O}_{G,e} \rightarrow K(V \times V)$  induced by  $\Phi$  satisfying  $\text{im}(\mathcal{O}_{G,e}) \subseteq \mathcal{O}_{V \times V, (x,x)}$ . For a nonzero function  $f$  on  $V \times V$ , write  $\text{div}(f) = \text{div}(f)_0 - \text{div}(f)_\infty$  which are effective divisors. Then

$$\mathcal{O}_{V \times V, (x,x)} = \{0\} \cup \{f \in K(V \times V) : \text{div}(f)_\infty \text{ does not contain } (x, x)\}.$$

Suppose  $\phi$  is not defined at  $x$ , then there exists  $f \in \text{im}(\mathcal{O}_{G,e})$  s.t.  $(x, x) \in \text{div}(f)_\infty$ . Then  $\Phi$  is not defined at any  $(y, y) \in \Delta \cap \text{div}(f)_\infty = \text{div}(f^{-1})_0$ , which is a pure codimension 1 subset of  $\Delta$  by Milne's AG thm 9.2. The corresponding subset in  $V$  is of pure codimension 1, and  $\phi$  is not defined there.  $\square$

**Theorem 1.3.7** (Milne 3.4). *Let  $\alpha: V \times W \rightarrow A$  be a morphism from a product of nonsingular varieties into an **abelian variety**. If  $\alpha(V \times \{w_0\}) = \{a_0\} = \alpha(\{v_0\} \times W)$  for some  $a_0 \in A$ ,  $v_0 \in V$ ,  $w_0 \in W$ , then  $\alpha(V \times W) = \{a_0\}$ .*

**Corollary 1.3.8** (Milne 3.7). *Every **rational map**  $\alpha: G \dashrightarrow A$  from a group variety into an **abelian variety** is the composition of a homomorphism and a translation in  $A$ .*

*Proof.* Since group varieties are nonsingular,  $\alpha: G \rightarrow A$  is a regular map by **Theorem 5**. The rest is as proof of Corollary 1.2.  $\square$

### 1.3.2 Dominating and birational maps

**Definition 1.3.9** (Dominating maps).  $\phi: V \dashrightarrow W$  is **dominating** if  $\text{im}(\phi_U)$  is dense in  $W$  for a representative  $(U, \phi_U) \in \phi$ .

Exercise: A **dominating**  $\phi: V \dashrightarrow W$  defines a homomorphism  $K(W) \rightarrow K(V)$  and any such homomorphism arises from a unique **dominating rational map**.

**Definition 1.3.10.**  $\phi: V \dashrightarrow W$  is **birational** if the corresponding  $K(W) \rightarrow K(V)$  is an isomorphism or, equivalently if there exists  $\psi: W \dashrightarrow V$  s.t.  $\phi \circ \psi$  and  $\psi \circ \phi$  are the identity wherever they are defined. In this case we say  $V$  and  $W$  are **birationally equivalent**.

**Note 1.3.11.** In general birational equivalence does not imply isomorphic. E.g.  $V$  a variety  $\emptyset \neq W \subsetneq V$  an open subset, or  $V = \mathbf{A}^1, W: y^2 = x^3$ .

**Theorem 1.3.12** (Milne 3.8). *If two **abelian varieties** are **birationally equivalent** then they are isomorphic as **abelian varieties**.*

*Proof.*  $A, B$  **abelian varieties** with  $\phi: A \dashrightarrow B$  a **birational** map with inverse  $\psi$ . Then by **Theorem 5**  $\phi, \psi$  extend to regular maps  $\phi: A \rightarrow B, \psi: B \rightarrow A$  and  $\phi \circ \psi, \psi \circ \phi$  are the identity everywhere. This implies that  $\phi$  is an isomorphism of algebraic varieties and after composition with a translation,  $\phi$  is also a group isomorphism.  $\square$

**Proposition 1.3.13** (Milne 3.9). *Any **rational map**  $\mathbf{A}^1 \dashrightarrow A$  or  $\mathbf{P}^1 \dashrightarrow A$ , for  $A$  an **abelian variety** is constant.*

*Proof.* **Theorem 5** implies  $\alpha: \mathbf{A}^1 \dashrightarrow A$  extends to  $\alpha: \mathbf{A}^1 \rightarrow A$  and we may assume  $\alpha(0) = e$ .  $(\mathbf{A}^1, +): \alpha(x + y) = \alpha(x) + \alpha(y)$  for all  $x, y \in \mathbf{A}^1(K) = K$ .  $(\mathbf{A}^1 \setminus \{0\}, \cdot): \alpha(xy) = \alpha(x) + \alpha(y) + c$  for all  $x, y \in K^\times$ . These can only hold at the same time if  $\alpha$  is constant.  $\mathbf{P}^1 \dashrightarrow A$  is constant, since its constant on affine patches.  $\square$

**Definition 1.3.14.**  $V/\bar{K}$  is **unirational** if there is a **dominating** map  $\mathbf{A}^n \dashrightarrow V$ , where  $n = \dim_{\bar{K}} V$ .  $V/K$  is **unirational** if  $V/\bar{K}$  is.

**Proposition 1.3.15** (Milne 3.10). *Every **rational map**  $V \dashrightarrow A$  from  $V$  **unirational** to  $A$  **abelian** is constant.*

*Proof.* Wlog  $K = \bar{K}$ . Since  $V$  is **unirational** we get  $\beta: \mathbf{P}^1 \times \cdots \times \mathbf{P}^1 \dashrightarrow V \dashrightarrow A$ , which extends to  $\beta: \mathbf{P}^1 \times \cdots \times \mathbf{P}^1 \rightarrow A$ . Then by Milne corollary 1.5, there exist regular maps  $\beta_i: \mathbf{P}^1 \rightarrow A$  s.t.  $\beta(x_1, \dots, x_n) = \sum \beta_i(x_i)$  and by **Proposition 13** each  $\beta_i$  map is constant.  $\square$

## 1.4 Theorem of the Cube (Ricky)

### 1.4.1 Crash Course in Line Bundles

Consider  $\mathbf{R}^2$ ,  $f: \mathbf{R} \rightarrow \mathbf{R}$ ,  $f(x, y) = x^2 + y^2 - 1$ , now  $S = \{f = 0\} \subseteq \mathbf{R}^2$  is a closed submanifold (in fact a circle). Question: Do all closed submanifolds arise in this way? Lets switch to  $\mathbf{C}$  better analogies with AG.

**Example 1.4.1.** Let  $X \in \mathbf{P}^n(\mathbf{C})$ , the answer here is no! (Because  $f: X \rightarrow \mathbf{C}^1$  is constant!) Want to define functions locally that give us level sets, but gluing such will give us a global section. Instead glue in a different way (i.e. into different “copies” of  $\mathbf{C}$ ) so that this doesn’t happen.

**Example 1.4.2.**  $X \in \mathbf{P}_{\mathbf{C}}^1$ ,  $\mathcal{O}_X$  the structure sheaf.

$$X = U_0 \cup U_1 = (\mathbf{A}^1, t) \cup (\mathbf{A}^1, s)$$

on  $U_0 \cap U_1$ ,  $t = s^{-1}$ . What is a global section of  $\mathcal{O}_X$ , a section of  $U_0$  and a section of  $U_1$  that glue.  $\mathcal{O}_X(U_0) = k[t]$ ,  $\mathcal{O}_X(U_1) = k[s]$  so given  $f(t)$ ,  $g(s)$  these glue to a global section iff  $f(t) = g(1/t)$  so  $f, g$  must be constant.

**Definition 1.4.3** (Line bundles). A **line bundle** on  $X$  is a locally free  $\mathcal{O}_X$ -module of rank 1, i.e.  $\exists \{U_i\}$  open cover along with isomorphisms  $\phi_i: \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_X|_{U_i}$ .

**Exercise 1.4.4.** Alternative definition: A **line bundle** on  $X$  is equivalent to the following data:

- An open cover of  $X$ .
- Transition maps  $\tau_{ij} \in \text{GL}_1(\mathcal{O}_X(U_i \cap U_j))$  satisfying  $\tau_{ij}\tau_{jk} = \tau_{ik}$  and  $\tau_{ii} = \text{id}$ .

**Example 1.4.5.** On  $X = \mathbf{P}_k^n$ , we have **line bundles**  $\mathcal{O}(d)$  for all  $d \in \mathbf{Z}$ . Just have to give cover and transition functions, use usual open cover  $\{U_i\}$  with  $U_i \cong \mathbf{A}^n$ . Then  $\tau_{ji}$  is given by multiplication by  $(x_i/x_j)^d$ .

**Exercise 1.4.6.**

$$H^0(X, \mathcal{O}(d)) (= \Gamma(X, \mathcal{O}(d)))$$

=  $k$ -vector space spanned by deg.  $d$  homogenous polynomials in  $k[x_0, \dots, x_n]$ .

**Exercise 1.4.7.** All **line bundles** on  $\mathbf{P}^n$  are isomorphic to some  $\mathcal{O}(d)$ .

We say a **line bundle**  $\mathcal{L}$  on  $X$  is trivial if  $\mathcal{L} \cong \mathcal{O}_X$ . Given  $\mathcal{L}_1$  and  $\mathcal{L}_2$  on  $X$  (line bundles) we can create a new **line bundle**  $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$ . So isomorphism classes of **line bundles** on  $X$  with  $\otimes$  form a group, denoted  $\text{Pic}(X)$  with identity  $\mathcal{O}_X$  and inverses  $\mathcal{L}^{-1} = \text{Hom}(\mathcal{L}, \mathcal{O}_X)$ .

**Example 1.4.8.** By previous exercise  $\text{Pic}(\mathbf{P}_k^n) \cong \mathbf{Z}$  since  $\mathcal{O}_X(d_1) \otimes \mathcal{O}_X(d_2) \cong \mathcal{O}_X(d_1 + d_2)$ .

**Fact 1.4.9.** If  $f: X \rightarrow Y$ , then given  $\mathcal{L}$  on  $Y$  we can pullback to a **line bundle**  $f^* \mathcal{L}$  on  $X$ , definition is complicated. We also know that  $f^*$  commutes with  $\otimes$  so in fact (as  $f^* \mathcal{O}_Y = \mathcal{O}_X$ ) we get a homomorphism  $f^*: \text{Pic}(Y) \rightarrow \text{Pic}(X)$ .

### 1.4.2 Relation to (Weil) divisors

Let  $X$  be a normal variety, call  $Z \subseteq X$ , a closed subvariety of codimension 1, a **prime divisor**. Then a divisor on  $X$  is a formal sum

$$D = \sum_{Z \subseteq X} n_Z \cdot Z$$

of **prime divisors**.

Let  $K = K(X)$  be the function field of  $X$ . Given  $f \in K^\times$  we can define

$$\operatorname{div}(f) = \sum v_Z(f) \cdot Z.$$

Given  $D \in \operatorname{Div}(X)$ , we can define a **line bundle**  $\mathcal{L}(D)$  on  $X$  via

$$\mathcal{L}(D)(U) = \{f \in K^\times : (D + \operatorname{div}(f))|_U \geq 0\} \cup \{0\}$$

where  $D|_U = \sum_{Z \cap U \neq \emptyset} n_Z \cdot (Z \cap U)$ .

**Proposition 1.4.10.** *The map*

$$\operatorname{Cl}(X) = \operatorname{Div}(X)/\operatorname{Princ}(X) \xrightarrow{\mathcal{L}(\cdot)} \operatorname{Pic}(X)$$

*is an isomorphism.*

### 1.4.3 Onto cubes

**Theorem 1.4.11** (Theorem of the cube). *Let  $U, V, W$  be **complete varieties**. If  $\mathcal{L}$  is a **line bundle** on  $U \times V \times W$  s.t.  $\mathcal{L}|_{\{u_0\} \times V \times W}, \mathcal{L}|_{U \times \{v_0\} \times W}, \mathcal{L}|_{U \times V \times \{w_0\}}$  are all trivial then  $\mathcal{L}$  is trivial.*

**Corollary 1.4.12** (Milne 5.2). *Let  $A$  be an **abelian variety**. Let  $p_i: A \times A \times A \rightarrow A$  be the projection onto the  $i$ th coordinate.  $p_{ij} = p_i + p_j$ ,  $p_{123} = p_1 + p_2 + p_3$ . Then for any  $\mathcal{L}$  on  $A$ , the **line bundle***

$$\mathcal{M} = p_{123}^* \mathcal{L} \otimes p_{12}^* \mathcal{L}^{-1} \otimes p_{23}^* \mathcal{L}^{-1} \otimes p_{13}^* \mathcal{L}^{-1} \otimes p_1^* \mathcal{L} \otimes p_2^* \mathcal{L} \otimes p_3^* \mathcal{L}$$

*is trivial.*

*Proof.* Let  $m: A \times A \rightarrow A$  be multiplication (addition?) and  $p, q$  the projections  $A \times A \rightarrow A$ . Then the composites of the maps  $\phi: A \times A \rightarrow A \times A \times A$ ,  $\phi(x, y) = (x, y, 0)$  with  $p_{123}, p_{12}, p_{23}, p_{13}, p_1, p_2, p_3$  are respectively  $m, m, q, p, p, q, 0$ . Hence the restriction of  $\mathcal{M}$  to  $A \times A \times \{0\}$  is

$$m^* \mathcal{L} \otimes m^* \mathcal{L}^{-1} \otimes q^* \mathcal{L}^{-1} \otimes p^* \mathcal{L}^{-1} \otimes p^* \mathcal{L} \otimes q^* \mathcal{L} \otimes O_{A \times A}$$

this is trivial by tensor commuting with pullback. Similarly  $\mathcal{M}$  restricts to a trivial bundle on  $A \times \{0\} \times A$  and  $\{0\} \times A \times A$ . So by **theorem of the cube 11**  $\mathcal{M}$  is trivial.  $\square$

**Corollary 1.4.13** (Milne 5.3). *Let  $f, g, h: V \rightarrow A$  ( $A$  abelian). Then for any  $\mathcal{L}$  on  $A$  the bundle*

$$\mathcal{M} = (f+g+h)^* \mathcal{L} \otimes (f+g)^* \mathcal{L}^{-1} \otimes (f+h)^* \mathcal{L}^{-1} \otimes (g+h)^* \mathcal{L}^{-1} \otimes f^* \mathcal{L} \otimes g^* \mathcal{L} \otimes h^* \mathcal{L}$$

*is trivial.*

*Proof.*  $\mathcal{M}$  is the pullback of the **line bundle** of **Corollary 12** via the map  $(f, g, h): V \rightarrow A \times A \times A$ .  $\square$

On  $A$  we have  $n_A: A \rightarrow A$  be  $n_A(a) = a + \cdots + a$  ( $n$  times) for  $n \in \mathbf{Z}$ .

**Corollary 1.4.14** (Milne 5.4). *For  $\mathcal{L}$  on  $A$  we have*

$$n_A^* \mathcal{L} \cong \mathcal{L}^{(n^2+n)/2} \otimes (-1)_A^* \mathcal{L}^{(n^2-n)/2}$$

*In particular if  $(-1)^* \mathcal{L} = \mathcal{L}$  (symmetric) then  $n_A^* \mathcal{L} = \mathcal{L}^{n^2}$ . And if  $(-1)^* \mathcal{L} = \mathcal{L}^{-1}$  (antisymmetric) then  $n_A^* \mathcal{L} = \mathcal{L}^n$ .*

*Proof.* Use [Corollary 13](#) with  $f = n_A, g = 1_A, h = (-1)_A$ . So the [line bundle](#)

$$(n)^* \mathcal{L} \otimes (n+1)^* \mathcal{L}^{-1} \otimes (n-1)^* \mathcal{L}^{-1} \otimes (1-1)^* \mathcal{L}^{-1} \otimes n^* \mathcal{L} \otimes 1^* \mathcal{L} \otimes (-1)^* \mathcal{L}$$

is trivial i.e.

$$(n+1)^* \mathcal{L} = (n-1)^* \mathcal{L}^{-1} \otimes n^* \mathcal{L}^2 \otimes \mathcal{L} \otimes (-1)^* \mathcal{L}$$

in statement  $n = 1$  is clear, so use  $n = 1$  in the above to get

$$2_A^* \mathcal{L} \cong \mathcal{L}^2 \otimes \mathcal{L} \otimes (-1)_A^* \mathcal{L} \cong \mathcal{L}^3 \otimes (-1)_A^* \mathcal{L}.$$

Then induct on  $n$  in above. □

**Theorem 1.4.15** (Theorem of the square (Milne 5.5)). *Let  $\mathcal{L}$  be an invertible sheaf (line bundle) on  $A$ . Let  $t_a: A \rightarrow A$  be translation by  $a \in A(k)$ . Then*

$$t_{a+b}^* \mathcal{L} \otimes \mathcal{L} \cong t_a^* \mathcal{L} \otimes t_b^* \mathcal{L}.$$

*Proof.* Use [Corollary 13](#) with  $f = \text{id}, g(x) = a, h(x) = b$  to get

$$t_{a+b}^* \mathcal{L} \otimes t_a^* \mathcal{L}^{-1} \otimes t_b^* \mathcal{L}^{-1} \otimes \mathcal{L}$$

is trivial. □

**Remark 1.4.16.** Tensor by  $\mathcal{L}^{-2}$  in the above equation to get

$$t_{a+b}^* \mathcal{L} \otimes \mathcal{L}^{-1} \cong (t_a^* \mathcal{L} \otimes \mathcal{L}^{-1}) \otimes (t_b^* \mathcal{L} \otimes \mathcal{L}^{-1}).$$

This gives a group homomorphism

$$A(k) \rightarrow \text{Pic}(A)$$

via

$$a \mapsto t_a^* \mathcal{L} \otimes \mathcal{L}^{-1}$$

for any  $\mathcal{L} \in \text{Pic}(A)$ .

## 1.5 The Adventures of BUNTES (Sachi)

### 1.5.1 In which we are introduced to an important homomorphism, review some concepts and our story begins

Abelian variety  $X$ , we know this is a complete group variety, our goal is to give an embedding  $X \rightarrow \mathbf{P}^N$  for some  $N$ . This motivates the study of [line bundles](#).

Last time Ricky proved theorem of [cube 1.4.11](#) and [square 1.4.15](#). For any [line bundle](#)  $L$  on  $X$ , there is a group homomorphism  $\Phi_L: X \rightarrow \text{Pic}(X)$  via  $x \mapsto T_x^* L \otimes L^{-1}$ . Be careful  $T_x^*$  is  $-x$ , convention, who knows why.

**Example 1.5.1.** Let  $X = E$  an elliptic curve,  $L = L((0))$ ,  $x \mapsto (x) - (0)$ , in this case this is in  $\text{Pic}^0(E) \cong E \cong \widehat{E}$ ,

**Proposition 1.5.2.** *This is translation invariant.*

*Proof.* Translate by  $q \in E$ .  $(x + q) - (q)$  take  $p$  to be the third point on the line with  $x, q$ ,  $(x) + (q) + (p) \cong 3(0)$  and  $(x + q) + (p) \cong 2(0)$  subtracting these gives  $(x) - (x + q) + (q) \cong (0)$  or  $(x) - (0) \cong (x + q) - (q)$ .  $\square$

What about the converse of this, what can we say about translation invariant [line bundles](#)

$$K(L) = \{x \in X : T_x^* L \cong L\}?$$

**Proposition 1.5.3.**  $K(L)$  is Zariski closed in  $X$ .

*Proof.* Consider  $m^* L \otimes p_2^* L^{-1}$  on  $X \times X$ , then

$$\{x : \text{this is trivial on } \{x\} \times X\}$$

is closed. [See-saw 1.6.6](#) implies restriction is pullback

$$T_x^* L \otimes L^{-1}$$

so this is  $K(L)$ .  $\square$

## 1.5.2 In which Pooh discovers our main theorem

**Proposition 1.5.4.** Let  $X$  be an [abelian variety](#) and  $L$  a [line bundle](#),  $L = L(D)$  then TFAE:

1.  $H(D) = \{x \in X : T_x^* D = D\}$  is finite.
2.  $K(L) = \{x \in X : T_x^* L \cong L\}$  is finite.
3.  $|2D|$  is basepoint free and defines a finite morphism  $X \rightarrow \mathbf{P}^N$ .
4.  $L$  is ample.

*Proof.* 3. to 4.. Is algebraic geometry.

2. to 1.. Follows as being equal is stronger than being linearly equivalent.

4. to 2.. [Section 3](#)

3. to 4.. [Section 4](#)  $\square$

## 1.5.3 In which Owl proves the ampleness of $L$ implies finiteness of $K(L)$

4. to 2. Assume  $L$  ample and  $K(L)$  is infinite. Let  $Y$  be the connected component at 0 of  $K(L)$ ,  $\dim Y > 0$ . Show trivial bundle is ample on  $Y$  implies  $Y$  is affine, But  $Y$  is closed and therefore [complete](#) so this is a contradiction.  $L|_Y$  ample  $[-1]^* L|_Y$  is ample.  $L|_Y \otimes [-1]^* L|_Y$  is ample, consider

$$\begin{aligned} d: Y &\rightarrow Y \times Y \\ y &\mapsto (y, -y) \end{aligned}$$

$m \circ d = \text{constant}$ ,  $d^* m^*(L) = \mathcal{O}_Y$ , LHS is  $L|_Y \otimes [-1]^* L|_Y$ .

### 1.5.4 In which Rabbbit sets out on a long journey to prove finiteness of $H(D)$ implies $|2D|$ is basepoint free and gives a finite map $X \rightarrow \mathbf{P}^N$

**Note 1.5.5.**  $|2D|$  is always basepoint free.

Apply the [theorem of the square 1.4.15](#):  $T_{x+y}^*D + D \cong T_x^*D + T_y^*D$ , let  $y = -x$ ,  $2D \cong T_x^*D + T_{-x}^*D$ . ( $D$  effective) For any  $y \in X$ , choose some  $x$  s.t. RHS doesn't contain  $y$ .  $E = 2D$

$$\psi_E: X \rightarrow \mathbf{P}^N$$

can we make this finite? If  $\psi_E$  is not finite then  $\psi(C) = \text{pt}$  for some irreducible curve  $C$  (Zariski's main theorem). For each divisor in  $|E|$  either it contains  $C$  or fails to intersect  $C$  by changing  $E$  if necessary, assume  $E \cap C = \emptyset$ .

**Claim 1.5.6.**  $T_x^*E \cap C = \emptyset$  or all of  $C$  for all  $x \in X$ .

*Proof.* Intersection numbers are constant. □

*Proof.*  $O(T_x^*E)|_{\bar{C}}$ , when  $x = 0$  this is trivial so  $\deg = 0$ . So  $\deg = 0$  for all [line bundles](#).  $E$  effective implies  $C \cap T_x^*E = \emptyset$  for all  $x$  s.t.  $\cap$  is not in  $C$ . □

**Claim 1.5.7.**  $E$  is invariant by translation by  $x - y$  for  $x, y \in C$ .

*Proof.* If  $e \in E$ ,  $T_{x-e}^*(E) \cap C \neq \emptyset$ . This is as  $x$  is in it,  $x - (x - e) = e$ , because it is nonempty it's all of  $C$ . So  $y$  is in it. So  $y - (x - e) \in E$ . This is also  $e - (x - y) \in E$ , so  $E$  is invariant under  $T_{x-y}^*$  □

Now assume  $H(E) = \{x \in X : T_x^*E = E\}$  is finite. But if  $\psi_E(C) = \text{pt}$  then  $T_{x-y}^*(E) = E$  for all  $x, y \in C$ . So  $H$  is not finite, a contradiction. So  $\psi_E$  can't collapse a curve so  $\psi_E$  is finite.

### 1.5.5 In which Piglet discovers a corollary

**Corollary 1.5.8.** *Abelian varieties are projective.*

*Proof.* Let  $X$  be an [abelian variety](#),  $U \subseteq X$  be an open affine set,  $0 \in U$ ,  $X \setminus U = D_1 \cup \dots \cup D_t$  irreducible divisors. Let  $D = \sum D_i$ , then claim:  $H(D) = \{x \in X : T_x^*D = D\}$  is finite. If  $H \subseteq U$ ,  $U$  affine, then  $H$  closed subvariety of an [abelian variety](#), hence [complete](#), so its finite. If  $x \in H$  then  $-x \in H$ . Now claim that if  $x \in H$  then  $T_x^*$  preserves  $U$ , if not let  $u \in U$ . Suppose  $u - x = d$  for some  $d \in D$  then  $u = d + x$  which is  $d$  translated by  $-x$  so  $d + x \in D$  so  $u \in D$ . But contradiction, oh no! So  $T_x^*$  preserves  $U$ , for all  $x \in H$ , as  $0 \in U$ , for all  $x \in H$  we have  $0 - x \in U$  and  $0 + x \in U$  so  $H \subseteq U$ . □

**Corollary 1.5.9.** *Abelian varieties are divisible.  $X[n]$  is finite for  $n \geq 1$ .*

*Proof.*  $[n]: X \rightarrow X$  and  $X[n]$  is the kernel of this. Note that for  $x \in X[n]$

$$[n] \circ T_x = [n]$$

$y \in X$ , then  $n(y - x) = ny - nx = ny$  so for all  $L \in \text{Pic } X$

$$T_x^*([n]^*L) \cong ([n]^*L)$$

which implies

$$K([n]^*L) \supseteq X[n]$$

and we just need to find  $L$  s.t. this is finite.  $X$  projective implies there exists an ample  $L$ . The [theorem of the cube 1.4.11](#) implies

$$[n]^*L \cong L^{\frac{n^2+n}{2}} \otimes L^{\frac{n^2-n}{2}}$$

where both terms on the right are ample, hence the left is also. □

### 1.5.6 Epilogue: In which we might discuss isogenies

**Definition 1.5.10.**  $f: X \rightarrow Y$  a morphism of varieties, get a field extension  $k(X)/f^*k(Y)$ , if  $\dim X = \dim Y$  and  $f$  is surjective. Then this is a finite field extension and  $\deg f$  is  $d = [k(X) : f^*k(Y)]$  and  $d = \#f^{-1}(y)$  for almost all  $y$ .

**Definition 1.5.11.** A homomorphism of [abelian varieties](#)  $f: X \rightarrow Y$  is an **isogeny** if  $f$  is surjective with finite kernel.

**Corollary 1.5.12.** Degree of  $[n]$  is  $n^{2g}$ , if  $n$  is prime to the characteristic of  $k$ ,  $k = \bar{k}$ ,  $g = \dim X$ .

*Proof.* Let  $D$  be an ample [symmetric](#) divisor, e.g.

$$D = D' + [-1]^*D'$$

know  $[n]^*D \sim n^2D$

$$\deg([n]^*(D \cdots D)) = ([n]^*D \cdots [n]^*D) = (n^2D \cdots n^2D) = n^{2g}(D \cdots D). \quad \square$$

## 1.6 Line Bundles and the Dual Abelian Variety (Angus)

**Meta-goal** Understand [line bundles](#) on [abelian varieties](#).

**Setup**  $A$  an [abelian variety](#)  $/k$ .

**Last time** For  $L$  a [line bundle](#) on  $A$  we get a map

$$\begin{aligned} \phi_L: A(k) &\rightarrow \text{Pic}(A) \\ a &\mapsto t_a^*L \otimes L^{-1} \end{aligned}$$

where

$$\text{Pic}(A) = \{\text{line bundles on } A\} / \sim.$$

This is a group homomorphism (by the [theorem of the square 1.4.15](#)). We define

$$K(L)(k) = \ker(\phi_L) = \{a \in A(k) : t_a^*L \simeq L\}.$$

**Today** We are going to package these into a big map

$$\begin{aligned} \phi: \text{Pic}(A) &\rightarrow \text{Hom}(A(k), \text{Pic}(A)) \\ L &\mapsto \phi_L. \end{aligned}$$

**Proposition 1.6.1.**

1.  $\phi$  is a group homomorphism
- 2.

$$\phi_{t_a^*L} = \phi_L$$



*Proof.* 1.

$$\begin{aligned}\phi_{L \otimes M}(a) &= t_a^*(L \otimes M) \otimes (L \otimes M)^{-1} \\ &= t_a^*L \otimes L^{-1}t_a^*M \otimes M^{-1} \\ &= \phi_L \otimes \phi_M\end{aligned}$$

2.

$$\begin{aligned}\phi_{t_b^*L}(a) &= t_a^*(t_b^*L) \otimes (t_b^*L)^{-1} \\ &= t_{a+b}^*L \otimes (t_b^*L)^{-1} \\ &= t_a^*L \otimes t_b^*L \otimes L^{-1} \otimes (t_b^*L)^{-1} \\ &= \phi_L(a)\end{aligned}$$

by the [theorem of the square 1.4.15](#) □

**Definition 1.6.2.**

$$\begin{aligned}\text{Pic}^0(A) &= \ker(\phi) \\ &= \{L \in \text{Pic}(A) : \phi_L = 0\} \\ &= \{L \in \text{Pic}(A) : t_a^*L \simeq L \ \forall a \in A(k)\} \\ &= \{\text{translation invariant line bundles}\}/\sim\end{aligned}$$

**Goals** Study  $\text{Pic}^0(A)$ , give it an [abelian variety](#) structure, solve a moduli problem, demonstrate some duality.

### 1.6.1 Aside: alternate description of $\text{Pic}^0(A)$

**Definition 1.6.3** (Algebraic Equivalence). Two [line bundles](#)  $L_1, L_2$  on an [abelian variety](#) are **algebraically equivalent** if there exists a variety  $Y$  with [line bundle](#)  $L$  on  $A \times Y$  and points  $y_1, y_2 \in Y$  s.t.  $L|_{A \times \{y_1\}} \simeq L_1, L|_{A \times \{y_2\}} \simeq L_2$ .

**Remark 1.6.4.** This looks like homotopy.

**Proposition 1.6.5.**

$$\text{Pic}^0(A) = \{\text{line bundles which are alg. equiv to } \mathcal{O}_A\}$$

*Proof.* [\[23\]](#). □

### 1.6.2 See-Saws

**Theorem 1.6.6** (See-saw theorem). Let  $X, T$  be varieties  $X$  [complete](#), let  $L$  be a [line bundle](#) on  $X \times T$ , let  $T_1 = \{t \in T : L|_{X \times \{t\}} \text{ is trivial}\}$  then  $T_1$  is closed in  $T$ . Further let  $p_2 : X \times T_1 \rightarrow T_1$ , then  $L|_{X \times T_1} \cong p_2^*M$  for some [line bundle](#)  $M$  on  $T_1$ .

**Remark 1.6.7.** In fact  $M = p_{2*}L$ .

**Corollary 1.6.8** (that no one states/only Milne). Let  $X, T$  be as above and let  $L, M$  be [line bundles](#) on  $X \times T$  s.t.

$$L|_{X \times \{t\}} \cong M|_{X \times \{t\}} \forall t \in T$$

$$L|_{\{t\} \times X} \cong M|_{\{t\} \times X} \text{ for some } x \in X$$

then  $L \cong M$ .

### 1.6.3 Properties of $\text{Pic}^0 A$

**Lemma 1.6.9.**  $L \in \text{Pic}^0(A)$  and  $m, p_1, p_2: A \times A \rightarrow A$

1. 
$$m^*L \cong p_1^*L \otimes p_2^*L$$
2. Given  $f, g: X \rightarrow A$  
$$(f + g)^*L \cong f^*L \otimes g^*L$$
3. 
$$[n]^*L \cong L^{\otimes n}$$
4. 
$$\phi_L(A(k)) \subseteq \text{Pic}^0(A)$$

for  $L \in \text{Pic}(A)$ .

*Proof.* 1.

$$(m^*L \otimes (p_1^*L)^{-1} \otimes (p_2^*L)^{-1})|_{A \times \{a\}} = t_a^*L \otimes L^{-1} = \mathcal{O}_A$$

$$(m^*L \otimes (p_1^*L)^{-1} \otimes (p_2^*L)^{-1})|_{\{a\} \times A} = t_a^*L \otimes L^{-1} = \mathcal{O}_A$$

by [see-saw 6](#) whole thing is trivial on  $A \times A$ .

2.

$$(f + g)^*L \cong (f \times g)^*m^*L \cong (f \times g)^*(p_1^*L \otimes p_2^*L) \cong f^*L \otimes g^*L$$

3. Induction of 3.

4.

$$\phi_{\phi_L(a)} = \phi_{t_a^*L} \otimes L^{-1} = \phi_{t_a^*L} \otimes L^{-1} = \phi_L \otimes \phi_{L^{-1}} = 0 \quad \square$$

**Proposition 1.6.10.** If  $L$  is nontrivial in  $\text{Pic}^0(A)$  then  $H^i(A, L) = 0 \forall i$ .

*Proof.* If  $H^0(A, L) \neq 0$ , we would have a nontrivial section  $s$  of  $L$  then  $[-1]^*s$  is a nontrivial section of  $[-1]^*L = L^{-1}$ . But if both  $L$  and  $L^{-1}$  have a nontrivial section then  $L \cong \mathcal{O}_A$ . So since  $L$  is nontrivial  $H^0(A, L) = 0$ . Now assume  $H^i(A, L) = 0$  for all  $i < j$ . Consider

$$\begin{aligned} A &\xrightarrow{\text{id} \times 0} A \times A \xrightarrow{m} A \\ a &\mapsto (a, 0) \mapsto a \end{aligned}$$

this gives

$$H^j(A, L) \rightarrow H^j(A \times A, m^*L) \rightarrow H^j(A, L)$$

which composes to the identity.

$$H^j(A \times A, m^*L) = H^j(A \times A, p_1^*L \otimes p_2^*L) = \bigoplus_{i=0}^j H^i(A, L) \otimes H^{j-i}(A, L)$$

by Künneth. The RHS is 0 by the inductive hypothesis. So the identity on  $H^j(A, L)$  factors through 0, hence the group is 0.  $\square$

We now think of  $\phi_L$  as a map  $\phi_L: A(k) \rightarrow \text{Pic}^0(A)$  with kernel  $K(L)(k)$ .

**Theorem 1.6.11.** If  $K(L)(k)$  is finite then  $\phi_L$  is surjective.

*Proof.* Idea is to study

$$\Lambda(L) = m^*L \otimes (p_1^*L)^{-1} \otimes (p_2^*L)^{-1}. \quad \square$$

Given an ample [line bundle](#)  $L$  on  $A$  we now have an isomorphism of groups

$$A(k)/K(L)(k) \cong \text{Pic}^0(A)$$

the LHS allows us to put an [abelian variety](#) structure on  $\text{Pic}^0(A)$ .

### 1.6.4 The Dual Abelian Variety

**Theorem 1.6.12.** *Let  $A$  be an **abelian variety** and  $L$  an ample **line bundle** on  $A$ , then the quotient scheme  $A/K(L)$  exists and is an **abelian variety** of the same dimension as  $A$ .*

*Proof.* (Sketch) (characteristic 0) Cover  $A$  by affine opens  $U_i = \text{Spec } R_i$  such that for all  $a \in A$  the orbit  $K(L)a \subseteq U_i$  for some  $i$ . We can do this because **abelian varieties** are projective. Then we say  $U_i/K(L) = \text{Spec}(R_i^{K(L)})$  then glue. (details in Mumford, II sec, 6 appendix). Since we are in characteristic 0, the quotient scheme is in fact a variety.  $\square$

**Definition 1.6.13** (Dual abelian varieties). The **dual abelian variety** is

$$\hat{A} = A/K(L).$$

**Remark 1.6.14.**

- 

$$\hat{A}(K) = \text{Pic}^0(A)$$

- We have an **isogeny**

$$\phi_L: A \rightarrow \hat{A}.$$

**Theorem 1.6.15.** *There is a unique **line bundle**  $\mathcal{P}$  on  $A \times \hat{A}$  called the **Poincaré bundle** such that*

1.

$$\mathcal{P}|_{A \times \{x\}} \in \text{Pic}^0(A) \text{ for all } x \in \hat{A}$$

2.

$$\mathcal{P}|_{0 \times \hat{A}} = 0$$

3. *If  $Z$  is a scheme with a **line bundle**  $R$  on  $A \times Z$  satisfying 1., 2., there exists a unique*

$$f: Z \rightarrow \hat{A}$$

*s.t.*

$$(\text{id} \times f)^* \mathcal{P} = R.$$

*That is  $(\hat{A}, \mathcal{P})$  represents the functor*

$$Z \mapsto \left\{ L \in \text{Pic}(A \times Z) : \begin{matrix} L|_{A \times \{z\}} \in \text{Pic}^0(A) \forall z \in Z \\ L|_{0 \times Z} = 0 \end{matrix} \right\} / \sim .$$

### 1.6.5 Dual morphisms

Let  $f: A \rightarrow B$  be a homomorphism of **abelian varieties**. Let  $\mathcal{P}_A, \mathcal{P}_B$  be the **Poincaré bundles** on  $A$  and  $B$ . Consider  $M = (F \times \text{id}_{\hat{B}})^* \mathcal{P}_B$  on  $A \times \hat{B}$ , then

1.

$$M|_{A \times \{x\}} \in \text{Pic}^0(A)$$

2.

$$M|_{\{0\} \times \hat{B}} = 0$$

thus by the universal property we get a unique morphism

$$\hat{f}: \hat{B} \rightarrow \hat{A}$$

satisfying

$$(\text{id}_A \times \hat{f})^* \mathcal{P}_A = (f \times \text{id}_{\hat{B}})^* \mathcal{P}_B.$$

**Definition 1.6.16** (Dual morphisms).  $\hat{f}$  as above is called the **dual morphism**.

**Remark 1.6.17.**

•

$$\begin{aligned} \hat{f}: \hat{B} = \text{Pic}^0(B) &\rightarrow \hat{A}(k) = \text{Pic}^0(A) \\ L &\mapsto f^*L \end{aligned}$$

•

$$[\hat{n}_A] = [n_{\hat{A}}]$$

Consider the **Poincaré bundle**  $\mathcal{P}_{\hat{A}}$  on  $\hat{A} \times \hat{A}$ , now think of  $\mathcal{P}_A$  as living on  $\hat{A} \times A$ . By the universal property of  $\mathcal{P}_{\hat{A}}$  get a unique morphism

$$\text{can}_A: A \rightarrow \hat{A}.$$

**Theorem 1.6.18.**  $\text{can}_A$  is an isomorphism.

**Lemma 1.6.19.**

$$\phi_{f^*L} = \hat{f} \circ \phi_L \circ f.$$

**Proposition 1.6.20.** If  $f: A \rightarrow B$  is an **isogeny**, then  $\hat{f}: \hat{B} \rightarrow \hat{A}$  is an **isogeny**. Further if  $N = \ker f$ , then  $\hat{N} = \ker \hat{f}$  is the Cartier dual of  $N$ .

**Definition 1.6.21** (Symmetric morphisms, (principal) polarizations). A morphism  $f: A \rightarrow \hat{A}$  is **symmetric** if  $f = \hat{f} \circ \text{can}_A$

A **polarization** is a **symmetric isogeny**  $f: A \rightarrow \hat{A}$  s.t.  $f = \phi_L$  for some ample **line bundle**  $L$  on  $A$ .

A **principal polarization** is a **polarization** of degree 1, i.e. an isomorphism.

**Remark 1.6.22.** Elliptic curves always admit **principal polarization**.

If one wishes to mimic the theory of elliptic curves, one should study principally polarized **abelian varieties**.

## 1.7 Endomorphisms and the Tate module

**Motivation**

$$\begin{aligned} f: \mathbf{P}^n \subseteq V_1 &\rightarrow V_2 \subseteq \mathbf{P}^m, V_i = V(I_i) \\ P &\mapsto \dots \end{aligned}$$

$$f = [f_1 : \dots : f_m], f_i \in \overline{K}(V_1)$$

this feels quite restrictive, an **isogeny** is even more so, rational, regular, homomorphism, surjective, finite kernel. It feels like there won't be too many but we have multiplication by  $n$  etc. so we should ask how many are there that will surprise us? I.e. what is

$$\text{rank}_{\mathbb{Z}} \text{Hom}(A, B) = ?$$

### 1.7.1 Poincaré's complete reducibility theorem

**Theorem 1.7.1** (Poincaré's complete reducibility theorem). *Let  $B \subseteq A$  then there is  $C \subseteq A$  s.t.  $B \cap C$  is finite and  $B + C = A$ . I.e.  $B \times C \rightarrow A$ ,  $(b, c) \mapsto b + c$  is an isogeny.*

*Proof.* Choose  $\mathcal{L}$  ample on  $A$

$$\begin{array}{ccc} B & \xrightarrow{i} & A \\ \phi_{i*} \mathcal{L} \downarrow & & \downarrow \phi_L \\ \hat{B} & \xleftarrow{\hat{i}} & \hat{A} \end{array}$$

$C$  is defined to be the connected component of  $\phi_L^{-1}(\ker \hat{i})$  in  $A$

$$\dim C = \dim \ker \hat{i} \geq \dim \hat{A} - \dim \hat{B} = \dim A - \dim B.$$

$B \cap C$  finite,  $z \in B$ ,  $z \in B \cap \phi_L^{-1}(\ker \hat{i}) = T_z^* L \otimes L^{-1}|_B$  is trivial if and only if  $z \in K(L|_B)$ . So  $L|_B$  ample implies  $K(L|_B)$  finite and so  $B \cap C$  is finite. So  $B \times C \rightarrow A$  has finite kernel and

$$\dim(B \times C) = \dim B + \dim C \geq \dim A$$

and surjective implies its an isogeny.  $\square$

**Definition 1.7.2** (Simple abelian varieties).  $A$  is called **simple** if there does not exists  $B \subseteq A$  other than  $B = 0, A$ .

**Corollary 1.7.3.**

$$A \sim A_1^{n_1} \times \cdots \times A_k^{n_k}$$

$A_i \not\sim A_j$  for  $i \neq j$  and  $A_i$  simple.

**Corollary 1.7.4.**  $\alpha \in \text{Hom}(A, B)$  for  $A, B$  simple then  $\alpha$  is an isogeny or 0.

*Proof.*  $\alpha(A) \subseteq B$  which implies  $\alpha(A) = B$  or 0. The connected component of 0 of  $\ker \alpha$  will be an abelian subvariety of  $A$ , denote it  $C$ . If  $C = 0$  then  $\ker \alpha$  is finite, if  $C = A$  then  $\alpha = 0$ . So  $\alpha$  is an isogeny or 0.  $\square$

**Corollary 1.7.5.** If  $A, B$  are simple and  $A \not\sim B$  then  $\text{Hom}(A, B) = 0$ .

**Definition 1.7.6.**

$$\text{End}^0(A) = \text{End}(A) \otimes \mathbb{Q}.$$

**Lemma 1.7.7.** If  $\alpha: A \rightarrow B$  is an isogeny, then there exists  $\beta: B \rightarrow A$  s.t.  $\beta \circ \alpha = n_A$  for some  $n \geq 1$ .

*Proof.*  $\alpha$  an isogeny implies  $\ker \alpha$  is finite. So there exists  $n$  with  $n \ker \alpha = 0$ .  $\ker \alpha \subseteq \ker n_A$

$$\begin{array}{ccccc} & & A & \xrightarrow{n_A} & A \\ & \swarrow \alpha & \downarrow & \nearrow \circ & \uparrow \\ B & \xrightarrow{\sim} & A/\ker \alpha & & \\ & \searrow & \downarrow \exists \beta & & \\ & & A/n_A & & \end{array}$$

so  $\beta \circ \alpha = n_A$ , also  $\alpha \circ \beta = n_B$ .  $\square$

**Corollary 1.7.8.** *A is simple then  $\text{End}^0(A)$  is a division ring,  $\alpha^{-1} = \beta \otimes \frac{1}{n}$ .*

**Corollary 1.7.9** (to Poincaré reducibility theorem). *If*

$$A \sim A_1^{n_1} \times \cdots \times A_k^{n_k}$$

*then*

$$\text{End}^0(A) \simeq \prod \text{End}^0(A_i)^{n_i^2}.$$

*Proof.*

$$\begin{aligned} \text{End}(A) \otimes \mathbf{Q} &\simeq \prod_{i,j} \text{Hom}(A_i^{n_i}, A_j^{n_j}) \otimes \mathbf{Q} \\ &\simeq \prod_i \text{End}(A_i)^{n_i^2} \otimes \mathbf{Q} \\ &\simeq \prod_i \text{End}^0(A_i)^{n_i^2} \quad \square \end{aligned}$$

**Theorem 1.7.10** (7.2). *If  $\dim A = g$  then  $\deg n_A = n^{2g}$ .*

**Corollary 1.7.11.**  *$\text{char } k \nmid n$  implies  $\ker(n_A) \simeq (\mathbf{Z}/n\mathbf{Z})^{2g}$ .*

*Proof.* If  $m|n$  then  $|\ker(m_A)| = m^{2g}$ , then use structure theorem.  $\square$

In particular if we let  $A[l^n] = A(k^{\text{sep}})[l^n]$ , then  $A[l^n] \simeq (\mathbf{Z}/l^n)^{2g}$  Define

$$T_l(A) = \varprojlim_n A[l^n], \quad A[l^{n+1}] \xrightarrow{l} A[l^n]$$

**Proposition 1.7.12.**

$$T_l \simeq (\mathbf{Z}_l)^{2g}$$

$\alpha: A \rightarrow B$  induces

$$\begin{aligned} T_l \alpha: T_l(A) &\rightarrow T_l(B) \\ (a_1, a_2, \dots) &\mapsto (\alpha(a_1), \alpha(a_2), \dots) \end{aligned}$$

**Lemma 1.7.13.**

$$\text{Hom}(A, B) \hookrightarrow \text{Hom}(T_l(A), T_l(B))$$

*Proof.* Let  $\alpha \in \text{Hom}(A, B)$  and assume  $T_l \alpha = 0$  then

$$\ker(\alpha|_{A_i}) \supseteq A_i[l^n] \forall n$$

for any simple component  $A_i$  of  $A$  so  $\alpha = 0$  on each  $A_i$  and hence  $\alpha = 0$  on  $A$ .  $\square$

**Corollary 1.7.14.**  *$\text{Hom}(A, B)$  is torsion free.*

Recall we are interested in knowing about  $\text{rank}_{\mathbf{Z}} \text{Hom}(A, B) = ?$ , can we bound this? If we could show that

$$\text{Hom}(A, B) \otimes \mathbf{Z}_l \hookrightarrow \text{Hom}(T_l(A), T_l(B))$$

we could conclude, so:

$$\begin{array}{ccc} \text{Hom}(A, B) \otimes \mathbf{Z}_l & \xhookrightarrow{\quad} & \text{Hom}(T_l A, T_l B) \\ \sim \downarrow & & \sim \downarrow \\ \prod_{i,j} (\text{Hom}(A_i, B_j) \otimes \mathbf{Z}_l) & \xhookrightarrow{\quad} & \prod_{i,j} \text{Hom}(T_l A_i, T_l B_j) \end{array}$$

$A_i + B_j = 0$ ,  $A_i \sim B_j$   $\text{Hom}(A_i, B_j) \hookrightarrow \text{End}(A_i)$ . Assume  $A = B$  and  $A$  simple, then  $\text{End}(A) \otimes \mathbf{Z}_l \hookrightarrow \text{End}(T_l(A))$ .

**Definition 1.7.15.**  $V/k$  then  $f: V \rightarrow k$  is called a (homogenous) polynomial function of degree  $d$  if  $\forall \{v_1, \dots, v_m\} \subseteq V$  linearly independent.

$$f(\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_m v_m)$$

is given by a homogenous polynomial of degree  $d$  in  $\lambda_i$  i.e.

$$f(\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_m v_m) = P(\lambda_1, \dots, \lambda_m)$$

for some  $P \in k[X_m]$  homogenous of degree  $d$ .

$$\deg: \text{End}(A) \rightarrow \mathbf{Z}$$

$\alpha$  an **isogeny** iff  $\deg \alpha, \alpha$  not an **isogeny** iff 0.

**Theorem 1.7.16.**  $\deg$  uniquely extends to a polynomial function of degree  $2g$  on  $\text{End}^0(A) \rightarrow \mathbf{Q}$ .

*Proof.* (of above continued)

$$\text{End}(A) \otimes \mathbf{Z}_l \hookrightarrow \text{End}(T_l(A))$$

for  $A$  **simple** iff for any finitely generated  $M \subseteq \text{End}(A)$

$$M \otimes \mathbf{Z}_l \hookrightarrow \text{End}(T_l(A))$$

Claim:

$$M^{\text{div}} = \{f \in \text{End}(A) : nf \in M \text{ for some } n \geq 1\}$$

is finitely generated.

Proof:  $M^{\text{div}} = (M \otimes \mathbf{Q}) \cap \text{End}(A)$   $\deg: M \otimes \mathbf{Q} \rightarrow \mathbf{Q}$  is a polynomial so it is continuous.

$$U = \{\phi \in M \otimes \mathbf{Q} : \deg \phi < 1\}$$

is open in  $M \otimes \mathbf{Q}$  but  $U \cap M^{\text{div}} = 0$  so  $M^{\text{div}}$  is a discrete subgroup of the finite dimensional  $\mathbf{Q}$ -vector space  $M \otimes \mathbf{Q}$  so  $M^{\text{div}}$  is finitely generated.  $M \hookrightarrow M^{\text{div}}$  so  $M \otimes \mathbf{Z}_l \hookrightarrow M^{\text{div}} \otimes \mathbf{Z}_l$  so we may assume  $M = M^{\text{div}}$ .

Let  $f_1, \dots, f_r$  be a  $\mathbf{Z}$ -basis for  $M$  and suppose that  $\sum a_i T_l(f_i) = 0$  for some  $a_i \in \mathbf{Z}_l$  not all 0. We can assume not all  $a_i$  are divisible by  $l$ . Choose  $a'_i \in \mathbf{Z}$  s.t.  $a'_i = a_i \pmod{l}$

$$f = \sum a'_i f_i \in \text{End}(A)$$

we then have

$$f = \sum a'_i T_l f_i$$

is 0 on the first coordinate of  $T_l$ . So  $A[l] \subseteq \ker f$  so there exists  $g$  with  $f = lg$   $f \in M$  implies  $g \in M^{\text{div}} = M$  so  $g = \sum b_i f_i$  and  $f = \sum lb_i f_i = \sum a_i f_i$  hence  $l \mid a_i$  for all  $i$  a contradiction. So  $\text{End}(A) \otimes \mathbf{Z}_l \hookrightarrow \text{End}(T_l(A))$ .

Therefore

$$\text{Hom}(A, B) \otimes \mathbf{Z}_l \hookrightarrow \text{Hom}(T_l(A), T_l(B))$$

$$\text{rank}_{\mathbf{Z}} \text{Hom}(A, B) \leq 4 \dim A \dim B.$$

□