## Part III Homological and Homotopical Algebra 2014

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### Elements of Homological Algebra

#### 1.1 Introduction

These are lecture notes for the 2014 Part III Homological and Homotopical Algebra course taught by Dr. Julian Holstein, these notes are part of Mjolnir.

The recommended books are:

- W. G. Dwyer and J. Spalinski, Homotopy theories and model categories
- S. I. Gelfand and Yu. I. Manin, Methods of Homological Algebra
- C. Weibel, An introduction to homological algebra

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#### 1.2 Motivation

Start with a graded ring  $\mathbf{C}[x_0,\ldots,x_n]$  with deg  $x_i=1$ . Consider a graded module  $M=\bigoplus_d M_d$  over R. Hilbert looked at the map  $d\mapsto H_M(d)=\dim_{\mathbf{C}} M_d$ . For example we can take R to be the homogeneous coordinate ring of  $\mathbf{P}^n$  and  $V(I)\subset \mathbf{P}^n$  a subvariety where I is a homogeneous ideal. We then take M=R/I, if V is a curve C then  $H_{R/I}(d)=\deg(V)\cdot d+(1-g(C))$ . Hilbert showed that the function  $H_M(d)$  is eventually polynomial. We can compute this function easily if M is free so we try to replace M by free modules. First we take

$$K_0 \to F_0 \to M$$

where  $K_0$  is the kernel of the surjective map from  $F_0$  to M. We can continue this getting

$$K_1 \to F_1 \to K_0$$

$$K_2 \to F_2 \to K_1$$
.

we can then write

$$\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$
,

this is a free resolution of M. We also have the following.

**Theorem 1.2.1** (Hilbert).  $F_{n+1} = 0$ .

Corollary 1.2.2.  $H_M(d) = \sum_i (-1)^i H_{F_i}(d)$ .

#### 1.3 Categorical notions

#### 1.3.1 Abelian Categories

**Example 1.3.1.** Rmod - the category of left R-modules for R an associative ring is an abelian category.

**Example 1.3.2.** The categories of sheaves of abelian groups on a topological space, sheaves of  $\mathcal{O}$ -modules on a scheme and (quasi-)coherent sheaves on a scheme are all abelian.

**Definition 1.3.3** (Additive categories). An **additive category** is a category in which:

- 1. Every hom-space has the structure of an abelian group.
- 2. There exists a 0-object (one with exactly one map to and from every other object).
- 3. Finite products exist (these are automatically equal to sums  $A \times B = A \oplus B = A \coprod B$ ).

In such a category we let

$$\ker(f) = \operatorname{eq}(A \xrightarrow{f \atop 0} B)$$

and

$$\operatorname{coker}(f) = \operatorname{coeq}(A \xrightarrow{f \atop 0} B).$$

**Definition 1.3.4** (Abelian categories). An **abelian category**  $\mathcal{A}$  is an additive category in which:

- 1. Every map f has a kernel and cokernel.
- 2. For all f we have  $\operatorname{coker}(\ker(f)) = \operatorname{im}(f) = \operatorname{coim}(f) = \ker(\operatorname{coker}(f))$ .

**Example 1.3.5.** Let  $\mathcal{B}$  be the category of pairs of vector spaces  $V \subset W$ , with morphisms the compatible linear maps. Consider the natural map  $f \colon 0 \subset V \to V \subset V$ , we then have im  $f \cong 0 \subset V$  but coim  $f \cong V \subset V$ . So this category is not abelian.

From now on we take A to be any abelian category.

#### 1.3.2 Exactness

**Definition 1.3.6** (Exact sequences). A sequence of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in  $\mathcal{A}$  is **exact at** B if im  $f = \ker g$ . A sequence is then exact if it is exact everywhere. An exact sequence of the form

$$0 \to A \to B \to C \to 0$$

is called a **short exact sequence**.

**Definition 1.3.7** (Mono and epi morphisms). A morphism f is a **monomorphism** if  $fg = fh \implies g = h$  and it is an **epimorphism** if  $gf = hf \implies g = h$ .

**Example 1.3.8.** In Ab the following are exact sequences:

$$0 \to \mathbf{Z}/2 \to \mathbf{Z}/2 \oplus \mathbf{Z}/2 \to \mathbf{Z}/2 \to 0$$
$$0 \to \mathbf{Z}/2 \to \mathbf{Z}/4 \to \mathbf{Z}/2 \to 0$$
$$0 \to \mathbf{Z} \xrightarrow{\cdot 3} \mathbf{Z} \to \mathbf{Z}/3 \to 0$$

**Definition 1.3.9** (Additive functors). A functor of additive categories is **additive** if it is a homomorphism on hom-sets.

#### 1.4 Chain complexes

**Definition 1.4.1** (Chain complexes). A **chain complex** $C_{\bullet}$  is a collection of objects  $(C_i)_{i \in \mathbb{Z}}$  in A with maps  $d_i : C_i \to C_{i-1}$  such that  $d_{i-1} \circ d_i = 0$ .

**Definition 1.4.2** (Cycles, boundaries, homology objects). We define the **cycles**  $Z_i = \ker d_i$  and **boundaries**  $B_i = \operatorname{im} d_{i+1}$  and the *i*th **homology object**  $H_i(C) = \operatorname{coker}(B_i \to Z_i)$ . A complex is **acyclic** if it is exact i.e.  $H_{\bullet}(C) = 0$ .

**Definition 1.4.3** (Cochain complexes). A **cochain complex** $C^{\bullet}$  is a collection of objects  $(C^i)_{i \in \mathbf{Z}}$  in  $\mathcal{A}$  with maps  $d_i \colon C_i \to C_{i+1}$  such that  $d_{i+1} \circ d_i = 0$ . We then have as above  $H^i$  the *i*th **cohomology object**.

We can switch between chain complexes and cochain complexes via  $C^i = C_{-i}$ .

**Example 1.4.4.** We have many such complexes:

- Singular (co-)chain complex on a top space.
- de Rahm complex.
- Cellular chain complex.
- Flabby resolution of a sheaf.
- Bar resolution of a group.
- Koszul complex.

**Definition 1.4.5** (Chain maps). Given B, C chain complexes, a **chain map**  $f: B \to C$  is a collection of maps  $f_i: B_i \to C_i$  such that df = fd.

We now have formed the **category of chain complexes** Ch(A) using these maps. We write Ch(R) for Ch(Rmod). Note that Ch(A) is an additive category moreover it is an abelian category, we can define and check everything level-wise. For example  $\ker(A \to B)_n = \ker(A_n \to B_n)$ . Note that the  $H_n$  form a functor  $Ch(A) \to A$ . Define  $f_* \colon H_nA \to H_nB$  in the natural wat and check it works.  $H_n$  is additive.

**Lemma 1.4.6** (Snake lemma). Let  $0 \to A \to B \to C \to 0$  be a short exact sequence then there exist natural boundary maps  $\partial_n$  which fit into a long exact sequence of homology objects

$$\cdots \longrightarrow H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C)$$

$$\xrightarrow{\partial_n} H_{n-1}(A) \longrightarrow H_{n-1}(B) \longrightarrow H_{n-1}(C) \longrightarrow \cdots$$

Proof. Exercise.

Naturality here means given two short exact sequences and compatible chain maps the induced maps on homology are compatible with  $\partial_n$ . (The obvious diagram commutes.)

Recall that f is a chain map if  $\partial f - f \partial = 0$ .

**Definition 1.4.8.** Let  $\underline{\text{Hom}}_n(A, B)$  consist of functions  $\{f_i : A_i \to B_{i+n}\}$  and define  $df = d \cdot f - (-1)^n f d$  if  $f \in \underline{\text{Hom}}_n$ . Check that

$$d^{2}f = d \cdot (d \cdot f - (-1)^{f}d) - (-1)(d \cdot f - (-1)^{n}f \cdot d) \cdot d = 0.$$

We use the "Sign rule" to help with definitions, this states that if a moves past b we pick a sign  $(-1)^{\deg a \deg b}$ .

Ch(A) can be enriched over  $Ch(\mathbf{Z})$ .

**Definition 1.4.9** (Shifted complexes). The **shifted complex**C[n] for  $C \in Ch(\mathcal{A})$  is defined by  $C[n]_i = C_{n+i}$  and  $d_i^{C[n]} = (-1)^n d_{n+i}^C$ .

Note that  $H_i(C) = H_0(C[i])$ .

So a chain map  $f: A \to B[n]$  is exactly a cycle in  $\underline{\mathrm{Hom}}_n(A,B)$ .

Now  $\operatorname{Hom}(A, B) = Z_0(\operatorname{\underline{Hom}}(A, B))$ , so what is  $H_0(\operatorname{\underline{Hom}}(A, B))$ ?

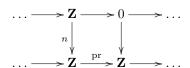
**Definition 1.4.10** (Chain homotopies). A **chain homotopy** S between chain maps  $f, g: A \to B$  is a collection  $S_i: A_i \to B_i$  such that  $\partial S + S \partial = f - g$ . Equivalently we could say a map  $A \to B[1]$  such that dS = g - f (note: not a chain map). We write  $f \simeq g$  to denote the fact that f is chain homotopic to g.

**Definition 1.4.11** (Chain homotopy equivalences). Two chain complexes A and B are said to be **chain homotopy equivalent** if there are some  $f: A \to b, g: B \to A$  such that  $gf \simeq 1_A$  and  $fg \simeq 1_B$ .

**Lemma 1.4.12.** If  $f \simeq g$  then  $f_* = g_*$  on homology.

Proof. Check. 
$$\Box$$

**Definition 1.4.13** (Quasi-isomorphisms). A chain map f inducing isomorphisms on homology is called a **quasi-isomorphism**. Two chains A, B are quasi-isomorphic if there is a quasi-isomorphism  $A \to B$  and  $B \to A$ .



**Example 1.4.14.** is a quasi-isomorphism.

Any chain homotopy equivalence is a quasi-isomorphism, the converse is false however.

**Definition 1.4.16** (Cones). Given  $f: A \to B$  we define a chain complex called the **cone** of f by  $cone(f)_n = A_{n-1} \oplus B_n$  with maps

$$d = \begin{pmatrix} -d_A & 0 \\ -f & d_B \end{pmatrix}.$$

Note that there exists a short exact sequence

$$B \xrightarrow[b\mapsto(0,b)]{} \operatorname{cone}(f) \underset{a,b)\mapsto-a}{\longrightarrow} A[-1].$$

Doing the diagram chase of the Snake lemma 1.4.6 we see that the boundary map is induced by f on homology i.e.

$$f_*: H_{n-1}A \to H_{n-1}B.$$

This proves the following.

**Lemma 1.4.17.** f is a quasi-isomorphism if and only if cone(f) is exact.

*Proof.* Look at the long exact sequence of  $B \to \text{cone}(f) \to A[-1]$ 

$$H_n(\operatorname{cone}(f)) \to H_n(A) \xrightarrow{f_*} H_{n-1}(B) \to H_{n-1}(\operatorname{cone}(f)).$$

#### 1.5 Exact Functors

**Definition 1.5.1** (Exact functors). An additive functor F is **exact** if it preserves short exact sequences. It is **left exact** if it sends a short exact sequence of the form

$$0 \to A \to B \to C \to 0$$

to an exact sequence

$$0 \to FA \to FB \to FC$$
.

We have a similar definition for **right exact**.

**Example 1.5.2.** The functor  $\operatorname{Hom}_{\mathcal{A}}(M,-)$  is left exact from  $\mathcal{A}$  to  $\operatorname{Ab} = \mathbf{Z} \operatorname{mod}$ . The functor  $\operatorname{Hom}_{\mathcal{A}}(-,M) \colon \mathcal{A}^{\operatorname{op}} \to \operatorname{Ab}$  is left exact.

Note that left adjoint functors are right exact as they preserve colimits.

**Example 1.5.3.** Let M be an R, S-bimodule (i.e. a left R-module and a right S-module). Then for  $A \in S \mod$ ,  $B \in R \mod$ 

$$\operatorname{Hom}_R(M \otimes_S A, B) \cong \operatorname{Hom}_S(M, \operatorname{Hom}_R(A, B))$$

Clearly not all functors are exact. However they all preserve split exact sequences, i.e. those of the form

$$0 \to A \to A \oplus C \to C$$
.

Because they preserve finite direct sums

(r,g),(f,s) are inverse isomorphisms if and only if g=fr+sg.

#### 1.6 Derived Functors, Introduction

We fix A, and Ch(A). If we have some right exact functor F we obtain exact sequences of the form

$$FA \rightarrow FB \rightarrow FC \rightarrow 0$$

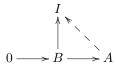
and the question arises, can we extend this exact sequence by placing objects to the left of it?

If F is exact on short exact sequences of complexes we get a long exact sequence of homology  $H_iFA$ . F is exact on complexes if it is level wise exact, but F is exact if it is level wise exact. We know F is exact on split exact sequences. So we can try to force a short exact sequence to be exact by replacing objects by complexes.

**Definition 1.6.1** (Projective and injective objects). An object M is **projective** if for all epimorphisms q and maps  $M \xrightarrow{f} B$  there exists a lift making

$$A \xrightarrow{f} B \longrightarrow 0$$

commute. The dual notion is called **injective** 



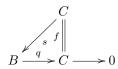
**Example 1.6.2.** Free modules in Rmod are projective. In  $Mat_n(R)$ -mod the column vectors  $R^n$  form a projective object.  $\mathbf{Q}$  is injective in Ab.

**Lemma 1.6.3.** If C is projective or A is injective then

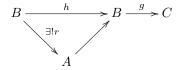
$$0 \to A \to B \to C \to 0$$

is split.

*Proof.* (We prove the C projective case) Consider



then gs = 1. Now produce r such that  $rf = 1_A$ ,  $fr + sg = 1_B$  and rs = 0. Let h = 1 - sg. Now gh = 0 giving that h = fr by the properties of the kernel.



Now check  $rf = 1_A$  and rs = 0.

Note that in Rmod this shows projectives are exactly summands of free modules.

**Definition 1.6.4** (Projective resolutions). A **projective resolution**  $P_{\bullet} \stackrel{\epsilon}{\to} A$  of A is a non-negative chain comlex such that all  $P_i$  are projective and  $\epsilon$  is a quasi-isomorphism. So  $H_iP = 0$  if i > 0 and  $H_0P = A$ .

**Definition 1.6.5** (Derived functors). The *i*th **left derived functor** $L_iF(A)$  of a right exact functor F is defined as  $H_iF(P)$  for some projective resolution P of A.

Dually we may define injective resolutions  $B \xrightarrow{\sim} I^{\bullet}$  with  $I \in \mathrm{Ch}^{\geq 0}(\mathcal{A})$  and we get **right derived functors** of a left exact functor,

$$R^i F(B) = H^i(FI).$$

Note  $L_{<0}F(A) = 0$  and  $L_0 = fP_0/FP_1 = F(P_0/P_1) = F(A)$ .

**Example 1.6.6** (Tor). Define  $\operatorname{Tor}_{i}^{R}(A,B)$  to be  $L_{i}(-\otimes_{R}B)(A)$ . Let  $\mathcal{A}=\operatorname{Ab}$ . What is  $\operatorname{Tor}_{i}(\mathbf{Z}/p,B)$ ?

$$\mathbf{Z}$$
 $p$ 
 $\mathbf{Z} \xrightarrow{p} \mathbf{Z} \xrightarrow{\sim} \mathbf{Z}_{p}$ 

is a projective resolution. So  $\operatorname{Tor}_* = H_*(B \xrightarrow{p} B)$  and we have  $\operatorname{Tor}_0^{\mathbf{Z}}(\mathbf{Z}/p, B) = B/pB$  and  $\operatorname{Tor}_1^{\mathbf{Z}}(\mathbf{Z}/p, B) = pB = \{b : pb = 0\}.$ 

**Example 1.6.7** (Ext). Define  $\operatorname{Ext}_R^i(A,B)$  to be  $R^i\operatorname{Hom}_R(-,B)(A)$ . Injective in Rmod<sup>op</sup> correspond to projectives in Rmod. So  $\operatorname{Ext}_{\mathbf{Z}}^*i(\mathbf{Z}/p\mathbf{Z},B)=H_*(B\overset{p}{\to}B)$  hence  $\operatorname{Ext}^0(\mathbf{Z}/p,B)={}_pB$  and  $\operatorname{Ext}^1(\mathbf{Z}/p,B)=B/pB$ .

#### 1.7 Derived Functors, Proofs

**Definition 1.7.1.**  $\mathcal{A}$  has **enough projectives** if for all  $M \in \mathcal{A}$  there exists a projective P such that  $P \to M \to 0$ .

Example 1.7.2. Rmod has enough projectives.

Warning: The category of abelian sheaves on a topological space does not have enough projectives in general.

**Lemma 1.7.3.** Projective resolutions exist in A if A has enough projectives.

*Proof.* Let  $A \in \mathcal{A}$ , then there exists

$$0 \to K_0 \to P_0 \to A \to 0$$

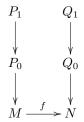
and inductively

$$0 \to K_{n+1} \to P_{n+1} \to K_n \to 0$$

with  $P_i$  projective. We can splice these together to get

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0.$$

**Theorem 1.7.4** (Comparison Theorem). Let  $\epsilon \colon P \to M$  and  $\eta \colon Q \to N$  be two projective resolutions and let  $f \colon M \to N$  then there exists a lift  $\tilde{f} \colon P \to Q$  (a chain map) unique up to chain homotopy.



*Proof.* Exercise.  $\Box$ 

Corollary 1.7.5. Projective resolutions are well defined up to chain homotopy equivalence and so derived functors are well defined.

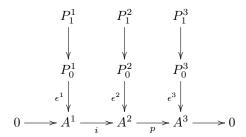
*Proof.* Lift the identity to get chain maps in both directions. Uniqueness implies that they are inverse up to homotopy.  $\Box$ 

Corollary 1.7.6.  $L_iF$  are functors.

Lemma 1.7.7 (Horseshoe Lemma). Given a short exact sequence

$$A^1 \rightarrow A^2 \rightarrow A^3$$

and projective resolutions  $P^1 \to A^1$  and  $P^3 \to A^3$  there exists a projective resolution  $P^2$  of  $A^2$  with  $P_i^2 = P_i^1 \oplus P_i^3$  and the inclusion and projection maps lift. So we have the following situation



*Proof.* By induction: To get  $\epsilon^2 \colon P_0^i \oplus P_0^3 \to A^2$  we use i and  $\epsilon^1$  and a lift of p. Now the Snake lemma 1.4.6 shows that  $\operatorname{coker} \epsilon^i$  and  $\ker \epsilon^i$  fit into a long exact sequence and hence  $\operatorname{coker} \epsilon^3 = 0$ . Now apply the induction assumption to the short exact sequence of kernels

$$0 \to \ker \epsilon^1 \to \ker \epsilon^2 \to \ker \epsilon^3 \to \operatorname{coker} \epsilon^1 = 0.$$

Corollary 1.7.8. A short exact sequence  $0 \to A \to B \to C \to 0$  in A gives a long exact sequence of left derived functors

$$\rightarrow L_2FC \rightarrow L_1FA \rightarrow L_1FB \rightarrow L_1FC \rightarrow FA \rightarrow FB \rightarrow FC \rightarrow 0.$$

*Proof.* Combine Horseshoe Lemma 1.7.7, 1.6.3 and Snake lemma 1.4.6. □

**Proposition 1.7.9.** The boundary map  $\partial$  is natural, i.e. given

we have lifts  $\partial \circ L_i f_3 = L_{i-1} f_1 \circ \partial$ .

Note that there is no extra work needed to do all of this for right derived functors.

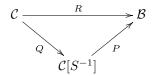
#### 1.8 The Derived Category

Idea: We want to talk about complexes up to quasi-isomorphism. We will reinterpret derived functors as ways of lifting functors to derived categories.

Remark 1.8.1. If we add simply inverses of quasi-isomorphisms we get nasty stuff!

**Definition 1.8.2** (Homotopy categories). Let the homotopy category  $K(\mathcal{A})$  of  $\mathcal{A}$  have objects the objects of  $Ch(\mathcal{A})$  and morphism the chain homotopy classes of chain maps. We can add boundedness conditions to our categories. So we let  $Ch_+(\mathcal{A})$  be only those chain complexes A with  $A_n = 0$  when n << 0, these are **bounded below** chain complexes. Similarly we define  $Ch_-(\mathcal{A})$  and  $Ch_b(\mathcal{A}) = Ch_-(\mathcal{A}) \cap Ch_+(\mathcal{A})$ . We also define  $Ch^+(\mathcal{A})$  etc. for cochain complexes. Finally we define  $K^+(\mathcal{A})$  etc. in the obvious way.

**Definition 1.8.3** (Localisations of categories). Given a category  $\mathcal{C}$  and a class of morphisms S define the localisation of  $\mathcal{C}$  at S to be a category  $\mathcal{C}[S^{-1}]$  with a functor  $\mathcal{C} \xrightarrow{Q} \mathcal{C}[S^{-1}]$  such that Q sends any  $s \in S$  to an isomorphism, and also such that Q is universal with respect to having this property. If  $\mathcal{C} \xrightarrow{R} \mathcal{B}$  sends S to isomorphisms then there exists some P so that we have



**Definition 1.8.4.** Let D(A), the **derived category** of A, be the localisation of K(A) at the quasi-isomorphisms. Similarly define the usual suspects  $D^b = K^b(A)$ [quasi-isomorphisms], etc.

Theorem 1.8.5. D(A) exists.

Proof. See Weibel 10.3, Gelfand-Manin III 2.

Although we didn't prove this we should note that we can write morphisms in D(A) as

$$A \xleftarrow{\sim} A' \xrightarrow{f} B$$

with  $f \in \operatorname{Hom}_{K(A)}(A', B)$  and  $q \in \operatorname{Hom}_{K(A)}(A', A)$ .

**Remark 1.8.6.**  $D^b(A)$  is equivalent to the subcategory of D(A) with cohomology in bounded degrees.

**Example 1.8.7.** Let X be a scheme, Coh(X) the abelian category of coherent sheaves on X. Then the derived category of X is defined to be  $D^b(X) = D^b(Coh(X))$ .

Note that D(A) is an additive, but not necessarily abelian category.

**Theorem 1.8.8.** Given a complex  $I^{\bullet} \in K^{+}(A)$  of injective objects and any chain complex  $A^{\bullet}$  then

$$\operatorname{Hom}_{D(\mathcal{A})}(A^{\bullet}, I^{\bullet}) \cong \operatorname{Hom}_{K(\mathcal{A})}(A^{\bullet}, I^{\bullet}).$$

*Proof.* (Sketch) Crucial ingredient:  $\operatorname{Hom}_{K(\mathcal{A})}(-,I^{\bullet})$  sends quasi-isomorphisms to quasi-isomorphisms. So we can replace

$$A \xleftarrow{\sim} A' \to I$$

by  $A \to I$ . By considering cones it suffices to check that  $\underline{\mathrm{Hom}}_{K(\mathcal{A})}(-,I)$  sends acyclics to complexes homotopy equivalent to 0. One can construct the homotopy equivalence by hand, using injectivity.

Corollary 1.8.9.

$$\operatorname{Hom}_{D(\mathcal{A})}(A, B[i]) = \operatorname{Ext}_{\mathcal{A}}^{i}(A, B).$$

*Proof.* Let  $B \to I^{\bullet}$  be an injective resolution. Then both sides are isomorphic to

$$\operatorname{Hom}_{K(\mathcal{A})}(A, I[i]) = H^0 \operatorname{\underline{Hom}}_{K(\mathcal{A})}(A, I[i]).$$

**Corollary 1.8.10.** Assume A has enough injectives and write  $\operatorname{inj} A \subset A$  for the full subcategory of injective objects. Then

$$K^+(\operatorname{inj} \mathcal{A}) \cong D^+(\mathcal{A}).$$

*Proof.* We have fully faithfullnes by 1.8.8. To see that it is essentially surjective write down injective resolutions for complexes (see later).

#### 1.9 Total derived functors

We now interpret/redefine derived functors as lifts to the derived category.

**Definition 1.9.1.** Let F:

# Applications

# Spectral Sequences

## **Model Categories**