

Part III Algebraic Topology 2014

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Chapter 1

Homology

1.1 Introduction

These are lecture notes for the 2014 Part III Algebraic Topology course taught by Dr. Jacob Rasmussen.

The recommended books are:

- [Algebraic Topology](#) - Allen Hatcher,
- Homology Theory - James W. Vick,
- Differential Forms in Algebraic Topology - Raoul Bott and Loring W. Tu.

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1.2 Homotopy

1.2.1 Homotopies

Definition 1.2.1 (Homotopic maps). Maps $f_0, f_1: X \rightarrow Y$ are said to be **homotopic** if there is a continuous map $F: X \times I \rightarrow Y$ such that

$$F(x, 0) = f_0(x) \text{ and } F(x, 1) = f_1(x) \quad \forall x \in X.$$

We let $\text{Map}(X, Y) = \{f: X \rightarrow Y \text{ continuous}\}$. Then letting $f_t(x) = F(x, t)$ in the above definition we see that f_t is a path from f_0 to f_1 in $\text{Map}(X, Y)$.

Example 1.2.2. 1. $X = Y = \mathbf{R}^n$, $f_0(\bar{x}) = \bar{0}$ and $f_1(\bar{x}) = \bar{x}$ are homotopic via $f_t(\bar{x}) = t\bar{x}$.

2. $S^1 = \{z \in \mathbf{C} : |z| = 1\}$ then

3. $S^n = \{\bar{x} \in \mathbf{R}^n : |\bar{x}| = 1\}$

Lemma 1.2.3. *Homotopy is an equivalence relation on $\text{Map}(X, Y)$.*

Lemma 1.2.4. *If $f_0 \sim f_1: X \rightarrow Y$ and $g_0 \sim g_1: Y \rightarrow Z$ then $g_0 \circ f_0 \sim g_1 \circ f_1$.*

Corollary 1.2.5. *For any space X the set $[X, \mathbf{R}^n]$ has one element.*

Proof. Define $0_X: X \rightarrow \mathbf{R}^n$ by $0_X(x) = 0 \in \mathbf{R}^n$ for any $x \in X$. □

Definition 1.2.6 (Contractible space). X is **contractible** if 1_X is homotopic to a constant map.

Proposition 1.2.7. Y is contractible $\iff [X, Y]$ has one element for any space X .

Proof. (\Rightarrow) as in corollary. (\Leftarrow) $[X, Y]$ has one element so $1_Y \sim$ a constant map. \square

Given a space X how can we tell if X is contractible? If X is contractible then it must be path connected for one.

Proof. Contractible implies that $[S^0, X]$ has one element and so $f: S^0 \rightarrow X$ extends to D^1 , and therefore X is path connected. \square

Similarly if $[S^1, X]$ has more than one element then X is not contractible.

Definition 1.2.8 (Simply connected). We say X is **simply connected** if $[S^1, X]$ has only one element.

We say two spaces X and Y are *homotopy equivalent* if there exists $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $g \circ f \sim 1_X$ and $f \circ g \sim 1_Y$.

Example 1.2.9. X is contractible if and only if $X \sim \{p\}$.

Proof. X contractible $\implies 1_X \sim c$, a constant map. Choose $f: X \rightarrow \{p\}$, $f(x) = p$ and $g: \{p\} \rightarrow X$, $g(p) = c$. Then $g \circ f = c \sim 1$ and $f \circ g = 1_{\{p\}}$. Converse: exercise. \square

Exercise 1.2.10. If $X_0 \sim X_1$ and $Y_0 \sim Y_1$ then $[X_0, Y_0]$ and $[X_1, Y_1]$ are in bijection.

Given X and Y how can we determine if $X \sim Y$? How do we determine $[X, Y]$? For example is $S^n \sim S^m$.

1.2.2 Homotopy groups

Definition 1.2.11 (Map of pairs). A **map of pairs** $f: (X, A) \rightarrow (Y, B)$ is a map $f: X \rightarrow Y$ with sets $A \subset X$ and $B \subset Y$ such that $f(A) \subset B$.

If we have maps of pairs $f_0, f_1: (X, A) \rightarrow (Y, B)$ then we write $f_0 \sim f_1$ if there exists $F: (X \times I, A \times I) \rightarrow (Y, B)$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$.

Definition 1.2.12 (Homotopy groups). If $*$ $\in X$ then the **n th homotopy group** is

$$\pi_n(X, *) = [(D^n, S^{n-1}) \rightarrow (X, \{*\})].$$

We now note several properties of this definition:

1. $\pi_0(X, *)$ = set of path components of X .
2. $\pi_1(X, *)$ is a group. $\pi_n(X, *)$ is an abelian group.
3. π_n is a functor

$$\left\{ \begin{array}{l} \text{pointed spaces} \\ \text{pointed maps} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{groups} \\ \text{group homomorphisms} \end{array} \right\}.$$

So given

$$f: (X, p) \rightarrow (Y, q)$$

we get

$$f_*: \pi_n(X, p) \rightarrow \pi_n(Y, q)$$

defined by

$$f_*(\gamma) = f \circ \gamma.$$

$$\begin{array}{cccccccc} n & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \pi_n(S^2) & 0 & \mathbf{Z} & \mathbf{Z} & \mathbf{Z}/2 & \mathbf{Z}/2 & \mathbf{Z}/12 & \mathbf{Z}/15 \end{array}$$

Example 1.2.13 (Homotopy groups of S^2).

1.3 Homology

Our goal is to construct a functor H_* from the category of topological spaces and continuous maps to the category of \mathbf{Z} -modules and \mathbf{Z} -linear maps. This means to each space X we associate an abelian group $H_*(X) = \bigoplus_{n \geq 0} H_n(X)$, and to each map $f: X \rightarrow Y$ a function $f_*: H_n(X) \rightarrow H_n(Y)$ satisfying $(1_X)_* = 1_{H_n(X)}$ and $(f \circ g)_* = f_* \circ g_*$.

Some properties we would like to have for our construction are:

1. Homotopy invariance, if $f \sim g: X \rightarrow Y$ then $f_* = g_*$.
2. The dimension axiom, $H_n(X) = 0$ for any $n > \dim X$.

1.3.1 Chain complexes

Definition 1.3.1 (Chain complex). If R is a commutative ring then a **chain complex** over R is a pair (C, d) satisfying:

1. $C = \bigoplus_{n \in \mathbf{Z}} C_n$ for R -modules C_n .
2. $d: C \rightarrow C$ where $d = \bigoplus d_n$ for R -linear maps d_n .
3. $d \circ d = 0$.

The indexing by n is called a **grading**. Usually we take $C_n = 0$ for $n < 0$. An element of $\ker d_n$ is called **closed** or a **cycle**. An element of $\operatorname{im} d_n$ is called a **boundary**. d is the **boundary map** or **differential**.

Definition 1.3.2 (Homology groups). If (C, d) is a chain complex, its n **th homology group** is

$$H_n(C, d) = \ker d_n / \operatorname{im} d_{n+1}.$$

If $x \in \ker d_n$ we write $[x]$ for its image in $H_n(C)$.

Example 1.3.3. 1. $C_0 = C_1 = \mathbf{Z}$, $C_i = 0$ otherwise,

$$0 \rightarrow \mathbf{Z} \xrightarrow{\cdot 3} \mathbf{Z} \rightarrow 0.$$

Then $H_1 = 0$, $H_0 = \mathbf{Z}/3$.

2.

$$\mathbf{Z} = \langle e \rangle \rightarrow \mathbf{Z}^2 = \langle f_1, f_2 \rangle \rightarrow \mathbf{Z} = \langle g \rangle \rightarrow 0$$

with $d(e) = f_1 - f_2$, $d(f_1) = d(f_2) = g$, then $H_*(C) = 0$ (exercise).

1.3.2 The chain complex of a simplex

Definition 1.3.4 (n -simplex). The n -dimensional **simplex** Δ^n is

$$\Delta^n = \left\{ (x_0, \dots, x_n) \in \mathbf{R}^n : \sum_i x_i = 1, x_i \geq 0 \forall i \right\}.$$

Δ^n has **vertices** v_0, \dots, v_n which are the intersections with the coordinate axes. The k -dimensional **faces** are in bijection with the $k+1$ element subsets of $\{0, \dots, n\}$.

Definition 1.3.5 (Simplicial chain complex). $S_*(\Delta^n)$ is the chain complex with $S_k(\Delta^n)$ the free \mathbf{Z} -module generated by the k -dimensional faces of Δ^n . So

$$S_k(\Delta^n) = \langle e_I : I = \{i_0, \dots, i_k : 0 \leq i_0 \leq \dots \leq i_k \leq n\} \rangle.$$

To define d it suffices to define $d(e_I)$, we let

$$d(e_I) = \sum_{j=0}^k (-1)^j e_{i_0, \dots, i_{j-1}, i_{j+1}, \dots, i_k} \in S_{k-1}(\Delta^n).$$

Example 1.3.6. 1. For Δ^1 we have $d(e_{0,1}) = e_1 - e_0$.

2. For Δ^2 we have $d(e_{0,1,2}) = e_{12} - e_{02} + e_{01}$ and so $d^2(e_I) = (e_2 - e_1) - (e_2 - e_0) + (e_1 - e_0) = 0$.

Lemma 1.3.7. $d^2 = 0$

Proof. It suffices to show $d^2(e_I) = 0$ for all I .

$$\begin{aligned} d^2(e_I) &= d \left(\sum_{j=0}^k (-1)^j e_{i_0, \dots, \hat{i}_j, \dots, i_k} \right) \\ &= \sum_{j=0}^k (-1)^j d(e_{i_0, \dots, \hat{i}_j, \dots, i_k}) \\ &= \sum_{j=0}^k (-1)^j \left(\sum_{l < j} (-1)^l e_{i_0, \dots, \hat{i}_l, \dots, \hat{i}_j, \dots, i_k} + \sum_{l > j} (-1)^{l-1} e_{i_0, \dots, \hat{i}_j, \dots, \hat{i}_l, \dots, i_k} \right) \\ &= \sum_{j=0}^k \left(\sum_{l < j} (-1)^{l+j} e_{I - i_l - i_j} - \sum_{l > j} (-1)^{l+j} e_{I - i_j - i_l} \right) = 0. \end{aligned}$$

□

Example 1.3.8 (Computing $H_*(S_*(\Delta^2))$).

$$0 \rightarrow \mathbf{Z} = \langle e_{012} \rangle \rightarrow \mathbf{Z}^3 = \langle e_{01}, e_{02}, e_{12} \rangle \rightarrow \mathbf{Z}^3 = \langle e_0, e_1, e_2 \rangle \rightarrow 0$$

with $d(e_{012}) = e_{12} - e_{02} + e_{01}$. So $\ker d_2 = 0 \implies H_2 = 0$.

$$d(ae_{01} + be_{02} + ce_{12}) = a(e_1 - e_0) + b(e_2 - e_1) + c(e_2 - e_0)$$

so $ae_{01} + be_{02} + ce_{12} \in \ker d_1 \iff -a - b = 0, a - c = 0, c + b = 0 \iff a = -b = c$ hence $\ker d_1 = \langle e_{01} - e_{02} + e_{12} \rangle = \text{im } d_2$ and so $H_1 = 0$. $\ker(d_0) = 0$, $\text{im } d_1 = \langle e_1 - e_0, e_2 - e_1, e_2 - e_0 \rangle$ so $H_0 = \mathbf{Z} = \langle [e_0] \rangle$.

Exercise 1.3.9. Show that $H_*(S_*(\Delta^n)) = 0$ if $k \neq 0$ and \mathbf{Z} if $k = 0$.

1.3.3 The singular chain complex

Definition 1.3.10 (Singular chain complex). If X is a space, the **singular chain complex** of X , $C_*(X)$ is defined by

$$C_n(x) = \langle e_\sigma : \sigma : \Delta^n \rightarrow X \text{ any continuous map} \rangle.$$

Where

$$d(e_\sigma) = \sum_{j=0}^n (-1)^j e_{\sigma \circ F_j} \in C_{n-1}(X)$$

where $F_j : \Delta^{n-1} \rightarrow \Delta^n$ is given by $F_j(x_0, \dots, x_{n-1}) = (x_0, \dots, 0, \dots, x_{n-1})$ with the 0 in the j th place.

1.3.3.1 Homotopy invariance

Definition 1.3.11 (Chain homotopic maps). Suppose $\phi, \psi: C_* \rightarrow C'_*$ are chain maps, we say that ϕ and ψ are **chain homotopic** if there exists an R -linear map $h: C_* \rightarrow C'_{*+1}$ such that $d' \circ h + h \circ d = \phi - \psi$. We denote this relation by $\phi \sim \psi$.

Lemma 1.3.12. *If $\phi \sim \psi$ then $\phi_* = \psi_*$.*

Proof.

$$\begin{aligned}\phi_*([x]) - \psi_*([x]) &= [\phi(x) - \psi(x)] \\ &= [d'hx + hdx] = [d'hx] = 0 \in H_*(C').\end{aligned}$$

□

Theorem 1.3.13. *Suppose $f \sim g: X \rightarrow Y$ via H then $f_{\#} \sim g_{\#} \implies f_* = g_*$.*

Proof.

□

Corollary 1.3.14. *If $X \sim Y$ then $H_*(X) \cong H_*(Y)$.*

Corollary 1.3.15. *If X is contractible then $H_*(X) \cong H_*(\{p\}) \cong \mathbf{Z}$ if $*$ = 0, 0 otherwise.*

1.4 Homology of a pair

1.4.1 Exact sequences

Definition 1.4.1 (Exact sequence). A sequence

$$\cdots \rightarrow A_{n+1} \rightarrow A_n \rightarrow A_{n-1} \rightarrow \cdots$$

of R -modules and R -linear maps is **exact at** A_n if $\ker f_n = \operatorname{im} f_{n+1}$.

A sequence is **exact** if it is exact at all A_n , then (A_*, f) is known as a **acyclic** chain complex (the homology is zero).

Example 1.4.2. 1. $0 \rightarrow A \xrightarrow{f} B$ is exact if and only if f is surjective.

2. $B \xrightarrow{g} C \rightarrow 0$ is exact if and only if g is injective.

3. $0 \rightarrow A \xrightarrow{f} A' \rightarrow 0$ is exact if and only if f is an isomorphism.

4. $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is exact if and only if $A \subset B$ and $C \cong B/A$.

Definition 1.4.3 (Short exact sequence). A sequence

$$0 \rightarrow A_* \xrightarrow{i} B_* \xrightarrow{\pi} C_* \rightarrow 0$$

is a **short exact sequence** of chain complexes if

1. A_*, B_*, C_* are chain complexes.

2. i, π are chain maps.

3.

$$0 \rightarrow A_n \xrightarrow{i} B_n \xrightarrow{\pi} C_n \rightarrow 0$$

is exact for all n .

Example 1.4.8 (Homology of a pair). Suppose $A \subset X$, then $C_*(A) \subset C_*(X)$ is a subcomplex. Define $C_*(X, A) = C_*(X)/C_*(A)$ and so we have a short exact sequence

$$0 \rightarrow C_*(A) \rightarrow C_*(X) \rightarrow C_*(X, A) \rightarrow 0.$$

1.4.2 Homology of a pair

Definition 1.4.9 (Homology of a pair). $H_*(X, A) = H_*(C_*(X, A))$ is the **homology of the pair** (X, A) .

From this we obtain:

Definition 1.4.10 (Long exact sequence of a pair). The **long exact sequence of the pair** (X, A) is

$$\cdots \rightarrow H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \rightarrow \cdots$$

If $f: (X, A) \rightarrow (Y, B)$ is a map of pairs then we get an **induced map** $f_\#: C_*(X) \rightarrow C_*(Y)$ defined by

$$(\sigma: \Delta^n \rightarrow A) \mapsto f \circ \sigma.$$

Observe that $f_\#(C_*(A)) \subset C_*(B)$ and so $f_\#$ descends to a map

$$f_\#: C_*(X)/C_*(A) \rightarrow C_*(Y)/C_*(B)$$

or equivalently $f_\#: C_*(X, A) \rightarrow C_*(Y, B)$ this then induces $f_*: H_*(X, A) \rightarrow H_*(Y, B)$.

Proposition 1.4.11 (Homotopy invariance). *If $f, g: (X, A) \rightarrow (Y, B)$ are homotopic as maps of pairs then $f_* = g_*: H_*(X, A) \rightarrow H_*(Y, B)$.*

Proof. Let $H: (X \times I, A \times I) \rightarrow (Y, B)$ be the homotopy, H induces a chain homotopy $h: C_*(X) \rightarrow C_{*+1}(Y)$ where $dh + hd = f_\# - g_\#$. $H(A \times I) \subset B$ so $h(C_*(A)) \subset C_{*+1}(B)$ this implies that h descends to a map

$$h: C_*(X)/C_*(A) \rightarrow C_{*+1}(Y)/C_{*+1}(B)$$

with $hd + dh = f_\# - g_\#$ as any relation satisfied will remain satisfied in the quotient. So we have $h: C_*(X, A) \rightarrow C_{*+1}(Y, A)$ and hence $f_\#, g_\#: C_*(X, A) \rightarrow C_*(Y, B)$ are chain homotopic. \square

1.4.2.1 Visualising relative homology classes

If W^{n+1} is a connected oriented compact manifold then we'll show that $H_{n+1}(W, \partial W) \cong \mathbf{Z} = \langle [W, \partial] \rangle$ (where the ∂ notation means relative to boundary). So given $f: (W, \partial) \rightarrow (X, A)$ we get $f_*([W, \partial W]) \in H_{n+1}(X, A)$.

Example 1.4.12. Let $X = \mathbf{R}^3$ and $A = S^1$ then $W = D^2$ defines a class in $H_2(\mathbf{R}^3, S^1)$ (the boundary of W lies inside of A).

1.4.3 Good pairs

Definition 1.4.13 (Good pair). (X, A) is a **good pair** if

1. $\exists U \subset X$ open and $A \subset U$
2. $\exists \pi: U \rightarrow A$ with $\pi|_A = \text{id}_A$.
3. $\pi \sim 1_{(U, A)}$ as maps of pairs from (U, A) to itself.
(i.e. A is a deformation retract of U).

- Example 1.4.14** (Good pairs). 1. (smooth manifold, closed submanifold).
 2. (simplicial complex, subcomplex).

Example 1.4.15 (Not good pairs). 1. (\mathbf{R}, \mathbf{Q}) .

2. Letting $H \subset \mathbf{R}^2$ be the Hawaiian earring then (\mathbf{R}^2, H) is not a good pair.

Theorem 1.4.16. *Take $A \subset X$ and let π be the natural map $\pi: (X, A) \rightarrow (X/A, A/A) \cong (X/A, *)$. Then if (X, A) is a good pair the induced map $\pi_*: H_*(X, A) \rightarrow H_*(X/A, *)$ is an isomorphism.*

Proof. Postponed. □

Exercise 1.4.17. The composite map ϕ in

$$\tilde{H}_*(X) \rightarrow H_*(X) \rightarrow H_*(X, *)$$

is an isomorphism. i.e.

$$H_*(X, *) = \begin{cases} H_*(X), & * > 0, \\ H_0(X)/\mathbf{Z}, & * \leq 0. \end{cases}$$

Proposition 1.4.18.

$$\tilde{H}_*(S^n) = \begin{cases} \mathbf{Z}, & * = n, \\ 0, & * \neq n. \end{cases}$$

Proof. We proceed by induction on n . For $n = 0$ we have $S^0 = \{p_+, p_-\}$ implying

$$H_*(S^0) = H_*(p_+) \oplus H_*(p_-) = \begin{cases} \mathbf{Z}^2, & * = 0, \\ 0, & * \neq 0, \end{cases}$$

giving

$$\tilde{H}_*(S^0) = \begin{cases} \mathbf{Z}, & * = 0, \\ 0, & * \neq 0. \end{cases}$$

Now consider the long exact sequence of the pair (D^n, S^{n-1})

$$H_*(D^n) \rightarrow H_*(D^n, S^{n-1}) \rightarrow H_{*-1}(S^{n-1}) \xrightarrow{\phi} H_{*-1}(D^n),$$

we can break this up using the kernel and cokernel to get

$$0 \rightarrow \text{coker } \phi_m \rightarrow H_m(D^n, S^{n-1}) \rightarrow \ker \phi_{m-1} \rightarrow 0. \quad (1.4.1)$$

D^n is contractible and so we get

$$H_*(D^n) = \begin{cases} \mathbf{Z}, & * = 0, \\ 0, & * \neq 0. \end{cases}$$

We have $\phi: H_0(S^{n-1}) \rightarrow H_0(D^n) \cong \mathbf{Z} = \langle e_p \rangle$ given by $\phi(e_p) = e_p$. We see that $\ker \phi = \tilde{H}_*(S^{n-1})$ and $\text{coker } \phi = 0$. Now by looking at (1.4.1) we see that

$$0 \rightarrow \tilde{H}_*(S^n) = H_*(S^n, *) = H_*(D^n, S^{n-1}) \rightarrow \ker \phi = \tilde{H}_*(S^{n-1}) \rightarrow 0$$

is exact giving $\tilde{H}_*(S^n) \cong \tilde{H}_*(S^{n-1})$. The claim then follows by induction. □

Corollary 1.4.19.

$$S^n \sim S^m \implies m = n.$$

Example 1.4.20 (Chains generating $H_n(S^n, *)$). 1. Choose $f: \Delta^n \rightarrow \Delta^n$ a homeomorphism and let $e: \Delta^n \rightarrow \Delta^n$ be the identity map. Then $a_{n-1} = f_{\#}(de) \in C_*(S^{n-1})$ and we have $da_{n-1} = df_{\#}(de) = f_{\#}(d^2e) = 0$ and so a_{n-1} is closed.

2. Choose $g: (D^n/S^{n-1}, S^{n-1}/S^{n-1}) \rightarrow (S^n, *)$. Now let $b_n = g_{\#}(f_{\#}(e)) \in C_n(S^n, *)$. Then $db_n = g_{\#}(a_{n-1}) \in C_*(*)$ implies b_n is closed in $C_*(S^n, *)$. For example if $n = 2$ we are crushing the boundary of the 2-simplex to a point.

Proposition 1.4.21. $[a_n]$ generates $\tilde{H}_n(S^n) = \mathbf{Z}$ (statement (A_n)).
 $[b_n]$ generates $\tilde{H}_n(S^n, *) = \mathbf{Z}$ (statement (B_n)).

Proof. Statement (A_0) : $S^0 = \{p_+, p_-\}$, $de = e_{p_+} - e_{p_-}$ so de generates $\tilde{H}_0(S^0)$. We'll show that $A_{n-1} \implies B_n$. Here $\partial: H_n(S^n, *) = H_n(D^n, S^{n-1}) \xrightarrow{\sim} \tilde{H}_{n-1}(S^{n-1})$ (by 1.4.18). So it suffices to check that $\partial[b_n] = [a_{n-1}]$ \square

Definition 1.4.22 (Wedge product). If (X_i, p_i) are pointed spaces the **wedge product**

$$\bigvee_{i \in I} (X_i, p_i) \text{ is } \coprod_{i \in I} X_i / \{p_i : i \in I\}.$$

If X_i is such that for any $p, q \in X_i$ there exists a homeomorphism $f: X_i \rightarrow X_i$ with $f(p) = q$ we can drop p_i from the notation (for example in the case of X_i a connected manifold we can do this).

Example 1.4.23.

Corollary 1.4.24. If (X_i, p_i) are good pairs then

$$\tilde{H}_* \left(\bigvee_i (X_i, p_i) \right) = \bigoplus_i \tilde{H}_*(X_i).$$

Proof.

$$\begin{aligned} \tilde{H}_* \left(\bigvee_{i \in I} X_i \right) &\cong H_* \left(\bigvee_{i \in I} X_i, p \right) \cong H_* \left(\coprod_{i \in I} X_i, \{p_i : i \in I\} \right) \\ &\cong \bigoplus_{i \in I} H_*(X_i, p_i) \cong \bigoplus_{i \in I} \tilde{H}_*(X_i). \end{aligned}$$

\square

Example 1.4.25.

$$H_*(S^1 \vee S^2) = \begin{cases} \mathbf{Z}, & * = 0, 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

1.5 Subdivision and Excision

Definition 1.5.1 (Singular chain complexes with respect to open covers.). If $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of X , let

$$C_n^{\mathcal{U}}(X) = \langle e_{\sigma} : \sigma : \Delta^n \rightarrow X, \text{ im } \sigma \subset U_i \text{ for some } i \rangle \subset C_n(X).$$

Observe that $\text{im } \sigma \subset U_i$ implies $\text{im } \sigma \circ F_j \subset U_i$ and so $C_*^{\mathcal{U}}$ is a subcomplex of C_* . Let $H_*^{\mathcal{U}}$ be the homology of this complex, then we have a map

$$i: C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X).$$

Lemma 1.5.2 (Subdivision). $C_*: H_*^{\mathcal{U}}(X) \rightarrow H_*(X)$ is an isomorphism.

1.6 Degree and Orientations

1.7 Cell Complexes

Definition 1.7.1 (Attaching of cells). If $f: S^{n-1} \rightarrow X$ then

$$X \cup_f D^n = X \amalg D^n / \sim$$

is the space obtained by **attaching** an n -dimensional cell to X via the map f .

Example 1.7.2. If $X = \{p\}$ and $f: S^{n-1} \rightarrow X$ then

$$X \cup_f D^n \cong D^n / S^{n-1} \cong S^n.$$

Definition 1.7.3 (Finite cell complex). A 0-dimensional **finite cell complex** is a finite disjoint union of points.

A k -dimensional finite cell complex is a space obtained by attaching finitely many k -cells to a $(k-1)$ -dimensional finite cell complex.

Example 1.7.4.

Example 1.7.5. If a finite cell complex X has one 0-cell and one n -cell then $X \cong S^n$. Similarly if X has one 0-cell and kn -cells then $X \cong \bigvee_{i=1}^k S^n$.

Example 1.7.6. T^2 is a finite cell complex with one 0-cell, two 1-cells and one 2-cell.

Example 1.7.7. Any simplicial complex is a finite cell complex and any closed manifold can be given the structure of a finite cell complex.

Chapter 2

Cohomology and Products

2.1 Homology with Coefficients and Cohomology

2.1.1 Hom and \otimes for modules

Definition 2.1.1 (Tensor product of R -modules). Let M, N be R -modules. Then the **tensor product** $M \otimes N$ is the R -modules generated by all pairs $m \otimes n$ for $m \in M, n \in N$ modulo the relations:

1. $(m_1 + m_2) \otimes (n_1 + n_2) = \sum m_i \otimes n_j$.
2. $r(m \otimes n) = (rm) \otimes n = m \otimes (rn)$.

We have the following properties of this definition:

1. $(M_1 \oplus M_2) \otimes (N_1 \oplus N_2) = \bigoplus M_i \otimes N_j$.
2. $M \otimes N \cong N \otimes M$.
3. $M \otimes R = M$

Example 2.1.2 (Tensor products). 1. $R^n \otimes R^m \cong R^{mn}$.

2. Letting $R = \mathbf{Z}, \mathbf{Q} \otimes \mathbf{Z}/a \cong 0$.
3. $\mathbf{Z}/a \otimes \mathbf{Z}/b \cong \mathbf{Z}/(a, b)$.

If $f: M_1 \rightarrow M_2$ and $g: N_1 \rightarrow N_2$ then there is a map

$$\begin{aligned} f \otimes g: M_1 \otimes N_1 &\rightarrow M_2 \otimes N_2, \\ m \otimes n &\mapsto f(m) \otimes g(n). \end{aligned}$$

Example 2.1.3. If $f: R^n \rightarrow R^m$ is given by multiplication by $A \in \text{Mat}_{n \times m}(R)$ then $f \otimes 1_M: M^n \rightarrow M^m$ is given by multiplication by A .

Definition 2.1.4 (Hom).

$$\text{Hom}(M, N) = \{f: M \rightarrow N : f \text{ is } R\text{-linear}\}$$

is an R -module, via $(f + rg)(m) = f(m) + rg(m)$.

From this definition we see that

1. $\text{Hom}(\bigoplus M_i, \bigoplus N_j) \cong \bigoplus_{i,j} \text{Hom}(M_i, N_j)$.
2. $\text{Hom}(R, M) \cong M$.

Note however that we do not have $\text{Hom}(M, N) = \text{Hom}(N, M)$ as for example $\text{Hom}(\mathbf{Z}/2, \mathbf{Z}) = 0$ but $\text{Hom}(\mathbf{Z}, \mathbf{Z}/2) = \mathbf{Z}/2$.

Definition 2.1.5 (Dual module). Given an R -module M the **dual** of M is $M^* = \text{Hom}(M, R)$.

Now if we have $f: M \rightarrow N$ we get a map

$$f^*: \text{Hom}(N, O) \rightarrow \text{Hom}(M, O)$$

given by $f^*g = g \circ f$.

Example 2.1.6. If $f: R^n \rightarrow R^m$ is multiplication by A then

$$f^*: \text{Hom}(R^m, O) \cong O^m \rightarrow \text{Hom}(R^n, O) \cong O^n$$

is multiplication by A^\top .

2.1.2 Hom and \otimes for chain complexes

If (C, d) is a chain complex defined over R then so are $(C_* \otimes M, d \otimes 1_M) = C_* \otimes M$ and $(\text{Hom}(C_*, M), d^*) = \text{Hom}(C_*, M)$.

2.2 Universal Coefficient Theorems

2.3 Products

Definition 2.3.1 (Product of chain complexes). If A, B are chain complexes the $C = A \otimes B$ is the chain complex with

$$C_i = \bigoplus_{j+k=i} A_j \otimes B_k$$

and

$$d(a \otimes b) = (da) \otimes b + (-1)^{|a|} a \otimes (db)$$

where $|a| = j$ if $a \in A_j$.

The following theorem is true but we will not prove it.

Theorem 2.3.2 (Eilenberg-Zilber).

$$C_*(X \times Y) \sim C_*(X) \otimes C_*(Y).$$

Given $H_*(A)$ and $H_*(B)$ how can we compute $H_*(A \otimes B)$.

Lemma 2.3.3. Suppose $A \sim A'$. Then $A \otimes B \sim A' \otimes B$.

Proof. Easy. □

Corollary 2.3.4. Suppose A, B are finitely generated chain complexes, defined over a field. Then

$$H_*(A \otimes B) = \bigoplus_{j+k=i} H_j(A) \otimes H_k(B)$$

i.e.

$$H_*(A \otimes B) \cong H_*(A) \otimes H_*(B).$$

Proof. A is a chain complex over a field, so $A \sim A'$ where $A'_i = H_i(A)$ and $d \equiv 0$ on A' . Similarly for B . Then

$$H_*(A \otimes B) \cong H_*(A' \otimes B') \cong A' \otimes B' = H_*(A) \otimes H_*(B).$$

□

Corollary 2.3.5. *If k is a field and X, Y are finite chain complexes then*

$$H_*(X \times Y; k) \cong H_*(X; k) \otimes_k H_*(Y; k).$$

Definition 2.3.6 (Poincare polynomial). The **Poincare polynomial** of X with coefficients in k is

$$P_k(X) = \sum_{i \geq 0} \dim H_i(X; k) t^i.$$

Example 2.3.7.

$$P_{\mathbf{Z}/2}(\mathbf{RP}^2) = 1 + t + t^2.$$

2.3.5 tells us that

$$P_k(X \times Y) = P_k(X)P_k(Y).$$

Example 2.3.8.

$$P_{\mathbf{Z}/2}(\mathbf{RP}^3 \times \mathbf{RP}^2) = (1 + t + t^2)(1 + t + t^2 + t^3).$$

For now we will suppose that A, B are finitely generated chain complexes over a PID R .

Lemma 2.3.9. *If $H_*(A)$ is a free R -module then $A \sim A'$ where $A'_i = H_i(A)$ and $d \equiv 0$ on A' .*

Proof. A is the direct sum of short and stupid complexes. $H_*(A)$ being torsion free implies that every short complex is of the form

$$C: R \xrightleftharpoons[\cdot a^{-1}]{\cdot a} R$$

with $a \in R^\times$ such a $C \sim 0$ (same proof as for a field). □

Corollary 2.3.10. *If $H_*(A)$ is free then $H_*(A \otimes B) \cong H_*(A) \otimes H_*(B)$.*

Proof. $A \otimes B \cong A' \otimes B$ where A' is as in 2.3.9. $A' \otimes B$ is a direct sum of copies of B , one for each generator of A' . This gives that $H_*(A' \otimes B)$ is a direct sum of copies of $H_*(B)$, one for each generator of $H_*(A)$ which implies that

$$H_*(A \otimes B) \cong H_*(A) \otimes H_*(B)$$

(since $H_*(A)$ is free).

$$R^n \otimes H_*(B) \cong (H_*(B))^n.$$

□

Corollary 2.3.11. *If X and Y are finite cell complexes and $H_*(X)$ is free over \mathbf{Z} , then $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$.*

Fact: If C_1 is a free resolution of M and C_2 is a free resolution of N . Then

$$\mathrm{Tor}_i(M, N) \cong H_i(C_1 \otimes C_2) \cong H_i(C_1 \otimes N) \cong H_i(M \otimes C_2).$$

Theorem 2.3.12 (Kunneth formula). *If A, B are free (finitely generated) chain complexes over a PID R then*

$$H_i(A \otimes B) \cong \bigoplus_{j+k=i} \operatorname{Tor}_0(H_j(A), H_k(B)) \oplus \bigoplus_{j+k=i-1} \operatorname{Tor}_1(H_j(A), H_k(B)).$$

Proof. It suffices to check the statement where A, B are short and/or stupid. We just did the case where both are short. The case where one or more is stupid was covered in the statement that

$$H_*(A \otimes B) \cong H_*(A) \otimes H_*(B)$$

if $H_*(A)$ is free. □

Corollary 2.3.13. *We can compute $H_*(X \times Y)$ from $H_*(X)$ and $H_*(Y)$.*

2.4 Cup product

Motivation: if k is a field

$$\begin{aligned} H^*(X \times X; k) &\cong (H_*(X \times X; k))^* \\ &\cong (H_*(X; k) \otimes H_*(X; k))^* \\ &\cong H^*(X; k) \otimes H^*(X; k) \end{aligned}$$

Definition 2.4.1 (Cup product). Suppose $a \in C^i(X)$ and $b \in C^j(X)$. If $\sigma: \Delta^{i+j} \rightarrow X$ then we define $a \smile b \in C^{i+j}(X)$ by

$$(a \smile b)(\sigma) = a(\sigma \circ F_i^+) b(\sigma \circ F_j^-)$$

where

$$\begin{aligned} F_i^+ : \Delta^i &\rightarrow \Delta^{i+j} \\ (x_0, \dots, x_i) &\mapsto (x_0, \dots, x_i, 0, \dots, 0) \end{aligned}$$

and

$$\begin{aligned} F_j^- : \Delta^j &\rightarrow \Delta^{i+j} \\ (x_0, \dots, x_j) &\mapsto (0, \dots, 0, x_0, \dots, x_j). \end{aligned}$$

Lemma 2.4.2.

$$d(a \smile b) = da \smile b + (-1)^{|a|} a \smile db.$$

Proof. Denote by $[v_0, \dots, v_k]$ the singular k -simplex that sends

$$\Delta^k \rightarrow \Delta^{i+j} \xrightarrow{\sigma} X$$

vertices of $\Delta^k \rightarrow$ vertices v_0, \dots, v_k of Δ^{i+j} .

Then taking $\sigma: \Delta^{i+j+1} \rightarrow X$

$$\begin{aligned}
d(a \smile b)(\sigma) &= (a \smile b)(d\sigma) \\
&= \sum_{k=0}^{i+j+1} (-1)^k a \smile b([v_0, \dots, \hat{v}_k, \dots, v_n]) \\
&= \sum_{k \leq i} (-1)^k a([v_0, \dots, \hat{v}_k, \dots, v_{i+1}]) b([v_{i+1}, \dots, v_n]) \\
&\quad + \sum_{k > i} (-1)^k a([v_0, \dots, v_i]) b([v_i, \dots, \hat{v}_k, \dots, v_n]) \\
&= \sum_{k \leq i+1} (-1)^k a([v_0, \dots, \hat{v}_k, \dots, v_{i+1}]) b([v_{i+1}, \dots, v_n]) \\
&\quad + \sum_{k \geq i} (-1)^k a([v_0, \dots, v_i]) b([v_i, \dots, \hat{v}_k, \dots, v_n]) \\
&= a(d([v_0, \dots, v_{i+1}])) b([v_{i+1}, \dots, v_n]) \\
&\quad + (-1)^i a([v_0, \dots, v_i]) b(d([v_i, \dots, v_n])) \\
&= da \smile b([v_0, \dots, v_n]) + (-1)^i a \smile db([v_0, \dots, v_n]).
\end{aligned}$$

□

Corollary 2.4.3. *There is a well-defined cup product*

$$\begin{aligned}
H^i(X) \times H^j(X) &\rightarrow H^{i+j}(X), \\
[a] \times [b] &\mapsto [a \smile b].
\end{aligned}$$

Proof. Must check

1. $da = c, db = 0 \implies d(a \smile b) = da \smile b \pm a \smile db = 0$ so ok.
2. $(a + dc) \smile b = a \smile b + dc \smile b = a \smile b + d(c \smile b)$ implies $[(a + dc) \smile b] = [a \smile b]$ and so \smile is well defined.

□

Proposition 2.4.4 (Properties of \smile). *Let $\alpha, \beta, \gamma \in H^*(X)$ then*

1. (Associative) $(\alpha \smile \beta) \smile \gamma = \alpha \smile (\beta \smile \gamma)$.
2. (Graded commutative) $a \smile b = (-1)^{|\alpha||\beta|} \beta \smile \alpha$.
3. (Identity) There is a class $1 \in H^0(X)$ such that $1 \smile \alpha = \alpha \smile 1 = \alpha$ for all $\alpha \in H^*(X)$.
4. (Functorial) If $f: X \rightarrow Y$, $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$.

Proof. 1. If $a \in C^i(X)$, $b \in C^j(X)$, $c \in C^k(X)$, $\sigma: \Delta^{i+j+k} \rightarrow X$ then

$$\begin{aligned}
((a \smile b) \smile c)(\sigma) &= a(\sigma \circ F_i^+) b(\sigma \circ F_j^{\text{mid}}) c(\sigma \circ F_k^-) \\
&= (a \smile (b \smile c))(\sigma).
\end{aligned}$$

Where $F_i^+(X) = (X^i \ 0)$, $F_k^-(X) = (0^{i+j} \ X^{i+j+k})$, $F_j^{\text{mid}}(X) = (0^i \ X^{i+j} \ 0)$.

2. We'll do later.

3. Let $e \in C^0(X)$ be given by

$$e(e_p) = 1 \text{ for all } p \in X$$

If $\gamma: [0, 1] \rightarrow X$ is a 1-chain then

$$de(e_\gamma) = e(de_\gamma) = e(e_{\gamma(1)} - e_{\gamma(0)}) = 1 - 1 = 0.$$

Easy to see $e \smile a = a \smile e = a$ for all $a \in C^*(X)$ then $1 = [e] \in H^0(X)$.

4.

$$\begin{aligned} f^\#(a \smile b)(\sigma) &= (a \smile b)(f \circ \sigma) \\ &= a(f \circ \sigma \circ F_i^+)b(f \circ \sigma \circ F_j^-) \\ &= f^\#(a)(\sigma \circ F_i^+)f^\#(b)(\sigma \circ F_j^-) \\ &= [f^\#(a) \smile f^\#(b)](\sigma). \end{aligned}$$

□

2.4.1 Digression: De Rahm Cohomology

Let M be a smooth manifold and $\omega \in \Omega^k(M)$. If $\sigma: \Delta^k \rightarrow M$ is a smooth we define $\omega(\sigma) = \int_\sigma \omega$. Stokes' theorem then gives us that $d\omega(\sigma) = \omega(d\sigma)$. Where on the left $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ is the differential map and on the right d is the boundary map in the sense of (smooth) singular chains. I.e. the two d 's are dual to each other on $C_*^{\text{smooth}}(M)$.

Theorem 2.4.5 (de Rahm).

$$H_k(\Omega^*(M), d) \cong H^k(M; \mathbf{R}).$$

Under this correspondence

$$[\omega] \smile [\nu] = [\omega \wedge \nu].$$

Example 2.4.6. 1. $H^*(S^n) = \langle 1, X_n \rangle$ where $\langle X_n \rangle = H^n(S^n)$ and we have

$$\begin{aligned} 1 \smile X_n &= X_n, \\ X_n \smile 1 &= X_n, \\ 1 \smile 1 &= 1, \\ X_n \smile X_n &= 0 \in H^{2n}(S^n) = 0. \end{aligned}$$

2. For arbitrary X, Y

$$C_*(X \amalg Y) \cong C_*(X) \oplus C_*(Y)$$

which implies

$$C^*(X \amalg Y) \cong C^*(X) \oplus C^*(Y)$$

$$\implies H^*(X \amalg Y) \cong H^*(X) \oplus H^*(Y) \text{ as rings}$$

with

$$(\alpha_1 \oplus \alpha_2) \smile (\beta_1 \oplus \beta_2) = (\alpha_1 \smile \beta_1) \oplus (\alpha_2 \smile \beta_2).$$

3. Considering $X \vee Y$ we have a map $p: X \amalg Y \rightarrow X \vee Y$ and so Mayer-Vietoris for $X \vee Y$ gives

$$\cdots \rightarrow H^*(X \vee Y) \xrightarrow{p^*} H^*(X) \oplus H^*(Y) \xrightarrow{i^*} H^*(\text{pt}) \rightarrow \cdots$$

Now assuming X and Y are path connected p^* is injective and an isomorphism in grading > 0 . In grading 0 we have

$$H^0(X \vee Y) \rightarrow H^0(X) \oplus H^0(Y) \rightarrow H^0(\text{pt})$$

with $1 \mapsto 1 \oplus 1$ and $a \oplus b \mapsto a - b$. And so as a ring we have

$$H^*(X \vee Y) \cong \ker i^* \subset H^*(X) \oplus H^*(Y)$$

where elements in grading > 0 look like (α, β) and in grading 0 look like $(n, n) \in \mathbf{Z} \times \mathbf{Z}$.

4. $S^2 \vee S^4$, $H^*(S^2 \vee S^4) = \langle 1, X_2, X_4 \rangle$ with $X_2^2 = 0$ since $X_2^2 = 0$ in $H^*(S^2)$.

2.4.2 Exterior Product

Let $\pi_1: X \times Y \rightarrow X$ and $\pi_2: X \times Y \rightarrow Y$.

Definition 2.4.7 (Exterior product). If $\alpha \in H^i(X)$, $\beta \in H^j(X)$ then their **exterior product** is $\alpha \times \beta = \pi_1^*(\alpha) \smile \pi_2^*(\beta) \in H^{i+j}(X \times Y)$.

Proposition 2.4.8. 1. $\alpha \smile \beta = \Delta^*(\alpha \times \beta)$ where $\Delta: X \rightarrow X \times X$ is the diagonal map.

2. $(\alpha \times \beta) \smile (\alpha' \times \beta') = (-1)^{|\beta||\alpha'|}(\alpha \smile \alpha') \times (\beta \smile \beta')$.

Proof. 1.

$$\Delta^*(\alpha \times \beta) = \Delta^*(\pi_1^*(\alpha) \smile \pi_2^*(\beta))$$

□

Chapter 3

Manifolds and Bundles

3.1 Vector Bundles

Definition 3.1.1 (Vector bundle). An n -dimensional **real vector bundle** is a triple (E, B, π) such that

1. $\pi: E \rightarrow B$.
2. Each fibre $\pi^{-1}(p)$ has the structure of an n -dimensional real vector space.
3. (Local triviality) For each $p \in B$ there is $U \subset B$ open containing p and a homeomorphism

$$f_U: \pi^{-1}(U) \rightarrow U \times \mathbf{R}^n$$

such that

- (a) $\pi_1 \circ f_U = \pi$,
- (b) $\pi_2 \circ f_U: \pi^{-1}(p) \rightarrow \mathbf{R}^n$ is a linear map (it must be an isomorphism of vector spaces).

A **complex vector bundle** is defined similarly with \mathbf{C} replacing \mathbf{R} .

3.1.1 Transition functions

If $f_1: \pi^{-1}(U_1) \rightarrow U_1 \times \mathbf{R}^n$ and $f_2: \pi^{-1}(U_2) \rightarrow U_2 \times \mathbf{R}^n$ are local trivialisations of E then the map $f_1 \circ f_2^{-1}: (U_1 \cap U_2) \times \mathbf{R}^n \rightarrow (U_1 \cap U_2) \times \mathbf{R}^n$ is linear on fibres. So there is a map $g_{12}: U_1 \cap U_2 \rightarrow \text{GL}_n(\mathbf{R})$ such that $f_1 \circ f_2^{-1}(p, v) = (p, g_{12}(p)v)$, g_{12} is called a **transition function**.

Exercise 3.1.2. If $f_3: U_3 \rightarrow U_3 \times \mathbf{R}^n$ is another local trivialisation then $g_{32}(p) \cdot g_{21}(p) = g_{31}(p)$, this is called the cocycle condition.

Example 3.1.3. 1. $E = B \times \mathbf{R}^n$ is the n -dimensional **trivial bundle** over B .

2. The Mobius band $M = [0, 1] \times \mathbf{R} / \sim$ where $(0, x) \sim (1, -x)$ is a bundle over S^1 via $\pi: M \rightarrow [0, 1] / \sim$. We have local trivialisations given by $U_1 = [0 - \epsilon, 1/2 + \epsilon] \times \mathbf{R}$ and $U_2 = [1/2 - \epsilon, 1 + \epsilon] \times \mathbf{R}$. The transition function is $g_{12}: \{0, 1/2\} \rightarrow \text{GL}_1(\mathbf{R})$ mapping $1/2 \rightarrow 1$ and $0 \rightarrow -1$.
3. If M is a smooth manifold with charts $\phi_i: U_i \rightarrow V_i \subset \mathbf{R}^n$ and transition functions $\psi_{ij} = \phi_i \circ \phi_j^{-1}$ where defined. Then the **tangent bundle** TM has local trivialisations $f_i = d\phi_i: TM|_{U_i} \rightarrow TV_i = V_i \times \mathbf{R}^n$ and transition functions

$$g_{ij} = d\phi_i \circ d\phi_j^{-1} = d(\phi_i \circ \phi_j^{-1}) = d\psi_{ij}.$$

