Geometric approaches to solving Diophantine equations

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Tomorrows Mathematicians Today 2013

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Introduction

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In this talk

Two general ideas:

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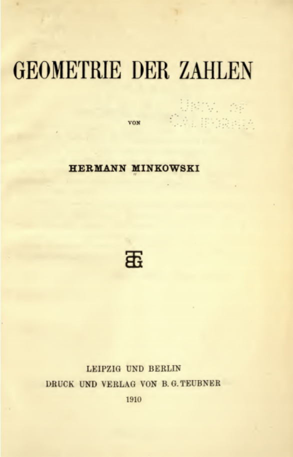
- Geometry of numbers
- Rational points on surfaces

Minkowski and the Geometry of Numbers

1910 - Hermann Minkowski publishes his paper “Geometrie der Zahlen” and sparks a new field called the Geometry of Numbers.



GEOMETRIE DER ZAHLEN

VON

HERMANN MINKOWSKI

LEIPZIG UND BERLIN
DRUCK UND VERLAG VON B. G. TEUBNER
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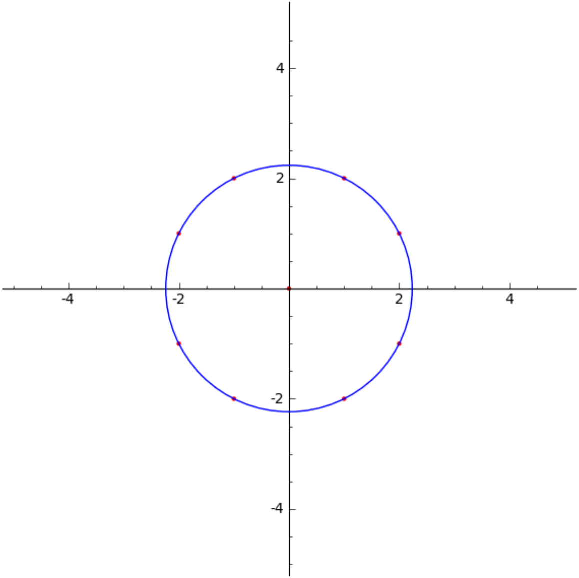
But 7, 11, 19, ... do not.
So maybe $x^2 + y^2 = p$ when $p \equiv 1 \pmod 4$.

Theorem (Fermat's Christmas Theorem)

An odd prime p can be written as $p = x^2 + y^2$ if and only if $p \equiv 1 \pmod 4$.

What can geometry do for us?

When $p = 5$ we can look at points satisfying our criteria:



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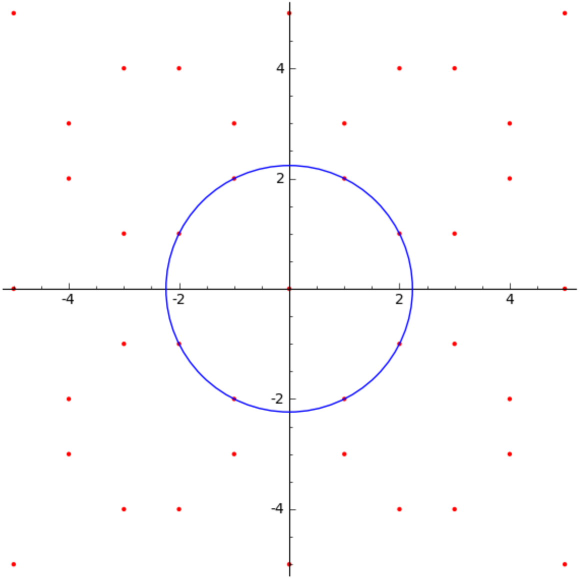

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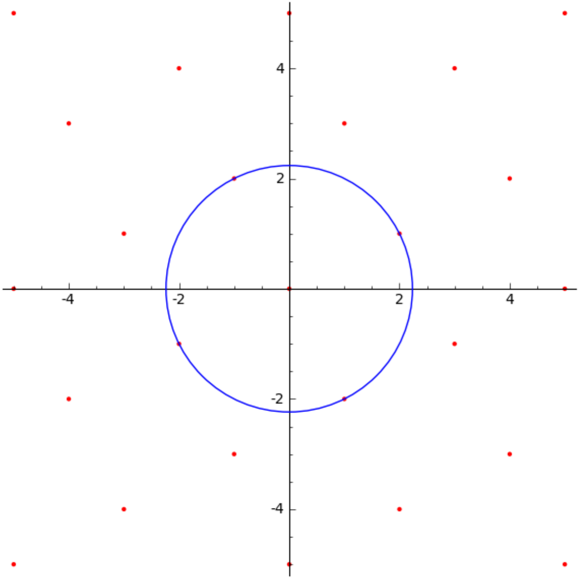

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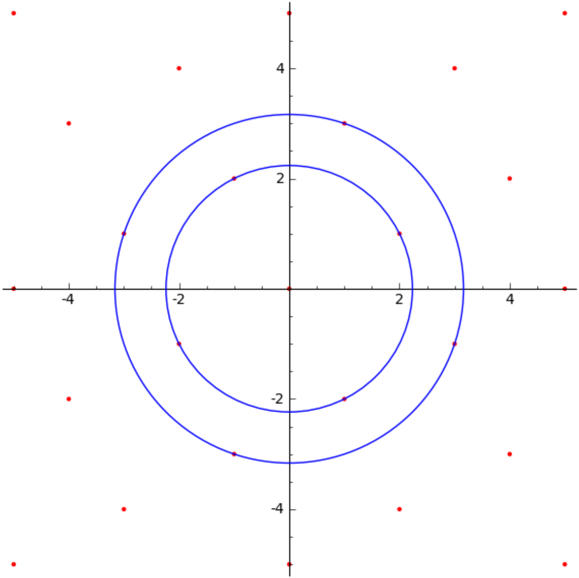

Minkowski's first theorem

We say a set B in $\mathbb{R}^n$ is symmetric if $x \in B \implies -x \in B$.
We say a set B is $\mathbb{R}^n$ is convex if
$x, y \in B \implies x + \lambda(y - x) \in B$ for $0 \leq \lambda \leq 1$.

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Theorem (Minkowski)

If B is a convex symmetric body and Λ a lattice in $\mathbb{R}^n$ then B contains a non-zero point of the lattice if:

$$\mathrm{Vol}(B) > 2^d i$$

Where i is the area of a single cell of the lattice Λ. We can find it using determinants, or the order of our lattice as a subgroup of $\mathbb{Z}^n$ for the more algebraically inclined.

Let's apply it

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The area of our lattice cell is p.
The area of our circle is $\pi 2p$
So Minkowski tells us that as:

$$\pi 2p = \mathrm{Vol}(B) > 2^d i = 2^2 p = 4p$$

We have a point which satisfies $x^2 + y^2 = kp$ for some $k \geq 1$, and also $x^2 + y^2 < 2p$. So $x^2 + y^2 = p$ and we are done.

Rational points on surfaces

Euler looked at

$$A^4 + B^4 = C^4 + D^4$$

with $A, B, C, D \in \mathbb{Q}$
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One parametrisation of solutions[1]:

$$\begin{aligned}
a(s,t) &= s^7 + s^5t^2 - 2s^3t^4 + 3s^2t^5 + st^6 \\
b(s,t) &= s^6t - 3s^5t^2 - 2s^4t^3 + s^2t^5 + t^7 \\
c(s,t) &= s^7 + s^5t^2 - 2s^3t^4 - 3s^2t^5 + st^6 \\
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Unfortunately this does not give us all solutions.

Counting solutions

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A: ∞
Can we find a better way of counting them?
We want to estimate the density of our solutions.

Heights

We want a different way of describing the size of a rational number.
Take

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Hurrah! There are only finitely many points with height less than a given value.

Counting points

We now want to count points in a set X with bounded height. We do this in a simple way and define:

$$N(X, B) := \#\{x \in X | H(x) \leq B\}$$

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We can analyse the growth of this function as B grows. Manin and others have conjectured that the growth of $N(X, B)$ is asymptotically governed by geometric properties of the surface for many problems[2].

Morals

Geometry can help us show solutions exist to Diophantine equations.
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Geometry can help us show solutions exist to Diophantine equations.
Geometric properties govern the density of the solutions to some problems.
And there is a lot more interplay between these two areas.

References

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