Chapter 1

Dessins d'Enfants

These are notes for BUNTES Spring 2018, the topic is Dessins d'Enfants, they were last updated January 27, 2018. For more details see the webpage. These notes are by Alex, feel free to email me at `alex.j.best@gmail.com` to report typos/suggest improvements, I'll be forever grateful.

1.1 Overview (Angus)

1.1.1 Belyi morphisms

Let X be an algebraic curve over $\mathbf{C}$ (i.e. a compact Riemann surface) when is X defined over $\overline{\mathbf{Q}}$?

Theorem 1.1.1 (Belyi)**.** *An algebraic curve $X/\mathbf{C}$ is defined over $\overline{\mathbf{Q}}$ $\iff$ there exists a morphism $\beta\colon X \to \mathbf{P}^1\,\mathbf{C}$ ramified only over $\{0, 1, \infty\}$.*

Definition 1.1.2 (Ramified)**.** (AG) A morphism $f\colon X \to Y$ is ramified at $x \in X$ if on local rings the induced map $f^\#\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ descended to

$$\mathcal{O}_{Y,f(x)}/\mathfrak{m} \to \mathcal{O}_{X,x}/f^\#(\mathfrak{m})$$

is not a finite inseparable field extension.

(RS) A morphism $f\colon X \to Y$ is ramified at $x \in X$ if there are charts around x and $f(x)$ such that $f(x) = x^n$.

Definition 1.1.3 (Belyi morphisms)**.** A **Belyi morphism** is one ramified only over $\{0, 1, \infty\}$

A **clean Belyi morphism** or **pure Belyi morphism** is a Belyi morphism where the ramification indices over 1 are all exactly 2.

Lemma 1.1.4. *A curve X admits a Belyi morphism iff it admits a clean Belyi morphism.*

Proof. If $\alpha\colon X \to \mathbf{P}^1\,\mathbf{C}$ is Belyi, then $\beta = 4\alpha(1-\alpha)$ is a clean Belyi morphism. □

1.1.2 Dessin d'Enfants

Definition 1.1.5. A **dessin d'Enfant** (or Grothendieck Dessin or just **Dessin**) is a triple (X_0, X_1, X_2) where X_2 is a compact Riemann surface, X_1 is a graph, $X_0 \subset X_1$ is a finite set of points, where $X_2 \smallsetminus X_1$ is a collection of open cells. $X_1 \smallsetminus X_0$ is a disjoint union of line segments

Lemma 1.1.6. *The data of a dessin is equivalent to a graph with an ordering on the edges coming out of each vertex.*

Definition 1.1.7 (Clean dessins)**.** A **clean dessin** is a dessin with a colouring (white and black) on the vertices such that adjacent vertices do not share a colour.

1.1.3 The Grothendieck correspondence

Given a Belyi morphism $\beta\colon X \to \mathbf{P}^1\,\mathbf{C}$ the graph $\beta^{-1}([0,1])$ defines a dessin.

Theorem 1.1.8. *The map*

$$\{(\textit{Clean})\ \textit{Belyi morphisms}\} \longrightarrow \{(\textit{clean})\ \textit{dessins}\}$$

$$\beta \mapsto \beta^{-1}([0,1])$$

is a bijection up to isomorphisms.

Example 1.1.9.

$$\mathbf{P}^1\,\mathbf{C} \to \mathbf{P}^1\,\mathbf{C}$$
$$x \mapsto x^3$$

$$\mathbf{P}^1\,\mathbf{C} \to \mathbf{P}^1\,\mathbf{C}$$
$$x \mapsto x^3 + 1$$

1.1.4 Covering spaces and Galois groups

A Belyi morphism defines a covering map.

$$\tilde{\beta}\colon \tilde{X} \to \mathbf{P}^1\,\mathbf{C} \setminus \{0, 1, \infty\}$$

the coverings are controlled by the profinite completion of

$$\pi_1(\mathbf{P}^1\,\mathbf{C} \setminus \{0, 1, \infty\}) = \mathbf{Z} * \mathbf{Z} = F_2$$

Theorem 1.1.10. *There is an injective homomorphism*

$$\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \hookrightarrow \hat{\pi}_1(\mathbf{P}^1\,\mathbf{C} \setminus \{0, 1, \infty\})$$

Proof. By Belyi's theorem every elliptic curve $E/\overline{\mathbf{Q}}$ admits a Belyi morphism. For each $j \in \overline{\mathbf{Q}}$ there exists an elliptic curve $E_j/\overline{\mathbf{Q}}$ with j-invariant j.

Given $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$,

$$\sigma(E_j) = E(\sigma(j))$$

assume $\sigma \mapsto 1$,

$$E_j \cong E_{\sigma(j)}\ \forall j$$
$$j = \sigma(j)\ \forall j$$

a contradiction. □

Corollary 1.1.11. *We have a faithful action of* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ *on dessins.*

Theorem 1.1.12. *We have a faithful action of* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ *on the set of dessins of any fixed genus.*