Part III Algebraic Topology - Michelmas 2014

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1 Introduction

These are lecture notes for the 2014 Part III Algebraic Topology course taught by Dr. Jacob Rasmussen.

The recommended books are:

- Algebraic Topology - Allen Hatcher,
- Homology Theory - James W. Vick,
- Differential Forms in Algebraic Topology - Raoul Bott and Loring W. Tu.

2 Homotopy

Lecture 1

Definition. Maps $f_0, f_1 \colon X \to Y$ are said to be *homotopic* if there is a continuous map $F \colon X \times I \to Y$ such that

$$F(x,0) = f_0(x) \text{ and } F(x,1) = f_1(x) \ \forall x \in X.$$

We let $\mathrm{Map}(X,Y) = \{f \colon X \to Y \text{ continuous}\}$. Then letting $f_t(x) = F(x,t)$ in the above definition we see that f_t is a path from f_0 to f_1 in $\mathrm{Map}(X,Y)$.

Examples.

1. $X = Y = \mathbf{R}^n$, $f_0(\overline{x}) = \overline{0}$ and $f_1(\overline{x}) = \overline{x}$ are homotopic via $f_t(\overline{x}) = t\overline{x}$.
2. $S^1 = \{z \in \mathbf{C} : |z| = 1\}$ then
3. $S^n = \{\overline{x} \in \mathbf{R}^n : |\overline{x}| = 1\}$

Lemma. *Homotopy is an equivalence relation on* $\mathrm{Map}(X,Y)$.

Definition.

$$[X,Y] = \mathrm{Map}(X,Y)/\sim = \text{set of homotopy classes of maps } X \to Y.$$

Lemma. *If $f_0 \sim f_1 \colon X \to Y$ and $g_0 \sim g_1 \colon Y \to Z$ then $g_0 \circ f_0 \sim g_1 \circ f_1$.*

Corollary. *For any space X the set $[X, \mathbf{R}^n]$ has one element.*

Proof. Define $0_X \colon X \to \mathbf{R}^n$ by $0_X(x) = 0 \in \mathbf{R}^n$ for any $x \in X$. □

Definition. X is *contractible* if 1_X is homotopic to a constant map.

Proposition. *Y is contractible $\iff$ $[X,Y]$ has one element for any space X.*

Proof. ($\Rightarrow$) as in corollary. ($\Leftarrow$) $[X,Y]$ has one element so $1_Y \sim$ a constant map. □

Given a space X how can we tell if X is contractible? If X is contractible then it must be path connected for one.

Proof. Contractible implies that $[S^0, X]$ has one element and so $f \colon S^0 \to X$ extends to D^1, and therefore X is path connected. □

Similarly if $[S^1, X]$ has more than one element then X is not contractible.

Definitions. We say X is *simply connected* if $[S^1, X]$ has only one element.

We say two space X and Y are *homotopy equivalent* if there exists $f \colon X \to Y$ and $g \colon Y \to X$ such that $g \circ f \sim 1_X$ and $f \circ g \sim 1_Y$.

Example. X is contractible if and only if $X \sim \{p\}$.

Proof. X contractible $\implies$ $1_X \sim c$, a constant map. Choose $f \colon X \to \{p\}$, $f(x) = p$ and $g \colon \{p\} \to X$, $g(p) = c$. Then $g \circ f = c \sim 1$ and $f \circ g = 1_{\{p\}}$. Converse: exercise. □

Exercise. If $X_0 \sim X_1$ and $Y_0 \sim Y_1$ then $[X_0, Y_0]$ and $[X_1, Y_1]$ are in bijection.

Given X and Y how can we determine if $X \sim Y$? How do we determine $[X,Y]$? For example is $S^n \sim S^m$.

3 Homotopy groups

Definitions. A *map of pairs* $f\colon (X,A) \to (Y,B)$ is a map $f\colon X \to Y$ with sets $A \subset X$ and $B \subset Y$ such that $f(A) \subset B$.

If we have maps of pairs $f_0, f_1\colon (X,A) \to (Y,B)$ then we write $f_0 \sim f_1$ if there exists $F\colon (X \times I, A \times I) \to (Y,B)$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$.

If $* \in X$ then the n^{th} *homology group*

$$\pi_n(X,*) = [(D^n, S^{n-1}) \to (X, \{*\})].$$

We now note several properties of this definition:

1. $\pi_0(X,*) =$ set of path components of X.
2. $\pi_1(X,*)$ is a group.

   $\pi_n(X,*)$ is an abelian group.
3. π_n is a functor

$$\left\{ \begin{smallmatrix} \text{pointed spaces} \\ \text{pointed maps} \end{smallmatrix} \right\} \to \left\{ \begin{smallmatrix} \text{groups} \\ \text{group homomorphisms} \end{smallmatrix} \right\}.$$