

# Deformation theory of Galois representations

MA842 at BU Spring 2020

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These are notes for Robert Pollack's course MA842 at BU Spring 2020.

The course webpage is <http://math.bu.edu/people/rpollack/Teach/842spring2020.html>.

## 1 Motivation

Lecture 1 21/1/2020

Let  $E_k$  denote the Eisenstein series of weight  $k$ ,  $k > 2$ .

$$E_k = \frac{-B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in M_k(\mathrm{SL}_2(\mathbf{Z})).$$

Where  $B_k$  are the Bernoulli numbers and

$$\sigma_{k-1}(n) = \sum_{d|n, d>0} d^{k-1}.$$

$E_2$  however is not holomorphic, so not a modular form.

Fix  $N$  a prime, notation has stuck from Mazur's Eisenstein ideal paper.

Then there exists a unique Eisenstein series on  $\Gamma_0(N)$  of weight 2.

$$E_2^{(N)} = \frac{N-1}{12} + \sum_{n=1}^{\infty} \sigma(n) q^n.$$

Funny observation: if  $N \equiv 1 \pmod{p}$  for prime  $p > 3$ . Then  $p | ((N-1)/12)$ , so  $E_2^{(N)}$  "looks cuspidal".

Then we hope that there exists a cuspidal eigenform  $f \in S_2(\Gamma_0(N))$  such that

$$f \equiv E_2^{(N)} \pmod{p''}.$$

This is in fact true, due to Koike in the 70's, there exists  $f \in S_2(\Gamma_0(N))$  such that

$$a_\ell(f) \equiv 1 + \ell \pmod{p}$$

for all  $\ell \neq N, p$ .

**Question 1.1** How many such  $f$  are there?

□

Merel '96:

$$f \text{ is unique} \iff \prod_{i=1}^{(N-1)/2} i^i \text{ is not a } p\text{-th power modulo } N.$$

Wake and Wang-Erickson describe the dimension of the space of such  $f$  using Massey products (higher cup products).

Method: Galois deformations!

## 1.1 Galois representations

We write

$$G_{\mathbf{Q}} = \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) = \varprojlim_{F/\mathbf{Q}, \text{fin. galois}} \text{Gal}(F/\mathbf{Q})$$

a profinite group.

$$\rho: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Q}_p)$$

a continuous homomorphism. Then view  $\text{GL}_2(\mathbf{Q}_p)$  as  $\text{Aut}(V)$  for a 2-dimensional  $\mathbf{Q}_p$  vector space and fix a 2-dimensional  $\mathbf{Z}_p$ -lattice

$$T \subseteq V$$

which is  $G_{\mathbf{Q}}$  stable. Then we can take

$$\bar{\rho}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{F}_p)$$

this is unique (w.r.t. the choice of  $T$ ) only up to semisimplification.

So we say two Galois representations  $\rho_1, \rho_2$  are congruent if

$$\bar{\rho}_1^{\text{ss}} \simeq \bar{\rho}_2^{\text{ss}}.$$

We say  $\rho_1, \rho_2$  are deformations of

$$\bar{\rho}_1 = \bar{\rho}_2$$

(imagine this is reducible).

Start with

$$\bar{\rho}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{F}_p)$$

consider “all” deformations of  $\bar{\rho}$  in good cases there exists a “universal” deformation of  $\bar{\rho}$ .

$R^{\text{univ}}$  a local ring with maximal ideal  $\mathfrak{m}_R$  such that

$$R/\mathfrak{m}_R = \mathbf{F}_p.$$

$$\rho^{\text{univ}}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(R^{\text{univ}})$$

such that if  $\rho: G_{\mathbf{Q}} \rightarrow \text{GL}_2(R)$  is a deformation of  $\bar{\rho}$  then there exists

$$R^{\text{univ}} \rightarrow R$$

such that

$$\begin{array}{ccc} G_{\mathbf{Q}} & \xrightarrow{\rho^{\text{univ}}} & \text{GL}_2(R^{\text{univ}}) \\ & \searrow \rho & \downarrow \\ & & \text{GL}_2(R) \end{array}$$

## 1.2 Modular forms

$$f = \sum a_n q^n \in S_k(\Gamma_0(N))$$

an eigenform leads to

$$\rho_f: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(K), K/\mathbf{Q}_p \text{ finite}$$

with the property that for all  $\ell \nmid Np$  we have

$$\mathrm{Tr}(\rho_f(\mathrm{Frob}_{\ell})) = a_{\ell}.$$

Modular forms can be congruent

$$a_{\ell}(f_1) \equiv a_{\ell}(f_2) \pmod{p} \text{ for all but finitely many } \ell$$

$$\Updownarrow$$

$$\bar{\rho}_{f_1}^{\mathrm{ss}} \simeq \bar{\rho}_{f_2}^{\mathrm{ss}}.$$

There exists a ring, the Hecke algebra  $\mathbf{T}$  parametrizing all  $f$ 's with the same  $\bar{\rho}$ .

$$f \rightsquigarrow \rho_f \implies R^{\mathrm{univ}} \rightarrow \mathbf{T}$$

so hope

$$R^{\mathrm{univ}} \simeq \mathbf{T}.$$

Wiles proof of FLT proved one of these.

Many more such theorems in the past couple of decades.

Wake and Wang-Erickson show that the dimension of

$$\{f : f \equiv E_2^{(N)}\} \leftrightarrow \mathrm{rank} \mathbf{T} = \mathrm{rank} R^{\mathrm{univ}}.$$

$$a_{\ell}(f) \equiv 1 + \ell \pmod{p}$$

$$\implies \bar{\rho}^{\mathrm{ss}} = \mathbf{1} \oplus \mu_p$$

but there does not exist  $R^{\mathrm{univ}}$  in this context.

The fix is to use pseudorepresentations instead of representations.

## 1.3 Pseudorepresentations

Let  $G$  be a group.

Then a pseudorepresentation  $T$  is a map

$$T: G \rightarrow A$$

for  $A$  a ring satisfying

1.

$$T(xy) = T(yx)$$

2.

$$T(x)T(y)T(z) - T(x)T(yz) - T(y)T(xz) - T(z)T(xy) + T(xyz) + T(xzy) = 0$$

and the analogous formulae for higher dimensions.

**Fact 1.2** If  $A$  is an algebraically closed field of characteristic  $\neq 2$ . Then for a given pseudorepresentation  $T$  there exists a true representation  $\rho$  such that

$$T = \text{Tr}(\rho).$$

But this does not hold in general.

Universal pseudodeformation rings always exist. Wake and Wang-Erickson use  $R^{\text{univ}}$  = universal pseudodeformation ring.

## 2 Definitions

### 2.1 Representations

Lecture 2 23/1/2020

**Definition 2.1 Representations.** Let  $G$  be a finite group and  $V$  a finite dimensional vector space over  $\mathbf{C}$  of dimension  $d$ . A **representation** of  $G$  is a homomorphism

$$G \xrightarrow{\rho} \text{Aut}(V) \simeq \text{GL}_d(\mathbf{C}),$$

$G$  acts linearly on  $V$ . ◇

Galois representations: Let  $G$  be a Galois group, possibly infinite.  $F$  be a field,

$$G_F = \text{Gal}(\bar{F}/F) = \varprojlim_{L/F, \text{ fin. gal.}} \text{Gal}(L/F)$$

profinite compact and totally disconnected.

Replace  $V$  with a finite free module over some topological ring  $A$

$$\rho: G_F \rightarrow \text{GL}_d(A)$$

a continuous homomorphism.

**Example 2.2**  $A = \mathbf{C}$  with the complex topology. □

**Fact 2.3** In this case  $\text{im}(\rho)$  is finite.

**Exercise 2.4** Prove this.

Then we can write

$$\begin{array}{ccc} G_F & \xrightarrow{\rho} & \text{GL}_d(\mathbf{C}) \\ & \searrow & \nearrow \\ & \text{Gal}(L/F) & \end{array}$$

where  $L/F$  is finite. There are many such representations.

**Conjecture 2.5** Every finite group is a quotient of  $G_{\mathbf{Q}}$ .

**Example 2.6**

$$F = \mathbf{Q}$$

$$L = \mathbf{Q}(\sqrt[4]{2}, i)$$

$$\text{Gal}(L/\mathbf{Q}) \simeq D_4$$

$$G_{\mathbf{Q}} \rightarrow \text{Gal}(L/\mathbf{Q}) \xrightarrow{\rho} \text{GL}_2(\mathbf{C})$$

with  $\rho$  the unique irreducible 2-dimensional representation of  $D_4$ . □

**Example 2.7** Let  $E/\mathbf{Q}$  be an elliptic curve

$$G_{\mathbf{Q}} \curvearrowright E[p] \simeq \mathbf{Z}/p \oplus \mathbf{Z}/p$$

$$\rho_{E,p}: G_{\mathbf{Q}} \rightarrow \text{Aut}(E[p]) \simeq \text{GL}_2(\mathbf{F}_p)$$

$$\rho_{E,p^n}: G_{\mathbf{Q}} \rightarrow \text{Aut}(E[p^n]) \simeq \text{GL}_2(\mathbf{Z}/p^n)$$

$$\rho_{E,p^\infty}: G_{\mathbf{Q}} \rightarrow \text{Aut}(E[p^\infty]) \simeq \text{GL}_2(\mathbf{Z}_p).$$

□

**Fact 2.8**  $\rho_{E,p^\infty}$  has finite image.

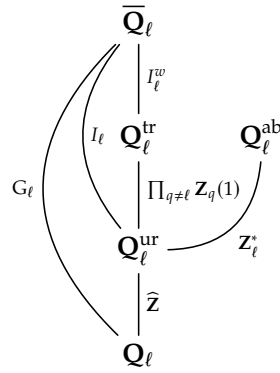
**Exercise 2.9** Prove this.

**Example 2.10** Let  $F = \mathbf{Q}_\ell$ ,  $A = \mathbf{Q}_p$

$$\rho: G_{\mathbf{Q}_\ell} \rightarrow \text{GL}_d(\mathbf{Q}_p).$$

□

The Galois theory of local fields looks like



where

$$\mathbf{Q}_\ell^{\text{tr}} = \mathbf{Q}_\ell(\{\sqrt[n]{\ell}\}_{\ell \nmid n})$$

the maximal tamely ramified extension

$$\mathbf{Q}_\ell^{\text{ur}} = \mathbf{Q}_\ell(\{\mu_n\}_{\ell \nmid n})$$

the maximal unramified extension

$$\mathbf{Q}_\ell^{\text{ab}} = \mathbf{Q}_\ell^{\text{un}}(\mu_{\ell^\infty})$$

the maximal abelian extension.

$$I_\ell = \text{Gal}(\overline{\mathbf{Q}}_\ell / \mathbf{Q}_\ell^{\text{ur}})$$

the inertia group

$$I_\ell^w = \text{Gal}(\overline{\mathbf{Q}}_\ell / \mathbf{Q}_\ell^{\text{tr}})$$

the wild inertia group

We say

$$\rho: G_{\mathbf{Q}_\ell} \rightarrow \text{GL}_d(\mathbf{Q}_p)$$

is **unramified** if

$$\rho(I_\ell) = \{1\}$$

is **tamely ramified** if

$$\rho(I_\ell^w) = \{1\}.$$

In the first case  $\rho$  is completely determined by  $\rho(\text{Frob}_\ell)$ . In the second case  $\rho$  is completely determined by  $\rho(\text{Frob}_\ell)$  and its value on a generator of

$$\text{Gal}(\mathbf{Q}_\ell^{\text{tr}}/\mathbf{Q}_\ell^{\text{ur}}).$$

The wild part:  $I_\ell^w$  is pro- $\ell$ ,  $\text{GL}_d(\mathbf{Z}_p)$  is almost pro- $p$  (it has a finite index pro- $p$  subgroup).

**Exercise 2.11** Prove this.

**Example 2.12** For  $d = 1$

$$\mathbf{Z}_p^* = \mathbf{F}_p^* \times (1 + p\mathbf{Z}_p).$$

□

Thus if  $\ell \neq p$  then

$$\rho(I_\ell^w) \subseteq \text{GL}_d(\mathbf{Q}_p)$$

is finite.

**Exercise 2.13** Prove this.

If  $\ell = p$  then this is handled by  $p$ -adic Hodge theory.

The connection to global representations is then that

$$\begin{array}{ccc} \overline{\mathbf{Q}} & \longrightarrow & \overline{\mathbf{Q}}_\ell \\ \uparrow & & \uparrow \\ \mathbf{Q} & \longrightarrow & \mathbf{Q}_\ell \end{array}$$

So

$$G_{\mathbf{Q}_\ell} \hookrightarrow G_{\mathbf{Q}}$$

via restriction to  $\overline{\mathbf{Q}}$ .

The image of this map is the **decomposition group** at  $\ell$ .

$$\rho: G_{\mathbf{Q}} \rightarrow \text{GL}_d(A),$$

we say that  $\rho$  is **unramified at  $\ell$**  if

$$\rho(I_\ell) = \{1\}.$$

In which case

$$\text{charpoly}(\rho(\text{Frob}_\ell))$$

is well-defined.

Returning to

$$\rho_{E,p^\infty}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Q}_p)$$

now.

**Fact 2.14**  $\rho_{E,p^\infty}$  is unramified outside of  $N_E \cdot p$  ( $N_E$  is the conductor of  $E$ ). i.e.

$$\rho_{E,p^\infty}$$

is unramified at  $\ell$  if and only if  $\ell \neq p$  and  $\ell$  is a prime of good reduction for  $E$ .

So  $\rho_{E,p^\infty}$  sees bad reduction.

From  $\rho_{E,p^\infty}$  you can recover  $E$  up to isogeny (Faltings).

**Example 2.15**

$$G_{\mathbf{Q}} \curvearrowright \mu_{p^n}$$

so we get

$$G_{\mathbf{Q}} \rightarrow \text{Aut}(\mu_{p^n}) \simeq (\mathbf{Z}/p^n \mathbf{Z})^* \simeq \text{GL}_1(\mathbf{Z}/p^n)$$

taking the inverse limit we get

$$\begin{array}{ccc} G_{\mathbf{Q}} & \xrightarrow{\epsilon_p} & \mathbf{Z}_p^* \\ & \searrow & \nearrow \sim \\ & \text{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) & \end{array}$$

this  $\epsilon_p$  is known as the  $p$ -adic cyclotomic character. This is unramified outside  $p$  and

$$\epsilon_p(\text{Frob}_\ell) = \ell$$

for  $\ell \neq p$ . □

**Remark 2.16**

$$\det(\rho_{E,p^\infty}) = \epsilon_p.$$

**Example 2.17**

$$f = \sum a_n q^n \in S_2(\Gamma_0(N), \mathbf{Q})$$

a weight 2 eigenform on  $\Gamma_0(N)$ , with rational fourier coefficients. Eichler-Shimura gives  $E_f/\mathbf{Q}$  an elliptic curve. Define

$$\rho_f = \rho_{E,p^\infty}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Q}_p)$$

unramified outside  $Np$

$$\text{Tr}(\rho(\text{Frob}_\ell)) = a_\ell$$

for  $\ell \nmid Np$ . More generally

$$f \in S_2(\Gamma_0(N))$$

an eigenform, Eichler-Shimura gives

$$A_f/\mathbf{Q}$$

an abelian variety which leads to

$$\rho_f: G_{\mathbf{Q}} \rightarrow \text{GL}_2(K)$$

$K/\mathbf{Q}_p$  finite. □

## 2.2 Congruences and elliptic curves

Let

$$E_1: y^2 = x^3 + x - 10$$

conductor  $2^2 \cdot 13$ , <https://www.lmfdb.org/EllipticCurve/Q/52a1/>

$$E_2: y^2 = x^3 - 584x + 5444$$

conductor  $2^2 \cdot 7 \cdot 13$ , <https://www.lmfdb.org/EllipticCurve/Q/364a1/>.

**Table 2.18**  $a_p$ 's for  $E_1, E_2$

$p$	2	3	5	7	11	13	17	19	23	29
$a_p(E_1)$	0	0	2	-2	-2	-1	6	-6	8	2
$a_p(E_2)$	0	0	-3	1	-2	-1	-4	-1	-7	7

Note that

$$a_\ell(E_1) \equiv a_\ell(E_2) \pmod{5}, \forall \ell \neq 7 \\ \implies \rho_{E_1,5} \simeq \rho_{E_2,5}(=\bar{\rho})$$

as Galois representations.

**Exercise 2.19** Prove this.

How common is this? We have 2 lifts of  $\bar{\rho}$

$$\rho_{E_1,5^\infty} \simeq \rho_{E_2,5^\infty}$$

how many other such?

## 2.3 Hida theory

$$\sum a_n(f)q^n = f \in S_{k_0}(\Gamma_0(N))$$

an eigenform.

$$a_p(f)$$

a  $p$ -adic unit.

$$\mathcal{F} = \sum_{n=1}^{\infty} a_n(k)q^n$$

with  $a_n$  a  $p$ -adic analytic function in  $k$ . The whole family gives Galois representations that reduce to the same  $\bar{\rho}$ .

Specialise  $k$  to some integer  $w$

$$\mathcal{F} = \sum a_n(w)q^n \in S_w(\Gamma_0(N))$$

take  $w = k_0$  to recover  $f$ .

Hida constructs

$$\rho^{Hida}: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{Z}_p[[x]])$$

unramified outside  $Np$ .

$$\ell \nmid Np \implies \mathrm{tr}(\rho^{Hida}(\mathrm{Frob}_\ell)) = a_\ell(x).$$

## 2.4 More examples of Galois representations in families

Lecture 3 28/1/2020

**A 1-dimensional family.** let

$$\epsilon_p = \text{cyclotomic character}$$

$$\epsilon_p: G_{\mathbf{Q}} \rightarrow \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \simeq \mathbf{Z}_p^* \simeq \mathbf{F}_p^* \times (1 + p\mathbf{Z}_p)$$

If we have  $k \in \mathbf{Z}$  we can take

$$\epsilon_p^k$$

its power.

Note that when  $k_1, k_2 \in \mathbf{Z}$  with

$$k_1 \equiv k_2 \pmod{p-1}$$



then

$$\epsilon_p^{k_1} \equiv \epsilon_p^{k_2} \pmod{p}$$

moreover if

$$k_1 \equiv k_2 \pmod{(p-1)p^N}$$

we have

$$\epsilon_p^{k_1} \equiv \epsilon_p^{k_2} \pmod{p^{N+1}}.$$

Set

$$\Lambda = \mathbf{Z}_p[[\mathbf{Z}_p^*]] = \varprojlim \mathbf{Z}_p[(\mathbf{Z}/p^N)^*]$$

$$\epsilon_p^{univ} : G_{\mathbf{Q}} \rightarrow \Lambda^* = \mathrm{GL}_1(\Lambda)$$

$$\sigma \mapsto [\epsilon_p(\sigma)]$$

$$\Lambda \xrightarrow{wt_k} \mathbf{Z}_p$$

$$wt_k \circ \epsilon_p^{univ}$$

where  $wt_k$  is defined by

$$\mathbf{Z}_p^* \rightarrow \mathbf{Z}_p^* \subseteq \mathbf{Z}_p$$

$$x \mapsto x^k$$

gives

$$\Lambda \xrightarrow{wt_k} \mathbf{Z}_p.$$

Take any

$$\phi \in \mathrm{Hom}_{cts}(\Lambda, \mathbf{C}_p) \simeq \mathrm{Hom}_{cts}(\mathbf{Z}_p^*, \mathbf{C}_p)$$

induces

$$G_{\mathbf{Q}} \rightarrow \mathbf{C}_p^*$$

$$\phi \circ \epsilon_p^{univ}.$$

$$\mathrm{Hom}_{cts}(\mathbf{Z}_p^*, \mathbf{C}_p)$$

is a union of  $p-1$  open disks as

$$\mathbf{Z}_p^* \simeq \mathbf{F}_p^* \times (1 + p\mathbf{Z}_p)$$

the rightmost factor is topologically generated by  $1+p$ . So

$$\phi : \mathbf{Z}_p^* \rightarrow \mathbf{C}_p^*$$

is determined by

$$\phi|_{\mathbf{F}_p^*}$$

and

$$\phi(\gamma) \text{ for any } \gamma \text{ a top. gen..}$$

We need

$$|\phi(\gamma) - 1|_p < 1.$$

We have  $p-1$  disks labelled by characters of  $\mathbf{F}_p^*$ . On each disk of  $\phi \in \mathrm{Hom}(\mathbf{Z}_p^*, \mathbf{C}_p^*)$  the reduction of

$$\phi \circ \epsilon_p^{univ} : G_{\mathbf{Q}} \rightarrow \mathbf{C}_p^*$$

is the same.

**A 2-d example.** Let  $E/\mathbf{Q}$  be a CM elliptic curve.

$$\rho_{E,p^\infty}: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{Z}_p)$$

then there exists  $K/\mathbf{Q}$  quadratic imaginary field and, if  $p$  is an ordinary prime

$$\psi: G_K \rightarrow \mathbf{Z}_p^*$$

s.t.

$$\rho_{E,p^\infty} \simeq \mathrm{Ind}_K^{\mathbf{Q}}(\psi).$$

So we get a “family”

$$\mathrm{Ind}_K^{\mathbf{Q}}(\psi^k),$$

more generally can make

$$\psi^{univ}: G_K \rightarrow \Lambda^*$$

$$\mathrm{Ind} \psi^{univ}: G_K \rightarrow \mathrm{GL}_2(\Lambda).$$

CM Hida family.

## 2.5 Deformation theory

$G$  a profinite group,  $k$  a finite field of characteristic  $p$  (simpler  $k = \mathbf{F}_p$ ).

$$\bar{\rho}: G \rightarrow \mathrm{GL}_d(k)$$

a continuous homomorphism.

To lift  $\bar{\rho}$  to be  $A$ -valued, we need a homomorphism

$$A \twoheadrightarrow k$$

we may as well assume  $A$  is local so it lifts to  $\mathrm{GL}_d(A)$ . We then only have one residual representation. Localize it at

$$\ker(A \rightarrow k).$$

If  $A$  is a local ring then  $A$  has a natural topology. Letting  $\mathfrak{m}_A$  be the maximal ideal

$$\{\mathfrak{m}_A^j\}$$

is a neighborhood base at 0, this is the  $\mathfrak{m}$ -adic topology.

So we have  $A$  local,  $\mathfrak{m}_A$  maximal  $A/\mathfrak{m}_A \simeq k$ , we need to fix this identification to ensure we don't have automorphisms.

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{GL}_d(A) \\ & \searrow \bar{\rho} & \downarrow \\ & & \mathrm{GL}_d(k) \end{array}$$

**Note 2.20** If we have  $M \in \mathrm{GL}_d(A)$  s.t.  $M \equiv 1 \pmod{\mathfrak{m}_A}$ , then

$$M\rho M^{-1}$$

lifts  $\bar{\rho}$  also.

**Definition 2.21** Call  $\rho$  and  $M\rho M^{-1}$  **strictly equivalent**.

◇

**Definition 2.22** A deformation of  $\bar{\rho}$  to  $A$  is a strict equivalence class of continuous homomorphisms

$$G \xrightarrow{\rho} \mathrm{GL}_d(A)$$

lifting  $\bar{\rho}$ . ◇

Want to rephrase this with less bases, instead of  $\bar{\rho}$  use  $\bar{V}$  a  $d$ -dimensional continuous representation of  $G$  over  $k$ .  $V$  a free  $A$ -module of rank  $d$  with continuous action of  $G$  s.t.

$$V \otimes A/\mathfrak{m}_A \simeq \bar{V}.$$

Naive hope: two such  $V$ 's are equivalent if they are isomorphic as  $G$ -modules.

Problem:  $R^{univ}$  won't exist, too many automorphisms. Ideally if  $V_A$  is a deformation of  $\bar{V}$  to  $A$  then there exists a unique

$$R^{univ} \rightarrow A$$

inducing

$$V^{univ} \rightarrow V_A.$$

Take II, let  $V$  be a free  $A$ -module with a continuous action of  $G$  and a fixed isomorphism.

$$V \otimes_A A/\mathfrak{m}_A \simeq \bar{V}$$

as representations of  $G$  over  $k \simeq A/\mathfrak{m}_A$ .

Two  $V$ 's are equivalent if there exists a  $G$ -module isomorphism  $V_1 \xrightarrow{\phi} V_2$  s.t.

$$\begin{array}{ccc} V_1 & \xrightarrow{\phi} & V_2 \\ \downarrow & & \downarrow \\ V_1 \otimes A/\mathfrak{m}_A & & V_2 \otimes A/\mathfrak{m}_A \\ \sim \downarrow & & \sim \downarrow \\ \bar{V} & \xrightarrow{\mathrm{id}} & \bar{V} \end{array}.$$

**Exercise 2.23** Check the two definitions are the same,  $\mathrm{GL}_d$  definition vs.  $\bar{V}$ .

Another approach

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{GL}_d(A) \\ & \searrow \bar{\rho} & \downarrow \\ & & \mathrm{GL}_d(k) \end{array}$$

lift of  $\bar{\rho}$  and say two  $\rho$ 's are the same if they are equal as maps, this is a **framed deformation**. We can write this more abstractly

$$\bar{V}$$

a continuous representation of  $G$  over  $k$  with fixed basis  $\beta$ .

$$V$$

a free  $A$  module with cont. action of  $G$  and a fixed isom.

$$V \otimes_A A/\mathfrak{m}_D \simeq \bar{V}$$

and a basis  $\beta_A$  lifting  $\beta$ .

Categories

$$C_k = C$$

the category of complete local Noetherian rings  $A$  with a fixed isomorphism  $A/\mathfrak{m}_A \simeq k$ . Where the maps are

$$A_1 \xrightarrow{\phi} A_2$$

a local ring homomorphism

$$\phi(\mathfrak{m}_{A_1}) \subseteq \mathfrak{m}_{A_2}$$

and the diagram

$$\begin{array}{ccc} A_1/\mathfrak{m}_{A_1} & \xrightarrow{\bar{\phi}} & A_2/\mathfrak{m}_{A_2} \\ & \searrow \sim & \swarrow \sim \\ & k & \end{array}$$

commutes.

Lecture 4 30/1/2020

Last time: multiplication by  $c$  on  $R^{univ}$  is not a ring map!

**Exercise 2.24** Show that the “bad” definition of deformation (without fixing the isomorphism to  $\bar{V}$ ) can’t be representable.

**Exercise 2.25** If  $\text{Aut}_G(\bar{\rho}) \supsetneq k^*$  show that the correct definition of deformation doesn’t lead to a representable functor.

$k$  finite field of characteristic  $p$ .  $C_k$  category of complete local Noetherian rings with a fixed identification

$$A/\mathfrak{m}_A \simeq k$$

with  $(A, \mathfrak{m}_A)$  a local ring, maps are local homomorphisms preserving the fixed identification.

**Witt vectors.**  $W(k)$  = Witt vectors of  $k$  and an object of  $C_k$ . The maximal ideal of  $W(k)$  is  $pW(k)$ .

$$W(k)/pW(k) \simeq k$$

$$W(k)^\times \rightarrow k^\times$$

this map has a section which is multiplicative.

$$[x] \mapsto x.$$

**Example 2.26** If  $k \simeq \mathbb{F}_q$ ,  $q = p^n$  then

$$W(k) = \mathcal{O}_{K_n}, K_n = \mathbb{Q}_p(\mu_{p^n-1})$$

unramified extension with

$$\mathfrak{m}_{W(k)} = (p)$$

take  $x^{q-1}$  and observe that its roots are distinct mod  $p$ . For  $W(\mathbb{F}_p) \simeq \mathbb{Z}_p$ .  $\square$

$\bar{x} \in k$ , lift to  $x \in W(k)$

$$\lim_{m \rightarrow \infty} x^{q^m} = [x].$$

**Exercise 2.27** Check this.

$A \in C_k$  there exists a unique ring hom

$$\begin{array}{ccc} W(k) & \longrightarrow & A \\ \downarrow & & \downarrow \\ k & \longrightarrow & A/\mathfrak{m}_A \end{array}$$

$A$  is a  $W(k)$ -module.

Moreover,  $A$  is a quotient of  $W(k)[[x_1, \dots]]$ . For  $A, A' \in C_k$

$$A \xrightarrow{\phi} A'$$

in  $C_k$  then  $\phi$  is automatically  $W(k)$ -linear.

**Deformation functor.**

$$\bar{\rho}: G \rightarrow \mathrm{GL}_d(k)$$

$$D_{\bar{\rho}}: C_k \rightarrow \mathrm{Set}$$

$$A \mapsto \{\text{deformations of } \bar{\rho} \text{ to } A\} = \{\rho: G \rightarrow \mathrm{GL}_d(A) \text{ lifting } \bar{\rho}\} / \text{strict equiv.}$$

Given

$$R \in C_k$$

we have

$$F_R: C_k \rightarrow \mathrm{Set}$$

$$A \mapsto \mathrm{Hom}(R, A).$$

**Exercise 2.28** If  $A \xrightarrow{\phi} A'$  is a local ring Hom which is  $W(k)$ -linear then  $\phi$  is a map in  $C_k$ .

**Definition 2.29 Representable functors.** Any functor

$$F: C_k \rightarrow \mathrm{Set}$$

is said to be representable if there exists  $R \in C_k$  such that

$$F \simeq F_R.$$

◇

To say  $D_{\bar{\rho}}$  is representable says that

$$\exists R_{\bar{\rho}}^{univ} \in C_k$$

such that

$$D_{\bar{\rho}}(A) \simeq \mathrm{Hom}(R_{\bar{\rho}}^{univ}, A)$$

which is functorial in  $A$ .

In this set-up take  $A = R_{\bar{\rho}}^{univ}$ . Then

$$D_{\bar{\rho}}(R_{\bar{\rho}}^{univ}) \simeq \mathrm{Hom}(R_{\bar{\rho}}^{univ}, R_{\bar{\rho}}^{univ}) \ni 1$$

so this corresponds to some  $V^{univ}$

Now given

$$\rho_A: G \rightarrow \mathrm{GL}_d(A) \simeq \mathrm{Aut}(V_A)$$

we have  $\rho_A \in D_{\bar{\rho}}(A) \simeq \text{Hom}(R_{\bar{\rho}}^{univ}, A)$ , so we get some  $\phi: R_{\bar{\rho}}^{univ} \rightarrow A$ . Giving a diagram

$$\begin{array}{ccc} D_{\bar{\rho}}(R_{\bar{\rho}}^{univ}) & \longrightarrow & \tilde{\text{Hom}}(R_{\bar{\rho}}^{univ}, R_{\bar{\rho}}^{univ}) \ni 1 \\ \phi \downarrow & & \downarrow \\ D_{\bar{\rho}}(A) & \longrightarrow & \tilde{\text{Hom}}(R_{\bar{\rho}}^{univ}, A) \ni \phi \end{array}$$

**Framed deformations.**

$$D_{\bar{\rho}}^{\square}(A) = \{\rho: G \rightarrow \text{GL}_d(A) \text{ lifting } \bar{\rho}\}$$

we will prove that

$$D_{\bar{\rho}}^{\square}$$

is in fact representable.

$$\bar{\rho}: G \rightarrow \text{GL}_d(k)$$

we build

$$R_{\bar{\rho}}^{\square, univ}$$

and

$$\rho_{\bar{\rho}}^{\square, univ}: G \rightarrow \text{GL}_d(R_{\bar{\rho}}^{\square, univ}).$$

Let  $\{g_{\alpha}\}_{\alpha}$  be a generating set of  $G$ . Make formal variables  $X_{ij}^{\alpha}$  for all  $\alpha$ ,  $1 \leq i, j \leq d$ .

$$R = W(k)[[\{X_{ij}^{\alpha}\}]]$$

try sending

$$g_{\alpha} \mapsto (X_{ij}^{\alpha}) \in M_d(R)$$

doesn't land in  $\text{GL}_d$ .

So "add on  $\bar{\rho}$ ".

$$k \rightarrow W(k)$$

$$x \mapsto [x]$$

so we get

$$\text{GL}_d(k) \xrightarrow{[\cdot]} \text{GL}_d(W(k))$$

$$[(a_{ij})_{ij}] \mapsto ([a_{ij}])_{ij}$$

now send

$$g_{\alpha} \mapsto (X_{ij}^{\alpha}) + [\bar{\rho}(g_{\alpha})] \in \text{GL}_d(R)$$

this does not give a map

$$G \rightarrow \text{GL}_d(R)$$

because of relations between  $g_{\alpha}$ .

**Example 2.30** Take the relation

$$g_1 g_2^{-1} = e$$

need

$$((X_{ij}^1) + [\bar{\rho}(g_1)])((X_{ij}^2) + [\bar{\rho}(g_2)])^{-1} - 1 = 0$$

this gives  $d^2$  equations in  $R$ . □

Can do this in general. In fact the LHS is in  $\mathfrak{m}_R$ . So if  $I$  is the ideal of all relations, then  $I \subseteq \mathfrak{m}_R$ . And let

$$R_{\bar{\rho}}^{univ, \square} = R/I.$$

$$\begin{aligned} \rho: G &\rightarrow \mathrm{GL}_d(R/I) \\ g_\alpha &\mapsto (X_{ij}^\alpha) + [\bar{\rho}(g_\alpha)] \pmod{I} \end{aligned}$$

is a well-defined homomorphism. And it is universal: If

$$\begin{aligned} \rho_A: G &\rightarrow \mathrm{GL}_d(A) \\ R &\xrightarrow{\pi} A \\ (X_{ij}^\alpha)_{ij} &\mapsto (\rho_A(g_\alpha) - [\bar{\rho}(g_\alpha)])_{ij} \end{aligned}$$

then  $\pi$  kills  $I$  (exercise).

$$R/I \xrightarrow{\bar{\pi}} A.$$

**Exercise 2.31** Check this induces  $\rho_A$  from the universal.

There are two problems with this, it is not necessary that the universal deformation we have constructed be continuous, or that the universal deformation ring be continuous.

The root of these problems is that there are potentially too many  $g_\alpha$ . introduce a condition  $\Phi_p$  that for all open finite index

$$G_0 \subseteq G$$

we have

$$\mathrm{Hom}(G_0, \mathbf{F}_p)$$

is finite.

**Exercise 2.32**

$$\mathrm{Hom}(G_0, \mathbf{F}_p)$$

is finite if and only if  $G^{pro-p}$  is topologically finitely generated. (this is Gouvea lemma 2.1)

If  $G = G_K$  with  $K/\mathbf{Q}_\ell$  finite, WLOG

$$G = G_0 = G_K$$

and  $\Phi_p$  holds in this case

$$G_K^{\mathrm{ab}} \simeq \hat{K}^* \simeq \hat{\mathbf{Z}} \times \mathcal{O}_K^*$$

an exercise in LCFT

$$G_0 \subseteq G_K$$

fin index open then

$$G_0 = G_L$$

with  $L/\mathbf{Q}_p$

Lecture 4 4/2/2020

**Redo.**

$$D_{\bar{\rho}}^{\square}: C \rightarrow \text{Set}$$

$$A \mapsto \left\{ \begin{array}{ccc} G & \xrightarrow{\rho} & \text{GL}_d(A) \\ & \searrow \bar{\rho} & \downarrow \\ & & \text{GL}_d(k) \end{array} \right\}$$

$$D_{\bar{\rho}}: C \rightarrow \text{Set}$$

$$A \mapsto D_{\bar{\rho}}^{\square} / \text{strict equiv.}$$

**Theorem 2.33**

1.  $D_{\bar{\rho}}^{\square}$  is representable (by  $R_{\bar{\rho}}^{\square}$ )
2. If  $\text{Aut}_G \bar{\rho} = k^{\times}$  then  $D_{\bar{\rho}}$  is representable ( $R_{\bar{\rho}}$ ).

Universal property: framed case:

$$\exists \rho_{univ}^{\square}: G \rightarrow \text{GL}_d(R_{\bar{\rho}}^{\square})$$

s.t. if  $\rho_A$  is a lift of  $\bar{\rho}$  to  $A$  then there exists a unique

$$R_{\bar{\rho}}^{\square} \xrightarrow{\phi} A$$

which is in  $C$  s.t.

$$\begin{array}{ccc} G & \xrightarrow{\rho_{univ}^{\square}} & \text{GL}_d(A) \\ & \searrow \rho_A & \downarrow \phi^* \\ & & \text{GL}_d(A) \end{array}$$

unframed case: there exists a deformation  $[\rho_{univ}]$  of  $\bar{\rho}$  to  $R_{\bar{\rho}}$  s.t. if  $[\rho_A]$  is a deformation of  $\bar{\rho}$  to  $A$  then there exists a unique

$$R_{\bar{\rho}} \xrightarrow{\phi} A$$

s.t

$$\phi^*([\rho_{univ}]) = [\rho_A].$$

$$D_{\bar{\rho}}^{\square}(R_{\bar{\rho}}^{\square}) = \text{Hom}(R_{\bar{\rho}}^{\square}, R_{\bar{\rho}}^{\square})$$

$$\rho_{univ}^{\square} \leftrightarrow 1.$$

**Exercise 2.34** Check this universal property (also unframed case).

**Last time.**

$$R_{\bar{\rho}}^{\square} = W(k)[[\{X_{ij}^{\alpha}\}_{i,j,\alpha}]]/I_{relations}.$$

Introduced the condition

$$\Phi_p \forall \text{ open finite index } G_0 \subseteq G, \text{ Hom}(G_0, \mathbb{F}_p) \text{ is finite.}$$



This condition holds for  $G_K$  when  $K/\mathbf{Q}_\ell$  is finite.

However it is false for  $G_K$  when  $K/\mathbf{Q}$  is finite. Such as  $K = \mathbf{Q}$ . As for  $\ell \equiv 1 \pmod{p}$  then

$$G_{\mathbf{Q}} \rightarrow \text{Gal}(\mathbf{Q}(\mu_\ell)/\mathbf{Q}) \rightarrow \mathbf{F}_p$$

and there are infinitely many such  $\ell$ 's

This is not a problem as the representations we are interested in, those coming from elliptic curves and modular forms, have good reduction outside a finite set of primes  $S$ . We may encode this into our deformation problem as follows: Let  $K/\mathbf{Q}$  finite and  $S$  a finite set of places of  $K$  and

$$K_S$$

be the maximal extension of  $K$  unramified outside of  $S$ . Then

$$G_{K,S} = \text{Gal}(K_S/K).$$

And  $\Phi_p$  is true of  $G_{K,S}$ . The proof is an exercise in Global CFT. Silverman VIII prop. 1.6.

$H$  profinite, let  $H^{\text{pro-}p}$  be the maximal pro- $p$  quotient of  $H$ .

**Exercise 2.35**  $\text{Hom}(H, \mathbf{F}_p)$  finite implies that  $H^{\text{pro-}p}$  is topologically finitely generated. This is an equivalence in fact (lemma 2.1 Gouvea).

**Existence of the framed deformation ring (take II).** Assume  $G$  satisfies  $\Phi_p$ .

$$\bar{\rho}: G \rightarrow \text{GL}_d(k),$$

$H = \ker(\bar{\rho})$  is finite index in  $G$ . Let

$$\rho_A: G \rightarrow \text{GL}_d(A)$$

lift  $\bar{\rho}$ .

$$\rho_A(H) \subseteq \ker(\text{GL}_d(A) \rightarrow \text{GL}_d(k)).$$

**Exercise 2.36** Prove this is pro- $p$ .

Hence  $\rho_A|_H$  factors through  $H^{\text{pro-}p}$ .

Pick topological generators  $g_1, \dots, g_s$  of  $H^{\text{pro-}p}$  (via  $\Phi_p$ ). Pick coset representatives  $g_{s+1}, \dots, g_t$  of  $G/H$ .

Run the argument from last time to get

$$R_{\bar{\rho}}^{\square} = W(k)[[\{X_{ij}^{\alpha}\}_{\alpha=1, \dots, t, i, j=1, \dots, d}]]/\overline{I_{\text{rel}}}.$$

**Space of universal framed deformations.**

$$\text{Hom}(R_{\bar{\rho}}^{\square}, A)$$

For unframed deformations it can be the case that we even have that

$$R_{\bar{\rho}} \simeq \mathbf{Z}_p[[X_1, X_2, X_3]]$$

$$A = \mathbf{Z}_p$$

$$D_{\bar{\rho}}(\mathbf{Z}_p) = \text{Hom}_C(\mathbf{Z}_p[[X_1, X_2, X_3]], \mathbf{Z}_p) \simeq (p\mathbf{Z}_p)^3.$$

To prove this space exists for unframed deformations, we have 3 options

1. We should have a bigger space of framed deformations in general, which should quotient down to that of unframed deformations, by forgetting the framing. Then we could construct  $R_{\bar{\rho}}$  using this quotient. Kisin's notes chapter 3.
2. Pseudo-deformations (for later).
3. Schlessinger's criterion.

**One dimensional examples.**

$$\begin{array}{ccc} G & \xrightarrow{\bar{\rho}=\bar{\chi}} & \mathrm{GL}_1(k) = k^\times \\ & \searrow \chi_A & \uparrow \\ & & A^\times \end{array}$$

$$1 \rightarrow 1 + \mathfrak{m}_A \rightarrow A^\times \rightarrow k^\times \rightarrow 1$$

$$A^\times \simeq k^\times \times (1 + \mathfrak{m}_A)$$

last term is free and  $k^\times$  is determined by  $\bar{\chi}$ .

Need

$$\begin{array}{ccc} G & \longrightarrow & 1 + \mathfrak{m}_A \\ & \searrow & \uparrow \\ & & G^{\mathrm{ab}, \mathrm{pro}-p} \end{array}$$

Any

$$\chi: G^{\mathrm{ab}, \mathrm{pro}-p} \rightarrow 1 + \mathfrak{m}_A$$

fixes a deformation of  $\bar{\chi}$ .

**Claim 2.37**

$$W(k)[[G^{\mathrm{ab}, \mathrm{pro}-p}]] \simeq R_{\bar{\chi}}$$

$$\mathrm{Hom}_C(W(k)[[G^{\mathrm{ab}, \mathrm{pro}-p}]], A) \simeq \mathrm{Hom}(G^{\mathrm{ab}, \mathrm{pro}-p}, A^\times).$$

The universal character is then

$$G \rightarrow G^{\mathrm{ab}, \mathrm{pro}-p} \rightarrow W(k)[[G^{\mathrm{ab}, \mathrm{pro}-p}]]$$

$$\sigma \mapsto \sigma \mapsto \langle \sigma \rangle [\bar{\chi}(\sigma)].$$

$$\chi_A: G \rightarrow A^* \text{????????}$$

check commutes

**Example 2.38**  $G = G_{\mathbf{Q}, \{p, \infty\}}$

$$G^{\mathrm{ab}} = G_{\mathbf{Q}, \{p, \infty\}}^{\mathrm{ab}} = \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \simeq \mathbf{Z}_p^*$$

$$G^{\mathrm{ab}, \mathrm{pro}-p} \simeq 1 + p\mathbf{Z}_p \simeq \mathbf{Z}_p$$

$$R_{\bar{\chi}} = \mathbf{Z}_p[[1 + p\mathbf{Z}_p]] \simeq \mathbf{Z}_p[[T]]$$

□

Lecture 5 6/2/2020

Today we will assume throughout that

$$\mathrm{Aut}_G(\bar{\rho}) \simeq k^*$$

so that there exists

$$R_{\bar{\rho}}$$

the universal deformation ring, although we didn't prove this yet.

Let  $A \in C$ , we define the cotangent space of  $A$  to be

$$t_A^* = \mathfrak{m}_A / \mathfrak{m}_A^2 + pA$$

the tangent space is then

$$t_A = \mathrm{Hom}(t_A^*, k).$$

**Example 2.39**

$$A = W(k)[[X_1, \dots, X_d]]$$

and  $\mathfrak{m}_A = (p, X_1, \dots, X_d)$  then

$$t_A^* = \frac{(p, X_1, \dots, X_d)}{(p, \{X_i X_j\}_{ij})} \simeq kX_1 \oplus \dots \oplus kX_d,$$

so this is  $d$ -dimensional and we see that we just have linear terms, which is consistent with the tangency interpretation.  $\square$

**Fact 2.40**

$$\dim(A/pA) \leq \dim_k t_A$$

where the LHS is the Krull dimension (the length of the longest chain of prime ideals), let  $d = \dim t_A$ , then Nakayama's lemma implies  $d$  is the minimal number of generators of

$$\mathfrak{m}_A/pA.$$

$$W(k)[[X_1, \dots, X_d]] \twoheadrightarrow A$$

$$X_i \mapsto \text{gen of } \mathfrak{m}_A/pA.$$

**Tangent spaces and deformations.** Let

$$k[\epsilon] = k[x]/(x^2)$$

then  $\epsilon = \bar{x}$  and we have  $\epsilon^2 = 0$ . These are known as the dual numbers.

**Proposition 2.41**

$$\text{Hom}_C(A, k[\epsilon]) \simeq t_A$$

for  $A \in C$ .

*Proof.*

$$\text{Hom}_C(A, k[\epsilon]) \simeq \text{Hom}_C(A/\mathfrak{m}_A^2 + pA, k[\epsilon])$$

since

$$(\mathfrak{m}_{k[\epsilon]})^2 = 0$$

and

$$p = 0 \in k[\epsilon].$$

$$A/\mathfrak{m}_A^2 + pA \simeq k \oplus \mathfrak{m}_A/\mathfrak{m}_A^2 + pA$$

$$x \mapsto (\bar{x}, x - [\bar{x}])$$

$$[\bar{x}] + y \mapsto (\bar{x}, y)$$

as  $k$ -vector spaces.

$$k[\epsilon] \simeq k \oplus \epsilon k.$$

$$\text{Hom}_C(A/\mathfrak{m}_A^2 + pA, k[\epsilon]) \simeq \text{Hom}_C(A/\mathfrak{m}_A^2 + pA, \epsilon k) \simeq t_A$$

where now we take homomorphisms of  $k$ -vector spaces.  $\blacksquare$

Take  $A = R_{\bar{\rho}}$  then the proposition implies

$$D_{\bar{\rho}}(k[\epsilon]) \simeq t_{R_{\bar{\rho}}}$$

to deform

$$\bar{\rho}$$

to  $k[\epsilon]$

$$\rho_\epsilon(g) = \bar{\rho}(g) + \epsilon \phi(g)$$

for some

$$g \mapsto \phi(g) \in M_d(k)$$

with

$$\begin{array}{ccc} G & \xrightarrow{\rho_\epsilon} & \mathrm{GL}_d(k[\epsilon]) \\ & \searrow \bar{\rho} & \downarrow \\ & & \mathrm{GL}_d(k) \end{array}$$

We know

$$\begin{aligned} \rho_\epsilon(gh) &= \rho_\epsilon(g)\rho_\epsilon(h) \\ \bar{\rho}(gh) + \epsilon\phi(gh) &= (\bar{\rho}(g) + \epsilon\phi(g))(\bar{\rho}(h) + \epsilon\phi(h)) \\ &= \bar{\rho}(g)\bar{\rho}(h) + \epsilon(\phi(g)\bar{\rho}(h) + \bar{\rho}(g)\phi(h)) \end{aligned}$$

so

$$\phi(gh) = \phi(g)\bar{\rho}(h) + \bar{\rho}(g)\phi(h)$$

scale on the right by

$$\begin{aligned} \bar{\rho}(h)^{-1}\bar{\rho}(g)^{-1} &= \bar{\rho}(gh)^{-1}. \\ \phi(gh)\bar{\rho}(gh)^{-1} &= \phi(g)\bar{\rho}(g)^{-1} + \bar{\rho}(g)\phi(h)\bar{\rho}(h)^{-1}\bar{\rho}(g)^{-1} \end{aligned}$$

set

$$\psi(g) = \phi(g)\bar{\rho}(g)^{-1}$$

so

$$\begin{aligned} \psi(gh) &= \psi(g) + \bar{\rho}(g)\psi(h)\bar{\rho}(g)^{-1} \\ \psi(h) &\in M_d(k) \simeq \mathrm{End}(\bar{\rho}) \cup G \\ \mathrm{End}(\bar{\rho}) &= \mathrm{Hom}(V_{\bar{\rho}}, V_{\bar{\rho}}) = \mathrm{ad}(\bar{\rho}) \end{aligned}$$

this is the representation where

$$\begin{aligned} g \bullet M &= \bar{\rho}(g)M\bar{\rho}(g)^{-1}. \\ \psi &\in Z^1(G, \mathrm{ad}(\bar{\rho})) \end{aligned}$$

so we get a map

$$\{\text{lifts of } \bar{\rho}\} \rightarrow Z^1(G, \mathrm{ad} \bar{\rho}).$$

**Exercise 2.42** Strictly equivalent lifts yield cocycles differing by coboundaries.

**Proposition 2.43**

$$\begin{aligned} t_{R_{\bar{\rho}}} &\simeq D_{\bar{\rho}}(k[\epsilon]) \simeq H^1(G, \mathrm{ad} \bar{\rho}) \\ D_{\bar{\rho}}^\square(k[\epsilon]) &\simeq Z^1(G, \mathrm{ad} \bar{\rho}) \end{aligned}$$

**Exercise 2.44** Prove this.

**Exercise 2.45** Show that  $\forall \bar{\rho}$ ,  $H^1(G, \mathrm{ad} \bar{\rho})$  is finite dimensional if and only if  $G$  satisfies  $\Phi_p$ .

Let

$$\begin{aligned} d_1 &= \dim_k H^1(G, \mathrm{ad} \bar{\rho}) \\ 0 \rightarrow I &\rightarrow W(k)[[X_1, \dots, X_{d_1}]] \rightarrow R_{\bar{\rho}} \rightarrow 0. \end{aligned}$$

How big is  $I$ ? This is equivalent to finding lower bounds on  $\dim R_{\bar{\rho}}$ . Obstruction to lifting  $\bar{\rho}$ .

$$\begin{array}{ccc}
 G & \xrightarrow{\exists \rho} & \mathrm{GL}_d(\mathbf{Z}/p^2) \\
 & \searrow \bar{\rho} & \downarrow \\
 & & \mathrm{GL}_d(\mathbf{F}_p)
 \end{array}$$

If we choose a set map  $\rho$  then we can set

$$\phi(g, h) = \rho(gh)\rho(h)^{-1}\rho(g)^{-1} \in 1 + pM_d(\mathbf{F}_p)$$

in the sequence

$$1 \rightarrow 1 + pM_d(\mathbf{F}_p) \rightarrow \mathrm{GL}_d(\mathbf{Z}/p^2) \rightarrow \mathrm{GL}_d(\mathbf{F}_p) \rightarrow 1$$

where

$$1 + pM_d(\mathbf{F}_p) \simeq M_d(\mathbf{F}_p) \simeq \mathrm{End}(\bar{\rho}).$$

**Claim 2.46**

$$\phi(g, h) \in Z^2(G, \mathrm{ad} \bar{\rho})$$

The condition to be such a cocycle is

$$a \bullet \phi(b, c) - \phi(ab, c) + \phi(a, bc) - \phi(a, b) = 0.$$

So we do a long calculation, using the fact that the kernel is abelian.

Exercise check that changing  $\rho$  changes by a coboundary.

$$\phi \in H^2(G, \mathrm{ad} \bar{\rho}).$$

More generally

$$A_1 \xrightarrow{\pi} A_0$$

$$I = \ker \pi$$

s.t.

$$\mathfrak{m}_{A_1} I = 0$$

so  $I$  is a  $k$ -vector space.

$$\phi(g, h) = \rho_1(gh)\rho_1(h)^{-1}\rho_1(g)^{-1} \equiv 1 \pmod{I}$$

so in

$$1 + M_d(I) \simeq 1 + M_d(k) \otimes I$$

$$\simeq \mathrm{ad} \bar{\rho} \otimes I.$$

$$\phi \in H^2(G, \mathrm{ad} \bar{\rho}) \otimes I.$$

Lecture 6 11/2/2020

$$\bar{\rho}: G \rightarrow \mathrm{GL}_d(k)$$

$$\mathrm{Aut}_D \bar{\rho} \simeq k^\times$$

implies that there exists

$$R_{\bar{\rho}}$$

a universal deformation ring representing

$$D_{\bar{\rho}}.$$

We defines

$$\begin{aligned} t_R^* &= \mathfrak{m}_R / \mathfrak{m}_R^2 + p. \\ t_R &= \text{Hom}(t_R^*, k). \\ t_{R_{\bar{\rho}}} &\simeq D_{\bar{\rho}}(k[\epsilon]) \simeq H^1(G, \text{ad } \bar{\rho}) \\ \dim &= d_1 \lambda \infty. \\ 0 \rightarrow I &\rightarrow W(k)[[X_1, \dots, X_{d_1}]] \rightarrow R_{\bar{\rho}} \rightarrow 0. \end{aligned}$$

$$\begin{aligned} A_0 &\in \mathcal{C} \\ \rho_0 &: G \rightarrow \text{GL}_d(A_0) \end{aligned}$$

deforming  $\bar{\rho}$ .

$$\begin{aligned} A_1 &\xrightarrow{\pi} A_0 \\ \mathfrak{m}_{A_1} \cdot \ker(\pi) &= 0 \\ O(\rho_0) &\in H^2(G, \text{ad } \bar{\rho}) \otimes \ker(\pi) \\ O(\rho_0) &= 0 \end{aligned}$$

if and only if there exists a deformation of  $\rho_0$  to  $A_1$  (via  $\pi$ ). If  $H^2(G, \text{ad } \bar{\rho}) = 0$  then we can always deform these small homs.

**Theorem 2.47** *Let*

$$d_i = \dim H^i(G, \text{ad } \bar{\rho}), \quad i = 1, 2$$

*then*

$$\text{krulldim}(R_{\bar{\rho}}/pR_{\bar{\rho}}) \geq d_1 - d_2. \quad (2.1)$$

*Additionally if  $d_2 = 0$  then*

$$R_{\bar{\rho}} \simeq W(k)[[X_1, \dots, X_{d_1}]].$$

**Conjecture 2.48** *We have equality in (2.1).*

If we have

$$\begin{aligned} R &\rightarrow S \in \mathcal{C} \\ t_R^* &\rightarrow t_S^* \\ t_S &\rightarrow t_R. \\ W(k)[[X_1, \dots, X_{d_1}]] &\rightarrow R_{\bar{\rho}} \rightarrow 0 \\ t_{R_{\bar{\rho}}} &\xrightarrow{\sim} t_{W(k)[[X_1, \dots, X_{d_1}]]}. \end{aligned}$$

Same dimensions and injective since surj on cotangent space.

*Of above.*

$$\begin{aligned} 0 \rightarrow I &\rightarrow W(k)[[X_1, \dots, X_{d_1}]] \rightarrow R_{\bar{\rho}} \rightarrow 0 \\ 0 \rightarrow J &\rightarrow k[[X_1, \dots, X_{d_1}]] \rightarrow R_{\bar{\rho}}/pR_{\bar{\rho}} \rightarrow 0 \\ 0 \rightarrow J/\mathfrak{m}J &\rightarrow k[[X_1, \dots, X_{d_1}]]/\mathfrak{m}J \rightarrow R_{\bar{\rho}}/pR_{\bar{\rho}} \rightarrow 0 \end{aligned}$$

Claim: there exists

$$\text{Hom}(J/\mathfrak{m}J, k) \hookrightarrow H^2(G, \text{ad } \bar{\rho})$$

this implies

$$\dim_k(J/\mathfrak{m}J) \leq d_2$$

so  $J$  has a set of generators of length  $\leq d_2$  by Nakayama. So

$$\text{krulldim}(R_{\bar{\rho}}/pR_{\bar{\rho}}) \geq d_1 - d_2.$$

Maybe this is Krull's principal ideal theorem.

The proof of the claim is as follows: Let

$$\tilde{\rho}$$

be  $\rho^{univ} \bmod p$

$$\tilde{\rho}: G \rightarrow \text{GL}_d(R_{\bar{\rho}}/pR_{\bar{\rho}})$$

$$O(\tilde{\rho}) \in H^2(G, \text{ad } \tilde{\rho}) \otimes J/\mathfrak{m}J$$

$$\text{Hom}(J/\mathfrak{m}J, k) \xrightarrow{\alpha} H^2(G, \text{ad } \tilde{\rho})$$

$$f \mapsto (1 \otimes f)(O(\tilde{\rho}))$$

$f \neq 0$  so that

$$\alpha(f) = 0$$

$$f: J/\mathfrak{m}J \rightarrow k$$

$$0 \rightarrow (J/\mathfrak{m}J)/\ker f \rightarrow ((k[[X_1, \dots, X_{d_1}]]/\mathfrak{m}J)/\ker f \rightarrow R_{\bar{\rho}}/pR_{\bar{\rho}} \rightarrow 0.$$

But this is

$$0 \rightarrow k \rightarrow R' \rightarrow R_{\bar{\rho}}/pR_{\bar{\rho}} \rightarrow 0.$$

Obstruction class is  $f(O(\tilde{\rho})) = 0$ . So there exists a deformation of  $\tilde{\rho}$  to  $R'$ . But  $\text{char } R' = p$  as

$$\tilde{\rho}$$

is the universal deformation of  $\bar{\rho}$  to a characteristic  $p$  ring.

So there exists

$$R_{\bar{\rho}}/pR_{\bar{\rho}} \rightarrow R'$$

hence

$$t_{R'}$$

has larger dimension than

$$t_{R_{\bar{\rho}}/pR_{\bar{\rho}}}$$

a contradiction.

For the second part

$$0 \rightarrow J \rightarrow k[[X_1, \dots, X_{d_1}]] \rightarrow R_{\bar{\rho}}/pR_{\bar{\rho}} \rightarrow 0$$

so

$$\dim(J/\mathfrak{m}J) \leq d_2 = 0$$

so

$$J/\mathfrak{m}J = 0$$

i.e.

$$J = 0.$$

Hence

$$R_{\bar{\rho}}/pR_{\bar{\rho}} \simeq k[[X_1, \dots, X_{d_1}]].$$

We need to know that  $p^n \neq 0$  in  $R_{\bar{\rho}}$ . Suffices to show that

$$D_{\bar{\rho}}(W(k)) \neq \emptyset.$$

Now we show that we can lift small homomorphisms

$$k \simeq W(k)/p \xleftarrow{G} W(k)/p^2 \xleftarrow{G} \dots$$

so we can lift to  $W(k)$ . ■

and we will introduce Tate's global Euler characteristic formula.

$$M = \text{finite cardinality } G_{K,S}\text{-module}$$

$$S \supseteq \{\text{inf. places}\} \cup \{\ell : \ell | \#M\}$$

$$\frac{\#H^0(G, M) \#H^2(G, M)}{\#H^1(G, M)} = (\#M)^{-[K:\mathbf{Q}]} \cdot \prod_{v \in S_\infty} \#H^0(G_{K_v}, M).$$

$$\text{Take } M = \text{ad } \bar{\rho} \simeq M_d(k).$$

$$S = \{\text{inf. places}\} \cup \{\mathfrak{p} : \mathfrak{p} | p\} \cup \{\text{ram. primes for } \bar{\rho}\}$$

$$d_0 - d_1 + d_2 = \sum_{v \in S_\infty} \dim H^0(G_{K_v}, \text{ad } \bar{\rho}) - [K : \mathbf{Q}] d^2$$

$$\text{where } d_0 = \dim H^0(G, \text{ad } \bar{\rho})$$

$$g \cdot M = M$$

$$\text{implies } d_0 = 1$$

$$\bar{\rho}(g) M \bar{\rho}(g)^{-1} = M$$

$$\implies M \in \text{End}_G(\bar{\rho}) = k.$$

$$d_0 - d_2 = [K : \mathbf{Q}] d^2 + 1 - \sum_{v \in S_\infty} \dim_k H^0(G_{K_v}, \text{ad } \bar{\rho}).$$

**Example 2.49**

$$\bar{\rho} = \bar{\chi} = \text{character of } = G_{K,S}$$

$$\text{ad } \bar{\rho} \simeq \mathbf{1}$$

so

$$\dim_k(H^0(G_{K_v}, \text{ad } \bar{\rho})) = 1$$

hence

$$\text{krulldim}(R_{\bar{\chi}}/pR_{\bar{\chi}}) \geq d_1 - d_2 = [K : \mathbf{Q}] + 1 - r_1 - r_2 = r_2 + 1 > 0.$$

$$R_{\bar{\chi}} \simeq W(k)[[G_{K,S}^{\text{ab},(p)}]]$$

$$R_{\bar{\chi}}/pR_{\bar{\chi}} \simeq k[[G_{K,S}^{\text{ab},(p)}]]$$

$$\text{krulldim}(R_{\bar{\chi}}/pR_{\bar{\chi}}) = \text{rk}_{\mathbf{Z}_p}(G_{K,S}^{\text{ab},(p)}) = r_2 + 1$$

which is exactly Leopoldt's conjecture. □

**Example 2.50**

$$\bar{\rho}: G_{K,S} \rightarrow \text{GL}_2(k)$$

$$S = \{p, \infty\}$$

$$\text{krulldim}(R_{\bar{\rho}}/pR_{\bar{\rho}}) = 4 + 1 - \dim H^0(G_{\mathbf{R}}, \text{ad } \bar{\rho})$$

take

$$c \in \text{Gal}(\mathbf{C}/\mathbf{R})$$

either

$$\bar{\rho}(c) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or

$$\bar{\rho}(c) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



the first case is called even the second odd. In the even case we have

$$\text{ad } \bar{\rho} = \mathbf{1}$$

and so  $H^0$  has dimension 4. And

$$\text{krulldim}(R_{\bar{\rho}}/pR_{\bar{\rho}}) = 1$$

$$\dim(R_{\bar{\rho}}/pR_{\bar{\rho}}) \geq 1.$$

In the odd case

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

so  $H^0$  is 2 dimensional and

$$\dim(R_{\bar{\rho}}/pR_{\bar{\rho}}) \geq 3.$$

□