Chapter 1

Dessins d'Enfants

These are notes for BUNTES Spring 2018, the topic is Dessins d'Enfants, they were last updated February 16, 2018. For more details see the webpage. These notes are by Alex, feel free to email me at `alex.j.best@gmail.com` to report typos/suggest improvements, I'll be forever grateful.

1.1 Overview (Angus)

1.1.1 Belyi morphisms

Let X be an algebraic curve over $\mathbf{C}$ (i.e. a compact Riemann surface) when is X defined over $\overline{\mathbf{Q}}$?

Theorem 1.1.1 (Belyi)**.** *An algebraic curve $X/\mathbf{C}$ is defined over $\overline{\mathbf{Q}}$ $\iff$ there exists a morphism $\beta\colon X \to \mathbf{P}^1\,\mathbf{C}$ ramified only over $\{0, 1, \infty\}$.*

Definition 1.1.2 (Ramified)**.** (AG) A morphism $f\colon X \to Y$ is **ramified** at $x \in X$ if on local rings the induced map $f^\#\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ descended to

$$\mathcal{O}_{Y,f(x)}/\mathfrak{m} \to \mathcal{O}_{X,x}/f^\#(\mathfrak{m})$$

is not a finite inseparable field extension.

(RS) A morphism $f\colon X \to Y$ is ramified at $x \in X$ if there are charts around x and $f(x)$ such that $f(x) = x^n$. This n is the **ramification index**.

Definition 1.1.3 (Belyi morphisms)**.** A **Belyi morphism** is one ramified only over $\{0, 1, \infty\}$

A **clean Belyi morphism** or **pure Belyi morphism** is a Belyi morphism where the ramification indices over 1 are all exactly 2.

Lemma 1.1.4. *A curve X admits a Belyi morphism iff it admits a clean Belyi morphism.*

Proof. If $\alpha\colon X \to \mathbf{P}^1\,\mathbf{C}$ is Belyi, then $\beta = 4\alpha(1-\alpha)$ is a clean Belyi morphism. ◻

1.1.2 Dessin d'Enfants

Definition 1.1.5. A **dessin d'Enfant** (or Grothendieck Dessin or just **Dessin**) is a triple (X_0, X_1, X_2) where X_2 is a compact Riemann surface, X_1 is a graph, $X_0 \subset X_1$ is a finite set of points, where $X_2 \smallsetminus X_1$ is a collection of open cells. $X_1 \smallsetminus X_0$ is a disjoint union of line segments

Lemma 1.1.6. *The data of a dessin is equivalent to a graph with an ordering on the edges coming out of each vertex.*

Definition 1.1.7 (Clean dessins)**.** A **clean dessin** is a dessin with a colouring (white and black) on the vertices such that adjacent vertices do not share a colour.

1.1.3 The Grothendieck correspondence

Given a Belyi morphism $\beta\colon X \to \mathbf{P}^1\,\mathbf{C}$ the graph $\beta^{-1}([0,1])$ defines a dessin.

Theorem 1.1.8. *The map*

$$\{(\textit{Clean})\ \textit{Belyi morphisms}\} \longrightarrow \{(\textit{clean})\ \textit{dessins}\}$$

$$\beta \mapsto \beta^{-1}([0,1])$$

is a bijection up to isomorphisms.

Example 1.1.9.

$$\mathbf{P}^1\,\mathbf{C} \to \mathbf{P}^1\,\mathbf{C}$$

$$x \mapsto x^3$$

$$\mathbf{P}^1\,\mathbf{C} \to \mathbf{P}^1\,\mathbf{C}$$

$$x \mapsto x^3 + 1$$

1.1.4 Covering spaces and Galois groups

A Belyi morphism defines a covering map.

$$\tilde{\beta}\colon \tilde{X} \to \mathbf{P}^1\,\mathbf{C} \setminus \{0,1,\infty\}$$

the coverings are controlled by the profinite completion of

$$\pi_1(\mathbf{P}^1\,\mathbf{C} \setminus \{0,1,\infty\}) = \mathbf{Z} * \mathbf{Z} = F_2.$$

Theorem 1.1.10. *There is a faithful action*

$$\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \circlearrowright \hat{\pi}_1(\mathbf{P}^1\,\mathbf{C} \setminus \{0,1,\infty\})$$

Proof. By Belyi's theorem every elliptic curve $E/\overline{\mathbf{Q}}$ admits a Belyi morphism. For each $j \in \overline{\mathbf{Q}}$ there exists an elliptic curve $E_j/\overline{\mathbf{Q}}$ with j-invariant j.

Given $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$,

$$\sigma(E_j) = E(\sigma(j))$$

assume $\sigma \mapsto 1$,

$$E_j \cong E_{\sigma(j)}\ \forall j$$

$$j = \sigma(j)\ \forall j$$

a contradiction. □

Corollary 1.1.11. *We have a faithful action of* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ *on dessins.*

Theorem 1.1.12. *We have a faithful action of* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ *on the set of dessins of any fixed genus.*

1.1.5 Exercises

Exercise 1.1.13. Compute the Dessins for the following Belyi morphisms

1. $$\mathbf{P}^1\,\mathbf{C} \to \mathbf{P}^1\,\mathbf{C}, \mapsto x^4$$

2. $$\mathbf{P}^1\,\mathbf{C} \to \mathbf{P}^1\,\mathbf{C}, \mapsto x^2(3-2x)$$

3. $$\mathbf{P}^1\,\mathbf{C} \to \mathbf{P}^1\,\mathbf{C}, \mapsto \frac{1}{x(2-x)}$$

Exercise 1.1.14. Give an alternate proof of the fact that X admts a Belyi morphism is and only if it admits a clean Belyi morphism using dessins and the Grothendieck correspondence.

Exercise 1.1.15. Prove that a Belyi morphism corresponding to a tree, that sends ∞ to ∞ is a polynomial.

1.2 Riemann Surfaces I (Ricky)

1.2.1 Definitions

Definition 1.2.1. A **topological surface** is a Hausdorff space X wich has a collection of charts

$$\{\phi_i \colon U_i \xrightarrow{\sim} \phi_i(U_i) \subseteq \mathbf{C}, \text{ open}\}_{i\in I}$$

such that

$$X = \bigcup_{i\in I} U_i.$$

We call X a **Riemann surface** if the transition functions $\phi_i \circ \phi_j^{-1}$ are holomorphic.

1.2.2 Examples

Example 1.2.2. Open subsets of $\mathbf{C}$, e.g.

$$\mathbf{C}$$

$$\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$$
$$\mathbf{H} = \{z \in \mathbf{C} : \operatorname{im} z > 0\}.$$

Example 1.2.3. $\hat{\mathbf{C}}$ = Riemann sphere = $\mathbf{C} \cup \{\infty\}$. A basis of neighborhoods of ∞ is given by

$$\{z \in \mathbf{C} : |z| > R\} \cup \{\infty\}.$$

Example 1.2.4.

$$\begin{aligned}
\mathbf{P}^1(\mathbf{C}) &= \{[z_0 : z_1] : (z_0, z_1) \neq (0,0)\} \\
U_0 &= \{[z_0, z_1] : z_0 \neq 0\} \to \mathbf{C} \\
[z_0 : z_1] &\mapsto \frac{z_1}{z_0} \\
U_1 &= \{[z_0, z_1] : z_1 \neq 0\} \to \mathbf{C} \\
[z_0 : z_1] &\mapsto \frac{z_0}{z_1}.
\end{aligned}$$

Example 1.2.5. Let $\Lambda = \mathbf{Z} \oplus \mathbf{Z}i \subseteq \mathbf{C}$ then $X = \mathbf{C}/\Lambda$ is a Riemann surface.

1.2.3 Morphisms

Definition 1.2.6 ((Holo/Mero)-morphisms of Riemann surfaces)**.** A **morphism of Riemann surfaces** is a continuous map

$$f : S \to S'$$

such that for all charts ϕ, ψ on S, S' respectively we have $\psi \circ f \circ \phi^{-1}$ is holomorphic.

We call a morphism $f : S \to \mathbf{C}$ a **holomorphic function** on S.

We say $f : S \to \mathbf{C}$ is a **meromorphic function** is $f \circ \phi^{-1}$ is meromorphic.

Exercise 1.2.7. The set of meromorphic functions on a Riemann surface form a field.

We denote the field of meromorphic functions by $\mathcal{M}(S)$.

Proposition 1.2.8 (1.26)**.**

$$\mathcal{M}(\hat{\mathbf{C}}) = \mathbf{C}(z).$$

Proof. Let $f : \hat{\mathbf{C}} \to \mathbf{C}$ be meromorphic. Then the number of poles of f is finite say at $a_1, \ldots, a_n$. So, locally at a_i we can write

$$f(z) = \sum_{j=1}^{j_i} \frac{\lambda_{j,i}}{(z - a_i)^j} + h_i(z)$$

with h_i holomorphic. Then

$$f(z) - \sum_{i=1}^{n} \sum_{j=1}^{j_i} \frac{\lambda_{j,i}}{(z - a_i)^j}$$

is holomorphic everywhere. By Liouville's theorem this is constant. □

We say S, S' are isomorphic if $\exists f : S \to S'$, $g : S' \to S$ morphisms such that $f \circ g = \mathrm{id}_{S'}$, $g \circ f = \mathrm{id}_S$.

Exercise 1.2.9. Show that

$$\hat{\mathbf{C}} \simeq \mathbf{P}^1(\mathbf{C}).$$

Remark 1.2.10. $\mathbf{C} \not\simeq \mathbf{D}$ by Liouville.

If S, S' are connected compact Riemann surfaces, then any nonconstant morphism $f : S \to S'$ is surjective. (Nonconstant holomorphic maps are open)

1.2.4 Ramification

Definition 1.2.11 (Orders of vanishing)**.** The **order of vanishing** at $P \in S$ of a holomorphic function on S is defined as follows: For ϕ a chart centered at P write

$$f \circ \phi^{-1}(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots, \; a_n \neq 0$$

then $\mathrm{ord}_P(f) = n$.

More generally, for $f : S \to S'$ we can define $m_P(f)$ (**multiplicity** of f at P) by using a chart ψ on S' and setting

$$m_P(f) = \mathrm{ord}_P(\psi \circ f).$$

If $m_P(f) \geq 2$ then we call P a **branch point** of f and call f ramified at P.

Example 1.2.12.

$$f : \mathbf{C} \to \mathbf{C},\ f(z) = z^2.$$

The chart $\phi_a(z) = z - a$ is centered at $a \in \mathbf{C}$. Then to compute $m_a(f)$ we compute

$$f \circ \phi_a^{-1}(z) = a^2 + 2az + z^2$$

hence

$$\mathrm{ord}_a(f) = \begin{cases} 0, & \text{if } a \neq 0 \\ 2, & \text{if } a = 0 \end{cases}.$$

1.2.5 Genus

Theorem 1.2.13 (Rado)**.** *Any orientable compact surface can be triangulated.*

Fact 1.2.14. *Riemann surfaces are orientable.*

Given such an oriented polygon coming from a Riemann surface, we can associate a word w to it from travelling around the perimeter.

Example 1.2.15. For the sphere $w = a^{-1}ab^{-1}bc^{-1}c$.

Fact 1.2.16. *Every such word can be normalised without changing the corresponding Riemann surface.*

$$w = \begin{cases} w_0 = aa^{-1}, \\ w_g = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \end{cases}$$

*The (uniquely determined) g is the **genus** of the surface.*

Example 1.2.17. $w_1 = a_1 b_1 a_1^{-1} b_1^{-1}$.
$w_2 = a_1 b_1 a_1^{-1} b_1 a_2 b_2 a_2^{-1} b_2^{-1}$.

Theorem 1.2.18.

$$\chi(S) = v - e + f = 2 - 2g(S).$$

1.3 Riemann Hurwitz Formula (Sachi)

Exercise 1.3.1 (Unimportant)**.** The genus is invariant under changing triangulation.

In particular there are at least two distinct ways of thinking about genus for Riemann surfaces R

1. $$\chi(R) = V - E + F = 2 - 2g$$
2. The dimension of the space of holomorphic differentials on R.

Goal: given R calculate genus

$$y^2 = (x+1)(x-1)(x+2)(x-2)$$

so in an ad hoc way

$$y = \sqrt{(x+1)(x-1)(x+2)(x-2)}$$

when x is not a root of the above we have two distinct values for y, we can imagine two copies of $\mathbf{C}$ sitting above each other and then square root will land in both copies. We have to make branch cuts between the roots and glue along these to account for the fact that going around a small loop surrounding a root will change the sign of our square root. We end up with something looking like a torus here.

Here we examined the value where there were not enough preimages when we plugged in a value for x. The idea is to project to x, and understand the number of preimages.

$$P(x, y) = y^n + p_{n-1}(x)y^{n-1} + \cdots + p_0(x)$$

an irreducible polynomial.

$$R = \{(x, y) : P(x, y) = 0\}.$$

If we fix $x_0 \in \mathbf{P}^1\,\mathbf{C}$ we can analyse how many y values lie over this x. If we have fixed our coefficients we expect n solutions in y over $\mathbf{C}$, i.e. points $(x_0, y) \in R$.

For some values of x_0 this will not be true, there will be fewer y-values, this occurs when we have a multiple root. This happens precisely when the discriminant of this polynomial vanishes, the discriminant is a polynomial and so has finitely many roots.

Definition 1.3.2 (Branch points)**.** Let $\pi\colon R \to \mathbf{P}^1\,\mathbf{C}$. We say x_0 is a **branch point** if there are ferwer than n distinct y-values above x. Then define the **total branching index**

$$b = \sum_{x \in \mathbf{P}^1\,\mathbf{C}} (\deg(\pi) - \#\pi^{-1}(x)).$$

Claim 1.3.3.

$$\chi(R) = \deg \pi \cdot \chi(\mathbf{P}^1\,\mathbf{C}) - b.$$

Lemma 1.3.4. *Locally given some choice of coordinate a non-constant morphism of Riemann surfaces*

$$f\colon R \to S$$

is given by $w \mapsto w^n$. More precisely given $r \in R$, $f(r) = s$ and $V_s \ni s$ a small neighbourhood choose an identification of

$$V_s \xrightarrow{\Psi} D$$

which sends $s \mapsto 0$ and we can find an analytic identification

$$r \in R_r \xrightarrow{\phi} D$$

such that

$$f(U_r) \subseteq V_s.$$

$$\begin{tikzcd} U_r \arrow[r, "f"] \arrow[d, "\phi"'] & V_s \arrow[d, "\Psi"] \\ D \arrow[r, "w \mapsto w^m"'] & D \end{tikzcd}$$

Proof. In Sachi's notes. $\square$

Proof. Of Claim 1.3.3.

Triangulate R so that every face lies in some small coordinate neighborhood s.t.

$$\pi : R \to \mathbf{P}^1\,\mathbf{C}$$

is given by $w \mapsto w^m$, s.t. every edge, all branch points are vertices. This ensures that each face edge and vertex has $n = \deg(\pi)$ preimages (except branch points). Then accounting for brach points we have $\deg(\pi) - \#\pi^{-1}(x_0)$ preimages. □

Example 1.3.5. $P(x, y)$ plane curve, classically have

$$g = \frac{(d-1)(d-2)}{2}$$

$\mathbf{P}^2 = \{[x : y : z]\}$ and $(\mathbf{P}^2)^* = [a : b : c]$, lines in $\mathbf{P}^2$

$$ax + by + cz = 0$$

and we have lines $\leftrightarrow$ points. We have C^* the dual curve in $\mathbf{P}^2$ cut out by the tangent lines t_Q for $Q \in C$. Claim $\deg C^* = (d-1)d$.

Want

$$R : \{P(x, y) = 0\} \xrightarrow{\pi} \mathbf{P}^1\,\mathbf{C}$$

compute b. In other words, if we fix an arbitrary point $Q \in C$ then there are $d(d-1)$ lines through Q which are tangent to C. Projecting to the x-coordinate $\iff$ family of lines through a point at ∞ $\iff$ $*$ line in $(\mathbf{P}^2)^*$. We have a new question: How many points does this line intersect (up to multiplicity). By bezout $\iff$ $\deg C^*$.

Proof (Matt emerton) Consider a point on C in $\mathbf{P}^2$ such that no tangent line to the curve at ∞ passes throught it. Move this point to the origin. If we write

$$P(x, y) = f_d + f_{d-1} + \cdots + f_0$$

then

$$(f_d, f_{d-1}) = 1$$

suppose they share a linear factor:

$$0 = (f_d)_x x + (f_d)_y y + f_{d-1},$$

then this defines a line through the origin. (Because this gives an equation of an asymptote, this is a contradiction).

$$f_d + f_{d-1} + \cdots + f_0 = 0$$

$$d f_d + (d-1) f_{d-1} + \cdots + f_1 = 0$$

$$\Longrightarrow$$

$$\begin{cases} f_d + f_{d-1} + \cdots + f_0 = 0 \\ f_{d-1} + 2 f_{d-2} + \cdots + (d-1) f_1 = 0 \end{cases} .$$

Now these have $d(d-1)$ common solutions. C^* has degree $d(d-1)$ so $b = d(d-1)$. Riemann-Hurwitz implies

$$\chi(R) = 2 \deg \pi - d(d-1)$$

$$\chi(R) = 2d - d(d-1)$$

so

$$g = \frac{(d-1)(d-2)}{2}.$$

A 3-fold equivalence of categories Amazing synthesis.

1. Analysis: Compact connected riemann surfaces.
2. Algebra: Field extensions $K/\mathbf{C}$ where K is finitely generated of transcendence degree 1 over $\mathbf{C}$.
3. Geometry: Complete nonsingular irreducble algebraic curves in $\mathbf{P}^n$.

3) curve $\longrightarrow$ 2) field extension. Over C all rational functions $\frac{P(x)}{Q(x)}$ $\deg P = \deg Q$, $P, Q \colon C \longrightarrow \mathbf{C} \cup \{\infty\}$.

3) $\longrightarrow$ 1) take complex structure induced by $\mathbf{P}^n$.

1) $\longrightarrow$ 2) associated field of meromorphic functions on X.

1) $\longrightarrow$ 3) Any curve which is holomorphic has an embedding into $\mathbf{P}^n$ (Riemann-Roch).

2) $\longrightarrow$ 1) $K/\mathbf{C}$ consider valuation rings R such that $K \supseteq R \supseteq \mathbf{C}$.

Example 1.3.6. $g = 0$, $\mathbf{P}^1\,\mathbf{C}$ $\mathbf{C}(t)$, $\mathbf{C} \cup \{\infty\}$.

Example 1.3.7. $g = 1$, elliptic curves, $f(x, y, z)$ smooth plane cubic, $f = 0$, $\mathbf{C}(\sqrt{f(x)}, x)$.

$$\mathbf{C}/\Lambda \longrightarrow \mathbf{P}^2$$
$$z \longmapsto (z, \wp(z), \wp'(z))$$
$$z \notin \Lambda$$

backwards

$$(x, y) \longmapsto \int_{(x_0,y_0)}^{(x,y)} \frac{\mathrm{d}x}{y}$$

Riemann-Hurwitz (generally) There's nothing that doesn't generalise about the previous proof.

Claim 1.3.8. *For $\pi \colon R \longrightarrow S$ a non-constant morphism of compact Riemann surfaces*

$$\chi(R) = \deg \pi \cdot \chi(S) - \sum_{x \in S} (\deg(\pi) - \#\pi^{-1}(x)).$$

Corollary 1.3.9. *There are no non-constant morphisms from a sphere to a surface of genus > 0.*

Proof.

$$f \colon \mathbf{P}^1\,\mathbf{C} \longrightarrow S$$
$$\chi(\mathbf{P}^1\,\mathbf{C}) = \deg f \, \chi(S) - b$$
$$2 = (+) \cdot (-) - b.$$

□

Exercise 1.3.10.

$$x^n + y^n + z^n = 0$$

is not solvable in non-constant polynomials for $n > 2$.

Exercise 1.3.11.

$$E = \mathbf{C}/\mathbf{Z} + \mathbf{Z}i$$

multiplication by i rotates $x \mapsto xi$ let $x \sim xi$. If we mod out by $\sim$ to get $E/{\sim}$ this is still a Riemann surface and the quotient map

$$f \colon E \longrightarrow E/{\sim}$$

is nice, compute the branch points of order 4 and order 2.

Exercise 1.3.12. X compact Riemann surface of $g \geq 2$ then there are at most $84(g-1)$ automorphisms of X.

Exercise 1.3.13. Klein quartic

$$x^3y + y^3z + z^3x = 0$$

has 168 automorphisms and is genus 3.

1.4 Riemann Surfaces and Discrete Groups (Rod)

Welcome to BUGLES (Boston university geometry learning expository seminar), the reason it is called bugles is because bugles are hyperbolic, and today we will see a lot of hyperbolic objects.

Plan

1. Uniformization
2. Fuchsian groups
3. Automorphisms of Riemann surfaces

Proposition 1.4.1.

$$\mathrm{Aut}(\hat{\mathbf{C}}) = \{z \mapsto \frac{az+b}{cz+d}\}$$

$$\mathrm{Aut})\mathbf{C}) = \{z \mapsto za + b\}$$

$$\mathrm{Aut}(\mathbf{H}) = \{z \mapsto \frac{az+b}{cz+d}, a, b, c, d \in \mathbf{R}\} = \mathrm{PSL}_2(\mathbf{R})$$

Theorem 1.4.2. *Σ has a universal cover $\widetilde{\Sigma}$ with $\pi_1(\Sigma) = 1$. $\widetilde{\Sigma} \to \Sigma$ holomorphic. $\Sigma = \widetilde{\Sigma}/G$ for $G = \pi_1(\Sigma)$. G acts freely and properly discontinuously.*

1.4.1 Uniformization

Theorem 1.4.3. *The only simply connected Riemann surfaces are $\hat{\mathbf{C}}$, $\mathbf{C}$, $\mathbf{H}$.*

Theorem 1.4.4. *Σ is a Riemann surface then*

$$g = 0 : \Sigma \cong \hat{\mathbf{C}}$$

$$g = 1 : \Sigma \cong \mathbf{C}/\Lambda$$

$$g \geq 2 : \Sigma \cong \mathbf{H}/K$$

Proof. $g = 0$ Uniformization.

$g \geq 1$ $\hat{\mathbf{C}}$ can't be a cover by Riemann-Hurwitz. $g = 1$ $\pi_1(\Sigma) = \mathbf{Z} \oplus \mathbf{Z}$ abelian.

Claim: no subgroup of $\mathrm{Aut}(\mathbf{H})$ is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ acting freely and properly discontinuously. So $\widetilde{\Sigma} = \hat{\mathbf{C}}$ $z \mapsto az + b$ free id $a = 1$ so $z \mapsto z + \lambda_1$ $z \mapsto z + \lambda_2$.

$g = 2$ $\pi_1(\Sigma)$ is not abelian but $z \mapsto z + \lambda_1$ is abelian!

$$\Sigma = \mathbf{H}/K,\ K \subseteq \mathrm{PSL}_2(\mathbf{R}).$$ □

Goal Understand Σ through $\widetilde{\Sigma}$ and G.

Fuchsian groups $g \geq 2$.

$$\mathrm{Aut}(\mathbf{H}) = \mathrm{PSL}_2(\mathbf{R}) = \mathrm{Isom}^+(\mathbf{H}, \frac{|dz|^2}{\Im Z})$$

hyperbolic $\mathbf{H}$, $\mathbf{D}$ and $\mathrm{PSL}_2(\mathbf{R})$ acts transitively on geodesics.

Definition 1.4.5 (Fuchsian groups)**.** A **Fuchsian group** is a discrete subgroup of $\mathrm{PSL}_2(\mathbf{R})$.

Remark 1.4.6. (proof in book) Even if Γ doesn't act freely the quotient

$$\mathbf{H} \to \mathbf{H}/\Gamma$$

is still a covering map and $\mathbf{H}/\Gamma$ is a Riemann surface.

Reflections on H Say μ is a geodesic in $\mathbf{H}$, i.e. a horocycle.There is $M \in \mathrm{PSL}_2(\mathbf{R})$ with $M\mu$ the imaginary axis. Then $R = -\bar{z}$ is the reflection over the imaginary axis. Now $R_\mu = M^{-1} \circ R \circ M$ is a reflection over μ.

$$R_\mu = \frac{a\bar{z} + b}{c\bar{z} + d} \notin \mathrm{PSL}_2(\mathbf{R})$$

this is a a problem for us.

Triangle groups Given $n, m, l \in \mathbf{Z} \cup \{\infty\}$ then there is a hyperbolic triangle with angles $\pi/n, \pi/m, \pi/l$ if

$$\frac{1}{n} + \frac{1}{m} + \frac{1}{l} < 1.$$

With area $\pi(1 - \frac{1}{n} - \frac{1}{m} - \frac{1}{l})$.

In the disk model we can start with a wedge of the disk and by adding a choice third geodesic with endpoints on the edge we can adjust the other angles to be what we like.So we can construct hyperbolic triangles with whatever angles we like. Then let R_1 be the reflection over 1 edge, R_2 , R_3 similarly. By reflecting our original triangle T with these reflections we can tesselate the disk, colouring alternately the triangles obtained using an odd or even number of reflections.

The only remaining problem is that R_i's are not in $\mathrm{PSL}_2(\mathbf{R})$. The solution is to define $x_1 = R_3 \circ R_1$, $x_2 = R_1 \circ R_2$, $x_3 = R_2 \circ R_3$ which are all in $\mathrm{PSL}_2(\mathbf{R})$ now. Now we need to take the union of two adjacent triangles before as a fundamental domain, some quadrilateral that still tesselates. So we have formed a Fuchsian group from our triangles.

A presentation for this group is

$$\langle x_1, x_2, x_3 | x_1^n = x_2^m = x_3^l = x_1 x_2 x_3 = 1 \rangle$$

note n, m, l can still be ∞.

Definition 1.4.7 (Triangle groups)**.** Let $\Gamma_{n,m,l}$ be the **triangle group** with signature $(1/n, 1/m, 1/l)$.

Remark 1.4.8.

$$\frac{1}{n} + \frac{1}{m} + \frac{1}{l} = 1$$
$$\frac{1}{n} + \frac{1}{m} + \frac{1}{l} > 1$$

still work on $\mathbf{C}$ and $\hat{\mathbf{C}}$ respectively.

Example 1.4.9 ($\mathrm{PSL}_2(\mathbf{Z})$)**.** Consider $\Gamma_{2,3,\infty}$ angles $\pi/2, \pi/3, 0$. We can draw such a triangle in the upper half plane with vertices $i, e^{\pi i/3}, \infty$. So a fundamental domain will be the region obtained by reflecting through the imaginary axis, given by $-\frac{1}{2} \leq \Re z \leq \frac{1}{2}$, $|z| \geq 1$. We have $R_1 = \frac{1}{\bar{z}}, R_2 = -\bar{z}+1, R_3 = -\bar{z}$ so $x_1 = \frac{-1}{z}, x_2 = \frac{1}{-z+1}, x_3 = z+1$. Then $\Gamma_{2,3,\infty} \cong \mathrm{PSL}_2(\mathbf{Z})$. Sometimes denoted $\Gamma(1)$.

Observation 1.4.10. If $\Gamma_1 < \Gamma_2$ and T is a fundamental domain of Γ_2 then if $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma_2$ are representatives of $\Gamma_1 \backslash \Gamma_2$ then

$$\bigcup \gamma_i(T)$$

is a fundamental domain for Γ_1.

Example 1.4.11 ($\Gamma(1)$)**.**

$$\Gamma(2) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{id} \pmod 2\}$$

then

$$[\Gamma(1) : \Gamma(2)] = 6$$

representatives of $\Gamma(2)\backslash\Gamma(1)$ are

$$x_1 = \mathrm{id},\ x_2 = \frac{-1}{z-1}, x_3 = \frac{z-1}{z},\ x_4 = \frac{z-1}{z},\ x_5 = \frac{-z}{x-1},\ x_6 = \frac{-1}{z}.$$

Lets see what these do, for example if $z = e^{i\theta}$

$$\Re(x_2(z)) = \frac{-1}{e^{i\theta}-1} = \frac{-e^{i\theta}+1}{2-2\cos\theta}) = \frac{1-\cos\theta}{2-2\cos\theta}\frac{1}{2}$$

if we plot this we see we get two copies of a 0,0,0 triangle so this corresponds to $\Gamma_{\infty,\infty,\infty}$.

$$\langle x_1, x_2, x_3 | x_1x_2x_3 = 1\rangle = \langle x_1, x_2\rangle = \pi_1(\mathbf{P}^1 \setminus \{0, 1, \infty\}).$$

Proposition 1.4.12. $S_1 = \mathbf{H}/\Gamma_1$, $S_2 = \mathbf{H}/\Gamma_2$ *then*

$$S_1 \cong S_2 \iff \Gamma_1 = T \circ \Gamma_2 \circ T^{-1},\ T \in \mathrm{PSL}_2(\mathbf{R}).$$

Proof. $\Leftarrow$ Define an $\phi\colon S_1 \to S_2$ via $\phi([z]_1) = [T(z)]_2$.
$\Rightarrow$ Take a lift

$$\begin{array}{ccc} \mathbf{H} & \xrightarrow{\tilde{\phi}} & \mathbf{H} \\ \downarrow & & \downarrow \\ \mathbf{H}/\Gamma_1 & \xrightarrow[\phi]{} & \mathbf{H}/\Gamma_2 \end{array}$$

then $T = \tilde{\phi}$. □

Proposition 1.4.13. Γ *a Fuchsian group acts freely*

$$\mathrm{Aut}(\mathbf{H}/\Gamma) = N(\Gamma)/\Gamma.$$

Proof. Previous proposition, set $\Gamma_1 = \Gamma_2$

$$N(\Gamma) \to \mathrm{Aut}(\mathbf{H}/\Gamma)$$

kernel is Γ. □

Corollary 1.4.14. *Let Σ be a Riemann surface with $g \geq 2$ then*

$$|\operatorname{Aut}(\Sigma)| < \infty.$$

Proof.

$$\begin{tikzcd} \mathbf{H} \arrow[d, "\phi_2"'] \arrow[dr, "\phi_1"] & \\ S = \mathbf{H}/\Gamma \arrow[r, "f"'] & \mathbf{H}/N(\Gamma) = S/\operatorname{Aut}(S) \end{tikzcd}$$

since ϕ_1, ϕ_2 are holomorphic then so is f. So $\deg f = |N(\Gamma)/\Gamma|$ and $\deg f < \infty$. ∎

Say Σ, $g \geq 2$, $G \subseteq \operatorname{Aut}(\Sigma)$. Let $\bar{g}$ be the genus of Σ/G

$$2g - 2 = |G|(2\bar{g} - 2) + \sum_p (I(p) - 1) = |G|(2\bar{g} - 2 + \sum_p (1 - \frac{1}{I(p)}))$$

where $I(p)$ is the stabiliser of p in G.

Exercise 1.4.15. Σ, $g \geq 2$ then $|\operatorname{Aut}(\Sigma)| \leq 84(g-1)$. Hint: cases.

Exercise 1.4.16. Consider

$$1 \to \Gamma(n) \to \Gamma(1) \to \mathrm{PSL}_2(\mathbf{Z}/n\mathbf{Z}) \to 1$$

compute genus of $\mathbf{H}/\Gamma(n)$.

Remark 1.4.17.