Chapter 1

Abelian Varieties

1.1 Introduction

1.1.1 Definitions

Definition 1.1.1 (Abelian varieties)**.** An **abelian variety** is a complete connected algebraic group.

Definition 1.1.2 (Algebraic groups)**.** An **algebraic group** is an algebraic variety G along with regular maps $m\colon G \times G \to G$, $e\colon * \to G$, $\mathrm{inv}\colon G \to G$ such that the following diagrams commute.

$$\begin{tikzcd}
* \times G \arrow[r, "e\times\mathrm{id}"] \arrow[dr, "\sim"'] & G \times G \arrow[d, "m"] & G \times * \arrow[l, "\mathrm{id}\times e"'] \arrow[dl, "\sim"] \\
& G &
\end{tikzcd}$$
identity

$$\begin{tikzcd}
G \arrow[r, "\mathrm{inv},\mathrm{id}"] \arrow[d] & G \times G \arrow[d, "m"] & G \arrow[l, "\mathrm{id},\mathrm{inv}"'] \arrow[d] \\
* \arrow[r, "e"'] & G & * \arrow[l, "e"']
\end{tikzcd}$$
Inverse

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\mathrm{id}\times m"] \arrow[d, "m\times\mathrm{id}"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$
Associativity

Definition 1.1.3 (Complete varieties)**.** A variety X is **complete** if every projection map

$$X \times Y \to Y$$

is closed.

Example 1.1.4.

- Elliptic curves.
- Weil restriction $\mathrm{Res}_{K/\mathbf{Q}}\, E$ of an elliptic curve E.
- Jacobian varieties of curves.

Plan:

- Some motivation via elliptic curves.
- Gathering some material about "completeness".
- Prove that abelian varieties are abelian.

1.1.2 Elliptic curves (char$(k) \neq 2, 3$)

Theorem 1.1.5. *TFAE for a projective curve E over k.*

1. *E is given by $Y^2Z = X^3 + aXZ^2 + bZ^3, 4a^3 + 27b^2ane0$.*
2. *E is nonsingular of genus 1 with a distinguished point P_0.*
3. *E is nonsingular with an algebraic group structure.*
4. *(if $k \subseteq \mathbf{C}$) such that $E(\mathbf{C}) = \mathbf{C}/\Lambda$ for some lattice $\Lambda \subseteq \mathbf{C}$.*

Proof. Strategy: Item 1 $\iff$ Item 2 $\iff$ Item 3 and Item 2 $\implies$ Item 4 $\implies$ Item 1.

Item 1 $\implies$ Item 2 is done.

Item 2 $\implies$ Item 1: Riemann-Roch states that $l(D) = l(K-D)+\deg(D)+1-g$ so here $l(D) = l(K-D) + \deg(D)$ further is $D > 0$ then $l(K-D) = 0$ in which case $l(D) = \deg(D)$. Consider $L(nP_0)$ for $n > 0$ Riemann-Roch implies that $l(nP_0) = n$ then it always contains the constants.

$$L(P_0) = k$$

$$L(2P_0) = k \oplus kx$$

$$L(3P_0) = k \oplus kx \oplus ky$$

$$\vdots$$

$$L(6P_0) = k \oplus kx \oplus ky \oplus kx^2 \oplus ky^2 \oplus kxy \oplus kx^3/\sim$$

so we must have a relation which after manipulation is of the desired form. We get an embedding

$$E \hookrightarrow \mathbf{P}^2$$

$$P \mapsto (x(P) : y(P) : 1)\,(P \neq P_0)$$

$$P_0 \mapsto (0 : 1 : 0)$$

and thus E is of the desired form. ◻

Definition 1.1.6 (Elliptic curves)**.** An **elliptic curve** over k is any/all of that 5.

Which of the above characterisations generalise to abelian varieties?

1. No, in general we don't know that the equations look like.
2. One could possibly replace "genus" with a condition on the dimension of cohomology groups.
3. Yes, this is essentially the definition.
4. Yes, stay tuned!

1.1.3 Complete varieties

Idea: if $X \times Y$ had product topology (instead of its Zariski topology) then complete is equivalent to compact.

We'd like to gather a few results about complete varieties we can use to access properties of abelian varieties (like abelianness).

Proposition 1.1.7. *Let V be a complete variety. Given any morphism $\phi \colon V \longrightarrow W$ $\phi(V)$ is closed.*

Proof. Let $\Gamma_\phi = \{(v, \phi(v)\} \subseteq V \times W$ be the graph of ϕ. Its a closed subvariety of $V \times W$. Under the projection $V \times W \longrightarrow W$, the image of Γ_ϕ is $\phi(V)$ and thus closed. □

Corollary 1.1.8. *If V is complete and connected, any regular function on V is constant.*

Proof. A regular function is a morphism $f \colon V \longrightarrow \mathbf{A}^1$. By the above $f(V) \subseteq \mathbf{A}^1$ is closed, and this is a finite set of points. But connected implies we just have one point. □

Corollary 1.1.9. *Let V be a complete connected variety. Let W be an affine variety. Given $\phi \colon V \longrightarrow W$, then $\phi(V)$ is a point.*

Proof. We have an embedding $W \hookrightarrow \mathbf{A}^n$. On $\mathbf{A}^n$ we have the coordinate functions $\mathbf{A}^n \xrightarrow{x_i} \mathbf{A}^1$. The composition

$$V \xrightarrow{\phi} W \hookrightarrow \mathbf{A}^n \longrightarrow \mathbf{A}^1$$

be the above is constant. Thus the coordinates of $\phi(V)$ are constant, so $\phi(V) = \{\mathrm{pt}\}$. □

A final result of interest that I won't prove today:

Theorem 1.1.10. *Projective varieties are complete.*

The main goal of this section is to prove the following theorem:

Theorem 1.1.11 (Rigidity)**.** *Let V, W be varieties such that V is complete and $V \times W$ is geometrically irreducible. Let $\alpha \colon V \times W \longrightarrow U$ be a morphism such that $\exists u_0 \in U(k), v_0 \in V(k), w_0 \in W(k)$ with $\alpha(V \times \{w_0\}) = \alpha(\{v_0\} \times W) = \{u_0\}$. Then $\alpha(V \times W) = \{u_0\}$.*

Proof. Since $V \times W$ is geometrically irreducible, V must be connected. Denote the projection $q \colon V \times W \longrightarrow W$. Let $U_0 \ni x_0$ be an open neighborhood. We consider the set

$$Z = \{w \in W : \alpha((v, w)) \notin U_0 \text{ for some } v \in V\} = q(\alpha^{-1}(U \setminus U_0))$$

Since q is closed, $Z \subseteq W$ is closed. Since $w_0 \in W \setminus Z$, $W \setminus Z$ is a nonempty open subset of W.

Consider $w \in W \setminus Z$. Since $V \times \{w\} \cong V$ it is complete and connected. Thus

$$\alpha(V \times \{w\}) = \{\mathrm{pt}\} = \alpha((v_0, w)) = \{u_0\}$$

which implies that

$$\alpha(V \times (W \setminus Z)) = \{u_0\}$$

Since $V \times (W \setminus Z) \subseteq V \times W$ is open and $V \times W$ is irreducible, it is dense. So $\alpha(V \times W) = \{u_0\}$. □

Proposition 1.1.12. *Let A, B be abelian varieties. Every morphism $\alpha\colon A \to B$ is the composition of a homomorphism and a translation.*

Proof. First compose by a translation on B such that $\alpha(0) = 0$. Consider the map

$$\begin{aligned}\phi\colon A \times A &\to B\\ (a, a') &\mapsto \alpha(a + a') - \alpha(A) - \alpha(a')\end{aligned}$$

Then

$$\begin{aligned}\phi(A \times \{0\}) &= \alpha(a + 0) - \alpha(a) - \alpha(0) = 0\\ \phi(\{0\} \times A) &= \alpha(0 + a) - \alpha(0) - \alpha(a) = 0.\end{aligned}$$

By the rigidity theorem 11 $\phi(A \times A) = \{0\}$ hence $\alpha(a + a') = \alpha(a) + \alpha(a')$. □

Corollary 1.1.13. *Abelian varieties are abelian.*

Proof. The inversion map $a \mapsto -a$ sends 0 to 0, thus is a homomorphism. Therefore

$$a + b - a - b = a + b - (a + b) = 0$$

and so

$$a + b = b + a.$$

□

1.2 Abelian varieties over C

The goal of this talk is to understand what abelian varieties look like over $\mathbf{C}$. The goal for me is to understand what a (principal) polarisation is and why it is important.

First immediate question: why study complex theory at all? The most classical field, algebraically closed, archimidean, characteristic 0.

Recall/rapidly learn the picture for elliptic curves, given E an elliptic curve we have for some Λ a rank 2 lattice in $\mathbf{C}$

$$\begin{aligned}\mathbf{C}/\Lambda &\xrightarrow{\sim} E(\mathbf{C}) \subseteq \mathbf{P}^2(\mathbf{C})\\ z &\mapsto (\wp(z) : \wp'(z) : 1)\\ 0 &\mapsto (0 : 1 : 0)\end{aligned}$$

where

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \smallsetminus \{0\}} \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2}.$$

This is a meromorphic function whose image lands in

$$y^2 = 4x^3 - g_2 x - g_3.$$

So the $\mathbf{C}$ points of an elliptic curve are topologically a tori.

Naturally one asks: does this generalise? Let A be an abelian variety over $\mathbf{C}$, what does $A(\mathbf{C})$ look like? Another torus?

Proposition 1.2.1. *$A(\mathbf{C})$ is a compact, connected, complex lie group.*

Proposition 1.2.2. *Let A be an abelian variety of dimension g over $\mathbf{C}$. Then we have*

$$A(\mathbf{C}) \cong V/\Lambda$$

where V is a g dimensional complex vector space and Λ is a full rank lattice of V (i.e Λ is a discrete subgroup of V s.t. $\mathbf{R} \otimes \Lambda = V$).

Proof. Differential geometry gives us a map of complex manifolds

$$\exp\colon \operatorname{Tgt}_0(A(\mathbf{C})) \to A(\mathbf{C})$$

this is a holomorphism. And since $A(\mathbf{C})$ is abelian, this is a homomorphism also. In general this is locally an isomorphism around 0.

Claim: exp is injective. There exists a neighborhood $U \supseteq 0$ s.t. $\exp(U) \cong U$. Consider the image $\exp(\operatorname{Tgt}_0 A(\mathbf{C}))$. For $x \in \exp(\operatorname{Tgt}_0 A(\mathbf{C}))$, $\{U + x\}$ are all open and give a cover. Thus $\exp(\operatorname{Tgt}_0 A(\mathbf{C}))$ is open. Since $A(\mathbf{C})$ is connected we are thus reduced to showing $\exp(\operatorname{Tgt}_0 A(\mathbf{C}))$ is closed also. Since exp is a homomorphism, the image is a subgroup. So its complement is the union of its non-trivial cosets, which is open. Thus $\exp(\operatorname{Tgt}_0 A(\mathbf{C}))$ is closed. Giving $\exp(\operatorname{Tgt}_0 A(\mathbf{C})) = A(\mathbf{C})$, which proves the claim.

exp is a local isomorphism, which gives that ker(exp) is discrete, i.e. a lattice. We now have

$$A(\mathbf{C}) \cong \operatorname{Tgt}_0 A(\mathbf{C})/\ker(\exp)$$

so as $A(\mathbf{C})$ is compact we cannot have a kernel which is not full rank, as otherwise the quotient could not be compact. □

Definition 1.2.3. We call any such V/Λ a **complex torus**.

From the above isomorphism we can now read off properties of $A(\mathbf{C})$ as a group.

Proposition 1.2.4. *$A(\mathbf{C})$ is divisible, and $A(\mathbf{C})[n] \cong (\mathbf{Z}/n\mathbf{Z})^{2g}$.*

Proof.
$$A(\mathbf{C}) \cong V/\Lambda \cong (\mathbf{R}/\mathbf{Z})^{2g}$$
isomorphisms as groups, thus $A(\mathbf{C})$ is divisible. Further, $(\mathbf{R}/\mathbf{Z})[n] = (\frac{1}{n}\mathbf{Z})/\mathbf{Z}$. □

Question Given a complex torus V/Λ, does there exist an abelian variety A such that $A(\mathbf{C}) \cong V/\Lambda$?

Example 1.2.5.

- $$\mathbf{C}/\Lambda \cong E(\mathbf{C}) \text{ always in dim } 1$$
- $$\mathbf{C}^2/\Lambda^2 \cong (E \times E)(\mathbf{C}) \text{ sometimes yes in higher dimension}$$
- $$\mathbf{C}^2/\langle (i,0),(i\sqrt{p},i),(1,0),(0,1)\rangle_{\mathbf{Z}}$$
  for p prime??? (I guess not, see Mumford)

Theorem 1.2.6 (Chow)**.** *If X is an analytic submanifold of $\mathbf{P}^n(\mathbf{C})$ then X is an algebraic subvariety.*

By this theorem it is enough to analytically imbed $V/\Lambda \hookrightarrow \mathbf{P}^m$. We can try and do this by mimicing the elliptic curve strategy, find enough functions $\theta\colon V/\Lambda \to \mathbf{C}$.

Proposition 1.2.7. *Let $X = V/\Lambda$. Then*

$$H^r(X,\mathbf{Z}) \cong \{\text{alternating } r\text{-forms } \Lambda \times \cdots \times \Lambda \to \mathbf{Z}\}.$$

Proof. $\pi\colon V \to V/\Lambda$ is a universal covering map, so

$$\Lambda = \pi^{-1}(0) \cong \pi_1(X, 0).$$

Because all these spaces are nice

$$H^1(X, \mathbf{Z}) \cong \mathrm{Hom}(\pi_1(X), \mathbf{Z}) \cong \mathrm{Hom}(\Lambda, \mathbf{Z}).$$

To extend to $r \neq 1$ use the Künneth formula:

$$\begin{tikzcd}
\bigwedge^r(H^1(X_1 \times X_2, \mathbf{Z})) \arrow[rr, equal] \arrow[d, equal, "\text{Künneth}"] & & H^r(X_1 \times X_2, \mathbf{Z}) \arrow[dd, equal, "\text{Künneth}"] \\
\bigwedge^r(H^1(X_1, \mathbf{Z}) \otimes H^1(X_2, \mathbf{Z})) \arrow[d, equal] & & \\
\bigoplus_{p+q=r}(\bigwedge^p(H^1(X_1, \mathbf{Z})) \otimes \bigwedge^q(H^1(X_2, \mathbf{Z}))) \arrow[rr, equal] & & \bigoplus_{p+q=r}(H^p(X_1, \mathbf{Z}) \otimes H^q(X_2, \mathbf{Z}))
\end{tikzcd}$$

Since we know the proposition for $S^1 = \mathbf{R}/\mathbf{Z}$ by taking products and applying the above we get it for all complex tori V/Λ. □

1.3 Rational Maps into Abelian Varieties

Note all varieties are irreducible today.

1.3.1 Rational maps

V, W varieties $/K$. Consider pairs (U, ϕ_U), where $\emptyset \neq U \subset V$ an open subset so U is dense, and $\phi_U\colon U \to W$ is a regular map.

Definition 1.3.1. (U, ϕ_U), $(U', \phi_{U'})$ are equivalent if ϕ_U and $\phi_{U'}$ agree on $U \cap U'$. An equivalence class ϕ of $\{(U, \phi_U)\}$ is a **rational map** $\phi\colon V \dashrightarrow W$ If $\phi\colon V \dashrightarrow W$ is defined at $v \in V$ if $v \in U$ for some $(U, \phi_U) \in \phi$.

Note 1.3.2. The set $U_1 = \bigcup U$ where ϕ is defined is open and $(U_1, \phi_1) \in \phi$ where $\phi_1\colon U_1 \to W$ restricts to ϕ_U on U.

Example 1.3.3.

1. Let $\emptyset \neq W \subseteq V$ be open. Then the rational map $V \dashrightarrow W$ induced by $\mathrm{id}\colon W \to W$ will not extend to V. To avoid this, assume W is complete (so $W = V$).

2. $C\colon y^2 = x^3$, then $\alpha\colon \mathbf{A}^1 \to C$, $a \mapsto (a^2, a^3)$ is a regular map, restricting to an isomorphism $\mathbf{A}^1 \setminus \{0\} \to C \setminus \{0\}$. The inverse of $\alpha|_{\mathbf{A}^1 \setminus \{0\}}$ represents $\beta\colon C \dashrightarrow \mathbf{A}^1$ which does not extend to C. This corresponds on function fields to
$$K(t) \to K(x, y)$$
$$t \mapsto y/x$$
which does not send $K[y]_{(t)}$ to $K[x, y]_{(x,y)}$.

3. Given a nonsingular surface V, $P \in V$ then $\exists \alpha\colon W \to V$ regular that induces an isomorphism $\alpha\colon W \setminus \alpha^{-1}(P) \to V \setminus P$, but $\alpha^{-1}(P)$ is a projective line. The rational map represented by α^{-1} is not regular on V (where to send P?).

Theorem 1.3.4 (Milne 3.1)**.** *A rational map $\phi\colon V \dashrightarrow W$ from a nonsingular variety V to a complete variety W is defined on an open subset $U \subseteq V$ whose complement has codimension ≥ 2.*

Proof. (V a curve) V nonsingular curve, $\emptyset \neq U \subseteq V$ open, $\phi\colon U \to W$ a regular map.

$$\begin{tikzcd}
 & & V \\
U \arrow[urr] \arrow[r] \arrow[drr] & U' \subseteq Z \subseteq V \times W \ni (v,w) \arrow[ur, "p"] \arrow[dr, "q"'] & \\
 & & W \ni w
\end{tikzcd}$$

U' is the image of U, $Z = \overline{U'}$. W is complete, Z closed implies $p(Z) \subseteq V$ is closed. Also, $U \subseteq p(Z) \implies p(Z) = V$.

$$U \xrightarrow{\sim} U' \to U$$

so

$$U' \xrightarrow{\sim} U$$

$$Z \twoheadrightarrow V$$

this implies $Z \xrightarrow{\sim} V$. Then $q|_Z\colon Z \to W$ is the extension of ϕ to V. ◻

Theorem 1.3.5 (Milne 3.2)**.** *A rational map $\phi\colon V \dashrightarrow A$ from a nonsingular variety V to an abelian variety W, extends to all of V.*

Proof. Theorem 4 Lemma 6 ◻

Lemma 1.3.6. *Let $\phi\colon V \dashrightarrow G$ be a map from a nonsingular variety to a group variety. Then either ϕ is defined on all of V or the set where ϕ is not defined is closed of pure codimension 1.*

Proof. Fix $(U, \phi_U) \in \phi$ and consider

$$\Phi\colon V \times V \dashrightarrow G$$

represented by

$$U \times U \xrightarrow{\phi_U \times \phi_U} G \times G \xrightarrow{\mathrm{id} \times \mathrm{inv}} G \times G \xrightarrow{m} G$$

$$(x, y) \mapsto \phi_U(x)\phi_U(y)^{-1}$$

Check ϕ is defined at x iff Φ is defined at (x,x) (and in this case $\Phi(x,x) = e$). This is equivalent to the map $\Phi^*\colon \mathcal{O}_{G,e} \to K(V \times V)$ induced by Φ satisfying $\mathrm{im}(\mathcal{O}_{G,e}) \subseteq \mathcal{O}_{V\times V,(x,x)}$ For a nonzero function f on $V \times V$, write $\mathrm{div}(f) = \mathrm{div}(f)_0 - \mathrm{div}(f)_\infty$ which are effective divisors. Then

$$\mathcal{O}_{V\times V,(x,x)} = \{0\} \cup \{f \in K(V \times V) : \mathrm{div}(f)_\infty \text{ does not contain } (x,x)\}.$$

Suppose ϕ is not defined at x, then there exists $f \in \mathrm{im}(\mathcal{O}_{G,e})$ s.t. $(x,x) \in \mathrm{div}(f)_\infty$. Then Φ is not defined at any $(y,y) \in \Delta \cap \mathrm{div}(f)_\infty = \mathrm{div}(f^{-1})_0$, which is a pure codimension 1 subset of Δ by Milne's AG thm 9.2. The corresponding subset in V is of pure codimension 1, and ϕ is not defined there. ◻

Theorem 1.3.7 (Milne 3.4)**.** *Let $\alpha\colon V \times W \to A$ be a morphism from a product of nonsingular varieties into an abelian variety. If $\alpha(V \times \{w_0\}) = \{a_0\} = \alpha(\{v_0\} \times W)$ for some $a_0 \in A$, $v_0 \in W$, $w_0 \in W$, then $\alpha(V \times W) = \{a_0\}$.*

Corollary 1.3.8 (Milne 3.7)**.** *Every rational map $\alpha\colon G \dashrightarrow A$ from a group variety into an abelian variety is the composition of a homomorphism and a translation in A.*

Proof. Since group varieties are nonsingular, $\alpha\colon G \to A$ is a regular map by Theorem 5. The rest is as proof of Corollary 1.2. □

1.3.2 Dominating and birational maps

Definition 1.3.9 (Dominating maps)**.** $\phi\colon V \dashrightarrow W$ is **dominating** if $\mathrm{im}(\phi_U)$ is dense in W for a representative $(U, \phi_U) \in \phi$.

Exercise: A dominating $\phi\colon V \dashrightarrow W$ defines a homomorphism $K(W) \to K(V)$ and any such homomorphism arises from a unique dominating rational map.

Definition 1.3.10. $\phi\colon V \dashrightarrow W$ is **birational** if the corresponding $K(W) \to K(V)$ is an isomorphism or, equivalently if there exists $\psi\colon W \dashrightarrow V$ s.t. $\phi \circ \psi$ and $\psi \circ \phi$ are the identity wherever they are defined. In this case we say V and W are **birationally equivalent**.

Note 1.3.11. In general birational equivalence does not imply isomorphic. E.g. V a variety $\emptyset \neq W \subsetneq V$ an open subset, or $V = \mathbf{A}^1$, $W\colon y^2 = x^3$.

Theorem 1.3.12 (Milne 3.8)**.** *If two abelian varieties are birationally equivalent then they are isomorphic as abelian varieties.*

Proof. A, B abelian varieties with $\phi\colon A \dashrightarrow B$ a birational map with inverse ψ. Then by Theorem 5 ϕ, ψ extend to regular maps $\phi\colon A \to B$, $\psi\colon B \to A$ and $\phi\circ\psi, \psi\circ\phi$ are the identity everywhere. This implies that ϕ is an isomorphism of algebraic varieties and after composition with a translation, ϕ is also a group isomorphism. □

Proposition 1.3.13 (Milne 3.9)**.** *Any rational map $\mathbf{A}^1 \dashrightarrow A$ or $\mathbf{P}^1 \dashrightarrow A$, for A an abelian variety is constant.*

Proof. Theorem 5 implies $\alpha\colon \mathbf{A}^1 \dashrightarrow A$ extends to $\alpha\colon \mathbf{A}^1 \to A$ and we may assume $\alpha(0) = e$. $(\mathbf{A}^1, +)$: $\alpha(x + y) = \alpha(x) + \alpha(y)$ for all $x, y \in \mathbf{A}^1(K) = K$. $(\mathbf{A}^1 \setminus \{0\}, \cdot)$: $\alpha(xy) = \alpha(x) + \alpha(y) + c$ for all $x, y \in K^\times$. These can only hold at the same time if α is constant. $\mathbf{P}^1 \dashrightarrow A$ is constant, since its constant on affine patches. □

Definition 1.3.14. $V/\overline{K}$ is **unirational** if there is a dominating map $\mathbf{A}^n \dashrightarrow V$, where $n = \dim_{\overline{K}} V$. V/K is unirational if V/K is.

Proposition 1.3.15 (Milne 3.10)**.** *Every rational map $V \dashrightarrow A$ from V unirational to A abelian is constant.*

Proof. Wlog $K = \overline{K}$. Since V is unirational we get $\beta\colon \mathbf{P}^1 \times \cdots \times \mathbf{P}^1 \dashrightarrow V \dashrightarrow A$, which extends to $\beta\colon \mathbf{P}^1 \times\cdots\times \mathbf{P}^1 \to A$. Then by Milne corollary 1.5, there exist regular maps $\beta_i\colon \mathbf{P}^1 \to A$ s.t. $\beta(x_1, \ldots, x_n) = \sum \beta_i(x_i)$ and by Proposition 13 each β_i map is constant. □

1.4 Theorem of the Cube

1.4.1 Crash Course in Line Bundles

Consider $\mathbf{R}^2$, $f\colon \mathbf{R} \to \mathbf{R}$, $f(x, y) = x^2 + y^2 - 1$, now $S = \{f = 0\} \subseteq \mathbf{R}^2$ is a closed submanifold (in fact a circle). Question: Do all closed submanifolds arise in this way? Lets switch to $\mathbf{C}$ better analogies with AG.

Example 1.4.1. Let $X \in \mathbf{P}^n(\mathbf{C})$, the answer here is no! (Because $f\colon X \to \mathbf{C}^1$ is constant!) Want to define functions locally that give us level sets, but gluing such will give us a global section. Instead glue in a different way (i.e. into different "copies" of $\mathbf{C}$) so that this doesn't happen.

Example 1.4.2. $X \in \mathbf{P}^1_{\mathbf{C}}$, $\mathcal{O}_X$ the structure sheaf.

$$X = U_0 \cup U_1 = (\mathbf{A}^1, t) \cup (\mathbf{A}^1, s)$$

on $U_0 \cap U_1$, $t = s^{-1}$. What is a global section of $\mathcal{O}_X$, a section of U_0 and a section of U_1 that glue. $\mathcal{O}_X(U_0) = k[t], \mathcal{O}_X(U_1) = k[s]$ so given $f(t), g(s)$ these glue to a global section iff $f(t) = g(1/t)$ so f, g must be constant.

Definition 1.4.3 (Line bundles)**.** A **line bundle** on X is a locally free $\mathcal{O}_X$-module of rank 1, i.e. $\exists\{U_i\}$ open cover along with isomorphisms $\phi_i\colon \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_X|_{U_i}$.

Exercise 1.4.4. Alternative definition: A line bundle on X is equivalent to the following data:

- An open cover of X.
- Transition maps $\tau_{ij} \in \mathrm{GL}_1(\mathcal{O}_X(U_i \cap U_j))$ satisfying $\tau_{ij}\tau_{jk} = \tau_{ik}$ and $\tau_{ii} = \mathrm{id}$.

Example 1.4.5. On $X = \mathbf{P}^n_k$, we have line bundles $\mathcal{O}(d)$ for all $d \in \mathbf{Z}$. Just have to give cover and transition functions, use usual open cover $\{U_i\}$ with $U_i \cong \mathbf{A}^n$. Then τ_{ji} is given by multiplication by $(x_i/x_j)^d$.

Exercise 1.4.6.

$$H^0(X, \mathcal{O}(d))(= \Gamma(X, \mathcal{O}(d)))$$

$= k$vector space spanned by deg. d homogenous polynomials in $k[x_0, \ldots, x_n]$.

Exercise 1.4.7. All line bundles on $\mathbf{P}^n$ are isomorphic to some $\mathcal{O}(d)$.

We say a line bundle $\mathcal{L}$ on X is trivial if $\mathcal{L} \cong \mathcal{O}_X$. Given $\mathcal{L}_1$ and $\mathcal{L}_2$ on X (line bundles) we can create a new line bundle $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$. So isomorphism classes of line bundles on X with $\otimes$ form a group, denoted $\mathrm{Pic}(X)$ with identity $\mathcal{O}_X$ and inverses $\mathcal{L}^{-1} = \mathrm{Hom}(\mathcal{L}, \mathcal{O}_X)$.

Example 1.4.8. By previous exercise $\mathrm{Pic}(\mathbf{P}^n_k) \cong \mathbf{Z}$ since $\mathcal{O}_X(d_1) \otimes \mathcal{O}_X(d_2) \cong \mathcal{O}_X(d_1 + d_2)$.

Fact 1.4.9. *If $f\colon X \to Y$, then given $\mathcal{L}$ on Y we can pullback to a line bundle $f^*\mathcal{L}$ on X, definition is complicated. We also know that f^* commutes with $\otimes$ so in fact (as $f^*\mathcal{O}_Y = \mathcal{O}_X$) we get a homomorphism $f^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$.*

1.4.2 Relation to (Weil) divisors

Let X be a normal variety, call $Z \subsetneq X$, a closed subvariety of codimension 1, a **prime divisor**. Then a divisor on X is a formal sum

$$D = \sum_{Z \subsetneq X} n_Z \cdot Z$$

of prime divisors.

Let $K = K(X)$ be the function field of X. Given $f \in K^\times$ we can define

$$\mathrm{div}(f) = \sum v_Z(f) \cdot Z.$$

Given $D \in \mathrm{Div}(X)$, we can define a line bundle $\mathcal{L}(D)$ on X via

$$\mathcal{L}(D)(U) = \{f \in K^\times : (D + \mathrm{div}(f))|_U \geq 0\} \cup \{0\}$$

where $D|_U = \sum_{Z \cap U \neq \emptyset} n_Z \cdot (Z \cap U)$.

Proposition 1.4.10. *The map*

$$\mathrm{Cl}(X) = \mathrm{Div}(X)/\mathrm{Princ}(X) \xrightarrow{\mathcal{L}(\cdot)} \mathrm{Pic}(X)$$

is an isomorphism.

1.4.3 Onto cubes

Theorem 1.4.11 (Theorem of the cube)**.** *Let U, V, W be complete varieties. If $\mathcal{L}$ is a line bundle on $U \times V \times W$ s.t. $\mathcal{L}|_{\{u_0\}\times V\times W}, \mathcal{L}|_{U\times\{v_0\}\times W}, \mathcal{L}|_{U\times V\times\{w_0\}}$ are all trivial then $\mathcal{L}$ is trivial.*

Corollary 1.4.12 (Milne 5.2)**.** *Let A be an abelian variety. Let $p_i\colon A \times A \times A \to A$ be the projection onto the ith coordinate. $p_{ij} = p_i + p_j$, $p_{123} = p_1 + p_2 + p_3$. Then for any $\mathcal{L}$ on A, the line bundle*

$$\mathcal{M} = p_{123}^* \mathcal{L} \otimes p_{12}^* \mathcal{L}^{-1} \otimes p_{23}^* \mathcal{L}^{-1} \otimes p_{13}^* \mathcal{L}^{-1} \otimes p_1^* \mathcal{L} \otimes p_2^* \mathcal{L} \otimes p_3^* \mathcal{L}$$

is trivial.

Proof. Let $m\colon A\times A \to A$ be multiplication (addition?) and p, q the projections $A\times A \to A$. Then the composites of the maps $\phi\colon A\times A \to A\times A\times A$, $\phi(x, y) = (x, y, 0)$ with $p_{123}, p_{12}, p_{23}, p_{13}, p_1, p_2, p_3$ are respectively $m, m, q, p, p, q, 0$. Hence the restriction of $\mathcal{M}$ to $A \times A \times \{0\}$ is

$$m^* \mathcal{L} \otimes m^* \mathcal{L}^{-1} \otimes q^* \mathcal{L}^{-1} \otimes p^* \mathcal{L}^{-1} \otimes p^* \mathcal{L} \otimes q^* \mathcal{L} \otimes \mathcal{O}_{A\times A}$$

this is trivial by tensor commuting with pullback. Similarly $\mathcal{M}$ restricts to a trivial bundle on $A \times \{0\} \times A$ and $\{0\} \times A \times A$. So by theorem of the cube 11 $\mathcal{M}$ is trivial. ◻

Corollary 1.4.13 (Milne 5.3)**.** *Let $f, g, h\colon V \to A$ (A abelian). Then for any $\mathcal{L}$ on A the bundle*

$$\mathcal{M} = (f+g+h)^* \mathcal{L} \otimes (f+g)^* \mathcal{L}^{-1} \otimes (f+h)^* \mathcal{L}^{-1} \otimes (g+h)^* \mathcal{L}^{-1} \otimes f^* \mathcal{L} \otimes g^* \mathcal{L} \otimes h^* \mathcal{L}$$

is trivial.

Proof. $\mathcal{M}$ is the pullback of the line bundle of Corollary 12 via the map $(f, g, h)\colon V \to A \times A \times A$. ◻

On A we have $n_A\colon A \to A$ be $n_A(a) = a + \cdots + a$ (n times) for $n \in \mathbf{Z}$.

Corollary 1.4.14 (Milne 5.4)**.** *For $\mathcal{L}$ on A we have*

$$n_A^* \mathcal{L} \cong \mathcal{L}^{(n^2+n)/2} \otimes (-1)_A^* \mathcal{L}^{(n^2-n)/2}$$

In particular if $(-1)^* \mathcal{L} = \mathcal{L}$ (symmetric) then $n_A^* \mathcal{L} = \mathcal{L}^{n^2}$. And if $(-1)^* \mathcal{L} = \mathcal{L}^{-1}$ (antisymmetric) then $n_A^* \mathcal{L} = \mathcal{L}^n$.

Proof. Use Corollary 13 with $f = n_A, g = 1_A, h = (-1)_A$. So the line bundle

$$(n)^* \mathcal{L} \otimes (n+1)^* \mathcal{L}^{-1} \otimes (n-1)^* \mathcal{L}^{-1} \otimes (1-1)^* \mathcal{L}^{-1} \otimes n^* \mathcal{L} \otimes 1^* \mathcal{L} \otimes (-1)^* \mathcal{L}$$

is trivial i.e.

$$(n+1)^* \mathcal{L} = (n-1)^* \mathcal{L}^{-1} \otimes n^* \mathcal{L}^2 \otimes \mathcal{L} \otimes (-1)^* \mathcal{L}$$

in statement $n = 1$ is clear, so use $n = 1$ in the above to get

$$2_A^* \mathcal{L} \cong \mathcal{L}^2 \otimes \mathcal{L} \otimes (-1)_A^* \mathcal{L} \cong \mathcal{L}^3 \otimes (-1)_A^* \mathcal{L}.$$

Then induct on n in above. □

Theorem 1.4.15 (Theorem of the square (Milne 5.5))**.** *Let $\mathcal{L}$ be an invertible sheaf (line bundle) on A. Let $t_a \colon A \longrightarrow A$ be translation by $a \in A(k)$. Then*

$$t_{a+b}^* \mathcal{L} \otimes \mathcal{L} \cong t_a^* \mathcal{L} \otimes t_b^* \mathcal{L}.$$

Proof. Use Corollary 13 with $f = \mathrm{id}$, $g(x) = a, h(x) = b$ to get

$$t_{a+b}^* \mathcal{L} \otimes t_a^* \mathcal{L}^{-1} \otimes t_b^* \mathcal{L}^{-1} \otimes \mathcal{L}$$

is trivial. □

Remark 1.4.16. Tensor by $\mathcal{L}^{-2}$ in the above equation to get

$$t_{a+b}^* \mathcal{L} \otimes \mathcal{L}^{-1} \cong (t_a^* \mathcal{L} \otimes \mathcal{L}^{-1}) \otimes (t_b^* \mathcal{L} \otimes \mathcal{L}^{-1}).$$

This gives a group homomorphism

$$A(k) \longrightarrow \mathrm{Pic}(A)$$

via

$$a \mapsto t_a^* \mathcal{L} \otimes \mathcal{L}^{-1}$$

for any $\mathcal{L} \in \mathrm{Pic}(A)$.

1.5 The Adventures of BUNTES

1.5.1 In which we are introduced to an important homomorphism, review some concepts and our story begins

Abelian variety X, we know this is a complete group variety, our goal is to give an embedding $X \longrightarrow \mathbf{P}^N$ for some N. This motivates the study of line bundles.

Last time Ricky proved theorem of cube 1.4.11 and square 1.4.15. For any line bundle L on X, there is a group homomorphism $\Phi_L \colon X \longrightarrow \mathrm{Pic}(X)$ via $x \mapsto T_x^* L \otimes L^{-1}$. Be careful T_x^* is $-x$, convention, who knows why.

Example 1.5.1. Let $X = E$ an elliptic curve, $L = L((0))$, $x \mapsto (x) - (0)$, in this case this is in $\mathrm{Pic}^0(E) \cong E \cong \widehat{E}$,

Proposition 1.5.2. *This is translation invariant.*

Proof. Translate by $q \in E$. $(x+q) - (q)$ take p to be the third point on the line with x, q, $(x) + (q) + (p) \cong 3(0)$ and $(x+q) + (p) \cong 2(0)$ subtracting these gives $(x) - (x+q) + (q) \cong (0)$ or $(x) - (0) \cong (x+q) - (q)$. □

What about the converse of this, what can we say about translation invariant line bundles

$$K(L) = \{x \in X : T_x^* L \cong L\}?$$

Proposition 1.5.3. *$K(L)$ is Zariski closed in X.*

Proof. Consider $m^*L \otimes p_2^* L^{-1}$ on $X \times X$, then

$$\{x : \text{this is trivial on } \{x\} \times X\}$$

is closed. Seesaw implies restriction is pullback

$$T_x^* L \otimes L^{-1}$$

so this is $K(L)$. □

1.5.2 In which Pooh discovers our main theorem

Proposition 1.5.4. *Let X be an abelian variety and L a line bundle, $L = L(D)$ then TFAE:*

1. *$H(D) = \{x \in X : T_x^* D = D\}$ is finite.*
2. *$K(L) = \{x \in X : T_x^* L \cong L\}$ is finite.*
3. *$|2D|$ is basepoint free and defines a finite morphism $X \longrightarrow \mathbf{P}^N$.*
4. *L is ample.*

Proof. 3. to 4.. Is algebraic geometry.

2. to 1.. Follows as being equal is stronger than being linearly equivalent.

4. to 2.. Section 3

3. to 4.. Section 4 □

1.5.3 In which Owl proves the ampleness of L implies finiteness of $K(L)$

4. to 2. Assume L ample and $K(L)$ is infinite. Let Y be the connected component at 0 of $K(L)$, $\dim Y > 0$. Show trivial bundle is ample on Y implies Y is affine, But Y is closed and therefore complete so this is a contradiction. $L|_Y$ ample $[-1]^* L|_Y]$is ample. $L|_Y \otimes [-1]^* L|_Y$ is ample, consider

$$\begin{aligned} d \colon Y &\longrightarrow Y \times Y \\ y &\longmapsto (y, -y) \end{aligned}$$

$m \circ d =$ constant, $d^* m^*(L) = \mathcal{O}_Y$, LHS is $L|_Y \otimes [-1]^* L|_Y$.

1.5.4 In which Rabbbit sets out on a long journey to prove finiteness of $H(D)$ implies $|2D|$ is basepoint free and gives a finite map $X \longrightarrow \mathbf{P}^N$

Note 1.5.5. $|2D|$ is always basepoint free.

Apply the theorem of the square 1.4.15: $T_{x+y}^*D + D \cong T_x^*D + T_y^*D$, let $y = -x$, $2D \cong T_x^*D + T_{-x}^*D$. ($D$ effective) For any $y \in X$, choose some x s.t. RHS doesn't contain y. $E = 2D$

$$\psi_E \colon X \longrightarrow \mathbf{P}^N$$

can we make this finite? If ψ_E is not finite then $\psi(C) = \mathrm{pt}$ for some irreducible curve C (Zariski's main theorem). For each divisor in $|E|$ either it contains C or fails to intersect C by changing E if necessary, assume $E \cap C = \emptyset$.

Claim 1.5.6. *$T_x^*E \cap C = \emptyset$ or all of C for all $x \in X$.*

Proof. Intersection numbers are constant. □

Proof. $\mathcal{O}(T_x^*E)|_{\tilde{C}}$, when $x = 0$ this is trivial so $\deg = 0$. So $\deg = 0$ for all line bundles. E effective implies $C \cap T_x^*E = \emptyset$ for all x s.t. $\cap$ is not in C. □

Claim 1.5.7. *E is invariant by translation by $x - y$ for $x, y \in C$.*

Proof. If $e \in E$, $T_{x-e}^*(E) \cap C \neq \emptyset$. This is as x is in it, $x - (x-e) = e$, because it is nonempty it's all of C. So y is in it. So $y - (x-e) \in E$. This is also $e - (x-y) \in E$, so E is invariant under T_{x-y}^* □

Now assume $H(E) = \{x \in X : T_x^*E = E\}$ is finite. But if $\psi_E(C) = \mathrm{pt}$ then $T_{x-y}^*(E) = E$ for all $x, y \in C$. So H is not finite, a contradiction. So ψ_E can't collapse a curve so ψ_E is finite.

1.5.5 In which Piglet discovers a corollary

Corollary 1.5.8. *Abelian varieties are projective.*

Proof. Let X be an abelian variety, $U \subseteq X$ be an open affine set, $0 \in U$, $X \setminus U = D_1 \cup \cdots \cup D_t$ irreducible divisors. Let $D = \sum D_i$, then claim: $H(D) = \{x \in X : T_x^*D = D\}$ is finite. If $H \subseteq U$, U affine, then H closed subvariety of an abelian variety, hence complete, so its finite. If $x \in H$ then $-x \in H$. Now claim that if $x \in H$ then T_x^* preserves U, if not let $u \in U$. Suppose $u - x = d$ for some $d \in D$ then $u = d + x$ which is d translated by $-x$ so $d + x \in D$ so $u \in D$. But contradiction, oh no! So T_x^* preserves U, for all $x \in H$, as $0 \in U$, for all $x \in H$ we have $0 - x \in U$ and $0 + x \in U$ so $H \subseteq U$. □

Corollary 1.5.9. *Abelian varieties are divisible. $X[n]$ is finite for $n \geq 1$.*

Proof. $[n] \colon X \longrightarrow X$ and $X[n]$ is the kernel of this. Note that for $x \in X[n]$

$$[n] \circ T_x = [n]$$

$y \in X$, then $n(y - x) = ny - nx = ny$ so for all $L \in \operatorname{Pic} X$

$$T_x^*([n]^*L) \cong ([n]^*L)$$

which implies

$$K([n]^*L) \supseteq X[n]$$

and we just need to find L s.t. this is finite. X projective implies there exists an ample L. The theorem of the cube 1.4.11 implies

$$[n]^*L \cong L^{\frac{n^2+n}{2}} \otimes L^{\frac{n^2-n}{2}}$$

where both terms on the right are ample, hence the left is also. □

1.5.6 Epilogue: In which we might discuss isogenies

Definition 1.5.10. $f\colon X \to Y$ a morphism of varieties, get a field extension $k(X)/f^*k(Y)$, if $\dim X = \dim Y$ and f is surjective. Then this is a finite field extension and $\deg f$ is $d = \left[k(X) : f^*k(Y)\right]$ and $d = \#f^{-1}(y)$ for almost all y.

Definition 1.5.11. A homomorphism of abelian varieties $f\colon X \to Y$ is an **isogeny** if f is surjective with finite kernel.

Corollary 1.5.12. *Degree of $[n]$ is n^{2g}, if n is prime to the characteristic of k, $k = \overline{k}$, $g = \dim X$.*

Proof. Let D be an ample symmetric divisor, e.g.

$$D = D' + [-1]^*D'$$

know $[n]^* D \sim n^2 D$

$$\deg([n]^*(D \cdot \ldots \cdot D)) = ([n]^*D \cdot \ldots \cdot [n]^*D) = (n^2 D \cdot \ldots \cdot n^2 D) = n^{2g}(D \cdot \ldots \cdot D). \qquad \square$$