

Deformation theory of Galois representations

MA842 at BU Spring 2020

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January 28, 2020

These are notes for Robert Pollack's course MA842 at BU Spring 2020.

The course webpage is <http://math.bu.edu/people/rpollack/Teach/842spring2020.html>.

Lecture 1 21/1/2018

1 Motivation

Let E_k denote the Eisenstein series of weight k , $k > 2$.

$$E_k = \frac{-B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in M_k(\mathrm{SL}_2(\mathbf{Z})).$$

Where B_k are the Bernoulli numbers and

$$\sigma_{k-1}(n) = \sum_{d|n, d>0} d^{k-1}.$$

E_2 however is not holomorphic, so not a modular form.

Fix N a prime, notation has stuck from Mazur's Eisenstein ideal paper.

Then there exists a unique Eisenstein series on $\Gamma_0(N)$ of weight 2.

$$E_2^{(N)} = \frac{N-1}{12} + \sum_{n=1}^{\infty} \sigma(n) q^n.$$

Funny observation: if $N \equiv 1 \pmod{p}$ for prime $p > 3$. Then $p | ((N-1)/12)$, so $E_2^{(N)}$ "looks cuspidal".

Then we hope that there exists a cuspidal eigenform $f \in S_2(\Gamma_0(N))$ such that

$$f \equiv E_2^{(N)} \pmod{p}.$$

This is in fact true, due to Koike in the 70's, there exists $f \in S_2(\Gamma_0(N))$ such that

$$a_\ell(f) \equiv 1 + \ell \pmod{p}$$

for all $\ell \neq N, p$.

Question 1.1 How many such f are there?

□

Merel '96:

$$f \text{ is unique} \iff \prod_{i=1}^{(N-1)/2} i^i \text{ is not a } p\text{-th power modulo } N.$$

Wake and Wang-Erickson describe the dimension of the space of such f using Massey products (higher cup products).

Method: Galois deformations!

1.1 Galois representations

We write

$$G_{\mathbf{Q}} = \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) = \varprojlim_{F/\mathbf{Q}, \text{ fin. galois}} \text{Gal}(F/\mathbf{Q})$$

a profinite group.

$$\rho: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Q}_p)$$

a continuous homomorphism. Then view $\text{GL}_2(\mathbf{Q}_p)$ as $\text{Aut}(V)$ for a 2-dimensional \mathbf{Q}_p vector space and fix a 2-dimensional \mathbf{Z}_p -lattice

$$T \subseteq V$$

which is $G_{\mathbf{Q}}$ stable. Then we can take

$$\bar{\rho}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{F}_p)$$

this is unique (w.r.t. the choice of T) only up to semisimplification.

So we say two Galois representations ρ_1, ρ_2 are congruent if

$$\bar{\rho}_1^{\text{ss}} \simeq \bar{\rho}_2^{\text{ss}}.$$

We say ρ_1, ρ_2 are deformations of

$$\bar{\rho}_1 = \bar{\rho}_2$$

(imagine this is reducible).

Start with

$$\bar{\rho}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{F}_p)$$

consider “all” deformations of $\bar{\rho}$ in good cases there exists a “universal” deformation of $\bar{\rho}$.

R^{univ} a local ring with maximal ideal \mathfrak{m}_R such that

$$R/\mathfrak{m}_R = \mathbf{F}_p.$$

$$\rho^{\text{univ}}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(R^{\text{univ}})$$

such that if $\rho: G_{\mathbf{Q}} \rightarrow \text{GL}_2(R)$ is a deformation of $\bar{\rho}$ then there exists

$$R^{\text{univ}} \rightarrow R$$

such that

$$\begin{array}{ccc} G_{\mathbf{Q}} & \xrightarrow{\rho^{\text{univ}}} & \text{GL}_2(R^{\text{univ}}) \\ & \searrow \rho & \downarrow \\ & & \text{GL}_2(R) \end{array}$$

1.2 Modular forms

$$f = \sum a_n q^n \in S_k(\Gamma_0(N))$$

an eigenform leads to

$$\rho_f: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(K), K/\mathbf{Q}_p \text{ finite}$$

with the property that for all $\ell \nmid Np$ we have

$$\mathrm{Tr}(\rho_f(\mathrm{Frob}_\ell)) = a_\ell.$$

Modular forms can be congruent

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{p} \text{ for all but finitely many } \ell$$

$$\Updownarrow$$

$$\bar{\rho}_{f_1}^{\mathrm{ss}} \simeq \bar{\rho}_{f_2}^{\mathrm{ss}}.$$

There exists a ring, the Hecke algebra \mathbf{T} parametrizing all f 's with the same $\bar{\rho}$.

$$f \leadsto \rho_f \implies R^{\mathrm{univ}} \rightarrow \mathbf{T}$$

so hope

$$R^{\mathrm{univ}} \simeq \mathbf{T}.$$

Wiles proof of FLT proved one of these.

Many more such theorems in the past couple of decades.

Wake and Wang-Erickson show that the dimension of

$$\{f : f \equiv E_2^{(N)}\} \leftrightarrow \mathrm{rank} \mathbf{T} = \mathrm{rank} R^{\mathrm{univ}}.$$

$$a_\ell(f) \equiv 1 + \ell \pmod{p}$$

$$\implies \bar{\rho}^{\mathrm{ss}} = \mathbf{1} \oplus \mu_p$$

but there does not exist R^{univ} in this context.

The fix is to use pseudorepresentations instead of representations.

1.3 Pseudorepresentations

Let G be a group.

Then a pseudorepresentation T is a map

$$T: G \rightarrow A$$

for A a ring satisfying

1.

$$T(xy) = T(yx)$$

2.

$$T(x)T(y)T(z) - T(x)T(yz) - T(y)T(xz) - T(z)T(xy) + T(xyz) + T(xzy) = 0$$

and the analogous formulae for higher dimensions.

Fact 1.2 If A is an algebraically closed field of characteristic $\neq 2$. Then for a given pseudorepresentation T there exists a true representation ρ such that

$$T = \text{Tr}(\rho).$$

But this does not hold in general.

Universal pseudodeformation rings always exist. Wake and Wang-Erickson use R^{univ} = universal pseudodeformation ring.

2 Definitions

2.1 Representations

Lecture 2 23/1/2018

Definition 2.1 Representations. Let G be a finite group and V a finite dimensional vector space over \mathbf{C} of dimension d . A **representation** of G is a homomorphism

$$G \xrightarrow{\rho} \text{Aut}(V) \simeq \text{GL}_d(\mathbf{C}),$$

G acts linearly on V . ◇

Galois representations: Let G be a Galois group, possibly infinite. F be a field,

$$G_F = \text{Gal}(\bar{F}/F) = \varprojlim_{L/F, \text{ fin. gal.}} \text{Gal}(L/F)$$

profinite compact and totally disconnected.

Replace V with a finite free module over some topological ring A

$$\rho: G_F \rightarrow \text{GL}_d(A)$$

a continuous homomorphism.

Example 2.2 $A = \mathbf{C}$ with the complex topology. □

Fact 2.3 In this case $\text{im}(\rho)$ is finite.

Exercise 2.4 Prove this.

Then we can write

$$\begin{array}{ccc} G_F & \xrightarrow{\rho} & \text{GL}_d(\mathbf{C}) \\ & \searrow & \nearrow \\ & \text{Gal}(L/F) & \end{array}$$

where L/F is finite. There are many such representations.

Conjecture 2.5 Every finite group is a quotient of $G_{\mathbf{Q}}$.

Example 2.6

$$F = \mathbf{Q}$$

$$L = \mathbf{Q}(\sqrt[4]{2}, i)$$

$$\text{Gal}(L/\mathbf{Q}) \simeq D_4$$

$$G_{\mathbf{Q}} \rightarrow \text{Gal}(L/\mathbf{Q}) \xrightarrow{\rho} \text{GL}_2(\mathbf{C})$$

with ρ the unique irreducible 2-dimensional representation of D_4 . □

Example 2.7 Let E/\mathbf{Q} be an elliptic curve

$$G_{\mathbf{Q}} \curvearrowright E[p] \simeq \mathbf{Z}/p \oplus \mathbf{Z}/p$$

$$\rho_{E,p}: G_{\mathbf{Q}} \rightarrow \text{Aut}(E[p]) \simeq \text{GL}_2(\mathbf{F}_p)$$

$$\rho_{E,p^n}: G_{\mathbf{Q}} \rightarrow \text{Aut}(E[p^n]) \simeq \text{GL}_2(\mathbf{Z}/p^n)$$

$$\rho_{E,p^\infty}: G_{\mathbf{Q}} \rightarrow \text{Aut}(E[p^\infty]) \simeq \text{GL}_2(\mathbf{Z}_p).$$

□

Fact 2.8 ρ_{E,p^∞} has finite image.

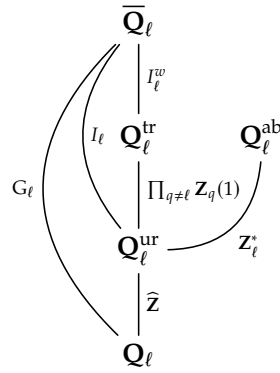
Exercise 2.9 Prove this.

Example 2.10 Let $F = \mathbf{Q}_\ell$, $A = \mathbf{Q}_p$

$$\rho: G_{\mathbf{Q}_\ell} \rightarrow \text{GL}_d(\mathbf{Q}_p).$$

□

The Galois theory of local fields looks like



where

$$\mathbf{Q}_\ell^{\text{tr}} = \mathbf{Q}_\ell(\{\sqrt[n]{\ell}\}_{\ell \nmid n})$$

the maximal tamely ramified extension

$$\mathbf{Q}_\ell^{\text{ur}} = \mathbf{Q}_\ell(\{\mu_n\}_{\ell \nmid n})$$

the maximal unramified extension

$$\mathbf{Q}_\ell^{\text{ab}} = \mathbf{Q}_\ell^{\text{un}}(\mu_{\ell^\infty})$$

the maximal abelian extension.

$$I_\ell = \text{Gal}(\overline{\mathbf{Q}}_\ell / \mathbf{Q}_\ell^{\text{ur}})$$

the inertia group

$$I_\ell^w = \text{Gal}(\overline{\mathbf{Q}}_\ell / \mathbf{Q}_\ell^{\text{tr}})$$

the wild inertia group

We say

$$\rho: G_{\mathbf{Q}_\ell} \rightarrow \text{GL}_d(\mathbf{Q}_p)$$

is **unramified** if

$$\rho(I_\ell) = \{1\}$$

is **tamely ramified** if

$$\rho(I_\ell^w) = \{1\}.$$

In the first case ρ is completely determined by $\rho(\text{Frob}_\ell)$. In the second case ρ is completely determined by $\rho(\text{Frob}_\ell)$ and its value on a generator of

$$\text{Gal}(\mathbf{Q}_\ell^{\text{tr}}/\mathbf{Q}_\ell^{\text{ur}}).$$

The wild part: I_ℓ^w is pro- ℓ , $\text{GL}_d(\mathbf{Z}_p)$ is almost pro- p (it has a finite index pro- p subgroup).

Exercise 2.11 Prove this.

Example 2.12 For $d = 1$

$$\mathbf{Z}_p^* = \mathbf{F}_p^* \times (1 + p\mathbf{Z}_p).$$

□

Thus if $\ell \neq p$ then

$$\rho(I_\ell^w) \subseteq \text{GL}_d(\mathbf{Q}_p)$$

is finite.

Exercise 2.13 Prove this.

If $\ell = p$ then this is handled by p -adic Hodge theory.

The connection to global representations is then that

$$\begin{array}{ccc} \overline{\mathbf{Q}} & \longrightarrow & \overline{\mathbf{Q}}_\ell \\ \uparrow & & \uparrow \\ \mathbf{Q} & \longrightarrow & \mathbf{Q}_\ell \end{array}$$

So

$$G_{\mathbf{Q}_\ell} \hookrightarrow G_{\mathbf{Q}}$$

via restriction to $\overline{\mathbf{Q}}$.

The image of this map is the **decomposition group** at ℓ .

$$\rho: G_{\mathbf{Q}} \rightarrow \text{GL}_d(A),$$

we say that ρ is **unramified at ℓ** if

$$\rho(I_\ell) = \{1\}.$$

In which case

$$\text{charpoly}(\rho(\text{Frob}_\ell))$$

is well-defined.

Returning to

$$\rho_{E,p^\infty}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Q}_p)$$

now.

Fact 2.14 ρ_{E,p^∞} is unramified outside of $N_E \cdot p$ (N_E is the conductor of E). i.e.

$$\rho_{E,p^\infty}$$

is unramified at ℓ if and only if $\ell \neq p$ and ℓ is a prime of good reduction for E .

So ρ_{E,p^∞} sees bad reduction.

From ρ_{E,p^∞} you can recover E up to isogeny (Faltings).

Example 2.15

$$G_{\mathbf{Q}} \curvearrowright \mu_{p^n}$$

so we get

$$G_{\mathbf{Q}} \rightarrow \text{Aut}(\mu_{p^n}) \simeq (\mathbf{Z}/p^n\mathbf{Z})^* \simeq \text{GL}_1(\mathbf{Z}/p^n)$$

taking the inverse limit we get

$$\begin{array}{ccc} G_{\mathbf{Q}} & \xrightarrow{\epsilon_p} & \mathbf{Z}_p^* \\ & \searrow & \nearrow \sim \\ & \text{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) & \end{array}$$

this ϵ_p is known as the p -adic cyclotomic character. This is unramified outside p and

$$\epsilon_p(\text{Frob}_\ell) = \ell$$

for $\ell \neq p$. □

Remark 2.16

$$\det(\rho_{E,p^\infty}) = \epsilon_p.$$

Example 2.17

$$f = \sum a_n q^n \in S_2(\Gamma_0(N), \mathbf{Q})$$

a weight 2 eigenform on $\Gamma_0(N)$, with rational fourier coefficients. Eichler-Shimura gives E_f/\mathbf{Q} an elliptic curve. Define

$$\rho_f = \rho_{E,p^\infty}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Q}_p)$$

unramified outside Np

$$\text{Tr}(\rho(\text{Frob}_\ell)) = a_\ell$$

for $\ell \nmid Np$. More generally

$$f \in S_2(\Gamma_0(N))$$

an eigenform, Eichler-Shimura gives

$$A_f/\mathbf{Q}$$

an abelian variety which leads to

$$\rho_f: G_{\mathbf{Q}} \rightarrow \text{GL}_2(K)$$

K/\mathbf{Q}_p finite. □

2.2 Congruences and elliptic curves

Let

$$E_1: y^2 = x^3 + x - 10$$

conductor $2^2 \cdot 13$, <https://www.lmfdb.org/EllipticCurve/Q/52a1/>

$$E_2: y^2 = x^3 - 584x + 5444$$

conductor $2^2 \cdot 7 \cdot 13$, <https://www.lmfdb.org/EllipticCurve/Q/364a1/>.

Table 2.18 a_p 's for E_1, E_2

p	2	3	5	7	11	13	17	19	23	29
$a_p(E_1)$	0	0	2	-2	-2	-1	6	-6	8	2
$a_p(E_2)$	0	0	-3	1	-2	-1	-4	-1	-7	7

Note that

$$a_\ell(E_1) \equiv a_\ell(E_2) \pmod{5}, \forall \ell \neq 7$$

$$\implies \rho_{E_1,5} \simeq \rho_{E_2,5}(=\bar{\rho})$$

as Galois representations.

Exercise 2.19 Prove this.

How common is this? We have 2 lifts of $\bar{\rho}$

$$\rho_{E_1,5^\infty} \simeq \rho_{E_2,5^\infty}$$

how many other such?

2.3 Hida theory

$$\sum a_n(f)q^n = f \in S_{k_0}(\Gamma_0(N))$$

an eigenform.

$$a_p(f)$$

a p -adic unit.

$$\mathcal{F} = \sum_{n=1}^{\infty} a_n(k)q^n$$

with a_n a p -adic analytic function in k . The whole family gives Galois representations that reduce to the same $\bar{\rho}$.

Specialise k to some integer w

$$\mathcal{F} = \sum a_n(w)q^n \in S_w(\Gamma_0(N))$$

take $w = k_0$ to recover f .

Hida constructs

$$\rho^{Hida}: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{Z}_p[[x]])$$

unramified outside Np .

$$\ell \nmid Np \implies \mathrm{tr}(\rho^{Hida}(\mathrm{Frob}_\ell)) = a_\ell(x).$$

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2.4 More examples of Galois representations in families

A 1-dimensional family. let

$$\epsilon_p = \text{cyclotomic character}$$

$$\epsilon_p: G_{\mathbf{Q}} \rightarrow \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \simeq \mathbf{Z}_p^* \simeq \mathbf{F}_p^* \times (1 + p\mathbf{Z}_p)$$

If we have $k \in \mathbf{Z}$ we can take

$$\epsilon_p^k$$

its power.

Note that when $k_1, k_2 \in \mathbf{Z}$ with

$$k_1 \equiv k_2 \pmod{p-1}$$

then

$$\epsilon_p^{k_1} \equiv \epsilon_p^{k_2} \pmod{p}$$

moreover if

$$k_1 \equiv k_2 \pmod{(p-1)p^N}$$

we have

$$\epsilon_p^{k_1} \equiv \epsilon_p^{k_2} \pmod{p^{N+1}}.$$

Set

$$\Lambda = \mathbf{Z}_p[[\mathbf{Z}_p^*]] = \varprojlim \mathbf{Z}_p[(\mathbf{Z}/p^N)^*]$$

$$\epsilon_p^{univ} : G_{\mathbf{Q}} \rightarrow \Lambda^* = \mathrm{GL}_1(\Lambda)$$

$$\sigma \mapsto [\epsilon_p(\sigma)]$$

$$\Lambda \xrightarrow{wt_k} \mathbf{Z}_p$$

$$wt_k \circ \epsilon_p^{univ}$$

where wt_k is defined by

$$\mathbf{Z}_p^* \rightarrow \mathbf{Z}_p^* \subseteq \mathbf{Z}_p$$

$$x \mapsto x^k$$

gives

$$\Lambda \xrightarrow{wt_k} \mathbf{Z}_p.$$

Take any

$$\phi \in \mathrm{Hom}_{cts}(\Lambda, \mathbf{C}_p) \simeq \mathrm{Hom}_{cts}(\mathbf{Z}_p^*, \mathbf{C}_p)$$

induces

$$G_{\mathbf{Q}} \rightarrow \mathbf{C}_p^*$$

$$\phi \circ \epsilon_p^{univ}.$$

$$\mathrm{Hom}_{cts}(\mathbf{Z}_p^*, \mathbf{C}_p)$$

is a union of $p-1$ open disks as

$$\mathbf{Z}_p^* \simeq \mathbf{F}_p^* \times (1 + p\mathbf{Z}_p)$$

the rightmost factor is topologically generated by $1+p$. So

$$\phi : \mathbf{Z}_p^* \rightarrow \mathbf{C}_p^*$$

is determined by

$$\phi|_{\mathbf{F}_p^*}$$

and

$$\phi(\gamma) \text{ for any } \gamma \text{ a top. gen..}$$

We need

$$|\phi(\gamma) - 1|_p < 1.$$

We have $p-1$ disks labelled by characters of \mathbf{F}_p^* . On each disk of $\phi \in \mathrm{Hom}(\mathbf{Z}_p^*, \mathbf{C}_p^*)$ the reduction of

$$\phi \circ \epsilon_p^{univ} : G_{\mathbf{Q}} \rightarrow \mathbf{C}_p^*$$

is the same.

A 2-d example. Let E/\mathbf{Q} be a CM elliptic curve.

$$\rho_{E,p^\infty}: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{Z}_p)$$

then there exists K/\mathbf{Q} quadratic imaginary field and, if p is an ordinary prime

$$\psi: G_K \rightarrow \mathbf{Z}_p^*$$

s.t.

$$\rho_{E,p^\infty} \simeq \mathrm{Ind}_K^{\mathbf{Q}}(\psi).$$

So we get a “family”

$$\mathrm{Ind}_K^{\mathbf{Q}}(\psi^k),$$

more generally can make

$$\psi^{univ}: G_K \rightarrow \Lambda^*$$

$$\mathrm{Ind} \psi^{univ}: G_K \rightarrow \mathrm{GL}_2(\Lambda).$$

CM Hida family.

2.5 Deformation theory

G a profinite group, k a finite field of characteristic p (simpler $k = \mathbf{F}_p$).

$$\bar{\rho}: G \rightarrow \mathrm{GL}_d(k)$$

a continuous homomorphism.

To lift $\bar{\rho}$ to be A -valued, we need a homomorphism

$$A \twoheadrightarrow k$$

we may as well assume A is local so it lifts to $\mathrm{GL}_d(A)$. We then only have one residual representation. Localize it at

$$\ker(A \rightarrow k).$$

If A is a local ring then A has a natural topology. Letting \mathfrak{m}_A be the maximal ideal

$$\{\mathfrak{m}_A^j\}$$

is a neighborhood base at 0, this is the \mathfrak{m} -adic topology.

So we have A local, \mathfrak{m}_A maximal $A/\mathfrak{m}_A \simeq k$, we need to fix this identification to ensure we don't have automorphisms.

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{GL}_d(A) \\ & \searrow \bar{\rho} & \downarrow \\ & & \mathrm{GL}_d(k) \end{array}$$

Note 2.20 If we have $M \in \mathrm{GL}_d(A)$ s.t. $M \equiv 1 \pmod{\mathfrak{m}_A}$, then

$$M\rho M^{-1}$$

lifts $\bar{\rho}$ also.

Definition 2.21 Call ρ and $M\rho M^{-1}$ **strictly equivalent**.

◇

Definition 2.22 A deformation of $\bar{\rho}$ to A is a strict equivalence class of continuous homomorphisms

$$G \xrightarrow{\rho} \mathrm{GL}_d(A)$$

lifting $\bar{\rho}$. ◇

Want to rephrase this with less bases, instead of $\bar{\rho}$ use \bar{V} a d -dimensional continuous representation of G over k . V a free A -module of rank d with continuous action of G s.t.

$$V \otimes A/\mathfrak{m}_A \simeq \bar{V}.$$

Naive hope: two such V 's are equivalent if they are isomorphic as G -modules.

Problem: R^{univ} won't exist, too many automorphisms. Ideally if V_A is a deformation of \bar{V} to A then there exists a unique

$$R^{univ} \rightarrow A$$

inducing

$$V^{univ} \rightarrow V_A.$$

Take II, let V be a free A -module with a continuous action of G and a fixed isomorphism.

$$V \otimes_A A/\mathfrak{m}_A \simeq \bar{V}$$

as representations of G over $k \simeq A/\mathfrak{m}_A$.

Two V 's are equivalent if there exists a G -module isomorphism $V_1 \xrightarrow{\phi} V_2$ s.t.

$$\begin{array}{ccc} V_1 & \xrightarrow{\phi} & V_2 \\ \downarrow & & \downarrow \\ V_1 \otimes A/\mathfrak{m}_A & & V_2 \otimes A/\mathfrak{m}_A \\ \sim \downarrow & & \sim \downarrow \\ \bar{V} & \xrightarrow{=} & \bar{V} \end{array} .$$

Exercise 2.23 Check the two definitions are the same, GL_d definition vs. \bar{V} .

Another approach

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{GL}_d(A) \\ & \searrow \bar{\rho} & \downarrow \\ & & \mathrm{GL}_d(k) \end{array}$$

lift of $\bar{\rho}$ and say two ρ 's are the same if they are equal as maps, this is a **framed deformation**. We can write this more abstractly

$$\bar{V}$$

a continuous representation of G over k with fixed basis β .

$$V$$

a free A module with cont. action of G and a fixed isom.

$$V \otimes_A A/\mathfrak{m}_D \simeq \bar{V}$$

and a basis β_A lifting β .

Categories

$$\mathcal{C}_k = \mathcal{C}$$

the category of complete local Noetherian rings A with a fixed isomorphism $A/\mathfrak{m}_A \simeq k$. Where the maps are

$$A_1 \xrightarrow{\phi} A_2$$

a local ring homomorphism

$$\phi(\mathfrak{m}_{A_1}) \subseteq \mathfrak{m}_{A_2}$$

and the diagram

$$\begin{array}{ccc} A_1/\mathfrak{m}_{A_1} & \xrightarrow{\bar{\phi}} & A_2/\mathfrak{m}_{A_2} \\ & \searrow \sim & \swarrow \sim \\ & k & \end{array}$$

commutes.