Part III Algebraic Geometry 2014

Part III Algebraic Geometry 2014

Contents

1 Sheaf Theory **1**
1.1 Introduction 1
1.2 Sheaves 1

2 Locally free and coherent $\mathcal{O}_X$-modules **7**
2.1 Locally free and coherent $\mathcal{O}_X$-modules 7

Chapter 1

Sheaf Theory

1.1 Introduction

These are lecture notes for the 2014 Part III Algebraic Geometry course taught by Dr. P.M.H. Wilson.

The recommended books are:

- Algebraic Geometry - ??

Generated: May 23, 2015, 07:31:00 (Z)

1.2 Sheaves

Let X be a topological space.

Definition 1.2.1 (Presheaves)**.** A **presheaf**$\mathcal{F}$ of abelian groups (resp. rings) on X consists of data:

1. For every open $U \subset X$ an abelian group (resp. ring) $\mathcal{F}(U)$.
2. For every inclusion of open sets $V \subset U$ a homomorphism called **restriction**, denoted $\rho^U_V \colon \mathcal{F}(U) \to \mathcal{F}(V)$ such that
   (a) $\mathcal{F}(\emptyset) = 0$.
   (b) $\rho^U_U \colon \mathcal{F}(U) \to \mathcal{F}(U)$ is $id_{\mathcal{F}(U)}$.
   (c) If $W \subset V \subset U$ open then $\rho^U_W = \rho^V_W \circ \rho^U_V$.

Remark 1.2.2. If $\mathcal{U}$ denotes the category of open sets in X (the morphisms are inclusions) then a presheaf of abelian groups over X is a contravariant functor $\mathcal{F} \colon \mathcal{U} \to \mathrm{Abgp}$. i.e. an element of $\mathrm{Abgp}^{\mathcal{U}^{\mathrm{op}}}$.

An element $s \in \mathcal{F}(U)$ is called a **section** of $\mathcal{F}$ over U. For $s \in \mathcal{F}(U)$ we denote $\rho^U_V(s)$ by $s|_V$.

Definition 1.2.3 (Sheaves)**.** A presheaf $\mathcal{F}$ on X is a **sheaf** if it satisfies two further conditions:

1. If U is open and has $U = \bigcup_i U_i$ an open cover and if $s \in \mathcal{F}(U)$ is such that $s|_{V_i} = 0$ for all i then $s = 0$. (A presheaf satisfying this condition is called a **monopresheaf**).
2. If $U = \bigcup_i V_i$ is an open cover and we have $s_i \in \mathcal{F}(V_i)$ such that $\forall i, j$ $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ then there exists $s \in \mathcal{F}(U)$ with $s|_{V_i} = s_i$ for all i.

Example 1.2.4. X a topological space, A any abelian group (resp. ring). The constant shead $\mathcal{A}$ determined by A is defined as follows: $\mathcal{A}(\emptyset) = \{0\}$, for $U \neq \emptyset$ open in X

$$\mathcal{A}(U) = \{\text{locally constant maps } U \to A\},$$

this is an abelian group (resp. ring) under pointwise operations. With obvious restriction maps we obtain a sheaf. If $U \neq \emptyset$ is open and connected, then $\mathcal{A}(U) = A$. If U an open set whose connected components are open (e.g. in a locally connected topological space X) then the section $\mathcal{A}(U)$ is a direct product of copies of A.

Example 1.2.5. If X is a differentiable (C^∞) manifold, we can define the sheaf of C^∞-functions ($\mathbf{R}$ or $\mathbf{C}$ valued) on X. Which is a sheaf of rings. Similarly if X is a complex manifold, we can define a sheaf of holomorphic functions on X. In both cases, the sheaf is called the **structure sheaf** of X, sometimes denoted $\mathcal{O}_X$.

Example 1.2.6. For V an (irreducible) variety (affine, projective, quasi-projective). We can consider V as a topological space with the Zariski topology. For U open in V set

$$\mathcal{O}_V(U) = \{\text{regular functions on } U\} = \{f \in k(V) \text{ s.t. } f \text{ regular on } U\}.$$

This is a sheaf of rings with respect to the Zariski topology, and is known as the **structure sheaf** for varieties. If V is affine we have that $\mathcal{O}_V(V) = k[V]$.

Definition 1.2.7 (Stalks)**.** If $\mathcal{F}$ is a presheaf on a topological space X and $P \in X$ we define the stalk $\mathcal{F}_P$ of $\mathcal{F}$ at P to be $\varinjlim_{U \ni P} \mathcal{F}(U)$ i.e. an element of $\mathcal{F}_P$ is represented by a pair (U, s) where U is an open neighbourhood of P and $s \in \mathcal{F}(U)$, where (U, s) and (V, t) define the same element of $\mathcal{F}_P$ if there exists an open neighbourhood $W \ni P$ with $W \subset V \cap U$ such that $s|_W = t|_W$ the elements s_P of $\mathcal{F}_P$ are called **germs**. If $\mathcal{F}$ is a sheaf of abelian groups or rings then $\mathcal{F}_P$ is an abelian group, ring, etc.

Example 1.2.8. For the constant sheaf $\mathcal{A}$ associated to A we have $\mathcal{A}_P = A$.

Example 1.2.9. For X a C^∞ manifold (resp. complex) with structure sheaf $\mathcal{O}_X$, the stalk $\mathcal{O}_{X,P}$ of $\mathcal{O}_X$ at $P \in V$ consists of germs of C^∞ (resp. holomorphic) functions.

Example 1.2.10. For V (irreducible) affine, projective or quasi-projective variety with structure sheaf $\mathcal{O}_V$ the stalk at $P \in V \mathcal{O}_{V,P} =$ local ring at P (defined before).

Definition 1.2.11 (Morphisms of (pre)sheaves)**.** If $\mathcal{F}$ and $\mathcal{G}$ are presheaves (resp. sheaves) on X a morphism $\Phi\colon \mathcal{F} \to \mathcal{G}$ consists of homomorphisms $\mathcal{F}(U) \xrightarrow{\Phi} \mathcal{G}(U)$ for all open U such that for $V \subseteq U$ open

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\phi(U)"] \arrow[d, "\rho_V^U"'] & \mathcal{G}(U) \arrow[d, "\rho'^U_V"] \\
\mathcal{F}(V) \arrow[r, "\phi(V)"'] & \mathcal{G}(V)
\end{tikzcd}$$

commutes.

A morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ induces a homomorphism $\phi_P\colon \mathcal{F}_P \to \mathcal{G}_P$ for each P, namely $\phi_P[(U, s)] = [(U, \phi(U)(s))]$, which is well defined.

Definition 1.2.12 (Injective and isomorphic sheaf morphisms)**.** A morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ of (pre)sheaves is **injective** if $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open U. e.g. sheaves of subgroups or subrings where $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ for all U. In this case $\mathcal{F}$ is called a **subsheaf** of $\mathcal{G}$.
A morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ is called an isomorphism if there exists an inverse morphism $\chi\colon \mathcal{G} \to \mathcal{F}$. This is equivalent to $\phi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ being bijective for all U since we can define $\chi(U) = \phi(U)^{-1}$ as the inverse.

Lemma 1.2.13. *Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves then*

1. *ϕ is injective iff $\phi_P\colon \mathcal{F}_P \to \mathcal{G}_P$ is injective for all $P \in X$,*
2. *ϕ is an isomorphism iff $\phi_P\colon \mathcal{F}_P \to \mathcal{G}_P$ is an isomorphism for all $P \in X$.*

Proof. ($\Rightarrow$) (true for presheaves too).

1. Suppose there exists a germ $s_P \in \mathcal{F}_P$ such that $\phi_P(s_P) = 0 \in \mathcal{G}_P$. i.e. there exists an open neighbourhood $W \subset U$ with $P \in W$ such that $\phi(U)(s|_W) = 0$. So by commutativity $\phi(W)(s|_W) = 0$ but ϕ injective implies $s|_W = 0$.
2. Clear.

($\Leftarrow$)

1. Needs first sheaf condition on $\mathcal{F}$. If ϕ_P injective for all P and U is open in X it remains to prove that $\phi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective. Suppose not and there exists $0 \neq s \in \mathcal{F}(u)$ such that $\phi(U)(s) = 0 \in \mathcal{G}(U)$. Let s_P denote the germ of s at $P \in U$, then $0 = \phi(U)(s)_P = \phi_P(s_P)$ for all $P \in U$. So s_P in $\mathcal{F}_P$ for all $P \in U$. Hence for all $P \in U$ we have an open neighbourhood $U \supset W \ni P$ such that $s|_W = 0$. So U is covered by open sets U_α such that $s|_{U_\alpha} = 0$ for all α, which implies that $s = 0$.
2. $\phi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an injection for all open U by the first part, so it remains to prove that it is surjective also. Suppose $t \in \mathcal{G}(U)$ and let $t_P \in \mathcal{G}_P$ be its germ at $P \in U$. Since ϕ_P is surjective we have some $s_P \in \mathcal{F}_P$ such that $\phi_P(s_P) = t_P$. Now suppose that s_P is represented by a pair (V, s) with $P \in V \subseteq U$ and $s \in \mathcal{F}(V)$. We then have that t_P is represented by $\phi(V)(s)$, i.e. $(U, t) \sim (V, \phi(V)(s))$. Shrinking V we may assume that we have an ope neighbourhood $V_P \ni P$ such that $\phi(V)(s)|_{V_P} = t|_{V_P}$. In this way we cover U by open sets giving $U == \bigcup_\alpha U_\alpha$ to obtain sections $s_\alpha \in \mathcal{F}_\alpha$ such that $\phi(U_\alpha)(s_\alpha) = t|_{U_\alpha}$. On the overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$ we have $\phi(U_{\alpha\beta})(s_\alpha|_{U_{\alpha\beta}}) = t|_{U_{\alpha\beta}} = \phi(U_{\alpha\beta})(s_\beta|_{U_{\alpha\beta}})$, therefore the injectivity of $\phi(U_{\alpha\beta})$ gives that $s_\alpha|_{U_{\alpha\beta}} = s_\beta|_{U_{\alpha\beta}}$. Since $\mathcal{F}$ is a sheaf the s_α patch together to give a section $s \in \mathcal{F}(U)$ wuch that $s|_{U_\alpha} = s_\alpha$ (using the second sheaf condition for $\mathcal{F}$). But then $\phi(U)(s)$ and t are sections of $\mathcal{G}(U)$ such that $\phi(U)(s)|_{U_\alpha} = \phi(U_\alpha)(s_\alpha) = t|_{U_\alpha}$ for all α. The first sheaf condition for $\mathcal{G}$ now gives that $\phi(U)(s) = t$ as required.

□

Definition 1.2.14 (Surjective sheaf morphisms)**.** A morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ is called **surjective** if $\phi_P\colon \mathcal{F}_P \to \mathcal{G}_P$ is surjective for all $P \in X$.

Definition 1.2.15 (Induced (pre)sheaves)**.** Given a (pre)sheaf $\mathcal{F}$ on a space X and a continuous map $f\colon X \to Y$ we have an **induced (pre)sheaf**, denoted $f_*\mathcal{F}$ on Y defined by

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$$

for U open in Y. With the restriction maps coming from $\mathcal{F}$ as $V \subset U$ implies $f^{-1}(V) \subset f^{-1}(U)$. It should be checked that indeed $\mathcal{F}$ being a (pre)sheaf implies $f_*\mathcal{F}$ is a (pre)sheaf.

Definition 1.2.16 (Ringed spaces)**.** A **ringed space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X.

Definition 1.2.17 (Morphisms of ringed spaces)**.** Given two ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ a **morphism of ringed spaces**$(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ where $f\colon X \to Y$ is continuous and $f^\#\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of rings. So $f^\#$ defines homomorphisms $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ for all U open in Y, compatible with restrictions. Hence we have homomorphisms on stalks too $\mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$.

Definition 1.2.18 (Ringed spaces over a ring)**.** If R is a commutative ring (e.g. a field), a **ringed space over** R is a ringed space with $\mathcal{O}_X$ a sheaf of R-algebras. (Therefore the restriction maps are homomorphisms of R-algebras.) A morphism of ringed spaces over R is defined in the obvious way.

Definition 1.2.19 (Locally ringed spaces)**.** A ringed space $(X, \mathcal{O}_X)$ is a **locally ringed space** (also known as a geometric space) if all $\mathcal{O}_{X,P}$ are local rings.

Definition 1.2.20 (Morphisms of locally ringed spaces)**.** A **morphism of locally ringed spaces** is a morphism of ringed spaces as above where all the induced maps $f^\#_P\colon \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ are local homomorphisms of local rings.

Example 1.2.21. $(X, \mathbf{Z})$ is a ringed space but not locally ringed.

Example 1.2.22. If X is a C^∞ (resp. complex) manifold with structure sheaf $\mathcal{O}_X$ then $(X, \mathcal{O}_X)$ is a locally ringed space over $\mathbf{R}$ (resp. over $\mathbf{C}$). A smooth (resp. holomorphic) map of manifolds $f\colon X \to Y$ yields a morphism of $\mathbf{R}$ (resp. $\mathbf{C}$)-algebras $f^\#\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ namely $f^\#\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$ given by $g \to g \circ f$ (smooth (resp. holomorphic) functions on Y pullback to ones on X). Clearly $g(f(P)) = 0 \iff f^\#(g)(P) = 0$ and so $f^\#(m_{Y,f(P)} \subseteq m_{X,P}$. So $(f, f^\#)$ is a morphism of locally ringed spaces over $\mathbf{R}$ (resp. $\mathbf{C}$).

Example 1.2.23. $(V, \mathcal{O}_V)$ for V an irreducible affine variety V is a ringed space via its structure sheaf, it is locally ringed over the base field k. If $\Phi\colon V \to W$ is a morphism of affine varieties then there exists a morphism of locally ringed spaces over $k(\phi, \phi^\#)\colon (V, \mathcal{O}_V) \to (W, \mathcal{O}_W)$ given by $\phi^\#(g) = g \circ \phi \in \mathcal{O}_V(\phi^{-1}U)$ for $g \in \mathcal{O}_W(U)$.

Lemma 1.2.24. *If V, W are (irreducible) affine varieties and $(f, f^\#)\colon (V, \mathcal{O}_V) \to (W, \mathcal{O}_W)$ is a morphism of locally ringed spaces over k then f is induced from a morphism of varieties $\phi\colon V \to W$ with $f^\# = \phi^\#$ defined as above.*

Proof. Suppose $V \subseteq \mathbf{A}^n$, $W \subseteq \mathbf{A}^m$ let y_j be the jth coordinate function on W and define $g_j = f^\#(y_j) \in \mathcal{O}_V(V) = k[V]$. Let $\phi = (g_1, \ldots, g_m)$, this is a morphism $V \to \mathbf{A}^m$. Suppose that $f(P) = (b_1, \ldots, b_m)$ for $P \in V$ then $y_j - b_j \in m_{W,f(P)}$ for all j which implies that $g_j(P) = b_j$ for all j. Since $f^\#$ is local we have that $\phi(P) = f(P)$ and so $\phi\colon V \to W$ is the same map as f on topological spaces. Moreover $y_1, \ldots, y_m$ generate $k[V]$ as a k-algebra and also generate $k(W)$ as a field over k. $f^\#(y_j) = g_j = y_j \circ \phi = \phi^\#(y_j)$ and it follows that $f^\# = \phi^\#$ on any $\mathcal{O}_W(U)$. $k[W] \subset \mathcal{O}_W(U) \subset k(W)$ for U open in W. □

Definition 1.2.25 (Morphisms of varieties)**.** Given V and W (irreducible) quasi-projective varieties, we define a **morphism** of varieties $V \to W$ to be a morphism of the corresponding locally ringed spaces over $k(V, \mathcal{O}_V) \to (W, \mathcal{O}_W)$.

1.2.1 $\mathcal{O}_X$-modules

Definition 1.2.26 ($\mathcal{O}_X$-modules)**.** Let M be a sheaf of abelian groups on a ringed space $(X, \mathcal{O}_X)$, M is said to be an $\mathcal{O}_X$**-module** if for every open set $U \subset X M(U)$ is an open $\mathcal{O}_X(U)$-module and for any $W \subseteq U$ open $\alpha \in \mathcal{O}_X(U)$, $m \in M(U)$, we have $(\alpha m)|_W = (\alpha|_W)(m|_W)$. Similarly we have the obvious definition for morphisms of $\mathcal{O}_X$-modules $\phi \colon M \to N$ (all maps respect the $\mathcal{O}_X$-module structure).

Example 1.2.27. For V a (irreducible) quasi-projective variety with structure sheaf $\mathcal{O}_V$ and $W \subset V$ a closed subvariety we have the **sheaf of ideals**$\mathcal{I}_W \subset \mathcal{O}_V$ a subsheaf of $\mathcal{O}_V$ given by

$$\mathcal{I}_W(U) = \{f \in \mathcal{O}_V(U) : f|_{W \cap U} \equiv 0\}.$$

This is clearly an $\mathcal{O}_V$-module.

Most things go through unchanged e.g. if M is an $\mathcal{O}_X$-module then any stalk M_P is an $\mathcal{O}_{X,P}$-module etc.
However there is the following technicality: If we take the pushforward of an $\mathcal{O}_X$-module M under a morphism of ringed spaces $(X, \mathcal{O}_X) \xrightarrow{\phi} (Y, \mathcal{O}_Y)$. The sheaf f_*M is naturally an $f_*\mathcal{O}_X$-module via the morphism $f^\# \colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ we can consider f_*M as an $\mathcal{O}_Y$-module which we denote by ϕ_*M. Explicitly for U open in Y we let

1.2.2 Sheafification

Definition 1.2.28 (Sheafification)**.** Given a preheaf $\mathcal{F}$ on X there exists an associated sheaf $\mathcal{F}^+$ and a morphism $\Theta \colon \mathcal{F} \to \mathcal{F}^+$ with the universal property that for any sheaf $\mathcal{G}$ on X and morphism $\phi \colon \mathcal{F} \to \mathcal{G}$ there is a unique morphism $\psi \colon \mathcal{F}^+ \to \mathcal{G}$ such that $\phi = \psi\Theta$. We construct $\mathcal{F}^+$ as follows: For U open in X we let

$$\mathcal{F}^+(U) = \left\{ \text{functions } s \colon U \to \coprod \mathcal{F}_p : \forall p \in U s(p) \in \mathcal{F}_p, \, \forall p \in U \exists \text{ open nbhd } W, \, t \in \mathcal{F}(W) \text{ with } s(q) = t_q \forall q \in Q \right\}$$

It is clear that this is a sheaf since sections are given in terms of functions.

There exists a morphism $\Theta \colon \mathcal{F} \to \mathcal{F}^+$ where $\Theta(U) \colon \mathcal{F}(U) \to \mathcal{F}^+(U)$ is given by $\sigma \mapsto s$ with $s(p) = \sigma_p$ for all $p \in U$.

Chapter 2

Locally free and coherent $\mathcal{O}_X$-modules

2.1 Locally free and coherent $\mathcal{O}_X$-modules

Definition 2.1.1 (Locally free $\mathcal{O}_X$-modules)**.** An $\mathcal{O}_X$-module M is locally free of rank r if for each $x \in X$ there exists an open neighbourhood $U \ni x$ s.t. $M|_U$ is isomorphic to $\mathcal{O}_U^r = \bigoplus_{i=1}^r \mathcal{O}_U$

Example 2.1.2. If $(X, \mathcal{O}_X)$ is a C^∞ (resp. complex) manifold and $E \to X$ a rank rC^∞ (resp. holomorphic) vector bundle over X we can define a corresponding $\mathcal{O}_X$-module $\mathcal{E}$ by letting $\mathcal{E}(U)$ be the C^∞_* (resp. holomorphic) sections σ of $\mathcal{E}$ over U. Since $\mathcal{E}$ is locally trivial it's clear that $\mathcal{E}$ is locally free of rank r.

Definition 2.1.3 (Transition functions)**.** For M a locally free $\mathcal{O}_X$-module of rank r we have an open cover $\{U_i\}$ of X (when X is a variety we may in fact take a finite open cover) and trivialisations $M|_{U_i} \cong \mathcal{O}_{U_i}^r$ this then gives rise to isomorphisms on the overlaps $U_{ij} = U_i \cap U_j$. We write ψ_{ji} for the map from the ith trivialisation to the jth. These functions are called the **transition functions**. They satisfy some compatibility conditions: $\psi_{ii} = id$, $\psi_{ij} = \psi_{ji}^{-1}$ and $\psi_{kj}\psi_{ji} = \psi_{ki}$.