

# Deformation theory of Galois representations

MA842 at BU Spring 2020

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January 23, 2020

These are notes for Robert Pollack's course MA842 at BU Spring 2020.

The course webpage is <http://math.bu.edu/people/rpollack/Teach/842spring2020.html>.

Lecture 1 21/1/2018

## 1 Motivation

Let  $E_k$  denote the Eisenstein series of weight  $k$ ,  $k > 2$ .

$$E_k = \frac{-B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in M_k(\mathrm{SL}_2(\mathbf{Z})).$$

Where  $B_k$  are the Bernoulli numbers and

$$\sigma_{k-1}(n) = \sum_{d|n, d>0} d.$$

$E_2$  however is not holomorphic, so not a modular form.

Fix  $N$  a prime, notation has stuck from Mazur's Eisenstein ideal paper.

Then there exists a unique Eisenstein series on  $\Gamma_0(N)$  of weight 2.

$$E_2^{(N)} = \frac{N-1}{12} + \sum_{n=1}^{\infty} \sigma(n) q^n.$$

Funny observation: if  $N \equiv 1 \pmod{p}$  for prime  $p > 3$ . Then  $p | ((N-1)/12)$ , so  $E_2^{(N)}$  "looks cuspidal".

Then we hope that there exists a cuspidal eigenform  $f \in S_2(\Gamma_0(N))$  such that

$$f \equiv E_2^{(N)} \pmod{p}.$$

This is in fact true, due to Koike in the 70's, there exists  $f \in S_2(\Gamma_0(N))$  such that

$$a_{\ell}(f) \equiv 1 + \ell \pmod{p}$$

for all  $\ell \neq N, p$ .

**Question 1.1** How many such  $f$  are there?

□

Merel '96:

$$f \text{ is unique} \iff \prod_{i=1}^{(N-1)/2} i^i \text{ is not a } p\text{-th power modulo } N.$$

Wake and Wang-Erickson describe the dimension of the space of such  $f$  using Massey products (higher cup products).

Method: Galois deformations!

## 1.1 Galois representations

We write

$$G_{\mathbf{Q}} = \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) = \varprojlim_{F/\mathbf{Q}, \text{ fin. galois}} \text{Gal}(F/\mathbf{Q})$$

a profinite group.

$$\rho: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Q}_p)$$

a continuous homomorphism. Then view  $\text{GL}_2(\mathbf{Q}_p)$  as  $\text{Aut}(V)$  for a 2-dimensional  $\mathbf{Q}_p$  vector space and fix a 2-dimensional  $\mathbf{Z}_p$ -lattice

$$T \subseteq V$$

which is  $G_{\mathbf{Q}}$  stable. Then we can take

$$\bar{\rho}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{F}_p)$$

this is unique (w.r.t. the choice of  $T$ ) only up to semisimplification.

So we say two Galois representations  $\rho_1, \rho_2$  are congruent if

$$\bar{\rho}_1^{\text{ss}} \simeq \bar{\rho}_2^{\text{ss}}.$$

We say  $\rho_1, \rho_2$  are deformations of

$$\bar{\rho}_1 = \bar{\rho}_2$$

(imagine this is reducible).

Start with

$$\bar{\rho}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{F}_p)$$

consider “all” deformations of  $\bar{\rho}$  in good cases there exists a “universal” deformation of  $\bar{\rho}$ .

$R^{\text{univ}}$  a local ring with maximal ideal  $\mathfrak{m}_R$  such that

$$R/\mathfrak{m}_R = \mathbf{F}_p.$$

$$\rho^{\text{univ}}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(R^{\text{univ}})$$

such that if  $\rho: G_{\mathbf{Q}} \rightarrow \text{GL}_2(R)$  is a deformation of  $\bar{\rho}$  then there exists

$$R^{\text{univ}} \rightarrow R$$

such that

$$\begin{array}{ccc} G_{\mathbf{Q}} & \xrightarrow{\rho^{\text{univ}}} & \text{GL}_2(R^{\text{univ}}) \\ & \searrow \rho & \downarrow \\ & & \text{GL}_2(R) \end{array}$$

## 1.2 Modular forms

$$f = \sum a_n q^n \in S_k(\Gamma_0(N))$$

an eigenform leads to

$$\rho_f: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(K), \quad K/\mathbf{Q}_p \text{ finite}$$

with the property that for all  $\ell \nmid Np$  we have

$$\mathrm{Tr}(\rho_f(\mathrm{Frob}_\ell)) = a_\ell.$$

Modular forms can be congruent

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{p} \text{ for all but finitely many } \ell$$

$$\Updownarrow$$

$$\bar{\rho}_{f_1}^{\mathrm{ss}} \simeq \bar{\rho}_{f_2}^{\mathrm{ss}}.$$

There exists a ring, the Hecke algebra  $\mathbf{T}$  parametrizing all  $f$ 's with the same  $\bar{\rho}$ .

$$f \leadsto \rho_f \implies R^{\mathrm{univ}} \rightarrow \mathbf{T}$$

so hope

$$R^{\mathrm{univ}} \simeq \mathbf{T}.$$

Wiles proof of FLT proved one of these.

Many more such theorems in the past couple of decades.

Wake and Wang-Erickson show that the dimension of

$$\{f : f \equiv E_2^{(N)}\} \leftrightarrow \mathrm{rank} \mathbf{T} = \mathrm{rank} R^{\mathrm{univ}}.$$

$$a_\ell(f) \equiv 1 + \ell \pmod{p}$$

$$\implies \bar{\rho}^{\mathrm{ss}} = \mathbf{1} \oplus \mu_p$$

but there does not exist  $R^{\mathrm{univ}}$  in this context.

The fix is to use pseudorepresentations instead of representations.

## 1.3 Pseudorepresentations

Let  $G$  be a group.

Then a pseudorepresentation  $T$  is a map

$$T: G \rightarrow A$$

for  $A$  a ring satisfying

1.

$$T(xy) = T(yx)$$

2.

$$T(x)T(y)T(z) - T(x)T(yz) - T(y)T(xz) - T(z)T(xy) + T(xyz) + T(xzy) = 0$$

and the analogous formulae for higher dimensions.

**Fact 1.2** If  $A$  is an algebraically closed field of characteristic  $\neq 2$ . Then for a given pseudorepresentation  $T$  there exists a true representation  $\rho$  such that

$$T = \text{Tr}(\rho).$$

But this does not hold in general.

Universal pseudodeformation rings always exist. Wake and Wang-Erickson use  $R^{\text{univ}}$  = universal pseudodeformation ring.

## 2 Definitions

### 2.1 Representations

Lecture 2 23/1/2018

**Definition 2.1 Representations.** Let  $G$  be a finite group and  $V$  a finite dimensional vector space over  $\mathbf{C}$  of dimension  $d$ . A **representation** of  $G$  is a homomorphism

$$G \xrightarrow{\rho} \text{Aut}(V) \simeq \text{GL}_d(\mathbf{C}),$$

$G$  acts linearly on  $V$ . ◇

Galois representations: Let  $G$  be a Galois group, possibly infinite.  $F$  be a field,

$$G_F = \text{Gal}(\overline{F}/F) = \varprojlim_{L/F, \text{ fin. gal.}} \text{Gal}(L/F)$$

profinite compact and totally disconnected.

Replace  $V$  with a finite free module over some topological ring  $A$

$$\rho: G_F \rightarrow \text{GL}_d(A)$$

a continuous homomorphism.

**Example 2.2**  $A = \mathbf{C}$  with the complex topology. □

**Fact 2.3** In this case  $\text{im}(\rho)$  is finite.

**Exercise 2.4** Prove this.

Then we can write

$$\begin{array}{ccc} G_F & \xrightarrow{\rho} & \text{GL}_d(\mathbf{C}) \\ & \searrow & \nearrow \\ & \text{Gal}(L/F) & \end{array}$$

where  $L/F$  is finite. There are many such representations.

**Conjecture 2.5** Every finite group is a quotient of  $G_{\mathbf{Q}}$ .

**Example 2.6**

$$F = \mathbf{Q}$$

$$L = \mathbf{Q}(\sqrt[4]{2}, i)$$

$$\text{Gal}(L/\mathbf{Q}) \simeq D_4$$

$$G_{\mathbf{Q}} \rightarrow \text{Gal}(L/\mathbf{Q}) \xrightarrow{\rho} \text{GL}_2(\mathbf{C})$$

with  $\rho$  the unique irreducible 2-dimensional representation of  $D_4$ . □

**Example 2.7** Let  $E/\mathbf{Q}$  be an elliptic curve

$$G_{\mathbf{Q}} \cup E[p] \simeq \mathbf{Z}/p \oplus \mathbf{Z}/p$$

$$\rho_{E,p}: G_{\mathbf{Q}} \rightarrow \text{Aut}(E[p]) \simeq \text{GL}_2(\mathbf{F}_p)$$

$$\rho_{E,p^n}: G_{\mathbf{Q}} \rightarrow \text{Aut}(E[p^n]) \simeq \text{GL}_2(\mathbf{Z}/p^n)$$

$$\rho_{E,p^\infty}: G_{\mathbf{Q}} \rightarrow \text{Aut}(E[p^\infty]) \simeq \text{GL}_2(\mathbf{Z}/p^\infty).$$

□

**Fact 2.8**  $\rho_{E,p^\infty}$  has finite image.

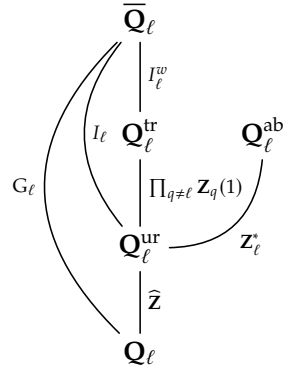
**Exercise 2.9** Prove this.

**Example 2.10** Let  $F = \mathbf{Q}_\ell$ ,  $A = \mathbf{Q}_p$

$$\rho: G_{\mathbf{Q}_\ell} \rightarrow \text{GL}_d(\mathbf{Q}_p).$$

□

The Galois theory of local fields looks like



where

$$\mathbf{Q}_\ell^{\text{tr}} = \mathbf{Q}_\ell(\{\sqrt[n]{\ell}\}_{\ell \nmid n})$$

the maximal tamely ramified extension

$$\mathbf{Q}_\ell^{\text{ur}} = \mathbf{Q}_\ell(\{\mu_n\}_{\ell \nmid n})$$

the maximal unramified extension

$$\mathbf{Q}_\ell^{\text{ab}} = \mathbf{Q}_\ell^{\text{un}}(\mu_{\ell^\infty})$$

the maximal abelian extension.

$$I_\ell = \text{Gal}(\overline{\mathbf{Q}}_\ell / \mathbf{Q}_\ell^{\text{ur}})$$

the inertia group

$$I_\ell^w = \text{Gal}(\overline{\mathbf{Q}}_\ell / \mathbf{Q}_\ell^{\text{tr}})$$

the wild inertia group

We say

$$\rho: G_{\mathbf{Q}_\ell} \rightarrow \text{GL}_d(\mathbf{Q}_p)$$

is unramified if

$$\rho(I_\ell) = \{1\}$$

is tamely ramified if

$$\rho(I_\ell^w) = \{1\}.$$

In the first case  $\rho$  is completely determined by  $\rho(\text{Frob}_\ell)$ . In the second case  $\rho$  is completely determined by  $\rho(\text{Frob}_\ell)$  and its value on a generator of

$$\text{Gal}(\mathbf{Q}_\ell^{\text{tr}}/\mathbf{Q}_\ell^{\text{ur}}).$$

The wild part:  $I_\ell^w$  is pro- $\ell$ ,  $\text{GL}_d(\mathbf{Z}_p)$  is almost pro- $p$  (it has a finite index pro- $p$  subgroup).

**Exercise 2.11** Prove this.

**Example 2.12** For  $d = 1$

$$\mathbf{Z}_p^* = \mathbf{F}_p^* \times (1 + p\mathbf{Z}_p).$$

□

Thus if  $\ell \neq p$  then

$$\rho(I_\ell^w) \subseteq \text{GL}_d(\mathbf{Q}_p)$$

is finite.

**Exercise 2.13** Prove this.

If  $\ell = p$  then this is handled by  $p$ -adic Hodge theory.

The connection to global representations is then that

$$\begin{array}{ccc} \overline{\mathbf{Q}} & \longrightarrow & \overline{\mathbf{Q}}_\ell \\ \uparrow & & \uparrow \\ \mathbf{Q} & \longrightarrow & \mathbf{Q}_\ell \end{array}$$

So

$$G_{\mathbf{Q}_\ell} \hookrightarrow G_{\mathbf{Q}}$$

via restriction to  $\overline{\mathbf{Q}}$ .

The image of this map is the **decomposition group** at  $\ell$ .

$$\rho: G_{\mathbf{Q}} \rightarrow \text{GL}_d(A),$$

we say that  $\rho$  is unramified at  $\ell$  if

$$\rho(I_\ell) = \{1\}.$$

In which case

$$\text{charpoly}(\rho(\text{Frob}_\ell))$$

is well-defined.

Returning to

$$\rho_{E,p^\infty}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Q}_p)$$

now.

**Fact 2.14**  $\rho_{E,p^\infty}$  is unramified outside of  $N_E \cdot p$  ( $N_E$  is the conductor of  $E$ ). i.e.

$$\rho_{E,p^\infty}$$

is unramified at  $\ell$  if and only if  $\ell \neq p$  and  $\ell$  is a prime of good reduction for  $E$ .

So  $\rho_{E,p^\infty}$  sees bad reduction.

From  $\rho_{E,p^\infty}$  you can recover  $E$  up to isogeny (Faltings).

**Example 2.15**

$$G_{\mathbf{Q}} \cup \mu_{p^n}$$

so we get

$$G_{\mathbf{Q}} \rightarrow \text{Aut}(\mu_{p^n} \simeq (\mathbf{Z}/p^n\mathbf{Z})^* \simeq \text{GL}_1(\mathbf{Z}/p^n))$$

taking the inverse limit we get

$$\begin{array}{ccc} G_{\mathbf{Q}} & \xrightarrow{\epsilon_p} & \mathbf{Z}_p^* \\ & \searrow & \nearrow \sim \\ & \text{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) & \end{array}$$

this  $\epsilon_p$  is known as the  $p$ -adic cyclotomic character. This is unramified outside  $p$  and

$$\epsilon_p(\text{Frob}_\ell) = \ell$$

for  $\ell \neq p$ . □

**Remark 2.16**

$$\det(\rho_{E,p^\infty}) = \epsilon_p.$$

**Example 2.17**

$$f = \sum a_n q^n \in S_2(\Gamma_0(N), \mathbf{Q})$$

a weight 2 eigenform on  $\Gamma_0(N)$ , with rational fourier coefficients. Eichler-Shimura gives  $E_f/\mathbf{Q}$  an elliptic curve. Define

$$\rho_f = \rho_{E,p^\infty}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Q}_p)$$

unramified outside  $Np$

$$\text{Tr}(\rho(\text{Frob}_\ell)) = a_\ell$$

for  $\ell \nmid Np$ . More generally

$$f \in S_2(\Gamma_0(N))$$

an eigenform, Eichler-Shimura gives

$$A_f/\mathbf{Q}$$

an abelian variety which leads to

$$\rho_f: G_{\mathbf{Q}} \rightarrow \text{GL}_2(K)$$

$K/\mathbf{Q}_p$  finite. □

## 2.2 Congruences and elliptic curves

Let

$$E_1: y^2 = x^3 + x - 10$$

conductor  $2^2 \cdot 13$ , <https://www.lmfdb.org/EllipticCurve/Q/52a1/>

$$E_2: y^2 = x^3 - 584x + 5444$$

conductor  $2^2 \cdot 7 \cdot 13$ , <https://www.lmfdb.org/EllipticCurve/Q/364a1/>.

**Table 2.18 Images**

$p$	2	3	5	7	11	13	17	19	23	29
$a_p(E_1)$	0	0	2	-2	-2	-1	6	-6	8	2
$a_p(E_2)$	0	0	-3	1	-2	-1	-4	-1	-7	7

Note that

$$\begin{aligned} a_\ell(E_1) &\equiv a_\ell(E_2) \pmod{5}, \forall \ell \neq 7 \\ \implies \rho_{E_1,5} &\simeq \rho_{E_2,5} (= \bar{\rho}) \end{aligned}$$

as Galois representations.

**Exercise 2.19** Prove this.

How common is this? We have 2 lifts of  $\bar{\rho}$

$$\rho_{E_1,5^\infty} \simeq \rho_{E_2,5^\infty}$$

how many other such?

## 2.3 Hida theory

$$\sum a_n(f)q^n = f \in S_{k_0}(\Gamma_0(N))$$

an eigenform.

$$a_p(f)$$

a  $p$ -adic unit.

$$\mathcal{F} = \sum_{n=1}^{\infty} a_n(k)q^n$$

with  $a_n$  a  $p$ -adic analytic function in  $k$ . The whole family gives Galois representations that reduce to the same  $\bar{\rho}$ .

Specialise  $k$  to some integer  $w$

$$\mathcal{F} = \sum a_n(w)q^n \in S_w(\Gamma_0(N))$$

take  $w = k_0$  to recover  $f$ .

Hida constructs

$$\rho^{Hida}: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{Z}_p[[x]])$$

unramified outside  $Np$ .

$$\ell \nmid Np \implies \mathrm{tr}(\rho^{Hida}(\mathrm{Frob}_\ell)) = a_\ell(x).$$