Chapter 1

Dessins d'Enfants

These are notes for BUNTES Spring 2018, the topic is Dessins d'Enfants, they were last updated February 4, 2018. For more details see the webpage. These notes are by Alex, feel free to email me at `alex.j.best@gmail.com` to report typos/suggest improvements, I'll be forever grateful.

1.1 Overview (Angus)

1.1.1 Belyi morphisms

Let X be an algebraic curve over $\mathbf{C}$ (i.e. a compact Riemann surface) when is X defined over $\overline{\mathbf{Q}}$?

Theorem 1.1.1 (Belyi)**.** *An algebraic curve $X/\mathbf{C}$ is defined over $\overline{\mathbf{Q}}$ $\iff$ there exists a morphism $\beta\colon X \to \mathbf{P}^1\,\mathbf{C}$ ramified only over $\{0, 1, \infty\}$.*

Definition 1.1.2 (Ramified)**.** (AG) A morphism $f\colon X \to Y$ is **ramified** at $x \in X$ if on local rings the induced map $f^{\#}\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ descended to

$$\mathcal{O}_{Y,f(x)}/\mathfrak{m} \to \mathcal{O}_{X,x}/f^{\#}(\mathfrak{m})$$

is not a finite inseparable field extension.

(RS) A morphism $f\colon X \to Y$ is ramified at $x \in X$ if there are charts around x and $f(x)$ such that $f(x) = x^n$. This n is the **ramification index**.

Definition 1.1.3 (Belyi morphisms)**.** A **Belyi morphism** is one ramified only over $\{0, 1, \infty\}$

A **clean Belyi morphism** or **pure Belyi morphism** is a Belyi morphism where the ramification indices over 1 are all exactly 2.

Lemma 1.1.4. *A curve X admits a Belyi morphism iff it admits a clean Belyi morphism.*

Proof. If $\alpha\colon X \to \mathbf{P}^1\,\mathbf{C}$ is Belyi, then $\beta = 4\alpha(1-\alpha)$ is a clean Belyi morphism. □

1.1.2 Dessin d'Enfants

Definition 1.1.5. A **dessin d'Enfant** (or Grothendieck Dessin or just **Dessin**) is a triple (X_0, X_1, X_2) where X_2 is a compact Riemann surface, X_1 is a graph, $X_0 \subset X_1$ is a finite set of points, where $X_2 \smallsetminus X_1$ is a collection of open cells. $X_1 \smallsetminus X_0$ is a disjoint union of line segments

Lemma 1.1.6. *The data of a dessin is equivalent to a graph with an ordering on the edges coming out of each vertex.*

Definition 1.1.7 (Clean dessins)**.** A **clean dessin** is a dessin with a colouring (white and black) on the vertices such that adjacent vertices do not share a colour.

1.1.3 The Grothendieck correspondence

Given a Belyi morphism $\beta\colon X \to \mathbf{P}^1\,\mathbf{C}$ the graph $\beta^{-1}([0,1])$ defines a dessin.

Theorem 1.1.8. *The map*

$$\{\text{(Clean) Belyi morphisms}\} \longrightarrow \{\text{(clean) dessins}\}$$

$$\beta \mapsto \beta^{-1}([0,1])$$

is a bijection up to isomorphisms.

Example 1.1.9.

$$\mathbf{P}^1\,\mathbf{C} \longrightarrow \mathbf{P}^1\,\mathbf{C}$$

$$x \mapsto x^3$$

$$\mathbf{P}^1\,\mathbf{C} \longrightarrow \mathbf{P}^1\,\mathbf{C}$$

$$x \mapsto x^3 + 1$$

1.1.4 Covering spaces and Galois groups

A Belyi morphism defines a covering map.

$$\tilde{\beta}\colon \tilde{X} \longrightarrow \mathbf{P}^1\,\mathbf{C} \setminus \{0, 1, \infty\}$$

the coverings are controlled by the profinite completion of

$$\pi_1(\mathbf{P}^1\,\mathbf{C} \setminus \{0,1,\infty\}) = \mathbf{Z} * \mathbf{Z} = F_2.$$

Theorem 1.1.10. *There is a faithful action*

$$\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \circlearrowright \hat{\pi}_1(\mathbf{P}^1\,\mathbf{C} \setminus \{0,1,\infty\})$$

Proof. By Belyi's theorem every elliptic curve $E/\overline{\mathbf{Q}}$ admits a Belyi morphism. For each $j \in \overline{\mathbf{Q}}$ there exists an elliptic curve $E_j/\overline{\mathbf{Q}}$ with j-invariant j.

Given $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$,

$$\sigma(E_j) = E(\sigma(j))$$

assume $\sigma \mapsto 1$,

$$E_j \cong E_{\sigma(j)}\ \forall j$$

$$j = \sigma(j)\ \forall j$$

a contradiction. □

Corollary 1.1.11. *We have a faithful action of* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ *on dessins.*

Theorem 1.1.12. *We have a faithful action of* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ *on the set of dessins of any fixed genus.*

1.1.5 Exercises

Exercise 1.1.13. Compute the Dessins for the following Belyi morphisms

1. $$\mathbf{P}^1\,\mathbf{C} \to \mathbf{P}^1\,\mathbf{C}, \mapsto x^4$$

2. $$\mathbf{P}^1\,\mathbf{C} \to \mathbf{P}^1\,\mathbf{C}, \mapsto x^2(3-2x)$$

3. $$\mathbf{P}^1\,\mathbf{C} \to \mathbf{P}^1\,\mathbf{C}, \mapsto frac1x(2-x)$$

Exercise 1.1.14. Give an alternate proof of the fact that X admts a Belyi morphism is and only if it admits a clean Belyi morphism using dessins and the Grothendieck correspondence.

Exercise 1.1.15. Prove that a Belyi morphism corresponding to a tree, that sends ∞ to ∞ is a polynomial.

1.2 Riemann Surfaces I (Ricky)

1.2.1 Definitions

Definition 1.2.1. A **topological surface** is a Hausdorff space X wich has a collection of charts

$$\{\phi_i\colon U_i \xrightarrow{\sim} \phi_i(U_i) \subseteq \mathbf{C}, \text{ open}\}_{i\in I}$$

such that

$$X = \bigcup_{i\in I} U_i.$$

We call X a **Riemann surface** if the transition functions $\phi_i \circ \phi_j^{-1}$ are holomorphic.

1.2.2 Examples

Example 1.2.2. Open subsets of $\mathbf{C}$, e.g.

$$\begin{gathered} \mathbf{C} \\ \mathbf{D} = \{z \in \mathbf{C} : |z| < 1\} \\ \mathbf{H} = \{z \in \mathbf{C} : \operatorname{im} z > 0\}. \end{gathered}$$

Example 1.2.3. $\hat{\mathbf{C}}$ = Riemann sphere = $\mathbf{C} \cup \{\infty\}$. A basis of neighborhoods of ∞ is given by

$$\{z \in \mathbf{C} : |z| > R\} \cup \{\infty\}.$$

Example 1.2.4.

$$\begin{aligned} \mathbf{P}^1(\mathbf{C}) &= \{[z_0 : z_1] : (z_0, z_1) \neq (0,0)\} \\ U_0 &= \{[z_0, z_1] : z_0 \neq 0\} \to \mathbf{C} \\ [z_0 : z_1] &\mapsto \frac{z_1}{z_0} \\ U_1 &= \{[z_0, z_1] : z_1 \neq 0\} \to \mathbf{C} \\ [z_0 : z_1] &\mapsto \frac{z_0}{z_1}. \end{aligned}$$

Example 1.2.5. Let $\Lambda = \mathbf{Z} \oplus \mathbf{Z}i \subseteq \mathbf{C}$ then $X = \mathbf{C}/\Lambda$ is a Riemann surface.

1.2.3 Morphisms

Definition 1.2.6 ((Holo/Mero)-morphisms of Riemann surfaces)**.** A **morphism of Riemann surfaces** is a continuous map

$$f : S \to S'$$

such that for all charts ϕ, ψ on S, S' respectively we have $\psi \circ f \circ \phi^{-1}$ is holomorphic.

We call a morphism $f : S \to \mathbf{C}$ a **holomorphic function** on S.

We say $f : S \to \mathbf{C}$ is a **meromorphic function** is $f \circ \phi^{-1}$ is meromorphic.

Exercise 1.2.7. The set of meromorphic functions on a Riemann surface form a field.

We denote the field of meromorphic functions by $\mathcal{M}(S)$.

Proposition 1.2.8 (1.26)**.**

$$\mathcal{M}(\hat{\mathbf{C}}) = \mathbf{C}(z).$$

Proof. Let $f : \hat{\mathbf{C}} \to \mathbf{C}$ be meromorphic. Then the number of poles of f is finite say at $a_1, \ldots, a_n$. So, locally at a_i we can write

$$f(z) = \sum_{j=1}^{j_i} \frac{\lambda_{j,i}}{(z - a_i)^j} + h_i(z)$$

with h_i holomorphic. Then

$$f(z) - \sum_{i=1}^{n} \sum_{j=1}^{j_i} \frac{\lambda_{j,i}}{(z - a_i)^j}$$

is holomorphic everywhere. By Liouville's theorem this is constant. □

We say S, S' are isomorphic if $\exists f : S \to S'$, $g : S' \to S$ morphisms such that $f \circ g = \mathrm{id}_{S'}$, $g \circ f = \mathrm{id}_S$.

Exercise 1.2.9. Show that

$$\hat{\mathbf{C}} \simeq \mathbf{P}^1(\mathbf{C}).$$

Remark 1.2.10. $\mathbf{C} \not\simeq \mathbf{D}$ by Liouville.

If S, S' are connected compact Riemann surfaces, then any nonconstant morphism $f : S \to S'$ is surjective. (Nonconstant holomorphic maps are open)

1.2.4 Ramification

Definition 1.2.11 (Orders of vanishing)**.** The **order of vanishing** at $P \in S$ of a holomorphic function on S is defined as follows: For ϕ a chart centered at P write

$$f \circ \phi^{-1}(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots, \; a_n \neq 0$$

then $\mathrm{ord}_P(f) = n$.

More generally, for $f : S \to S'$ we can define $m_P(f)$ (**multiplicity** of f at P) by using a chart ψ on S' and setting

$$m_P(f) = \mathrm{ord}_P(\psi \circ f).$$

If $m_P(f) \geq 2$ then we call P a **branch point** of f and call f ramified at P.

Example 1.2.12.

$$f\colon \mathbf{C} \to \mathbf{C},\ f(z) = z^2.$$

The chart $\phi_a(z) = z - a$ is centered at $a \in \mathbf{C}$ Then to compute $m_a(f)$ we compute

$$f \circ \phi_a^{-1}(z) = a^2 + 2az + z^2$$

hence

$$\mathrm{ord}_a(f) = \begin{cases} 0, & \text{if } a \neq 0 \\ 2, & \text{if } a = 0 \end{cases}.$$

1.2.5 Genus

Theorem 1.2.13 (Rado)**.** *Any orientable compact surface can be triangulated.*

Fact 1.2.14. *Riemann surfaces are orientable.*

Given such an oriented polygon coming from a Riemann surface, we can associate a word w to it from travelling around the perimeter.

Example 1.2.15. For the sphere $w = a^{-1}ab^{-1}bc^{-1}c$.

Fact 1.2.16. *Every such word can be normalised without changing the corresponding Riemann surface.*

$$w = \begin{cases} w_0 = aa^{-1}, \\ w_g = a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1} \end{cases}$$

*The (uniquely determined) g is the **genus** of the surface.*

Example 1.2.17. $w_1 = a_1b_1a_1^{-1}b_1^{-1}$.
$w_2 = a_1b_1a_1^{-1}b_1a_2b_2a_2^{-1}b_2^{-1}$.

Theorem 1.2.18.

$$\chi(S) = v - e + f = 2 - 2g(S).$$