Deformation theory of Galois representations

MA842 at BU Spring 2020

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These are notes for Robert Pollack's course MA842 at BU Spring 2020.

The course webpage is http://math.bu.edu/people/rpollack/Teach/842spring2020.html.

1 Motivation

Lecture 1 21/1/2020

Let E_k denote the Eisenstein series of weight k, $k > 2$.

$$E_k = \frac{-B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in M_k(\mathrm{SL}_2(\mathbf{Z})).$$

Where B_k are the Bernoulli numbers and

$$\sigma_{k-1}(n) = \sum_{d|n,\, d>0} d^{k-1}.$$

E_2 however is not holomorphic, so not a modular form.

Fix N a prime, notation has stuck from Mazur's Eisenstein ideal paper.

Then there exists a unique Eisenstein series on $\Gamma_0(N)$ of weight 2.

$$E_2^{(N)} = \frac{N-1}{12} + \sum_{n=1}^{\infty} \sigma(n) q^n.$$

Funny observation: if $N \equiv 1 \pmod p$ for prime $p > 3$. Then $p | ((N-1)/12)$, so $E_2^{(N)}$ "looks cuspidal".

Then we hope that there exists a cuspidal eigenform $f \in S_2(\Gamma_0(N))$ such that

$$f \equiv E_2^{(N)} \qquad \text{“} \mod p\text{”}.$$

This is in fact true, due to Koike in the 70's, there exists $f \in S_2(\Gamma_0(N))$ such that

$$a_\ell(f) \equiv 1 + \ell \pmod p$$

for all $\ell \neq N, p$.

Question 1.1 How many such f are there? □

Merel '96:

$$f \text{ is unique } \iff \prod_{i=1}^{(N-1)/2} i^i \text{ is not a } p\text{-th power modulo } N.$$

Wake and Wang-Erickson describe the dimension of the space of such f using Massey products (higher cup products).

Method: Galois deformations!

1.1 Galois representations

We write

$$G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) = \varprojlim_{F/\mathbf{Q},\text{fin. galois}} \mathrm{Gal}(F/\mathbf{Q})$$

a profinite group.

$$\rho\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Q}_p)$$

a continuous homomorphism. Then view $\mathrm{GL}_2(\mathbf{Q}_p)$ as $\mathrm{Aut}(V)$ for a 2-dimensional $\mathbf{Q}_p$ vector space and fix a 2-dimensional $\mathbf{Z}_p$-lattice

$$T \subseteq V$$

which is $G_{\mathbf{Q}}$ stable. Then we can take

$$\bar{\rho}\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_p)$$

this is unique (w.r.t. the choice of T) only up to semisimplification.

So we say two Galois representations ρ_1, ρ_2 are congruent if

$$\bar{\rho}_1^{\mathrm{ss}} \simeq \bar{\rho}_2^{\mathrm{ss}}.$$

We say ρ_1, ρ_2 are deformations of

$$\bar{\rho}_1 = \bar{\rho}_2$$

(imagine this is reducible).

Start with

$$\bar{\rho}\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_p)$$

consider "all" deformations of $\bar{\rho}$ in good cases there exists a "universal" deformation of $\bar{\rho}$.

R^{univ} a local ring with maximal ideal $\mathfrak{m}_R$ such that

$$R/\mathfrak{m}_R = \mathbf{F}_p.$$

$$\rho^{\mathrm{univ}}\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(R^{\mathrm{univ}})$$

such that if $\rho\colon G_{\mathbf{Q}} \to GL_2(R)$ is a deformation of $\bar{\rho}$ then there exists

$$R^{\mathrm{univ}} \to R$$

such that

$$\begin{tikzcd} G_{\mathbf{Q}} \arrow[r, "\rho^{\mathrm{univ}}"] \arrow[dr, "\rho"'] & \mathrm{GL}_2(R^{\mathrm{univ}}) \arrow[d] \\ & GL_2(R) \end{tikzcd}.$$

1.2 Modular forms

$$f = \sum a_n q^n \in S_k(\Gamma_0(N))$$

an eigenform leads to

$$\rho_f \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(K),\ K/\mathbf{Q}_p \text{ finite}$$

with the property that for all $\ell \nmid Np$ we have

$$\mathrm{Tr}(\rho_f(\mathrm{Frob}_\ell)) = a_\ell.$$

Modular forms can be congruent

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod p \text{ for all but finitely many } \ell$$

$$\Updownarrow$$

$$\bar\rho_{f_1}^{\mathrm{ss}} \simeq \bar\rho_{f_2}^{\mathrm{ss}}.$$

There exists a ring, the Hecke algebra $\mathbf{T}$ parametrizing all f's with the same $\bar\rho$.

$$f \rightsquigarrow \rho_f \implies R^{\mathrm{univ}} \to \mathbf{T}$$

so hope

$$R^{\mathrm{univ}} \simeq \mathbf{T}.$$

Wiles proof of FLT proved one of these.

Many more such theorems in the past couple of decades.

Wake and Wang-Erickson show that the dimension of

$$\{f : f \equiv E_2^{(N)}\} \leftrightarrow \operatorname{rank} \mathbf{T} = \operatorname{rank} R^{\mathrm{univ}}.$$

$$a_\ell(f) \equiv 1 + \ell \pmod p$$

$$\implies \bar\rho^{\mathrm{ss}} = \mathbf{1} \oplus \mu_p$$

but there does not exist R^{univ} in this context.

The fix is to use pseudorepresentations instead of representations.

1.3 Pseudorepresentations

Let G be a group.

Then a pseudorepresentation T is a map

$$T \colon G \to A$$

for A a ring satisfying

1. $$T(xy) = T(yx)$$

2. $$T(x)T(y)T(z) - T(x)T(yz) - T(y)T(xz) - T(z)T(xy) + T(xyz) + T(xzy) = 0$$

and the analogous formulae for higher dimensions.

Fact 1.2 *If A is an algebraically closed field of characteristic $\neq 2$. Then for a given pseudorepresentation T there exists a true representation ρ such that*

$$T = \mathrm{Tr}(\rho).$$

But this does not hold in general.

Universal pseudodeformation rings always exist. Wake and Wang-Erickson use R^{univ} = universal pseudodeformation ring.

2 Definitions

2.1 Representations

Lecture 2 23/1/2020

Definition 2.1 Representations. Let G be a finite group and V a finite dimensional vector space over $\mathbf{C}$ of dimension d. A **representation** of G is a homomorphism

$$G \xrightarrow{\rho} \mathrm{Aut}(V) \simeq \mathrm{GL}_d(\mathbf{C}),$$

G acts linearly on V. ◊

Galois representations: Let G be a Galois group, possibly infinite. F be a field,

$$G_F = \mathrm{Gal}(\overline{F}/F) = \varprojlim_{L/F,\text{ fin. gal.}} \mathrm{Gal}(L/F)$$

profinite compact and totally disconnected.

Replace V with a finite free module over some topological ring A

$$\rho\colon G_F \to \mathrm{GL}_d(A)$$

a continuous homomorphism.

Example 2.2 $A = \mathbf{C}$ with the complex topology. □

Fact 2.3 *In this case* $\mathrm{im}(\rho)$ *is finite.*

Exercise 2.4 Prove this.

Then we can write

$$\begin{tikzcd} G_F \arrow[rr, "\rho"] \arrow[dr] & & \mathrm{GL}_d(\mathbf{C}) \\ & \mathrm{Gal}(L/F) \arrow[ur] & \end{tikzcd}$$

where L/F is finite. There are many such representations.

Conjecture 2.5 *Every finite group is a quotient of* $G_{\mathbf{Q}}$.

Example 2.6

$$F = \mathbf{Q}$$

$$L = \mathbf{Q}(\sqrt[4]{2}, i)$$

$$\mathrm{Gal}(L/\mathbf{Q}) \simeq D_4$$

$$G_{\mathbf{Q}} \to \mathrm{Gal}(L/\mathbf{Q}) \xrightarrow{\rho} \mathrm{GL}_2(\mathbf{C})$$

with ρ the unique irreducible 2-dimensional representation of D_4. □

Example 2.7 Let $E/\mathbf{Q}$ be an elliptic curve

$$G_{\mathbf{Q}} \circlearrowright E[p] \simeq \mathbf{Z}/p \oplus \mathbf{Z}/p$$

$$\begin{aligned} \rho_{E,p} \colon G_{\mathbf{Q}} &\to \operatorname{Aut}(E[p]) \simeq \mathrm{GL}_2(\mathbf{F}_p) \\ \rho_{E,p^n} \colon G_{\mathbf{Q}} &\to \operatorname{Aut}(E[p^n]) \simeq \mathrm{GL}_2(\mathbf{Z}/p^n) \\ \rho_{E,p^\infty} \colon G_{\mathbf{Q}} &\to \operatorname{Aut}(E[p^\infty]) \simeq \mathrm{GL}_2(\mathbf{Z}_p). \end{aligned}$$

□

Fact 2.8 *ρ_{E,p^∞} has finite image.*

Exercise 2.9 Prove this.

Example 2.10 Let $F = \mathbf{Q}_\ell$, $A = \mathbf{Q}_p$

$$\rho \colon G_{\mathbf{Q}_\ell} \to \mathrm{GL}_d(\mathbf{Q}_p).$$

□

The Galois theory of local fields looks like

$$\begin{tikzcd}
& \overline{\mathbf{Q}}_\ell \arrow[d, dash, "I_\ell^w"] \arrow[dd, dash, bend right, "I_\ell"'] \arrow[ddd, dash, bend right=60, "G_\ell"'] & \\
& \mathbf{Q}_\ell^{\mathrm{tr}} \arrow[d, dash, "\prod_{q \neq \ell} \mathbf{Z}_q(1)"] & \mathbf{Q}_\ell^{\mathrm{ab}} \\
& \mathbf{Q}_\ell^{\mathrm{ur}} \arrow[d, dash, "\widehat{\mathbf{Z}}"] \arrow[ur, dash, bend right, "\mathbf{Z}_\ell^*"'] & \\
& \mathbf{Q}_\ell &
\end{tikzcd}$$

where

$$\mathbf{Q}_\ell^{\mathrm{tr}} = \mathbf{Q}_\ell(\{\sqrt[n]{\ell}\}_{\ell \nmid n})$$

the maximal tamely ramified extension

$$\mathbf{Q}_\ell^{\mathrm{ur}} = \mathbf{Q}_\ell(\{\mu_n\}_{\ell \nmid n})$$

the maximal unramified extension

$$\mathbf{Q}_\ell^{\mathrm{ab}} = \mathbf{Q}_\ell^{\mathrm{un}}(\mu_{\ell^\infty})$$

the maximal abelian extension.

$$I_\ell = \operatorname{Gal}(\overline{\mathbf{Q}}_\ell/\mathbf{Q}_\ell^{\mathrm{ur}})$$

the intertia group

$$I_\ell^w = \operatorname{Gal}(\overline{\mathbf{Q}}_\ell/\mathbf{Q}_\ell^{\mathrm{tr}})$$

the wild intertia group

We say

$$\rho \colon G_{\mathbf{Q}_\ell} \to \mathrm{GL}_d(\mathbf{Q}_p)$$

is **unramified** if

$$\rho(I_\ell) = \{1\}$$

is **tamely ramified** if

$$\rho(I_\ell^w) = \{1\}.$$

In the first case ρ is completely determined by $\rho(\mathrm{Frob}_\ell)$. In the second case ρ is completely determined by $\rho(\mathrm{Frob}_\ell)$ and its value on a generator of

$$\mathrm{Gal}(\mathbf{Q}_\ell^{\mathrm{tr}}/\mathbf{Q}_\ell^{\mathrm{ur}}).$$

The wild part: I_ℓ^w is pro-ℓ, $\mathrm{GL}_d(\mathbf{Z}_p)$ is almost pro-p (it has a finite index pro-p subgroup).

Exercise 2.11 Prove this.

Example 2.12 For $d = 1$

$$\mathbf{Z}_p^* = \mathbf{F}_p^* \times (1 + p\mathbf{Z}_p).$$

□

Thus if $\ell \neq p$ then

$$\rho(I_\ell^w) \subseteq \mathrm{GL}_d(\mathbf{Q}_p)$$

is finite.

Exercise 2.13 Prove this.

If $\ell = p$ then this is handled by p-adic Hodge theory.

The connection to global representations is then that

$$\begin{tikzcd} \overline{\mathbf{Q}} \arrow[r] & \overline{\mathbf{Q}}_\ell \\ \mathbf{Q} \arrow[u] \arrow[r] & \mathbf{Q}_\ell \arrow[u] \end{tikzcd}$$

So

$$G_{\mathbf{Q}_\ell} \hookrightarrow G_{\mathbf{Q}}$$

via restriction to $\overline{\mathbf{Q}}$.

The image of this map is the **decomposition group** at ℓ.

$$\rho\colon G_{\mathbf{Q}} \to \mathrm{GL}_d(A),$$

we say that ρ is **unramified at ℓ** if

$$\rho(I_\ell) = \{1\}.$$

In which case

$$\mathrm{charpoly}(\rho(\mathrm{Frob}_\ell))$$

is well-defined.

Returning to

$$\rho_{E,p^\infty}\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Q}_p)$$

now.

Fact 2.14 *ρ_{E,p^∞} is unramified outside of $N_E \cdot p$ (N_E is the conductor of E). i.e.*

$$\rho_{E,p^\infty}$$

is unramified at ℓ if and only if $\ell \neq p$ and ℓ is a prime of good reduction for E.

So ρ_{E,p^∞} sees bad reduction.

From ρ_{E,p^∞} you can recover E up to isogeny (Faltings).

Example 2.15

$$G_{\mathbf{Q}} \circlearrowright \mu_{p^n}$$

so we get

$$G_{\mathbf{Q}} \to \mathrm{Aut}(\mu_{p^n}) \simeq (\mathbf{Z}/p^n\mathbf{Z})^* \simeq \mathrm{GL}_1(\mathbf{Z}/p^n)$$

taking the inverse limit we get

$$\begin{tikzcd} G_{\mathbf{Q}} \arrow[rr, "\epsilon_p"] \arrow[dr] & & \mathbf{Z}_p^* \\ & \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \arrow[ur, "\sim"'] & \end{tikzcd}$$

this ϵ_p is known as the p-adic cyclotomic character. This is unramified outside p and

$$\epsilon_p(\mathrm{Frob}_\ell) = \ell$$

for $\ell \neq p$. $\square$

Remark 2.16

$$\det(\rho_{E,p^\infty}) = \epsilon_p.$$

Example 2.17

$$f = \sum a_n q^n \in S_2(\Gamma_0(N), \mathbf{Q})$$

a weight 2 eigenform on $\Gamma_0(N)$, with rational fourier coefficients. Eichler-Shimura gives $E_f/\mathbf{Q}$ an elliptic curve. Define

$$\rho_f = \rho_{E,p^\infty} \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Q}_p)$$

unramified outside Np

$$\mathrm{Tr}(\rho(\mathrm{Frob}_\ell)) = a_\ell$$

for $\ell \nmid Np$. More generally

$$f \in S_2(\Gamma_0(N))$$

an eigenform, Eichler-Shimura gives

$$A_f/\mathbf{Q}$$

an abelian variety which leads to

$$\rho_f \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(K)$$

$K/\mathbf{Q}_p$ finite. $\square$

2.2 Congruences and elliptic curves

Let

$$E_1 \colon y^2 = x^3 + x - 10$$

conductor $2^2 \cdot 13$, https://www.lmfdb.org/EllipticCurve/Q/52a1/

$$E_2 \colon y^2 = x^3 - 584x + 5444$$

conductor $2^2 \cdot 7 \cdot 13$, https://www.lmfdb.org/EllipticCurve/Q/364a1/.

Table 2.18 a_p**'s for** E_1, E_2

p	2	3	5	7	11	13	17	19	23	29
$a_p(E_1)$	0	0	2	-2	-2	-1	6	-6	8	2
$a_p(E_2)$	0	0	-3	1	-2	-1	-4	-1	-7	7

Note that
$$a_\ell(E_1) \equiv a_\ell(E_2) \pmod 5, \forall \ell \neq 7$$
$$\implies \rho_{E_1,5} \simeq \rho_{E_2,5} (= \bar{\rho})$$
as Galois representations.

Exercise 2.19 Prove this.

How common is this? We have 2 lifts of $\bar{\rho}$
$$\rho_{E_1,5^\infty} \simeq \rho_{E_2,5^\infty}$$
how many other such?

2.3 Hida theory

$$\sum a_n(f)q^n = f \in S_{k_0}(\Gamma_0(N))$$
an eigenform.
$$a_p(f)$$
a p-adic unit.
$$\mathscr{F} = \sum_{n=1}^{\infty} a_n(k)q^n$$
with a_n a p-adic analytic function in k. The whole family gives Galois representations that reduce to the same $\bar{\rho}$.

Specialise k to some integer w
$$\mathscr{F} = \sum a_n(w)q^n \in S_w(\Gamma_0(N))$$
take $w = k_0$ to recover f.

Hida constructs
$$\rho^{Hida} \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_p[[x]])$$
unramified outside Np.
$$\ell \nmid Np \implies \mathrm{tr}(\rho^{Hida}(\mathrm{Frob}_\ell)) = a_\ell(x).$$

2.4 More examples of Galois representations in families

Lecture 3 28/1/2020

A 1-dimensional family. let
$$\epsilon_p = \text{cyclotomic character}$$
$$\epsilon_p \colon G_{\mathbf{Q}} \to \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \simeq \mathbf{Z}_p^* \simeq \mathbf{F}_p^* \times (1 + p\mathbf{Z}_p)$$
If we have $k \in \mathbf{Z}$ we can take
$$\epsilon_p^k$$
its power.

Note that when $k_1, k_2 \in \mathbf{Z}$ with
$$k_1 \equiv k_2 \pmod{p-1}$$

then

$$\epsilon_p^{k_1} \equiv \epsilon_p^{k_2} \pmod{p}$$

moreover if

$$k_1 \equiv k_2 \pmod{(p-1)p^N}$$

we have

$$\epsilon_p^{k_1} \equiv \epsilon_p^{k_2} \pmod{p^{N+1}}.$$

Set

$$\Lambda = \mathbf{Z}_p[[\mathbf{Z}_p^*]] = \varprojlim \mathbf{Z}_p[(\mathbf{Z}/p^N)^*]$$

$$\epsilon_p^{univ} \colon G_{\mathbf{Q}} \to \Lambda^* = \mathrm{GL}_1(\Lambda)$$

$$\sigma \mapsto [\epsilon_p(\sigma)]$$

$$\Lambda \xrightarrow{wt_k} \mathbf{Z}_p$$

$$wt_k \circ \epsilon_p^{univ}$$

where wt_k is defined by

$$\mathbf{Z}_p^* \to \mathbf{Z}_p^* \subseteq \mathbf{Z}_p$$

$$x \mapsto x^k$$

gives

$$\Lambda \xrightarrow{wt_k} \mathbf{Z}_p.$$

Take any

$$\phi \in \mathrm{Hom}_{cts}(\Lambda, \mathbf{C}_p) \simeq \mathrm{Hom}_{cts}(\mathbf{Z}_p^*, \mathbf{C}_p)$$

induces

$$G_{\mathbf{Q}} \to \mathbf{C}_p^*$$

$$\phi \circ \epsilon_p^{univ}.$$

$$\mathrm{Hom}_{cts}(\mathbf{Z}_p^*, \mathbf{C}_p)$$

is a union of $p-1$ open disks as

$$\mathbf{Z}_p^* \simeq \mathbf{F}_p^* \times (1 + p\mathbf{Z}_p)$$

the rightmost factor is topologically generated by $1+p$. So

$$\phi \colon \mathbf{Z}_p^* \to \mathbf{C}_p^*$$

is determined by

$$\phi|_{\mathbf{F}_p^*}$$

and

$$\phi(\gamma) \text{ for any } \gamma \text{ a top. gen..}$$

We need

$$|\phi(\gamma) - 1|_p < 1.$$

We have $p-1$ disks labelled by characters of $\mathbf{F}_p^*$. On each disk of $\phi \in \mathrm{Hom}(\mathbf{Z}_p^*, \mathbf{C}_p^*)$ the reduction of

$$\phi \circ \epsilon_p^{univ} \colon G_{\mathbf{Q}} \to \mathbf{C}_p^*$$

is the same.

A 2-d example. Let $E/\mathbf{Q}$ be a CM elliptic curve.

$$\rho_{E,p^\infty}\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_p)$$

then there exists $K/\mathbf{Q}$ quadratic imaginary field and, if p is an ordinary prime

$$\psi\colon G_K \to \mathbf{Z}_p^*$$

s.t.

$$\rho_{E,p^\infty} \simeq \mathrm{Ind}_K^{\mathbf{Q}}(\psi).$$

So we get a "family"

$$\mathrm{Ind}_K^{\mathbf{Q}}(\psi^k),$$

more generally can make

$$\psi^{univ}\colon G_K \to \Lambda^*$$

$$\mathrm{Ind}\,\psi^{univ}\colon G_K \to \mathrm{GL}_2(\Lambda).$$

CM Hida family.

2.5 Deformation theory

G a profinite group, k a finite field of characteristic p (simpler $k = \mathbf{F}_p$).

$$\bar\rho\colon G \to \mathrm{GL}_d(k)$$

a continuous homomorphism.

To lift $\bar\rho$ to be A-valued, we need a homomorphism

$$A \twoheadrightarrow k$$

we may as well assume A is local so it lifts to $GL_d(A)$. We then only have one residual representation. Localize it at

$$\ker(A \to k).$$

If A is a local ring then A has a natural topology. Letting $\mathfrak{m}_A$ be the maximal ideal

$$\{\mathfrak{m}_A^j\}$$

is a neighborhood base at 0, this is the $\mathfrak{m}$-adic topology.

So we have A local, $\mathfrak{m}_A$ maximal $A/\mathfrak{m}_A \simeq k$, we need to fix this identification to ensure we don't have automorphisms.

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[dr, "\bar\rho"'] & \mathrm{GL}_d(A) \arrow[d] \\ & \mathrm{GL}_d(k) \end{tikzcd} .$$

Note 2.20 If we have $M \in \mathrm{GL}_d(A)$ s.t. $M \equiv 1 \pmod{\mathfrak{m}_A}$, then

$$M\rho M^{-1}$$

lifts $\bar\rho$ also.

Definition 2.21 Call ρ and $M\rho M^{-1}$ **strictly equivalent**. ◊

Definition 2.22 A **deformation of $\bar{\rho}$ to A** is a strict equivalence class of continuous homomorphisms

$$G \xrightarrow{\rho} \mathrm{GL}_d(A)$$

lifting $\bar{\rho}$. ◊

Want to rephrase this with less bases, instead of $\bar{\rho}$ use $\overline{V}$ a d-dimensional continuous representation of G over k. V a free A-module of rank d with continuous action of G s.t.

$$V \otimes A/\mathfrak{m}_A \simeq \overline{V}.$$

Naive hope: two such V's are equivalent if they are isomorphic as G-modules.

Problem: R^{univ} won't exist, too many automorphisms. Ideally if V_A is a deformation of $\overline{V}$ to A then there exists a unique

$$R^{univ} \to A$$

inducing

$$V^{univ} \to V_A.$$

Take II, let V be a free A-module with a continuous action of G and a fixed isomorphism.

$$V \otimes_A A/\mathfrak{m}_A \simeq \overline{V}$$

as representations of G over $k \simeq A/\mathfrak{m}_A$.

Two V's are equivalent if there exists a G-module isomorphism $V_1 \xrightarrow{\phi} V_2$ s.t.

$$\begin{tikzcd}
V_1 \arrow[r, "\phi"] \arrow[d] & V_2 \arrow[d] \\
V_1 \otimes A/\mathfrak{m}_A \arrow[d, "\sim"'] & V_2 \otimes A/\mathfrak{m}_A \arrow[d, "\sim"'] \\
\overline{V} \arrow[r, "\mathrm{id}"] & \overline{V}
\end{tikzcd}.$$

Exercise 2.23 Check the two definitions are the same, GL_d definition vs. $\overline{V}$.

Another approach

$$\begin{tikzcd}
G \arrow[r, "\rho"] \arrow[dr, "\bar{\rho}"'] & \mathrm{GL}_d(A) \arrow[d] \\
& \mathrm{GL}_d(k)
\end{tikzcd}$$

lift of $\bar{\rho}$ and say two ρ's are the same if they are equal as maps, this is a **framed deformation**. We can write this more abstractly

$$\overline{V}$$

a continuous representation of G over k with fixed basis β.

$$V$$

a free A module with cont. action of G and a fixed isom.

$$V \otimes_A A/\mathfrak{m}_D \simeq \overline{V}$$

and a basis β_A lifting β.

Categories

$$\mathcal{C}_k = \mathcal{C}$$

the category of complete local Noetherian rings A with a fixed isomorphism $A/\mathfrak{m}_A \simeq k$. Where the maps are

$$A_1 \xrightarrow{\phi} A_2$$

a local ring homomorphism

$$\phi(\mathfrak{m}_{A_1}) \subseteq \mathfrak{m}_{A_2}$$

and the diagram

$$\begin{tikzcd} A_1/\mathfrak{m}_{A_1} \arrow[rr, "\bar{\phi}"] \arrow[dr, "\sim"'] & & A_2/\mathfrak{m}_{A_2} \arrow[dl, "\sim"] \\ & k & \end{tikzcd}$$

commutes.

Lecture 4 30/1/2020

Last time: multiplication by c on R^{univ} is not a ring map!

Exercise 2.24 Show that the "bad" definition of deformation (without fixing the isomorphism to $\overline{V}$) can't be representable.

Exercise 2.25 If $\mathrm{Aut}_G(\bar{\rho}) \supsetneq k^*$ show that the correct definition of deformation doesn't lead to a representable functor.

k finite field of characteristic p. $\mathcal{C}_k$ category of complete local Noetherian rings with a fixed identification

$$A/\mathfrak{m}_A \simeq k$$

with $(A, \mathfrak{m}_A)$ a local ring, maps are local homomorphisms preserving the fixed identification.

Witt vectors. $W(k)$ = Witt vectors of k and an object of $\mathcal{C}_k$. The maximal ideal of $W(k)$ is $pW(k)$.

$$W(k)/pW(k) \simeq k$$

$$W(k)^\times \to k^\times$$

this map has a section which is multiplicative.

$$[x] \leftarrow\!\shortmid x.$$

Example 2.26 If $k \simeq \mathbf{F}_q$, $q = p^n$ then

$$W(k) = \mathcal{O}_{K_n},\ K_n = \mathbf{Q}_p(\mu_{p^n-1})$$

unramified extension with

$$\mathfrak{m}_{W(k)} = (p)$$

take x^{q-1} and observe that its roots are distinct mod p. For $W(\mathbf{F}_p) \simeq \mathbf{Z}_p$. $\square$

$\bar{x} \in k$, lift to $x \in W(k)$

$$\lim_{m\to\infty} x^{q^m} = [x].$$

Exercise 2.27 Check this.

$A \in \mathcal{C}_k$ there exists a unique ring hom

$$\begin{tikzcd} W(k) \arrow[r] \arrow[d] & A \arrow[d] \\ k \arrow[r] & A/\mathfrak{m}_A \end{tikzcd}$$

A is a $W(k)$-module.

Moreover, A is a quotient of $W(k)[[x_1, \dots,]]$. For $A, A' \in \mathcal{C}_k$

$$A \xrightarrow{\phi} A'$$

in $\mathcal{C}_k$ then ϕ is automatically $W(k)$-linear.

Deformation functor.

$$\bar{\rho}\colon G \to \mathrm{GL}_d(k)$$

$$D_{\bar{\rho}}\colon \mathcal{C}_k \to \mathrm{Set}$$

$$A \mapsto \{\text{deformations of } \bar{\rho} \text{ to } A\} = \{\rho\colon G \to \mathrm{GL}_d(A) \text{ lifting } \bar{\rho}\}/\text{strict equiv..}$$

Given

$$R \in \mathcal{C}_k$$

we have

$$F_R\colon \mathcal{C}_k \to \mathrm{Set}$$

$$A \mapsto \mathrm{Hom}(R, A).$$

Exercise 2.28 If $A \xrightarrow{\phi} A'$ is a local ring Hom which is $W(k)$-linear then ϕ is a map in $\mathcal{C}_k$.

Definition 2.29 Representable functors. Any functor

$$F\colon \mathcal{C}_k \to \mathrm{Set}$$

is said to be representable if there exists $R \in \mathcal{C}_k$ such that

$$F \simeq F_R.$$

◊

To say $D_{\bar{\rho}}$ is representable says that

$$\exists R_{\bar{\rho}}^{univ} \in \mathcal{C}_k$$

such that

$$D_{\bar{\rho}}(A) \simeq \mathrm{Hom}(R_{\bar{\rho}}^{univ}, A)$$

which is functorial in A.

In this set-up take $A = R_{\bar{\rho}}^{univ}$. Then

$$D_{\bar{\rho}}(R_{\bar{\rho}}^{univ}) \simeq \mathrm{Hom}(R_{\bar{\rho}}^{univ}, R_{\bar{\rho}}^{univ}) \ni \mathbf{1}$$

so this corresponds to some V^{univ}

Now given

$$\rho_A\colon G \to \mathrm{GL}_d(A) \simeq \mathrm{Aut}(V_A)$$

we have $\rho_A \in D_{\bar\rho}(A) \simeq \mathrm{Hom}(R^{univ}_{\bar\rho}, A)$, so we get some $\phi\colon R^{univ}_{\bar\rho} \to A$. Giving a diagram

$$\begin{tikzcd} D_{\bar\rho}(R^{univ}_{\bar\rho}) \arrow[r, "\sim"] \arrow[d, "\phi"'] & \mathrm{Hom}(R^{univ}_{\bar\rho}, R^{univ}_{\bar\rho}) \ni \mathbf{1} \arrow[d] \\ D_{\bar\rho}(A) \arrow[r, "\sim"] & \mathrm{Hom}(R^{univ}_{\bar\rho}, A) \ni \phi \end{tikzcd}.$$

Framed deformations.

$$D^{\square}_{\bar\rho}(A) = \{\rho\colon G \to \mathrm{GL}_d(A) \text{ lifting } \bar\rho\}$$

we will prove that

$$D^{\square}_{\bar\rho}$$

is in fact representable.

$$\bar\rho\colon G \to \mathrm{GL}_d(k)$$

we build

$$R^{\square,univ}_{\bar\rho}$$

and

$$\rho^{\square,univ}_{\bar\rho}\colon G \to \mathrm{GL}_d(R^{\square,univ}_{\bar\rho}).$$

Let $\{g_\alpha\}_\alpha$ be a generating set of G. Make formal variables X^α_{ij} for all α, $1 \le i, j \le d$.

$$R = W(k)[[\{X^\alpha_{ij}\}]]$$

try sending

$$g_\alpha \mapsto (X^\alpha_{ij}) \in M_d(R)$$

doesn't land in GL_d.

So "add on $\bar\rho$".

$$k \to W(k)$$

$$x \mapsto [x]$$

so we get

$$\mathrm{GL}_d(k) \xrightarrow{[\cdot]} \mathrm{GL}_d(W(k))$$

$$[(a_{ij})_{ij}] \mapsto ([a_{ij}])_{ij}$$

now send

$$g_\alpha \mapsto (X^\alpha_{ij}) + [\bar\rho(g_\alpha)] \in \mathrm{GL}_d(R)$$

this does not give a map

$$G \to \mathrm{GL}_d(R)$$

because of relations between g_α.

Example 2.30 Take the relation

$$g_1 g_2^{-1} = e$$

need

$$((X^1_{ij}) + [\bar\rho(g_1)])((X^2_{ij}) + [\bar\rho(g_2)])^{-1} - 1 = 0$$

this gives d^2 equations in R. □

Can do this in general. In fact the LHS is in $\mathfrak{m}_R$. So if I is the ideal of all relations, then $I \subseteq \mathfrak{m}_R$. And let

$$R_{\bar{\rho}}^{univ,\square} = R/I.$$

$$\rho\colon G \to \mathrm{GL}_d(R/I)$$
$$g_\alpha \mapsto (X_{ij}^\alpha) + [\bar{\rho}(g_\alpha)] \pmod{I}$$

is a well-defined homomorphism. And it is universal: If

$$\rho_A\colon G \to \mathrm{GL}_d(A)$$

$$R \xrightarrow{\pi} A$$

$$(X_{ij}^\alpha)_{ij} \mapsto (\rho_A(g_\alpha) - [\bar{\rho}(g_\alpha)])_{ij}$$

then π kills I (exercise).

$$R/I \xrightarrow{\bar{\pi}} A.$$

Exercise 2.31 Check this induces ρ_A from the universal.

There are two problems with this, it is not necessary that the universal deformation we have constructed be continuous, or that the universal deformation ring be continuous.

The root of these problems is that there are potentially too many g_α. introduce a condition Φ_p that for all open finite index

$$G_0 \subseteq G$$

we have

$$\mathrm{Hom}(G_0, \mathbf{F}_p)$$

is finite.

Exercise 2.32

$$\mathrm{Hom}(G_0, \mathbf{F}_p)$$

is finite if and only if G^{pro-p} is topologically finitely generated. (this is Gouvea lemma 2.1)

If $G = G_K$ with $K/\mathbf{Q}_\ell$ finite, WLOG

$$G = G_0 = G_K$$

and Φ_p holds in this case

$$G_K^{\mathrm{ab}} \simeq \hat{K}^* \simeq \hat{\mathbf{Z}} \times \mathcal{O}_K^*$$

an exercise in LCFT

$$G_0 \subseteq G_K$$

fin index open then

$$G_0 = G_L$$

with $L/\mathbf{Q}_p$

Lecture 4 4/2/2020

Redo.

$$D_{\bar\rho}^{\square}\colon \mathcal{C} \to \mathrm{Set}$$

$$A \mapsto \left\{ \begin{tikzcd} G \arrow[r, "\rho"] \arrow[dr, "\bar\rho"'] & \mathrm{GL}_d(A) \arrow[d] \\ & \mathrm{GL}_d(k) \end{tikzcd} \right\}$$

$$D_{\bar\rho}\colon \mathcal{C} \to \mathrm{Set}$$

$$A \mapsto D_{\bar\rho}^{\square}/\text{strict equiv.}$$

Theorem 2.33

1. *$D_{\bar\rho}^{\square}$ is representable (by $R_{\bar\rho}^{\square}$)*

2. *If $\mathrm{Aut}_G\,\bar\rho = k^\times$ then $D_{\bar\rho}$ is representable ($R_{\bar\rho}$).*

Universal property: framed case:

$$\exists \rho_{univ}^{\square}\colon G \to \mathrm{GL}_d(R_{\bar\rho}^{\square})$$

s.t. if ρ_A is a lift of $\bar\rho$ to A then there exists a unique

$$R_{\bar\rho}^{\square} \xrightarrow{\phi} A$$

which is in $\mathcal{C}$ s.t.

$$\begin{tikzcd} G \arrow[r, "\rho_{univ}^{\square}"] \arrow[dr, "\rho_A"'] & \mathrm{GL}_d(A) \arrow[d, "\phi^*"] \\ & \mathrm{GL}_d(A) \end{tikzcd}$$

unframed case: there exists a deformation $[\rho_{univ}]$ of $\bar\rho$ to $R_{\bar\rho}$ s.t. if $[\rho_A]$ is a deformation of $\bar\rho$ to A then there exists a unique

$$R_{\bar\rho} \xrightarrow{\phi} A$$

s.t

$$\phi^*([\rho_{univ}]) = [\rho_A].$$

$$D_{\bar\rho}^{\square}(R_{\bar\rho}^{\square}) = \mathrm{Hom}(R_{\bar\rho}^{\square}, R_{\bar\rho}^{\square})$$

$$\rho_{univ}^{\square} \leftrightarrow \mathbf{1}.$$

Exercise 2.34 Check this universal property (also unframed case).

Last time.

$$R_{\bar\rho}^{\square} = W(k)[[\left\{X_{ij}^{\alpha}\right\}_{i,j,\alpha}]]/I_{relations}.$$

Introduced the condition

$$\Phi_p \forall \text{ open finite index } G_0 \subseteq G,\ \mathrm{Hom}(G_0, \mathbf{F}_p) \text{ is finite.}$$

This condition holds for G_K when $K/\mathbf{Q}_\ell$ is finite.

However it is false for G_K when $K/\mathbf{Q}$ is finite. Such as $K = \mathbf{Q}$. As for $\ell \equiv 1 \pmod p$ then

$$G_{\mathbf{Q}} \to \mathrm{Gal}(\mathbf{Q}(\mu_\ell)/\mathbf{Q}) \to \mathbf{F}_p$$

and there are infinitely many such ℓ's

This is not a problem as the representations we are interested in, those coming from elliptic curves and modular forms, have good reduction outside a finite set of primes S. We may encode this into our deformation problem as follows: Let $K/\mathbf{Q}$ finite and S a finite set of places of K and

$$K_S$$

be the maximal extension of K unramified outside of S. Then

$$G_{K,S} = \mathrm{Gal}(K_S/K).$$

And Φ_p is true of $G_{K,S}$. The proof is an exercise in Global CFT. Silverman $VIII$ prop. 1.6.

H profinite, let $H^{\text{pro-}p}$ be the maximal pro-p quotient of H.

Exercise 2.35 $\mathrm{Hom}(H, \mathbf{F}_p)$ finite implies that $H^{\text{pro-}p}$ is topologically finitely generated. This is an equivalence in fact (lemma 2.1 Gouvea).

Existence of the framed deformation ring (take II). Assume G satisfies Φ_p.

$$\bar{\rho}\colon G \to \mathrm{GL}_d(k),$$

$H = \ker(\bar{\rho})$ is finite index in G. Let

$$\rho_A\colon G \to \mathrm{GL}_d(A)$$

lift $\bar{\rho}$.

$$\rho_A(H) \subseteq \ker(\mathrm{GL}_d(A) \to \mathrm{GL}_d(k)).$$

Exercise 2.36 Prove this is pro-p.

Hence $\rho_A|_H$ factors through $H^{\text{pro-}p}$.

Pick topological generators $g_1, \dots, g_s$ of $H^{\text{pro-}p}$ (via Φ_p). Pick coset representatives $g_{s+1}, \dots, g_t$ of G/H.

Run the argument from last time to get

$$R^{\square}_{\bar{\rho}} = W(k)[[\left\{X^{\alpha}_{ij}\right\}_{\alpha=1,\dots,t,\, i,j=1,\dots,d}]]/\overline{I_{rel}}.$$

Space of universal framed deformations.

$$\mathrm{Hom}(R^{\square}_{\bar{\rho}}, A)$$

For unframed deformations it can be the case that we even have that

$$R_{\bar{\rho}} \simeq \mathbf{Z}_p[[X_1, X_2, X_3]]$$

$$A = \mathbf{Z}_p$$

$$D_{\bar{\rho}}(\mathbf{Z}_p) = \mathrm{Hom}_C(\mathbf{Z}_p[[X_1, X_2, X_3]], \mathbf{Z}_p) \simeq (p\mathbf{Z}_p)^3.$$

To prove this space exists for unframed deformations, we have 3 options

1. We should have a bigger space of framed deformations in general, which should quotient down to that of unframed deformations, by forgetting the framing. Then we could construct $R_{\bar{\rho}}$ using this quotient. Kisin's notes chapter 3.
2. Pseudo-deformations (for later).
3. Schlessinger's criterion.

One dimensional examples.

$$\begin{tikzcd} G \arrow[r, "\bar\rho=\bar\chi"] \arrow[dr, "\chi_A"'] & \mathrm{GL}_1(k) = k^\times . \\ & A^\times \arrow[u] \end{tikzcd}$$

$$1 \to 1 + \mathfrak{m}_A \to A^\times \to k^\times \to 1$$

$$A^\times \simeq k^\times \times (1 + \mathfrak{m}_A)$$

last term is free and $k^\times$ is determined by $\bar\chi$.

Need

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & 1 + \mathfrak{m}_A \\ & G^{\text{ab,pro-}p} \arrow[u] \end{tikzcd}$$

Any

$$\chi\colon G^{\text{ab,pro-}p} \to 1 + \mathfrak{m}_A$$

fixes a deformation of $\bar\chi$.

Claim 2.37

$$W(k)[[G^{\text{ab,}pro\text{-}p}]] \simeq R_{\bar\chi}$$

$$\mathrm{Hom}_{\mathcal{C}}(W(k)[[G^{\text{ab,}pro\text{-}p}]], A) \simeq \mathrm{Hom}(G^{\text{ab,}pro\text{-}p}, A^\times).$$

The universal character is then

$$G \to G^{\text{ab,pro-}p} \to W(k)[[G^{\text{ab,pro-}p}]]$$

$$\sigma \mapsto \sigma \mapsto \langle\sigma\rangle[\bar\chi(\sigma)].$$

$$\chi_A\colon G \to A^*?????????$$

check commutes

Example 2.38 $G = G_{\mathbf{Q},\{p,\infty\}}$

$$G^{\text{ab}} = G^{\text{ab}}_{\mathbf{Q},\{p,\infty\}} = \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \simeq \mathbf{Z}_p^*$$

$$G^{\text{ab,pro-}p} \simeq 1 + p\mathbf{Z}_p \simeq \mathbf{Z}_p$$

$R_{\bar\chi} = \mathbf{Z}_p[[1 + p\mathbf{Z}_p]] \simeq \mathbf{Z}_p[[T]]$ □

Lecture 5 6/2/2020

Today we will assume throughout that

$$\mathrm{Aut}_G(\bar\rho) \simeq k^*$$

so that there exists

$$R_{\bar\rho}$$

the universal deformation ring, although we didn't prove this yet.

Let $A \in \mathcal{C}$, we define the cotangent space of A to be

$$t_A^* = \mathfrak{m}_A/\mathfrak{m}_A^2 + pA$$

the tangent space is then

$$t_A = \mathrm{Hom}(t_A^*, k).$$

Example 2.39

$$A = W(k)[[X_1, \ldots, X_d]]$$

and $\mathfrak{m}_A = (p, X_1, \ldots, X_d)$ then

$$t_A^* = \frac{(p, X_1, \ldots, X_d)}{(p, \{X_i X_j\}_{ij})} \simeq kX_1 \oplus \cdots kX_d,$$

so this is d-dimensional and we see that we just have linear terms, which is consistent with the tangency interpretation. □

Fact 2.40

$$\dim(A/pA) \leq \dim_k t_A$$

where the LHS is the Krull dimension (the length of the longest chain of prime ideals), let $d = \dim t_A$, then Nakayama's lemma implies d is the minimal number of generators of

$$\mathfrak{m}_A/pA.$$

$$W(k)[[X_1, \ldots, X_d]] \twoheadrightarrow A$$

$$X_i \mapsto \textit{gen of } \mathfrak{m}_A/pA.$$

Tangent spaces and deformations. Let

$$k[\epsilon] = k[x]/(x^2)$$

then $\epsilon = \bar{x}$ and we have $\epsilon^2 = 0$. These are known as the dual numbers.

Proposition 2.41

$$\mathrm{Hom}_{\mathcal{C}}(A, k[\epsilon]) \simeq t_A$$

for $A \in \mathcal{C}$.

Proof.

$$\mathrm{Hom}_{\mathcal{C}}(A, k[\epsilon]) \simeq \mathrm{Hom}_{\mathcal{C}}(A/\mathfrak{m}_A^2 + pA, k[\epsilon])$$

since

$$(\mathfrak{m}_{k[\epsilon]})^2 = 0$$

and

$$p = 0 \in k[\epsilon].$$

$$A/\mathfrak{m}_A^2 + pA \simeq k \oplus \mathfrak{m}_A/\mathfrak{m}_A^2 + pA$$

$$x \mapsto (\bar{x}, x - [\bar{x}])$$

$$[\bar{x}] + y \leftarrow\!\shortmid (\bar{x}, y)$$

as k-vector spaces.

$$k[\epsilon] \simeq k \oplus \epsilon k.$$

$$\mathrm{Hom}_{\mathcal{C}}(A/\mathfrak{m}_A^2 + pA, k[\epsilon]) \simeq \mathrm{Hom}_{\mathcal{C}}(A/\mathfrak{m}_A^2 + pA, \epsilon k) \simeq t_A$$

where now we take homomorphisms of k-vector spaces. ■

Take $A = R_{\bar{\rho}}$ then the proposition implies

$$D_{\bar{\rho}}(k[\epsilon]) \simeq t_{R_{\bar{\rho}}}$$

to deform

$$\bar{\rho}$$

to $k[\epsilon]$

$$\rho_\epsilon(g) = \bar{\rho}(g) + \epsilon\phi(g)$$

for some

$$g \mapsto \phi(g) \in M_d(k)$$

with

$$\begin{tikzcd} G \arrow[r, "\rho_\epsilon"] \arrow[dr, "\bar{\rho}"'] & \mathrm{GL}_d(k[\epsilon]) \arrow[d] \\ & \mathrm{GL}_d(k) \end{tikzcd} .$$

We know

$$\rho_\epsilon(gh) = \rho_\epsilon(g)\rho_\epsilon(h)$$

$$\bar{\rho}(gh) + \epsilon\phi(gh) = (\bar{\rho}(g) + \epsilon\phi(g))(\bar{\rho}(h) + \epsilon\phi(h))$$

$$= \bar{\rho}(g)\bar{\rho}(h) + \epsilon(\phi(g)\bar{\rho}(h) + \bar{\rho}(g)\phi(h))$$

so

$$\phi(gh) = \phi(g)\bar{\rho}(h) + \bar{\rho}(g)\phi(h)$$

scale on the right by

$$\bar{\rho}(h)^{-1}\bar{\rho}(g)^{-1} = \bar{\rho}(gh)^{-1}.$$

$$\phi(gh)\bar{\rho}(gh)^{-1} = \phi(g)\bar{\rho}(g)^{-1} + \bar{\rho}(g)\phi(h)\bar{\rho}(h)^{-1}\bar{\rho}(g)^{-1}$$

set

$$\psi(g) = \phi(g)\bar{\rho}(g)^{-1}$$

so

$$\psi(gh) = \psi(g) + \bar{\rho}(g)\psi(h)\bar{\rho}(g)^{-1}$$

$$\psi(h) \in M_d(k) \simeq \mathrm{End}(\bar{\rho}) \circlearrowleft G$$

$$\mathrm{End}(\bar{\rho}) = \mathrm{Hom}(V_{\bar{\rho}}, V_{\bar{\rho}}) = \mathrm{ad}(\bar{\rho})$$

this is the representation where

$$g \bullet M = \bar{\rho}(g)M\bar{\rho}(g)^{-1}.$$

$$\psi \in Z^1(G, \mathrm{ad}(\bar{\rho}))$$

so we get a map

$$\{\text{lifts of } \bar{\rho}\} \longrightarrow Z^1(G, \mathrm{ad}\,\bar{\rho}).$$

Exercise 2.42 Strictly equivalent lifts yield cocyles differing by coboundaries.

Proposition 2.43

$$t_{R_{\bar{\rho}}} \simeq D_{\bar{\rho}}(k[\epsilon]) \simeq H^1(G, \mathrm{ad}\,\bar{\rho})$$

$$D^{\square}_{\bar{\rho}}(k[\epsilon]) \simeq Z^1(G, \mathrm{ad}\,\bar{\rho})$$

Exercise 2.44 Prove this.

Exercise 2.45 Show that $\forall\bar{\rho}$, $H^1(G, \mathrm{ad}\,\bar{\rho})$ is finite dimensional if and only if G satisfies Φ_p.

Let

$$d_1 = \dim_k H^1(G, \mathrm{ad}\,\bar{\rho})$$

$$0 \to I \to W(k)[[X_1, \ldots, X_{d_1}]] \to R_{\bar{\rho}} \to 0.$$

How big is I? This is equivalent to finding lower bounds on $\dim R_{\bar{\rho}}$. Obstruction to lifting $\bar{\rho}$.

$$\begin{tikzcd} G \arrow[r, "\exists ? \rho"] \arrow[dr, "\bar\rho"'] & \mathrm{GL}_d(\mathbf{Z}/p^2) \arrow[d] \\ & \mathrm{GL}_d(\mathbf{F}_p) \end{tikzcd}.$$

If we choose a set map ρ then we can set

$$\phi(g,h) = \rho(gh)\rho(h)^{-1}\rho(g)^{-1} \in 1 + pM_d(\mathbf{F}_p)$$

in the sequence

$$1 \to 1 + pM_d(\mathbf{F}_p) \to \mathrm{GL}_d(\mathbf{Z}/p^2) \to \mathrm{GL}_d(\mathbf{F}_p) \to 1$$

where

$$1 + pM_d(\mathbf{F}_p) \simeq M_d(\mathbf{F}_p) \simeq \mathrm{End}(\bar\rho).$$

Claim 2.46

$$\phi(g,h) \in Z^2(G, \mathrm{ad}\,\bar\rho)$$

The condition to be such a cocycle is

$$a \bullet \phi(b,c) - \phi(ab,c) + \phi(a,bc) - \phi(a,b) = 0.$$

So we do a long calculation, using the fact that the kernel is abelian.

Exercise check that changing ρ changes by a coboundary.

$$\phi \in H^2(G, \mathrm{ad}\,\bar\rho).$$

More generally

$$A_1 \xrightarrow{\pi} A_0$$

$$I = \ker \pi$$

s.t.

$$\mathfrak{m}_{A_1} I = 0$$

so I is a k-vector space.

$$\phi(g,h) = \rho_1(gh)\rho_1(h)^{-1}\rho_1(g)^{-1} \equiv 1 \pmod{I}$$

so in

$$\begin{aligned} 1 + M_d(I) &\simeq 1 + M_d(k) \otimes I \\ &\simeq \mathrm{ad}\,\bar\rho \otimes I. \end{aligned}$$

$$\phi \in H^2(G, \mathrm{ad}\,\bar\rho) \otimes I.$$

Lecture 6 11/2/2020

$$\bar\rho \colon G \to \mathrm{GL}_d(k)$$

$$\mathrm{Aut}_G\,\bar\rho \simeq k^\times$$

implies that there exists

$$R_{\bar\rho}$$

a universal deformation ring representing

$$D_{\bar\rho}.$$

We defines

$$t_R^* = \mathfrak{m}_R/\mathfrak{m}_R^2 + p.$$
$$t_R = \mathrm{Hom}(t_R^*, k).$$
$$t_{R_{\bar\rho}} \simeq D_{\bar\rho}(k[\epsilon]) \simeq H^1(G, \mathrm{ad}\,\bar\rho)$$
$$\dim = d_1 < \infty.$$
$$0 \to I \to W(k)[[X_1, \dots, X_{d_1}]] \to R_{\bar\rho} \to 0.$$

$$A_0 \in \mathcal{C}$$
$$\rho_0 \colon G \to \mathrm{GL}_d(A_0)$$

deforming $\bar\rho$.

$$A_1 \xrightarrow{\pi} A_0$$
$$\mathfrak{m}_{A_1} \cdot \ker(\pi) = 0$$
$$\mathcal{O}(\rho_0) \in H^2(G, \mathrm{ad}\,\bar\rho) \otimes \ker(\pi)$$
$$\mathcal{O}(\rho_0) = 0$$

if and only if there exists a deformation of ρ_0 to A_1 (via π). If $H^2(G, \mathrm{ad}\,\bar\rho) = 0$ then we can always deform these small homs.

Theorem 2.47 *Let*

$$d_i = \dim H^i(G, \mathrm{ad}\,\bar\rho),\ i = 1, 2$$

then

$$\mathrm{krulldim}(R_{\bar\rho}/pR_{\bar\rho}) \geq d_1 - d_2. \tag{2.1}$$

Additionally if $d_2 = 0$ then

$$R_{\bar\rho} \simeq W(k)[[X_1, \dots, X_{d_1}]].$$

Conjecture 2.48 *We have equality in (2.1).*

If we have

$$R \to S \in \mathcal{C}$$
$$t_R^* \to t_S^*$$
$$t_S \to t_R.$$
$$W(k)[[X_1, \dots, X_{d_1}]] \to R_{\bar\rho} \to 0$$
$$t_{R_{\bar\rho}} \xrightarrow{\sim} t_{W(k)[[X_1, \dots, X_{d_1}]]}.$$

Same dimensions and injective since surj on cotangent space.

proof of theorem.

$$0 \to I \to W(k)[[X_1, \dots, X_{d_1}]] \to R_{\bar\rho} \to 0$$
$$0 \to J \to k[[X_1, \dots, X_{d_1}]] \to R_{\bar\rho}/pR_{\bar\rho} \to 0$$
$$0 \to J/\mathfrak{m}J \to k[[X_1, \dots, X_{d_1}]]/\mathfrak{m}J \to R_{\bar\rho}/pR_{\bar\rho} \to 0$$

Claim: there exists

$$\mathrm{Hom}(J/\mathfrak{m}J, k) \hookrightarrow H^2(G, \mathrm{ad}\,\bar\rho)$$

this implies

$$\dim_k(J/\mathfrak{m}J) \leq d_2$$

so J has a set of generators of length $\leq d_2$ by Nakayama. So

$$\operatorname{krulldim}(R_{\bar\rho}/pR_{\bar\rho}) \geq d_1 - d_2.$$

Maybe this is Krull's principal ideal theorem.

The proof of the claim is as follows: Let

$$\tilde\rho$$

be ρ^{univ} mod p

$$\tilde\rho\colon G \to \mathrm{GL}_d(R_{\bar\rho}/pR_{\bar\rho})$$

$$\mathcal{O}(\tilde\rho) \in H^2(G, \operatorname{ad}\bar\rho) \otimes J/\mathfrak{m}J$$

$$\operatorname{Hom}(J/\mathfrak{m}J, k) \xrightarrow{\alpha} H^2(G, \operatorname{ad}\bar\rho)$$

$$f \mapsto (1 \otimes f)(\mathcal{O}(\tilde\rho))$$

$f \neq 0$ so that

$$\alpha(f) = 0$$

$$f : J/\mathfrak{m}J \to k$$

$$0 \to (J/\mathfrak{m}J)/\ker f \to ((k[[X_1, \ldots, X_{d_1}]])/\mathfrak{m}J)/\ker f \to R_{\bar\rho}/pR_{\bar\rho} \to 0.$$

But this is

$$0 \to k \to R' \to R_{\bar\rho}/pR_{\bar\rho} \to 0.$$

Obstruction class is $f(\mathcal{O}(\tilde\rho)) = 0$. So there exists a deformation of $\tilde\rho$ to R'. But char $R' = p$ as

$$\tilde\rho$$

is the universal deformation of $\bar\rho$ to a characteristic p ring.

So there exists

$$R_{\bar\rho}/pR_{\bar\rho} \to R'$$

hence

$$t_{R'}$$

has larger dimension than

$$t_{R_{\bar\rho}/pR_{\bar\rho}}$$

a contradiction.

For the second part

$$0 \to J \to k[[X_1, \ldots, X_{d_1}]] \to R_{\bar\rho}/pR_{\bar\rho} \to 0$$

so

$$\dim(J/\mathfrak{m}J) \leq d_2 = 0$$

so

$$J/\mathfrak{m}J = 0$$

i.e.

$$J = 0.$$

Hence

$$R_{\bar\rho}/pR_{\bar\rho} \simeq k[[X_1, \ldots, X_{d_1}]].$$

We need to know that $p^n \neq 0$ in $R_{\bar\rho}$. Suffices to show that

$$D_{\bar\rho}(W(k)) \neq \emptyset.$$

Having we switch to

$$G = G_{K,S}$$

Now $H^2 = 0$ implies we can lift small homomorphisms

$$k \simeq W(k)/p \leftarrow W(k)/p^2 \leftarrow \cdots$$

so we can lift to $W(k)$. ■

and we will introduce Tate's global Euler characteristic formula.

$$M = \text{finite cardinality} G_{K,S}\text{-module}$$

$$S \supseteq \{\text{inf. places}\} \cup \{\ell : \ell | \# M\}$$

$$\frac{\# H^0(G,M) \# H^2(G,M)}{\# H^1(G,M)} = (\# M)^{-[K:\mathbf{Q}]} \cdot \prod_{v \in S_\infty} \# H^0(G_{K_v}, M).$$

Take $M = \operatorname{ad} \bar{\rho} \simeq M_d(k)$.

$$S = \{\text{inf. places}\} \cup \{\mathfrak{p} : \mathfrak{p} | p\} \cup \{\text{ram. primes for } \bar{\rho}\}$$

$$d_0 - d_1 + d_2 = \sum_{v \in S_\infty} \dim H^0(G_{K_v}, \operatorname{ad} \bar{\rho}) - [K : \mathbf{Q}] d^2$$

where $d_0 = \dim H^0(G, \operatorname{ad} \bar{\rho})$

$$g \cdot M = M$$

implies $d_0 = 1$

$$\bar{\rho}(g) M \bar{\rho}(g)^{-1} = M$$

$$\implies M \in \operatorname{End}_G(\bar{\rho}) = k.$$

$$d_1 - d_2 = [K : \mathbf{Q}] d^2 + 1 - \sum_{v \in S_\infty} \dim_k H^0(G_{K_v}, \operatorname{ad} \bar{\rho}).$$

Example 2.49

$$\bar{\rho} = \bar{\chi} = \text{character of } = G_{K,S}$$

$$\operatorname{ad} \bar{\rho} \simeq \mathbf{1}$$

so

$$\dim_k (H^0(G_{K_v}, \operatorname{ad} \bar{\rho})) = 1$$

hence

$$\operatorname{krulldim}(R_{\bar{\chi}} / p R_{\bar{\chi}}) \geq d_1 - d_2 = [K : \mathbf{Q}] + 1 - r_1 - r_2 = r_2 + 1 > 0.$$

$$R_{\bar{\chi}} \simeq W(k)[[G_{K,S}^{\mathrm{ab},(p)}]]$$

$$R_{\bar{\chi}} / p R_{\bar{\chi}} \simeq k[[G_{K,S}^{\mathrm{ab},(p)}]]$$

$$\operatorname{krulldim}(R_{\bar{\chi}} / p R_{\bar{\chi}}) = \operatorname{rk}_{\mathbf{Z}_p}(G_{K,S}^{\mathrm{ab},(p)}) = r_2 + 1$$

which is exactly Leopoldt's conjecture. □

Example 2.50

$$\bar{\rho} \colon G_{K,S} \longrightarrow \mathrm{GL}_2(k)$$

$$S = \{p, \infty\}$$

$$\operatorname{krulldim}(R_{\bar{\rho}} / p R_{\bar{\rho}}) \geq 4 + 1 - \dim H^0(G_{\mathbf{R}}, \operatorname{ad} \bar{\rho})$$

take

$$c \in \operatorname{Gal}(\mathbf{C}/\mathbf{R})$$

either

$$\bar{\rho}(c) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or

$$\bar{\rho}(c) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the first case is called even the second odd. In the even case we have

$$\operatorname{ad}\bar\rho = \mathbf{1}$$

and so H^0 has dimension 4. And

$$\operatorname{krulldim}(R_{\bar\rho}/pR_{\bar\rho}) = 1.$$

In the odd case

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}a & -b\\ -c & d\end{pmatrix}$$

so H^0 is 2 dimensional and

$$\operatorname{krulldim}(R_{\bar\rho}/pR_{\bar\rho}) \geq 3.$$

□

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When $\bar\rho = \bar\rho_f$ for some $f \in S_k(\Gamma_0(N))$.

Theorem 2.51 Weston. *When $k \geq 3$ then $H^2(G_{\mathbf{Q},S}, \operatorname{ad}\bar\rho) = 0$ for all but finitely many p.*

So $R_{\bar\rho} \simeq W(k)[[X,Y,Z]]$ and

$$D_{\bar\rho}(W(k)) = \operatorname{Hom}(R_{\bar\rho}, W(k))$$
$$k = \mathbf{F}_q = \operatorname{Hom}(R_{\bar\rho}, \mathbf{Z}_p)$$
$$\phi\colon \mathbf{Z}_p[[X,Y,Z]] \to \mathbf{Z}_p$$

is determined by the image of X, Y, Z

$$\phi(X), \phi(Y), \phi(Z) \in p\mathbf{Z}_p.$$

The question is, why three dimensional?

One dimension comes from twists

$$\mathbf{1}^{univ}\colon G_{\mathbf{Q},S} \to \mathbf{Z}_p[[G_{\mathbf{Q},S}^{\mathrm{ab},pro-p}]]^* \simeq \mathbf{Z}_p[[T]]$$
$$\bar\rho \otimes \mathbf{1}^{univ}\colon G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathbf{F}_p[[T]]).$$

A second dimension is from p-adic variation, given f of weight k_0 there exists

$$\mathscr{F} = \sum_{n\geq 1} a_n(x) q^n$$

where $a_n(x)$ is a p-adic analytic function on a disc around k_0 s.t.

$$k \in \mathbf{Z}^{\geq 2}$$

in that disc.

$$\mathscr{F}_k = \sum_{n\geq 1} a_n(k) q^n \in S_k(\Gamma_0(Np))$$

an eigenform.

$$\rho_{\mathscr{F}}\colon G_{\mathbf{Q},S} \to \mathrm{GL}_2(A)$$

where A are analytic functions in a disk.

Different families.

The third dimension comes from the infinite fern, we can place a form in a family in two distinct ways, get a space filling curve.

We will also want to add conditions to our deformations, getting subfunctors with conditions until we get finite sets of points.

2.6 Deformation theory II

Let C be the category of complete local Noetherian rings $(A, \mathfrak{m}_A)$ with finite residue field k and an identification $A/\mathfrak{m}_A \simeq k$. The morphisms are local homomorphisms preserving the identification of residue fields.

This is the same as the category of complete local Noetherian $W(k)$-algebras with residue field k and morphisms are local $W(k)$-algebra homomorphisms.

This is as we have

$$W(k) \longrightarrow A$$

local, leading to

$$k \xrightarrow{\sim} A/\mathfrak{m}_A \simeq k$$

the $W(k)$-algebra structure provides a canonical identification of $A/\mathfrak{m}_A$ with k.

$$\begin{tikzcd} A \arrow[rr, "\phi"] & & B \\ & W(k) \arrow[ul] \arrow[ur] & \end{tikzcd} \rightsquigarrow \begin{tikzcd} A/\mathfrak{m}_A \arrow[rr] & & B/\mathfrak{m}_B \\ & k \arrow[ul, "\sim"] \arrow[ur, "\sim"'] & \end{tikzcd}.$$

So from now on we will use the second definition.

Given $A \in C$ the ring A is a topological ring under the $\mathfrak{m}$-adic topology.

$$A \simeq \varprojlim_n A/\mathfrak{m}_A^n.$$

Lemma 2.52 $A, B \in C$

$$f : A \longrightarrow B$$

then f is continuous.

Proof. We need

$$f^{-1}(\mathfrak{m}_B^n) \supseteq \mathfrak{m}_A^m$$

$$f(\mathfrak{m}_A) \subseteq \mathfrak{m}_B \implies f(\mathfrak{m}_A^n) \subseteq \mathfrak{m}_B^n \implies f^{-1}(\mathfrak{m}_B^n) \supseteq \mathfrak{m}_A^n.$$

∎

Lemma 2.53 $A, B \in C$

$$f : A \longrightarrow B$$

a ring homomorphism, then f is local.

Proof.

$$A \xrightarrow{f} B \supseteq \mathfrak{m}_B$$

where the containment is finite index.

$$A/f^{-1}(\mathfrak{m}_B) \hookrightarrow B/\mathfrak{m}_B$$

as k is finite we have $A/f^{-1}(\mathfrak{m}_B)$ is finite hence artinian. So

$$\mathfrak{m}_A^n \subseteq f^{-1}(\mathfrak{m}_B).$$

Since finite artinian rings have nilpotent maximal ideals.

So

$$f(\mathfrak{m}_A^n) \subseteq \mathfrak{m}_B$$

assume

$$f(\mathfrak{m}_A) \not\subseteq \mathfrak{m}_B.$$

Take

$$x \in \mathfrak{m}_A$$

s.t.

$$f(x) \notin \mathfrak{m}_B$$

hence

$$f(x)$$

is a unit. So

$$f(x^n)$$

is a unit, a contradiction. ■

Deformations. $A \in C$ then

$$D_{\bar{\rho}}(A) = \left\{(V_A, i_A) : V_A \simeq A^d \circlearrowleft G,\ i_A \colon V_A \otimes k \simeq \overline{V}\right\} / \sim .$$

$$(V_A, i_A) \simeq (V'_A, i'_A)$$

when

$$\exists \phi \colon V_A \xrightarrow{\sim} V'_A$$

as G-reps making the diagram commute.

A funny observation is that

$$(V_A, i_A) \simeq (V_A, i'_A).$$

$$(i'_A)^{-1} \circ i_A \in \mathrm{End}(V_A \otimes k) = \mathrm{End}(\bar{\rho}) \simeq k$$

so we are just scaling by $c \in k^*$. But we can take $\phi = [c]$.

Larger category. C' the category of local profinite $W(k)$-algebras with residue field k

$$A \in C' \implies A = \varprojlim_I A/I$$

for I finite index. The profinite topology on A has $\{I\}_{I \text{ finite index}}$ base of opens around 0. Morphisms are $W(k)$-algebra homomorphisms.

If $A, B \in C'$ with

$$f \colon A \longrightarrow B$$

a ring hom, then f is continuous.

Two topologies (profinite, $\mathfrak{m}$-adic). Need not be the same.

Example 2.54

$$A = k[[x_1, \dots, x_n, \dots]]$$

have

$$\mathfrak{m}_A = (x_1, \dots)$$
$$\mathfrak{m}_A^2 = (x_1, \dots)^2$$

not finite index. □

Claim 2.55 *Generally I open for the profinite topology implies it is open $\mathfrak{m}$-adically.*

Proof. If

$$A/I$$

is finite (thus artinian) we have

$$\mathfrak{m}_A^n \subseteq I$$

so I is $\mathfrak{m}$-adically open. ■

Claim 2.56 *If A is in addition noetherian I open $\mathfrak{m}$-adically then it is open in the profinite topology.*

Proof. If I is open $\mathfrak{m}$-adically I is open profinitely $\iff$ ($I \supseteq m_A^n$ implies finite index). Suffices to see $A/\mathfrak{m}_A^n$ is finite. For A noetherian have $\mathfrak{m}_A^j$ finitely generated for all j. So $\mathfrak{m}_A^j/\mathfrak{m}_A^{j+1}$ is finite (since fin. dim. $/k$).

$$A/\mathfrak{m}_A^n \supseteq \mathfrak{m}/\mathfrak{m}^2 \supseteq \cdots \supseteq 0$$

■