Deformation theory of Galois representations

MA842 at BU Spring 2020

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These are notes for Robert Pollack's course MA842 at BU Spring 2020.

The course webpage is http://math.bu.edu/people/rpollack/Teach/842spring2020.html.

Lecture 1 21/1/2018

1 Motivation

Let E_k denote the Eisenstein series of weight k, k > 2.

$$E_k = \frac{-B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n \in M_k(\mathrm{SL}_2(\mathbf{Z})).$$

Where B_k are the Bernoulli numbers and

$$\sigma_{k-1}(n) = \sum_{d|n, d>0} d^{k-1}.$$

 E_2 however is not holomorphic, so not a modular form.

Fix *N* a prime, notation has stuck from Mazur's Eisenstein ideal paper.

Then there exists a unique Eisenstein series on $\Gamma_0(N)$ of weight 2.

$$E_2^{(N)} = \frac{N-1}{12} + \sum_{n=1}^{\infty} \sigma(n)q^n.$$

Funny observation: if $N \equiv 1 \pmod{p}$ for prime p > 3. Then $p \mid ((N-1)/12)$, so $E_2^{(N)}$ "looks cuspidal".

Then we hope that there exists a cuspidal eigenform $f \in S_2(\Gamma_0(N))$ such that

$$f \equiv E_2^{(N)} \qquad \text{''} \mod p''.$$

This is in fact true, due to Koike in the 70's, there exists $f \in S_2(\Gamma_0(N))$ such that

$$a_{\ell}(f) \equiv 1 + \ell \pmod{p}$$

for all $\ell \neq N$, p.

Question 1.1 How many such *f* are there?

Merel '96:

$$f$$
 is unique $\iff \prod_{i=1}^{(N-1)/2} i^i$ is not a p -th power modulo N .

Wake and Wang-Erickson describe the dimension of the space of such *f* using Massey products (higher cup products).

Method: Galois deformations!

1.1 Galois representations

We write

$$G_{\mathbf{Q}} = \operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) = \varprojlim_{F/\mathbf{Q}, \text{fin. galois}} \operatorname{Gal}(F/\mathbf{Q})$$

a profinite group.

$$\rho \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Q}_p)$$

a continuous homomorphism. Then view $\operatorname{GL}_2(\mathbf{Q}_p)$ as $\operatorname{Aut}(V)$ for a 2-dimensional \mathbf{Q}_p vector space and fix a 2-dimensional \mathbf{Z}_p -lattice

$$T \subset V$$

which is $G_{\mathbf{O}}$ stable. Then we can take

$$\bar{\rho} \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_p)$$

this is unique (w.r.t. the choice of T) only up to semisimplification. So we say two Galois representations ρ_1 , ρ_2 are congruent if

$$\bar{\rho}_1^{\text{ss}} \simeq \bar{\rho}_2^{\text{ss}}$$
.

We say ρ_1 , ρ_2 are deformations of

$$\bar{\rho}_1 = \bar{\rho}_2$$

(imagine this is reducible).

Start with

$$\bar{\rho}: G_{\mathbf{O}} \to \mathrm{GL}_2(\mathbf{F}_v)$$

consider "all" deformations of $\bar{\rho}$ in good cases there exists a "universal" deformation of $\bar{\rho}$.

 R^{univ} a local ring with maximal ideal \mathfrak{m}_R such that

$$R/\mathfrak{m}_R = \mathbf{F}_p$$
.

$$\rho^{\text{univ}} : G_{\mathbf{Q}} \to GL_2(R^{\text{univ}})$$

such that if $\rho: G_{\mathbb{Q}} \to GL_2(R)$ is a deformation of $\bar{\rho}$ then there exists

$$R^{\mathrm{univ}} \to R$$

such that

$$G_{\mathbf{Q}} \xrightarrow{\rho^{\text{univ}}} GL_2(R^{\text{univ}})$$
.
$$GL_2(R)$$

1.2 Modular forms

$$f = \sum a_n q^n \in S_k(\Gamma_0(N))$$

an eigenform leads to

$$\rho_f \colon G_{\mathbf{Q}} \to \operatorname{GL}_2(K), K/\mathbf{Q}_p$$
 finite

with the property that for all $\ell \nmid Np$ we have

$$\operatorname{Tr}(\rho_f(\operatorname{Frob}_{\ell})) = a_{\ell}.$$

Modular forms can be congruent

 $a_{\ell}(f_1) \equiv a_{\ell}(f_2) \pmod{p}$ for all but finitely many ℓ

1

$$\bar{\rho}_{f_1}^{\mathrm{ss}} \simeq \bar{\rho}_{f_2}^{\mathrm{ss}}.$$

There exists a ring, the Hecke algebra T parametrizing all f's with the same $\bar{\rho}$.

$$f \rightsquigarrow \rho_f \implies R^{\text{univ}} \to \mathbf{T}$$

so hope

$$R^{\mathrm{univ}} \simeq \mathbf{T}$$
.

Wiles proof of FLT proved one of these.

Many more such theorems in the past couple of decades.

Wake and Wang-Erickson show that the dimension of

$${f: f \equiv E_2^{(N)}} \leftrightarrow \operatorname{rank} \mathbf{T} = \operatorname{rank} R^{\operatorname{univ}}.$$

$$a_{\ell}(f) \equiv 1 + \ell \pmod{p}$$

 $\implies \bar{\rho}^{ss} = \mathbf{1} \oplus \mu_{p}$

but there does not exist R^{univ} in this context.

The fix is to use pseudorepresentations instead of representations.

1.3 Pseudorepresentations

Let *G* be a group.

Then a pseudorepresentation T is a map

$$T: G \rightarrow A$$

for A a ring satisfying

1.

$$T(xy) = T(yx)$$

2.

$$T(x)T(y)T(z) - T(x)T(yz) - T(y)T(xz) - T(z)T(xy) + T(xyz) + T(xzy) = 0$$

and the analogous formulae for higher dimensions.

Fact 1.2 If A is an algebraically closed field of characteristic \neq 2. Then for a given pseudorepresentation T there exists a true representation ρ such that

$$T = \text{Tr}(\rho)$$
.

But this does not hold in general.

Universal pseudodeformation rings always exist. Wake and Wang-Erickson use $R^{\rm univ}$ = universal pseudodeformation ring.

2 Definitions

2.1 Representations

Lecture 2 23/1/2018

Definition 2.1 Representations. Let G be a finite group and V a finite dimensional vector space over \mathbf{C} of dimension d. A **representation** of G is a homomorphism

$$G \xrightarrow{\rho} \operatorname{Aut}(V) \simeq \operatorname{GL}_d(\mathbf{C}),$$

G acts linearly on *V*.

Galois representations: Let G be a Galois group, possibly infinite. F be a field,

 \Diamond

$$G_F = \operatorname{Gal}(\overline{F}/F) = \varprojlim_{L/F, \text{ fin. gal.}} \operatorname{Gal}(L/F)$$

profinite compact and totally disconnected.

Replace *V* with a finite free module over some topological ring *A*

$$\rho\colon G_F\to \mathrm{GL}_d(A)$$

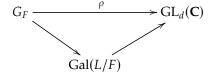
a continuous homomorphism.

Example 2.2 $A = \mathbf{C}$ with the complex topology.

Fact 2.3 *In this case* $im(\rho)$ *is finite.*

Exercise 2.4 Prove this.

Then we can write



where L/F is finite. There are many such representations.

Conjecture 2.5 *Every finite group is a quotient of* $G_{\mathbf{O}}$.

Example 2.6

$$F = \mathbf{Q}$$

$$L = \mathbf{Q}(\sqrt[4]{2}, i)$$

$$Gal(L/\mathbf{Q}) \simeq D_4$$

$$G_{\mathbf{Q}} \to Gal(L/\mathbf{Q}) \xrightarrow{\rho} GL_2(\mathbf{C})$$

with ρ the unique irreducible 2-dimensional representation of D_4 .

Example 2.7 Let E/\mathbf{Q} be an elliptic curve

$$G_{\mathbf{Q}} \cup E[p] \simeq \mathbf{Z}/p \oplus \mathbf{Z}/p$$

$$\rho_{E,p} \colon G_{\mathbf{Q}} \to \operatorname{Aut}(E[p]) \simeq \operatorname{GL}_{2}(\mathbf{F}_{p})$$

$$\rho_{E,p^{n}} \colon G_{\mathbf{Q}} \to \operatorname{Aut}(E[p^{n}]) \simeq \operatorname{GL}_{2}(\mathbf{Z}/p^{n})$$

$$\rho_{E,p^{\infty}} \colon G_{\mathbf{Q}} \to \operatorname{Aut}(E[p^{\infty}]) \simeq \operatorname{GL}_{2}(\mathbf{Z}_{p}).$$

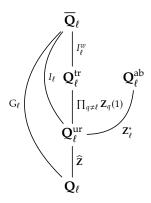
Fact 2.8 $\rho_{E,p^{\infty}}$ has finite image.

Exercise 2.9 Prove this.

Example 2.10 Let $F = \mathbf{Q}_{\ell}$, $A = \mathbf{Q}_{p}$

$$\rho \colon G_{\mathbf{Q}_{\ell}} \to \mathrm{GL}_d(\mathbf{Q}_p).$$

The Galois theory of local fields looks like



where

$$\mathbf{Q}^{\mathrm{tr}}_{\ell} = \mathbf{Q}_{\ell}(\{\sqrt[n]{\ell}\}_{\ell \nmid n})$$

the maximal tamely ramified extension

$$\mathbf{Q}_{\ell}^{\mathrm{ur}} = \mathbf{Q}_{\ell}(\{\mu_n\}_{\ell \nmid n})$$

the maximal unramified extension

$$\mathbf{Q}_{\ell}^{\mathrm{ab}} = \mathbf{Q}_{\ell}^{\mathrm{un}}(\mu_{\ell^{\infty}})$$

the maximal abelian extension.

$$I_{\ell} = \operatorname{Gal}(\overline{\mathbf{Q}}_{\ell}/\mathbf{Q}_{\ell}^{\operatorname{ur}})$$

the intertia group

$$I_{\ell}^{w} = \operatorname{Gal}(\overline{\mathbf{Q}}_{\ell}/\mathbf{Q}_{\ell}^{\operatorname{tr}})$$

the wild intertia group

We say

$$\rho: G_{\mathbf{Q}_{\ell}} \to \mathrm{GL}_d(\mathbf{Q}_p)$$

is unramified if

$$\rho(I_\ell) = \{1\}$$

is tamely ramified if

$$\rho(I_\ell^w)=\{1\}.$$

In the first case ρ is completely determined by $\rho(\operatorname{Frob}_{\ell})$. In the second case ρ is completely determined by $\rho(\operatorname{Frob}_{\ell})$ and its value on a generator of

$$Gal(\mathbf{Q}_{\ell}^{tr}/\mathbf{Q}_{\ell}^{ur}).$$

The wild part: I_{ℓ}^{w} is pro- ℓ , $GL_{d}(\mathbf{Z}_{p})$ is almost pro-p (it has a finite index pro-p subgroup).

Exercise 2.11 Prove this.

Example 2.12 For d = 1

$$\mathbf{Z}_p^* = \mathbf{F}_p^* \times (1 + p\mathbf{Z}_p).$$

Thus if $\ell \neq p$ then

$$\rho(I_{\ell}^{w}) \subseteq \operatorname{GL}_{d}(\mathbf{Q}_{p})$$

is finite.

Exercise 2.13 Prove this.

If $\ell = p$ then this is handled by p-adic Hodge theory. The connection to global representations is then that



So

$$G_{\mathbf{Q}_{\ell}} \hookrightarrow G_{\mathbf{Q}}$$

via restriction to $\overline{\mathbf{Q}}$.

The image of this map is the **decomposition group** at ℓ .

$$\rho: G_{\mathbf{O}} \to \mathrm{GL}_d(A)$$
,

we say that ρ is **unramified at** ℓ if

$$\rho(I_{\ell}) = \{1\}.$$

In which case

charpoly(
$$\rho(\text{Frob}_{\ell})$$
)

is well-defined.

Returning to

$$\rho_{E,p^{\infty}} \colon G_{\mathbf{O}} \to \mathrm{GL}_2(\mathbf{Q}_p)$$

now.

Fact 2.14 $\rho_{E,p^{\infty}}$ is unramified outside of $N_E \cdot p$ (N_E is the conductor of E). i.e.

$$\rho_{E,p^\infty}$$

is unramified at ℓ if and only if $\ell \neq p$ and ℓ is a prime of good reduction for E.

So $\rho_{E,p^{\infty}}$ sees bad reduction.

From $\rho_{E,p^{\infty}}$ you can recover *E* up to isogeny (Faltings).

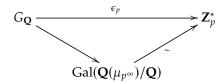
Example 2.15

$$G_{\mathbf{Q}} \cup \mu_{p^n}$$

so we get

$$G_{\mathbf{O}} \to \operatorname{Aut}(\mu_{p^n}) \simeq (\mathbf{Z}/p^n\mathbf{Z})^* \simeq \operatorname{GL}_1(\mathbf{Z}/p^n)$$

taking the inverse limit we get



this ϵ_p is known as the p-adic cyclotomic character. This is unramified outside p and

$$\epsilon_p(\operatorname{Frob}_{\ell}) = \ell$$

for $\ell \neq p$.

Remark 2.16

$$\det(\rho_{E,p^{\infty}}) = \epsilon_p.$$

Example 2.17

$$f = \sum a_n q^n \in S_2(\Gamma_0(N), \mathbf{Q})$$

a weight 2 eigenform on $\Gamma_0(N)$, with rational fourier coefficients. Eichler-Shimura gives E_f/\mathbf{Q} an elliptic curve. Define

$$\rho_f = \rho_{E,p^\infty} \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Q}_p)$$

unramified outside Np

$$Tr(\rho(Frob_{\ell})) = a_{\ell}$$

for $\ell \nmid Np$. More generally

$$f \in S_2(\Gamma_0(N))$$

an eigenform, Eichler-Shimura gives

$$A_f/\mathbf{Q}$$

an abelian variety which leads to

$$\rho_f \colon G_{\mathbf{O}} \to \operatorname{GL}_2(K)$$

 K/\mathbf{Q}_p finite.

2.2 Congruences and elliptic curves

Let

$$E_1$$
: $y^2 = x^3 + x - 10$

 $conductor\ 2^2 \cdot 13, \texttt{https://www.lmfdb.org/EllipticCurve/Q/52a1/}$

$$E_2$$
: $y^2 = x^3 - 584x + 5444$

conductor $2^2 \cdot 7 \cdot 13$, https://www.lmfdb.org/EllipticCurve/Q/364a1/.

Table 2.18 a_p 's for E_1 , E_2

р	2	3	5	7	11	13	17	19	23	29
$a_p(E_1)$	0	0	2	-2	-2	-1	6	-6	8	2
$a_p(E_2)$	0	0	-3	1	-2	-1	-4	-1	-7	7

Note that

$$a_{\ell}(E_1) \equiv a_{\ell}(E_2) \pmod{5}, \ \forall \ell \neq 7$$

 $\Longrightarrow \rho_{E_1,5} \simeq \rho_{E_2,5} (= \bar{\rho})$

as Galois representations.

Exercise 2.19 Prove this.

How common is this? We have 2 lifts of $\bar{\rho}$

$$\rho_{E_1,5^{\infty}} \simeq \rho_{E_2,5^{\infty}}$$

how many other such?

2.3 Hida theory

$$\sum a_n(f)q^n=f\in S_{k_0}(\Gamma_0(N))$$

an eigenform.

$$a_p(f)$$

a p-adic unit.

$$\mathscr{F} = \sum_{n=1}^{\infty} a_n(k) q^n$$

with a_n a p-adic analytic function in k. The whole family gives Galois representations that reduce to the same $\bar{\rho}$.

Specialise k to some integer w

$$\mathcal{F} = \sum a_n(w)q^n \in S_w(\Gamma_0(N))$$

take $w = k_0$ to recover f.

Hida constructs

$$\rho^{Hida} \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_p[[x]])$$

unramified outside Np.

$$\ell \nmid Np \implies \operatorname{tr}(\rho^{Hida}(\operatorname{Frob}_{\ell})) = a_{\ell}(x).$$

Lecture 3 28/1/2018

2.4 More examples of Galois representations in families

A 1-dimensional family. let

 ϵ_p = cyclotomic character

$$\epsilon_p \colon G_{\mathbf{Q}} \to \operatorname{Gal}(\mathbf{Q}(\mu_{p^{\infty}})/\mathbf{Q}) \simeq \mathbf{Z}_p^* \simeq \mathbf{F}_p^* \times (1 + p\mathbf{Z}_p)$$

If we have $k \in \mathbf{Z}$ we can take

 ϵ_{v}^{k}

its power.

Note that when $k_1, k_2 \in \mathbf{Z}$ with

$$k_1 \equiv k_2 \pmod{p-1}$$

then

$$\epsilon_p^{k_1} \equiv \epsilon_p^{k_2} \pmod{p}$$

moreover if

$$k_1 \equiv k_2 \pmod{(p-1)p^N}$$

we havee

$$\epsilon_p^{k_1} \equiv \epsilon_p^{k_2} \pmod{p^{N+1}}.$$

Set

$$\begin{split} \Lambda &= \mathbf{Z}_p[[\mathbf{Z}_p^*]] = \varprojlim \mathbf{Z}_p[(\mathbf{Z}/p^N)^*] \\ & \epsilon_p^{univ} \colon G_{\mathbf{Q}} \to \Lambda^* = \mathrm{GL}_1(\Lambda) \\ & \sigma \mapsto [epsilon_p(\sigma)] \\ & \Lambda \xrightarrow{wt_k} \mathbf{Z}_p \\ & wt_k \circ \epsilon_p^{univ} \end{split}$$

where wt_k is defined by

$$\mathbf{Z}_p^* \to \mathbf{Z}_p^* \subseteq \mathbf{Z}_p$$
$$x \mapsto x^k$$

gives

$$\Lambda \xrightarrow{wt_k} \mathbf{Z}_p.$$

Take any

$$\phi \in \operatorname{Hom}_{cts}(\Lambda, \mathbb{C}_p) \simeq \operatorname{Hom}_{cts}(\mathbb{Z}_p^*, \mathbb{C}_p)$$

induces

$$G_{\mathbf{Q}} \to \mathbf{C}_p^*$$

 $\phi \circ \epsilon_p^{univ}$.

$$\operatorname{Hom}_{cts}(\mathbf{Z}_p^*, \mathbf{C}_p)$$

is a union of p-1 open disks as

$$\mathbf{Z}_p^* \simeq \mathbf{F}_p^* \times (1 + p\mathbf{Z}_p)$$

the rightmost factor is topologically generated by 1 + p. So

$$\phi \colon \mathbf{Z}_p^* \to \mathbf{C}_p^*$$

is determined by

$$\phi|_{\mathbf{F}_n^*}$$

and

$$\phi(\gamma)$$
 for any γ a top. gen..

We need

$$|\phi(\gamma) - 1|_p < 1.$$

We have p-1 disks labelled by characters of \mathbf{F}_p^* . On each disk of $\phi \in \mathrm{Hom}(\mathbf{Z}_p^*, \mathbf{C}_p^*)$ the reduction of

$$\phi \circ \epsilon_p^{univ} \colon G_{\mathbf{Q}} \to \mathbf{C}_p^*$$

is the same.

A 2-d example. Let E/\mathbf{Q} be a CM elliptic curve.

$$\rho_{E,p^{\infty}} \colon G_{\mathbf{Q}} \to \operatorname{GL}_2(\mathbf{Z}_p)$$

then there exists K/\mathbf{Q} quadratic imaginary field and , p-ordinary

$$\psi\colon G_K\to \mathbf{Z}_p^*$$

s.t.

$$\rho_{E,p^{\infty}} \simeq \operatorname{Ind}_{K}^{\mathbf{Q}}(\psi).$$

So we get a "family"

$$\operatorname{Ind}_K^{\mathbf{Q}}(\psi^k)$$
,

more generally can make

$$\psi^{univ}\colon G_K\to \Lambda^*$$

Ind
$$\psi^{univ}$$
: $G_K \to GL_2(\Lambda)$.

CM Hida family.

2.5 Deformation theory

G a profinite group, *k* a finite field of characteristic *p* (simpler $k = \mathbf{F}_p$).

$$\bar{\rho} \colon G \to \operatorname{GL}_d(k)$$

a continuous homomorphism.

To lift $\bar{\rho}$ to be *A*-valued, we need a homomorphism

$$A \rightarrow k$$

we may as well assume A is local so it lifts to $GL_d(A)$. We then only have one residual representation. Localize it at

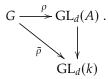
$$\ker(A \to k)$$
.

If A is a local ring then A has a natural topology. Letting \mathfrak{m}_A be the maximal ideal

$$\{\mathfrak{m}_A^j\}$$

is a neighborhood base at 0, this is the m-adic topology.

So we have A local, \mathfrak{m}_A maximal $A/\mathfrak{m}_A \simeq k$, we need to fix this identification to ensure we don't have automorphisms.



Note 2.20 If we have $M \in GL_d(A)$ s.t. $M \equiv 1 \pmod{\mathfrak{m}_A}$, then

$$M\rho M^{-1}$$

lifts $\bar{\rho}$ also.

Definition 2.21 Call ρ and $M\rho M^{-1}$ strictly equivalent.

Definition 2.22 A **deformation of** $\bar{\rho}$ **to A** is a strict equivalence class of continuous homomorphisms

$$G \xrightarrow{\rho} GL_d(A)$$

lifting $\bar{\rho}$.

Want to rephrase this with less bases, instead of $\bar{\rho}$ use \overline{V} a d-dimensional continuous representation of G over k. V a free A-module of rank d with continuous action of G s.t.

$$V \otimes A/\mathfrak{m}_A \simeq \overline{V}$$
.

Naive hope: two such V's are equivalent if they are isomorphic as G-modules.

Problem: R^{univ} won't exist, too many automorphisms. Ideally if V_A is a deformation of \overline{V} to A then there exists a unique

$$R^{univ} \rightarrow A$$

inducing

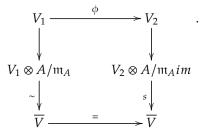
$$V^{univ} \rightarrow V_A$$
.

Take II, let V be a free A-module with a continuous action of G and a fixed isomorphism.

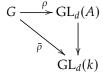
$$V \otimes_A A/\mathfrak{m}_A \simeq \overline{V}$$

as representations of *G* over $k \simeq A/\mathfrak{m}_A$.

Two V's are equivalent if there exists a G-module isomorphism $V_1 \xrightarrow{\phi} V_2$ s.t.



Exercise 2.23 Check the two definitions are the same, GL_d definition vs. \overline{V} . Another approach



lift of $\bar{\rho}$ and say two ρ 's are the same if they are equal as maps, this is a **framed deformation**. We can write this more abstractly

$$\overline{V}$$

a continuous representation of G over k with fixed basis β .

V

a free A module with cont. action of G and a fixed isom.

$$V \otimes_A A/\mathfrak{m}_D \simeq \overline{V}$$

and a basis β_A lifting β .

$$C_k = C$$

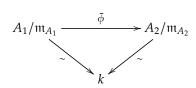
the category of complete local Noetherian rings A with a fixed isomorphism $A/\mathfrak{m}_A\simeq k$. Where the maps are

$$A_1 \xrightarrow{\phi} A_2$$

a local ring homomorphism

$$\phi(\mathfrak{m}_{A_1}) \subseteq \mathfrak{m}_{A_2}$$

and the diagram



commutes.