Equidistribution in Number Theory

MA841 BU Fall 2018

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October 18, 2018

1 Equidistribution in Number Theory

Lecture 1 6/9/2018

These are notes for Ali Altuğ's course MA841 at BU Fall 2018.

The course webpage is http://math.bu.edu/people/saaltug/2018_3/2018_3_sem.html.

Course overview: Upon Akshay Venkatesh winning the fields medal I decided to dedicate this term to various aspects of equidistribution results in number theory and their relations to L-functions. I am aiming to cover basic results like Linnik's and Duke's theorems, as well as certain aspects of subconvexity. As we move along we may briefly touch several other aspects like quantum unique ergodicity (QUE) or equidistribution in other settings (e.g. over function fields). There is an abundance of material on this topic. We will not follow a single book or an article (although the first book and the survey articles following that will be the main reference), however here are a bunch of helpful papers/books. I will update these references as we move along.

1.1 Introduction

> I wasn't going to do this at all, but then Akshay won the Fields medal.
>
> —Ali Altuğ

Some topics we will try and cover:

1. Linnik's and Duke's theorem 1970-1990
2. lattice points and hyperbolic geometry
3. modular forms
4. spectral theory
5. modular forms of half-integral weight.
6. harmonic analysis
7. Kuznetsov formula
8. Other topics!

1.2 Linnik and Duke

Akshay's first breakthrough, according to Sarnak, was subconvexity. He became interested in ergodic theory, because they could prove hard theorems, equidistribution is a powerful tool in number theory.

Our main goal will be to talk about: Some problems stated (and proved) by Linnik. In the book, Ergodic properties of algebraic fields, 1968. He considered lattice points on a sphere of radius n, these are points (say in $\mathbf{R}^3$) whose coordinates are integral of a fixed distance from the origin. Analogous to the circle problem. If n is fixed there is nothing to distribute, but if we vary n and project down we can ask do they accumulate miss any patches, generally how do they distribute. Of course this can be generalised.

Setup. Sphere S^2.

Lattice points $\alpha = (x_1, x_2, x_3) \in \mathbf{Z}^3$.

$$|\alpha|^2 = x_1^2 + x_2^2 + x_3^2$$

this is where the number theory comes, we are looking at representability of numbers n by this ternary quadratic form.

Various methods exist for studying this for varying n, quadratic reciprocity for $n = 2$, circle method/Vinogradov for $n = 4$. $n = 3$ is the cut off, here half-integral weight modular forms are relevant.

Set

$$\Omega_N = \left\{ x = \frac{\alpha}{|\alpha|} : \alpha \in \mathbf{Z}^2, \, |\alpha|^2 = N \right\} \subseteq S^2.$$

Question 1.1 Are Ω_N "equidistributed" as $N \to \infty$? □

There is an immediate obstruction to N being a sum of 3 squares, e.g. $N = 7$ implies $\Omega_7 = \emptyset$.

Recall: (Gauss/Legendre)

$$N = x_1^2 + x_2^2 + x_3^2, \, x_i \in \mathbf{Z} \iff N \neq 4^a(8b + 7),$$

so we avoid these sets and ask the same question.

Theorem 1.2 Linnik. *Let $f \in C^\infty(S^2)$, then as $N \to \infty$, N squarefree $N \not\equiv 7 \pmod 8$, $\left(\frac{N}{p}\right) = 1$ for some fixed odd prime p*

$$\frac{1}{\#\Omega_N} \sum_{x \in \Omega_N} f(x) \to \int_{S^2} f \, \mathrm{d}\sigma$$

where $\mathrm{d}\sigma$ is the Lebesgue measure on S^2.

This is saying that the points are equidistributed with respect to the Lebesgue measure.

The last condition is a defect of the method, known as a Linnik condition.

Remark 1.3 Linnik's proof is ergodic theoretic.

After this came Duke in 1988, in the mean time, Weil conjectures were proved, Iwaniec gave bounds for Kloosterman sums. Duke was a graduate student of Sarnak at Courant. He gave a more direct proof of 1.2 which does not have the $\left(\frac{N}{p}\right) = 1$ condition. His proof is based on the theory of (half-integral weight) modular forms, and a good bound Iwaniec on certain exponential sums.

Why do exponential sums enter the picture? We are trying to prove that we have a sum converging to an integral. Generally we work with a basis of functions first, we could try using fourier analysis, using harmonics as our basis, this is when exponential sums appear. That requires us to work out harmonics on the sphere (spherical harmonics) which leads to representation theory, which as the sphere is compact involves Weyl representations etc.

Duke also proved "the same" theorem over modular surfaces. Instead of looking at expanding spheres we study expanding hyperboloids.

Setup. Space

$$\pm\Gamma\backslash\mathbf{H}$$

$$\mathbf{H} = \left\{x + iy \in CC : y > 0\right\}$$

$$\Gamma = \mathrm{SL}_2(\mathbf{Z})$$

$$\mathcal{F} = \left\{z \in \mathbf{H} : -\frac{1}{2} \leq \Re(z) \leq 0, |z| \geq 1 \text{ or } \frac{1}{2} > \Re(z) > 0, |z| > 1\right\}$$

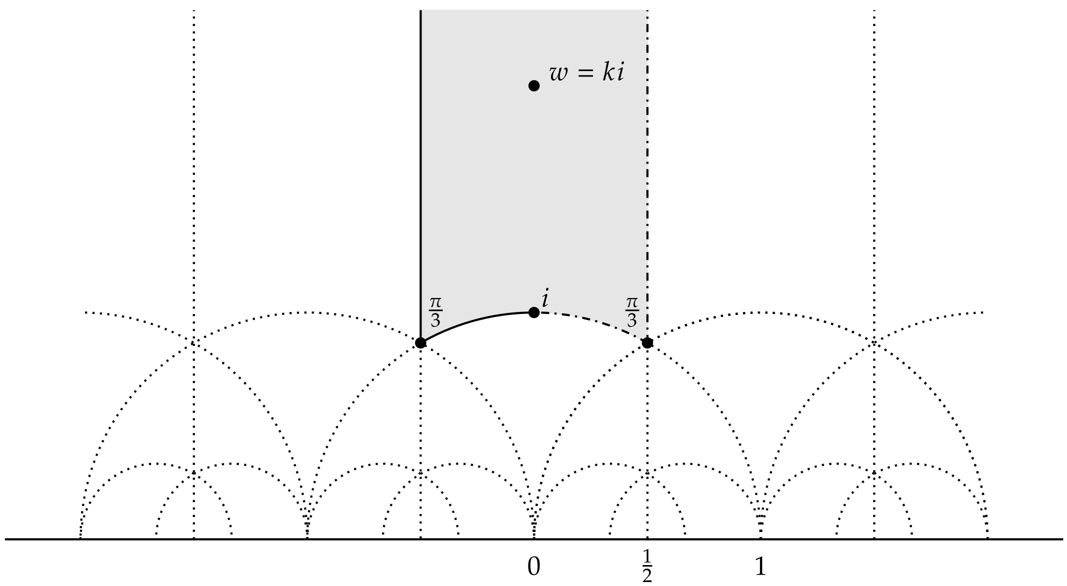

Figure 1.4: $\mathcal{F}$

Ω_N's are now replaced with CM points.

Digression (CM points). Q a binary quadratic form

$$Q(x, y) = ax^2 + bxy + cy^2$$

of discriminant $d = b^2 - 4ac < 0$, $a, b, c \in \mathbf{Z}$.

Theorem 1.5 Gauss. *There are only finitely many equivalence classes of such forms for fixed d.*

The number of such is given by the Hurwitz class number.

Given Q we associate a CM point

$$z_Q = \frac{-b + \sqrt{d}}{2a} \in \mathbf{H}.$$

The action of Γ on Q is the same as the action of Γ on z_Q.

$$z_{\gamma Q} = \gamma z_Q.$$

Let

$$\Lambda_d = \{z_Q \in \mathcal{F} : \mathrm{disc}(Q) = d\}.$$

We need to sum properly to take automorphisms into account.

Definition 1.6

$$\sum\nolimits^*_{z_Q \in \Lambda_d}$$

is the sum weighted by $\frac{1}{2}$ if $Q = a(x^2 + y^2)$ $(d = -4)$, $\frac{1}{3}$ if $Q = a(x^2 + xy + y^2)$ $(d = -3)$. ◊

If we want to be fancy we can say the word stack here.

Remark 1.7

$$\sum_{z_Q \in \Lambda_d}{}^{*} 1 = H(d)$$

where $H(d)$ is the Hurwitz class number.

Remark 1.8 If d is fundamental, i.e. a discriminant of some $\mathbf{Q}(\sqrt{d})$ then $H(d) = h(d)$ the regular class number.

For the measure on $\mathcal{F}$ we take

$$\mathrm{d}\mu = \frac{3}{\pi} \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

Theorem 1.9 Duke. *Let $f \in C^\infty(\mathbf{H})$, that is Γ invariant and bounded on $\mathcal{F}$, then as $d \to \infty$ over fundamental discriminants*

$$\frac{1}{\#\Lambda_d} \sum_{z \in \Lambda_d}{}^{*} f(z) \to \iint_{\mathcal{F}} f \,\mathrm{d}\mu.$$

Equidistribution implies density, but is so much more, for example we cannot have dense points but which happen to cluster towards some line for example.

1.3 Basics

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Question: Let $\alpha \in \mathbf{R}$ and consider $\{\alpha n\}$ where $\{x\} = x \pmod 1$ so $\left\{\frac{3}{2}\right\} = \frac{1}{2}$. How are these distributed?

Example 1.10 If $\alpha = \frac{2}{7}$ then we have $\{\{\alpha n\} : n \in \mathbf{N}\} = \{\frac{i}{7} : i \in \{0, \ldots, 6\}\}$ and in fact it hits each evenly. □

Example 1.11 If $\alpha = \sqrt{2}$ so $\{\alpha\} \approx 0.4142?...$ $\{\alpha 2\} \approx 0.8284?...$ $\{\alpha 3\} \approx 0.24264...$ $\{\alpha 4\} \approx 0.656854...$ These spread out densely, but there is a difference between density and equidistribution. In this example, equidistribution says that the proportion of time the sequence spends in each interval (a, b) is $b - a$. □

So questions are: is $\{n\alpha\}$ dense?

Is $\{n\alpha\}$ uniformly distributed (equidistributed with respect to the standard measure)?

The answer to both questions is yes.

Theorem 1.12 Kronecker. *Let $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ then $\{n\alpha\}$ is dense in $[0, 1)$.*

Digression (Diophantine approximation). This is a very tough area of number theory, not so many definitive results here.

Theorem 1.13 Dirichlet. *Let $\alpha \in \mathbf{R}, N \in \mathbf{Z}_{>0}$ then there exists p, q with $q > 0$.*

$$\left|q\alpha - p\right| < \frac{1}{N}.$$

Proof. (Pidgeonhole) divide $[0,1)$ into even N subintervals of width $\frac{1}{N}$, consider

$$\alpha_0 = 0, \alpha_1 = \{\alpha 1\}, \alpha_2 = \{\alpha 2\}, \ldots \in [0,1)$$

as soon as we get to α_N we must have two in one subinterval say $|\alpha_{n_1} - \alpha_{n_2}| < \frac{1}{N}$. So there exists p_{n_1}, p_{n_2} such that

$$|n_1\alpha - k_{n_1} - (n_2\alpha - k_{n_2})| < \frac{1}{N}$$

$$|(n_1 - n_2)\alpha - (k_{n_1} - k_{n_2})| < \frac{1}{N}. \qquad \blacksquare$$

Corollary 1.14 *Let $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ then there exists infinitely many coprime p, q with $q > 0$ satisfying*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Proof. Exercise. $\blacksquare$

This is very strong, Roth's theorem tells us that even $q^{2+\epsilon}$ here is enough to force finiteness.
Note 1.15 $\alpha \notin \mathbf{Q}$ is necessary! Otherwise

$$\left|\frac{p_0}{q_0} - \frac{p}{q}\right| = \left|\frac{p_0 q - p q_0}{q q_0}\right| \geq \frac{\left|p_0 q - p q_0\right|}{\max\left\{q^2, q_0^2\right\}}$$

so choose $q > q_0$ implies $\frac{p_0}{q_0} = \frac{p}{q}$.

One can do better:
Theorem 1.16 Hurwitz. *Let $\alpha \in \mathbf{R} \setminus \mathbf{Q}$*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$$

then there exists infinitely many coprime p, q with $q > 0$ satisfying.

Note 1.17 $\sqrt{5}$ is the best possible without further restriction on α.

Example 1.18 If $\alpha = \frac{1-\sqrt{5}}{2}$ then

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Aq^2}$$

has only finitely many solutions for $A > \sqrt{5}$. $\square$

What if we allow further restriction?
Theorem 1.19 Liouville. *Let $\alpha \in \mathbf{R}$ algebraic of degree $n > 1$. Then there exists $A > 0$ such that for all p, q with $g > 0$*

$$\left|\alpha - \frac{p}{q}\right| > \frac{A}{q^n}.$$

Proof. Let $f \in \mathbf{Z}[x]$ be the minimal polynomial of α. Gauss's lemma implies that f is irreducible over $\mathbf{Q}$ so

$$q^n f\left(\frac{p}{q}\right) \in \mathbf{Z} \setminus \{0\} \, \forall \frac{p}{q} \in \mathbf{Q}.$$

The mean value theorem says that there exists $x_0 \in [\alpha, p/q)$ s.t.

$$\frac{f(p/q) - \cancelto{0}{f(\alpha)}}{p/q - \alpha} = f'(x_0)$$

so

$$\frac{q^n f(p/q)}{q^n f'(x_0)} = |p/q - \alpha|,$$

notice how n appears here. ∎

This theme of using some calculus is repeated across diophantine analysis.

Remark 1.20

- Thue: can replace n with $(\deg(\alpha) + 2)/2$ (This already has implications to integral solutions of degree ≥ 3 polynomials $f \in \mathbf{Z}[x]$, e.g. elliptic curves with bounded integral discriminant)
- Roth ($\sim$ 1958): for all $\epsilon > 0$, there are only finitely many p/q satisfying

$$|\alpha - \frac{p}{q}| < \frac{1}{q^{2+\epsilon}}.$$

Back to equidistribution. $a_n = \{n\alpha\}, \{a_n\}_{n=1}^{\infty}$.

Theorem 1.21 Kronecker. *a_n is dense in $[0,1)$ if $\alpha \notin \mathbf{Q}$.*

Proof. Will show for any $x \in [0,1)$ there exists

$$\{a_{n_j}\}_{j=1}^{\infty} \text{ s.t. } a_{n_j} \to x.$$

Notation $\|x\|$ means the distance to the nearest integer. Dirichlet implies that infinitely many p/q have $\alpha - p/q| < 1/q^2$.Given $\epsilon > 0$ let q be such that

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2} \implies |q\alpha - p| < \frac{1}{q}$$

$$\frac{1}{q} < \epsilon$$

Choose j such that $j(\alpha q - p)$ is within $1/q$ of x (why?). So

$$\|j(\alpha q - p) - x\| < \frac{1}{q} < \epsilon.$$

(Fill in the gaps here). ∎

Now let's define what is means to be uniformly distributed.

Definition 1.22 Uniformly distributed sequence. $a_1, a_2, \ldots$ is called **uniformly distributed** if for all $(b,c) \subseteq [0,1)$

$$\#\{n \leq N : \{a_n\} \in (b,c)\} \sim N(c-b)$$

or

$$\lim_{N\to\infty} \frac{\#\{n \le N : \{a_n\} \in (b,c)\}}{N} = c - b.$$

$\Diamond$

Uniformly distributed or not?

When $\alpha \in \mathbf{R}\setminus\mathbf{Q}$: $\{n\alpha\}$, $\{n^2\alpha\}$, $\{(3n^2+2n+1)\alpha\}$ $\{(\sqrt{5}n^3+2n-(\zeta_{10}+\bar{\zeta}_{10}))\alpha\}$ all are.

$\{n!e\}$, $\{\log(n)\}$ and $\{\log(p_n)\}$ are not.

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Example 1.23 Several.

1. $\alpha \in \mathbf{R}\setminus\mathbf{Q}$, $a_n = \{n\alpha\}$. Yes
2. $\alpha \in \mathbf{R}\setminus\mathbf{Q}$, $a_n = \{n^2\alpha\}$. Yes
3. $a_n = \{\log(n)\}$. No
4. $a_n = \{n!e\}$. No (not dense).
5. $\{\log(p_n)\}$. No.
6. $a_n = \{\sqrt{n}\}$. Yes.
7. $a_n = \{e^n\}$. ?
8. $a_n = \{\log n!\}$. Yes ?
9. $a_n = \{\log\log n!\}$. No

To begin let's show $a_n = \{n!e\}$ is not equidistributed as it has only one limit point 0.

$$e = \sum_{n=1}^{\infty} \frac{1}{n!}$$

so $n!e \in \mathbf{Z} + \frac{1}{n+1} + \frac{(n+1)(n+2)}{+} \cdots \le \frac{e}{n+1} \to 0$, as $n \to \infty$. So its not dense, hence certainly not equidistributed. $\square$

Theorem 1.24 *The sequence $a_n = \{n\alpha\}$ is equidistributed.*

Proof. Let $\epsilon > 0$, choose $M > 0$ such that $\frac{1}{M} < \epsilon$, furthermore choose a $\delta < \frac{1}{M}$. Dirichlet implies there exists $m \in \mathbf{Z}$ s.t. $\|m\alpha\| = \delta$.

$$0 < \delta < \frac{1}{M} < \epsilon$$

recall $\|x\|$ is the distance to the nearest integer. Consider the set $S_i = \{\{n\alpha\} : n < N,\ n \equiv i \pmod m\}$ then

$$S = \{\{n\alpha\} : n < N\} = \bigsqcup_{i=1}^{m} S_i$$

moreover,

$$S_i = \{\{km\alpha + i\alpha\} : 0 \le k \le N_i\}$$

where

$$N_i = \frac{N}{m} + O(1)$$

(why is the $O(1)$ here?). We can rewrite these S_i as follows:

$$S_i = \{\delta k + \gamma_i \pmod 1 : 0 \le k \le N_i\}$$

where

$$\begin{cases} i\alpha \pmod 1 & \text{if } \delta = \{m\alpha\} \\ i\alpha - \delta(N_i+1) \pmod 1 & \text{if } \delta = 1 - \{m\alpha\} \end{cases}$$

replace k by $N_i + 1 - k$.

Now $0 \le \gamma < 1$ and let

$$K_i = \lfloor \delta N_i + \gamma_i \rfloor$$

$$\#\{k < N_i : \{\delta k + \gamma_i\} \in [b,c]\} = \sum_{j=0}^{K_i} \#\{k \le N_i : \delta k + \gamma \in [j+b, j+c]\}$$

$$= (K_i + O(1))\left(\frac{c-b}{\delta} + O(1)\right)$$

$$N_i(c-b) + O\left(\frac{c-b}{\delta} + \delta N_i + 1\right)$$

so

$$S = \bigsqcup_{i=1}^{m} S_i$$

$$\#S = N(c-b) + O\left(\frac{M}{\delta} + \delta N\right)$$

(check!). Finally choose $N > \frac{M}{\delta^2}$. So that

$$\frac{M}{\delta} + \delta N < \epsilon N.$$

Which implies

$$\#\{n < N : \{n\alpha\} \in [b,c]\} = \#S = N(c-b) + \epsilon N. \qquad \blacksquare$$

We followed our nose essentially, but we needed to put ourselves in a favourable position so that we can get a good handle on the error.

Example 1.25 10. Consider the sequence

$$b_{m,i} = \frac{i}{m}$$

and define

$$a_{\binom{m}{2}+i} = b_{m,i}$$

so that

$$b_{m,i}$$

$$0, 1$$

$$0, \frac{1}{2}, 1$$

$$0, \frac{1}{3}, \frac{2}{3}, 1$$

$$0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$$

$$a_m = 0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$$

this is uniformly distributed! $\square$

Exercise 1.26 Prove this (in a similar way to before).

Proposition 1.27

1. *Any sequence has a rearrangement that is not uniformly distributed.*
2. *Any dense sequence has a rearrangement that is uniformly distributed.*

Proof.

1. If not sense we are done already. Otherwise let $I_1 = [0, \frac{1}{2}), I_2 = [\frac{1}{2}, 1]$ Let $X = \{a_n \in I_1\}, Y = \{a_n \in I_2\}$, so both sets are infinite. Let $b_n = x_1, \ldots, x_{10^{10}}, y_1, x_{10^{10}+1}, \ldots, x_{20^{20}}, \ldots$ so $\#\{n : b_n \in [4/5, 1]\} \sim \frac{N}{10^{10}}$.
2. If a_n is dense then, let
$$b_{n,i} = a_m \text{ s.t. } \frac{i-1}{n} < a_m < \frac{i}{n}$$
by the example above this is equidistributed.

∎

What can we say about the space of uniformly distributed sequences? Is it closed under addition? No ($a_n = -b_n$). $\{ka_n\}$ neither ($a_n = \alpha n$, $\frac{1}{\alpha}a_n = n$). $a_n b_n$ doesn't work either, but if b_n converges then $a_n b_n$ is uniformly distributed.

1.4 Weyl's criterion (1916)

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Theorem 1.28 Weyl's criterion. $\{a_n\}_{n=1}^{\infty} \in \mathbf{R}$ *then TFAE:*

1. $\{a_n\}_{n=1}^{\infty}$ *is uniformly distributed* mod 1.
2. $\forall f \in C([0,1])$
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{a_n\}) = \int_0^1 f \,\mathrm{d}x.$$
3. $$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(ma_n) = \delta_m, \forall m \in \mathbf{Z}$$

where $e(x) = e^{2\pi i x}$,
$$\delta_m = \begin{cases} 0 & m \neq 0 \\ 1 & m = 0 \end{cases},$$
collectively this is known as ***Weyl's criterion***.

Proof. 1. $\implies$ 2.. 1. means 2. is true for the characteristic function of $[b, c]$, hence true for step functions (because everything is linear).

Recall that any $f \in C[0, 1])$ is the uniform limit of such functions.

Exercise: Choose one that is ϵ close to f and finish the proof.

2. $\implies$ 1.. Find $f_1, f_2 \in C([0, 1])$ such that $f_1(x) \leq \chi_{(b,c)}(x) \leq f_2(x)$ s.t.
$$\int f_1 - \chi_{(b,c)}) \leq \epsilon$$

Exercise, finish this proof.

2. $\implies$ 3.. Obvious.

3. $\implies$ 2.. Recall Fejér's theorem that $f \in C[0,1])$ is the uniform limit of its fourier series. Pick an $n \in \mathbf{N}$ s.t.

$$f_n(x) - \epsilon \leq f(x) \leq f_n(x) + \epsilon.$$

Exercise, finish this proof. ■

Using this we can tackle more examples, e.g.:

1. $\{\alpha p_n\}$, n-th primes, $\alpha \in \mathbf{R} \setminus \mathbf{Q}$.
2. $\{\sqrt{n}\}$
3. $\{\log(n)\}$
4. $\{\log(p_n)\}$
5. $\{\beta^n\}$, $\beta \in \mathbf{R}$.
6. $\{\alpha a_n\}$, $a_n \in \mathbf{Z}$

1.5 Some applications of Weyl's criterion

Application 1. $\{n\alpha\}$ is uniformly distributed iff $\alpha \in \mathbf{R} \setminus \mathbf{Q}$. Let

$$S_N(h,\alpha) = \sum_{n=0}^{N} e(h\alpha n)$$

then

$$S_N(h,\alpha) = \frac{e((N+1)h\alpha) - 1}{e(h\alpha) - 1}.$$

Then $e(h\alpha) - 1 \neq 0$ for all $h \in \mathbf{Z} \setminus \{0\}$ iff $\alpha \notin \mathbf{Q}$ so $S_N(h,\alpha) = O_{h,\alpha}(1)$ iff $\alpha \notin \mathbf{Q}$. So $\lim_{N\to\infty} S_N(h,\alpha) = 0$, $\forall h \neq 0 \iff \alpha \notin \mathbf{Q}$. And $S_N(0,\alpha) = N+1$ which implies

$$\lim_{N\to\infty} \frac{1}{N} S_N(0,\alpha) = 1$$

so by Weyl's criterion $\{n\alpha\}$ is uniformly distributed if $\alpha \notin \mathbf{Q}$.

Digression.
Theorem 1.29 Vinogradov 1937. *$\{\alpha p_n\}$ is uniformly distributed iff $\alpha \notin \mathbf{Q}$.*

He actually showed weak Goldbach conjecture, Every (sufficiently large) *odd* integer is a sum of 3 primes. No bound by Vinogradov, Borozdin gave a large bound, Helfgott brought it down to reality.

Application 2.

$$\{\beta^n\}$$

Theorem 1.30 Koksma 1935. *For almost every (Lebesgue) $\beta \in \mathbf{R}_{>1}$, $\left\{\beta^n\right\}$ is equidistributed.*

Koksma was a Dutch student of Van der Corput.

Theorem 1.31 Weyl. *Let $|a_n| \to \infty$ be a sequence of distinct integers, the set of $\alpha \in \mathbf{R}$ such that αa_n is not uniformly distributed has Lebesgue measure 0.*

Application 3.
$$\{\log(n)\}$$
is not equidistributed.

Exercise 1.32 Prove this. Hint : assume it was

$$\#\{n < 1001N :\} = \#\{n < 1000N :\} + \#\{1000N \le n \le 1001N :\}$$

then
$$\log(n) - \log(100N + n_1) = \log(1000N) + \log(1 + n_1/1000N)$$

Application 4.
$$\{\sqrt{n}\}$$

Before this, lets introduce

Definition 1.33 Discrepancy. Let $a_n \in [0,1)$ be a sequence and $N \in \mathbf{Z}_{\ge 0}$ we define
$$D_N = \sup_{0\le b\le c\le 1} \left\{ \left| \frac{\#\{n < N : a_n \in [b,c]\}}{N} - (c-b) \right| \right\}.$$

◊

Some useful lemmas:

Lemma 1.34 *Let*

$$D_N^* = \sup_{0\le c\le 1} \left\{ \left| \frac{\#\{n < N : a_n \in [0,c]\}}{N} - c \right| \right\}$$

then
$$D_N^* \to 0 \iff D_N \to 0.$$

Proof. Exercise (hint squeeze). ■

Lemma 1.35 a_n *is uniformly distributed* $\iff D_N \to 0 \iff D_N^* \to 0.$

Proof. Ommitted/exercise, 2.1.1 of Kuipers and Niederreiter. ■

For $\sqrt{n}$ now.
$$S_N(c) = \#\{n \le N : 0 \le \{\sqrt{n}\} \le c\}.$$

Note 1.36 If $d = \lfloor \sqrt{n} \rfloor$ then

$$\sqrt{n} = d + \{\sqrt{n}\}$$

and $d \le \sqrt{n} \le d+1$ and $\#\{n : d \le \sqrt{n} \le d+1\} = 2d+2$.

Similarly
$$\#\{n : d \le \sqrt{n} \le d + \alpha\} = 2\alpha d + \alpha^2 + 1$$

then
$$S_N(\alpha) = \sum_{j=0}^{\lfloor \sqrt{N} \rfloor} \#\{n : j + \alpha \le \sqrt{n} \le j + \alpha\}$$

$$= \sum_{j=0}^{\lfloor \sqrt{N} \rfloor} 2\alpha d + \alpha^2 + O(1)$$

$$= 2\alpha \frac{\lfloor \sqrt{N} \rfloor (\lfloor \sqrt{N} \rfloor + 1)}{2} + O(\sqrt{N}) = \alpha N + O(\sqrt{N})$$

so

$$D_N^* = \sum_{0\le\alpha\le 1}\left\{\left|\frac{S_N(\alpha)}{N}-\alpha\right|\right\} = \sup_{0\le\alpha\le 1} O_\alpha(1/\sqrt{N}) = O(1/\sqrt{N}).$$

Exercise 1.37 Make the dependence on α explicit and make the proof rigorous.

Application 5.
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Proposition 1.38

$$\{\log(p_n)\}$$

is not equidistributed.

Proof. Assume otherwise, let $k \in \mathbf{Z}_{>0}$.

$$I_k = \min\{n : p_n > e^k\}$$

$$I_{k-1/2} = \min\{n : p_n > e^{k-1/2}\}$$

$$\chi_{[0,1/2)} = \text{char. fn. of } [0,1/2).$$

Consider

$$S_k = \sum_{n<I_k} \chi_{[0,1/2)}(\{\log(p_n)\})$$

$$S_{k-1/2} = \sum_{n<I_{k-1/2}} \chi_{[0,1/2]}(\{\log(p_n)\})$$

Note that $S_k = S_{k-1/2}$.
If uniformly distributed

$$\lim_{k\to\infty}\frac{S_k}{I_k} = \lim_{k\to\infty}\frac{S_{k-1/2}}{I_{k-1/2}} = L$$

$$L > 0 \implies I_k/I_{k-1/2} \to 1.$$

But the prime number theorem says

$$\pi(N) = \#\{p < N\} \sim \frac{N}{\log N}$$

so

$$I_k/I_{k-1/2} \sim \sqrt{e} \neq 1$$

so $L = 0$ but this cannot happen either.

$$S_{k-1/2} \ge \pi(e^{k-1/2}) - \pi(e^{k-1})$$

$$\ge e^{k-1}\frac{\sqrt{e}-1}{k} > 0.$$

∎

1.6 Weyl differencing and the Van der Corput inequality

Theorem 1.39 Weyl. *Let*

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbf{R}[x]$$

then

$$\{P(n)\}$$

is uniformly distributed mod 1 iff there exists $n_0 \in \{1, \dots, n\}$ s.t. $a_{n_0} \notin \mathbf{Q}$.

Lemma 1.40 Weyl differencing. *Let $y_n \in \mathbf{C}$, such that $y_n = 0$ if $n < 1$ or $n > N$ then we would like to bound*

$$\sum_m \sum_n y_n \overline{y}_m = \sum_{m,n=1}^{N} = S_N \overline{S}_N = |S_N|^2,$$

***Weyl differencing** gives*

$$\sum_{m=1}^{N} \sum_{n=1}^{N} y_n \overline{y}_m = \underbrace{\sum_n |y_n|^2}_{\text{diagonal contribution}} \underbrace{+2\Re \left(\sum_{n=1}^{N-1} \sum_{m=1}^{N-n} y_{n+m} \overline{y}_m \right)}_{\text{off diagonal terms}}.$$

Proof. Idea: Use the transformation

$$n \mapsto n + m$$

$$m \mapsto m$$

$$\begin{pmatrix} n \\ m \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

switch orders

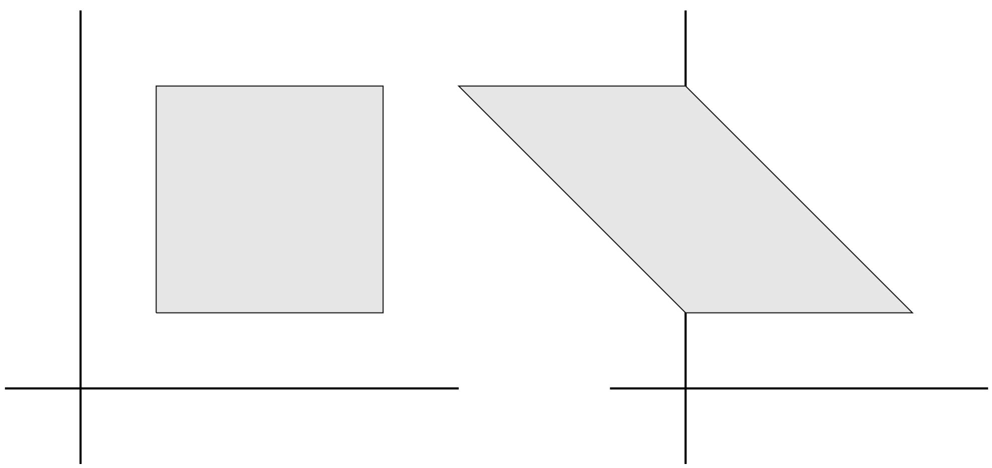

Figure 1.41: Regions

$$\sum_m \sum_n y_n \overline{y}_m = \sum_m \sum_n y_{n+m} \overline{y}_m$$

Note: if $n \geq 0$ then $m = 1, \dots, N - n$, $n = 0, 1, \dots, N - 1$. If $n < 0$ then

$$m = 1 - n, \dots, N$$

$$n = 1 - N, \dots, -1$$

$$\sum_{m=1}^{N} |y_m|^2 + \sum_{n=1}^{N-1} \sum_{m=1}^{N-n} y_{n+m} \overline{y}_m + \sum_{n=-N+1}^{-1} \sum_{m=1-n}^{N} y_{m+n} \overline{y}_m$$

$$= \sum_{m=1}^{N} |y_m|^2 + 2\Re \left(\sum_{n=1}^{N-1} \sum_{m=1}^{N-n} y_{n+m} \overline{y}_m \right)$$

∎

Theorem 1.42 Weyl. *Let $|x_n| \to \infty$ be a sequence of distinct integers, the set of $\alpha \in \mathbf{R}$ such that αx_n is not uniformly distributed has Lebesgue measure 0.*

Proof. Let $\alpha \in [0,1)$, let $h \in \mathbf{Z} \setminus \{0\}$

$$S_h(N,\alpha) = \frac{1}{N}\sum_{n=1}^{N} e(hx_n\alpha)$$

the trick is now to bound $|S_h(N,\alpha)|^2$

$$N^2|S_h(N,\alpha)|^2 = \sum_{n=1}^{N} 1 + 2\Re\left(\sum_{n=1}^{N-1}\sum_{m=1}^{N-n} e(h(x_{n+m}-x_m)\alpha)\right)$$

so

$$\int_0^1 |S_h(N,\alpha)|^2 \, \mathrm{d}\alpha = \frac{1}{N}.$$

Recall Fatou's lemma: Let $f_n > 0$ be sequence of positive measurable functions, then if

$$f(x) = \liminf f_n(x)$$

we have

$$\int f \leq \liminf \int f_n.$$

Take

$$f_n(\alpha) = \sum_{N=1}^{n} |S_h(N^2,\alpha)|^2$$

Fatou implies

$$\int_0^1 \sum_{n=1}^{\infty} |S_h(N^2,\alpha)|^2 \, \mathrm{d}\alpha \leq \sum_{N=1}^{\infty} \int_0^1 |S_h(N^2,\alpha)|^2 \, \mathrm{d}\alpha < \infty$$

so

$$\sum_{N=1}^{\infty} |S_h(N^2,\alpha)|^2$$

is finite for almost all α and for any $h \neq 0$. Then by Weyl's criterion we are done. ∎

Lemma 1.43 Van der Corput. *Let $y_n \in \mathbf{C}$, such that $y_n = 0$ if $n < 1$ or $n > N$ then*

$$\forall H \in \mathbf{Z}_{>0}$$

we have

$$\left|\sum_{n=1}^{N} y_n\right|^2 \leq \frac{N+H}{H+1}\sum_{n=1}^{N}|y_n|^2 + \frac{2(N+H)}{H+1}\sum_{h=1}^{H}\left(1-\frac{h}{H+1}\right)\left|\sum_{n=1}^{N-h} y_{n+h}\overline{y}_n\right|.$$

Proof.

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Consider

$$(H+1)^2\left|\sum_{n=1}^{N} y_n\right|^2$$

then

$$
\begin{aligned}
(H+1)^2\left|\sum_{n=1}^{N} y_n\right|^2 &= \left|\sum_{h=0}^{H}\sum_{n=1}^{N} y+n\right|^2 \\
&= \left|\sum_{h=0}^{H}\sum_{n\in\mathbf{Z}} y_{n+h}\right|^2 \\
&= \left|\sum_{n\in\mathbf{Z}}\sum_{h=0}^{H} y_{n+h}\right|^2 \qquad (1.1)
\end{aligned}
$$

note n-sum is nonzero for at most $N+H$ terms $1-H \le n \le N$ then Cauchy-Schwarz implies

$$
\begin{aligned}
&\le (N+H)\sum_n \left|\sum_{h=0}^{H} y_{n-h}\right|^2 \\
&= (N+H)\sum_n \sum_{h_1=0}^{H}\sum_{h_2=0}^{H} y_{n+h_1}\overline{y_{n+h_2}} \qquad (1.2)
\end{aligned}
$$

let $m = n+h_2$, $l = h_1 - h_2$. Then

$$
= (N+H)\sum_{-H\le l\le H}\sum_m \overline{y}_m y_{m-l}\underbrace{\left(\sum_{0\le h_1,h_2\le H} 1\right)}_{=I(H,l)}
$$

note $I(H,l) = H+1-|l|$, so

$$
= (N+H)\sum_{-H\le l\le H} H+1-|l|\sum_m \overline{y}_m y_{m-l}
$$

then if $l=0$ we get the first term of the statement, $l \neq 0$ the other. ■

What is the difference between this and Weyl differencing? When H is large, not so much, but we can take H small now, shifting the weighting around. We change the balance to make one part shorter and the other longer.

Theorem 1.44 Van der Corput differencing. *If for each $h \in \mathbf{Z}_{\ge 1}$ the sequence $b_h(n) = a_{n+h} - a_n$ then so is a_n.*

Proof. We'll use Weyl's criterion and the van der Corput lemma. Fix, $N \in \mathbf{Z}_{\ge 1}$ a $H \le N$. Then for any $k \in \mathbf{Z}\setminus\{0\}$

$$
\left|\sum e(ka_n)\right|^2 \le \frac{N+H}{H+1}N + 2\frac{N+H}{H+1}\sum\left(1-\frac{h}{H+1}\right)\left|\sum_{n=1}^{N-h} e(b_h(n)k)\right|
$$

$$
\implies \lim_{N\to\infty}\frac{1}{N^2}\left|\sum_{n=1}^{N} e(ka_n)\right|^2 \le \frac{1}{H+1}
$$

since H is arbitrary

$$
\lim_{N\to\infty}\frac{1}{N^2}\left|\sum_{n=1}^{N} e(ka_n)\right|^2 = 0.
$$

■

Proof of Weyl's theorem. Let $\deg P = d$ then, for $d = 1$ we are done by Weyl's criterion, for $d \le D$ use van der Corput differencing. Note that for fixed h, $P(n+h) - P(n)$ is of lower degree. ■

1.7 A different perspective (Ergodic)

Furstenberg (1981 book) gives a different proof that $\{n^2\alpha\}$ is uniformly distributed.

Ergodic theory 101.
Definition 1.45 Ergodic measures. Let X be a locally compact space and H a non-compact group, $H \circlearrowright X$. μ a probability measure on X, H-invariant. We say that μ is an **ergodic measure** if any of the following equivalent conditions hold

1. $A \subseteq X$ and A is H-invariant ($hA = A$ for all $h \in H$). Then
$$\mu(A) = 1 \text{ or } \mu(A) = 0.$$

2. For f measurable μ almost everywhere H-invariant
$$f(hx) = f(x)$$
for almost all x then f is constant μ almost everywhere.

3. μ is an extreme point on the convex set of H-invariant probability measures.

$\Diamond$

Definition 1.46 Uniquely ergodic actions. An action of a group H on a locally compact space X is **uniquely ergodic** if there is only one invariant probability measure on X. $\Diamond$

Example 1.47 $x \mapsto 5x$, or $x \mapsto \sqrt{2} + x$ on T^1 are ergodic, let's show that $x \mapsto \sqrt{2} + x$ is uniquely ergodic. $\square$

Lemma 1.48 *Let $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ then $T_\alpha \colon x \to x + \alpha$ is uniquely ergodic.*

Proof. Let μ be an invariant probability measure on $[0, 1]$ the nth Fourier coefficient
$$\hat{\mu}(n) = \int e(nx)\,\mathrm{d}\mu(x) = e(n\alpha)\int e(nx)\,\mathrm{d}\mu(x)$$
so $\hat{\mu}(n) = \delta(n)$. ∎

Example 1.49 For $x \mapsto 5x$ $f = \sum a_n e(nx) = \sum a_n e(5nx)$ so $a_n = a_{5n}$
$$\infty > \|f\|^2 = \|a_n\|^2 \implies a_n = 0 \forall n \neq 0$$
$$\implies f \equiv C \text{ a.e.}$$
$\square$

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Definition 1.50 Equidistribution. A sequence of probability measures μ_n on a locally compact space X is called μ-**equidistributed** if they converge to μ in the weak $*$ topology. I.e.
$$\forall f \in C_c(X), \int f\,\mathrm{d}\mu_n \to \int f\,\mathrm{d}\mu.$$
$\Diamond$

Remark 1.51 If we have a sequence a_n these define a sequence of measures

$$\mu_N = \frac{1}{N} \sum_{n \leq N} \delta(x - a_n)$$

if these are equidistributed then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(a_n) = \int_X f(x)\, \mathrm{d}\mu(x).$$

Theorem 1.52 Birkhoff ergodic theorem. *(X, B, μ, T) a measure preserving system (as in the definition of ergodic) with μ a probability measure. Then for any $f \in L^1(X, \mu)$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i(x) = \bar{f}(x)$$

for some $\bar{f} \in L^1(X, \mu)$ moreover

$$\int \bar{f}\, \mathrm{d}\mu = \int f\, \mathrm{d}\mu$$

and if T is ergodic then $\bar{f}(x) = \int f\, \mathrm{d}\mu$ almost everywhere.

Remark 1.53 This does not help us!
But the following does:

Theorem 1.54 *If X is compact and $H \cong \mathbf{R}$ is uniquely ergodic with the unique H-invariant measure μ then the statement of Birkhoff holds for every $x \in X$.*

Proof. Sketch

For any x construct

$$\mu_L = \frac{1}{L} \int_0^L h_t x\, \mathrm{d}t$$

$$\mu_L \to \mu,\ L \to \infty$$

in the weak $*$ sense. ∎

Remark 1.55 Funny side remark (Benford's law). First digit of a set of observations are usually 1 (roughly 30%).
Why? Assumption, the process follows a power law, if b^n is the first digit of this is

$$\log_{10} b^n = n \log_{10} b$$

this is k iff $n \log_{10} b \in [\log_{10} k, \log_{10} k + 1]$. If these are equidistributed then the probability of k is

$$\int_{\log_{10} k}^{\log_{10} k+1} 1\, \mathrm{d}x = \log(1 + 1/k).$$

Theorem 1.56 *$\{n^2\alpha\}$ is equidistributed for $\alpha \in \mathbf{R} \setminus \mathbf{Q}$.*

Proof. Furstenberg

1. Construct a suitable dynamical system on T^2 (2-torus), for which a specific orbit gives the sequence $\{n^2\alpha\}$.
2. Will show that this is uniquely ergodic.

$$T : T^2 \to T^2$$
$$(x, y) \mapsto (x + \alpha, y + 2x + \alpha)$$
$$T^n_\alpha(x, y) = (x + n\alpha, y + 2nx + n^2\alpha)$$

in particular the orbit of $(0, 0)$ is

$$T^n(0, 0) = (n\alpha, n^2\alpha)$$

so if we show that $T^n(0, 0)$ is equidistributed we are done.

So we show that X, T is uniquely ergodic.

1. Lebesgue is ergodic.
2. Only ergodic measure

1. $$f \in L^2(T^2, \mathrm{d}m)$$

   T-invariant
   $$f(x, y) = \sum_{m,n} a_{m,n} e(mx + ny)$$

   T-invariance implies $a_{m,n} = e((m + n)\alpha) a_{m+2n,n}$ in particular

   $$|a_{m,n}| = |a_{m+2n,n}|$$

   so
   $$a_{m,n} = 0$$

   if $n \neq 0$. By Riemann-Lebesgue lemma. So for $n = 0$

   $$\sum a_{m,0} e(mx)$$

   T-invariance implies $a_{m,0} = e(m\alpha) a_{m,0}$ so $a_{m,0} = 0$ for all $m \neq 0$. So f is constant almost everywhere.

2. We use the following:

   Claim:

   $g : T^1 \to T^1$ measurable, let $T_g : T^2 \to T^2$

   $$(x, y) \mapsto (x + \alpha, y + g(\alpha))$$

   Then if the Lebesgue measure m is T_g-ergodic then it indeed is uniquely ergodic.

   Proof: exercise.

■

1.8 Duke's theorem

Recall.
Lecture 8 4/10/2018

1.2 1.9.

We will now prove this, not via Linnik's proof, but via an analogue of Weyl's criterion. Strategy:

1. Weyl's criterion $\rightsquigarrow$ bounding exponential sums.
2. To bound these sums we will use automorphic methods.

We will begin with 1.2.

So we will be working on S^2. Recall 1.28 which was for $[0,1)$. There we used exponentials, why? Because they are a dense, convenient basis, fourier theory.

To replace this on the sphere S^2 we use the *spherical harmonics*, these are homogeneous harmonic polynomials. Analogous with the S^1 case.

$$\left(\frac{x+iy}{|x+iy|}\right)^m = e(m\theta)$$

the same spherical harmonic construction works for S^n.

We will show

$$\frac{1}{\#\Omega_n}\sum_{x\in\Omega_n} P(x) \longrightarrow 0$$

for $\deg P(x) > 0$. i.e.

$$\sum_{\alpha\in\mathbf{Z}^3,|\alpha|^2=n} P\left(\frac{\alpha}{|\alpha|}\right) = o(r_3(n)) \tag{1.3}$$

where

$$r_3(n) = \#\{a^2+b^2+c^2 = n : a,b,c\in\mathbf{Z}\}.$$

Connection to automorphic forms. θ functions, P spherical harmonics as before.

Definition 1.57

$$\theta_P(z) = \sum_{\alpha\in\mathbf{Z}^3} P(\alpha)e(|\alpha|^2 z)$$

for $z\in\mathbf{H}$ this converges.

$$\theta_P(z) = \sum_{n=0}^{\infty} r_3(n,P)e(nz)$$

$$r_3(n,P) = \sum_{|\alpha|^2=n} P(\alpha).$$

◊

Fact 1.58

1. *This is a modular form of weight* $\frac{3}{2}+\deg P$ *for* $\Gamma_0(4)$.
2. *It is a cusp form if* $\deg P > 0$.

Observation 1.59 If $\deg P \not\equiv 0 \pmod 2$ this function is 0.

To show (1.3):

1. Show that
$$r_3(n) \gg_\epsilon n^{1/2-\epsilon}$$
(Gauss-Siegel)
2. Show that
$$r_3(n,P) \ll_\delta n^{k/2-1/4-\delta}$$
for some $\delta > 0$.

Why?
$$\begin{aligned}\sum_{|\alpha|^2=n} P\left(\frac{\alpha}{|\alpha|}\right) &= n^{-\deg(P)/2} \sum_{|\alpha|^2=n} P(\alpha)\\ &= n^{-\deg(P)/2} r_3(n,P)\end{aligned}$$
note:
$$\frac{k}{2} - \frac{1}{4} - \delta = \frac{1}{2} + \frac{\deg(P)}{2} - \delta.$$
So if $r_3(n,P) \ll n^{(1+\deg(P))/2-\delta}$ which implies
$$\sum_{|\alpha|^2=n} P\left(\frac{\alpha}{|\alpha|}\right) \ll n^{\frac{1}{2}-\delta}.$$

If we knew a half-integral weight Ramanujan conjecture:
$$f \in S_k(N),\ k \in \frac{1}{2} + \mathbf{Z}_{>0}$$
for squarefree n
$$a_f(n) = O(n^{(k-1)/2+\epsilon}).$$

Digression (θ-functions). Recall: Integral weight modular forms
$$\Gamma = \mathrm{SL}_2(\mathbf{Z}) = \Gamma(1)$$
$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbf{Z} \right\}$$
In general Γ congruence subgroup $\Gamma(N) \subseteq \Gamma \subseteq \Gamma(1)$ for some N.
$$f : \mathbf{H} \to \mathbf{C}$$
is called a modular form of weight k for Γ if f is holomorphic everywhere, including at the cusps.
$$f(\gamma z) = (cz+d)^k f(z)\, \forall \gamma \in \Gamma.$$
If $f(z) = \sum_{n=1}^\infty a_n q^n$ then f is cuspidal, the space of such is $S_k(\Gamma)$ where S stands for the German Spitzenform. Spitze means cusp, kinda like a pointy spit?

Example 1.60
$$\Delta(q) = q \prod_{n=1}^\infty (1-q^n)^{24}$$

$$E_k(z) = \sum_{(c,d)=1} \frac{1}{(cz+d)^{2k}}$$

$\square$

Conjecture 1.61 Ramanujan (here theorem of Deligne). $f \in S_k(\Gamma)$

$$a_n = O_z(n^{(k-1)/2+\epsilon}) \,\forall \epsilon > 0.$$

Classically

$$\widetilde{\theta}(z) = \sum_{m \in \mathbf{Z}} e^{i\pi m^2 z}$$

converges absolutely on $z \in \mathbf{H}$

$$\widetilde{\theta}(z+2) = \widetilde{\theta}(z)$$

$$\widetilde{\theta}(-1/z) = \sqrt{-iz}\,\widetilde{\theta}(z)$$

in general for $\gamma \in \Gamma_0(4)$

$$\widetilde{\theta}(\gamma z) = j(\gamma; z)\widetilde{\theta}(z)$$

where $j(\gamma; z) = \left(\frac{c}{d}\right) \epsilon_d^{-1}(cz+d)^{1/2}$ where $\epsilon_d = 1$ if $d \equiv 1 \pmod 4$, -1 if $d \equiv 3 \pmod 4$ sign of Gauss sum. $\left(\frac{c}{d}\right)$ is a sibling of Legendre form.

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Recall: P spherical harmonic degree l. Aim to show:

$$\sum_{|x|^2=n} P(x) = o(r_3(n))$$

where

$$r_3(n) = \#\{a^2 + b^2 + c^2 = n : a, b, c \in \mathbf{Z}\}.$$

Took

$$\theta(z; P) = \sum_{n \in \mathbf{Z}} r(n; P) e(nz).$$

A modular form of weight $3/2 + l$ for $\Gamma_0(4)$. Cusp form if $l > 0$.

The strategy is then to show

$$r_3(n) \gg n^{1/2-\epsilon} \,\forall \epsilon > 0$$

$$r_3(n) \ll n^{k/2-1/4-\delta} \text{ for some } \delta > 0.$$

Definition 1.62 Half-integral weight modular forms. Let $N \equiv 0 \pmod 4$. A modular form of **half-integral** weight $k \in \frac{1}{2} + \mathbf{Z}_{\geq 0}$. For $\Gamma_0(N)$ is a holomorphic function on $\mathbf{H}$ s.t.

1. $$f(\gamma z) = j(\gamma; z)^{2k} f(z)$$

2. f is holomorphic at the cusps

if $\chi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d)$, $f(\gamma z) = \chi(d) j(\gamma; z)^{2k} f(z)$ gives the space $M_k(\Gamma_0(N), \chi)$. $\diamond$

Where do these things come from?

A construction due to Schoenberg 1939, Pfetzer 1953, Shimura 1973 is as follows:

A an $n \times n$ positive definite integral matrix, $N \in \mathbf{Z}$ s.t. NA^{-1} is integral, P a spherical harmonic relative to A i.e. for P homogeneous of degree v.

$$\sum_{i,j} \tilde{a}_{ij} \frac{\partial^2 P}{\partial x_i x_j} = 0$$

$$[a_{ij}] = A^{-1}$$

Definition 1.63 Let $h \in \mathbf{Z}^n$, set

$$\tilde{\theta}_P(z,h,N) = \sum_{m \equiv h \pmod N} P(m) e\left(\frac{(m^{\mathrm{T}}Am)z}{2N^2}\right).$$

◊

Fact 1.64 Poisson summation.

$$\tilde{\theta}_P(\gamma z, h, N) = e\left(\frac{ab(h^{\mathrm{T}}Ah)}{2N^2}\right)\left(\frac{\det A}{d}\right)\left(\frac{2c}{d}\right)^n \epsilon_d^{-n}(cz+d)^{k/2}\tilde{\theta}_P(z;ah,N)$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : b \equiv 0 \pmod 2, c \equiv 0 \pmod{2N}$$

$$k = n + 2v$$

Example 1.65 $n = 1, N = 1$ $P(m) = m^v$ $v = 0, 1$

$$\tilde{\theta}_P(z) = \sum_{m \in \mathbf{Z}} m^v e(m^2 z/2)$$

for $v = 0$ this is classical θ for $v = 1$ cusp form on $\Gamma_0(8)$ of weight $3/2$, we could also twist by a character mod 4. □

Example 1.66 $A = I_{n \times n}$ P spherical harmonic of degree v

$$\tilde{\theta}_P(z;0,1) = \sum_{m \in \mathbf{Z}^n} P(m) e(|m|^2 z/2)$$

set $z = 2z$

$$\theta_P(z) = \sum_{m \in \mathbf{Z}} P(m) e(|m|^2 z).$$

□

Note 1.67

$$\theta_P(z) \in S_k(\Gamma_0(4))$$

$$k = \frac{h}{2} + v,\ v > 0$$

$$\theta_P(z) \in M_{h/2}(\Gamma_0(4))$$

not cuspidal.

Example 1.68 $A = 4 \times 4$ integral positive definite

$$Q = Q_A(x),\ x^{\mathrm{T}}Ax$$

$$r_Q(n) = \#\{x \in \mathbf{Z}^4 : Q(x) = n\}$$

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n) e(nz) \in M_2(\Gamma_0(N))$$

for some $N = 4\det(A)$. □

Half-integral weight Ramanujan conjecture (Metaplectic). Naively we would like to mimic the integral case and say

$$a_f(n) = O(n^{(k-1)/2+\epsilon})$$

where $f \in S_k(\Gamma)$.

But this is not true as stated here!:

Example 1.69 Let

$$\theta(z;\chi) = \sum_{m\in\mathbf{Z}} m\chi(m)e(m^2 z)$$

for odd χ this is a cusp form in $S_k(\Gamma,\chi)$. For $k = 3/2$, let $f = \theta$ and

$$a_f(m^2) \sim m\chi(m) \implies a_f(n) = O(\sqrt{n})$$

for n square. But $(k-1)/2 = 1/4$ but $O(\sqrt{n})$ is not $O(n^{1/4+\epsilon})$ for any $\epsilon \geq 1/4$.□

So we avoid these or stick to squarefree n.

Conjecture 1.70 Half-integral weight Ramanujan conjecture.

$$a_f(n) = O(n^{(k-1)/2+\epsilon})$$

for all n square-free $f \in S_k(\Gamma,\chi)$.

Why this exponent? For integral weight: Representation theory gives $a_f(m)$'s which correspond to Hecke eigenvalues which under Langlands are tempered for GL_N.

Digression:

Proposition 1.71 Hecke bound. *$f \in S_k(\Gamma)$ then $a_f(n) = O(n^{k/2})$.*

Proof. $y = \Im(z)$, $y(\gamma z) = |cz+d|^{-2}y$ so

$$F(z) = y^{k/2}|f(z)|$$

Invariant on Γ bounded at the cusps $|F(z)| < M$. So

$$\int_0^1 f(z)e(nx) = e^{-2\pi i n y}a_f(n)$$

$$\leq \int_0^1 y^{-k/2}M = O(y^{-k/2})$$

so $a_f(n) = O(y^{-k/2}e^{2\pi i n y})$ with $y = 1/n$. ■

Another digression:

What is the distribution of rational points on S^2?

$$\{x_1^2 + x_2^2 + x_3^2 = 1 : x_i \in \mathbf{Q}\}$$

$$\iff r_3(n^2)$$

Proposition 1.72 Hurwitz. *A generating function for this is given by*

$$\sum_{n=1}^{\infty} \frac{r_3(n^2)}{n^s} = 6(1-2^{1-s})\frac{\overbrace{\zeta(s)\zeta(1-s)}^{=\sum \sigma(n)/n^s}}{L\left(s,\left(\frac{-1}{\cdot}\right)\right)}$$

Exercise 1.73 Prove this!

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Corollary 1.74 *This shows*

$$r_3(n^2) \gg n.$$

Proof. (Sketch) expand the formula, take $n = p$ an odd prime

$$\zeta(s)\zeta(1-s) = \sum \frac{\sigma(n)}{n^s}$$

(exercise)

$$L(s,\chi_{-4})^{-1} = \prod_p (1 - \frac{\chi_{-4}(p)}{p^s})$$

so

$$\sum \frac{r_3(n^2)}{n^s} = \sum \frac{\sigma(m_1)}{m_1^s} \prod (1 - \chi_{-4}(m_p)/p^s) 6(1-2^{1-s})$$

$$= \sum \frac{\sigma(m_1)}{m_1^s} \sum \mu(m_)\chi_{-4}(m_2)/m_2^s 6(1-2^{1-s})$$

so

$$r_3(p^2) = (\sigma(p) - \chi_{-4}(p))6 \geq 6p$$

■

The next big breakthrough came much later, showing something similar for $r_3(n,P)$, but now we hope to see cancellation.

Theorem 1.75 Shimura '71. *Let*

$$f \in S_{k/2}(N,\chi)$$

be a half integral weight cusp form, k odd, $4|N$.

$$f(z) = \sum_{n=1}^{\infty} a_f(n)e(nz),$$

assume f is a common eigenfunction for all T_{p^2}.
Let $\widetilde{\chi}(m) = \chi(m)\chi_{-4}(m)$ and $\lambda = (k-1)/2$. Set

$$\sum_{n=1}^{\infty} \frac{A(n)}{n^s} = L(s+1-\lambda, \widetilde{\chi}) \cdot \left(\sum_{m=1}^{\infty} \frac{a_f(m^2)}{m^s} \right)$$

$$F(z) = \sum_{n=1}^{\infty} A(n)e(nz).$$

Then

$$F(z) \in S_{2\lambda}(N_1, \chi^2).$$

Recall:

$$\theta_P(z) = \sum r_3(n;P)e(nz) \in S_{3/2+l}(\Gamma_0(4))$$

is a weight $3/2 + l$ form if $l = \deg P$.

Shimura gives us

$$F_P(z) \in S_{2l+2}(\Gamma_0(2))$$

then we get

Corollary 1.76

$$r_3(n^2,P) = \sum_{d|n} A_P(d)\mu\left(\frac{n}{d}\right)\chi_{-4}\left(\frac{n}{d}\right)\left(\frac{n}{d}\right)^l$$

hence if $A_P(d) \ll d^{l+1-\delta}$ then we are done.

The Hecke bound implies $A_P(d) \ll d^{l+1}$.

Detour: bounding Fourier coefficients. There are various approaches, such as Poincare series, Kloosterman sums, and bound these, to get a bound on the Fourier coefficient.

If $f \in S_{k_1}(N_1)$ we will write a spanning set for $S_{k_1}(N_1)$, then bound fourier coefficients on each of these guys, which will suffice.

Poincare series:

If we were just looking for a modular form we might try Eisenstein series

$$E_k(z) = \sum_{c,d} \frac{1}{(cz+d)^{2k}}$$

these are not cuspidal. But if we twist this a little bit

$$P_m(z;k) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma;z)^{-2k} e(m\gamma z).$$

Where $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\}$.

$$j(\gamma;z) = \left(\frac{c}{d}\right) \epsilon_d^{-1} (cz+d)^{1/2}$$

they converge absolutely for $k > 2$.

Some properties:

Lemma 1.77

$$P_m(\tau z;k) = (cz+d)^{2k} P_m(z;k)$$

Proof. Exercise, j satisifes a cocycle relation

$$j(\gamma;\tau z) j(\tau;z) = j(\gamma\tau;z).$$ ∎

Proposition 1.78

$$P_m(z) \in S_k(\Gamma)$$

Proof. To follow. ∎

Lemma 1.79 *Let* $f(z) = \sum a_f(n) e(nz)$

$$\left\langle P_m(z;k), f \right\rangle_{\mathrm{Pet}} = \frac{\overline{a}_f(m)}{(4\pi m)^{k-1}} \Gamma(k-1).$$

Proof.

$$\begin{aligned}
\left\langle P_m(z;k), f \right\rangle_{\mathrm{Pet}} &= \int_{\Gamma \backslash \mathbf{H}} P_m(z;k) \overline{f}(z) y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \int_{\Gamma \backslash \mathbf{H}} \sum_{\gamma \in \pm\Gamma_\infty \backslash \Gamma} j(\gamma;z)^{-2k} e(m\gamma z) \overline{f}(z) y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \sum_{\gamma \in \pm\Gamma_\infty \backslash \Gamma} \int_{\Gamma \backslash \mathbf{H}} j(\gamma;z)^{-2k} e(m\gamma z) \overline{f}(z) y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \int_{\Gamma_\infty \backslash \mathbf{H}} e(mz) \overline{f}(z) y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \int_0^1 \int_0^\infty e(mz) \overline{f}(z) y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}
\end{aligned}$$

$$= \overline{a}_f(m) \int_0^\infty e(2miy) y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

$$= \frac{\overline{a}_f(m)}{(4\pi m)^{k-1}} \Gamma(k-1) \qquad \blacksquare$$

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Corollary 1.80 *$P_m(z;k)$ span $S_k(\Gamma)$ as m varies through integers.*

Proof. If $f \perp P_m(z,k)$ for all m then $f = 0$. $\blacksquare$

Observation 1.81 The corollary implies that bounding fourier coefficients of $f \in S_k(\Gamma)$ is equivalent to bounding fourier coefficients of $P_m(z,k)$.

Proposition 1.82

$$\langle P_m(z,k), P_n(z,k) \rangle = \left(\frac{m}{n}\right)^{(k-1)/2} \left(\delta_{m,n} + \frac{2\pi}{i^k} \sum_{c \equiv 0 \pmod N} J_{k=1}\left(\frac{2\pi\sqrt{mn}}{c}\right) \frac{K(m,n;c)}{c} \right)$$

where

J_k is the Bessel function of the first kind

$$N \text{ s.t. } \Gamma = \Gamma_0(N)$$

$$K(m,n;c) = \sum_{d \in (\mathbf{Z}/c)^\times} \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} e\left(\frac{m\overline{d} + nd}{c}\right)$$

$$\delta_{m,n} = \begin{cases} 1 & m = n, \\ 0 & m \neq n \end{cases}.$$

Proof.

$$\int_0^1 P_m(z,k) e(-nz)\,\mathrm{d}z = \int_0^1 \left(\sum_{\pm\Gamma_\infty \backslash \Gamma} j(\gamma;z)^{-2k} e(m\gamma z) \right) e(-nz)\,\mathrm{d}z$$

$$= int_0^1 \underbrace{j(1;z)^{-2k} e((m-n)z)}_{\delta_{m,n}} \mathrm{d}z + \text{rest}$$

$$= \int_0^1 j(1;z)^{-2k} e((m-n))\,\mathrm{d}z + \text{rest}$$

Note:

$$e(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} z) = e(z).$$

So

$$\int_0^1 \sum_{\Gamma_\infty \backslash \Gamma} j(\gamma;z)^{-2k} e(m\gamma z)\,\mathrm{d}z$$

$$= \int_0^1 sum_{\Gamma_\infty \backslash \Gamma / \Gamma_\infty} \sum_{\gamma_\infty \in \Gamma_\infty} j(\gamma\gamma_\infty z)^{-2k} e(m\gamma\gamma_\infty z) e(-nz)\,\mathrm{d}z$$

$$\int_0^1 \sum_{\Gamma_\infty \backslash \Gamma / \Gamma_\infty} \sum_{\alpha \in \mathbf{Z}} j(\gamma; z+\alpha)^{-2k} e(m\gamma(z+\alpha)) e(-nz)\,\mathrm{d}z$$

$$\int_0^1 \sum_{\Gamma_\infty\backslash\Gamma/\Gamma_\infty} j(\gamma;z)^{-2k} e(m\gamma z - nz)\,\mathrm{d}z$$

Note:

$$m\gamma z = m\frac{az+b}{cz+d} = m\left(\frac{a}{c} - \frac{1}{c(cz+d)}\right)$$

$$\pm\Gamma_\infty\backslash\Gamma/\Gamma_\infty = \{(a,d,c) : c > 0,\ a,d \in (\mathbf{Z}/c)^\times,\ ad \equiv 1 \pmod{c}\}$$

So our main integral is

$$\sum_{c>0,\, d\in\mathbf{Z}/c^\times} \epsilon_d^{-2k}\left(\frac{c}{d}\right)^{2k} \int_{-\infty}^{\infty} \frac{1}{(cz+d)^k} e(m\frac{a}{c} - \frac{m}{c(cz+d)} - nz)\,\mathrm{d}z$$

$$\sum_{c>0,\, d\in\mathbf{Z}/c^\times,\, ad\equiv 1 \pmod{c}} \frac{e((ma+nd)/c)}{c^k}\epsilon_d^{-2k}\left(\frac{c}{d}\right)^{2k} \int_{-\infty}^{\infty} \frac{1}{z^k} e(-(zn + \frac{m}{c^2 z})\,\mathrm{d}z$$

Then use the following integral representation

$$\int_{-\infty+iA}^{\infty+iA} w^{-k} e^{-(\mu_1 w + \mu_2 m^{-1})}\,\mathrm{d}w$$

$$= 2\pi\left(\frac{\mu_1}{\mu_2}\right)^{(k-1)/2} e^{ik\pi/2} J_{k-1}(4\pi\sqrt{\mu_1\mu_2}) \qquad \blacksquare$$

To recap:

$$P_m(z) = \sum_{n=1}^{\infty} \hat{P}_m(n) e(nz)$$

$$\hat{P}_m(n) = \left(\frac{n}{m}\right)^{(k-1)/2}\left\{\delta_{m,n} + \frac{2\pi}{i^k}\sum_{c\equiv 0 \pmod{N},\, c>0} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\frac{K(m,n;c)}{c}\right)\right\}$$

Digression: Let $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, $k = 12$ then $S_k(\Gamma) = \langle\Delta\rangle$

$$\Delta(z) = (2\pi)^{12}\sum_{n=1}^{\infty} \tau(n) e(nz).$$

As $\Delta = \eta^{24}$ we have $\tau(n) \in \mathbf{Z}$.
As

$$P_m(z;12) \in S_{12}(1) = \langle\Delta\rangle$$
$$\implies P_m(z;12) = \kappa(m)\Delta$$

to calculate $\kappa(m)$

$$\langle P_m(z), \Delta\rangle = \frac{\overline{a}_m}{(4\pi m)^{k-1}}\Gamma(k-1) = \frac{2\pi\overline{\tau(m)}}{(4\pi m)^{k-1}}\Gamma(k-1) = \frac{2\pi 10!}{(4\pi m)^{11}}$$

so

$$P_m(z) = \frac{2\pi 10!}{(4\pi m)^{11}}\frac{\Delta(z)}{\|\Delta\|^2_{\mathrm{Pet}}}\tau(m)$$

Another consequence:

Theorem 1.83 Petersson trace formula. *For any $m, n \geq 1$*

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}} \sum_{f \text{ o.n. basis for } S_k(\Gamma)} \overline{a}_f(m) a_f(n) = \delta_{m,n} + \frac{2\pi}{i^k} \sum_{c \equiv 0 \pmod{N},\, c>0} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c} \frac{K(m,n;c)}{c}\right).$$

Proof. Let $P_m(z) = \sum_f \left\langle P_m(z), f \right\rangle f$ then

$$\langle P_m(z), P_n(z) \rangle = \sum_f \overline{a_f(m)} a_f(n) \frac{\Gamma(k-1)^2}{(4\pi m)^{k-1}(4\pi n)^{k-1}}$$

on the other hand

$$\langle P_m(z), P_n(z) \rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} \left(\frac{m}{n}\right)^{(k-1)/2} \{\cdots\}. \qquad \blacksquare$$