

Chapter 1

Abelian Varieties

1.1 Introduction (Angus)

1.1.1 Definitions

Definition 1.1.1 (Abelian varieties). An **abelian variety** is a **complete** connected **algebraic group**.

Definition 1.1.2 (Algebraic groups). An **algebraic group** is an algebraic variety G along with regular maps $m: G \times G \rightarrow G$, $e: * \rightarrow G$, $\text{inv}: G \rightarrow G$ such that the following diagrams commute.

$$\begin{array}{ccccc} * \times G & \xrightarrow{e \times \text{id}} & G \times G & \xleftarrow{\text{id} \times e} & G \times * \text{ identity} \\ & \searrow \sim & \downarrow m & \swarrow \sim & \\ & & G & & \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{\text{inv}, \text{id}} & G \times G & \xleftarrow{\text{id}, \text{inv}} & G \text{ Inverse} \\ \downarrow & & \downarrow m & & \downarrow \\ * & \xrightarrow{e} & G & \xleftarrow{e} & * \end{array}$$

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\text{id} \times m} & G \times G \text{ Associativity} \\ m \times \text{id} \downarrow & & \downarrow m \\ G \times G & \xrightarrow{m} & G \end{array}$$

Definition 1.1.3 (Complete varieties). A variety X is **complete** if every projection map

$$X \times Y \rightarrow Y$$

is closed.

Example 1.1.4.

- Elliptic curves.
- Weil restriction $\text{Res}_{K/\mathbb{Q}} E$ of an elliptic curve E .
- Jacobian varieties of curves.

Plan:

- Some motivation via elliptic curves.
- Gathering some material about “completeness”.
- Prove that [abelian varieties](#) are abelian.

1.1.2 Elliptic curves ($\text{char}(k) \neq 2, 3$)

Theorem 1.1.5. TFAE for a projective curve E over k .

1. E is given by $Y^2Z = X^3 + aXZ^2 + bZ^3$, $4a^3 + 27b^2 \neq 0$.
2. E is nonsingular of genus 1 with a distinguished point P_0 .
3. E is nonsingular with an [algebraic group](#) structure.
4. (if $k \subseteq \mathbb{C}$) such that $E(\mathbb{C}) = \mathbb{C}/\Lambda$ for some lattice $\Lambda \subseteq \mathbb{C}$.

Proof. Strategy: [Item 1](#) \iff [Item 2](#) \iff [Item 3](#) and [Item 2](#) \implies [Item 4](#) \implies [Item 1](#).

[Item 1](#) \implies [Item 2](#) is done.

[Item 2](#) \implies [Item 1](#): Riemann-Roch states that $l(D) = l(K-D) + \deg(D) + 1 - g$ so here $l(D) = l(K-D) + \deg(D)$ further is $D > 0$ then $l(K-D) = 0$ in which case $l(D) = \deg(D)$. Consider $L(nP_0)$ for $n > 0$ Riemann-Roch implies that $l(nP_0) = n$ then it always contains the constants.

$$L(P_0) = k$$

$$L(2P_0) = k \oplus kx$$

$$L(3P_0) = k \oplus kx \oplus ky$$

$$\vdots$$

$$L(6P_0) = k \oplus kx \oplus ky \oplus kx^2 \oplus ky^2 \oplus kxy \oplus kx^3 / \sim$$

so we must have a relation which after manipulation is of the desired form. We get an embedding

$$E \hookrightarrow \mathbb{P}^2$$

$$P \mapsto (x(P) : y(P) : 1) (P \neq P_0)$$

$$P_0 \mapsto (0 : 1 : 0)$$

and thus E is of the desired form. \square

Definition 1.1.6 (Elliptic curves). An **elliptic curve** over k is any/all of [that 5](#).

Which of the above characterisations generalise to abelian varieties?

1. No, in general we don't know that the equations look like.
2. One could possibly replace “genus” with a condition on the dimension of cohomology groups.
3. Yes, this is essentially the definition.
4. Yes, stay tuned!

1.1.3 Complete varieties

Idea: if $X \times Y$ had product topology (instead of its Zariski topology) then **complete** is equivalent to compact.

We'd like to gather a few results about **complete** varieties we can use to access properties of **abelian varieties** (like abelianness).

Proposition 1.1.7. *Let V be a **complete** variety. Given any morphism $\phi: V \rightarrow W$ $\phi(V)$ is closed.*

Proof. Let $\Gamma_\phi = \{(v, \phi(v))\} \subseteq V \times W$ be the graph of ϕ . It's a closed subvariety of $V \times W$. Under the projection $V \times W \rightarrow W$, the image of Γ_ϕ is $\phi(V)$ and thus closed. \square

Corollary 1.1.8. *If V is **complete** and connected, any regular function on V is constant.*

Proof. A regular function is a morphism $f: V \rightarrow \mathbf{A}^1$. By the above $f(V) \subseteq \mathbf{A}^1$ is closed, and this is a finite set of points. But connected implies we just have one point. \square

Corollary 1.1.9. *Let V be a **complete** connected variety. Let W be an affine variety. Given $\phi: V \rightarrow W$, then $\phi(V)$ is a point.*

Proof. We have an embedding $W \hookrightarrow \mathbf{A}^n$. On \mathbf{A}^n we have the coordinate functions $\mathbf{A}^n \xrightarrow{x_i} \mathbf{A}^1$. The composition

$$V \xrightarrow{\phi} W \hookrightarrow \mathbf{A}^n \rightarrow \mathbf{A}^1$$

be the above is constant. Thus the coordinates of $\phi(V)$ are constant, so $\phi(V) = \{\text{pt}\}$. \square

A final result of interest that I won't prove today:

Theorem 1.1.10. *Projective varieties are **complete**.*

The main goal of this section is to prove the following theorem:

Theorem 1.1.11 (Rigidity). *Let V, W be varieties such that V is **complete** and $V \times W$ is geometrically irreducible. Let $\alpha: V \times W \rightarrow U$ be a morphism such that $\exists u_0 \in U(k), v_0 \in V(k), w_0 \in W(k)$ with $\alpha(V \times \{w_0\}) = \alpha(\{v_0\} \times W) = \{u_0\}$. Then $\alpha(V \times W) = \{u_0\}$.*

Proof. Since $V \times W$ is geometrically irreducible, V must be connected. Denote the projection $q: V \times W \rightarrow W$. Let $U_0 \ni x_0$ be an open neighborhood. We consider the set

$$Z = \{w \in W : \alpha((v, w)) \notin U_0 \text{ for some } v \in V\} = q(\alpha^{-1}(U \setminus U_0))$$

Since q is closed, $Z \subseteq W$ is closed. Since $w_0 \in W \setminus Z$, $W \setminus Z$ is a nonempty open subset of W .

Consider $w \in W \setminus Z$. Since $V \times \{w\} \cong V$ it is **complete** and connected. Thus

$$\alpha(V \times \{w\}) = \{\text{pt}\} = \alpha((v_0, w)) = \{u_0\}$$

which implies that

$$\alpha(V \times (W \setminus Z)) = \{u_0\}$$

Since $V \times (W \setminus Z) \subseteq V \times W$ is open and $V \times W$ is irreducible, it is dense. So $\alpha(V \times W) = \{u_0\}$. \square

Proposition 1.1.12. Let A, B be *abelian varieties*. Every morphism $\alpha: A \rightarrow B$ is the composition of a homomorphism and a translation.

Proof. First compose by a translation on B such that $\alpha(0) = 0$. Consider the map

$$\begin{aligned}\phi: A \times A &\rightarrow B \\ (a, a') &\mapsto \alpha(a + a') - \alpha(a) - \alpha(a')\end{aligned}$$

Then

$$\begin{aligned}\phi(A \times \{0\}) &= \alpha(a + 0) - \alpha(a) - \alpha(0) = 0 \\ \phi(\{0\} \times A) &= \alpha(0 + a) - \alpha(0) - \alpha(a) = 0.\end{aligned}$$

By the *rigidity theorem 11* $\phi(A \times A) = \{0\}$ hence $\alpha(a + a') = \alpha(a) + \alpha(a')$. \square

Corollary 1.1.13. *Abelian varieties are abelian.*

Proof. The inversion map $a \mapsto -a$ sends 0 to 0, thus is a homomorphism. Therefore

$$a + b - a - b = a + b - (a + b) = 0$$

and so

$$a + b = b + a. \quad \square$$

1.2 Abelian varieties over \mathbb{C} (Alex)

The goal of this talk is to understand what *abelian varieties* look like over \mathbb{C} . The goal for me is to understand what a (principal) polarisation is and why it is important.

First immediate question: why study complex theory at all? The most classical field, algebraically closed, archimidean, characteristic 0.

Recall/rapidly learn the picture for elliptic curves, given E an elliptic curve we have for some Λ a rank 2 lattice in \mathbb{C}

$$\begin{aligned}\mathbb{C}/\Lambda &\xrightarrow{\sim} E(\mathbb{C}) \subseteq \mathbb{P}^2(\mathbb{C}) \\ z &\mapsto (\wp(z) : \wp'(z) : 1) \\ 0 &\mapsto (0 : 1 : 0)\end{aligned}$$

where

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right).$$

This is a meromorphic function whose image lands in

$$y^2 = 4x^3 - g_2x - g_3.$$

So the \mathbb{C} points of an elliptic curve are topologically a torus.

Naturally one asks: does this generalise? Let A be an *abelian variety* over \mathbb{C} , what does $A(\mathbb{C})$ look like? Another torus?

Proposition 1.2.1. $A(\mathbb{C})$ is a compact, connected, complex lie group.

Proposition 1.2.2. Let A be an *abelian variety* of dimension g over \mathbb{C} . Then we have

$$A(\mathbb{C}) \cong V/\Lambda$$

where V is a g dimensional complex vector space and Λ is a full rank lattice of V (i.e Λ is a discrete subgroup of V s.t. $\mathbb{R} \otimes \Lambda = V$).

Proof. Differential geometry gives us a map of complex manifolds

$$\exp: \mathrm{Tgt}_0(A(\mathbf{C})) \rightarrow A(\mathbf{C})$$

this is a holomorphism. And since $A(\mathbf{C})$ is abelian, this is a homomorphism also. In general this is locally an isomorphism around 0.

Claim: \exp is injective. There exists a neighborhood $U \ni 0$ s.t. $\exp(U) \cong U$. Consider the image $\exp(\mathrm{Tgt}_0 A(\mathbf{C}))$. For $x \in \exp(\mathrm{Tgt}_0 A(\mathbf{C}))$, $\{U + x\}$ are all open and give a cover. Thus $\exp(\mathrm{Tgt}_0 A(\mathbf{C}))$ is open. Since $A(\mathbf{C})$ is connected we are thus reduced to showing $\exp(\mathrm{Tgt}_0 A(\mathbf{C}))$ is closed also. Since \exp is a homomorphism, the image is a subgroup. So its complement is the union of its non-trivial cosets, which is open. Thus $\exp(\mathrm{Tgt}_0 A(\mathbf{C}))$ is closed. Giving $\exp(\mathrm{Tgt}_0 A(\mathbf{C})) = A(\mathbf{C})$, which proves the claim.

\exp is a local isomorphism, which gives that $\ker(\exp)$ is discrete, i.e. a lattice. We now have

$$A(\mathbf{C}) \cong \mathrm{Tgt}_0 A(\mathbf{C}) / \ker(\exp)$$

so as $A(\mathbf{C})$ is compact we cannot have a kernel which is not full rank, as otherwise the quotient could not be compact. \square

Definition 1.2.3. We call any such V/Λ a **complex torus**.

From the above isomorphism we can now read off properties of $A(\mathbf{C})$ as a group.

Proposition 1.2.4. $A(\mathbf{C})$ is divisible, and $A(\mathbf{C})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{2g}$.

Proof.

$$A(\mathbf{C}) \cong V/\Lambda \cong (\mathbf{R}/\mathbf{Z})^{2g}$$

isomorphisms as groups, thus $A(\mathbf{C})$ is divisible. Further, $(\mathbf{R}/\mathbf{Z})[n] = (\frac{1}{n}\mathbf{Z})/\mathbf{Z}$. \square

Question Given a **complex torus** V/Λ , does there exist an **abelian variety** A such that $A(\mathbf{C}) \cong V/\Lambda$?

Example 1.2.5.

•

$$\mathbf{C}/\Lambda \cong E(\mathbf{C}) \text{ always in dim 1}$$

•

$$\mathbf{C}^2/\Lambda^2 \cong (E \times E)(\mathbf{C}) \text{ sometimes yes in higher dimension}$$

•

$$\mathbf{C}^2 / \langle (i, 0), (i\sqrt{p}, i), (1, 0), (0, 1) \rangle_{\mathbf{Z}}$$

for p prime??? (I guess not, see Mumford)

Theorem 1.2.6 (Chow). *If X is an analytic submanifold of $\mathbf{P}^m(\mathbf{C})$ then X is an algebraic subvariety.*

By this theorem it is enough to analytically imbed $V/\Lambda \hookrightarrow \mathbf{P}^m$. We can try and do this by mimicing the elliptic curve strategy, find enough functions $\theta: V/\Lambda \rightarrow \mathbf{C}$.

Proposition 1.2.7. *Let $X = V/\Lambda$. Then*

$$H^r(X, \mathbf{Z}) \cong \{\text{alternating } r\text{-forms } \Lambda \times \cdots \times \Lambda \rightarrow \mathbf{Z}\}.$$

Proof. $\pi: V \rightarrow V/\Lambda$ is a universal covering map, so

$$\Lambda = \pi^{-1}(0) \cong \pi_1(X, 0).$$

Because all these spaces are nice

$$H^1(X, \mathbf{Z}) \cong \text{Hom}(\pi_1(X), \mathbf{Z}) \cong \text{Hom}(\Lambda, \mathbf{Z}).$$

To extend to $r \neq 1$ use the Künneth formula:

$$\begin{array}{ccc} \wedge^r(H^1(X_1 \times X_2, \mathbf{Z})) & \xlongequal{\quad\quad\quad} & H^r(X_1 \times X_2, \mathbf{Z}) \\ \parallel \text{Künneth} & & \parallel \text{Künneth} \\ \wedge^r(H^1(X_1, \mathbf{Z}) \otimes H^1(X_2, \mathbf{Z})) & & \\ \parallel & & \\ \bigoplus_{p+q=r} (\wedge^p(H^1(X_1, \mathbf{Z})) \otimes \wedge^q(H^1(X_2, \mathbf{Z}))) & \xlongequal{\quad\quad\quad} & \bigoplus_{p+q=r} (H^p(X_1, \mathbf{Z}) \otimes H^q(X_2, \mathbf{Z})) \end{array}$$

Since we know the proposition for $S^1 = \mathbf{R}/\mathbf{Z}$ by taking products and applying the above we get it for all complex tori V/Λ . \square

Proposition 1.2.8. *There is a correspondence*

$$\begin{aligned} \{\text{Hermitian forms } H \text{ on } V\} &\leftrightarrow \{\text{Alternating forms } E: V \times V \rightarrow \mathbf{R}, E(iu, iv) = E(u, v)\} \\ H &\mapsto \text{im } H \\ E(iu, v) + iE(u, v) &\leftarrow E. \end{aligned}$$

Now we will consider **line bundles** on $X = V/\Lambda$, that is

$$L \xrightarrow{\pi} X$$

such that for any $x \in X$ there exists $U \ni x$ with $\pi^{-1}(U) \cong \mathbf{C} \times U$. We can obtain these from hermitian forms and some auxilliary data as follows.

Definition 1.2.9. If H is a hermitian form on V such that $E(\Lambda \times \Lambda) \subseteq \mathbf{Z}$ there exists a map

$$\alpha: \Lambda \rightarrow \mathbf{C}^* = \{z \in \mathbf{C}^* : |z| = 1\}$$

such that

$$\alpha(u + v) = e^{i\pi E(u, v)} \alpha(u) \alpha(v).$$

Further, there is a **line bundle** $L(H, \alpha)$ on X which is defined by quotienting $\mathbf{C} \times V$ by Λ which acts via

$$\phi_u(\lambda, v) = (\alpha(u) e^{\pi H(v, u) + \frac{1}{2} \pi H(u, u)} \lambda, v + u) \text{ for } u \in \Lambda,$$

we'll denote by e_u the factor $\alpha(u) e^{\pi H(v, u) + \frac{1}{2} \pi H(u, u)}$ for brevity.

Theorem 1.2.10 (Appell-Humbert). *Any **line bundle** on X is of the form $L(H, \alpha)$ for some H, α as above. Further*

$$L(H_1, \alpha_1) \otimes L(H_2, \alpha_2) = L(H_1 + H_2, \alpha_1 \alpha_2).$$

In fact we have the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(\Lambda, \mathbf{C}) & \longrightarrow & \{\text{data } (H, \alpha)\} & \longrightarrow & \{\text{gp. of Herm. } H \text{ w/ } E(\Lambda \times \Lambda) \subseteq \mathbf{Z}\} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Pic}^0(X) & \longrightarrow & \text{Pic}(X) & \longrightarrow & \ker(H^2(X, \mathbf{Z}) \rightarrow H^2(X, \mathcal{O}_X)) \longrightarrow 0 \end{array}$$

where $\text{Pic}(X)$ is the group of all **line bundles** on X and Pic^0 is the subgroup of those which are topologically trivial.

We wanted functions $X \rightarrow \mathbf{C}$. Now we can instead consider sections s of $L(H, \alpha) \xrightarrow{\pi} X$ i.e. maps $s: X \rightarrow L(H, \alpha)$ with $\pi \circ s = \text{id}$. Denote the space of such sections $H^0(X, L(H, \alpha))$.

Definition 1.2.11 (Theta functions). The sections of $L(H, \alpha)$ correspond to holomorphic functions

$$\theta: V \rightarrow \mathbf{C}$$

such that $\theta(z + u) = e_u \theta(z)$, we will call such a θ a **theta function** for (H, α) .

If H is not positive definite the space of such functions is 0!

Proposition 1.2.12. *If H is positive definite, then the dimension of $H^0(X, L(H, \alpha))$ is $\sqrt{\det E}$ where we really mean the determinant of a matrix for E with respect to an integral basis.*

Theorem 1.2.13 (Lefschetz). *Given a positive definite H , there exists an imbedding $X \hookrightarrow \mathbf{P}^m$.*

Proof. Sketch: Let $L = L(H, \alpha)$, consider $L(H, \alpha)^{\otimes 3} = L(3H, \alpha^3)$, take a basis of $\theta_0, \dots, \theta_d$ of $H^0(X, L^{\otimes 3})$.

Claim: $\Theta: z \mapsto (\theta_0(z) : \dots : \theta_d(z)) \subseteq \mathbf{P}^d$ is an embedding.

To see that this is well defined, we must give a section of $L^{\otimes 3}$ not vanishing at z for all $z \in X$. Let $\theta \in H^0(X, L) \setminus \{0\}$. Then pick a, b such that the section of $L^{\otimes 3}$ given by

$$\theta(z - a)\theta(z - b)\theta(z + a + b)$$

does not vanish. This is possible and thus we have a nonvanishing section of $L^{\otimes 3}$.

For injectivity, show that if the above section has the same values on z_1, z_2 then it is a **theta function** for some sublattice. Almost all sections aren't **theta functions** for a sublattice (this uses [Proposition 12](#)).

Something similar must be done for tangent vectors. \square

Definition 1.2.14 (Riemann forms). A **Riemann form** is $E: \Lambda \times \Lambda \rightarrow \mathbf{Z}$ alternating such that

$$E_{\mathbf{R}}: V \times V \rightarrow \mathbf{R}$$

has the property that $E(iu, iv) = E(u, v)$ and the corresponding Hermitian form is positive definite.

Definition 1.2.15 (Polarizable tori). A **complex torus** $X = V/\Lambda$ is **polarizable** if there exists a **Riemann form** E on Λ .

Example 1.2.16 (Proposition). Every \mathbf{C}/Λ where $\Lambda = \langle 1, \tau \rangle_{\mathbf{Z}}$ is **polarizable**.

To see this take

$$E(u, v) = \frac{uv}{\text{im } \tau}$$

as a **Riemann form**.

Putting everything together we have obtained an equivalence of categories

$$\{\text{abelian varieties over } \mathbf{C}\} \leftrightarrow \{\text{polarizable complex tori}\}.$$

Definition 1.2.17 (Isogenies of complex tori). An **isogeny** of complex tori is a homomorphism $V/\Lambda \rightarrow V'/\Lambda'$ with finite kernel.

Definition 1.2.18 (Dual vector spaces). Given V a complex vector space, let

$$V^* = \{f: V \rightarrow \mathbf{C} : f(u+v) = f(u) + f(v), f(\alpha v) = \bar{\alpha}f(v)\}$$

and given $\Lambda \subset V$ a lattice, let

$$\Lambda^* = \{f \in V^* : f(\lambda) \in \mathbf{Z} \forall \lambda \in \Lambda\}.$$

Definition 1.2.19 (Dual tori). If $X = V/\Lambda$, $X^\vee = V^*/\Lambda^*$ is the **dual torus**.

Proposition 1.2.20 (Existence of Weil pairing).

$$X \times X^\vee \rightarrow \mathbf{C}$$

so

$$X[n] \times X^\vee[n] \rightarrow (\frac{1}{n^2}/\frac{1}{n}\mathbf{Z}) \cong \mathbf{Z}/n\mathbf{Z}$$

this is called the **Weil pairing**.

Can a **complex torus** be isogenous to its own dual? If X is **polarizable** then

$$\begin{aligned} X &\rightarrow X^\vee \\ v &\mapsto H(v, -) \end{aligned}$$

is an **isogeny**.

Definition 1.2.21. A **polarization** is an **isogeny** $X \rightarrow X^\vee$.

1.3 Rational Maps into Abelian Varieties (Maria)

Note all varieties are irreducible today.

1.3.1 Rational maps

V, W varieties $/K$. Consider pairs (U, ϕ_U) , where $\emptyset \neq U \subset V$ an open subset so U is dense, and $\phi_U: U \rightarrow W$ is a regular map.

Definition 1.3.1. $(U, \phi_U), (U', \phi_{U'})$ are equivalent if ϕ_U and $\phi_{U'}$ agree on $U \cap U'$. An equivalence class ϕ of $\{(U, \phi_U)\}$ is a **rational map** $\phi: V \dashrightarrow W$. If $\phi: V \dashrightarrow W$ is defined at $v \in V$ if $v \in U$ for some $(U, \phi_U) \in \phi$.

Note 1.3.2. The set $U_1 = \bigcup U$ where ϕ is defined is open and $(U_1, \phi_1) \in \phi$ where $\phi_1: U_1 \rightarrow W$ restricts to ϕ_U on U .

Example 1.3.3.

1. Let $\emptyset \neq W \subseteq V$ be open. Then the **rational map** $V \dashrightarrow W$ induced by $\text{id}: W \rightarrow W$ will not extend to V . To avoid this, assume W is **complete** (so $W = V$).
2. $C: y^2 = x^3$, then $\alpha: \mathbf{A}^1 \rightarrow C, a \mapsto (a^2, a^3)$ is a regular map, restricting to an isomorphism $\mathbf{A}^1 \setminus \{0\} \rightarrow C \setminus \{0\}$. The inverse of $\alpha|_{\mathbf{A}^1 \setminus \{0\}}$ represents $\beta: C \dashrightarrow \mathbf{A}^1$ which does not extend to C . This corresponds on function fields to

$$K(t) \rightarrow K(x, y)$$

$$t \mapsto y/x$$

which does not send $K[y]_{(t)}$ to $K[x, y]_{(x, y)}$.

3. Given a nonsingular surface V , $P \in V$ then $\exists \alpha: W \rightarrow V$ regular that induces an isomorphism $\alpha: W \setminus \alpha^{-1}(P) \rightarrow V \setminus P$, but $\alpha^{-1}(P)$ is a projective line. The **rational map** represented by α^{-1} is not regular on V (where to send P ?).

Theorem 1.3.4 (Milne 3.1). A **rational map** $\phi: V \dashrightarrow W$ from a nonsingular variety V to a **complete** variety W is defined on an open subset $U \subseteq V$ whose complement has codimension ≥ 2 .

Proof. (V a curve) V nonsingular curve, $\emptyset \neq U \subseteq V$ open, $\phi: U \rightarrow W$ a regular map.

$$\begin{array}{ccc}
 & & V \\
 & \nearrow & \uparrow p \\
 U & \longrightarrow & U' \subseteq Z \subseteq V \times W \ni (v, w) \\
 & \searrow & \downarrow q \\
 & & W \ni w
 \end{array}$$

U' is the image of U , $Z = \overline{U'}$. W is **complete**, Z closed implies $p(Z) \subseteq V$ is closed. Also, $U \subseteq p(Z) \implies p(Z) = V$.

$$U \xrightarrow{\sim} U' \rightarrow U$$

so

$$U' \xrightarrow{\sim} U$$

$$Z \twoheadrightarrow V$$

this implies $Z \xrightarrow{\sim} V$. Then $q|_Z: Z \rightarrow W$ is the extension of ϕ to V . \square

Theorem 1.3.5 (Milne 3.2). A **rational map** $\phi: V \dashrightarrow A$ from a nonsingular variety V to an **abelian variety** W , extends to all of V .

Proof. **Theorem 4 Lemma 6** \square

Lemma 1.3.6. Let $\phi: V \dashrightarrow G$ be a map from a nonsingular variety to a group variety. Then either ϕ is defined on all of V or the set where ϕ is not defined is closed of pure codimension 1.

Proof. Fix $(U, \phi_U) \in \phi$ and consider

$$\Phi: V \times V \dashrightarrow G$$

represented by

$$\begin{aligned}
 U \times U &\xrightarrow{\phi_U \times \phi_U} G \times G \xrightarrow{\text{id} \times \text{inv}} G \times G \xrightarrow{m} G \\
 (x, y) &\mapsto \phi_U(x) \phi_U(y)^{-1}
 \end{aligned}$$

Check ϕ is defined at x iff Φ is defined at (x, x) (and in this case $\Phi(x, x) = e$). This is equivalent to the map $\Phi^*: \mathcal{O}_{G,e} \rightarrow K(V \times V)$ induced by Φ satisfying $\text{im}(\mathcal{O}_{G,e}) \subseteq \mathcal{O}_{V \times V, (x,x)}$. For a nonzero function f on $V \times V$, write $\text{div}(f) = \text{div}(f)_0 - \text{div}(f)_\infty$ which are effective divisors. Then

$$\mathcal{O}_{V \times V, (x,x)} = \{0\} \cup \{f \in K(V \times V) : \text{div}(f)_\infty \text{ does not contain } (x, x)\}.$$

Suppose ϕ is not defined at x , then there exists $f \in \text{im}(\mathcal{O}_{G,e})$ s.t. $(x, x) \in \text{div}(f)_\infty$. Then Φ is not defined at any $(y, y) \in \Delta \cap \text{div}(f)_\infty = \text{div}(f^{-1})_0$, which is a pure codimension 1 subset of Δ by Milne's AG thm 9.2. The corresponding subset in V is of pure codimension 1, and ϕ is not defined there. \square

Theorem 1.3.7 (Milne 3.4). *Let $\alpha: V \times W \rightarrow A$ be a morphism from a product of nonsingular varieties into an **abelian variety**. If $\alpha(V \times \{w_0\}) = \{a_0\} = \alpha(\{v_0\} \times W)$ for some $a_0 \in A$, $v_0 \in V$, $w_0 \in W$, then $\alpha(V \times W) = \{a_0\}$.*

Corollary 1.3.8 (Milne 3.7). *Every **rational map** $\alpha: G \dashrightarrow A$ from a group variety into an **abelian variety** is the composition of a homomorphism and a translation in A .*

Proof. Since group varieties are nonsingular, $\alpha: G \rightarrow A$ is a regular map by **Theorem 5**. The rest is as proof of Corollary 1.2. \square

1.3.2 Dominating and birational maps

Definition 1.3.9 (Dominating maps). $\phi: V \dashrightarrow W$ is **dominating** if $\text{im}(\phi_U)$ is dense in W for a representative $(U, \phi_U) \in \phi$.

Exercise: A **dominating** $\phi: V \dashrightarrow W$ defines a homomorphism $K(W) \rightarrow K(V)$ and any such homomorphism arises from a unique **dominating rational map**.

Definition 1.3.10. $\phi: V \dashrightarrow W$ is **birational** if the corresponding $K(W) \rightarrow K(V)$ is an isomorphism or, equivalently if there exists $\psi: W \dashrightarrow V$ s.t. $\phi \circ \psi$ and $\psi \circ \phi$ are the identity wherever they are defined. In this case we say V and W are **birationally equivalent**.

Note 1.3.11. In general birational equivalence does not imply isomorphic. E.g. V a variety $\emptyset \neq W \subsetneq V$ an open subset, or $V = \mathbf{A}^1, W: y^2 = x^3$.

Theorem 1.3.12 (Milne 3.8). *If two **abelian varieties** are **birationally equivalent** then they are isomorphic as **abelian varieties**.*

Proof. A, B **abelian varieties** with $\phi: A \dashrightarrow B$ a **birational** map with inverse ψ . Then by **Theorem 5** ϕ, ψ extend to regular maps $\phi: A \rightarrow B, \psi: B \rightarrow A$ and $\phi \circ \psi, \psi \circ \phi$ are the identity everywhere. This implies that ϕ is an isomorphism of algebraic varieties and after composition with a translation, ϕ is also a group isomorphism. \square

Proposition 1.3.13 (Milne 3.9). *Any **rational map** $\mathbf{A}^1 \dashrightarrow A$ or $\mathbf{P}^1 \dashrightarrow A$, for A an **abelian variety** is constant.*

Proof. **Theorem 5** implies $\alpha: \mathbf{A}^1 \dashrightarrow A$ extends to $\alpha: \mathbf{A}^1 \rightarrow A$ and we may assume $\alpha(0) = e$. $(\mathbf{A}^1, +): \alpha(x + y) = \alpha(x) + \alpha(y)$ for all $x, y \in \mathbf{A}^1(K) = K$. $(\mathbf{A}^1 \setminus \{0\}, \cdot): \alpha(xy) = \alpha(x) + \alpha(y) + c$ for all $x, y \in K^\times$. These can only hold at the same time if α is constant. $\mathbf{P}^1 \dashrightarrow A$ is constant, since its constant on affine patches. \square

Definition 1.3.14. V/\bar{K} is **unirational** if there is a **dominating** map $\mathbf{A}^n \dashrightarrow V$, where $n = \dim_{\bar{K}} V$. V/K is **unirational** if V/\bar{K} is.

Proposition 1.3.15 (Milne 3.10). *Every **rational map** $V \dashrightarrow A$ from V **unirational** to A **abelian** is constant.*

Proof. Wlog $K = \bar{K}$. Since V is **unirational** we get $\beta: \mathbf{P}^1 \times \cdots \times \mathbf{P}^1 \dashrightarrow V \dashrightarrow A$, which extends to $\beta: \mathbf{P}^1 \times \cdots \times \mathbf{P}^1 \rightarrow A$. Then by Milne corollary 1.5, there exist regular maps $\beta_i: \mathbf{P}^1 \rightarrow A$ s.t. $\beta(x_1, \dots, x_n) = \sum \beta_i(x_i)$ and by **Proposition 13** each β_i map is constant. \square

1.4 Theorem of the Cube (Ricky)

1.4.1 Crash Course in Line Bundles

Consider \mathbf{R}^2 , $f: \mathbf{R} \rightarrow \mathbf{R}$, $f(x, y) = x^2 + y^2 - 1$, now $S = \{f = 0\} \subseteq \mathbf{R}^2$ is a closed submanifold (in fact a circle). Question: Do all closed submanifolds arise in this way? Lets switch to \mathbf{C} better analogies with AG.

Example 1.4.1. Let $X \in \mathbf{P}^n(\mathbf{C})$, the answer here is no! (Because $f: X \rightarrow \mathbf{C}^1$ is constant!) Want to define functions locally that give us level sets, but gluing such will give us a global section. Instead glue in a different way (i.e. into different “copies” of \mathbf{C}) so that this doesn’t happen.

Example 1.4.2. $X \in \mathbf{P}_{\mathbf{C}}^1$, \mathcal{O}_X the structure sheaf.

$$X = U_0 \cup U_1 = (\mathbf{A}^1, t) \cup (\mathbf{A}^1, s)$$

on $U_0 \cap U_1$, $t = s^{-1}$. What is a global section of \mathcal{O}_X , a section of U_0 and a section of U_1 that glue. $\mathcal{O}_X(U_0) = k[t]$, $\mathcal{O}_X(U_1) = k[s]$ so given $f(t)$, $g(s)$ these glue to a global section iff $f(t) = g(1/t)$ so f, g must be constant.

Definition 1.4.3 (Line bundles). A **line bundle** on X is a locally free \mathcal{O}_X -module of rank 1, i.e. $\exists \{U_i\}$ open cover along with isomorphisms $\phi_i: \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_X|_{U_i}$.

Exercise 1.4.4. Alternative definition: A **line bundle** on X is equivalent to the following data:

- An open cover of X .
- Transition maps $\tau_{ij} \in \text{GL}_1(\mathcal{O}_X(U_i \cap U_j))$ satisfying $\tau_{ij}\tau_{jk} = \tau_{ik}$ and $\tau_{ii} = \text{id}$.

Example 1.4.5. On $X = \mathbf{P}_k^n$, we have **line bundles** $\mathcal{O}(d)$ for all $d \in \mathbf{Z}$. Just have to give cover and transition functions, use usual open cover $\{U_i\}$ with $U_i \cong \mathbf{A}^n$. Then τ_{ji} is given by multiplication by $(x_i/x_j)^d$.

Exercise 1.4.6.

$$H^0(X, \mathcal{O}(d)) (= \Gamma(X, \mathcal{O}(d)))$$

= k vector space spanned by deg. d homogenous polynomials in $k[x_0, \dots, x_n]$.

Exercise 1.4.7. All **line bundles** on \mathbf{P}^n are isomorphic to some $\mathcal{O}(d)$.

We say a **line bundle** \mathcal{L} on X is trivial if $\mathcal{L} \cong \mathcal{O}_X$. Given \mathcal{L}_1 and \mathcal{L}_2 on X (line bundles) we can create a new **line bundle** $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$. So isomorphism classes of **line bundles** on X with \otimes form a group, denoted $\text{Pic}(X)$ with identity \mathcal{O}_X and inverses $\mathcal{L}^{-1} = \text{Hom}(\mathcal{L}, \mathcal{O}_X)$.

Example 1.4.8. By previous exercise $\text{Pic}(\mathbf{P}_k^n) \cong \mathbf{Z}$ since $\mathcal{O}_X(d_1) \otimes \mathcal{O}_X(d_2) \cong \mathcal{O}_X(d_1 + d_2)$.

Fact 1.4.9. If $f: X \rightarrow Y$, then given \mathcal{L} on Y we can pullback to a **line bundle** $f^* \mathcal{L}$ on X , definition is complicated. We also know that f^* commutes with \otimes so in fact (as $f^* \mathcal{O}_Y = \mathcal{O}_X$) we get a homomorphism $f^*: \text{Pic}(Y) \rightarrow \text{Pic}(X)$.

1.4.2 Relation to (Weil) divisors

Let X be a normal variety, call $Z \subseteq X$, a closed subvariety of codimension 1, a **prime divisor**. Then a divisor on X is a formal sum

$$D = \sum_{Z \subseteq X} n_Z \cdot Z$$

of **prime divisors**.

Let $K = K(X)$ be the function field of X . Given $f \in K^\times$ we can define

$$\operatorname{div}(f) = \sum v_Z(f) \cdot Z.$$

Given $D \in \operatorname{Div}(X)$, we can define a **line bundle** $\mathcal{L}(D)$ on X via

$$\mathcal{L}(D)(U) = \{f \in K^\times : (D + \operatorname{div}(f))|_U \geq 0\} \cup \{0\}$$

where $D|_U = \sum_{Z \cap U \neq \emptyset} n_Z \cdot (Z \cap U)$.

Proposition 1.4.10. *The map*

$$\operatorname{Cl}(X) = \operatorname{Div}(X)/\operatorname{Princ}(X) \xrightarrow{\mathcal{L}(\cdot)} \operatorname{Pic}(X)$$

is an isomorphism.

1.4.3 Onto cubes

Theorem 1.4.11 (Theorem of the cube). *Let U, V, W be **complete** varieties. If \mathcal{L} is a **line bundle** on $U \times V \times W$ s.t. $\mathcal{L}|_{\{u_0\} \times V \times W}, \mathcal{L}|_{U \times \{v_0\} \times W}, \mathcal{L}|_{U \times V \times \{w_0\}}$ are all trivial then \mathcal{L} is trivial.*

Corollary 1.4.12 (Milne 5.2). *Let A be an **abelian variety**. Let $p_i: A \times A \times A \rightarrow A$ be the projection onto the i th coordinate. $p_{ij} = p_i + p_j$, $p_{123} = p_1 + p_2 + p_3$. Then for any \mathcal{L} on A , the **line bundle***

$$\mathcal{M} = p_{123}^* \mathcal{L} \otimes p_{12}^* \mathcal{L}^{-1} \otimes p_{23}^* \mathcal{L}^{-1} \otimes p_{13}^* \mathcal{L}^{-1} \otimes p_1^* \mathcal{L} \otimes p_2^* \mathcal{L} \otimes p_3^* \mathcal{L}$$

is trivial.

Proof. Let $m: A \times A \rightarrow A$ be multiplication (addition?) and p, q the projections $A \times A \rightarrow A$. Then the composites of the maps $\phi: A \times A \rightarrow A \times A \times A$, $\phi(x, y) = (x, y, 0)$ with $p_{123}, p_{12}, p_{23}, p_{13}, p_1, p_2, p_3$ are respectively $m, m, q, p, p, q, 0$. Hence the restriction of \mathcal{M} to $A \times A \times \{0\}$ is

$$m^* \mathcal{L} \otimes m^* \mathcal{L}^{-1} \otimes q^* \mathcal{L}^{-1} \otimes p^* \mathcal{L}^{-1} \otimes p^* \mathcal{L} \otimes q^* \mathcal{L} \otimes O_{A \times A}$$

this is trivial by tensor commuting with pullback. Similarly \mathcal{M} restricts to a trivial bundle on $A \times \{0\} \times A$ and $\{0\} \times A \times A$. So by **theorem of the cube 11** \mathcal{M} is trivial. \square

Corollary 1.4.13 (Milne 5.3). *Let $f, g, h: V \rightarrow A$ (A abelian). Then for any \mathcal{L} on A the bundle*

$$\mathcal{M} = (f+g+h)^* \mathcal{L} \otimes (f+g)^* \mathcal{L}^{-1} \otimes (f+h)^* \mathcal{L}^{-1} \otimes (g+h)^* \mathcal{L}^{-1} \otimes f^* \mathcal{L} \otimes g^* \mathcal{L} \otimes h^* \mathcal{L}$$

is trivial.

Proof. \mathcal{M} is the pullback of the **line bundle** of **Corollary 12** via the map $(f, g, h): V \rightarrow A \times A \times A$. \square

On A we have $n_A: A \rightarrow A$ be $n_A(a) = a + \cdots + a$ (n times) for $n \in \mathbf{Z}$.

Corollary 1.4.14 (Milne 5.4). *For \mathcal{L} on A we have*

$$n_A^* \mathcal{L} \cong \mathcal{L}^{(n^2+n)/2} \otimes (-1)_A^* \mathcal{L}^{(n^2-n)/2}$$

In particular if $(-1)^ \mathcal{L} = \mathcal{L}$ (symmetric) then $n_A^* \mathcal{L} = \mathcal{L}^{n^2}$. And if $(-1)^* \mathcal{L} = \mathcal{L}^{-1}$ (antisymmetric) then $n_A^* \mathcal{L} = \mathcal{L}^n$.*

Proof. Use [Corollary 13](#) with $f = n_A, g = 1_A, h = (-1)_A$. So the [line bundle](#)

$$(n)^* \mathcal{L} \otimes (n+1)^* \mathcal{L}^{-1} \otimes (n-1)^* \mathcal{L}^{-1} \otimes (1-1)^* \mathcal{L}^{-1} \otimes n^* \mathcal{L} \otimes 1^* \mathcal{L} \otimes (-1)^* \mathcal{L}$$

is trivial i.e.

$$(n+1)^* \mathcal{L} = (n-1)^* \mathcal{L}^{-1} \otimes n^* \mathcal{L}^2 \otimes \mathcal{L} \otimes (-1)^* \mathcal{L}$$

in statement $n = 1$ is clear, so use $n = 1$ in the above to get

$$2_A^* \mathcal{L} \cong \mathcal{L}^2 \otimes \mathcal{L} \otimes (-1)_A^* \mathcal{L} \cong \mathcal{L}^3 \otimes (-1)_A^* \mathcal{L}.$$

Then induct on n in above. □

Theorem 1.4.15 (Theorem of the square (Milne 5.5)). *Let \mathcal{L} be an invertible sheaf (line bundle) on A . Let $t_a: A \rightarrow A$ be translation by $a \in A(k)$. Then*

$$t_{a+b}^* \mathcal{L} \otimes \mathcal{L} \cong t_a^* \mathcal{L} \otimes t_b^* \mathcal{L}.$$

Proof. Use [Corollary 13](#) with $f = \text{id}, g(x) = a, h(x) = b$ to get

$$t_{a+b}^* \mathcal{L} \otimes t_a^* \mathcal{L}^{-1} \otimes t_b^* \mathcal{L}^{-1} \otimes \mathcal{L}$$

is trivial. □

Remark 1.4.16. Tensor by \mathcal{L}^{-2} in the above equation to get

$$t_{a+b}^* \mathcal{L} \otimes \mathcal{L}^{-1} \cong (t_a^* \mathcal{L} \otimes \mathcal{L}^{-1}) \otimes (t_b^* \mathcal{L} \otimes \mathcal{L}^{-1}).$$

This gives a group homomorphism

$$A(k) \rightarrow \text{Pic}(A)$$

via

$$a \mapsto t_a^* \mathcal{L} \otimes \mathcal{L}^{-1}$$

for any $\mathcal{L} \in \text{Pic}(A)$.

1.5 The Adventures of BUNTES (Sachi)

1.5.1 In which we are introduced to an important homomorphism, review some concepts and our story begins

Abelian variety X , we know this is a complete group variety, our goal is to give an embedding $X \rightarrow \mathbf{P}^N$ for some N . This motivates the study of [line bundles](#).

Last time Ricky proved theorem of [cube 1.4.11](#) and [square 1.4.15](#). For any [line bundle](#) L on X , there is a group homomorphism $\Phi_L: X \rightarrow \text{Pic}(X)$ via $x \mapsto T_x^* L \otimes L^{-1}$. Be careful T_x^* is $-x$, convention, who knows why.

Example 1.5.1. Let $X = E$ an elliptic curve, $L = L((0))$, $x \mapsto (x) - (0)$, in this case this is in $\text{Pic}^0(E) \cong E \cong \widehat{E}$,

Proposition 1.5.2. *This is translation invariant.*

Proof. Translate by $q \in E$. $(x + q) - (q)$ take p to be the third point on the line with x, q , $(x) + (q) + (p) \cong 3(0)$ and $(x + q) + (p) \cong 2(0)$ subtracting these gives $(x) - (x + q) + (q) \cong (0)$ or $(x) - (0) \cong (x + q) - (q)$. \square

What about the converse of this, what can we say about translation invariant [line bundles](#)

$$K(L) = \{x \in X : T_x^* L \cong L\}?$$

Proposition 1.5.3. $K(L)$ is Zariski closed in X .

Proof. Consider $m^* L \otimes p_2^* L^{-1}$ on $X \times X$, then

$$\{x : \text{this is trivial on } \{x\} \times X\}$$

is closed. [See-saw 1.6.6](#) implies restriction is pullback

$$T_x^* L \otimes L^{-1}$$

so this is $K(L)$. \square

1.5.2 In which Pooh discovers our main theorem

Proposition 1.5.4. Let X be an [abelian variety](#) and L a [line bundle](#), $L = L(D)$ then TFAE:

1. $H(D) = \{x \in X : T_x^* D = D\}$ is finite.
2. $K(L) = \{x \in X : T_x^* L \cong L\}$ is finite.
3. $|2D|$ is basepoint free and defines a finite morphism $X \rightarrow \mathbf{P}^N$.
4. L is ample.

Proof. 3. to 4.. Is algebraic geometry.

2. to 1.. Follows as being equal is stronger than being linearly equivalent.

4. to 2.. [Section 3](#)

3. to 4.. [Section 4](#) \square

1.5.3 In which Owl proves the ampleness of L implies finiteness of $K(L)$

4. to 2. Assume L ample and $K(L)$ is infinite. Let Y be the connected component at 0 of $K(L)$, $\dim Y > 0$. Show trivial bundle is ample on Y implies Y is affine, But Y is closed and therefore [complete](#) so this is a contradiction. $L|_Y$ ample $[-1]^* L|_Y$ is ample. $L|_Y \otimes [-1]^* L|_Y$ is ample, consider

$$\begin{aligned} d: Y &\rightarrow Y \times Y \\ y &\mapsto (y, -y) \end{aligned}$$

$m \circ d = \text{constant}$, $d^* m^*(L) = \mathcal{O}_Y$, LHS is $L|_Y \otimes [-1]^* L|_Y$.

1.5.4 In which Rabbbit sets out on a long journey to prove finiteness of $H(D)$ implies $|2D|$ is basepoint free and gives a finite map $X \rightarrow \mathbf{P}^N$

Note 1.5.5. $|2D|$ is always basepoint free.

Apply the [theorem of the square 1.4.15](#): $T_{x+y}^*D + D \cong T_x^*D + T_y^*D$, let $y = -x$, $2D \cong T_x^*D + T_{-x}^*D$. (D effective) For any $y \in X$, choose some x s.t. RHS doesn't contain y . $E = 2D$

$$\psi_E: X \rightarrow \mathbf{P}^N$$

can we make this finite? If ψ_E is not finite then $\psi(C) = \text{pt}$ for some irreducible curve C (Zariski's main theorem). For each divisor in $|E|$ either it contains C or fails to intersect C by changing E if necessary, assume $E \cap C = \emptyset$.

Claim 1.5.6. $T_x^*E \cap C = \emptyset$ or all of C for all $x \in X$.

Proof. Intersection numbers are constant. \square

Proof. $O(T_x^*E)|_{\bar{C}}$, when $x = 0$ this is trivial so $\deg = 0$. So $\deg = 0$ for all [line bundles](#). E effective implies $C \cap T_x^*E = \emptyset$ for all x s.t. \cap is not in C . \square

Claim 1.5.7. E is invariant by translation by $x - y$ for $x, y \in C$.

Proof. If $e \in E$, $T_{x-e}^*(E) \cap C \neq \emptyset$. This is as x is in it, $x - (x - e) = e$, because it is nonempty it's all of C . So y is in it. So $y - (x - e) \in E$. This is also $e - (x - y) \in E$, so E is invariant under T_{x-y}^* \square

Now assume $H(E) = \{x \in X : T_x^*E = E\}$ is finite. But if $\psi_E(C) = \text{pt}$ then $T_{x-y}^*(E) = E$ for all $x, y \in C$. So H is not finite, a contradiction. So ψ_E can't collapse a curve so ψ_E is finite.

1.5.5 In which Piglet discovers a corollary

Corollary 1.5.8. *Abelian varieties are projective.*

Proof. Let X be an [abelian variety](#), $U \subseteq X$ be an open affine set, $0 \in U$, $X \setminus U = D_1 \cup \dots \cup D_i$ irreducible divisors. Let $D = \sum D_i$, then claim: $H(D) = \{x \in X : T_x^*D = D\}$ is finite. If $H \subseteq U$, U affine, then H closed subvariety of an [abelian variety](#), hence [complete](#), so its finite. If $x \in H$ then $-x \in H$. Now claim that if $x \in H$ then T_x^* preserves U , if not let $u \in U$. Suppose $u - x = d$ for some $d \in D$ then $u = d + x$ which is d translated by $-x$ so $d + x \in D$ so $u \in D$. But contradiction, oh no! So T_x^* preserves U , for all $x \in H$, as $0 \in U$, for all $x \in H$ we have $0 - x \in U$ and $0 + x \in U$ so $H \subseteq U$. \square

Corollary 1.5.9. *Abelian varieties are divisible. $X[n]$ is finite for $n \geq 1$.*

Proof. $[n]: X \rightarrow X$ and $X[n]$ is the kernel of this. Note that for $x \in X[n]$

$$[n] \circ T_x = [n]$$

$y \in X$, then $n(y - x) = ny - nx = ny$ so for all $L \in \text{Pic } X$

$$T_x^*([n]^*L) \cong ([n]^*L)$$

which implies

$$K([n]^*L) \supseteq X[n]$$

and we just need to find L s.t. this is finite. X projective implies there exists an ample L . The [theorem of the cube 1.4.11](#) implies

$$[n]^*L \cong L^{\frac{n^2+n}{2}} \otimes L^{\frac{n^2-n}{2}}$$

where both terms on the right are ample, hence the left is also. \square

1.5.6 Epilogue: In which we might discuss isogenies

Definition 1.5.10. $f: X \rightarrow Y$ a morphism of varieties, get a field extension $k(X)/f^*k(Y)$, if $\dim X = \dim Y$ and f is surjective. Then this is a finite field extension and $\deg f$ is $d = [k(X) : f^*k(Y)]$ and $d = \#f^{-1}(y)$ for almost all y .

Definition 1.5.11. A homomorphism of [abelian varieties](#) $f: X \rightarrow Y$ is an **isogeny** if f is surjective with finite kernel.

Corollary 1.5.12. Degree of $[n]$ is n^{2g} , if n is prime to the characteristic of k , $k = \bar{k}$, $g = \dim X$.

Proof. Let D be an ample [symmetric](#) divisor, e.g.

$$D = D' + [-1]^*D'$$

know $[n]^*D \sim n^2D$

$$\deg([n]^*(D \cdots D)) = ([n]^*D \cdots [n]^*D) = (n^2D \cdots n^2D) = n^{2g}(D \cdots D). \quad \square$$

1.6 Line Bundles and the Dual Abelian Variety (Angus)

Meta-goal Understand [line bundles](#) on [abelian varieties](#).

Setup A an [abelian variety](#) $/k$.

Last time For L a [line bundle](#) on A we get a map

$$\begin{aligned} \phi_L: A(k) &\rightarrow \text{Pic}(A) \\ a &\mapsto t_a^*L \otimes L^{-1} \end{aligned}$$

where

$$\text{Pic}(A) = \{\text{line bundles on } A\} / \sim.$$

This is a group homomorphism (by the [theorem of the square 1.4.15](#)). We define

$$K(L)(k) = \ker(\phi_L) = \{a \in A(k) : t_a^*L \simeq L\}.$$

Today We are going to package these into a big map

$$\begin{aligned} \phi: \text{Pic}(A) &\rightarrow \text{Hom}(A(k), \text{Pic}(A)) \\ L &\mapsto \phi_L. \end{aligned}$$

Proposition 1.6.1.

1. ϕ is a group homomorphism
- 2.

$$\phi_{t_a^*L} = \phi_L$$

Proof. 1.

$$\begin{aligned}\phi_{L \otimes M}(a) &= t_a^*(L \otimes M) \otimes (L \otimes M)^{-1} \\ &= t_a^*L \otimes L^{-1}t_a^*M \otimes M^{-1} \\ &= \phi_L \otimes \phi_M\end{aligned}$$

2.

$$\begin{aligned}\phi_{t_b^*LM}(a) &= t_a^*(t_b^*L) \otimes (t_b^*L)^{-1} \\ &= t_{a+b}^*L \otimes (t_b^*L)^{-1} \\ &= t_a^*L \otimes t_b^*L \otimes L^{-1} \otimes (t_b^*L)^{-1} \\ &= \phi_L(a)\end{aligned}$$

by the [theorem of the square 1.4.15](#) □

Definition 1.6.2.

$$\begin{aligned}\text{Pic}^0(A) &= \ker(\phi) \\ &= \{L \in \text{Pic}(A) : \phi_L = 0\} \\ &= \{L \in \text{Pic}(A) : t_a^*L \simeq L \ \forall a \in A(k)\} \\ &= \{\text{translation invariant line bundles}\}/\sim\end{aligned}$$

Goals Study $\text{Pic}^0(A)$, give it an [abelian variety](#) structure, solve a moduli problem, demonstrate some duality.

1.6.1 Aside: alternate description of $\text{Pic}^0(A)$

Definition 1.6.3 (Algebraic Equivalence). Two [line bundles](#) L_1, L_2 on an [abelian variety](#) are **algebraically equivalent** if there exists a variety Y with [line bundle](#) L on $A \times Y$ and points $y_1, y_2 \in Y$ s.t. $L|_{A \times \{y_1\}} \simeq L_1, L|_{A \times \{y_2\}} \simeq L_2$.

Remark 1.6.4. This looks like homotopy.

Proposition 1.6.5.

$$\text{Pic}^0(A) = \{\text{line bundles which are alg. equiv to } \mathcal{O}_A\}$$

Proof. [\[23\]](#). □

1.6.2 See-Saws

Theorem 1.6.6 (See-saw theorem). Let X, T be varieties X [complete](#), let L be a [line bundle](#) on $X \times T$, let $T_1 = \{t \in T : L|_{X \times \{t\}} \text{ is trivial}\}$ then T_1 is closed in T . Further let $p_2 : X \times T_1 \rightarrow T_1$, then $L|_{X \times T_1} \cong p_2^*M$ for some [line bundle](#) M on T_1 .

Remark 1.6.7. In fact $M = p_{2*}L$.

Corollary 1.6.8 (that no one states/only Milne). Let X, T be as above and let L, M be [line bundles](#) on $X \times T$ s.t.

$$L|_{X \times \{t\}} \cong M|_{X \times \{t\}} \forall t \in T$$

$$L|_{\{t\} \times X} \cong M|_{\{t\} \times X} \text{ for some } x \in X$$

then $L \cong M$.

1.6.3 Properties of $\text{Pic}^0 A$

Lemma 1.6.9. $L \in \text{Pic}^0(A)$ and $m, p_1, p_2: A \times A \rightarrow A$

1.
$$m^*L \cong p_1^*L \otimes p_2^*L$$
2. Given $f, g: X \rightarrow A$
$$(f + g)^*L \cong f^*L \otimes g^*L$$
3.
$$[n]^*L \cong L^{\otimes n}$$
4.
$$\phi_L(A(k)) \subseteq \text{Pic}^0(A)$$

for $L \in \text{Pic}(A)$.

Proof. 1.

$$(m^*L \otimes (p_1^*L)^{-1} \otimes (p_2^*L)^{-1})|_{A \times \{a\}} = t_a^*L \otimes L^{-1} = \mathcal{O}_A$$

$$(m^*L \otimes (p_1^*L)^{-1} \otimes (p_2^*L)^{-1})|_{\{a\} \times A} = t_a^*L \otimes L^{-1} = \mathcal{O}_A$$

by [see-saw 6](#) whole thing is trivial on $A \times A$.

2.

$$(f + g)^*L \cong (f \times g)^*m^*L \cong (f \times g)^*(p_1^*L \otimes p_2^*L) \cong f^*L \otimes g^*L$$

3. Induction of 3.

4.

$$\phi_{\phi_L(a)} = \phi_{t_a^*L} \otimes L^{-1} = \phi_{t_a^*L} \otimes L^{-1} = \phi_L \otimes \phi_{L^{-1}} = 0 \quad \square$$

Proposition 1.6.10. If L is nontrivial in $\text{Pic}^0(A)$ then $H^i(A, L) = 0 \forall i$.

Proof. If $H^0(A, L) \neq 0$, we would have a nontrivial section s of L then $[-1]^*s$ is a nontrivial section of $[-1]^*L = L^{-1}$. But if both L and L^{-1} have a nontrivial section then $L \cong \mathcal{O}_A$. So since L is nontrivial $H^0(A, L) = 0$. Now assume $H^i(A, L) = 0$ for all $i < j$. Consider

$$\begin{array}{ccc} A & \xrightarrow{\text{id} \times 0} & A \times A \xrightarrow{m} A \\ & & a \mapsto (a, 0) \mapsto a \end{array}$$

this gives

$$H^j(A, L) \rightarrow H^j(A \times A, m^*L) \rightarrow H^j(A, L)$$

which composes to the identity.

$$H^j(A \times A, m^*L) = H^j(A \times A, p_1^*L \otimes p_2^*L) = \bigoplus_{i=0}^j H^i(A, L) \otimes H^{j-i}(A, L)$$

by Künneth. The RHS is 0 by the inductive hypothesis. So the identity on $H^j(A, L)$ factors through 0, hence the group is 0. \square

We now think of ϕ_L as a map $\phi_L: A(k) \rightarrow \text{Pic}^0(A)$ with kernel $K(L)(k)$.

Theorem 1.6.11. If $K(L)(k)$ is finite then ϕ_L is surjective.

Proof. Idea is to study

$$\Lambda(L) = m^*L \otimes (p_1^*L)^{-1} \otimes (p_2^*L)^{-1}. \quad \square$$

Given an ample [line bundle](#) L on A we now have an isomorphism of groups

$$A(k)/K(L)(k) \cong \text{Pic}^0(A)$$

the LHS allows us to put an [abelian variety](#) structure on $\text{Pic}^0(A)$.

1.6.4 The Dual Abelian Variety

Theorem 1.6.12. *Let A be an **abelian variety** and L an ample **line bundle** on A , then the quotient scheme $A/K(L)$ exists and is an **abelian variety** of the same dimension as A .*

Proof. (Sketch) (characteristic 0) Cover A by affine opens $U_i = \text{Spec } R_i$ such that for all $a \in A$ the orbit $K(L)a \subseteq U_i$ for some i . We can do this because **abelian varieties** are projective. Then we say $U_i/K(L) = \text{Spec}(R_i^{K(L)})$ then glue. (details in Mumford, II sec, 6 appendix). Since we are in characteristic 0, the quotient scheme is in fact a variety. \square

Definition 1.6.13 (Dual abelian varieties). The **dual abelian variety** is

$$\hat{A} = A/K(L).$$

Remark 1.6.14.

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$$\hat{A}(K) = \text{Pic}^0(A)$$

- We have an isogney

$$\phi_L: A \rightarrow \hat{A}.$$

Theorem 1.6.15. *There is a unique **line bundle** \mathcal{P} on $A \times \hat{A}$ called the **Poincaré bundle** such that*

1.

$$\mathcal{P}|_{A \times \{x\}} \in \text{Pic}^0(A) \text{ for all } x \in \hat{A}$$

2.

$$\mathcal{P}|_{0 \times \hat{A}} = 0$$

3. *If Z is a scheme with a **line bundle** R on $A \times Z$ satisfying 1., 2., there exists a unique*

$$f: Z \rightarrow \hat{A}$$

s.t.

$$(\text{id} \times f)^* \mathcal{P} = R.$$

That is (\hat{A}, \mathcal{P}) represents the functor

$$Z \mapsto \left\{ L \in \text{Pic}(A \times Z) : \begin{matrix} L|_{A \times \{z\}} \in \text{Pic}^0(A) \forall z \in Z \\ L|_{0 \times Z} = 0 \end{matrix} \right\} / \sim .$$

1.6.5 Dual morphisms

Let $f: A \rightarrow B$ be a homomorphism of **abelian varieties**. Let $\mathcal{P}_A, \mathcal{P}_B$ be the **Poincaré bundles** on A and B . Consider $M = (F \times \text{id}_{\hat{B}})^* \mathcal{P}_B$ on $A \times \hat{B}$, then

1.

$$M|_{A \times \{x\}} \in \text{Pic}^0(A)$$

2.

$$M|_{\{0\} \times \hat{B}} = 0$$

thus by the universal property we get a unique morphism

$$\hat{f}: \hat{B} \rightarrow \hat{A}$$

satisfying

$$(\text{id}_A \times \hat{f})^* \mathcal{P}_A = (f \times \text{id}_{\hat{B}})^* \mathcal{P}_B.$$

Definition 1.6.16 (Dual morphisms). \hat{f} as above is called the **dual morphism**.

Remark 1.6.17.

•

$$\begin{aligned} \hat{f}: \hat{B} = \text{Pic}^0(B) &\rightarrow \hat{A}(k) = \text{Pic}^0(A) \\ L &\mapsto f^*L \end{aligned}$$

•

$$[\hat{n}_A] = [n_{\hat{A}}]$$

Consider the **Poincaré bundle** $\mathcal{P}_{\hat{A}}$ on $\hat{A} \times \hat{\hat{A}}$, now think of \mathcal{P}_A as living on $\hat{A} \times A$. By the universal property of $\mathcal{P}_{\hat{A}}$ get a unique morphism

$$\text{can}_A: A \rightarrow \hat{\hat{A}}.$$

Theorem 1.6.18. can_A is an isomorphism.

Lemma 1.6.19.

$$\phi_{f^*L} = \hat{f} \circ \phi_L \circ f.$$

Proposition 1.6.20. If $f: A \rightarrow B$ is an **isogeny**, then $\hat{f}: \hat{B} \rightarrow \hat{A}$ is an **isogeny**. Further if $N = \ker f$, then $\hat{N} = \ker \hat{f}$ is the Cartier dual of N .

Definition 1.6.21 (Symmetric morphisms, (principal) polarizations). A morphism $f: A \rightarrow \hat{A}$ is **symmetric** if $f = \hat{f} \circ \text{can}_A$

A **polarization** is a **symmetric isogeny** $f: A \rightarrow \hat{A}$ s.t. $f = \phi_L$ for some ample **line bundle** L on A .

A **principal polarization** is a **polarization** of degree 1, i.e. an isomorphism.

Remark 1.6.22. Elliptic curves always admit **principal polarization**.

If one wishes to mimic the theory of elliptic curves, one should study principally polarized **abelian varieties**.