

Deformation theory of Galois representations

MA842 at BU Spring 2020

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These are notes for Robert Pollack's course MA842 at BU Spring 2020.

The course webpage is <http://math.bu.edu/people/rpollack/Teach/842spring2020.html>.

Lecture 1 21/1/2018

1 Background

Let E_k denote the Eisenstein series of weight k , $k > 2$.

$$E_k = \frac{-B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n \in M_k(\mathrm{SL}_2(\mathbf{Z})).$$

Where B_k are the Bernoulli numbers and

$$\sigma_{k-1}(n) = \sum_{d|n, d>0} d.$$

E_2 however is not holomorphic, so not a modular form.

Fix N a prime, notation has stuck from Mazur's Eisenstein ideal paper.

Then there exists a unique Eisenstein series on $\Gamma_0(N)$ of weight 2.

$$E_2^{(N)} = \frac{N-1}{12} + \sum_{n=1}^{\infty} \sigma(n)q^n.$$

Funny observation: if $N \equiv 1 \pmod{p}$ for prime $p > 3$. Then $p | ((N-1)/12)$, so $E_2^{(N)}$ "looks cuspidal".

Then we hope that there exists a cuspidal eigenform $f \in S_2(\Gamma_0(N))$ such that

$$f \equiv E_2^{(N)} \pmod{p}.$$

This is in fact true, due to Koike in the 70's, there exists $f \in S_2(\Gamma_0(N))$ such that

$$a_\ell(f) \equiv 1 + \ell \pmod{p}$$

for all $\ell \neq N, p$.

Question 1.1 How many such f are there?

□

Merel '96:

$$f \text{ is unique} \iff \prod_{i=1}^{(N-1)/2} i^i \text{ is not a } p\text{-th power modulo } N.$$

Wake and Wang-Erickson describe the dimension of the space of such f using Massey products (higher cup products).

Method: Galois deformations!

1.1 Galois representations

We write

$$G_{\mathbf{Q}} = \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) = \varprojlim_{F/\mathbf{Q}, \text{ fin. galois}} \text{Gal}(F/\mathbf{Q})$$

a profinite group.

$$\rho: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Q}_p)$$

a continuous homomorphism. Then view $\text{GL}_2(\mathbf{Q}_p)$ as $\text{Aut}(V)$ for a 2-dimensional \mathbf{Q}_p vector space and fix a 2-dimensional \mathbf{Z}_p -lattice

$$T \subseteq V$$

which is $G_{\mathbf{Q}}$ stable. Then we can take

$$\bar{\rho}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{F}_p)$$

this is unique (w.r.t. the choice of T) only up to semisimplification.

So we say two Galois representations ρ_1, ρ_2 are congruent if

$$\bar{\rho}_1^{\text{ss}} \simeq \bar{\rho}_2^{\text{ss}}.$$

We say ρ_1, ρ_2 are deformations of

$$\bar{\rho}_1 = \bar{\rho}_2$$

(imagine this is reducible).

Start with

$$\bar{\rho}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{F}_p)$$

consider “all” deformations of $\bar{\rho}$ in good cases there exists a “universal” deformation of $\bar{\rho}$.

R^{univ} a local ring with maximal ideal \mathfrak{m}_R such that

$$R/\mathfrak{m}_R = \mathbf{F}_p.$$

$$\rho^{\text{univ}}: G_{\mathbf{Q}} \rightarrow \text{GL}_2(R^{\text{univ}})$$

such that if $\rho: G_{\mathbf{Q}} \rightarrow \text{GL}_2(R)$ is a deformation of $\bar{\rho}$ then there exists

$$R^{\text{univ}} \rightarrow R$$

such that

$$\begin{array}{ccc} G_{\mathbf{Q}} & \xrightarrow{\rho^{\text{univ}}} & \text{GL}_2(R^{\text{univ}}) \\ & \searrow \rho & \downarrow \\ & & \text{GL}_2(R) \end{array}$$

1.2 Modular forms

$$f = \sum a_n q^n \in S_k(\Gamma_0(N))$$

an eigenform leads to

$$\rho_f: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(K), \quad K/\mathbf{Q}_p \text{ finite}$$

with the property that for all $\ell \nmid Np$ we have

$$\mathrm{Tr}(\rho_f(\mathrm{Frob}_\ell)) = a_\ell.$$

Modular forms can be congruent

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{p} \text{ for all but finitely many } \ell$$

$$\Updownarrow$$

$$\bar{\rho}_{f_1}^{\mathrm{ss}} \simeq \bar{\rho}_{f_2}^{\mathrm{ss}}.$$

There exists a ring, the Hecke algebra \mathbf{T} parametrizing all f 's with the same $\bar{\rho}$.

$$f \leadsto \rho_f \implies R^{\mathrm{univ}} \rightarrow \mathbf{T}$$

so hope

$$R^{\mathrm{univ}} \simeq \mathbf{T}.$$

Wiles proof of FLT proved one of these.

Many more such theorems in the past couple of decades.

Wake and Wang-Erickson show that the dimension of

$$\{f : f \equiv E_2^{(N)}\} \leftrightarrow \mathrm{rank} \mathbf{T} = \mathrm{rank} R^{\mathrm{univ}}.$$

$$a_\ell(f) \equiv 1 + \ell \pmod{p}$$

$$\implies \bar{\rho}^{\mathrm{ss}} = \mathbf{1} \oplus \mu_p$$

but there does not exist R^{univ} in this context.

The fix is to use pseudorepresentations instead of representations.

1.3 Pseudorepresentations

Let G be a group.

Then a pseudorepresentation T is a map

$$T: G \rightarrow A$$

for A a ring satisfying

1.

$$T(xy) = T(yx)$$

2.

$$T(x)T(y)T(z) - T(x)T(yz) - T(y)T(xz) - T(z)T(xy) + T(xyz) + T(xzy) = 0$$

and the analogous formulae for higher dimensions.

Fact 1.2 *If A is an algebraically closed field of characteristic $\neq 2$. Then for a given pseudorepresentation T there exists a true representation ρ such that*

$$T = \mathrm{Tr}(\rho).$$

But this does not hold in general.

Universal pseudodeformation rings always exist. Wake and Wang-Erickson use R^{univ} = universal pseudodeformation ring.