Part III Algebraic Topology - Michelmas 2014

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Contents

1 Introduction

These are lecture notes for the 2014 Part III Algebraic Topology course taught by Dr. Jacob Rasmussen.

The recommended books are:

- Algebraic Topology - Allen Hatcher,
- Homology Theory - James W. Vick,
- Differential Forms in Algebraic Topology - Raoul Bott and Loring W. Tu.

2 Homotopy

Lecture 1

Definition. Maps $f_0, f_1 \colon X \to Y$ are said to be *homotopic* if there is a continuous map $F \colon X \times I \to Y$ such that

$$F(x,0) = f_0(x) \text{ and } F(x,1) = f_1(x) \ \forall x \in X.$$

We let $\mathrm{Map}(X,Y) = \{f \colon X \to Y \text{ continuous}\}$. Then letting $f_t(x) = F(x,t)$ in the above definition we see that f_t is a path from f_0 to f_1 in $\mathrm{Map}(X,Y)$.

Examples.

1. $X = Y = \mathbf{R}^n$, $f_0(\overline{x}) = \overline{0}$ and $f_1(\overline{x}) = \overline{x}$ are homotopic via $f_t(\overline{x}) = t\overline{x}$.
2. $S^1 = \{z \in \mathbf{C} : |z| = 1\}$ then
3. $S^n = \{\overline{x} \in \mathbf{R}^n : |\overline{x}| = 1\}$

Lemma. *Homotopy is an equivalence relation on* $\mathrm{Map}(X,Y)$.

Definition.

$$[X,Y] = \mathrm{Map}(X,Y)/\sim = \text{set of homotopy classes of maps } X \to Y.$$

Lemma. *If* $f_0 \sim f_1 \colon X \to Y$ *and* $g_0 \sim g_1 \colon Y \to Z$ *then* $g_0 \circ f_0 \sim g_1 \circ f_1$.

Corollary. *For any space* X *the set* $[X, \mathbf{R}^n]$ *has one element.*

Proof. Def ☐

Definition. X is *contractible* if 1_X is homotopic to a constant map.

Proposition. Y *is contractible* $\iff$ $[X,Y]$ *has one element for any space* X.

Proof. Define $0_X \colon X \to \mathbf{R}^n$ by $0_X(x) = 0 \in \mathbf{R}^n$ for any $x \in X$. ☐