MA3H6 Algebraic Topology - Lecture Notes

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Contents

1 Introduction

These are lecture notes that I typeset for MA3H6 Algebraic Topology in 2014, they are currently full of gaps, mistakes, wrong statements, notation abuse and lots of other badness. However they might be useful to someone, despite the fact they lack very many pictures at present. If you find anything else that can be improved send me an email at a.j.best@warwick.ac.uk, thanks.

2 Basics

2.1 Topological review

Notation.

$$\begin{aligned} \mathbb{R}^n &= \{(x_1, \dots, x_n) \mid x_j \in \mathbb{R}\} \text{ with the product topology of open intervals.} \\ \|x\| &= \sqrt{\sum x_i^2}. \\ B^n &= \{x \in \mathbb{R}^n \mid \|x\| \le 1\} \text{ the } n-1 \text{ sphere.} \\ S^{n-1} &= \{x \in \mathbb{R}^n \mid \|x\| = 1\}. \\ \mathbb{R}^0 &= \{()\}. \end{aligned}$$

Exercise.

$$B^n \times B^m \cong B^{n+m}.$$

Exercise.

$$S^n \times S^m \ncong S^{n+m}.$$

Hint. Find an invariant of topological spaces that distinguishes them.

Invariants Connectedness, Hausdorffness, π_1, compactness, Euler characteristic. But none of these work.

Quotients We recall that the quotient topology is defined by $a \subseteq X/\sim$ is open iff its preimage under the map $f\colon X \to X/\sim$ is open. This topology makes as many of the sets of the quotient as possible open while keeping the quotient map continuous.

There are more ways to produce S^1, for example

$$S^1 \cong [0,1]/0 \sim 1$$

when equipped with the quotient topology.

Another way is to consider $\mathbb{R}/\mathbb{Z} = \mathbb{R}/\{x \sim y \iff x - y \in \mathbb{Z}\}$. So there is a map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ which is the covering map of $\mathbb{R}/\mathbb{Z}$ by its universal cover.

3 Simplicial homology

3.1 Simplices

Definition. We define the n-simplex to be

$$\Delta^n = \left\{x \in \mathbb{R}^{n+1} \mid x_i \ge 0 \ \forall i, \ \sum x_i = 1\right\}.$$

In general if $v_i \in \mathbb{R}^m$ are a collection of $n+1$ affinely independent points (do not lie in an $n-1$ dimensional subspace) then we define

$$[v] = [v_0, v_1, \ldots, v_n] = \left\{ \sum x_i v_i \mid x_i \in \Delta^n \right\}.$$

If we omit some of the v_i we obtain a facet of $[v]$. If we only omit one of them we get a face. This is denoted by

$$[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n]$$

where the v_i is read to be omitted.

The vertices are ordered and if $[v]$, $[w]$ are simplices of the same dimension then there exists a unique affine map extending the ordering of the vertices. The standard map $f\colon [v] \to [w]$ sends v_i to w_i and respects barycentric coordinates.

Definition. A facet of Δ is a subsimplex (i.e. pick some x_i and set them to zero).

Definition. A face is a codimension one facet.

Definition. The boundary of Δ^n is denoted by $\partial\Delta^n$ and consists of the union of its faces.

We have that $\mathring{\Delta} = \Delta - \partial\Delta$.

Example.

Exercise. Count the k-dimensional faces of Δ^n.

3.2 Δ-complexes

Definition. Fix X a topological space and a collection of maps

$$\{\sigma_\alpha\colon \Delta_\alpha \to X \mid \alpha \in A\}.$$

This is known as a Δ-*complex structure* on X if:

(i) (Partition) for all α $\sigma_\alpha|\mathring{\Delta}_\alpha$ is injective and for $x \in X$ there is a *unique* $\alpha \in A$ s.t. $x \in \sigma_\alpha(\mathring{\Delta}_\alpha)$.

(ii) (Tiling) If $\Delta \subset \Delta_\alpha$ is a face then there is a unique $\beta \in A$ s.t. $\sigma_\backslash|\Delta = \sigma_\beta \circ f$ where $f\colon \Delta \to \Delta_\beta$ is the canonical map.

(iii) (Topology) $U \subset X$ is open iff $\forall\alpha$ $\sigma_\alpha^{-1}(U) \subset \Delta_\alpha$ is open.

We can state this equivalently as: X must be homeomorphic to the quotient space

$$\bigsqcup_{\alpha\in A} \Delta_\alpha/\text{face gluings}.$$

Example.

Example. $\partial\Delta^n$ gives a Δ-complex structure on S^{n-1}.

Example. If we double Δ^n across $\partial\Delta$ we get a Δ-complex structure on S^n.

Example. Check these are homeomorphic to S^n.

Non Example. Violates tiling on the edge marked $[0, 2]$ and so is not a Δ-complex structure.

Exercise. 1. Find a Δ-complex structure on the space in the non-example above.

2. Show that every graph admits a Δ-complex structure.

Example. Here the indexing set $A = \mathbb{R}$ (very big!).

Definition. A Δ-complex is *finite dimensional* if there exists n s.t. for all α $\dim(\Delta_\alpha) \leq n$.

Definition. A Δ-complex structure is *finite* if $|A| < \infty$ (where as above A is the index set).

Exercise. Show that if X admits a Δ-complex structure then X is Hausdorff.

Exercise. Show that if $\{\sigma_\alpha\}$ is a Δ-complex structure on X and $K \subset X$ is compact then K meets the interiors of only finitely many of the σ_α's.

Exercise. If X, Y admit Δ-complex structures then so does $X \times Y$.

3.3 Abelian groups

Fix A a set. Define $\mathbb{Z}[A]$ to be the *free abelian group* on A given by

$$\begin{aligned}\mathbb{Z}[A] &= \left\{ \sum_{\alpha \in A} n_\alpha \cdot \alpha \,\middle|\, n_\alpha \in \mathbb{Z} \text{ and all but finitely many are non-zero} \right\} \\ &= \text{all finite } \mathbb{Z}\text{-linear sums of elements of } A.\end{aligned}$$

Example.

$$\mathbb{Z}[\{\alpha, \beta\}] \cong \mathbb{Z}^2 = \{n\alpha + m\beta \mid m, n \in \mathbb{Z}\}.$$

If A is finite then $\mathbb{Z}[A] \cong \mathbb{Z}^A$. But if $|A| = \infty$ then this is false.

Exercise. $\mathbb{Q}$ is *not* a free abelian group.

3.4 Chains

Suppose $(X, \{\sigma\})$ is a space equipped with a Δ-complex structure.

Definition. We define the set of *n-chains* to be

$$C_n^\Delta = \mathbb{Z}[\{\sigma_\alpha \mid \dim(\Delta_\alpha) = n\}].$$

Example.

3.5 Boundary operators

Recall $\Delta_v = [v_0, v_1, \ldots, v_n]$ is an n-simplex.

The ith face of Δ is $[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n]$.

Definition. We define the *boundary operator* as follows. First suppose $\sigma\colon \Delta \to X$ is a map.

We then define

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma \mid [v_0, \ldots, \hat{v}_i, \ldots, v_n].$$

Which is an $(n-1)$-chain.

So we extend linearly to define

$$\partial\colon C_n^{\Delta}(X) \to C_{n-1}^{\Delta}(X)$$

given by

$$\sum n_\alpha \sigma_\alpha \mapsto \sum n_\alpha \partial\sigma_\alpha.$$

Example.

Lemma.

$$\partial_{n-1} \circ \partial_n = 0.$$

"The extremes of the extremes are empty".

Proof. It suffices to check this on a basis element

$$\sigma\colon \Delta^n \to X$$

so

$$\partial_n\sigma = \sum_{i=0}^{n} (-1)^i \sigma \mid [v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

now we apply ∂_{n-1}:

$$\begin{aligned}
\partial_{n-1}\partial_n\sigma &= \partial_{n-1}\left(\sum_{i=0}^{n} (-1)^i \sigma \mid [v_0, \ldots, \hat{v}_i, \ldots, v_n]\right) \\
&= \sum_{i=0}^{n} (-1)^i \partial_{n-1}\left(\sigma \mid [v_0, \ldots, \hat{v}_i, \ldots, v_n]\right) \\
&= \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{n-1} (-1)^j \left(\sigma \mid [v_0, \ldots, \hat{v}_i, \ldots, v_n]\right) \mid [w_0, \ldots, \hat{w}_j, \ldots, w_{n-1}] \\
&= \sum_{i=0}^{n} (-1)^i \left(\sum_{j<i} (-1)^j \sigma \mid [v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_n]\right.
\end{aligned}$$

$$
\begin{aligned}
&\left.+\sum_{j>i}(-1)^{j+1}\sigma \mid [v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]\right) \\
&=\sum_{j<i}(-1)^{j+i}\sigma \mid [v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n] \\
&-\sum_{j>i}(-1)^{j+i}\sigma \mid [v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n] \\
&=0
\end{aligned}
$$

□

3.6 Chain complexes

Definition. A sequence $\{C_n\}_{n=0}^{\infty}$ of abelian groups with homomorphisms

$$\partial_n \colon C_n \to C_{n-1}$$

such that $\partial^2 = 0$ is called a *chain complex.*

By convention we take C_{-1} to be 0.

Example.

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to 0.$$

Given two chain complexes we can form the direct sum by taking the direct sum of each of the groups and letting the operators act elementwise.

Terminology If $c \in C_n$ we call c an *n-chain.*
If $z \in Z_n = \ker(\partial_n)$ we call z an *n-cycle.*
If $b \in B_n = \operatorname{im}(\partial_{n-1})$ we call b an *n-boundary.*
If $h \in Z_n/B_n = H_n$ we call h a *homology class.*
Since $\partial^2 = 0$ we deduce that $B_n \le Z_n$ and $H_n = Z_n/B_n$ makes sense.

Example. For

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to 0$$

we have $H_1 = 0$, $H_0 = \mathbb{Z}/2\mathbb{Z}$ and $H_k = 0$ for all $k \ge 1$.

Definition. If (X, σ) is a Δ-complex then set $C_n^{\Delta}(X) = \mathbb{Z}[\{\sigma_\alpha \mid \dim(\Delta_\alpha) = n\}]$ and $\partial_n \colon C_n^{\Delta}(X) \to C_{n-1}^{\Delta}(X)$ is the boundary operator.
Then $H_n^{\Delta}(X)$ are called the *simplicial homology groups* of X.

Theorem. This is independent of the choice of Δ-complex structure on X.

3.7 Computations

1. $X = \{\mathrm{pt}\}$. $C_0^{\Delta}(X) \cong \mathbb{Z}$ and all others are 0, so we have the chain complex:

$$\cdots \to 0 \to 0 \to \mathbb{Z} \to 0.$$

So $H_0^{\Delta}(\mathrm{pt}) \cong \mathbb{Z}$ and $H_k^{\Delta}(\mathrm{pt}) \cong 0$ if $k \geq 1$.

2. $X = S^1$. $C_0^{\Delta}(X) \cong \mathbb{Z}$, $C_1^{\Delta}(X) \cong \mathbb{Z}$ and all others are 0, so we have the chain complex:

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \to 0.$$

We see that $\partial e = \sum_{i=0}^{1}(-1)^i e|[v_0, \ldots, \hat{v}_i, \ldots, v_1] = e|[v_1] - e|[v_0] = v - v = 0$. So

$$H_k^{\Delta}(S^1) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } 1, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise. Compute $H_*^{\Delta}(S^1)$ for the Δ-complex structure on S^1 with k vertices and k edges.

Exercise. Compute $H_*^{\Delta}(X)$ for the $X = B^2,\ S^1$ and K^2 (the Klein bottle).

Exercise. Using the fact that Δ^n is a Δ-complex structure on B^n compute $H_*^{\Delta}(B^n)$.
In general you'll want to make use of the Smith normal form.

4 Singular homology

Definition. A *singular n-simplex* in X is a map $\sigma\colon \Delta^n \to X$.

Definition.

$$C_n^{\mathrm{sing}}(X) = \mathbb{Z}[\{\sigma\colon \Delta^n \to X\}].$$

We call $c \in C_n^{\mathrm{sing}}(X)$ a singular n-chain.

Definition. We define $\partial\colon C_n^{\mathrm{sing}}(X) \to C_{n-1}^{\mathrm{sing}}(X)$ exactly as before by

$$\partial\sigma = \sum_{i=0}^{n}(-1)^i \sigma|[v_0, \ldots, \hat{v}_i, \ldots, v_n].$$

And again we define $Z_n^{\mathrm{sing}}(X)$ (resp. $B_n^{\mathrm{sing}}(X)$) exactly as above and call it the group of singular n-cycles (resp. n-boundaries).

Definition. Now $H_n^{\mathrm{sing}}(X) = Z_n^{\mathrm{sing}}(X)/B_n^{\mathrm{sing}}(X)$ is the n-th *singular homology group.*

Remark. We have that $\partial_{n-1} \circ \partial_n = 0$ exactly as before.

Example. Suppose X is a single point, then there is a unique Δ-complex structure on X. We say $\sigma^0\colon \mathbb{Z} \to X$ is the "constant map". So $0 \to \mathbb{Z} \to 0$ is the chain complex $C_*^\Delta(X)$. So

$$H_n^\Delta = \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n \geq 1. \end{cases}$$

Suppose X is as above again, then we can compute

$$H_n^{\text{sing}} = \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n \geq 1. \end{cases}$$

This is as in dimension n there is only the constant map

$$\sigma^n\colon \Delta^n \to X$$

so $C_n^{\text{sing}}(X) \cong \mathbb{Z}$ and we also have that

$$\begin{aligned} \partial\sigma^n &= \sum_{i=0}^{n} (-1)^i \sigma|[v_0, \ldots, \hat{v}_i, \ldots, v_n] = \sum_{i=0}^{n} (-1)^i \sigma^{n-1} \\ &= \left(\sum_{i=0}^{n} (-1)^i \right) \sigma^{n-1} = \begin{cases} 0 & n \text{ odd}, \\ \sigma^{n-1} & n \text{ even}, \end{cases} \end{aligned}$$

except if $n = 0$. So C_n is

$$\cdots \to \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \to 0$$

and so the singular homology groups are as claimed.

Challenge Compute $H_n^{\text{sing}}(S^1)$ from the definitions.

Theorem. If X admits a Δ-complex structure then

$$H_*^\Delta(X) \cong H_*^{\text{sing}}(X).$$

The left hand side is generally easier to compute, but the right can be easier to prove theorems with.

Proposition. If $X = \bigsqcup X_\alpha$ where all X_α are path connected spaces then

$$H_n^{\text{sing}}(X) = \bigoplus_\alpha H_n^{\text{sing}}(X_\alpha).$$

Proof.

$$C_n^{\text{sing}}(X) = \bigoplus_\alpha C_n^{\text{sing}}(X_\alpha).$$

and ∂ respects this "splitting". □

Proposition. If $X \neq \emptyset$ and X is path connected then $H_0^{\text{sing}}(X) \cong \mathbb{Z}$.

Proof. Define $\epsilon\colon C_0(X) \to \mathbb{Z}$ by $\sum n_\alpha v_\alpha \mapsto \sum n_\alpha$, the augmentation map, then ϵ is surjective. We claim that $\ker(\epsilon) = \operatorname{im}(\partial_1)$. Given any $\tau\colon \Delta^1 \to X$ that goes from v to w we have that $\partial\tau = w - v$ so $\epsilon(\partial\tau) = 1 - 1 = 0$ and the image is contained in the kernel. Now fix $\sum n_\alpha v_\alpha$ s.t. $\epsilon(\sum n_\alpha v_\alpha) = 0$. Also fix some $u \in X$ and for all α pick $\tau_\alpha\colon \Delta^1 \to X$ a path from u to v_α. Consider $\sum n_\alpha \tau_\alpha \in C_1(X)$

$$\begin{aligned}
\partial\left(\sum n_\alpha \tau_\alpha\right) &= \sum \partial(n_\alpha \tau_\alpha) \\
&= \sum n_\alpha \partial\tau_\alpha \\
&= \sum n_\alpha (v_\alpha - u) \\
&= \sum n_\alpha v_\alpha - \sum n_\alpha u \\
&= \sum n_\alpha v_\alpha - u \sum n_\alpha \\
&= \sum n_\alpha v_\alpha - u \cdot 0 \in \operatorname{im}
\end{aligned}$$

Hence im = ker as claimed.

And so $H_0 = \ker(\partial_0)/\operatorname{im}(\partial_1) = \ker(\partial_0)/\ker(\epsilon) = C_0(X)/\ker(\epsilon) \cong \mathbb{Z}$. $\square$

4.1 Reduced Homology

Definition. If X has k path components, then $H_0(X) \cong \mathbb{Z}^k$ so we define the *augmented chain complex*

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where ϵ is the augmentation map from above. Define the *reduced homology groups* $\tilde{H}_*(X)$ to be the homology groups of this chain complex. So $\tilde{H}_n(X) = H_n(X)$ is $n > 0$ and $\tilde{H}_0(X) = \ker(\epsilon)/\operatorname{im}(\partial_1)$. Hence if X has k path components

$$\tilde{H}_0(X) \cong \mathbb{Z}^{k-1}.$$

Recall that $H_*(X \sqcup Y) = H_*(X) \oplus H_*(Y)$, the reduced homology groups behave nicely with respect to many operations such as 1-point unions. In a 1-point union $X \vee Y = X \sqcup Y/x \sim y$ for some designated point $x \in X$ and $y \in Y$. So $\tilde{H}_*(X \vee Y) = \tilde{H}_*(X) \oplus \tilde{H}_*(Y)$.

4.2 Functoriality

Definition. Suppose $f\colon X \to Y$ is a (continuous) map. Let $f_n\colon C_n(X) \to C_n(Y)$ by $\sigma \mapsto f \circ \sigma$. The function $f \circ \sigma$ is again a map from Δ^n to Y and so still lies in $C_n(Y)$.

The key property of this definition is that $\partial_n \circ f_n = f_{n-1} \circ \partial_n$. This is saying that the square

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1}(X) \arrow[d, "f_{n-1}"] \\ C_n(Y) \arrow[r, "\partial_n"] & C_{n-1}(Y) \end{tikzcd}$$

commutes. We denote the family of these maps f_n as $f_\#$,

Exercise. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ then $(g \circ f)_n = g_n \circ f_n$.

4.3 Chain maps

Definition. If C_*, D_* are chain complexes we say that a family of homomorphisms $f_\#\colon C_* \to D_*$ is a *chain map* if

$$f_\# \circ \partial = \partial \circ f_\#$$

Example. If $f\colon X \to Y$ is continuous then $f_\#$ is a chain map from $C_*(X) \to C_*(Y)$.

Example. Suppose (X, σ) is a Δ-complex then

$$i\colon C_n^\Delta(X) \to C_n^{\text{sing}}(X)$$

is also a chain map.

Proposition. If $f_\#\colon C_* \to D_*$ is a chain map then $f_\#$ induces a homomorphism

$$f_*\colon H_*(C) \to H_*(D)$$

given by

$$f_*([z]) = [f_\#(z)].$$

Proof. Check that $f_\#(Z_n^C) \leq Z_n^D$ (exercise) and that $f_\#(B_n^C) \leq B_n^D$. So $f_\#(b) = f_\#(\partial c) = \partial f_\#(c)$. □

Remark. If $f\colon X \to Y$ is a homomorphism then there exists a continuous inverse $g\colon Y \to X$ such that

$$f_*\colon H_*(X) \to H_*(Y)$$

is inverse to

$$g_*\colon H_*(Y) \to H_*(X).$$

4.4 Homotopic spaces

Definition. We say two maps f and g from $X \to Y$ are *homotopic* if there is a map $F\colon X \times [0,1] \to Y$ such that $f(x) = F(x,0)$ and $g(x) = F(x,1)$. We write $f \sim g$.

We then say two spaces X and Y are *homotopy equivalent* if there exists maps $f\colon X \to Y$ and $g\colon Y \to X$ such that

$$(g \circ f) \sim \mathrm{Id}_X \text{ and } (f \circ g) \sim \mathrm{Id}_Y.$$

Example.

$$S^n \sim \mathbb{R}^{n+1} \setminus \{0\}$$

via (for $n = 1$)

$$\begin{aligned} i\colon S^1 &\to \mathbb{R}^2 \setminus \{0\} \\ x &\mapsto x \end{aligned}$$

and

$$\begin{aligned} r\colon \mathbb{R}^2 \setminus \{0\} &\to S^1 \\ x &\mapsto \frac{x}{\|x\|}. \end{aligned}$$

We also have

$$S^n \sim B^{n+1} \setminus \{0\}$$

Theorem. If $f \sim g\colon X \to Y$ then

$$f_* = g_*\colon H_*(X) \to H_*(Y).$$

Corollary. If X is homotopy equivalent to Y via f then

$$f_*\colon H_*(X) \to H_*(Y)$$

is an isomorphism.

Proof.

$$(\mathrm{Id}_X)_* = \mathrm{Id}_{H_*}$$

□

Definition. Suppose $f_\#, g_\#\colon C_* \to D_*$ are chain maps. A sequence of homomorphisms $P_n\colon C_n \to D_{n+1}$ is called a *chain homotopy* if

$$\partial_{n-1} P_n + P_{n-1}\partial_n = g_\# - f_\#$$

in there is a chain homotopy between two chain maps $f_\#, g_\#$ we write $f_\# \sim g_\#$.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial"] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & C_n \arrow[r, "\partial"] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & C_{n-1} \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] \\
D_{n+1} \arrow[r, "\partial"] & D_n \arrow[r, "\partial"] & D_{n-1}
\end{tikzcd}$$

Proposition. If $f_\# \sim g_\# \colon C_* \to D_*$ then

$$f_* = g_* \colon H_*(C) \to H_*(D).$$

Proof. Pick any $h \in H_*(C)$, we want to show $(g_* - f_*)(h) = 0$. Choose some $x \in Z_n(C)$ such that $h = [z]$ and compute

$$\begin{aligned}
(g_* - f_*)(h) &= (g_* - f_*)([z]) \\
&= [(g_\# - f_\#)(z)] \\
&= [(P\partial + \partial P)(z)] \\
&= [P\partial z + \partial P z] \\
&= [P0 + \partial P z] \\
&= [\partial(Pz)] \\
&= 0 \text{ (as } B_n = 0 \text{ in homology).}
\end{aligned}$$

□

4.5 Prisms

Definition. A *prism* is a copy of $\Delta^n \times I$.

We can subdivide $\Delta \times I$ into $n+1$ dimensional simplices of the form

$$[v_0, v_1, \ldots, v_i, w_{i+1}, \ldots, w_n],$$

where v are the vertices of the simplex at one end of the interval and w are the vertices at the other.

If we have $F \colon X \times I \to Y$ and $\sigma \colon \Delta^n \to X$ we let

$$F\sigma = F \circ (\sigma \times \mathrm{Id}_I) \colon \Delta^n \times I \to Y.$$

Proof (of above theorem). $f \sim g \colon X \to Y$, let $F \colon X \times I \to Y$ be the homotopy. Then define

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i F\sigma|[v_0, \ldots, v_i, w_{i+1}, \ldots, w_n]$$

this is the prism operator. We now claim that P is a chain homotopy from $f_\#$ to $g_\#$. To see this fix $\sigma \colon \Delta^n \to X$ and compute

$$\begin{aligned}
\partial P\sigma &= \partial\left(\sum_{i=0}^{n} (-1)^i F\sigma|[v_0, \ldots, v_i, w_{i+1}, \ldots, w_n]\right) \\
&= \sum_{j \le i} (-1)^{i+j} F\sigma|[v_0, \ldots, \hat{v}_j, \ldots, w_n] + \sum_{i \le j} (-1)^{i+j+1} F\sigma|[v_0, \ldots, \hat{w}_j, \ldots, w_n]
\end{aligned}$$

and

$$\begin{aligned}
P\partial\sigma &= P\left(\sum (-1)^j \sigma|[v_0, \ldots, \hat{v}_j, \ldots, v_n]\right) \\
&= \sum_{i<j} (-1)^{i+j} F\sigma|[v_0, \ldots, \hat{w}_j, \ldots, w_n] + \sum_{j<i} (-1)^{i+j-1} F\sigma|[v_0, \ldots, \hat{v}_j, \ldots, w_n].
\end{aligned}$$

So

$$\begin{aligned}
\partial P\sigma + P\partial\sigma &= \sum (-1)^{2i} F\sigma|[v_0, \ldots, \hat{v}_i, w_i, \ldots, w_n] + \sum (-1)^{2i+1} F\sigma|[v_0, \ldots, v_i, \hat{w}_i, \ldots, w_n] \\
&= F\sigma|[\hat{v}_0, w_0, \ldots, w_n] - F\sigma|[v_0, \ldots, v_n, \hat{w}_n] \\
&= g_\#\sigma - f_\#\sigma.
\end{aligned}$$

□

4.6 Exact sequences

Definition. We say a complex C_* is *exact* if $H_*(C) \equiv 0$ (or equivalently if $Z_n = B_n$ for all n).

We also say that a sequence is *short* if it has at most 3 non-zero terms.

Example.

$$0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0.$$
$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z} \to 0.$$

Definition. A short sequence of chain complexes

$$0 \to A_* \xrightarrow{i_\#} B_* \xrightarrow{j_\#} C_* \to 0$$

is *exact* if for all n

$$0 \to A_n \xrightarrow{i_n} B_n \xrightarrow{j_n} C_n \to 0$$

is exact and $i_\#$, $j_\#$ are chain maps.

Definition. We say that (X, A) is a *good pair* if $A \subseteq X$ is non-empty, closed and there exists an open V with $X \supset V \supset A$ such that V deformation retracts to A.

Example. $(\mathbb{R}^2, S^1)$ is a good pair.

We say $f\colon (X, A) \to (Y, B)$ is a *map of pairs* if $f\colon X \to Y$ is a map and $f(A) \subset B$.

Example.

$$f\colon (I, \partial I) \to (\mathbb{R}^2, S^1).$$

Similarly we can define a *homotopy of maps of pairs* to be a function $F\colon X \times I \to Y$ where each $F_t\colon (X, A) \to (Y, B)$ is a map of pairs and $F_0 = f$, $F_1 = g$.

4.7 Relative homology

Suppose (X, A) is a pair. Note that

$$i_{\#} \colon C_*(A) \to C_*(X)$$

is an inclusion. Define $C_*(X, A) = C_*(X)/C_*(A)$ and we then have that $C_n(X, A) = C_n(X)/C_n(A)$ and as ∂^X preserves $C_*(A)$ it descends to give $\partial^{(X,A)}$. We have that $\partial^{(X,A)}[c] = [\partial^X c]$.

Exercise. Show $\partial^{(X,A)}$ is well defined and $(\partial^{(X,A)})^2 = 0$.

Note that

$$0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$$

is a short exact sequence of chain complexes.

Definition. $H_n(X, A) = Z_n(X, A)/B_n(X, A)$, we also say that $[z] \in Z_n(X, A)$ is a *relative cycle* and $[b] \in B_n(X, A)$ is a *relative boundary.*

If $z \in [z] \in Z_n(X, A)$ then $\partial^{(X,A)}[z] = 0 \in C_n(X, A)$ i.e. $\partial^{(X,A)}[z] = [a][\text{any } a \in C_n(A)]$. $[\partial^X z] = [a]$ i.e. $\partial^X z = C_n(A)$.

Example.

$$\begin{aligned} H_*(X, X) &= 0, \\ H_*(X, \emptyset) &= H_*(X), \\ H_*(X, \{\mathrm{pt}\}) &= \tilde{H}_*(X) \text{ (exercise).} \end{aligned}$$

Proposition. If f is homotopic, $f \sim g \colon (X, A) \to (Y, B)$ then $f_* = g_* \colon H_*(X, A) \to H_*(Y, B)$.

Proof. The prism operator gives a chain homotopy. □

Corollary. If $A \subset V$ and V deformation retracts to A then $H_*(V, A) = 0$.

Proof. □

4.8 Long exact sequences

Theorem. Suppose $0 \to A_* \xrightarrow{i_{\#}} B_* \xrightarrow{j_{\#}} C_* \to 0$ is exact then there is a $\partial_* \colon H_{*+1}(C) \to H_*(A)$ making the following triangle exact

$$\begin{tikzcd} H_*(A) \arrow[rr, "i_*"] & & H_*(B) \arrow[dl, "j_*"] \\ & H_*(C) \arrow[ul, "\partial_*"] & \end{tikzcd}$$

that is

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r] & H_n(B) \arrow[r] & H_n(C) \arrow[dll, rounded corners, "\partial_n"] \\ & H_{n-1}(A) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(C) \arrow[r] & \cdots \end{tikzcd}$$

is a long exact sequence of groups (an exact complex).

Proof. We define δ_*. Fix some $[c] \in H_n(C)$, so $c \in Z_n(C) \leq C_n$. The map j is surjective so pick $b \in B_n$ such that $j(b) = c$. Since $c \in Z_n$ we have $\partial c = 0$ and so $j\partial b = \partial jb = 0$. Now since $\ker(j) = \operatorname{im}(i)$ there is some $a \in A_{n-1}$ such that $ia = \partial b$. We then define $\delta_*[c] = [a]$, we have a few things to check:

1. $a \in Z_{n-1(A)}$: $i\partial a = 0 \iff \partial a = 0$, $i\partial a = \partial ia = \partial\partial b = 0$ as required.
2. δ_* is well defined:
   (i) Suppose we pick $c + \partial c'$ instead of c. Pick any b' such that $jb' = c'$ so that $j(b + \partial b') = c + \partial c'$, however then $\partial(b + \partial b') = \partial b$ is the same.
   (ii) Suppose we picked b'' such that $jb'' = c$. Then $j(b - b'') = 0$ and so there is some $a' \in A_n$ such that $ia' = b - b''$, i.e. $b'' = b - ia'$. We then have $\partial b'' = \partial b - \partial ia' = ia - i\partial a' = i(a - \partial a')$ so b'' gives $a - \partial a'$ and b gives a. Since $[a] = [a - \partial a']$ we have that δ_* is well defined.
3. δ_* is a homomorphism as i, j and ∂ are.
4. To see that the chain complex is as claimed we have some more checks to make:
   (i) $j_*i_*[a] = 0$: $j_*i_*[a] = j_*[i_\# a] = [j_\# i_\# a] = [0]$.
   (ii) $\delta_* j_*[b] = 0$: Set $jb = c$, suppose $\delta_*[c] = [a]$, we know $\partial b = 0$ so $ia = \partial b = 0$ but also that i is injective. So $a = 0$ and hence $[a] = 0$ as required.
   (iii) $j_*\delta_*[c] = 0$: Set $\delta_*[c] = [a]$ and let $ia = \partial b$ and $jb = c$ as usual. So $i_*[a] = [ia] = [\partial b] = 0$.
5. To see that the complex is exact we must show the opposite inclusions of images and kernels to the ones demonstrated above, we only show (i) here and (ii) & (iii) are left as exercises.

   $\ker(j_*) \leq \operatorname{im}(i_*)$: Pick $[b] \in \ker(j_*)$, suppose $j_*[b] = 0$ i.e. $[jb] = 0$ and if $jb = c$ then there is a c' such that $c = \partial c'$. We know j is surjective so there is b' such that $jb' = c'$ and therefore $j(b - \partial b') = c - \partial c' = 0$. So there exists a with $ia = b - \partial b'$ so $i_*[a] = [b - \partial b'] = [b]$.

□

Example. If (X, A) is a pair then

$$\begin{tikzcd} H_*(A) \arrow[rr, "i_*"] & & H_*(X) \arrow[dl, "q_*"] \\ & H_*(X,A) \arrow[ul, "\partial_*"] & \end{tikzcd}$$

is exact.

Remark. If $A = \emptyset$ then j_* is an isomorphism.

Theorem.

$$\begin{tikzcd} \tilde{H}_*(A) \arrow[rr, "i_*"] & & \tilde{H}_*(X) \arrow[dl, "q_*"] \\ & H_*(X,A) \arrow[ul, "\partial_*"] & \end{tikzcd}$$

is also exact, thus $\tilde{H}_*(X) \cong H_*(X, X)$.

Example. If (X, B, A) is a triple then

$$\begin{tikzcd} H_*(B,A) \arrow[rr, "i_*"] & & H_*(X,A) \arrow[dl, "q_*"] \\ & H_*(X,B) \arrow[ul, "\partial_*"] & \end{tikzcd}$$

is exact.

Example. Set $(X, A) = (B^2, S^1)$ then applying the snake lemma to

$$0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$$

gives that

$$H_k(B^2, S^1) = \begin{cases} \mathbb{Z} & \text{if } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise. For H_*^{sing} suppose that $A \xrightarrow{i} X$ and $X \xrightarrow{r} S$ is a *retraction*, i.e. $r \circ i = \text{id}_A$. Prove that

$$H_*(X) \cong H_*(A) \oplus H_*(X, A).$$

Theorem (Excision)**. Version 1** Suppose $Z \subset A \subset X$ with closure$(A) \subseteq$ interior(A), then

$$H_*^{\text{sing}}(X \setminus Z, A \setminus Z) \cong H_*^{\text{sing}}(X, A).$$

Version 2 Suppose $A, B \subseteq X$ and $X \subseteq \text{interior}(A) \cup \text{interior}(B)$ then

$$H_*^{\text{sing}}(B, B \cap A) \cong H_*^{\text{sing}}(X, A).$$