1. How many subsets of $\{1, 2, \ldots, 2n\}$ are there which contain:

   (i) All of the odd numbers up to $2n$.

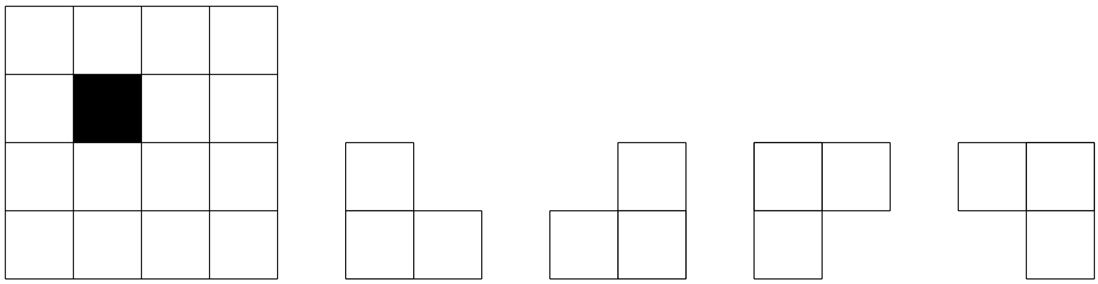
   (ii) At least one odd number.

   (iii) As many odd numbers as even numbers.

   (iv) At least as many odd numbers as even numbers.

   (You don't need a closed expression for the last two).

2. Given a checkerboard of dimension $2^n \times 2^n$ with one square already covered, show that we can cover all the remaining squares with L shaped pieces:



3. Given sets A and B of size m and n respectively find, in all cases, the number of

   (i) Bijections $f\colon A \to B$.

   (ii) Injections $f\colon A \to B$.

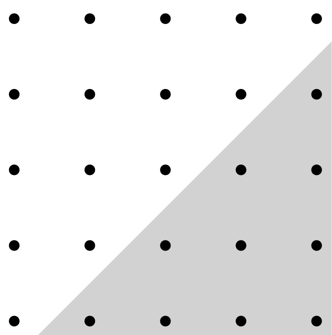
   We have not considered surjections above, and with good reason, the formulae are far harder to determine!

4. What is wrong with the following proof by induction that all cows are the same colour:

   Base case: 1 cow is the same colour as itself.

   Inductive step: given n cows, consider two different groups of $n-1$ cows. By the inductive assumption, all cows within these groups are the same colour, and since the groups overlap all n cows must be the same colour.

5. (i) Given a grid of $(n+1) \times (n+1)$ dots find an expression for the number of dots in the bottom right half (not including the lead diagonal).



(ii) Prove the formula

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

(iii) (Bonus) Use this technique to find closed forms for the sums

$$\sum_{i=1}^{n} i^2 \text{ and } \sum_{i=1}^{n} i^3.$$

6. Show that given a full binary tree T we have the inequality

$$\text{size}(T) \leq 2^{\text{height}(T)+1} - 1.$$

7. Prove, using the well-ordering principle, that $\sqrt{k}$ is irrational for any prime number k.

8. Given a set P of n people in how many ways can a delegation $D \subset P$ with a specified leader $l \in D$ be chosen from the set.

   For example with two people A and B the answer is 4 as we can either take just A or B as the delegation and they are then the only possible leaders, or we can send them both as the delegation and have either as the leader.

9. In how many ways can n rooks be placed on a chess board of size $n \times n$ such that no two are attacking each other (a rook can only attack pieces in the same row or column as itself).

10. The Cantor-Schröder-Bernstein theorem states that if there is an injection $f\colon A \to B$ and an injection $g\colon B \to A$, then there is a bijection $h\colon A \to B$. Using this fact, prove that the set of all subsets of natural numbers and the set of all infinite subsets of natural numbers have the same cardinality.

11. In how many different ways can the following be done:

    (i) A tower built from 10 white bricks and 10 black bricks?

    (ii) A tower built from n white bricks and m black bricks?

    (iii) n rooks be placed on a chess board of size $n \times n$ such that no two are attacking each other (a rook can only attack pieces in the same row or column as itself).

12. Show that given a *full* binary tree T we have the inequality

$$\text{size}(T) \leq 2^{\text{height}(T)+1} - 1.$$

13. Prove the following identities (try to find both a combinatorial and algebraic proof for both of them):

    (i)

$$\binom{n}{2} + \binom{n+1}{2} = n^2.$$

    (ii)

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}.$$

14. Alice has 10 distinct balls. She first splits them into two piles, then chooses a pile with at least two balls in and splits it into two more piles. She repeats this until she has all of the balls in different piles.

    (i) How many steps does it take for Alice to finish doing this.

    (ii) Show that the number of different ways she could do this is

$$\binom{10}{2}\binom{9}{2}\cdots\binom{3}{2}\binom{2}{2}.$$

15. Show that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}.$$

    (Bonus) Find and prove a similar identity for products of three consecutive numbers.

16. For each of the following pairs of numbers decide which is larger:

    (a)

$$8,000,000,000 \text{ and } \binom{100}{10}.$$

    (b)

$$200 \text{ and } \binom{8}{6}.$$

17. You are invited to bet on the result of a lottery. There are 18 differently numbered balls from which you are asked to choose 6 of. The balls are then mixed up and 6 are drawn at random, if you guessed all 6 correctly you win £1,000,000, but get anything wrong and you will win nothing!

    It costs £1 to enter the lottery, should you do so?

18. The bounds written above are good, but not perfect, in fact for fixed n they are the worst when $\binom{n}{k}$ is at a maximum, i.e. for coefficients of the form $\binom{n}{n/2}$. Using the fact that the maximum element of a set of natural numbers is at most as large as the sum of them derive a better upper bound for

$$\binom{2n}{n}.$$

19. Alice has 10 distinct balls. She first splits them into two piles, then chooses a pile with at least two balls in and splits it into two more piles. She repeats this until she has all of the balls in different piles.

    (i) How many steps does it take for Alice to finish doing this.

    (ii) Show that the number of different ways she could do this is

$$\binom{10}{2}\binom{9}{2}\cdots\binom{3}{2}\binom{2}{2}.$$

20. Show that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}.$$

    (Bonus) Find and prove a similar identity for products of three consecutive numbers.

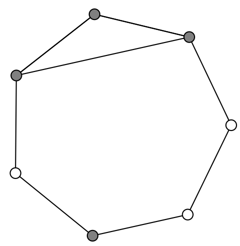
21. n people are at a party. Show that at least two people must have the same number of friends at the party (assume friendship is symmetric).

22. How many permutations of the 26 letters of the English alphabet contain none of the words fish, rat or bird?

23. (i) Fix some n and consider $A \subseteq \{1, 2, \ldots, 2n\}$ with $|A| = n+1$. Show that there exists at least two elements of A which are coprime.

    (ii) (Harder!) Prove that there is at least one element of A which divides another.

24. Each of the vertices of a regular heptagon (seven sides) are coloured with two colours. Prove that there exists an isosceles triangle within it with all vertices the same colour.



25. How many natural numbers less than or equal to one million are neither a square, a cube nor a fifth power?

26. Let $N = \{1, 2, \ldots, 100\}$ and $A \subset N$ with A of size 55.

    (i) Show that there must be a pair of elements of A whose difference is 9.

    (ii) (Bonus) Is the same result true if $|A| = 54$?

27. Find a nice form for the generating functions for the following sequences:

    (i) $(a_n) = (1, 2, 1, 0, 0, 0, \ldots)$,

    (ii) $(b_n) = (1, 2, 3, 4, 5, 6, \ldots)$,

    (iii) $(c_n) = (1, -2, 4, -8, 16, -32, \ldots)$,

    (iv) $d_n = \binom{2014}{n}$,

    (v) $e_n = (-1)^n n$.

28. Assume the generating function of a sequence $(a_0, a_1, a_2, \ldots)$ is $F(x)$. Find the generating functions of:

    (i) $(0, a_0, 0, a_1, 0, a_2, \ldots)$,

    (ii) $(a_0, 0, -2a_1, 0, 4a_2, 0, -8a_3, \ldots)$.

29. Find the sequences given by each of these generating functions:

    (i)
    $$\frac{1}{1-3x},$$

    (ii)
    $$\frac{2x-3}{1-7x+10x^2},$$

(iii)
$$\frac{x}{(1-x)^2}.$$

30. Solve the recurrence
$$Q_n = \begin{cases} 0 & n \leq 0, \\ 1 & n = 1, \\ Q_{n-1} + 2 \cdot Q_{n-2} & n \geq 2, \end{cases}$$
by first finding a generating function for Q_n.

31. Suppose you want to place 675 identical balls into four different bins, in such a way that each bin has either 150, 175 or 200 balls in it. Using a generating function, find the number of ways of doing this.

32. Suppose you have infinitely many red balls, one white ball and infinitely many blue balls.
    (i) Find a generating function for the number of distinct ways of picking n balls from this set.
    (ii) What is the generating function for this set if we make the additional restriction that an even number of blue balls must be taken? Simplify this function as much as you can and find the number of ways to pick the balls in this manner.

33. Solve the recurrence
$$a_0 = 1,$$
$$a_k = 3a_{k-1} + 4^k.$$

34. Find a generating function for the number of ways to make k pence with 1p, 2p 5p 10p, 20p and 50p coins.

35. Let F_n be the Fibonacci sequence ($F_0 = F_1 = 1,\ F_n = F_{n-1} + F_{n-2}\ \forall n \geq 2$).
    (i) Find a simpler form for the generating function $F(x)$ of F_n.
    (ii) Show that
$$F_n = \binom{n}{0} + \binom{n-1}{1} + \cdots.$$

36. Let F_n be the Fibonacci numbers (as above). Show that:
    (i) $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$,
    (ii) F_n divides F_{2n} for all n,
    (iii) (Bonus) F_n divides F_{kn} for all n and k.

37. Given a positive integer n let $p(n)$ be the number ways (ignoring orderings) of splitting n up as a sum of positive numbers. These are called *partitions* of n.

    For example $p(4) = 5$ as it can be written as $4,\ 3+1,\ 2+2,\ 2+1+1$ and $1+1+1+1$.
    (i) Find a simpler form for the generating function of $p(n)$.
    (ii) Find a simpler form for the generating function of $p_o(n)$, the number of partitions of n as a sum of only odd numbers.

(iii) Prove that $p_o(n) = p_d(n)$, the number of partitions of n into sums of distinct terms (each summand is used at most once).

(iv) (Bonus) Show that the number of partitions into numbers not divisible by 3 is the same as the number of partitions where each summand is used no more than twice.

38. Given a sorted list of numbers the binary search algorithm finds the position of a specified number (if it is in the list). It works by selecting the midpoint of the given list, checking whether the element searched for is before or after this position and then recursing on the half of the list that the target must be in. Therefore the running time of this algorithm is given by

$$T(n) \leq \begin{cases} T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + c & \text{if } n > 1, \\ 1 & \text{if } n = 1, \end{cases}$$

for some constant c.

(a) Find bounds for $T(3)$, $T(5)$, $T(10)$ and $T(15)$.

(b) Solve the recurrence to find an explicit function f such that $T(n) = O(f(n))$.

39. The median of a set of n numbers is the element in the middle position if they placed in increasing order (assume n is odd if you like). In this question we will look at how quickly we can find the median of a set of numbers.

(a) Give a $O(n \log n)$ algorithm to find the median of a set of numbers.

(b) Give a $O(n)$ algorithm that will find the kth largest element of a set of numbers for k fixed.

(c) It turns out it is possible to use a divide and conquer algorithm to find the median of a set of numbers quicker than the algorithm above[1]. This algorithm's runtime is given by a function $T(n)$ satisfying the following for some constant c:

$$T(n) \leq \begin{cases} T\left(\left\lfloor \frac{n}{5} \right\rfloor\right) + T\left(\left\lfloor \frac{3n}{4} \right\rfloor\right) + cn & \text{if } n > 5, \\ c & \text{if } n \leq 5. \end{cases}$$

Find bounds for $T(8)$, $T(11)$ and $T(20)$.

(d) Prove that $T(n) = O(n)$ and so the median can in fact be found in linear time!

(e) Give an argument for why we cannot hope to beat this asymptotically.

40. Multiplying two $n \times n$ matrices can be done with $O(n^3)$ multiplications (using the normal method), however a different method called Strassen's algorithm uses asymptotically less multiplications. Strassen's algorithm divides the matrix into four pieces, recurses on those and then uses 7 multiplications (and some additions) to combine these into the product wanted, so it uses

$$T(n) = \begin{cases} 1 & \text{if } n = 1, \\ 7T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \Theta(n^2) & \end{cases}$$

operations. How many operations does this algorithm need asymptotically?

41. Solve the recurrence

$$T(n) = \begin{cases} T\left(\sqrt[3]{n}\right) + n & \text{if } n > 1, \\ 1 & \text{if } n = 1. \end{cases}$$

[1] Google for "median of medians" or "linear time selection"

42. Solve the recurrence

$$T(n) = \begin{cases} 3T\left(\frac{n}{4}\right) + n\log n & \text{if } n > 1, \\ 1 & \text{if } n = 1. \end{cases}$$

43. If

$$T(n) = \begin{cases} 2T\left(\frac{n}{7}\right) + 5T\left(\frac{n}{8}\right) + n & \text{if } n > 1, \\ 1 & \text{if } n = 1. \end{cases}$$

Prove that $T(n) = O(n \log n)$.

44. Design a divide and conquer algorithm for question 2, analyse its asymptotic runtime.