

## Part III Algebraic Topology 2014



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# Chapter 1

## Homology

### 1.1 Introduction

These are lecture notes for the 2014 Part III Algebraic Topology course taught by Dr. Jacob Rasmussen.

The recommended books are:

- [Algebraic Topology](#) - Allen Hatcher,
- Homology Theory - James W. Vick,
- Differential Forms in Algebraic Topology - Raoul Bott and Loring W. Tu.

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### 1.2 Homotopy

#### 1.2.1 Homotopies

**Definition 1.2.1** (Homotopic maps). Maps  $f_0, f_1: X \rightarrow Y$  are said to be **homotopic** if there is a continuous map  $F: X \times I \rightarrow Y$  such that

$$F(x, 0) = f_0(x) \text{ and } F(x, 1) = f_1(x) \quad \forall x \in X.$$

We let  $\text{Map}(X, Y) = \{f: X \rightarrow Y \text{ continuous}\}$ . Then letting  $f_t(x) = F(x, t)$  in the above definition we see that  $f_t$  is a path from  $f_0$  to  $f_1$  in  $\text{Map}(X, Y)$ .

**Example 1.2.2.** 1.  $X = Y = \mathbf{R}^n$ ,  $f_0(\bar{x}) = \bar{0}$  and  $f_1(\bar{x}) = \bar{x}$  are homotopic via  $f_t(\bar{x}) = t\bar{x}$ .

2.  $S^1 = \{z \in \mathbf{C} : |z| = 1\}$  then

3.  $S^n = \{\bar{x} \in \mathbf{R}^n : |\bar{x}| = 1\}$

**Lemma 1.2.3.** *Homotopy is an equivalence relation on  $\text{Map}(X, Y)$ .*

**Lemma 1.2.4.** *If  $f_0 \sim f_1: X \rightarrow Y$  and  $g_0 \sim g_1: Y \rightarrow Z$  then  $g_0 \circ f_0 \sim g_1 \circ f_1$ .*

**Corollary 1.2.5.** *For any space  $X$  the set  $[X, \mathbf{R}^n]$  has one element.*

*Proof.* Define  $0_X: X \rightarrow \mathbf{R}^n$  by  $0_X(x) = 0 \in \mathbf{R}^n$  for any  $x \in X$ . □

**Definition 1.2.6** (Contractible space).  $X$  is **contractible** if  $1_X$  is homotopic to a constant map.

**Proposition 1.2.7.**  $Y$  is contractible  $\iff [X, Y]$  has one element for any space  $X$ .

*Proof.*  $(\implies)$  as in corollary.  $(\impliedby)$   $[X, Y]$  has one element so  $1_Y \sim$  a constant map.  $\square$

Given a space  $X$  how can we tell if  $X$  is contractible? If  $X$  is contractible then it must be path connected for one.

*Proof.* Contractible implies that  $[S^0, X]$  has one element and so  $f: S^0 \rightarrow X$  extends to  $D^1$ , and therefore  $X$  is path connected.  $\square$

Similarly if  $[S^1, X]$  has more than one element then  $X$  is not contractible.

**Definition 1.2.8** (Simply connected). We say  $X$  is **simply connected** if  $[S^1, X]$  has only one element.

We say two space  $X$  and  $Y$  are *homotopy equivalent* if there exists  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  such that  $g \circ f \sim 1_X$  and  $f \circ g \sim 1_Y$ .

**Example 1.2.9.**  $X$  is contractible if and only if  $X \sim \{p\}$ .

*Proof.*  $X$  contractible  $\implies 1_X \sim c$ , a constant map. Choose  $f: X \rightarrow \{p\}$ ,  $f(x) = p$  and  $g: \{p\} \rightarrow X$ ,  $g(p) = c$ . Then  $g \circ f = c \sim 1$  and  $f \circ g = 1_{\{p\}}$ . Converse: exercise.  $\square$

**Exercise 1.2.10.**

Given  $X$  and  $Y$  how can we determine if  $X \sim Y$ ? How do we determine  $[X, Y]$ ? For example is  $S^n \sim S^m$ .

## 1.2.2 Homotopy groups

**Definition 1.2.11** (Map of pairs). A **map of pairs**  $f: (X, A) \rightarrow (Y, B)$  is a map  $f: X \rightarrow Y$  with sets  $A \subset X$  and  $B \subset Y$  such that  $f(A) \subset B$ .

If we have maps of pairs  $f_0, f_1: (X, A) \rightarrow (Y, B)$  then we write  $f_0 \sim f_1$  if there exists  $F: (X \times I, A \times I) \rightarrow (Y, B)$  such that  $F(x, 0) = f_0(x)$  and  $F(x, 1) = f_1(x)$ .

**Definition 1.2.12** (Homotopy groups). If  $* \in X$  then the  $n$ th **homotopy group** is

$$\pi_n(X, *) = [(D^n, S^{n-1}) \rightarrow (X, \{*\})].$$

We now note several properties of this definition:

1.  $\pi_0(X, *)$  = set of path components of  $X$ .
2.  $\pi_1(X, *)$  is a group.  $\pi_n(X, *)$  is an abelian group.
3.  $\pi_n$  is a functor

$$\left\{ \begin{array}{c} \text{pointed spaces} \\ \text{pointed maps} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{groups} \\ \text{group homomorphisms} \end{array} \right\}.$$

So given

$$f: (X, p) \rightarrow (Y, q)$$

we get

$$f_*: \pi_n(X, p) \rightarrow \pi_n(Y, q)$$

defined by

$$f_*(\gamma) = f \circ \gamma.$$

$n$	1	2	3	4	5	6	7
$\pi_n(S^2)$	0	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/12$	$\mathbf{Z}/15$

**Example 1.2.13** (Homotopy groups of  $S^2$ ).



## 1.3 Homology

Our goal is to construct a functor  $H_*$  from the category of topological spaces and continuous maps to the category of  $\mathbf{Z}$ -modules and  $\mathbf{Z}$ -linear maps. This means to each space  $X$  we associate an abelian group  $H_*(X) = \bigoplus_{n \geq 0} H_n(X)$ , and to each map  $f: X \rightarrow Y$  a function  $f_*: H_n(X) \rightarrow H_n(Y)$  satisfying  $(1_X)_* = 1_{H_n(X)}$  and  $(f \circ g)_* = f_* \circ g_*$ .

Some properties we would like to have for our construction are:

1. Homotopy invariance, if  $f \sim g: X \rightarrow Y$  then  $f_* = g_*$ .
2. The dimension axiom,  $H_n(X) = 0$  for any  $n > \dim X$ .

### 1.3.1 Chain complexes

**Definition 1.3.1** (Chain complex). If  $R$  is a commutative ring then a **chain complex** over  $R$  is a pair  $(C, d)$  satisfying:

1.  $C = \bigoplus_{n \in \mathbf{Z}} C_n$  for  $R$ -modules  $C_n$ .
2.  $d: C \rightarrow C$  where  $d = \bigoplus d_n$  for  $R$ -linear maps  $d_n$ .
3.  $d \circ d = 0$ .

The indexing by  $n$  is called a **grading**. Usually we take  $C_n = 0$  for  $n < 0$ . An element of  $\ker d_n$  is called **closed** or a **cycle**. An element of  $\operatorname{im} d_n$  is called a **boundary**.  $d$  is the **boundary map** or **differential**.

**Definition 1.3.2** (Homology groups). If  $(C, d)$  is a chain complex, its  $n$ **th homology group** is

$$H_n(C, d) = \ker d_n / \operatorname{im} d_{n+1}.$$

If  $x \in \ker d_n$  we write  $[x]$  for its image in  $H_n(C)$ .

**Example 1.3.3.** 1.  $C_0 = C_1 = \mathbf{Z}$ ,  $C_i = 0$  otherwise,

$$0 \rightarrow \mathbf{Z} \xrightarrow{-3} \mathbf{Z} \rightarrow 0.$$

Then  $H_1 = 0$ ,  $H_0 = \mathbf{Z}/3$ .

2.

$$\mathbf{Z} = \langle e \rangle \rightarrow \mathbf{Z}^2 = \langle f_1, f_2 \rangle \rightarrow \mathbf{Z} = \langle g \rangle \rightarrow 0$$

with  $d(e) = f_1 - f_2$ ,  $d(f_1) = d(f_2) = g$ , then  $H_*(C) = 0$  (exercise).

### 1.3.2 The chain complex of a simplex

**Definition 1.3.4** ( $n$ -simplex). The  $n$ -dimensional **simplex**  $\Delta^n$  is

$$\Delta^n = \left\{ (x_0, \dots, x_n) \in \mathbf{R}^n : \sum_i x_i = 1, x_i \geq 0 \forall i \right\}.$$

$\Delta^n$  has **vertices**  $v_0, \dots, v_n$  which are the intersections with the coordinate axes. The  $k$ -dimensional **faces** are in bijection with the  $k+1$  element subsets of  $\{0, \dots, n\}$ .

**Definition 1.3.5** (Simplicial chain complex).  $S_*(\Delta^n)$  is the chain complex with  $S_k(\Delta^n)$  the free  $\mathbf{Z}$ -module generated by the  $k$ -dimensional faces of  $\Delta^n$ . So

$$S_k(\Delta^n) = \langle e_I : I = \{i_0, \dots, i_k : 0 \leq i_0 \leq \dots \leq i_k \leq n\} \rangle.$$

To define  $d$  it suffices to define  $d(e_I)$ , we let

$$d(e_I) = \sum_{j=0}^k (-1)^j e_{i_0, \dots, i_{j-1}, i_{j+1}, \dots, i_k} \in S_{k-1}(\Delta^n).$$

**Example 1.3.6.** 1. For  $\Delta^1$  we have  $d(e_{0,1}) = e_1 - e_0$ .

2. For  $\Delta^2$  we have  $d(e_{0,1,2}) = e_{12} - e_{02} + e_{01}$  and so  $d^2(e_I) = (e_2 - e_1) - (e_2 - e_0) + (e_1 - e_0) = 0$ .

**Lemma 1.3.7.**  $d^2 = 0$

*Proof.* It suffices to show  $d^2(e_I) = 0$  for all  $I$ .

$$\begin{aligned} d^2(e_I) &= d \left( \sum_{j=0}^k (-1)^j e_{i_0, \dots, \hat{i}_j, \dots, i_k} \right) \\ &= \sum_{j=0}^k (-1)^j d(e_{i_0, \dots, \hat{i}_j, \dots, i_k}) \\ &= \sum_{j=0}^k (-1)^j \left( \sum_{l < j} (-1)^l e_{i_0, \dots, \hat{i}_l, \dots, \hat{i}_j, \dots, i_k} + \sum_{l > j} (-1)^{l-1} e_{i_0, \dots, \hat{i}_j, \dots, \hat{i}_l, \dots, i_k} \right) \\ &= \sum_{j=0}^k \left( \sum_{l < j} (-1)^{l+j} e_{I - i_l - i_j} - \sum_{l > j} (-1)^{l+j} e_{I - i_j - i_l} \right) = 0. \end{aligned}$$

□

**Example 1.3.8** (Computing  $H_*(S_*(\Delta^2))$ ).

$$0 \rightarrow \mathbf{Z} = \langle e_{012} \rangle \rightarrow \mathbf{Z}^3 = \langle e_{01}, e_{02}, e_{12} \rangle \rightarrow \mathbf{Z}^3 = \langle e_0, e_1, e_2 \rangle \rightarrow 0$$

with  $d(e_{012}) = e_{12} - e_{02} + e_{01}$ . So  $\ker d_2 = 0 \implies H_2 = 0$ .

$$d(ae_{01} + be_{02} + ce_{12}) = a(e_1 - e_0) + b(e_2 - e_1) + c(e_2 - e_0)$$

so  $ae_{01} + be_{02} + ce_{12} \in \ker d_1 \iff -a - b = 0, a - c = 0, c + b = 0 \iff a = -b = c$  hence  $\ker d_1 = \langle e_{01} - e_{02} + e_{12} \rangle = \text{im } d_2$  and so  $H_1 = 0$ .  $\ker(d_0) = 0$ ,  $\text{im } d_1 = \langle e_1 - e_0, e_2 - e_1, e_2 - e_0 \rangle$  so  $H_0 = \mathbf{Z} = \langle [e_0] \rangle$ .

**Exercise 1.3.9.** Show that  $H_*(S_*(\Delta^n)) = 0$  if  $k \neq 0$  and  $\mathbf{Z}$  if  $k = 0$ .

### 1.3.3 The singular chain complex

**Definition 1.3.10** (Singular chain complex). If  $X$  is a space, the **singular chain complex** of  $X$ ,  $C_*(X)$  is defined by

$$C_n(x) = \langle e_\sigma : \sigma : \Delta^n \rightarrow X \text{ any continuous map} \rangle.$$

Where

$$d(e_\sigma) = \sum_{j=0}^n (-1)^j e_{\sigma \circ F_j} \in C_{n-1}(X)$$

where  $F_j : \Delta^{n-1} \rightarrow \Delta^n$  is given by  $F_j(x_0, \dots, x_{n-1}) = (x_0, \dots, 0, \dots, x_{n-1})$  with the 0 in the  $j$ th place.

#### 1.3.3.1 Homotopy invariance

**Definition 1.3.11** (Chain homotopic maps). Suppose  $\phi, \psi : C_* \rightarrow C'_*$  are chain maps, we say that  $\phi$  are **chain homotopic** if there exists an  $R$ -linear map  $h : C_* \rightarrow C'_{*+1}$  such that  $d' \circ h + h \circ d = \phi - \psi$ . We denote this relation by  $\phi \sim \psi$ .

**Lemma 1.3.12.** *If  $\phi \sim \psi$  then  $\phi_* = \psi_*$ .*

*Proof.*

$$\begin{aligned}\phi_*([x]) - \psi_*([x]) &= [\phi(x) - \psi(x)] \\ &= [d'hx + hdx] = [d'hx] = 0 \in H_*(C').\end{aligned}$$

□

**Theorem 1.3.13.** *Suppose  $f \sim g: X \rightarrow Y$  via  $H$  then  $f_{\#} \sim g_{\#} \implies f_* = g_*$ .*

*Proof.*

□

**Corollary 1.3.14.** *If  $X \sim Y$  then  $H_*(X) \cong H_*(Y)$ .*

**Corollary 1.3.15.** *If  $X$  is contractible then  $H_*(X) \cong H_*(\{p\}) \cong \mathbf{Z}$  if  $*$  = 0, 0 otherwise.*

## 1.4 Homology of a pair

### 1.4.1 Exact sequences

**Definition 1.4.1** (Exact sequence). A sequence

$$\cdots \rightarrow A_{n+1} \rightarrow A_n \rightarrow A_{n-1} \rightarrow \cdots$$

of  $R$ -modules and  $R$ -linear maps is **exact at**  $A_n$  if  $\ker f_n = \operatorname{im} f_{n+1}$ .

A sequence is **exact** if it is exact at all  $A_n$ , then  $(A_*, f)$  is known as a **acyclic** chain complex (the homology is zero).

**Example 1.4.2.** 1.  $0 \rightarrow A \xrightarrow{f} B$  is exact if and only if  $f$  is surjective.

2.  $B \xrightarrow{g} C \rightarrow 0$  is exact if and only if  $g$  is injective.

3.  $0 \rightarrow A \xrightarrow{f} A' \rightarrow 0$  is exact if and only if  $f$  is an isomorphism.

4.  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is exact if and only if  $A \subset B$  and  $C \cong B/A$ .

**Definition 1.4.3** (Short exact sequence). A sequence

$$0 \rightarrow A_* \xrightarrow{i} B_* \xrightarrow{\pi} C_* \rightarrow 0$$

is a **short exact sequence** of chain complexes if

1.  $A_*, B_*, C_*$  are chain complexes.

2.  $i, \pi$  are chain maps.

3.

$$0 \rightarrow A_n \xrightarrow{i} B_n \xrightarrow{\pi} C_n \rightarrow 0$$

is exact for all  $n$ .

**Lemma 1.4.4** (Snake lemma). *If*

$$0 \rightarrow A_* \xrightarrow{i} B_* \xrightarrow{\pi} C_* \rightarrow 0$$

*is a short exact sequence of chain complexes then there is an associated **long exact sequence** of homology groups*



### 1.4.2 Homology of a pair

**Definition 1.4.9** (Homology of a pair).  $H_*(X, A) = H_*(C_*(X, A))$  is the **homology of the pair**  $(X, A)$ .

From this we obtain:

**Definition 1.4.10** (Long exact sequence of a pair). The **long exact sequence of the pair**  $(X, A)$  is

$$\cdots \rightarrow H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \rightarrow \cdots$$

If  $f: (X, A) \rightarrow (Y, B)$  is a map of pairs then we get an **induced map**  $f_\#: C_*(X) \rightarrow C_*(Y)$  defined by

$$(\sigma: \Delta^n \rightarrow A) \mapsto f \circ \sigma.$$

Observe that  $f_\#(C_*(A)) \subset C_*(B)$  and so  $f_\#$  descends to a map

$$f_\#: C_*(X)/C_*(A) \rightarrow C_*(Y)/C_*(B)$$

or equivalently  $f_\#: C_*(X, A) \rightarrow C_*(Y, B)$  this then induces  $f_*: H_*(X, A) \rightarrow H_*(Y, B)$ .

**Proposition 1.4.11** (Homotopy invariance). *If  $f, g: (X, A) \rightarrow (Y, B)$  are homotopic as maps of pairs then  $f_* = g_*: H_*(X, A) \rightarrow H_*(Y, B)$ .*

*Proof.* Let  $H: (X \times I, A \times I) \rightarrow (Y, B)$  be the homotopy,  $H$  induces a chain homotopy  $h: C_*(X) \rightarrow C_{*+1}(Y)$  where  $dh + hd = f_\# - g_\#$ .  $H(A \times I) \subset B$  so  $h(C_*(A)) \subset C_{*+1}(B)$  this implies that  $h$  descends to a map

$$h: C_*(X)/C_*(A) \rightarrow C_{*+1}(Y)/C_{*+1}(B)$$

with  $hd + dh = f_\# - g_\#$  as any relation satisfied will remain satisfied in the quotient. So we have  $h: C_*(X, A) \rightarrow C_{*+1}(Y, A)$  and hence  $f_\#, g_\#: C_*(X, A) \rightarrow C_*(Y, B)$  are chain homotopic.  $\square$

#### 1.4.2.1 Visualising relative homology classes

If  $W^{n+1}$  is a connected oriented compact manifold then we'll show that  $H_{n+1}(W, \partial W) \cong \mathbf{Z} = \langle [W, \partial] \rangle$  (where the  $\partial$  notation means relative to boundary). So given  $f: (W, \partial) \rightarrow (X, A)$  we get  $f_*([W, \partial W]) \in H_{n+1}(X, A)$ .

**Example 1.4.12.** Let  $X = \mathbf{R}^3$  and  $A = S^1$  then  $W = D^2$  defines a class in  $H_2(\mathbf{R}^3, S^1)$  (the boundary of  $W$  lies inside of  $A$ ).

### 1.4.3 Good pairs

**Definition 1.4.13** (Good pair).  $(X, A)$  is a **good pair** if

1.  $\exists U \subset X$  open and  $A \subset U$
2.  $\exists \pi: U \rightarrow A$  with  $\pi|_A = \text{id}_A$ .
3.  $\pi \sim 1_{(U, A)}$  as maps of pairs from  $(U, A)$  to itself.  
(i.e.  $A$  is a deformation retract of  $U$ ).

**Example 1.4.14** (Good pairs). 1. (smooth manifold, closed submanifold).

2. (simplicial complex, subcomplex).

**Example 1.4.15** (Not good pairs). 1.  $(\mathbf{R}, \mathbf{Q})$ .

2. Letting  $H \subset \mathbf{R}^2$  be the Hawaiian earring then  $(\mathbf{R}^2, H)$  is not a good pair.

**Theorem 1.4.16.** *Take  $A \subset X$  and let  $\pi$  be the natural map  $\pi: (X, A) \rightarrow (X/A, A/A) \cong (X/A, *)$ . Then if  $(X, A)$  is a good pair the induced map  $\pi_*: H_*(X, A) \rightarrow H_*(X/A, *)$  is an isomorphism.*

*Proof.* Postponed. □

**Exercise 1.4.17.** The composite map  $\phi$  in

$$\tilde{H}_*(X) \rightarrow H_*(X) \rightarrow H_*(X, *)$$

is an isomorphism. i.e.

$$H_*(X, *) = \begin{cases} H_*(X), & * > 0, \\ H_0(X)/\mathbf{Z}, & * \leq 0. \end{cases}$$

**Proposition 1.4.18.**

$$\tilde{H}_*(S^n) = \begin{cases} \mathbf{Z}, & * = n, \\ 0, & * \neq n. \end{cases}$$

*Proof.* We proceed by induction on  $n$ . For  $n = 0$  we have  $S^0 = \{p_+, p_-\}$  implying

$$H_*(S^0) = H_*(p_+) \oplus H_*(p_-) = \begin{cases} \mathbf{Z}^2, & * = 0, \\ 0, & * \neq 0, \end{cases}$$

giving

$$\tilde{H}_*(S^0) = \begin{cases} \mathbf{Z}, & * = 0, \\ 0, & * \neq 0. \end{cases}$$

Now consider the long exact sequence of the pair  $(D^n, S^{n-1})$

$$H_*(D^n) \rightarrow H_*(D^n, S^{n-1}) \rightarrow H_{*-1}(S^{n-1}) \xrightarrow{\phi} H_{*-1}(D^n),$$

we can break this up using the kernel and cokernel to get

$$0 \rightarrow \text{coker } \phi_m \rightarrow H_m(D^n, S^{n-1}) \rightarrow \ker \phi_{m-1} \rightarrow 0. \quad (1.4.1)$$

$D^n$  is contractible and so we get

$$H_*(D^n) = \begin{cases} \mathbf{Z}, & * = 0, \\ 0, & * \neq 0. \end{cases}$$

We have  $\phi: H_0(S^{n-1}) \rightarrow H_0(D^n) \cong \mathbf{Z} = \langle e_p \rangle$  given by  $\phi(e_p) = e_p$ . We see that  $\ker \phi = \tilde{H}_*(S^{n-1})$  and  $\text{coker } \phi = 0$ . Now by looking at (1.4.1) we see that

$$0 \rightarrow \tilde{H}_*(S^n) = H_*(S^n, *) = H_*(D^n, S^{n-1}) \rightarrow \ker \phi = \tilde{H}_*(S^{n-1}) \rightarrow 0$$

is exact giving  $\tilde{H}_*(S^n) \cong \tilde{H}_*(S^{n-1})$ . The claim then follows by induction. □

**Corollary 1.4.19.**

$$S^n \sim S^m \implies m = n.$$

**Example 1.4.20** (Chains generating  $H_n(S^n, *)$ ). 1. Choose  $f: \Delta^n \rightarrow \Delta^n$  a homeomorphism and let  $e: \Delta^n \rightarrow \Delta^n$  be the identity map. Then  $a_{n-1} = f_{\#}(de) \in C_*(S^{n-1})$  and we have  $da_{n-1} = df_{\#}(de) = f_{\#}(d^2e) = 0$  and so  $a_{n-1}$  is closed.

2. Choose  $g: (D^n/S^{n-1}, S^{n-1}/S^{n-1}) \rightarrow (S^n, *)$ . Now let  $b_n = g_\#(f_\#(e)) \in C_n(S^n, *)$ . Then  $db_n = g_\#(a_{n-1}) \in C_*(*)$  implies  $b_n$  is closed in  $C_*(S^n, *)$ . For example if  $n = 2$  we are crushing the boundary of the 2-simplex to a point.

**Proposition 1.4.21.**  $[a_n]$  generates  $\tilde{H}_n(S^n) = \mathbf{Z}$  (statement  $(A_n)$ ).  
 $[b_n]$  generates  $\tilde{H}_n(S^n, *) = \mathbf{Z}$  (statement  $(B_n)$ ).

*Proof.* Statement  $(A_0)$ :  $S^0 = \{p_+, p_-\}$ ,  $de = e_{p_+} - e_{p_-}$  so  $de$  generates  $\tilde{H}_0(S^0)$ . We'll show that  $A_{n-1} \implies B_n$ . Here  $\partial: H_n(S^n, *) = H_n(D^n, S^{n-1}) \xrightarrow{\sim} \tilde{H}_{n-1}(S^{n-1})$  (by 1.4.18). So it suffices to check that  $\partial[b_n] = [a_{n-1}]$   $\square$

**Definition 1.4.22** (Wedge product). If  $(X_i, p_i)$  are pointed spaces the **wedge product**

$$\bigvee_{i \in I} (X_i, p_i) \text{ is } \coprod_{i \in I} X_i / \{p_i : i \in I\}.$$

If  $X_i$  is such that for any  $p, q \in X_i$  there exists a homeomorphism  $f: X_i \rightarrow X_i$  with  $f(p) = q$  we can drop  $p_i$  from the notation (for example in the case of  $X_i$  a connected manifold we can do this).

**Example 1.4.23.**

**Corollary 1.4.24.** If  $(X_i, p_i)$  are good pairs then

$$\tilde{H}_* \left( \bigvee_i (X_i, p_i) \right) = \bigoplus_i \tilde{H}_*(X_i).$$

*Proof.*

$$\begin{aligned} \tilde{H}_* \left( \bigvee_{i \in I} X_i \right) &\cong H_* \left( \bigvee_{i \in I} X_i, p \right) \cong H_* \left( \coprod_{i \in I} X_i, \{p_i : i \in I\} \right) \\ &\cong \bigoplus_{i \in I} H_*(X_i, p_i) \cong \bigoplus_{i \in I} \tilde{H}_*(X_i). \end{aligned}$$

$\square$

**Example 1.4.25.**

$$H_*(S^1 \vee S^2) = \begin{cases} \mathbf{Z}, & * = 0, 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

## 1.5 Subdivision and Excision

**Definition 1.5.1.** If  $\mathcal{U} = \{U_i\}_{i \in I}$  is an open cover of  $X$ , let

$$C_n^{\mathcal{U}}(X) = \langle e_\sigma : \sigma : \Delta^n \rightarrow X, \text{ im } \sigma \subset U_i \text{ for some } i \rangle \subset C_n(X).$$

Observe that  $\text{im } \sigma \subset U_i$  implies  $\text{im } \sigma \circ F_j \subset U_i$  and so  $C_*^{\mathcal{U}}$  is a subcomplex of  $C_*$ . Let  $H_*^{\mathcal{U}}$  be the homology of this complex, then we have a map

$$i: C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X).$$

**Lemma 1.5.2** (Subdivision).  $C_*: H_*^{\mathcal{U}}(X) \rightarrow H_*(X)$  is an isomorphism.

## 1.6 Degree and Orientations

## 1.7 Cell Complexes

**Definition 1.7.1** (Attaching of cells). If  $f: S^{n-1} \rightarrow X$  then

$$X \cup_f D^n = X \amalg D^n / \sim$$

is the space obtained by **attaching** an  $n$ -dimensional cell to  $X$  via the map  $f$ .

**Example 1.7.2.** If  $X = \{p\}$  and  $f: S^{n-1} \rightarrow X$  then

$$X \cup_f D^n \cong D^n / S^{n-1} \cong S^n.$$

**Definition 1.7.3** (Finite cell complex). A 0-dimensional **finite cell complex** is a finite disjoint union of points.

A  $k$ -dimensional finite cell complex is a space obtained by attaching finitely many  $k$ -cells to a  $(k-1)$ -dimensional finite cell complex.

**Example 1.7.4.**

**Example 1.7.5.** If a finite cell complex  $X$  has one 0-cell and one  $n$ -cell then  $X \cong S^n$ . Similarly if  $X$  has one 0-cell and  $kn$ -cells then  $X \cong \bigvee_{i=1}^k S^n$ .

**Example 1.7.6.**  $T^2$  is a finite cell complex with one 0-cell, two 1-cells and one 2-cell.

**Example 1.7.7.** Any simplicial complex is a finite cell complex and any closed manifold can be given the structure of a finite cell complex.



## Chapter 2

# Cohomology and Products

### 2.1 Homology with Coefficients and Cohomology

#### 2.1.1 Hom and $\otimes$ for modules

**Definition 2.1.1** (Tensor product of  $R$ -modules). Let  $M, N$  be  $R$ -modules. Then the **tensor product**  $M \otimes N$  is the  $R$ -modules generated by all pairs  $m \otimes n$  for  $m \in M, n \in N$  modulo the relations:

1.  $(m_1 + m_2) \otimes (n_1 + n_2) = \sum m_i \otimes n_j$ .
2.  $r(m \otimes n) = (rm) \otimes n = m \otimes (rn)$ .

We have the following properties of this definition:

1.  $(M_1 \oplus M_2) \otimes (N_1 \oplus N_2) = \bigoplus M_i \otimes N_j$ .
2.  $M \otimes N \cong N \otimes M$ .
3.  $M \otimes R = M$

**Example 2.1.2** (Tensor products). 1.  $R^n \otimes R^m \cong R^{mn}$ .

2. Letting  $R = \mathbf{Z}, \mathbf{Q} \otimes \mathbf{Z}/a \cong 0$ .
3.  $\mathbf{Z}/a \otimes \mathbf{Z}/b \cong \mathbf{Z}/(a, b)$ .

If  $f: M_1 \rightarrow M_2$  and  $g: N_1 \rightarrow N_2$  then there is a map

$$\begin{aligned} f \otimes g: M_1 \otimes N_1 &\rightarrow M_2 \otimes N_2, \\ m \otimes n &\mapsto f(m) \otimes g(n). \end{aligned}$$

**Example 2.1.3.** If  $f: R^n \rightarrow R^m$  is given by multiplication by  $A \in \text{Mat}_{n \times m}(R)$  then  $f \otimes 1_M: M^n \rightarrow M^m$  is given by multiplication by  $A$ .

**Definition 2.1.4** (Hom).

$$\text{Hom}(M, N) = \{f: M \rightarrow N : f \text{ is } R\text{-linear}\}$$

is an  $R$ -module, via  $(f + rg)(m) = f(m) + rg(m)$ .

From this definition we see that

1.  $\text{Hom}(\bigoplus M_i, \bigoplus N_j) \cong \bigoplus_{i,j} \text{Hom}(M_i, N_j)$ .
2.  $\text{Hom}(R, M) \cong M$ .

Note however that we do not have  $\text{Hom}(M, N) = \text{Hom}(N, M)$  as for example  $\text{Hom}(\mathbf{Z}/2, \mathbf{Z}) = 0$  but  $\text{Hom}(\mathbf{Z}, \mathbf{Z}/2) = \mathbf{Z}/2$ .

**Definition 2.1.5** (Dual module). Given an  $R$ -module  $M$  the **dual** of  $M$  is  $M^* = \text{Hom}(M, R)$ .

Now if we have  $f: M \rightarrow N$  we get a map

$$f^*: \text{Hom}(N, O) \rightarrow \text{Hom}(M, O)$$

given by  $f^*g = g \circ f$ .

**Example 2.1.6.** If  $f: R^n \rightarrow R^m$  is multiplication by  $A$  then

$$f^*: \text{Hom}(R^m, O) \cong O^m \rightarrow \text{Hom}(R^n, O) \cong O^n$$

is multiplication by  $A^\top$ .

### 2.1.2 Hom and $\otimes$ for chain complexes

If  $(C, d)$  is a chain complex defined over  $R$  then so are  $(C_* \otimes M, d \otimes 1_M) = C_* \otimes M$  and  $(\text{Hom}(C_*, M), d^*) = \text{Hom}(C_*, M)$ .

## 2.2 Universal Coefficient Theorems

### 2.3 Products

**Definition 2.3.1** (Product of chain complexes). If  $A, B$  are chain complexes the  $C = A \otimes B$  is the chain complex with

$$C_i = \bigoplus_{j+k=i} A_j \otimes B_k$$

and

$$d(a \otimes b) = (da) \otimes b + (-1)^{|a|} a \otimes (db)$$

where  $|a| = j$  if  $a \in A_j$ .

The following theorem is true but we will not prove it.

**Theorem 2.3.2** (Eilenberg-Zilber).

$$C_*(X \times Y) \sim C_*(X) \otimes C_*(Y).$$

Given  $H_*(A)$  and  $H_*(B)$  how can we compute  $H_*(A \otimes B)$ .

**Lemma 2.3.3.** Suppose  $A \sim A'$ . Then  $A \otimes B \sim A' \otimes B$ .

*Proof.* Easy. □

**Corollary 2.3.4.** Suppose  $A, B$  are finitely generated chain complexes, defined over a field. Then

$$H_*(A \otimes B) = \bigoplus_{j+k=i} H_j(A) \otimes H_k(B)$$

i.e.

$$H_*(A \otimes B) \cong H_*(A) \otimes H_*(B).$$

*Proof.*  $A$  is a chain complex over a field, so  $A \sim A'$  where  $A'_i = H_i(A)$  and  $d \equiv 0$  on  $A'$ . Similarly for  $B$ . Then

$$H_*(A \otimes B) \cong H_*(A' \otimes B') \cong A' \otimes B' = H_*(A) \otimes H_*(B).$$

□

**Corollary 2.3.5.** *If  $k$  is a field and  $X, Y$  are finite chain complexes then*

$$H_*(X \times Y; k) \cong H_*(X; k) \otimes_k H_*(Y; k).$$

**Definition 2.3.6** (Poincare polynomial). The **Poincare polynomial** of  $X$  with coefficients in  $k$  is

$$P_k(X) = \sum_{i \geq 0} \dim H_i(X; k) t^i.$$

**Example 2.3.7.**

$$P_{\mathbf{Z}/2}(\mathbf{RP}^2) = 1 + t + t^2.$$

2.3.5 tells us that

$$P_k(X \times Y) = P_k(X)P_k(Y).$$

**Example 2.3.8.**

$$P_{\mathbf{Z}/2}(\mathbf{RP}^3 \times \mathbf{RP}^2) = (1 + t + t^2)(1 + t + t^2 + t^3).$$

For now we will suppose that  $A, B$  are finitely generated chain complexes over a PID  $R$ .

**Lemma 2.3.9.** *If  $H_*(A)$  is a free  $R$ -module then  $A \sim A'$  where  $A'_i = H_i(A)$  and  $d \equiv 0$  on  $A'$ .*

*Proof.*  $A$  is the direct sum of short and stupid complexes.  $H_*(A)$  being torsion free implies that every short complex is of the form

$$C: R \xrightleftharpoons[\cdot a^{-1}]{\cdot a} R$$

with  $a \in R^\times$  such a  $C \sim 0$  (same proof as for a field). □

**Corollary 2.3.10.** *If  $H_*(A)$  is free then  $H_*(A \otimes B) \cong H_*(A) \otimes H_*(B)$ .*

*Proof.*  $A \otimes B \cong A' \otimes B$  where  $A'$  is as in 2.3.9.  $A' \otimes B$  is a direct sum of copies of  $B$ , one for each generator of  $A'$ . This gives that  $H_*(A' \otimes B)$  is a direct sum of copies of  $H_*(B)$ , one for each generator of  $H_*(A)$  which implies that

$$H_*(A \otimes B) \cong H_*(A) \otimes H_*(B)$$

(since  $H_*(A)$  is free).

$$R^n \otimes H_*(B) \cong (H_*(B))^n.$$

□

**Corollary 2.3.11.** *If  $X$  and  $Y$  are finite cell complexes and  $H_*(X)$  is free over  $\mathbf{Z}$ , then  $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$ .*

Fact: If  $C_1$  is a free resolution of  $M$  and  $C_2$  is a free resolution of  $N$ . Then

$$\mathrm{Tor}_i(M, N) \cong H_i(C_1 \otimes C_2) \cong H_i(C_1 \otimes N) \cong H_i(M \otimes C_2)$$

.

**Theorem 2.3.12** (Kunneth formula). *If  $A, B$  are free (finitely generated) chain complexes over a PID  $R$  then*

$$H_i(A \otimes B) \cong \bigoplus_{j+k=i} \text{Tor}_0(H_j(A), H_k(B)) \oplus \bigoplus_{j+k=i-1} \text{Tor}_1(H_j(A), H_k(B)).$$

*Proof.* It suffices to check the statement where  $A, B$  are short and/or stupid. We just did the case where both are short. The case where one or more is stupid was covered in the statement that

$$H_*(A \otimes B) \cong H_*(A) \otimes H_*(B)$$

if  $H_*(A)$  is free. □

**Corollary 2.3.13.** *We can compute  $H_*(X \times Y)$  from  $H_*(X)$  and  $H_*(Y)$ .*

## 2.4 Cup product

Motivation: if  $k$  is a field

$$\begin{aligned} H^*(X \times X; k) &\cong (H_*(X \times X; k))^* \\ &\cong (H_*(X; k) \otimes H_*(X; k))^* \\ &\cong H^*(X; k) \otimes H^*(X; k) \end{aligned}$$

**Definition 2.4.1** (Cup product). Suppose  $a \in C^i(X)$  and  $b \in C^j(X)$ . If  $\sigma: \Delta^{i+j} \rightarrow X$  then we define  $a \smile b \in C^{i+j}(X)$  by

$$(a \smile b)(\sigma) = a(\sigma \circ F_i^+) b(\sigma \circ F_j^-)$$

where

$$\begin{aligned} F_i^+ : \Delta^i &\rightarrow \Delta^{i+j} \\ (x_0, \dots, x_i) &\mapsto (x_0, \dots, x_i, 0, \dots, 0) \end{aligned}$$

and

$$\begin{aligned} F_j^- : \Delta^j &\rightarrow \Delta^{i+j} \\ (x_0, \dots, x_j) &\mapsto (0, \dots, 0, x_0, \dots, x_j). \end{aligned}$$

**Lemma 2.4.2.**

$$d(a \smile b) = da \smile b + (-1)^{|a|} a \smile db.$$

*Proof.* Denote by  $[v_0, \dots, v_k]$  the singular  $k$ -simplex that sends

$$\Delta^k \rightarrow \Delta^{i+j} \xrightarrow{\sigma} X$$

vertices of  $\Delta^k \rightarrow$  vertices  $v_0, \dots, v_k$  of  $\Delta^{i+j}$ .

Then taking  $\sigma: \Delta^{i+j+1} \rightarrow X$

$$\begin{aligned}
d(a \smile b)(\sigma) &= (a \smile b)(d\sigma) \\
&= \sum_{k=0}^{i+j+1} (-1)^k a \smile b([v_0, \dots, \hat{v}_k, \dots, v_n]) \\
&= \sum_{k \leq i} (-1)^k a([v_0, \dots, \hat{v}_k, \dots, v_{i+1}]) b([v_{i+1}, \dots, v_n]) \\
&\quad + \sum_{k > i} (-1)^k a([v_0, \dots, v_i]) b([v_i, \dots, \hat{v}_k, \dots, v_n]) \\
&= \sum_{k \leq i+1} (-1)^k a([v_0, \dots, \hat{v}_k, \dots, v_{i+1}]) b([v_{i+1}, \dots, v_n]) \\
&\quad + \sum_{k \geq i} (-1)^k a([v_0, \dots, v_i]) b([v_i, \dots, \hat{v}_k, \dots, v_n]) \\
&= a(d([v_0, \dots, v_{i+1}])) b([v_{i+1}, \dots, v_n]) \\
&\quad + (-1)^i a([v_0, \dots, v_i]) b(d([v_i, \dots, v_n])) \\
&= da \smile b([v_0, \dots, v_n]) + (-1)^i a \smile db([v_0, \dots, v_n]).
\end{aligned}$$

□

**Corollary 2.4.3.** *There is a well-defined cup product*

$$\begin{aligned}
H^i(X) \times H^j(X) &\rightarrow H^{i+j}(X), \\
[a] \times [b] &\mapsto [a \smile b].
\end{aligned}$$

*Proof.* Must check

1.  $da = c, db = 0 \implies d(a \smile b) = da \smile b \pm a \smile db = 0$  so ok.
2.  $(a + dc) \smile b = a \smile b + dc \smile b = a \smile b + d(c \smile b)$  implies  $[(a + dc) \smile b] = [a \smile b]$  and so  $\smile$  is well defined.

□

**Proposition 2.4.4** (Properties of  $\smile$ ). *Let  $\alpha, \beta, \gamma \in H^*(X)$  then*

1. (Associative)  $(\alpha \smile \beta) \smile \gamma = \alpha \smile (\beta \smile \gamma)$ .
2. (Graded commutative)  $a \smile b = (-1)^{|\alpha||\beta|} \beta \smile \alpha$ .
3. (Identity) *There is a class  $1 \in H^0(X)$  such that  $1 \smile \alpha = \alpha \smile 1 = \alpha$  for all  $\alpha \in H^*(X)$ .*
4. (Functorial) *If  $f: X \rightarrow Y$ ,  $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$ .*

*Proof.* 1. If  $a \in C^i(X), b \in C^j(X), c \in C^k(X), \sigma: \Delta^{i+j+k} \rightarrow X$  then

$$\begin{aligned}
((a \smile b) \smile c)(\sigma) &= a(\sigma \circ F_i^+) b(\sigma \circ F_j^{\text{mid}}) c(\sigma \circ F_k^-) \\
&= (a \smile (b \smile c))(\sigma).
\end{aligned}$$

Where  $F_i^+(X) = (X^i \ 0), F_k^-(X) = (0^{i+j} \ X^{i+j+k}), F_j^{\text{mid}}(X) = (0^i \ X^{i+j} \ 0)$ .

2. We'll do later.

3. Let  $e \in C^0(X)$  be given by

$$e(e_p) = 1 \text{ for all } p \in X$$

If  $\gamma: [0, 1] \rightarrow X$  is a 1-chain then

$$de(e_\gamma) = e(de_\gamma) = e(e_{\gamma(1)} - e_{\gamma(0)}) = 1 - 1 = 0.$$

Easy to see  $e \smile a = a \smile e = a$  for all  $a \in C^*(X)$  then  $1 = [e] \in H^0(X)$ .

4.

$$\begin{aligned} f^\#(a \smile b)(\sigma) &= (a \smile b)(f \circ \sigma) \\ &= a(f \circ \sigma \circ F_i^+)b(f \circ \sigma \circ F_j^-) \\ &= f^\#(a)(\sigma \circ F_i^+)f^\#(b)(\sigma \circ F_j^-) \\ &= [f^\#(a) \smile f^\#(b)](\sigma). \end{aligned}$$

□

### 2.4.1 Digression: De Rahm Cohomology

Let  $M$  be a smooth manifold and  $\omega \in \Omega^k(M)$ . If  $\sigma: \Delta^k \rightarrow M$  is a smooth we define  $\omega(\sigma) = \int_\sigma \omega$ . Stokes' theorem then gives us that  $d\omega(\sigma) = \omega(d\sigma)$ . Where on the left  $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$  is the differential map and on the right  $d$  is the boundary map in the sense of (smooth) singular chains. I.e. the two  $d$ 's are dual to each other on  $C_*^{\text{smooth}}(M)$ .

**Theorem 2.4.5** (de Rahm).

$$H_k(\Omega^*(M), d) \cong H^k(M; \mathbf{R}).$$

Under this correspondence

$$[\omega] \smile [\nu] = [\omega \wedge \nu].$$

**Example 2.4.6.** 1.  $H^*(S^n) = \langle 1, X_n \rangle$  where  $\langle X_n \rangle = H^n(S^n)$  and we have

$$\begin{aligned} 1 \smile X_n &= X_n, \\ X_n \smile 1 &= X_n, \\ 1 \smile 1 &= 1, \\ X_n \smile X_n &= 0 \in H^{2n}(S^n) = 0. \end{aligned}$$

2. For arbitrary  $X, Y$

$$C_*(X \amalg Y) \cong C_*(X) \oplus C_*(Y)$$

which implies

$$C^*(X \amalg Y) \cong C^*(X) \oplus C^*(Y)$$

$$\implies H^*(X \amalg Y) \cong H^*(X) \oplus H^*(Y) \text{ as rings}$$

with

$$(\alpha_1 \oplus \alpha_2) \smile (\beta_1 \oplus \beta_2) = (\alpha_1 \smile \beta_1) \oplus (\alpha_2 \smile \beta_2).$$

3. Considering  $X \vee Y$  we have a map  $p: X \amalg Y \rightarrow X \vee Y$  and so Mayer-Vietoris for  $X \vee Y$  gives

$$\cdots \rightarrow H^*(X \vee Y) \xrightarrow{p^*} H^*(X) \oplus H^*(Y) \xrightarrow{i^*} H^*(\text{pt}) \rightarrow \cdots$$

Now assuming  $X$  and  $Y$  are path connected  $p^*$  is injective and an isomorphism in grading  $> 0$ . In grading 0 we have

$$H^0(X \vee Y) \rightarrow H^0(X) \oplus H^0(Y) \rightarrow H^0(\text{pt})$$

with  $1 \mapsto 1 \oplus 1$  and  $a \oplus b \mapsto a - b$ . And so as a ring we have

$$H^*(X \vee Y) \cong \ker i^* \subset H^*(X) \oplus H^*(Y)$$

where elements in grading  $> 0$  look like  $(\alpha, \beta)$  and in grading 0 look like  $(n, n) \in \mathbf{Z} \times \mathbf{Z}$ .

4.  $S^2 \vee S^4$ ,  $H^*(S^2 \vee S^4) = \langle 1, X_2, X_4 \rangle$  with  $X_2^2 = 0$  since  $X_2^2 = 0$  in  $H^*(S^2)$ .

### 2.4.2 Exterior Product

Let  $\pi_1: X \times Y \rightarrow X$  and  $\pi_2: X \times Y \rightarrow Y$ .

**Definition 2.4.7** (Exterior product). If  $\alpha \in H^i(X)$ ,  $\beta \in H^j(X)$  then their **exterior product** is  $\alpha \times \beta = \pi_1^*(\alpha) \smile \pi_2^*(\beta) \in H^{i+j}(X \times Y)$ .

**Proposition 2.4.8.** 1.  $\alpha \smile \beta = \Delta^*(\alpha \times \beta)$  where  $\Delta: X \rightarrow X \times X$  is the diagonal map.

2.  $(\alpha \times \beta) \smile (\alpha' \times \beta') = (-1)^{|\beta||\alpha'|}(\alpha \smile \alpha') \times (\beta \smile \beta')$ .

*Proof.* 1.

$$\Delta^*(\alpha \times \beta) = \Delta^*(\pi_1^*(\alpha) \smile \pi_2^*(\beta))$$

□





2.5 Notation

Symbol	Description	Page
		<a href="#">1</a>