

Singular Moduli

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- 1 Introduction
- 2 Background
- 3 The Hilbert class field
- 4 Singular moduli
- 5 Modern work
- 6 Conclusion

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$$\approx 12^3(231^2 - 1)^3 + 744 - 10^{-13} \cdot 7.499 \dots$$

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$$K = \mathbf{Q}(\zeta_n), \quad \mathbf{Z}_K = \mathbf{Z}[\zeta_n].$$

- $K = \mathbf{Q}(\sqrt{d})$, d squarefree then

$$\mathbf{Z}_K = \begin{cases} \mathbf{Z}[\sqrt{d}] & \text{if } d \equiv 2, 3 \pmod{4}, \\ \mathbf{Z}[(1 + \sqrt{d})/2] & \text{if } d \equiv 1 \pmod{4}. \end{cases}$$

The ideal class group

Given a number field K we let $I(\mathbf{Z}_K)$ be the set

$$\{M \text{ subgroup of } K : \mathbf{Z}_K M \subset M, \exists a \in \mathbf{Z}_k \text{ s.t. } aM \subset \mathbf{Z}_K, M \neq 0\}$$

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$\mathrm{cl}(\mathbf{Z}_K)$ measures how far \mathbf{Z}_K is from having unique factorisation.

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The Hilbert class field (of an imaginary quadratic field)

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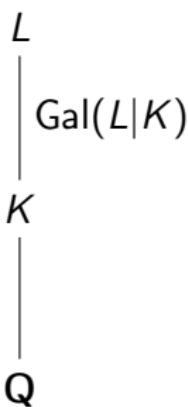
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$\mathbb{Q}(\sqrt{-163})$	$\mathbb{Q}(\sqrt{-163})$	1

The Artin reciprocity theorem for the Hilbert class field

Theorem

If K is a number field and L is its Hilbert class field then

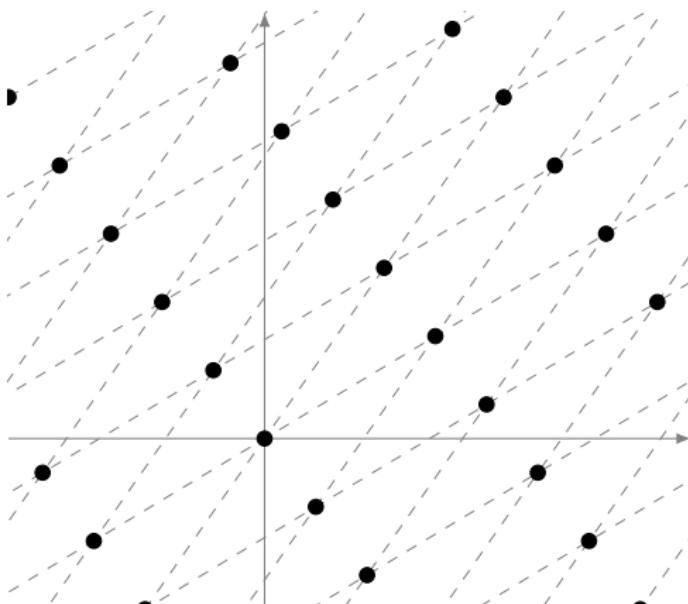
$$\text{cl}(\mathbb{Z}_K) \cong \text{Gal}(L|K).$$



Lattices

Definition

A **lattice** is an additive subgroup of \mathbb{C} that is isomorphic to \mathbb{Z}^2 .



Homothety

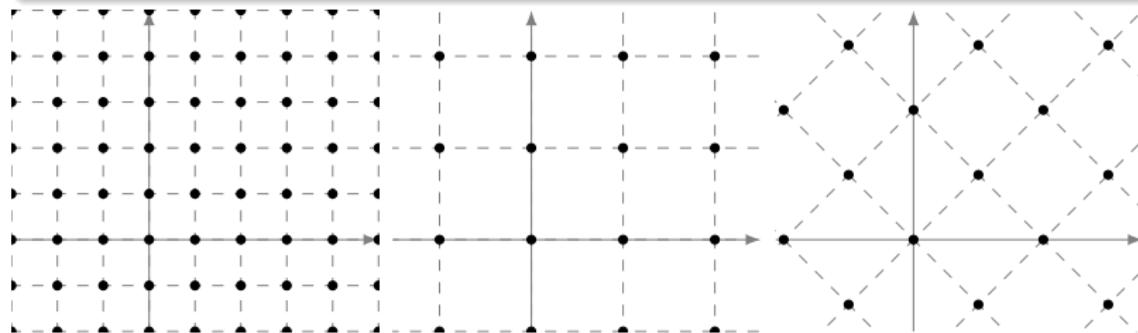
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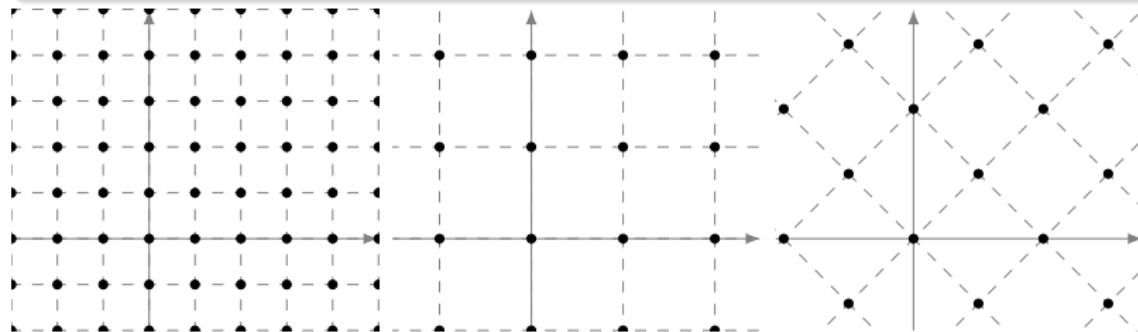
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Every lattice is homothetic to one of the form $\mathbf{Z} + \mathbf{Z}\tau$ for some $\tau \in \mathbf{C}$ with positive imaginary part.

The j -invariant

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$$j: \{\text{lattices}\} \rightarrow \mathbf{C}$$

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We can define j on the upper half plane by $j(\tau) = j(\mathbf{Z} + \mathbf{Z}\tau)$.

Letting $q = e^{2\pi i\tau}$ it turns out that

$$\begin{aligned} j(\tau) &= q^{-1} + 744 + 196884q + 21493760q^2 \\ &\quad + 864299970q^3 + 20245856256q^4 + \dots \end{aligned}$$

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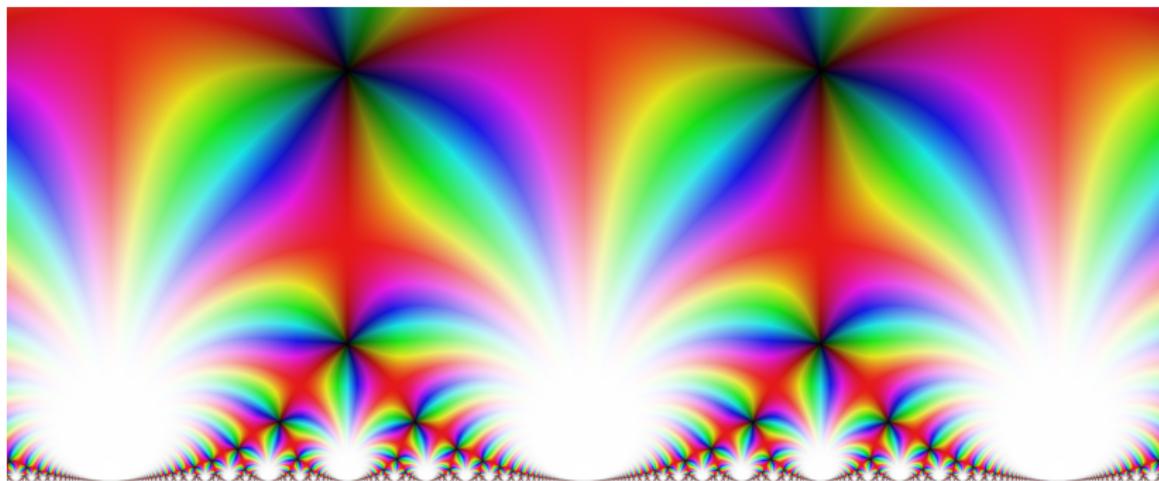


Figure : The j -invariant, picture by Fredrik Johansson

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$$j\left(\sqrt{-14}\right) = 2^3 \left(323 + 228\sqrt{2} + (231 + 161\sqrt{2})\sqrt{\sqrt{2} - 1}\right)^3.$$

(A corollary of) The first main theorem of class field theory

Theorem

If K is an imaginary quadratic field, $\mathbb{Z}_K = \mathbb{Z} + \mathbb{Z}\tau$ then:

- ① $j(\tau)$ is an algebraic integer.

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A (kind of) converse (Schneider)

If τ is an algebraic number that is not imaginary quadratic then $j(\tau)$ is transcendental.

Explaining Hermite's observations

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$$\begin{aligned} j(\tau) &= e^{-\pi i(1+i\sqrt{163})} + 744 + 196884e^{\pi i(1+i\sqrt{163})} + \dots \\ &= -e^{\pi\sqrt{163}} + 744 - 196884e^{-\pi\sqrt{163}} + \dots \end{aligned}$$

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The trailing terms are tiny (of order 10^{-13}) here giving

$$e^{\pi\sqrt{163}} \approx -j(\tau) + 744.$$

The class number 1 problem

Theorem (Stark-Heegner)

The only imaginary quadratic number fields with trivial class group are $\mathbf{Q}(\sqrt{-d})$ for

$$d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}.$$

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isn't really. The value is not as close to the corresponding singular modulus as $e^{-\pi\sqrt{d}}$ has larger absolute value for smaller d .

A formula of Gross-Zagier

We have that $j((1 + \sqrt{-67})/2) = -12^3(21^2 - 1)^3$ and
 $j((1 + \sqrt{-163})/2) = -12^3(231^2 - 1)^3$ and so

$$j\left(\frac{1 + \sqrt{-163}}{2}\right) - j\left(\frac{1 + \sqrt{-67}}{2}\right) = -2^{15} \cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 139 \cdot 331.$$

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Definition

The discriminant of an imaginary quadratic number τ is the discriminant of its minimal polynomial over \mathbb{Z} . i.e. if $a\tau^2 + b\tau + c = 0$ then the discriminant of τ is $b^2 - 4ac$.

A formula of Gross-Zagier

Theorem (Gross-Zagier, '84)

Given imaginary quadratic integers τ_1, τ_2 of coprime discriminants $d_1, d_2 < -4$ then we have

$$N(j(\tau_1) - j(\tau_2))^2 = \pm \prod_{\substack{x, n, n' \in \mathbb{Z} \\ n, n' > 0 \\ x^2 + 4nn' = d_1d_2}} n^{\epsilon(n')},$$

where

$$\epsilon(p) = \begin{cases} 1 & \text{if } p \nmid d_1, \text{ } d_1 \text{ is a square } \pmod{p}, \\ -1 & \text{if } p \nmid d_1, \text{ } d_1 \text{ is not a square } \pmod{p}, \\ 1 & \text{if } p \nmid d_2, \text{ } d_2 \text{ is a square } \pmod{p}, \\ -1 & \text{if } p \nmid d_2 = 1, \text{ } d_2 \text{ is not a square } \pmod{p}, \end{cases}$$

for p prime and ϵ is defined multiplicatively for all n .

Closing remarks

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- Singular moduli are not particularly complex objects in and of themselves.
- But their relation between different areas of mathematics ensures that they are still a research topic to this day.

Sources

I used some of the following when preparing this talk, and so they are probably good places to look to learn more about the topic:

- “Primes of the form $x^2 + ny^2$ ” – David A. Cox
- “Don Zagier’s work on singular moduli” – Benedict Gross
- “Complex multiplication and singular moduli” – Chao Li
- “Properties of Singular Moduli” - Ken Ono