

# Chapter 1

## Dessins d'Enfants

These are notes for BUNTES Spring 2018, the topic is Dessins d'Enfants, they were last updated March 17, 2018. For more details see [the webpage](#). These notes are by Alex, feel free to email me at [alex.j.best@gmail.com](mailto:alex.j.best@gmail.com) to report typos/suggest improvements, I'll be forever grateful.

### 1.1 Overview (Angus)

#### 1.1.1 Belyi morphisms

Let  $X$  be an algebraic curve over  $\mathbb{C}$  (i.e. a compact Riemann surface) when is  $X$  defined over  $\overline{\mathbb{Q}}$ ?

**Theorem 1.1.1** (Belyi). *An algebraic curve  $X/\mathbb{C}$  is defined over  $\overline{\mathbb{Q}}$   $\iff$  there exists a morphism  $\beta: X \rightarrow \mathbb{P}^1 \mathbb{C}$  ramified only over  $\{0, 1, \infty\}$ .*

**Definition 1.1.2** (Ramified). (AG) A morphism  $f: X \rightarrow Y$  is **ramified** at  $x \in X$  if on local rings the induced map  $f^\#: \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$  descended to

$$\mathcal{O}_{Y,f(x)}/\mathfrak{m} \rightarrow \mathcal{O}_{X,x}/f^\#(\mathfrak{m})$$

is not a finite inseparable field extension.

(RS) A morphism  $f: X \rightarrow Y$  is ramified at  $x \in X$  if there are charts around  $x$  and  $f(x)$  such that  $f(x) = x^n$ . This  $n$  is the **ramification index**.

**Definition 1.1.3** (Belyi morphisms). A **Belyi morphism** is one ramified only over  $\{0, 1, \infty\}$

A **clean Belyi morphism** or **pure Belyi morphism** is a Belyi morphism where the ramification indices over 1 are all exactly 2.

**Lemma 1.1.4.** *A curve  $X$  admits a Belyi morphism iff it admits a clean Belyi morphism.*

*Proof.* If  $\alpha: X \rightarrow \mathbb{P}^1 \mathbb{C}$  is Belyi, then  $\beta = 4\alpha(1-\alpha)$  is a clean Belyi morphism.  $\square$

#### 1.1.2 Dessin d'Enfants

**Definition 1.1.5.** A **dessin d'Enfant** (or Grothendieck Dessin or just **Dessin**) is a triple  $(X_0, X_1, X_2)$  where  $X_2$  is a compact Riemann surface,  $X_1$  is a graph,  $X_0 \subset X_1$  is a finite set of points, where  $X_2 \setminus X_1$  is a collection of open cells.  $X_1 \setminus X_0$  is a disjoint union of line segments

**Lemma 1.1.6.** *The data of a dessin is equivalent to a graph with an ordering on the edges coming out of each vertex.*

**Definition 1.1.7** (Clean dessins). A **clean dessin** is a dessin with a colouring (white and black) on the vertices such that adjacent vertices do not share a colour.

### 1.1.3 The Grothendieck correspondence

Given a Belyi morphism  $\beta: X \rightarrow \mathbf{P}^1 \mathbf{C}$  the graph  $\beta^{-1}([0, 1])$  defines a dessin.

**Theorem 1.1.8.** *The map*

$$\{(\text{Clean}) \text{ Belyi morphisms}\} \rightarrow \{(\text{clean}) \text{ dessins}\}$$

$$\beta \mapsto \beta^{-1}([0, 1])$$

*is a bijection up to isomorphisms.*

**Example 1.1.9.**

$$\mathbf{P}^1 \mathbf{C} \rightarrow \mathbf{P}^1 \mathbf{C}$$

$$x \mapsto x^3$$

$$\mathbf{P}^1 \mathbf{C} \rightarrow \mathbf{P}^1 \mathbf{C}$$

$$x \mapsto x^3 + 1$$

### 1.1.4 Covering spaces and Galois groups

A Belyi morphism defines a covering map.

$$\tilde{\beta}: \tilde{X} \rightarrow \mathbf{P}^1 \mathbf{C} \setminus \{0, 1, \infty\}$$

the coverings are controlled by the profinite completion of

$$\pi_1(\mathbf{P}^1 \mathbf{C} \setminus \{0, 1, \infty\}) = \mathbf{Z} * \mathbf{Z} = F_2.$$

**Theorem 1.1.10.** *There is a faithful action*

$$\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \curvearrowright \hat{\pi}_1(\mathbf{P}^1 \mathbf{C} \setminus \{0, 1, \infty\})$$

*Proof.* By Belyi's theorem every elliptic curve  $E/\overline{\mathbf{Q}}$  admits a Belyi morphism. For each  $j \in \overline{\mathbf{Q}}$  there exists an elliptic curve  $E_j/\overline{\mathbf{Q}}$  with  $j$ -invariant  $j$ .

Given  $\sigma \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ ,

$$\sigma(E_j) = E(\sigma(j))$$

assume  $\sigma \mapsto 1$ ,

$$E_j \cong E_{\sigma(j)} \quad \forall j$$

$$j = \sigma(j) \quad \forall j$$

a contradiction. □

**Corollary 1.1.11.** *We have a faithful action of  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$  on dessins.*

**Theorem 1.1.12.** *We have a faithful action of  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$  on the set of dessins of any fixed genus.*

### 1.1.5 Exercises

**Exercise 1.1.13.** Compute the Dessins for the following Belyi morphisms

1.

$$\mathbf{P}^1 \mathbf{C} \rightarrow \mathbf{P}^1 \mathbf{C}, \mapsto x^4$$

2.

$$\mathbf{P}^1 \mathbf{C} \rightarrow \mathbf{P}^1 \mathbf{C}, \mapsto x^2(3 - 2x)$$

3.

$$\mathbf{P}^1 \mathbf{C} \rightarrow \mathbf{P}^1 \mathbf{C}, \mapsto \frac{1}{x(2-x)}$$

**Exercise 1.1.14.** Give an alternate proof of the fact that  $X$  admits a Belyi morphism if and only if it admits a clean Belyi morphism using dessins and the Grothendieck correspondence.

**Exercise 1.1.15.** Prove that a Belyi morphism corresponding to a tree, that sends  $\infty$  to  $\infty$  is a polynomial.

## 1.2 Riemann Surfaces I (Ricky)

### 1.2.1 Definitions

**Definition 1.2.1.** A **topological surface** is a Hausdorff space  $X$  which has a collection of charts

$$\{\phi_i: U_i \xrightarrow{\sim} \phi_i(U_i) \subseteq \mathbf{C}, \text{ open}\}_{i \in I}$$

such that

$$X = \bigcup_{i \in I} U_i.$$

We call  $X$  a **Riemann surface** if the transition functions  $\phi_i \circ \phi_j^{-1}$  are holomorphic.

### 1.2.2 Examples

**Example 1.2.2.** Open subsets of  $\mathbf{C}$ , e.g.

$$\mathbf{C}$$

$$\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$$

$$\mathbf{H} = \{z \in \mathbf{C} : \text{im } z > 0\}.$$

**Example 1.2.3.**  $\hat{\mathbf{C}}$  = Riemann sphere =  $\mathbf{C} \cup \{\infty\}$ . A basis of neighborhoods of  $\infty$  is given by

$$\{z \in \mathbf{C} : |z| > R\} \cup \{\infty\}.$$

**Example 1.2.4.**

$$\mathbf{P}^1(\mathbf{C}) = \{[z_0 : z_1] : (z_0, z_1) \neq (0, 0)\}$$

$$U_0 = \{[z_0, z_1] : z_0 \neq 0\} \rightarrow \mathbf{C}$$

$$[z_0 : z_1] \mapsto \frac{z_1}{z_0}$$

$$U_1 = \{[z_0, z_1] : z_1 \neq 0\} \rightarrow \mathbf{C}$$

$$[z_0 : z_1] \mapsto \frac{z_0}{z_1}.$$

**Example 1.2.5.** Let  $\Lambda = \mathbf{Z} \oplus \mathbf{Z}i \subseteq \mathbf{C}$  then  $X = \mathbf{C}/\Lambda$  is a Riemann surface.

### 1.2.3 Morphisms

**Definition 1.2.6** ((Holo/Mero)-morphisms of Riemann surfaces). A **morphism of Riemann surfaces** is a continuous map

$$f: S \rightarrow S'$$

such that for all charts  $\phi, \psi$  on  $S, S'$  respectively we have  $\psi \circ f \circ \phi^{-1}$  is holomorphic.

We call a morphism  $f: S \rightarrow \mathbf{C}$  a **holomorphic function** on  $S$ .

We say  $f: S \rightarrow \mathbf{C}$  is a **meromorphic function** if  $f \circ \phi^{-1}$  is meromorphic.

**Exercise 1.2.7.** The set of meromorphic functions on a Riemann surface form a field.

We denote the field of meromorphic functions by  $\mathcal{M}(S)$ .

**Proposition 1.2.8** (1.26).

$$\mathcal{M}(\hat{\mathbf{C}}) = \mathbf{C}(z).$$

*Proof.* Let  $f: \hat{\mathbf{C}} \rightarrow \mathbf{C}$  be meromorphic. Then the number of poles of  $f$  is finite say at  $a_1, \dots, a_n$ . So, locally at  $a_i$  we can write

$$f(z) = \sum_{j=1}^{j_i} \frac{\lambda_{j,i}}{(z - a_i)^j} + h_i(z)$$

with  $h_i$  holomorphic. Then

$$f(z) - \sum_{i=1}^n \sum_{j=1}^{j_i} \frac{\lambda_{j,i}}{(z - a_i)^j}$$

is holomorphic everywhere. By Liouville's theorem this is constant.  $\square$

We say  $S, S'$  are isomorphic if  $\exists f: S \rightarrow S', g: S' \rightarrow S$  morphisms such that  $f \circ g = \text{id}_{S'}, g \circ f = \text{id}_S$ .

**Exercise 1.2.9.** Show that

$$\hat{\mathbf{C}} \simeq \mathbf{P}^1(\mathbf{C}).$$

**Remark 1.2.10.**  $\mathbf{C} \neq \mathbf{D}$  by Liouville.

If  $S, S'$  are connected compact Riemann surfaces, then any nonconstant morphism  $f: S \rightarrow S'$  is surjective. (Nonconstant holomorphic maps are open)

### 1.2.4 Ramification

**Definition 1.2.11** (Orders of vanishing). The **order of vanishing** at  $P \in S$  of a holomorphic function on  $S$  is defined as follows: For  $\phi$  a chart centered at  $P$  write

$$f \circ \phi^{-1}(z) = a_n z^n + a_{n+1} z^{n+1} + \dots, a_n \neq 0$$

then  $\text{ord}_P(f) = n$ .

More generally, for  $f: S \rightarrow S'$  we can define  $m_P(f)$  (**multiplicity** of  $f$  at  $P$ ) by using a chart  $\psi$  on  $S'$  and setting

$$m_P(f) = \text{ord}_P(\psi \circ f).$$

If  $m_P(f) \geq 2$  then we call  $P$  a **branch point** of  $f$  and call  $f$  ramified at  $P$ .

**Example 1.2.12.**

$$f: \mathbb{C} \rightarrow \mathbb{C}, f(z) = z^2.$$

The chart  $\phi_a(z) = z - a$  is centered at  $a \in \mathbb{C}$ . Then to compute  $m_a(f)$  we compute

$$f \circ \phi_a^{-1}(z) = a^2 + 2az + z^2$$

hence

$$\text{ord}_a(f) = \begin{cases} 0, & \text{if } a \neq 0 \\ 2, & \text{if } a = 0 \end{cases}.$$

### 1.2.5 Genus

**Theorem 1.2.13** (Rado). *Any orientable compact surface can be triangulated.*

**Fact 1.2.14.** *Riemann surfaces are orientable.*

Given such an oriented polygon coming from a Riemann surface, we can associate a word  $w$  to it from travelling around the perimeter.

**Example 1.2.15.** For the sphere  $w = a^{-1}ab^{-1}bc^{-1}c$ .

**Fact 1.2.16.** *Every such word can be normalised without changing the corresponding Riemann surface.*

$$w = \begin{cases} w_0 = aa^{-1}, \\ w_g = a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1} \end{cases}$$

The (uniquely determined)  $g$  is the **genus** of the surface.

**Example 1.2.17.**  $w_1 = a_1b_1a_1^{-1}b_1^{-1}$ .

$$w_2 = a_1b_1a_1^{-1}b_1a_2b_2a_2^{-1}b_2^{-1}.$$

**Theorem 1.2.18.**

$$\chi(S) = v - e + f = 2 - 2g(S).$$

## 1.3 Riemann Hurwitz Formula (Sachi)

**Exercise 1.3.1** (Unimportant). The genus is invariant under changing triangulation.

In particular there are at least two distinct ways of thinking about genus for Riemann surfaces  $R$

1.

$$\chi(R) = V - E + F = 2 - 2g$$

2. The dimension of the space of holomorphic differentials on  $R$ .

Goal: given  $R$  calculate genus

$$y^2 = (x+1)(x-1)(x+2)(x-2)$$

so in an ad hoc way

$$y = \sqrt{(x+1)(x-1)(x+2)(x-2)}$$

when  $x$  is not a root of the above we have two distinct values for  $y$ , we can imagine two copies of  $\mathbb{C}$  sitting above each other and then square root will

land in both copies. We have to make branch cuts between the roots and glue along these to account for the fact that going around a small loop surrounding a root will change the sign of our square root. We end up with something looking like a torus here.

Here we examined the value where there were not enough preimages when we plugged in a value for  $x$ . The idea is to project to  $x$ , and understand the number of preimages.

$$P(x, y) = y^n + p_{n-1}(x)y^{n-1} + \cdots + p_0(x)$$

an irreducible polynomial.

$$R = \{(x, y) : P(x, y) = 0\}.$$

If we fix  $x_0 \in \mathbf{P}^1 \mathbf{C}$  we can analyse how many  $y$  values lie over this  $x$ . If we have fixed our coefficients we expect  $n$  solutions in  $y$  over  $\mathbf{C}$ , i.e. points  $(x_0, y) \in R$ .

For some values of  $x_0$  this will not be true, there will be fewer  $y$ -values, this occurs when we have a multiple root. This happens precisely when the discriminant of this polynomial vanishes, the discriminant is a polynomial and so has finitely many roots.

**Definition 1.3.2** (Branch points). Let  $\pi: R \rightarrow \mathbf{P}^1 \mathbf{C}$ . We say  $x_0$  is a **branch point** if there are fewer than  $n$  distinct  $y$ -values above  $x$ . Then define the **total branching index**

$$b = \sum_{x \in \mathbf{P}^1 \mathbf{C}} (\deg(\pi) - \#\pi^{-1}(x)).$$

**Claim 1.3.3.**

$$\chi(R) = \deg \pi \cdot \chi(\mathbf{P}^1 \mathbf{C}) - b.$$

**Lemma 1.3.4.** *Locally given some choice of coordinates a non-constant morphism of Riemann surfaces*

$$f: R \rightarrow S$$

is given by  $w \mapsto w^n$ . More precisely given  $r \in R$ ,  $f(r) = s$  and  $V_s \ni s$  a small neighbourhood choose an identification of

$$V_s \xrightarrow{\Psi} D$$

which sends  $s \mapsto 0$  and we can find an analytic identification

$$r \in R_r \xrightarrow{\phi} D$$

such that

$$f(U_r) \subseteq V_s.$$

$$\begin{array}{ccc} U_r & \xrightarrow{f} & V_s \\ \phi \downarrow & & \downarrow \Psi \\ D & \xrightarrow{w \mapsto w^n} & D \end{array}$$

*Proof.* In Sachi's notes. □

*Proof.* Of [Claim 1.3.3](#).

Triangulate  $R$  so that every face lies in some small coordinate neighborhood s.t.

$$\pi: R \rightarrow \mathbf{P}^1 \mathbf{C}$$

is given by  $w \mapsto w^m$ , s.t. every edge, all branch points are vertices. This ensures that each face edge and vertex has  $n = \deg(\pi)$  preimages (except branch points). Then accounting for branch points we have  $\deg(\pi) - \#\pi^{-1}(x_0)$  preimages.  $\square$

**Example 1.3.5.**  $P(x, y)$  plane curve, classically have

$$g = \frac{(d-1)(d-2)}{2}$$

$\mathbf{P}^2 = \{[x : y : z]\}$  and  $(\mathbf{P}^2)^* = [a : b : c]$ , lines in  $\mathbf{P}^2$

$$ax + by + cz = 0$$

and we have lines  $\leftrightarrow$  points. We have  $C^*$  the dual curve in  $\mathbf{P}^2$  cut out by the tangent lines  $t_Q$  for  $Q \in C$ . Claim  $\deg C^* = (d-1)d$ .

Want

$$R : \{P(x, y) = 0\} \xrightarrow{\pi} \mathbf{P}^1 \mathbf{C}$$

compute  $b$ . In other words, if we fix an arbitrary point  $Q \in C$  then there are  $d(d-1)$  lines through  $Q$  which are tangent to  $C$ . Projecting to the  $x$ -coordinate  $\iff$  family of lines through a point at  $\infty \iff$  \* line in  $(\mathbf{P}^2)^*$ . We have a new question: How many points does this line intersect (up to multiplicity). By bezout  $\iff \deg C^*$ .

Proof (Matt emerton) Consider a point on  $C$  in  $\mathbf{P}^2$  such that no tangent line to the curve at  $\infty$  passes through it. Move this point to the origin. If we write

$$P(x, y) = f_d + f_{d-1} + \cdots + f_0$$

then

$$(f_d, f_{d-1}) = 1$$

suppose they share a linear factor:

$$0 = (f_d)_x x + (f_d)_y y + f_{d-1},$$

then this defines a line through the origin. (Because this gives an equation of an asymptote, this is a contradiction).

$$f_d + f_{d-1} + \cdots + f_0 = 0$$

$$df_d + (d-1)f_{d-1} + \cdots + f_1 = 0$$

$$\implies$$

$$\begin{cases} f_d + f_{d-1} + \cdots + f_0 = 0 \\ f_{d-1} + 2f_{d-2} + \cdots + (d-1)f_1 = 0 \end{cases}.$$

Now these have  $d(d-1)$  common solutions.  $C^*$  has degree  $d(d-1)$  so  $b = d(d-1)$ . Riemann-Hurwitz implies

$$\chi(R) = 2 \deg \pi - d(d-1)$$

$$\chi(R) = 2d - d(d-1)$$

so

$$g = \frac{(d-1)(d-2)}{2}.$$

**A 3-fold equivalence of categories** Amazing synthesis.

1. Analysis: Compact connected riemann surfaces.
  2. Algebra: Field extensions  $K/\mathbf{C}$  where  $K$  is finitely generated of transcendence degree 1 over  $\mathbf{C}$ .
  3. Geometry: Complete nonsingular irreducible algebraic curves in  $\mathbf{P}^n$ .
- 3) curve  $\rightarrow$  2) field extension. Over  $\mathbf{C}$  all rational functions  $\frac{P(x)}{Q(x)}$   $\deg P = \deg Q$ ,  $P, Q: \mathbf{C} \rightarrow \mathbf{C} \cup \{\infty\}$ .
- 3)  $\rightarrow$  1) take complex structure induced by  $\mathbf{P}^n$ .
- 1)  $\rightarrow$  2) associated field of meromorphic functions on  $X$ .
- 1)  $\rightarrow$  3) Any curve which is holomorphic has an embedding into  $\mathbf{P}^n$  (Riemann-Roch).
- 2)  $\rightarrow$  1)  $K/\mathbf{C}$  consider valuation rings  $R$  such that  $K \supseteq R \supseteq \mathbf{C}$ .

**Example 1.3.6.**  $g = 0$ ,  $\mathbf{P}^1 \mathbf{C} \mathbf{C}(t)$ ,  $\mathbf{C} \cup \{\infty\}$ .

**Example 1.3.7.**  $g = 1$ , elliptic curves,  $f(x, y, z)$  smooth plane cubic,  $f = 0$ ,  $\mathbf{C}(\sqrt{f(x)}, x)$ .

$$\begin{aligned} \mathbf{C}/\Lambda &\rightarrow \mathbf{P}^2 \\ z &\mapsto (z, \wp(z), \wp'(z)) \\ z &\notin \Lambda \end{aligned}$$

backwards

$$(x, y) \mapsto \int_{(x_0, y_0)}^{(x, y)} \frac{dx}{y}$$

**Riemann-Hurwitz (generally)** There's nothing that doesn't generalise about the previous proof.

**Claim 1.3.8.** For  $\pi: R \rightarrow S$  a non-constant morphism of compact Riemann surfaces

$$\chi(R) = \deg \pi \cdot \chi(S) - \sum_{x \in S} (\deg(\pi) - \#\pi^{-1}(x)).$$

**Corollary 1.3.9.** There are no non-constant morphisms from a sphere to a surface of genus  $> 0$ .

*Proof.*

$$\begin{aligned} f: \mathbf{P}^1 \mathbf{C} &\rightarrow S \\ \chi(\mathbf{P}^1 \mathbf{C}) &= \deg f \chi(S) - b \\ 2 &= (+) \cdot (-) - b. \end{aligned}$$

□

**Exercise 1.3.10.**

$$x^n + y^n + z^n = 0$$

is not solvable in non-constant polynomials for  $n > 2$ .

**Exercise 1.3.11.**

$$E = \mathbf{C}/\mathbf{Z} + \mathbf{Z}i$$

multiplication by  $i$  rotates  $x \mapsto xi$  let  $x \sim xi$ . If we mod out by  $\sim$  to get  $E/\sim$  this is still a Riemann surface and the quotient map

$$f: E \rightarrow E/\sim$$

is nice, compute the branch points of order 4 and order 2.



**Exercise 1.3.12.**  $X$  compact Riemann surface of  $g \geq 2$  then there are at most  $84(g-1)$  automorphisms of  $X$ .

**Exercise 1.3.13.** Klein quartic

$$x^3y + y^3z + z^3x = 0$$

has 168 automorphisms and is genus 3.

## 1.4 Riemann Surfaces and Discrete Groups (Rod)

Welcome to BUGLES (Boston university geometry learning expository seminar), the reason it is called bugles is because bugles are hyperbolic, and today we will see a lot of hyperbolic objects.

Plan

1. Uniformization
2. Fuchsian groups
3. Automorphisms of Riemann surfaces

**Proposition 1.4.1.**

$$\text{Aut}(\hat{\mathbf{C}}) = \{z \mapsto \frac{az+b}{cz+d}\}$$

$$\text{Aut}(\mathbf{C}) = \{z \mapsto za+b\}$$

$$\text{Aut}(\mathbf{H}) = \{z \mapsto \frac{az+b}{cz+d}, a, b, c, d \in \mathbf{R}\} = \text{PSL}_2(\mathbf{R})$$

**Theorem 1.4.2.**  $\Sigma$  has a universal cover  $\tilde{\Sigma}$  with  $\pi_1(\Sigma) = 1$ .  $\tilde{\Sigma} \rightarrow \Sigma$  holomorphic.  $\Sigma = \tilde{\Sigma}/G$  for  $G = \pi_1(\Sigma)$ .  $G$  acts freely and properly discontinuously.

### 1.4.1 Uniformization

**Theorem 1.4.3.** The only simply connected Riemann surfaces are  $\hat{\mathbf{C}}$ ,  $\mathbf{C}$ ,  $\mathbf{H}$ .

**Theorem 1.4.4.**  $\Sigma$  is a Riemann surface then

$$g = 0 : \Sigma \cong \hat{\mathbf{C}}$$

$$g = 1 : \Sigma \cong \mathbf{C}/\Lambda$$

$$g \geq 2 : \Sigma \cong \mathbf{H}/K.$$

*Proof.*  $g = 0$  Uniformization.

$g \geq 1$   $\hat{\mathbf{C}}$  can't be a cover by Riemann-Hurwitz.  $g = 1$   $\pi_1(\Sigma) = \mathbf{Z} \oplus \mathbf{Z}$  abelian.

Claim: no subgroup of  $\text{Aut}(\mathbf{H})$  is isomorphic to  $\mathbf{Z} \oplus \mathbf{Z}$  acting freely and properly discontinuously. So  $\tilde{\Sigma} = \hat{\mathbf{C}}$   $z \mapsto az+b$  free id  $a = 1$  so  $z \mapsto z + \lambda_1$   
 $z \mapsto z + \lambda_2$ .

$g = 2$   $\pi_1(\Sigma)$  is not abelian but  $z \mapsto z + \lambda_1$  is abelian!

$$\Sigma = \mathbf{H}/K, K \subseteq \text{PSL}_2(\mathbf{R}).$$

□

**Goal** Understand  $\Sigma$  through  $\tilde{\Sigma}$  and  $G$ .

**Fuchsian groups**  $g \geq 2$ .

$$\text{Aut}(\mathbf{H}) = \text{PSL}_2(\mathbf{R}) = \text{Isom}^+(\mathbf{H}, \frac{|dz|^2}{\Im Z})$$

hyperbolic  $\mathbf{H}$ ,  $\mathbf{D}$  and  $\text{PSL}_2(\mathbf{R})$  acts transitively on geodesics.

**Definition 1.4.5** (Fuchsian groups). A **Fuchsian group** is a discrete subgroup of  $\text{PSL}_2(\mathbf{R})$ .

**Remark 1.4.6.** (proof in book) Even if  $\Gamma$  doesn't act freely the quotient

$$\mathbf{H} \rightarrow \mathbf{H}/\Gamma$$

is still a covering map and  $\mathbf{H}/\Gamma$  is a Riemann surface.

**Reflections on  $\mathbf{H}$**  Say  $\mu$  is a geodesic in  $\mathbf{H}$ , i.e. a horocycle. There is  $M \in \text{PSL}_2(\mathbf{R})$  with  $M\mu$  the imaginary axis. Then  $R = -\bar{z}$  is the reflection over the imaginary axis. Now  $R_\mu = M^{-1} \circ R \circ M$  is a reflection over  $\mu$ .

$$R_\mu = \frac{a\bar{z} + b}{c\bar{z} + d} \notin \text{PSL}_2(\mathbf{R})$$

this is a problem for us.

**Triangle groups** Given  $n, m, l \in \mathbf{Z} \cup \{\infty\}$  then there is a hyperbolic triangle with angles  $\pi/n, \pi/m, \pi/l$  if

$$\frac{1}{n} + \frac{1}{m} + \frac{1}{l} < 1.$$

With area  $\pi(1 - \frac{1}{n} - \frac{1}{m} - \frac{1}{l})$ .

In the disk model we can start with a wedge of the disk and by adding a choice third geodesic with endpoints on the edge we can adjust the other angles to be what we like. So we can construct hyperbolic triangles with whatever angles we like. Then let  $R_1$  be the reflection over 1 edge,  $R_2, R_3$  similarly. By reflecting our original triangle  $T$  with these reflections we can tessellate the disk, colouring alternately the triangles obtained using an odd or even number of reflections.

The only remaining problem is that  $R_i$ 's are not in  $\text{PSL}_2(\mathbf{R})$ . The solution is to define  $x_1 = R_3 \circ R_1, x_2 = R_1 \circ R_2, x_3 = R_2 \circ R_3$  which are all in  $\text{PSL}_2(\mathbf{R})$  now. Now we need to take the union of two adjacent triangles before as a fundamental domain, some quadrilateral that still tessellates. So we have formed a Fuchsian group from our triangles.

A presentation for this group is

$$\langle x_1, x_2, x_3 | x_1^n = x_2^m = x_3^l = x_1 x_2 x_3 = 1 \rangle$$

note  $n, m, l$  can still be  $\infty$ .

**Definition 1.4.7** (Triangle groups). Let  $\Gamma_{n,m,l}$  be the **triangle group** with signature  $(1/n, 1/m, 1/l)$ .

**Remark 1.4.8.**

$$\begin{aligned} \frac{1}{n} + \frac{1}{m} + \frac{1}{l} &= 1 \\ \frac{1}{n} + \frac{1}{m} + \frac{1}{l} &> 1 \end{aligned}$$

still work on  $\mathbf{C}$  and  $\hat{\mathbf{C}}$  respectively.

**Example 1.4.9** ( $\text{PSL}_2(\mathbf{Z})$ ). Consider  $\Gamma_{2,3,\infty}$  angles  $\pi/2, \pi/3, 0$ . We can draw such a triangle in the upper half plane with vertices  $i, e^{\pi i/3}, \infty$ . So a fundamental domain will be the region obtained by reflecting through the imaginary axis, given by  $-\frac{1}{2} \leq \Re z \leq \frac{1}{2}, |z| \geq 1$ . We have  $R_1 = \frac{1}{\bar{z}}, R_2 = -\bar{z} + 1, R_3 = -\bar{z}$  so  $x_1 = \frac{-1}{z}, x_2 = \frac{1}{-z+1}, x_3 = z + 1$ . Then  $\Gamma_{2,3,\infty} \cong \text{PSL}_2(\mathbf{Z})$ . Sometimes denoted  $\Gamma(1)$ .

**Observation 1.4.10.** If  $\Gamma_1 < \Gamma_2$  and  $T$  is a fundamental domain of  $\Gamma_2$  then if  $\gamma_1, \gamma_2, \dots, \gamma_n \in \Gamma_2$  are representatives of  $\Gamma_1 \backslash \Gamma_2$  then

$$\bigcup \gamma_i(T)$$

is a fundamental domain for  $\Gamma_1$ .

**Example 1.4.11** ( $\Gamma(1)$ ).

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{id} \pmod{2} \right\}$$

then

$$[\Gamma(1) : \Gamma(2)] = 6$$

representatives of  $\Gamma(2) \backslash \Gamma(1)$  are

$$x_1 = \text{id}, x_2 = \frac{-1}{z-1}, x_3 = \frac{z-1}{z}, x_4 = \frac{z-1}{z}, x_5 = \frac{-z}{x-1}, x_6 = \frac{-1}{z}.$$

Lets see what these do, for example if  $z = e^{i\theta}$

$$\Re(x_2(z)) = \frac{-1}{e^{i\theta}-1} = \frac{-e^{i\theta}+1}{2-2\cos\theta} = \frac{1-\cos\theta}{2-2\cos\theta} \frac{1}{2}$$

if we plot this we see we get two copies of a 0,0,0 triangle so this corresponds to  $\Gamma_{\infty,\infty,\infty}$ .

$$\langle x_1, x_2, x_3 | x_1 x_2 x_3 = 1 \rangle = \langle x_1, x_2 \rangle = \pi_1(\mathbf{P}^1 \setminus \{0, 1, \infty\}).$$

**Proposition 1.4.12.**  $S_1 = \mathbf{H}/\Gamma_1, S_2 = \mathbf{H}/\Gamma_2$  then

$$S_1 \cong S_2 \iff \Gamma_1 = T \circ \Gamma_2 \circ T^{-1}, T \in \text{PSL}_2(\mathbf{R}).$$

*Proof.*  $\Leftarrow$  Define an  $\phi: S_1 \rightarrow S_2$  via  $\phi([z]_1) = [T(z)]_2$ .

$\Rightarrow$  Take a lift

$$\begin{array}{ccc} \mathbf{H} & \xrightarrow{\tilde{\phi}} & \mathbf{H} \\ \downarrow & & \downarrow \\ \mathbf{H}/\Gamma_1 & \xrightarrow{\phi} & \mathbf{H}/\Gamma_2 \end{array}$$

then  $T = \tilde{\phi}$ . □

**Proposition 1.4.13.**  $\Gamma$  a Fuchsian group acts freely

$$\text{Aut}(\mathbf{H}/\Gamma) = N(\Gamma)/\Gamma.$$

*Proof.* Previous proposition, set  $\Gamma_1 = \Gamma_2$

$$N(\Gamma) \rightarrow \text{Aut}(\mathbf{H}/\Gamma)$$

kernel is  $\Gamma$ . □

**Corollary 1.4.14.** *Let  $\Sigma$  be a Riemann surface with  $g \geq 2$  then*

$$|\text{Aut}(\Sigma)| < \infty.$$

*Proof.*

$$\begin{array}{ccc} \mathbf{H} & & \\ \phi_2 \downarrow & \searrow \phi_1 & \\ S = \mathbf{H}/\Gamma & \xrightarrow{f} & \mathbf{H}/N(\Gamma) = S/\text{Aut}(S) \end{array}$$

since  $\phi_1, \phi_2$  are holomorphic then so is  $f$ . So  $\deg f = |N(\Gamma)/\Gamma|$  and  $\deg f < \infty$ .  $\square$

Say  $\Sigma, g \geq 2, G \subseteq \text{Aut}(\Sigma)$ . Let  $\bar{g}$  be the genus of  $\Sigma/G$

$$2g - 2 = |G|(2\bar{g} - 2) + \sum_p (I(p) - 1) = |G|(2\bar{g} - 2 + \sum_{i=1}^n (1 - \frac{1}{|I(p_i)|}))$$

where  $I(p)$  is the stabiliser of  $p$  in  $G$  and  $\{p_i\}$  area maximal set of fixed points of  $G$  inequivalent under the action of  $G$ .

**Exercise 1.4.15.**  $\Sigma, g \geq 2$  then  $|\text{Aut}(\Sigma)| \leq 84(g - 1)$ . Hint: cases.

**Exercise 1.4.16.** Consider

$$1 \rightarrow \Gamma(n) \rightarrow \Gamma(1) \rightarrow \text{PSL}_2(\mathbf{Z}/n\mathbf{Z}) \rightarrow 1$$

compute genus of  $\mathbf{H}/\Gamma(n)$ .

## 1.5 Riemann Surfaces and Discrete Groups II (Jim)

### 1.5.1 Moduli space of compact Riemann surfaces with genus $g$

$g = 0$ . Uniformization tells us that up to isomorphisms all Riemann surfaces of genus 0 are  $\mathbf{P}^1$  hence the moduli space  $\mathcal{M}_0 = \{\text{pt}\}$ .

$g = 1$ . Uniformization tells us that each Riemann surface of genus 1 is a torus and can be written as  $\mathbf{C}/\omega_1\mathbf{Z} + \omega_2\mathbf{Z} \rightarrow \mathbf{C}/(\mathbf{Z} \oplus \tau\mathbf{Z})$ , with  $\tau = \pm\omega_1/\omega_2$ .

**Proposition 1.5.1** (2.54).

$$\mathcal{M}_1 \simeq \mathbf{H}/\text{PSL}_2(\mathbf{Z}) \simeq \mathbf{C}.$$

*Proof.* Idea: Existence of

$$\mathbf{C}/\Lambda_{\tau_1} \xrightarrow{\sim} \mathbf{C}/\Lambda_{\tau_2}$$

with  $\bar{T}([0]) = [0]$  is equivalent to the existence of  $T \in \text{Aut}(\mathbf{C})$  (choose  $T(z) = wz$ ) such that  $w(\mathbf{Z} \oplus \tau_1\mathbf{Z}) = \mathbf{Z} \oplus \tau_2\mathbf{Z}$ . This in turn is equivalent to the existence of

$$A, A' \in \text{GL}_2(\mathbf{Z})$$

s.t.  $\det(A) = \det(A') = \pm 1$  so that

$$\begin{aligned} \begin{pmatrix} w \\ w\tau_1 \end{pmatrix} A \begin{pmatrix} 1 \\ \tau_2 \end{pmatrix} &= A' \begin{pmatrix} w \\ w\tau_1 \end{pmatrix} \\ \implies \tau_q &= A\psi_2 = \frac{a\tau_2 + b}{c\tau_2 + d} \end{aligned}$$

and  $A \in \mathrm{PSL}_2(\mathbf{R})$ . Implies  $A \in \mathrm{PSL}_2(\mathbf{Z})$  as both  $\tau_1, \tau_2 \in \mathbf{H}$ . Conversely if

$$\tau_1 = \frac{a\tau_2 + b}{c\tau_2 + d}$$

isomorphism is induced by  $T(z) = (c\tau_2 + d)z$ .  $\square$

$g > 1$   $\mathcal{M}_g$  is a complex variety of dimension  $3g - 3$ . Uniformization tells us that describing a Riemann surface amounts to specifying  $2g$  real  $2 \times 2$  matrices  $\{\gamma_i\}_{i=1}^{2g}$  such that

1.  $\det(\gamma_i) = 1$  which implies that  $\gamma_i$  depends on 3 real parameters so we have a total of  $6g$ .
2.  $\prod_{i=1}^g [\gamma_i, \gamma_{g+i}] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  3 relations, so  $6g - 3$ . Since for any  $\gamma \in \mathrm{PSL}_2(\mathbf{R})$   $\Gamma = \langle \gamma_i \rangle$  and  $\gamma\Gamma\gamma^{-1}$  uniformize isomorphic Riemann surfaces implies  $6g - 6$  real parameters, so  $3g - 3$  complex parameters.

### 1.5.2 Monodromy

Let  $f: S_1 \rightarrow S$  be a morphism of degree  $d$  ramified over  $y_1, \dots, y_n \in S$ . For  $y \in S \setminus \{y_1, \dots, y_n\}$  we have a group homomorphism

$$M_f: \pi_1(S \setminus \{y_1, \dots, y_n\}) \rightarrow \mathrm{Bij}(f^{-1}(y))$$

$$\gamma \mapsto M_f(\gamma) = \sigma_\gamma^{-1}.$$

$\sigma_\gamma$  is defined as follows:

$$\gamma \in \pi_1(S \setminus \{y_1, \dots, y_n\})$$

lifts to a path  $\tilde{\gamma}$  from  $x \in f^{-1}(y)$  to another  $x' \in f^{-1}(y)$  set  $\sigma_\gamma(x) = x'$ . If we number the points in  $f^{-1}(y)$  we may think of  $M_f(\pi_1) \subseteq \Sigma_d$ , via some  $\phi: \{1, \dots, d\} \rightarrow f^{-1}(y)$ .  $\mathrm{Mon}(f)$  is the image of  $M_f(\pi_1)$  in  $\Sigma_d$ .

**Monodromy and Fuchsian groups** Let

$$\pi: \mathbf{H}/\Gamma_1 \rightarrow \mathbf{H}/\Gamma$$

be the Fuchsian group representation of the map

$$f: S_1 \rightarrow S \ni y.$$

Identifications  $y = [z_0]_\Gamma$  for some  $z_0 \in \mathbf{H}$ .

$$\pi_1(S \setminus \{y_1, \dots, y_n\}) \simeq \Gamma$$

$$f^{-1}(y) = \{[\beta z_0]_{\Gamma_1}\}$$

where  $\beta$  runs along a set of representatives of  $\Gamma_1 \backslash \Gamma$ .

$$M_f: \Gamma \rightarrow \mathrm{Bij}(\Gamma_1 \backslash \Gamma)$$

$$\gamma \mapsto M_f(\gamma)$$

$$\implies \gamma \sim \pi_1([z_0, \gamma(z_0)])$$

where  $[z_0, \gamma(z_0)]$  is a path in  $\mathbf{H}$ . Lift this loop to  $\mathbf{H}/\Gamma_1$  is the path  $\pi_{\Gamma_1}(\beta[z_0, \gamma_0(z_0)])$ . which corresponds to  $\Gamma_1\beta\gamma$ , this implies  $\sigma_\gamma(\Gamma_1\beta) = \Gamma_1\beta\gamma$ .

**Corollary 1.5.2** (2.59).

$$M_\pi : \Gamma \rightarrow \text{Bij}(\Gamma_1 \backslash \Gamma)$$

induces an isomorphism

$$\frac{\Gamma}{\bigcap_{\beta \in \Gamma_1} \beta^{-1} \Gamma_1 \beta} \simeq \text{Mon}(\pi)$$

characterize morphisms by monodromy. Let  $f_i$  have degree 2, non conjugate.

**Proposition 1.5.3** (2.63). *For  $S$  a compact Riemann surface and  $\beta = \{a_1, \dots, a_n\} \subset S$  for some  $d \geq 1$  there are only finitely many pairs  $(\tilde{S}, f)$  where  $\tilde{S}$  is a compact Riemann surface and*

$$f : \tilde{S} \rightarrow S$$

*is a degree  $d$  morphism with branching value set  $\beta$ .*

*Proof.* Special case: Assume  $S = \mathbf{P}^1$  and  $n = 3$ .

$$\begin{aligned} \Gamma = \Gamma(2) &= \{A \in \text{PSL}_2(\mathbf{Z}) : A = \text{id} \pmod{2}\} \\ &= \pi_1(S' \setminus \{0, 1, \infty\}) \end{aligned}$$

is generated by  $\gamma_1, \gamma_2$  so any map  $M_f : \Gamma(2) \rightarrow \Sigma_d$  is determined by images of  $\gamma_1, \gamma_2$ .  $\square$

### 1.5.3 Galois coverings

**Definition 1.5.4.** A covering  $f : S_1 \rightarrow S_2$  is Galois (or regular, or normal) if the covering group

$$\text{Aut}(S, f) = \{h \in \text{Aut}(S_1) : f \circ h = f\} = G$$

acts transitively on each fibre. With this notion we can think of  $S_1 \rightarrow S_1/G$ .

**Proposition 1.5.5** (2.65).

$$f : S_1 \rightarrow S_2$$

is Galois if and only if

$$f^* : M(S_2) \rightarrow M(S_1)$$

is a Galois extension. In this case  $\text{Aut}(S_1, f) \simeq \text{Gal}(M(S_1)/M(S_2))$ .

**Example 1.5.6.** Hyperelliptic covers of  $\mathbf{P}^1$  given by

$$S = \{y^2 = \prod_{i=1}^N (x - a_i)\} \rightarrow \mathbf{P}^1$$

$$(x, y) \rightarrow x$$

covering group  $G$  is order 2 generated by the involution  $J(x, y) = (x, -y)$ .

**Proposition 1.5.7** (2.66). *A covering*

$$f : S_1 \rightarrow S_2$$

*is normal/Galois iff*

$$\deg(f) = |\text{Mon}(f)|.$$

### 1.5.4 Normalization of coverings of $\mathbf{P}^1$

Let  $f: S \rightarrow \mathbf{P}^1$  be a cover of  $\deg d > 0$  with  $\text{Mon}(f) \leq \Sigma_d$ .

The normalisation

$$\tilde{f}: \tilde{S} \rightarrow \mathbf{P}^1$$

associated to  $f$  has  $\text{Mon}(f) \cong \text{Aut}(\tilde{S}, \tilde{g})$  and  $\tilde{f}^*: M(\mathbf{P}^1) \rightarrow M(\mathbf{P}^1)$  is the normalisation of the extension

$$f^*: M(\mathbf{P}^1) \hookrightarrow M(S)$$

Normalization of extensions  $K \hookrightarrow L$  is a Galois extension of  $K$  of lowest possible degree containing  $L$ .

**Definition 1.5.8.** Normalization of  $f: S \rightarrow \mathbf{P}^1$   $\deg d > 0$  is a Galois covering  $\tilde{f}: \tilde{S} \rightarrow \mathbf{P}^1$  of lowest possible degree s.t.  $\exists \pi: \tilde{S} \rightarrow S$  with the diagram commuting.

**Corollary 1.5.9** (2.73).

$$\text{Mon}(f) \simeq \text{Aut}(\tilde{S}, \tilde{f})$$

Interpretation in terms of Fuchsian groups:

**Proposition 1.5.10.** Let  $f: S_1 \rightarrow S$  be a covering of Riemann surfaces  $S_1 \setminus f^{-1}\{y_1, \dots, y_n\} \rightarrow S \setminus \{y_1, \dots, y_n\}$ . The unramified cover and  $\pi: \mathbf{H}/\Gamma_1 \rightarrow \mathbf{H}/\Gamma$  the Fuchsian group representatives. The normalisation of  $f$  can be represented as the compactification of

$$\mathbf{H} / \bigcap_{\gamma \in \Gamma} \gamma^{-1} \Gamma_1 \gamma \rightarrow \mathbf{H}/\Gamma_1 \rightarrow \mathbf{H}/\Gamma$$

so the covering group is isomorphic to  $\Gamma / \bigcap \gamma^{-1} \Gamma_1 \gamma \simeq \text{Mon}(f)$ .

**Example 1.5.11.** Let  $F(x, y) = y^2x - (y - 1)^3$  consider

$$S_F \rightarrow \mathbf{P}^1$$

$$(x, y) \rightarrow x$$

$S_F$  has genus 0.  $S_F \rightarrow \mathbf{P}^1$  is of degree 3 and ramified at most over  $0, \frac{-27}{4}, \infty$ .  $\text{Mon}(x) \simeq \Sigma_3$  so not a normal covering. Normalization of  $(S_F, x)$  is  $(\tilde{S}_F, \tilde{x})$  where

$$\tilde{F}(x, y) = y^2(1 - y)^2x + (1 - y + y^2)$$

## 1.6 Belyi's theorem (Maria)

**Theorem 1.6.1.** Let  $S$  be a compact riemann surface, then the following are equivalent.

1.  $S$  is defined over  $\overline{\mathbf{Q}}$  (iff over a number field)
2.  $S$  admits a morphism  $f: S \rightarrow \mathbf{P}^1$  with at most 3 branching values.

**Definition 1.6.2** (Belyi functions). A meromorphic function with less than 4 branching values is a **Belyi function**.

**Remark 1.6.3.**

1. Branching values can be taken to be in  $\{0, 1, \infty\}$ .
2. If  $S \neq \mathbf{P}^1$ , then  $f: S \rightarrow \mathbf{P}^1$  has at least 3 branching values

**Definition 1.6.4** (Belyi polynomials). Let  $m, n \in \mathbf{N}$ ,  $\lambda = m/(m+n)$ , define

$$P_\lambda(x) = P_{m,n}(x) = \frac{(m+1)^{m+n}}{m^m n^n} x^m (1-x)^n$$

**Belyi polynomials.**

**Proposition 1.6.5.**  $P_\lambda$  satisfies

1.  $P_\lambda$  ramifies at exactly  $0, 1, \lambda, \infty$ .
2.  $P_\lambda(0) = P_\lambda(1) = 0, P_\lambda(\lambda) = 1, P_\lambda(\infty) = \infty$ .

**Example 1.6.6.**

$$S_\lambda : y^2 = x(x-1)(x-\lambda)$$

with  $\lambda = m/(m+n)$ . From ex. 1.32

$$x : S_\lambda \rightarrow \mathbf{P}^1$$

$$(x, y) \mapsto x$$

$$\infty \mapsto \infty$$

ramifies over  $0, 1, \lambda, \infty$ . Then  $f = P_\lambda \circ x : S_\lambda \rightarrow \mathbf{P}^1$  ramifies exactly at  $(0, 0), (1, 0), (\lambda, 0), \infty$ . With branching values  $0, 0, 1, \infty$  so that  $f$  is a Belyi function.

### 1.6.1 Proof of a) implies b)

**Note 1.6.7.** Its enough to show  $\exists f : S \rightarrow \mathbf{P}^1$  ramified over  $\{0, 1, \infty, \lambda_1, \dots, \lambda_n\} \subseteq \mathbf{Q} \cup \{\infty\}$ . Given this we can repeatedly use Belyi polynomials to obtain  $g : S \rightarrow \mathbf{P}^1$  ramified over  $\{0, 1, \infty\}$ .

Write  $S = S_F$

$$F(x, y) = p_0(x)y^n + \dots + p_n(x)$$

defined over  $\overline{\mathbf{Q}}[x, y]$ . Let  $B_0 = \{\mu_1, \dots, \mu_s\}$  be the branching values of  $x : S_F \rightarrow \mathbf{P}^1$ .

Theorem 1.86 says that the each  $\mu_i$  is  $\infty$ , a root of  $p_0(x)$  or a common root of  $F, F_y$  which implies by lemma 1.84 that  $B_0 \subseteq \overline{\mathbf{Q}} \cup \{\infty\}$ . If  $B_0 \subseteq \mathbf{Q} \cup \{\infty\}$  we are done otherwise let  $m_1(T) \in \mathbf{Q}[T]$  be the minimal polynomial of  $\{\mu_1, \dots, \mu_s\}$ . Let  $\{\beta_1, \dots, \beta_d\}$  be the roots of  $m'_1(T)$  and  $p'(T)$  their min. poly. Note :  $\deg P(t) < \deg m'_1(T)$

Note:  $\text{Branch}(g \circ f) = \text{Branch}(g) \cup g(\text{Branch}(f))$  branching values.

So  $B_1 \text{ Branch}(m_1 \circ x) = m_1(\{\text{roots of } m'_1\}) \cup \{0, \infty\}$ .

$$S \xrightarrow{x} \mathbf{P}^1 \xrightarrow{m_1} \mathbf{P}^1$$

If  $B_1 \subseteq \mathbf{Q} \cup \{\infty\}$  done. Otherwise let  $m_2(T)$  be the minimal polynomial /  $\mathbf{Q}$  of  $\{m_1(\beta_1), \dots, m_1(\beta_d)\}$ ,  $B_2 = \text{Branch}(m_2 \circ m_1 \circ x)$ . Fact:  $\deg(m(t)) < \deg(m_1(T))$ .

Repeat inductively until  $B_k \subseteq \mathbf{Q} \cup \{\infty\}$  which is guaranteed by the decreasing degrees.



### 1.6.2 Algebraic characterization of morphisms

**Proposition 1.6.8.** *Defining a morphism  $f: S_F \rightarrow S_G$  is equivalent to giving a pair of rational functions*

$$f = (R_1, R_2), \quad R_i = \frac{P_i}{Q_i}, \quad P_i, Q_i \in \mathbb{C}[x, y], \quad Q_i \notin (F)$$

*such that  $Q_1^{\deg_x(G)} Q_2^{\deg_y(G)} G(R_1, R_2) = HF$  for some  $H \in \mathbb{C}[x, y]$ .  $f(R_1, R_2)$  is an isomorphism if there exists an inverse morphism  $h: S_G \rightarrow S_F$ .*

**Remark 1.6.9.**

$$\begin{array}{ccc} S_f & \xrightarrow{f} & S_G \\ & \searrow m & \downarrow h \\ & & S_D \end{array}$$

The fact that this diagram commutes can be expressed by polynomial identities.

### 1.6.3 Galois action

Let  $\text{Gal}(\mathbb{C}) = \text{Gal}(\mathbb{C}/\mathbb{Q})$ .

**Definition 1.6.10.** For  $\sigma \in \text{Gal}(\mathbb{C})$ ,  $a \in \mathbb{C}$  denote  $a^\sigma = \sigma(a)$ ,

1. If  $P = \sum a_{ij} x^i y^j \in \mathbb{C}[x, y]$  set

$$P^\sigma = \sum a_{ij}^\sigma x^i y^j \in \mathbb{C}[x, y]$$

if  $R = P/Q$  set  $R^\sigma = P^\sigma/Q^\sigma$ .

2. If  $S \simeq S_F$ ,  $S^\sigma = S_{F^\sigma}$ .
3. If  $\Psi = (R_1, R_2): S_F \rightarrow S_G$  is a morphism, set  $\Psi^\sigma = (R_1^\sigma, R_2^\sigma): S_{F^\sigma} \rightarrow S_{G^\sigma}$ .
4. For an equivalence class  $(S, f) = (S_F, R(x, y))$  of ramified covers of  $\mathbb{P}^1$  set  $(S, f)^\sigma = (S^\sigma, f^\sigma) = (S_{F^\sigma}, R^\sigma(x, y))$ .

**Exercise 1.6.11.** Verify this Galois action is well-defined (lemma 3.12).

Recall:  $S_F$  is constructed from a noncompact Riemann surface  $S_F^\times \subseteq \mathbb{C}^2$  by adding finitely many points, (theorem 1.86). If  $P = (a, b) \in S_F^\times$  then  $P^\sigma = (a^\sigma, b^\sigma)$ . What about the other points?

### 1.6.4 Points and valuations

**Definition 1.6.12.** Let  $\mathcal{M}$  be a function field. A (discrete) valuation of  $\mathcal{M}$  is  $v: \mathcal{M}^* \rightarrow \mathbb{Z}$  s.t.

1.  $v(\phi\psi) = v(\phi) + v(\psi)$
2.  $v(\phi \pm \psi) \geq \min\{v(\phi), v(\psi)\}$
3.  $v(\phi) = 0$  if  $\phi \in \mathbb{C}^*$
4.  $v$  is nontrivial  $\exists \phi: v(\phi) \neq 0$

set  $v(0) = \infty$ .

Facts:

$$A_v = \{\phi \in \mathcal{M} : v(\phi) \geq 0\} \subseteq \mathcal{M}$$

is a subring that is a local ring with a maximal ideal

$$M_v = \{\phi \in \mathcal{M} : v(\phi) > 0\} = (\phi)$$

for some  $\phi$  a uniformizer.

If  $v(\phi) = 1$   $v$  is normalised.

**Proposition 1.6.13 (3.15).** *Every point  $P \in S$  a compact Riemann surface defines a valuation on  $\mathcal{M}(S)$  by  $v_P(\phi) = \text{ord}_P(\phi)$ .*

*Proof.* Easy exercise. □

**Theorem 1.6.14 (3.23).** *For any compact Riemann surface  $S$*

$$P \in S \mapsto v_P = \text{ord}_P$$

*gives a 1-1 correspondence between points of  $S$  and normalised valuations on  $\mathcal{M}(S)$ .*

*Proof.* Sketch: First prove it for  $S = \mathbf{P}^1$ .

Inductively meromorphic functions separate points.

Surjectivity study behaviour of valuations in finite extensions of fields and use a nonconstant morphism  $f: S \rightarrow \mathbf{P}^1$  to reduce to the case of  $\mathbf{P}^1$ . □

### Galois action on points

**Definition 1.6.15.**

1. Given a valuation  $v$  on  $\mathcal{M}(S)$  define a valuation  $v^\sigma$  on  $\mathcal{M}(S^\sigma)$  by  $v^\sigma = v \circ \sigma^{-1}$  i.e.  $v^\sigma(\psi^\sigma) = v(\psi)$  for all  $\psi \in \mathcal{M}(S)$ .
2. For  $P \in S$  define  $P^\sigma \in S^\sigma$  as the unique point in  $S^\sigma$  s.t.  $v_{P^\sigma} = (v_P)^\sigma$ .

**Proposition 1.6.16 (3.25).**

1. For  $\sigma \in \text{Gal}(\mathbf{C})$ ,  $P \mapsto P^\sigma$  is a bijection  $S \rightarrow S^\sigma$ .
2. On  $P \in S_F^\times$  this agrees with the previous definition of  $P^\sigma$ .
3.  $a^\sigma = a$  for all  $a \in \mathbf{Q} \cup \{\infty\}$  for all  $\sigma \in \text{Gal}(\mathbf{C})$ .

*Proof.* Sketch

1.  $a \mapsto a^{\sigma^{-1}}$ .
2. Follows as in proof of 3.22
3. Obvious for  $a \in \mathbf{Q}$ , for  $\infty$ :

$$(v_\infty)^\sigma(x-1) = v_\infty(x - a^{\sigma^{-1}}) = 1 = v_\infty(x-1)$$

for all  $a \in \mathbf{C}$  implies  $(v_\infty)^{\sigma^{-1}} = v_\infty$  implies  $\infty^\sigma = \infty$ .

□

### 1.6.5 Elementary invariants of the action of $\text{Gal}(\mathbf{C})$ .

**Remark 1.6.17.** The bijection  $S \leftrightarrow S^\sigma$  is not holomorphic. In general  $S$  and  $S^\sigma$  are not isomorphic.

**Theorem 1.6.18.** *The action of  $\text{Gal}(\mathbf{C})$  on pairs  $(S, f)$  satisfies*

1. 
$$\deg(f^\sigma) = \deg(f)$$
2. 
$$(f(P))^\sigma = f^\sigma(P^\sigma)$$
3. 
$$\text{ord}_{P^\sigma}(f^\sigma) = \text{ord}_P(f)$$
4.  $a \in \hat{\mathbf{C}}$  is a branching value of  $f$  iff  $a^\sigma$  is a branching value of  $f^\sigma$ .
5.  $\text{genus}(S) = \text{genus}(S^\sigma)$  i.e. they are homeomorphic.
6.  $\text{Aut}(S, f) \rightarrow \text{Aut}(S^\sigma, f^\sigma)$  via  $h \mapsto h^\sigma$  is a group homomorphism.
7. The monodromy group  $\text{Mon}(f)$  of  $(S, f)$  is isomorphic to  $\text{Mon}(f^\sigma)$  of  $(S^\sigma, f^\sigma)$ .

We will use properties 1 and 4 at least.

**Proposition 1.6.19** (Criterion 3.29). *For a compact Riemann surface  $S$  the following are equivalent*

1.  $S$  is defined over  $\overline{\mathbf{Q}}$ .
2.  $\{S^\sigma\}_{\sigma \in \text{Gal}(\mathbf{C})}$  contains only finitely many isomorphism classes of Riemann surfaces.

*Proof.* 1 implies 2:  $S = S_F$ ,  $F = K[x, y]$  for  $K$  a number field then

$$|\{F^\sigma\}_{\sigma \in \text{Gal}(\mathbf{C})}| \leq [K : \mathbf{Q}]$$

2 implies 1 is section 3.7. □

**Proof of b implies a in Belyi's theorem (3.61)** Suppose  $f: S \rightarrow \mathbf{P}^1$  is a morphism of degree  $d$  with branching values  $\{0, 1, \infty\}$ . By theorem 3.28  $\forall \sigma \in \text{Gal}(\mathbf{C})$

$$f^\sigma: S^\sigma \rightarrow \mathbf{P}^1$$

is a morphism of degree  $d$  and branching values are

$$\{\sigma(0), \sigma(1), \sigma(\infty)\} = \{0, 1, \infty\}.$$

So  $\{f^\sigma\}_{\sigma \in \text{Gal}(\mathbf{C})}$  gives rise to only finitely many monodromy homomorphisms.

$$F_{f^\sigma}: \pi_1(\mathbf{P}^1 \setminus \{0, 1, \infty\}) \rightarrow \Sigma_d$$

the fundamental group is free on two generators so there are only finitely many such maps. Theorem 2.61 implies  $\{S^\sigma\}_{\sigma \in \text{Gal}(\mathbf{C})}$  contains only finitely many equivalence classes so by the criterion  $S$  is defined over  $\overline{\mathbf{Q}}$ .

### 1.6.6 The field of definition of Belyi functions (3.8)

**Proposition 1.6.20.** *Belyi functions are defined over  $\overline{\mathbf{Q}}$ .*

*Proof.* Use the same methods as in 3.7. □