Deformation theory of Galois representations

MA842 at BU Spring 2020

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These are notes for Robert Pollack's course MA842 at BU Spring 2020.
The course webpage is http://math.bu.edu/people/rpollack/Teach/842spring2020.html.

Lecture 1 21/1/2018

1 Background

Let E_k denote the Eisenstein series of weight k, $k > 2$.

$$E_k = \frac{-B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in M_k(\mathrm{SL}_2(\mathbf{Z})).$$

Where B_k] are the Bernoulli numbers and

$$\sigma_{k-1}(n) = \sum_{d|n,\, d>0} .$$

E_2 however is not holomorphic, so not a modular form.
Fix N a prime, notation has stuck from Mazur's Eisenstein ideal paper.
Then there exists a unique Eisenstein series on $\Gamma_0(N)$ of weight 2.

$$E_2^{(N)} = \frac{N-1}{12} + \sum_{n=1}^{\infty} \sigma(n) q^n.$$

Funny observation: if $N \equiv 1 \pmod{p}$ for prime $p > 3$. Then $p|((N-1)/12)$, so $E_2^{(N)}$ "looks cuspidal".
Then we hope that there exists a cuspidal eigenform $f \in S_2(\Gamma_0(N))$ such that

$$f \equiv E_2^{(N)}\text{“} \pmod{p}\text{”}.$$

This is in fact true, due to Koike in the 70's, there exists $f \in S_2(\Gamma_0(N))$ such that

$$a_\ell(f) \equiv 1 + \ell \pmod{p}$$

for all $\ell \neq N, p$.

Question 1.1 How many such f are there? □

Merel '96:

$$f \text{ is unique} \iff \prod_{i=1}^{(N-1)/2} i^i \text{ is not a } p\text{-th power modulo } N.$$

Wake and Wang-Erickson describe the dimension of the space of such f using Massey products (higher cup products).

Method: Galois deformations!

1.1 Galois representations

We write

$$G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) = \varprojlim_{F/\mathbf{Q}, \text{fin. galois}} \mathrm{Gal}(F/\mathbf{Q})$$

a profinite group.

$$\rho \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Q}_p)$$

a continuous homomorphism. Then view $\mathrm{GL}_2(\mathbf{Q}_p)$ as $\mathrm{Aut}(V)$ for a 2-dimensional $\mathbf{Q}_p$ vector space and fix a 2-dimensional $\mathbf{Z}_p$-lattice

$$T \subseteq V$$

which is $G_{\mathbf{Q}}$ stable. Then we can take

$$\bar{\rho} \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_p)$$

this is unique (w.r.t. the choice of T) only up to semisimplification.

So we say two Galois representations ρ_1, ρ_2 are congruent if

$$\bar{\rho}_1^{\mathrm{ss}} \simeq \bar{\rho}_2^{\mathrm{ss}}.$$

We say ρ_1, ρ_2 are deformations of

$$\bar{\rho}_1 = \bar{\rho}_2$$

(imagine this is reducible).

Start with

$$\bar{\rho} \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_p)$$

consider "all" deformations of $\bar{\rho}$ in good cases there exists a "universal" deformation of $\bar{\rho}$.

R^{univ} a local ring with maximal ideal $\mathfrak{m}_R$ such that

$$R/\mathfrak{m}_R = \mathbf{F}_p.$$

$$\rho^{\mathrm{univ}} \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(R^{\mathrm{univ}})$$

such that if $\rho \colon G_{\mathbf{Q}} \to GL_2(R)$ is a deformation of $\bar{\rho}$ then there exists

$$R^{\mathrm{univ}} \to R$$

such that

$$\begin{tikzcd} G_{\mathbf{Q}} \arrow[r, "\rho^{\mathrm{univ}}"] \arrow[dr, "\rho"'] & \mathrm{GL}_2(R^{\mathrm{univ}}) \arrow[d] \\ & GL_2(R) \end{tikzcd}.$$

1.2 Modular forms

$$f = \sum a_n q^n \in S_k(\Gamma_0(N))$$

an eigenform leads to

$$\rho_f \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(K),\ K/\mathbf{Q}_p \text{ finite}$$

with the property that for all $\ell \nmid Np$ we have

$$\mathrm{Tr}(\rho_f(Frob_\ell)) = a_\ell.$$

Modular forms can be congruent

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{p} \text{ for all but finitely many } \ell$$

$$\Updownarrow$$

$$\bar{\rho}^{\mathrm{ss}}_{f_1} \simeq \bar{\rho}^{\mathrm{ss}}_{f_2}.$$

There exists a ring, the Hecke algebra $\mathbf{T}$ parametrizing all f's with the same $\bar{\rho}$.

$$f \rightsquigarrow \rho_f \implies R^{\mathrm{univ}} \to \mathbf{T}$$

so hope

$$R^{\mathrm{univ}} \simeq \mathbf{T}.$$

Wiles proof of FLT proved one of these.

Many more such theorems in the past couple of decades.

Wake and Wang-Erickson show that the dimension of

$$\{f : f \equiv E_2^{(N)}\} \leftrightarrow \operatorname{rank} \mathbf{T} = \operatorname{rank} R^{\mathrm{univ}}.$$

$$a_\ell(f) \equiv 1 + \ell \pmod{p}$$

$$\implies \bar{\rho}^{\mathrm{ss}} = \mathbf{1} \oplus \mu_p$$

but there does not exist R^{univ} in this context.

The fix is to use pseudorepresentations instead of representations.

1.3 Pseudorepresentations

Let G be a group.

Then a pseudorepresentation T is a map

$$T \colon G \to A$$

for A a ring satisfying

1. $$T(xy) = T(yx)$$

2. $$T(x)T(y)T(z) - T(x)T(yz) - T(y)T(xz) - T(z)T(xy) + T(xyz) + T(xzy) = 0$$

and the analogous formulae for higher products.

Fact 1.2 *If A is an algebraically closed field of characteristic $\neq 2$. Then for a given pseudorepresentation T there exists a true representation ρ such that*

$$T = \mathrm{Tr}(\rho).$$

But this does not hold in general.

Universal pseudodeformation rings always exist. Wake and Wang-Erickson use R^{univ} = universal pseudodeformation ring.