Part III Homological and Homotopical Algebra 2014

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Contents

Chapter 1

Elements of Homological Algebra

1.1 Introduction

These are lecture notes for the 2014 Part III Homological and Homotopical Algebra course taught by Dr. Julian Holstein, these notes are part of Mjolnir.

The recommended books are:

- W. G. Dwyer and J. Spalinski, Homotopy theories and model categories
- S. I. Gelfand and Yu. I. Manin, Methods of Homological Algebra
- C. Weibel, An introduction to homological algebra

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1.2 Motivation

Start with a graded ring $\mathbf{C}[x_0, \ldots, x_n]$ with $\deg x_i = 1$. Consider a graded module $M = \bigoplus_d M_d$ over R. Hilbert looked at the map $d \mapsto H_M(d) = \dim_{\mathbf{C}} M_d$. For example we can take R to be the homogeneous coordinate ring of $\mathbf{P}^n$ and $V(I) \subset \mathbf{P}^n$ a subvariety where I is a homogeneous ideal. We then take $M = R/I$, if V is a curve C then $H_{R/I}(d) = \deg(V) \cdot d + (1 - g(C))$. Hilbert showed that the function $H_M(d)$ is eventually polynomial. We can compute this function easily if M is free so we try to replace M by free modules. First we take

$$K_0 \to F_0 \to M$$

where K_0 is the kernel of the surjective map from F_0 to M. We can continue this getting

$$\begin{gathered} K_1 \to F_1 \to K_0 \\ K_2 \to F_2 \to K_1 \\ \vdots \end{gathered}$$

we can then write

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0,$$

this is a free resolution of M. We also have the following.

Theorem 1.2.1 (Hilbert)**.** $F_{n+1} = 0$.

Corollary 1.2.2. $H_M(d) = \sum_i (-1)^i H_{F_i}(d)$.

1.3 Categorical notions

1.3.1 Abelian Categories

Example 1.3.1. Rmod - the category of left R-modules for R an associative ring is an abelian category.

Example 1.3.2. The categories of sheaves of abelian groups on a topological space, sheaves of $\mathcal{O}$-modules on a scheme and (quasi-)coherent sheaves on a scheme are all abelian.

Definition 1.3.3 (Additive categories)**.** An **additive category** is a category in which:

1. Every hom-space has the structure of an abelian group.
2. There exists a 0-object (one with exactly one map to and from every other object).
3. Finite products exist (these are automatically equal to sums $A \times B = A \oplus B = A \amalg B$).

In such a category we let

$$\ker(f) = \mathrm{eq}\left(A \overset{f}{\underset{0}{\rightrightarrows}} B \right)$$

and

$$\mathrm{coker}(f) = \mathrm{coeq}\left(A \overset{f}{\underset{0}{\rightrightarrows}} B \right).$$

Definition 1.3.4 (Abelian categories)**.** An **abelian category**$\mathcal{A}$ is an additive category in which:

1. Every map f has a kernel and cokernel.
2. For all f we have $\mathrm{coker}(\ker(f)) = \mathrm{im}(f) = \mathrm{coim}(f) = \ker(\mathrm{coker}(f))$.

Example 1.3.5. Let $\mathcal{B}$ be the category of pairs of vector spaces $V \subset W$, with morphisms the compatible linear maps. Consider the natural map $f\colon 0 \subset V \to V \subset V$, we then have $\mathrm{im}\, f \cong 0 \subset V$ but $\mathrm{coim}\, f \cong V \subset V$. So this category is not abelian.

From now on we take $\mathcal{A}$ to be any abelian category.

1.3.2 Exactness

Definition 1.3.6 (Exact sequences)**.** A sequence of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\mathcal{A}$ is **exact at** B if $\mathrm{im}\, f = \ker g$. A sequence is then exact if it is exact everywhere. An exact sequence of the form

$$0 \to A \to B \to C \to 0$$

is called a **short exact sequence**.

Definition 1.3.7 (Mono and epi morphisms)**.** A morphism f is a **monomorphism** if $fg = fh \implies g = h$ and it is an **epimorphism** if $gf = hf \implies g = h$.

Example 1.3.8. In Abgp the following are exact sequences:

$$0 \to \mathbf{Z}/2 \to \mathbf{Z}/2 \oplus \mathbf{Z}/2 \to \mathbf{Z}/2 \to 0$$
$$0 \to \mathbf{Z}/2 \to \mathbf{Z}/4 \to \mathbf{Z}/2 \to 0$$
$$0 \to \mathbf{Z} \xrightarrow{\cdot 3} \mathbf{Z} \to \mathbf{Z}/3 \to 0$$

Definition 1.3.9 (Additive functors)**.** A functor of additive categories is **additive** if it is a homomorphism on hom-sets.

1.4 Chain complexes

Definition 1.4.1 (Chain complexes)**.** A **chain complex**$C_\bullet$ is a collection of objects $(C_i)_{i\in\mathbf{Z}}$ in $\mathcal{A}$ with maps $d_i \colon C_i \to C_{i-1}$ such that $d_{i-1} \circ d_i = 0$.

Definition 1.4.2 (Cycles, boundaries, homology objects)**.** We define the **cycles**$Z_i = \ker d_i$ and **boundaries**$B_i = \operatorname{im} d_{i+1}$ and the ith **homology object**$H_i(C) = \operatorname{coker}(B_i \to Z_i)$. A complex is **acyclic** if it is exact i.e. $H_\bullet(C) = 0$.

Definition 1.4.3 (Cochain complexes)**.** A **cochain complex**$C^\bullet$ is a collection of objects $(C^i)_{i\in\mathbf{Z}}$ in $\mathcal{A}$ with maps $d_i \colon C_i \to C_{i+1}$ such that $d_{i+1} \circ d_i = 0$. We then have as above H^i the ith **cohomology object**.

We can switch between chain complexes and cochain complexes via $C^i = C_{-i}$.

Example 1.4.4. We have many such complexes:

- Singular (co-)chain complex on a top space.
- de Rahm complex.
- Cellular chain complex.
- Flabby resolution of a sheaf.
- Bar resolution of a group.
- Koszul complex.

Definition 1.4.5 (Chain maps)**.** Given B, C chain complexes, a **chain map**$f \colon B \to C$ is a collection of maps $f_i \colon B_i \to C_i$ such that $df = fd$.

We now have formed the **category of chain complexes** $\mathrm{Ch}(\mathcal{A})$ using these maps. We write $\mathrm{Ch}(R)$ for $\mathrm{Ch}(\mathrm{Rmod})$. Note that $\mathrm{Ch}(\mathcal{A})$ is an additive category moreover it is an abelian category, we can define and check everything level-wise. For example $\ker(A \to B)_n = \ker(A_n \to B_n)$. Note that the H_n form a functor $\mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$. Define $f_* \colon H_n A \to H_n B$ in the natural wat and check it works. H_n is additive.

Lemma 1.4.6 (Snake lemma)**.** *Let $0 \to A \to B \to C \to 0$ be a short exact sequence then there exist natural boundary maps ∂_n which fit into a long exact sequence of homology objects*

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "f_*"] & H_n(B) \arrow[r, "g_*"] & H_n(C) \arrow[dll, rounded corners, "\partial_n"] & \\
& H_{n-1}(A) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(C) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. Exercise. □

Naturality here means given two short exact sequences and compatible chain maps the induced maps on homology are compatible with ∂_n. (The obvious diagram commutes.)

Recall that f is a chain map if $\partial f - f \partial = 0$.

Definition 1.4.8. Let $\underline{\mathrm{Hom}}_n(A, B)$ consist of functions $\{f_i \colon A_i \to B_{i+n}\}$ and define $df = d \cdot f - (-1)^n f d$ if $f \in \underline{\mathrm{Hom}}_n$. Check that

$$d^2 f = d \cdot (d \cdot f - (-1)^f d) - (-1)(d \cdot f - (-1)^n f \cdot d) \cdot d = 0.$$

We use the "Sign rule" to help with definitions, this states that if a moves past b we pick a sign $(-1)^{\deg a \deg b}$.

$\mathrm{Ch}(\mathcal{A})$ can be enriched over $\mathrm{Ch}(\mathbf{Z})$.

Definition 1.4.9 (Shifted complexes)**.** The **shifted complex**$C[n]$ for $C \in \mathrm{Ch}(\mathcal{A})$ is defined by $C[n]_i = C_{n+i}$ and $d_i^{C[n]} = (-1)^n d_{n+i}^C$.

Note that $H_i(C) = H_0(C[i])$.

So a chain map $f \colon A \to B[n]$ is exactly a cycle in $\underline{\mathrm{Hom}}_n(A, B)$.

Now $\mathrm{Hom}(A, B) = Z_0(\underline{\mathrm{Hom}}(A, B))$, so what is $H_0(\underline{\mathrm{Hom}}(A, B))$?

Definition 1.4.10 (Chain homotopies)**.** A **chain homotopy**S between chain maps $f, g \colon A \to B$ is a collection $S_i \colon A_i \to B_i$ such that $\partial S + S \partial = f - g$. Equivalently we could say a map $A \to B[1]$ such that $dS = g - f$ (note: not a chain map). We write $f \simeq g$ to denote the fact that f is chain homotopic to g.

Definition 1.4.11 (Chain homotopy equivalences)**.** Two chain complexes A and B are said to be **chain homotopy equivalent** if there are some $f \colon A \to b$, $g \colon B \to A$ such that $gf \simeq 1_A$ and $fg \simeq 1_B$.

Lemma 1.4.12. *If $f \simeq g$ then $f_* = g_*$ on homology.*

Proof. Check. □

Definition 1.4.13 (Quasi-isomorphisms)**.** A chain map f inducing isomorphisms on homology is called a **quasi-isomorphism**. Two chains A, B are quasi-isomorphic if there is a quasi-isomorphism $A \to B$ and $B \to A$.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbf{Z} \arrow[r] \arrow[d, "n"'] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathbf{Z} \arrow[r, "\mathrm{pr}"] & \mathbf{Z} \arrow[r] & \cdots
\end{tikzcd}$$

Example 1.4.14. is a quasi-isomorphism.

Any chain homotopy equivalence is a quasi-isomorphism, the converse is false however.

Definition 1.4.16 (Cones)**.** Given $f \colon A \to B$ we define a chain complex called the **cone** of f by $\mathrm{cone}(f)_n = A_{n-1} \oplus B_n$ with maps

$$d = \begin{pmatrix} -d_A & 0 \\ -f & d_B \end{pmatrix}.$$

Chapter 2

Applications

Chapter 3

Spectral Sequences

Chapter 4

Model Categories