MA3H6 Algebraic Topology - Lecture Notes

Based on lectures by Dr. Saul Schleimer

Typeset by Alex J. Best

May 16, 2014

Contents

1 Introduction

These are lecture notes that I typeset for MA3H6 Algebraic Topology in 2014, they are currently full of gaps, mistakes, wrong statements, notation abuse and lots of other badness. However they might be useful to someone, despite the fact they lack very many pictures at present. If you find anything else that can be improved send me an email at a.j.best@warwick.ac.uk, thanks.

2 Topological review

Notation.

$$\begin{aligned}
\mathbb{R}^n &= \{(x_1, \ldots, x_n) \mid x_j \in \mathbb{R}\} \text{ with the product topology of open intervals.}\\
\|x\| &= \sqrt{\sum x_i^2}.\\
B^n &= \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \text{ the } n-1 \text{ sphere.}\\
S^{n-1} &= \{x \in \mathbb{R}^n \mid \|x\| = 1\}.\\
\mathbb{R}^0 &= \{()\}.
\end{aligned}$$

Exercise.

$$B^n \times B^m \cong B^{n+m}.$$

Exercise.

$$S^n \times S^m \ncong S^{n+m}.$$

Hint. Find an invariant of topological spaces that distinguishes them.

Invariants Connectedness, Hausdorffness, π_1, compactness, Euler characteristic. But none of these work.

Quotients We recall that the quotient topology is defined by $a \subseteq X/\sim$ is open iff its preimage under the map $f\colon X \to X/\sim$ is open. This topology makes as many of the sets of the quotient as possible open while keeping the quotient map continuous.

There are more ways to produce S^1, for example

$$S^1 \cong [0,1]/0 \sim 1$$

when equipped with the quotient topology.

Another way is to consider $\mathbb{R}/\mathbb{Z} = \mathbb{R}/\{x \sim y \iff x - y \in \mathbb{Z}\}$. So there is a map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ which is the covering map of $\mathbb{R}/\mathbb{Z}$ by its universal cover.

3 Simplicial homology

3.1 Simplices

Definition 3.1.1. We define the n-simplex to be

$$\Delta^n = \left\{ x \in \mathbb{R}^{n+1} \middle| x_i \geq 0 \ \forall i, \ \sum x_i = 1 \right\}.$$

In general if $v_i \in \mathbb{R}^m$ are a collection of $n+1$ affinely independent points (do not lie in an $n-1$ dimensional subspace) then we define

$$[v] = [v_0, v_1, \ldots, v_n] = \left\{ \sum x_i v_i \middle| x_i \in \Delta^n \right\}.$$

If we omit some of the v_i we obtain a facet of $[v]$. If we only omit one of them we get a face. This is denoted by

$$[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n]$$

where the v_i is read to be omitted.

The vertices are ordered and if $[v]$, $[w]$ are simplices of the same dimension then there exists a unique affine map extending the ordering of the vertices. The standard map $f \colon [v] \to [w]$ sends v_i to w_i and respects barycentric coordinates.

Definition 3.1.2. A facet of Δ is a subsimplex (i.e. pick some x_i and set them to zero).

Definition 3.1.3. A face is a codimension one facet.

Definition 3.1.4. The boundary of Δ^n is denoted by $\partial\Delta^n$ and consists of the union of its faces.

We have that $\mathring{\Delta} = \Delta - \partial\Delta$.

Example 3.1.5.

Exercise. Count the k-dimensional faces of Δ^n.

3.2 Δ-complexes

Definition 3.2.1. Fix X a topological space and a collection of maps

$$\{\sigma_\alpha \colon \Delta_\alpha \to X \mid \alpha \in A\}.$$

This is known as a Δ-*complex structure* on X if:

(i) (Partition) for all α $\sigma_\alpha|\mathring{\Delta}_\alpha$ is injective and for $x \in X$ there is a *unique* $\alpha \in A$ s.t. $x \in \sigma_\alpha(\mathring{\Delta}_\alpha)$.

(ii) (Tiling) If $\Delta \subset \Delta_\alpha$ is a face then there is a unique $\beta \in A$ s.t. $\sigma_\alpha|\Delta = \sigma_\beta \circ f$ where $f \colon \Delta \to \Delta_\beta$ is the canonical map.

(iii) (Topology) $U \subset X$ is open iff $\forall \alpha\ \sigma_\alpha^{-1}(U) \subset \Delta_\alpha$ is open.

We can state this equivalently as: X must be homeomorphic to the quotient space

$$\bigsqcup_{\alpha \in A} \Delta_\alpha / \text{face gluings.}$$

Example 3.2.2.

Example 3.2.3. $\partial \Delta^n$ gives a Δ-complex structure on S^{n-1}.

Example 3.2.4. If we double Δ^n across $\partial \Delta$ we get a Δ-complex structure on S^n.

Example 3.2.5. Check these are homeomorphic to S^n.

Non Example 3.2.6. Violates tiling on the edge marked $[0, 2]$ and so is not a Δ-complex structure.

Exercise. 1. Find a Δ-complex structure on the space in the non-example above.

2. Show that every graph admits a Δ-complex structure.

Example 3.2.7. Here the indexing set $A = \mathbb{R}$ (very big!).

Definition 3.2.8. A Δ-complex is *finite dimensional* if there exists n s.t. for all α $\dim(\Delta_\alpha) \leq n$.

Definition 3.2.9. A Δ-complex structure is *finite* if $|A| < \infty$ (where as above A is the index set).

Exercise. Show that if X admits a Δ-complex structure then X is Hausdorff.

Exercise. Show that if $\{\sigma_\alpha\}$ is a Δ-complex structure on X and $K \subset X$ is compact then K meets the interiors of only finitely many of the σ_α's.

Exercise. If X, Y admit Δ-complex structures then so does $X \times Y$.

3.3 Abelian groups

Fix A a set. Define $\mathbb{Z}[A]$ to be the *free abelian group* on A given by

$$\begin{aligned}\mathbb{Z}[A] &= \left\{ \sum_{\alpha \in A} n_\alpha \cdot \alpha \,\middle|\, n_\alpha \in \mathbb{Z} \text{ and all but finitely many are non-zero} \right\} \\ &= \text{all finite } \mathbb{Z}\text{-linear sums of elements of } A.\end{aligned}$$

Example 3.3.1.

$$\mathbb{Z}[\{\alpha, \beta\}] \cong \mathbb{Z}^2 = \{n\alpha + m\beta \mid m, n \in \mathbb{Z}\}.$$

If A is finite then $\mathbb{Z}[A] \cong \mathbb{Z}^A$. But if $|A| = \infty$ then this is false.

Exercise. $\mathbb{Q}$ is *not* a free abelian group.

3.4 Chains

Suppose $(X, \{\sigma\})$ is a space equipped with a Δ-complex structure.

Definition 3.4.1. We define the set of *n-chains* to be

$$C_n^\Delta = \mathbb{Z}[\{\sigma_\alpha \mid \dim(\Delta_\alpha) = n\}].$$

Example 3.4.2.

3.5 Boundary operators

Recall $\Delta_v = [v_0, v_1, \ldots, v_n]$ is an n-simplex.

The ith face of Δ is $[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n]$.

Definition 3.5.1. We define the *boundary operator* as follows. First suppose $\sigma \colon \Delta \to X$ is a map.

We then define

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma \mid [v_0, \ldots, \hat{v}_i, \ldots, v_n].$$

Which is an $(n-1)$-chain.

So we extend linearly to define

$$\partial \colon C_n^\Delta(X) \to C_{n-1}^\Delta(X)$$

given by

$$\sum n_\alpha \sigma_\alpha \mapsto \sum n_\alpha \partial\sigma_\alpha.$$

Example 3.5.2.

Lemma 3.5.3.

$$\partial_{n-1} \circ \partial_n = 0.$$

"The extremes of the extremes are empty".

Proof. It suffices to check this on a basis element

$$\sigma \colon \Delta^n \to X$$

so

$$\partial_n\sigma = \sum_{i=0}^{n} (-1)^i \sigma \mid [v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

now we apply ∂_{n-1}:

$$\partial_{n-1}\partial_n\sigma = \partial_{n-1}\left(\sum_{i=0}^{n} (-1)^i \sigma \mid [v_0, \ldots, \hat{v}_i, \ldots, v_n]\right)$$

$$
\begin{aligned}
&= \sum_{i=0}^{n} (-1)^i \partial_{n-1} \left(\sigma \mid [v_0, \ldots, \hat{v}_i, \ldots, v_n]\right) \\
&= \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{n-1} (-1)^j \left(\sigma \mid [v_0, \ldots, \hat{v}_i, \ldots, v_n]\right) \mid [w_0, \ldots, \hat{w}_j, \ldots, w_{n-1}] \\
&= \sum_{i=0}^{n} (-1)^i \left(\sum_{j<i} (-1)^j \sigma \mid [v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_n] \right. \\
&\quad \left. + \sum_{j>i} (-1)^{j+1} \sigma \mid [v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_n] \right) \\
&= \sum_{j<i} (-1)^{j+i} \sigma \mid [v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_n] \\
&\quad - \sum_{j>i} (-1)^{j+i} \sigma \mid [v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_n] \\
&= 0
\end{aligned}
$$

□

3.6 Chain complexes

Definition 3.6.1. A sequence $\{C_n\}_{n=0}^{\infty}$ of abelian groups with homomorphisms

$$\partial_n \colon C_n \to C_{n-1}$$

such that $\partial^2 = 0$ is called a *chain complex.*

By convention we take C_{-1} to be 0.

Example 3.6.2.

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to 0.$$

Given two chain complexes we can form the direct sum by taking the direct sum of each of the groups and letting the operators act elementwise.

Terminology If $c \in C_n$ we call c an *n-chain.*
If $z \in Z_n = \ker(\partial_n)$ we call z an *n-cycle.*
If $b \in B_n = \operatorname{im}(\partial_{n-1})$ we call b an *n-boundary.*
If $h \in Z_n/B_n = H_n$ we call h a *homology class.*
Since $\partial^2 = 0$ we deduce that $B_n \leq Z_n$ and $H_n = Z_n/B_n$ makes sense.

Example 3.6.3. For

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to 0$$

we have $H_1 = 0$, $H_0 = \mathbb{Z}/2\mathbb{Z}$ and $H_k = 0$ for all $k \geq 1$.

Definition 3.6.4. If (X,σ) is a Δ-complex then set $C_n^{\Delta}(X) = \mathbb{Z}[\{\sigma_\alpha \mid \dim(\Delta_\alpha) = n\}]$ and $\partial_n \colon C_n^{\Delta}(X) \to C_{n-1}^{\Delta}(X)$ is the boundary operator.

Then $H_n^{\Delta}(X)$ are called the *simplicial homology groups* of X.

Theorem 3.6.5. This is independent of the choice of Δ-complex structure on X.

3.7 Computations

1. $X = \{\text{pt}\}$. $C_0^{\Delta}(X) \cong \mathbb{Z}$ and all others are 0, so we have the chain complex:
$$\cdots \to 0 \to 0 \to \mathbb{Z} \to 0.$$
So $H_0^{\Delta}(\text{pt}) \cong \mathbb{Z}$ and $H_k^{\Delta}(\text{pt}) \cong 0$ if $k \geq 1$.

2. $X = S^1$. $C_0^{\Delta}(X) \cong \mathbb{Z}$, $C_1^{\Delta}(X) \cong \mathbb{Z}$ and all others are 0, so we have the chain complex:
$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \to 0.$$
We see that $\partial e = \sum_{i=0}^{1}(-1)^i e|[v_0,\ldots,\hat{v}_i,\ldots,v_1] = e|[v_1] - e|[v_0] = v - v = 0$. So
$$H_k^{\Delta}(S^1) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } 1, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise. Compute $H_*^{\Delta}(S^1)$ for the Δ-complex structure on S^1 with k vertices and k edges.

Exercise. Compute $H_*^{\Delta}(X)$ for the $X = B^2,\ S^1$ and K^2 (the Klein bottle).

Exercise. Using the fact that Δ^n is a Δ-complex structure on B^n compute $H_*^{\Delta}(B^n)$.

In general you'll want to make use of the Smith normal form.

4 Singular homology

Definition 4.0.1. A *singular n-simplex* in X is a map $\sigma \colon \Delta^n \to X$.

Definition 4.0.2.
$$C_n^{\text{sing}}(X) = \mathbb{Z}[\{\sigma \colon \Delta^n \to X\}].$$

We call $c \in C_n^{\text{sing}}(X)$ a singular n-chain.

Definition 4.0.3. We define $\partial \colon C_n^{\text{sing}}(X) \to C_{n-1}^{\text{sing}}(X)$ exactly as before by
$$\partial\sigma = \sum_{i=0}^{n}(-1)^i \sigma|[v_0,\ldots,\hat{v}_i,\ldots,v_n].$$

And again we define $Z_n^{\text{sing}}(X)$ (resp. $B_n^{\text{sing}}(X)$) exactly as above and call it the group of singular n-cycles (resp. n-boundaries).

Definition 4.0.4. Now $H_n^{\text{sing}}(X) = Z_n^{\text{sing}}(X)/B_n^{\text{sing}}(X)$ is the n-th *singular homology group.*

Remark. We have that $\partial_{n-1} \circ \partial_n = 0$ exactly as before.

Example 4.0.5. Suppose X is a single point, then there is a unique Δ-complex structure on X. We say $\sigma^0 \colon \mathbb{Z} \to X$ is the "constant map". So $0 \to \mathbb{Z} \to 0$ is the chain complex $C_*^\Delta(X)$. So

$$H_n^\Delta = \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n \geq 1. \end{cases}$$

Suppose X is as above again, then we can compute

$$H_n^{\text{sing}} = \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n \geq 1. \end{cases}$$

This is as in dimension n there is only the constant map

$$\sigma^n \colon \Delta^n \to X$$

so $C_n^{\text{sing}}(X) \cong \mathbb{Z}$ and we also have that

$$\begin{aligned} \partial \sigma^n &= \sum_{i=0}^n (-1)^i \sigma|[v_0, \ldots, \hat{v}_i, \ldots, v_n] = \sum_{i=0}^n (-1)^i \sigma^{n-1} \\ &= \left(\sum_{i=0}^n (-1)^i \right) \sigma^{n-1} = \begin{cases} 0 & n \text{ odd}, \\ \sigma^{n-1} & n \text{ even}, \end{cases} \end{aligned}$$

except if $n = 0$. So C_n is

$$\cdots \to \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \to 0$$

and so the singular homology groups are as claimed.

Challenge Compute $H_n^{\text{sing}}(S^1)$ from the definitions.

Theorem 4.0.6. If X admits a Δ-complex structure then

$$H_*^\Delta(X) \cong H_*^{\text{sing}}(X).$$

The left hand side is generally easier to compute, but the right can be easier to prove theorems with.

Proposition 4.0.7. If $X = \bigsqcup X_\alpha$ where all X_α are path connected spaces then

$$H_n^{\text{sing}}(X) = \bigoplus_\alpha H_n^{\text{sing}}(X_\alpha).$$

Proof.

$$C_n^{\text{sing}}(X) = \bigoplus_\alpha C_n^{\text{sing}}(X_\alpha).$$

and ∂ respects this "splitting". □

Proposition 4.0.8. If $X \neq \emptyset$ and X is path connected then $H_0^{\text{sing}}(X) \cong \mathbb{Z}$.

Proof. Define $\epsilon \colon C_0(X) \to \mathbb{Z}$ by $\sum n_\alpha v_\alpha \mapsto \sum n_\alpha$, the augmentation map, then ϵ is surjective. We claim that $\ker(\epsilon) = \operatorname{im}(\partial_1)$. Given any $\tau \colon \Delta^1 \to X$ that goes from v to w we have that $\partial\tau = w - v$ so $\epsilon(\partial\tau) = 1 - 1 = 0$ and the image is contained in the kernel. Now fix $\sum n_\alpha v_\alpha$ s.t. $\epsilon(\sum n_\alpha v_\alpha) = 0$. Also fix some $u \in X$ and for all α pick $\tau_\alpha \colon \Delta^1 \to X$ a path from u to v_α. Consider $\sum n_\alpha \tau_\alpha \in C_1(X)$

$$\begin{aligned}
\partial\left(\sum n_\alpha \tau_\alpha\right) &= \sum \partial(n_\alpha \tau_\alpha) \\
&= \sum n_\alpha \partial\tau_\alpha \\
&= \sum n_\alpha (v_\alpha - u) \\
&= \sum n_\alpha v_\alpha - \sum n_\alpha u \\
&= \sum n_\alpha v_\alpha - u \sum n_\alpha \\
&= \sum n_\alpha v_\alpha - u \cdot 0 \in \operatorname{im}
\end{aligned}$$

Hence im = ker as claimed.

And so $H_0 = \ker(\partial_0)/\operatorname{im}(\partial_1) = \ker(\partial_0)/\ker(\epsilon) = C_0(X)/\ker(\epsilon) \cong \mathbb{Z}$. □

4.1 Reduced Homology

Definition 4.1.1. If X has k path components, then $H_0(X) \cong \mathbb{Z}^k$ so we define the *augmented chain complex*

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where ϵ is the augmentation map from above. Define the *reduced homology groups* $\tilde{H}_*(X)$ to be the homology groups of this chain complex. So $\tilde{H}_n(X) = H_n(X)$ is $n > 0$ and $\tilde{H}_0(X) = \ker(\epsilon)/\operatorname{im}(\partial_1)$. Hence if X has k path components

$$\tilde{H}_0(X) \cong \mathbb{Z}^{k-1}.$$

Recall that $H_*(X \sqcup Y) = H_*(X) \oplus H_*(Y)$, the reduced homology groups behave nicely with respect to many operations such as 1-point unions. In a 1-point union $X \vee Y = X \sqcup Y / x \sim y$ for some designated point $x \in X$ and $y \in Y$. So $\tilde{H}_*(X \vee Y) = \tilde{H}_*(X) \oplus \tilde{H}_*(Y)$.

4.2 Functoriality

Definition 4.2.1. Suppose $f\colon X \to Y$ is a (continuous) map. Let $f_n\colon C_n(X) \to C_n(Y)$ by $\sigma \mapsto f \circ \sigma$. The function $f \circ \sigma$ is again a map from Δ^n to Y and so still lies in $C_n(Y)$.

The key property of this definition is that $\partial_n \circ f_n = f_{n-1} \circ \partial_n$. This is saying that the square

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1}(X) \arrow[d, "f_{n-1}"] \\ C_n(Y) \arrow[r, "\partial_n"] & C_{n-1}(Y) \end{tikzcd}$$

commutes. We denote the family of these maps f_n as $f_\#$,

Exercise. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ then $(g \circ f)_n = g_n \circ f_n$.

4.3 Chain maps

Definition 4.3.1. If C_*, D_* are chain complexes we say that a family of homomorphisms $f_\#\colon C_* \to D_*$ is a *chain map* if

$$f_\# \circ \partial = \partial \circ f_\#$$

Example 4.3.2. If $f\colon X \to Y$ is continuous then $f_\#$ is a chain map from $C_*(X) \to C_*(Y)$.

Example 4.3.3. Suppose (X, σ) is a Δ-complex then

$$i\colon C_n^\Delta(X) \to C_n^{\text{sing}}(X)$$

is also a chain map.

Proposition 4.3.4. If $f_\#\colon C_* \to D_*$ is a chain map then $f_\#$ induces a homomorphism

$$f_*\colon H_*(C) \to H_*(D)$$

given by

$$f_*([z]) = [f_\#(z)].$$

Proof. Check that $f_\#(Z_n^C) \leq Z_n^D$ (exercise) and that $f_\#(B_n^C) \leq B_n^D$. So $f_\#(b) = f_\#(\partial c) = \partial f_\#(c)$. $\square$

Remark. If $f\colon X \to Y$ is a homomorphism then there exists a continuous inverse $g\colon Y \to X$ such that

$$f_*\colon H_*(X) \to H_*(Y)$$

is inverse to

$$g_*\colon H_*(Y) \to H_*(X).$$

4.4 Homotopic spaces

Definition 4.4.1. We say two maps f and g from $X \to Y$ are *homotopic* if there is a map $F\colon X \times [0,1] \to Y$ such that $f(x) = F(x,0)$ and $g(x) = F(x,1)$. We write $f \sim g$.

We then say two spaces X and Y are *homotopy equivalent* if there exists maps $f\colon X \to Y$ and $g\colon Y \to X$ such that

$$(g \circ f) \sim \mathrm{Id}_X \text{ and } (f \circ g) \sim \mathrm{Id}_Y.$$

Example 4.4.2.

$$S^n \sim \mathbb{R}^{n+1} \setminus \{0\}$$

via (for $n = 1$)

$$\begin{aligned} i\colon S^1 &\to \mathbb{R}^2 \setminus \{0\} \\ x &\mapsto x \end{aligned}$$

and

$$\begin{aligned} r\colon \mathbb{R}^2 \setminus \{0\} &\to S^1 \\ x &\mapsto \frac{x}{\|x\|}. \end{aligned}$$

We also have

$$S^n \sim B^{n+1} \setminus \{0\}$$

Theorem 4.4.3. If $f \sim g\colon X \to Y$ then

$$f_* = g_*\colon H_*(X) \to H_*(Y).$$

Corollary 4.4.4. If X is homotopy equivalent to Y via f then

$$f_*\colon H_*(X) \to H_*(Y)$$

is an isomorphism.

Proof.

$$(\mathrm{Id}_X)_* = \mathrm{Id}_{H_*}$$

□

Definition 4.4.5. Suppose $f_\#, g_\#\colon C_* \to D_*$ are chain maps. A sequence of homomorphisms $P_n\colon C_n \to D_{n+1}$ is called a *chain homotopy* if

$$\partial_{n-1} P_n + P_{n-1}\partial_n = g_\# - f_\#$$

in there is a chain homotopy between two chain maps $f_\#, g_\#$ we write $f_\# \sim g_\#$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & C_n \arrow[r, "\partial"] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & C_{n-1} \arrow[r] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial"] & D_n \arrow[r, "\partial"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Proposition 4.4.6. If $f_\# \sim g_\# \colon C_* \to D_*$ then

$$f_* = g_* \colon H_*(C) \to H_*(D).$$

Proof. Pick any $h \in H_*(C)$, we want to show $(g_* - f_*)(h) = 0$. Choose some $x \in Z_n(C)$ such that $h = [z]$ and compute

$$\begin{aligned}
(g_* - f_*)(h) &= (g_* - f_*)([z]) \\
&= [(g_\# - f_\#)(z)] \\
&= [(P\partial + \partial P)(z)] \\
&= [P\partial z + \partial P z] \\
&= [P0 + \partial P z] \\
&= [\partial(Pz)] \\
&= 0 \text{ (as } B_n = 0 \text{ in homology).}
\end{aligned}$$

□

4.5 Prisms

Definition 4.5.1. A *prism* is a copy of $\Delta^n \times I$.

We can subdivide $\Delta \times I$ into $n+1$ dimensional simplices of the form

$$[v_0, v_1, \ldots, v_i, w_i, w_{i+1}, \ldots, w_n],$$

where v are the vertices of the simplex at one end of the interval and w are the vertices at the other.

If we have $F \colon X \times I \to Y$ and $\sigma \colon \Delta^n \to X$ we let

$$F\sigma = F \circ (\sigma \times \mathrm{Id}_I) \colon \Delta^n \times I \to Y.$$

Proof of theorem 4.4.3. $f \sim g \colon X \to Y$, let $F \colon X \times I \to Y$ be the homotopy. Then define

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i F\sigma|[v_0, \ldots, v_i, w_i, w_{i+1}, \ldots, w_n]$$

this is the prism operator. We now claim that P is a chain homotopy from $f_\#$ to $g_\#$. To see this fix $\sigma \colon \Delta^n \to X$ and compute

$$\begin{aligned}
\partial P\sigma &= \partial\left(\sum_{i=0}^{n} (-1)^i F\sigma|[v_0, \ldots, v_i, w_{i+1}, \ldots, w_n]\right) \\
&= \sum_{j \le i} (-1)^{i+j} F\sigma|[v_0, \ldots, \hat{v}_j, \ldots, w_n] + \sum_{i \le j} (-1)^{i+j+1} F\sigma|[v_0, \ldots, \hat{w}_j, \ldots, w_n]
\end{aligned}$$

and

$$
\begin{aligned}
P\partial\sigma &= P\left(\sum(-1)^j\sigma|[v_0,\dots,\hat{v}_j,\dots,v_n]\right) \\
&= \sum_{i<j}(-1)^{i+j}F\sigma|[v_0,\dots,\hat{w}_j,\dots,w_n] + \sum_{j<i}(-1)^{i+j-1}F\sigma|[v_0,\dots,\hat{v}_j,\dots,w_n].
\end{aligned}
$$

So

$$
\begin{aligned}
\partial P\sigma + P\partial\sigma &= \sum(-1)^{2i}F\sigma|[v_0,\dots,\hat{v}_i,w_i,\dots,w_n] + \sum(-1)^{2i+1}F\sigma|[v_0,\dots,v_i,\hat{w}_i,\dots,w_n] \\
&= F\sigma|[\hat{v}_0,w_0,\dots,w_n] - F\sigma|[v_0,\dots,v_n,\hat{w}_n] \\
&= g_\#\sigma - f_\#\sigma.
\end{aligned}
$$

□

4.6 Exact sequences

Definition 4.6.1. We say a complex C_* is *exact* if $H_*(C) \equiv 0$ (or equivalently if $Z_n = B_n$ for all n).

We also say that a sequence is *short* if it has at most 3 non-zero terms.

Example 4.6.2.

$$0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0.$$

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z} \to 0.$$

Definition 4.6.3. A short sequence of chain complexes

$$0 \to A_* \xrightarrow{i_\#} B_* \xrightarrow{j_\#} C_* \to 0$$

is *exact* if for all n

$$0 \to A_n \xrightarrow{i_n} B_n \xrightarrow{j_n} C_n \to 0$$

is exact and $i_\#$, $j_\#$ are chain maps.

Definition 4.6.4. We say that (X, A) is a *good pair* if $A \subseteq X$ is non-empty, closed and there exists an open V with $X \supset V \supset A$ such that V deformation retracts to A.

Example 4.6.5. $(\mathbb{R}^2, S^1)$ is a good pair.

We say $f\colon (X, A) \to (Y, B)$ is a *map of pairs* if $f\colon X \to Y$ is a map and $f(A) \subset B$.

Example 4.6.6.

$$f\colon (I, \partial I) \to (\mathbb{R}^2, S^1).$$

Similarly we can define a *homotopy of maps of pairs* to be a function $F\colon X \times I \to Y$ where each $F_t\colon (X, A) \to (Y, B)$ is a map of pairs and $F_0 = f$, $F_1 = g$.

4.7 Relative homology

Suppose (X, A) is a pair. Note that

$$i_\# \colon C_*(A) \to C_*(X)$$

is an inclusion. Define $C_*(X, A) = C_*(X)/C_*(A)$ and we then have that $C_n(X, A) = C_n(X)/C_n(A)$ and as ∂^X preserves $C_*(A)$ it descends to give $\partial^{(X,A)}$. We have that $\partial^{(X,A)}[c] = [\partial^X c]$.

Exercise. Show $\partial^{(X,A)}$ is well defined and $(\partial^{(X,A)})^2 = 0$.

Note that

$$0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$$

is a short exact sequence of chain complexes.

Definition 4.7.1. $H_n(X, A) = Z_n(X, A)/B_n(X, A)$, we also say that $[z] \in Z_n(X, A)$ is a *relative cycle* and $[b] \in B_n(X, A)$ is a *relative boundary.*

If $z \in [z] \in Z_n(X, A)$ then $\partial^{(X,A)}[z] = 0 \in C_n(X, A)$ i.e. $\partial^{(X,A)}[z] = [a][\text{any } a \in C_n(A)]$. $[\partial^X z] = [a]$ i.e. $\partial^X z = C_n(A)$.

Example 4.7.2.

$$\begin{aligned} H_*(X, X) &= 0, \\ H_*(X, \emptyset) &= H_*(X), \\ H_*(X, \{\text{pt}\}) &= H_*\tilde{(}X) \text{ (exercise)}. \end{aligned}$$

Proposition 4.7.3. If f is homotopic, $f \sim g \colon (X, A) \to (Y, B)$ then $f_* = g_* \colon H_*(X, A) \to H_*(Y, B)$.

Proof. The prism operator gives a chain homotopy. □

Corollary 4.7.4. If $A \subset V$ and V deformation retracts to A then $H_*(V, A) = 0$.

Proof. □

4.8 Long exact sequences

Theorem 4.8.1. Suppose $0 \to A_* \xrightarrow{i_\#} B_* \xrightarrow{j_\#} C_* \to 0$ is exact then there is a $\partial_* \colon H_{*+1}(C) \to H_*(A)$ making the following triangle exact

$$\begin{tikzcd} H_*(A) \arrow[rr, "i_*"] & & H_*(B) \arrow[dl, "j_*"] \\ & H_*(C) \arrow[ul, "\partial_*"] & \end{tikzcd}$$

that is

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] & H_n(B) \arrow[r] & H_n(C) \arrow[dll, rounded corners, "\partial_n"'] & \\
& H_{n-1}(A) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(C) \arrow[r] & \cdots
\end{tikzcd}$$

is a long exact sequence of groups (an exact complex).

Proof. We define δ_*. Fix some $[c] \in H_n(C)$, so $c \in Z_n(C) \leq C_n$. The map j is surjective so pick $b \in B_n$ such that $j(b) = c$. Since $c \in Z_n$ we have $\partial c = 0$ and so $j\partial b = \partial jb = 0$. Now since $\ker(j) = \operatorname{im}(i)$ there is some $a \in A_{n-1}$ such that $ia = \partial b$. We then define $\delta_*[c] = [a]$, we have a few things to check:

1. $a \in Z_{n-1(A)}$: $i\partial a = 0 \iff \partial a = 0$, $i\partial a = \partial ia = \partial\partial b = 0$ as required.

2. δ_* is well defined:

   (i) Suppose we pick $c + \partial c'$ instead of c. Pick any b' such that $jb' = c'$ so that $j(b + \partial b') = c + \partial c'$, however then $\partial(b + \partial b') = \partial b$ is the same.

   (ii) Suppose we picked b'' such that $jb'' = c$. Then $j(b - b'') = 0$ and so there is some $a' \in A_n$ such that $ia' = b - b''$, i.e. $b'' = b - ia'$. We then have $\partial b'' = \partial b - \partial ia' = ia - i\partial a' = i(a - \partial a')$ so b'' gives $a - \partial a'$ and b gives a. Since $[a] = [a - \partial a']$ we have that δ_* is well defined.

3. δ_* is a homomorphism as i, j and ∂ are.

4. To see that the chain complex is as claimed we have some more checks to make:

   (i) $j_*i_*[a] = 0$: $j_*i_*[a] = j_*[i_\# a] = [j_\# i_\# a] = [0]$.

   (ii) $\delta_* j_*[b] = 0$: Set $jb = c$, suppose $\delta_*[c] = [a]$, we know $\partial b = 0$ so $ia = \partial b = 0$ but also that i is injective. So $a = 0$ and hence $[a] = 0$ as required.

   (iii) $j_*\delta_*[c] = 0$: Set $\delta_*[c] = [a]$ and let $ia = \partial b$ and $jb = c$ as usual. So $i_*[a] = [ia] = [\partial b] = 0$.

5. To see that the complex is exact we must show the opposite inclusions of images and kernels to the ones demonstrated above, we only show (i) here and (ii) & (iii) are left as exercises.

   $\ker(j_*) \leq \operatorname{im}(i_*)$: Pick $[b] \in \ker(j_*)$, suppose $j_*[b] = 0$ i.e. $[jb] = 0$ and if $jb = c$ then there is a c' such that $c = \partial c'$. We know j is surjective so there is b' such that $jb' = c'$ and therefore $j(b - \partial b') = c - \partial c' = 0$. So there exists a with $ia = b - \partial b'$ so $i_*[a] = [b - \partial b'] = [b]$.

□

Example 4.8.2. If (X, A) is a pair then

$$\begin{tikzcd} H_*(A) \arrow[rr, "i_*"] & & H_*(X) \arrow[dl, "q_*"] \\ & H_*(X,A) \arrow[ul, "\partial_*"] & \end{tikzcd}$$

is exact.

Remark. If $A = \emptyset$ then j_* is an isomorphism.

Theorem 4.8.3.

$$\begin{tikzcd} \tilde{H}_*(A) \arrow[rr, "i_*"] & & \tilde{H}_*(X) \arrow[dl, "q_*"] \\ & H_*(X,A) \arrow[ul, "\partial_*"] & \end{tikzcd}$$

is also exact, thus $\tilde{H}_*(X) \cong H_*(X, X)$.

Example 4.8.4. If (X, B, A) is a triple then

$$\begin{tikzcd} H_*(B,A) \arrow[rr, "i_*"] & & H_*(X,A) \arrow[dl, "q_*"] \\ & H_*(X,B) \arrow[ul, "\partial_*"] & \end{tikzcd}$$

is exact.

Example 4.8.5. Set $(X, A) = (B^2, S^1)$ then applying the snake lemma to

$$0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$$

gives that

$$H_k(B^2, S^1) = \begin{cases} \mathbb{Z} & \text{if } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise. For H_*^{sing} suppose that $A \xrightarrow{i} X$ and $X \xrightarrow{r} S$ is a *retraction*, i.e. $r \circ i = \text{id}_A$. Prove that

$$H_*(X) \cong H_*(A) \oplus H_*(X, A).$$

4.9 Excision

Theorem 4.9.1 (Excision)**.**
Version 1. Suppose $Z \subset A \subset X$ with closure$(A) \subseteq$ interior(A), then

$$H_*^{\text{sing}}(X \setminus Z, A \setminus Z) \cong H_*^{\text{sing}}(X, A).$$

Version 2. Suppose $A, B \subseteq X$ and $X \subseteq \text{interior}(A) \cup \text{interior}(B)$ then

$$H_*^{\text{sing}}(B, B \cap A) \cong H_*^{\text{sing}}(X, A).$$

We would like to know that

$$H_*(X, A) = H_*(\tilde{X/A})$$

and we can prove this for good pairs (X, A) using excision.

Also if X has a Δ-complex structure excision gives that

$$H_*^{\Delta} \cong H_*^s(X).$$

These isomorphisms are induced by inclusion. We will use both versions of excision but prove version 2.

Exercise. Show both versions of excision are equivalent.

4.9.1 Covers

Definition 4.9.2. $\mathcal{U} = \{U_\alpha\}$ is an open cover of X if all U_α are open subsets of X and X is contained in the union of all U_α.

Suppose $A \subset X$ we say $\sigma \colon \Delta \to X$ is subordinate to A if $\sigma(\Delta) \subset A$. We also say that $\mathcal{U} = \{A_\alpha\}$ is a *cover* of X if $\{\text{interior}\, A_\alpha\}$ is an open cover of X.

Definition 4.9.3. With the notation as above let

$$C_n^{\mathcal{U}}(X) = \left\{ \sum n_\beta \sigma_\beta \in C_n^s(X) \middle| \forall \beta\ \exists \alpha \text{ s.t. } \sigma_\beta \text{ is subordinate to } A_\alpha \right\}.$$

Note that $\partial \colon C_n(X) \to C_{n-1}(X)$ respects subordination i.e. restricts to give

$$\partial \colon C_n^{\mathcal{U}}(X) \to C_{n-1}^{\mathcal{U}}(X).$$

So $C_*^{\mathcal{U}}(X) = \{C_n^{\mathcal{U}}(X), \partial\}$ is a chain complex. The map

$$i_\# \colon C_*^{\mathcal{U}} \to C_*^s$$

is an injective chain map. And we define $H_*^{\mathcal{U}}$ in the usual way.

Proposition 4.9.4. Suppose $\mathcal{U} = \{A, B\}$ is a cover of X, then there is a chain map

$$p_\#\colon C_*(X) \to C_*^{\mathcal{U}}(X)$$

called subdivision such that $p_\# \circ i_\# = \mathrm{id}_{C_*^{\mathcal{U}}}$, $i_\# \circ p_\# \sim \mathrm{id}_{C_*^{\mathcal{U}}}$ is a chain homotopy equivalence to C_* and $H_*^{\mathcal{U}} \cong H_*$.

Assuming the above proposition we now prove excision version 2.

Proof. $\mathcal{U} = \{A, B\}$ is a cover of X so

$$i_\#\colon C_*^{\mathcal{U}} \to C_*$$

induces isomorphism on homology by the above proposition. Also i induces

$$C_*^{\mathcal{U}}(X)/C_*^s(A) \xrightarrow{i} C_*^s(X)/C_*^s(A)$$

so we also get an isomorphism

$$C_*^s(B)/C_*^s(B \cap A) \xrightarrow{\cong} C_*^{\mathcal{U}}(X)/C_*(A).$$

Thus

$$C_n(B)/C_n(A \cap B) \cong C_n^{\mathcal{U}}(X)/C_n(A) \xrightarrow{\sim \text{h.e.}} C_n(X)/C_n(A).$$

So all three have isomorphisms. □

The subdivision operator, used to prove the above proposition is similar to the prism operator (it breaks a large object into pieces).

Exercise. Suppose X is a Δ-complex and $A \in X$ is a subcomplex ($A \neq \emptyset$) show (X, A) is a good pair.

Exercise. Show that if

$$A = \{0\} \cup \left\{ \frac{1}{n} \middle| n \in \mathbb{Z}_{\geq 0} \right\}, \; X = [0, 1]$$

then (X, A) is not a good pair.

4.9.2 Coning

Suppose $Y \subset \mathbb{R}^m$ is convex, that is if $x, y \in Y$ then the interval $[x, y] \subset Y$.

Definition 4.9.5. With Y as above we let

$$C_n^l(Y) = \left\{ \sum n_\alpha \sigma_\alpha \middle| \sigma_\alpha\colon \Delta^n \to Y \text{ is affine linear} \right\}.$$

Suppose that $\sigma = [v_0, \ldots, v_n]$ is a simplex in $\mathbb{R}^m$, i.e. that the v_i do not lie in any affine $(n-1)$-dimensional subspace. Suppose $b \in \mathbb{R}^m$ and that $\{b, v_0, \ldots, v_n\}$ is again a set of affinely independent points.

Definition 4.9.6.

$$b\sigma = [b, v_0, \ldots, v_n]$$

i.e. $b\sigma$ is the *cone* of σ (now called the *base*) to b (the *apex*).

Fixing a b we can then define

$$b\colon C_n^l(Y) \to C_{n+1}^l(Y) \text{ by } \sigma \mapsto b\sigma.$$

Exercise. Show that

$$\partial b + b\partial = 1.$$

4.9.3 Barycenters

Definition 4.9.7. Given $\sigma \subset Y$ a simplex ($\sigma = [v_0, \ldots, v_n]$) define the *barycenter*

$$b_\sigma = \sum_{i=0}^{n} \frac{1}{n+1} v_i.$$

Now define the *subdivision operator* by $Sv = v$ in dimension 0 and otherwise

$$S\sigma = b_\sigma S\partial\sigma.$$

This then gives a map $S\colon C_n^l(Y) \to C_n^l(Y)$ defined by $\sigma \mapsto \sigma_\# \circ S \circ \mathrm{id}_{\Delta^n}$.

As this definition is recursive all proofs using it will be inductive.

Exercise. If σ is an n-simplex count the number of n-simplices in $S\sigma$. Challenge: count all of the faces.

Proposition 4.9.8. The operator S is a chain map.

Proof. Inductively

$$\begin{aligned}
\partial S\sigma &= \partial(bS\partial\sigma) \\
&= (\partial b)S\partial\sigma \\
&= (1 - b\partial)S\partial\sigma \\
&= (S\partial - b\partial S\partial)\sigma \\
&= (S\partial - bS\partial^2)\sigma \\
&= S\partial\sigma.
\end{aligned}$$

$\square$

Lemma 4.9.9 (Fine lemma)**.** If τ is a simplex in $S\sigma$ then $\operatorname{diam}(\tau) \leq \frac{n}{n+1}\operatorname{diam}(\sigma)$. So if $\tau \in S^k\sigma$ then $\operatorname{diam}(\tau) \leq \left(\frac{n}{n+1}\right)^k \operatorname{diam}(\sigma)$.

Note that $\left(\frac{n}{n+1}\right)^k \to 0$ as $k \to \infty$.

We want to prove that S is chain homotopic to the identity. To do this we again subdivide $\Delta^n \times I$ (as for P, the prism operator).

Definition 4.9.10. We recursively define a new prism operator

$$T\sigma = b_\sigma(\sigma - T\partial\sigma).$$

Which gives the map

$$T\colon C_n(X) \to C_{n+1}(X) \text{ defined by } \sigma \mapsto \sigma_\# \circ T \circ \mathrm{id}_{\Delta^n}.$$

We also define another map D by $D_0 = 0$ and $D_{m+1} = D_m + TS^m$ so

$$D_m = T\left(\sum_{i=0}^{m-1} S^i\right)$$

Proposition 4.9.11.

$$\partial D_m + D_m\partial = 1 - S^m.$$

Proof.

$$\begin{aligned}
\partial D_{m+1} &= \partial(D_m + TS^m)\\
&= \partial D_m + \partial TS^m\\
&= 1 - S^m - D_m\partial + \partial TS^m\\
&= 1 - S^m - D_m\partial + (1 - S - T\partial)S^m\\
&= 1 - S^{m+1} - (D_m + TS^m)\partial\\
&= 1 - S^{m+1} - D_{m+1}\partial.
\end{aligned}$$

□

This gives that $1 \sim S^m$ for all m.

Lemma 4.9.12 (Fine lemma 2)**.** Fix $\mathcal{U}$ a cover of X and $\sigma\colon \Delta^n \to X$, then there exists m s.t. $S^m\sigma$ is subordinate to $\mathcal{U}$.

Proof. Define $\mathcal{U}^\sigma = \{\sigma^{-1}(U) \mid U \in \mathcal{U}\}$ this is a cover of Δ^n. Δ^n is compact so by the Lebesgue covering lemma there is some ϵ for all $x \in \Delta^n$ so that there exists $U \in \mathcal{U}^\sigma$ such that $B_\epsilon(x) \subset \operatorname{interior}(U)$. Pick m such that $\sqrt{2}\left(\frac{n}{n+1}\right)^m < \frac{\epsilon}{100}$. Now for all $\tau \in S^m \circ \mathrm{id}_A$ we have some $U \in \mathcal{U}^\sigma$ such that $\tau \subset U$ so $\sigma \circ \tau$ is in some element of U i.e. $S^m\sigma$ is subordinate to $\mathcal{U}$. Hence $S^m\sigma \in C_n^{\mathcal{U}}(X)$. □

We now finish proving the main proposition above.

Definition 4.9.13. We define a function

$$m\colon \{\sigma\colon \Delta^n \to x\} \to \mathbb{N} \text{ by } \sigma \mapsto m(\sigma).$$

where $m(\sigma)$ is the least natural number such that $S_\sigma^m \in C_n^{\mathcal{U}}(X)$. The map m is well defined by Fine lemma 2.

Definition 4.9.14. We now let $D_\sigma = D_{m(\sigma)}\sigma$.

We want D to be the desired chain homotopy for proving the proposition. So we want $\rho\colon C_*(X) \to C_*^{\mathcal{U}}(X)$ so that

$$\rho \circ i \sim 1^{\mathcal{U}} \text{ and } i \circ p \sim 1.$$

Definition 4.9.15. Define

$$\rho(\sigma) = S_\sigma^{m(\sigma)} + D_{m(\sigma)}\partial\sigma - D\partial\sigma$$

and extend linearly.

Proposition 4.9.16.

$$\rho\colon C_*(X) \to C_*^{\mathcal{U}}(X).$$

Proof. Fix $\sigma \in C_*(X)$, $S_\sigma^{m(\sigma)} \in C_n^{\mathcal{U}}(X)$. The difference $(D_{m(\sigma)}\partial - D\partial)\sigma$ lies in $C_*^{\mathcal{U}}(X)$ as well, the second term subtracts off insufficiently subdivided simplices. □

We call this ρ a *roof line.* If $\sum \sigma_i$ is a chain then $\rho\left(\sum \sigma_i\right)$ is called a *skyline.*

Proposition 4.9.17.

$$\rho\sigma = \sigma - \partial D\sigma - D\partial\sigma.$$

Proof.

$$\begin{aligned}
(S^{m(\sigma)} + D_{m(\sigma)}\partial - D\partial)\sigma &= S^{m(\sigma)} + D_{m(\sigma)}\partial\sigma - D\partial\sigma \\
&= (1 - \partial D_{m(\sigma)})\sigma - D\partial\sigma \\
&= -D_{m(\sigma)}\sigma + \sigma - D\partial\sigma \\
&= -\partial D\sigma + \sigma - D\partial\sigma.
\end{aligned}$$

□

Proposition 4.9.18.

$$\partial\rho = \rho\partial$$

i.e. ρ is a chain map.

Proof. Exercise (straightforward). □

Note that $\partial D\sigma + D\partial\sigma = \sigma - \rho\sigma$, so

$$\partial D + D\partial = 1 - i \circ \rho.$$

Also by inspection we see that

$$1^{\mathcal{U}} = \rho \circ i.$$

This then completes the proof of the above proposition and hence completes the proof of excision.

4.10 Quotients

Suppose $A \subset X$, so (X, A) is a pair. Define the quotient X/A to be the sect of equivalence classes

$$X/A = \{[x] \mid x \sim y \iff x = y \text{ or } x, y \in A\}.$$

We equip X/A with the quotient topology by defining $q\colon X \to X/A$ by $x \mapsto [x]$ and letting $U \subset X/A$ be open if and only if $q^{-1}(U)$ is open.

Exercise.

$$B^n/S^{n-1} \cong S^n.$$

Proposition 4.10.1. Suppose (X, A) is a good pair with $A \neq \emptyset$. Then

$$H_*(X, A) = \tilde{H}_*(X/A).$$

Proof. Fix $V \subset X$ s.t. V deformation retracts to A, then we have the following diagram

$$\begin{tikzcd}
H_*(X,A) \arrow[r, "q_*(7)"] \arrow[d, "j_*(1)"'] & H_*(X/A, A/A) \cong \tilde{H}_*(X/A) \arrow[d, "j_*(5)"] \\
H_*(X,V) \arrow[r, "q_*(6)"] & H_*(X/A, V/A) \\
H_*(X \smallsetminus A, V \smallsetminus A) \arrow[r, "q_*(3)"] \arrow[u, "i_*(2)"] & H_*(X/A \smallsetminus A/A, V/A \smallsetminus A/A) \arrow[u, "i_*(4)"']
\end{tikzcd}$$

Now both the top and bottom squares are commutative so to show that (6) is an isomorphism it suffices to show (2), (3) and (4) are. Similarly to show that (7) is an isomorphism it then suffices to show that (1) and (5) are. For (1) consider the triple (X, V, A) so

$$\begin{tikzcd}
H_*(V,A) \arrow[rr, "i_*"] & & H_*(X,A) \arrow[dl, "j_*"] \\
& H_*(X,V) \arrow[ul, "\delta_*"] &
\end{tikzcd}$$

is an exact triangle. But V deformation retracts to A so $H_*(V, A) = 0$ (exercise). Thus j_* is an isomorphism.

For (2) we excise A from V (check that $\mathrm{cl}(A) \subseteq \mathrm{int}(V)$).

For (3) we note that the map $q\colon (X \smallsetminus A, V \smallsetminus A) \to (X/A \smallsetminus A/A, V/A \smallsetminus A/A)$ is a homeomorphism of pairs and so q_* is an isomorphism.

For (4) argue as in (2) after (exercise) proving that if (X, A) is a good pair then $(X/A, A/A)$ is too.

For (5) note that $(X/A, V/A, A/A)$ is a triple and $H_*(V/A, A/A) = 0$ because V/A deformation retracts to A/A (by the exercise above). $\square$

Theorem 4.10.2. The following triangle is exact

$$\begin{tikzcd} \tilde{H}_*(A) \arrow[rr, "i_*"] & & \tilde{H}_*(X) \arrow[dl, "q_*"] \\ & \tilde{H}_*(X/A) \arrow[ul, "\delta_*"] & \end{tikzcd}$$

for good pairs (X, A).

Proof. $H_*(X, A) \cong \tilde{H}_*(X/A)$ proposition 4.10.1 and the exact triangle for the relative homology of a pair.

$$\begin{tikzcd} \tilde{H}_*(A) \arrow[rr, "i_*"] & & \tilde{H}_*(X) \arrow[dl, "q_*"] \\ & H_*(X/A) \arrow[ul, "\delta_*"] & \end{tikzcd}$$

□

Corollary 4.10.3.

$$H_*(S^n) = \begin{cases} \mathbb{Z}^2 & \text{if } n = k = 0, \\ \mathbb{Z} & \text{if } k = 0,\, n > 0, \\ \mathbb{Z} & \text{if } n = k > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First we review some facts.

1. $\tilde{H}_*(B^n) = 0$ because $B^n \sim \{\mathrm{pt}\}$.
2. $B^n / S^{n-1} \cong S^n$.
3. (B^n, S^{n-1}) is a good pair.

So

$$\tilde{H}_*(S^n) \cong \tilde{H}_*(B^n / S^{n-1}) \cong H_*(B^n, S^{n-1})$$

since $\tilde{H}_*(B^n) \cong 0$ we have

$$H_*(B^n, S^{n-1}) \cong H_{*-1}(S^{n-1})$$

and the triangle

$$\begin{tikzcd} \tilde{H}_*(S^{n-1}) \arrow[rr] & & \tilde{H}_*(B^n) \arrow[dl] \\ & H_*(B^n, S^{n-1}) \arrow[ul, "\delta_*"] & \end{tikzcd}$$

The corollary then follows from induction on n. □

Note that in reduced homology we have

$$\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

The spheres are like atoms of homology.

4.11 Applications

The intermediate value theorem states that if $f\colon I \to I$ has $f(0) = 0$ and $f(1) = 1$ then f is surjective. This follows from the fact that there is no retraction of I to ∂I (exercise).

Corollary 4.11.1. *B^n does not retract to ∂B^n.*

Proof. Suppose that it did. So we have $r\colon B^n \to S^{n-1}$ such that $r \circ i\colon S^{n-1} \to S^{n-1}$ is the identity. So

$$(r \circ i)_* = \mathrm{id}_{\tilde{H}_*(S^{n-1}}$$

and $(r \circ i)_* = r_* \circ i_*$. We have the triangle

$$\begin{tikzcd}
\mathbb{Z} = \tilde{H}_{n-1}(S^{n-1}) \arrow[rr, "(r\circ i)_{n-1}"] \arrow[dr, "i_{n-1}"'] & & \tilde{H}_{n-1}(S^{n-1}) \\
& \tilde{H}_*(B^n) = 0 \arrow[ur, "r_{n-1}"'] &
\end{tikzcd}$$

but this is a contradiction. ☐

Theorem 4.11.2 (Brouwer fixed point theorem)**.** *Any $f\colon B^n \to B^n$ has a fixed point.*

Challenge: prove this directly for $n = 1$ and that this is false for B^∞.

Proof. Let $h\colon B^n \to B^n$ and suppose that for all x we have $h(x) \neq x$. Now let L_x be the line through x and $h(x)$ and let $f(x) \in S^{n-1} \cap L_x$ be the point nearer to x. Now (exercise) show that f is well defined, continuous and $f\colon B^n \to S^{n-1}$ is a retraction. ☐

4.12 Local homology

Definition 4.12.1. If $x \in X$ then $H_*(X, X \setminus x)$ are called the *local homology groups at x.*

These groups are called local as if we have an open neighbourhood U of x then by excision (of $X \setminus U$)

$$i_*\colon H_*(U, U \setminus x) \xrightarrow{\cong} H_*(X, X \setminus x).$$

Exercise. Suppose X is a finite graph, compute $H_*(X, X \setminus x)$ for all $x \in X$.

Example 4.12.2. Recall that $\mathbb{R}^n \setminus 0$ is homotopy equivalent to S^{n-1}, so $H_*(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong \tilde{H}_{*-1}(S^{n-1})$. We then use the exact triangle for the pair

$$\begin{tikzcd}
\tilde{H}_*(S^{n-1}) = \tilde{H}_*(\mathbb{R}^n \setminus 0) \arrow[rr, "i_*"] & & \tilde{H}_*(\mathbb{R}^n) = 0 \arrow[dl, "j_*"] \\
& H_*(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \arrow[ul, "\delta_*"] &
\end{tikzcd}$$

since

$$\tilde{H}_k(S^{n-1}) = \begin{cases} \mathbb{Z} & \text{if } k = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

we find

$$H_k(\mathbb{R}^n, \mathbb{R}^n \setminus 0) = \begin{cases} \mathbb{Z} & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

4.13 Invariance of domain

Theorem 4.13.1 (Invariance of domain, Brouwer)**.** If $V \subseteq \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$ are open and $h\colon V \to U$ is a homeomorphism then $m = n$.

Proof. If h is a homeomorphism then pick $x \in V$ and $y = h(x) \in U$ so we have

$$\begin{tikzcd}
H_*(V, V \setminus x) \arrow[r, "\cong", "h_*"'] & H_*(U, U \setminus y) \\
H_*(\mathbb{R}^n, \mathbb{R}^n \setminus x) \arrow[u, "\cong"] & H_*(\mathbb{R}^m, \mathbb{R}^m \setminus y) \arrow[u, "\cong"'] \\
H_*(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \arrow[u, "\cong"] & H_*(\mathbb{R}^m, \mathbb{R}^m \setminus 0) \arrow[u, "\cong"']
\end{tikzcd}$$

this is only possible if $n = m$. □

An important corollary of this result is the fact that the dimension of a manifold is well defined.

4.14 Explicit generators for $H_n(S^n)$

We will now find explicit generators for the infinite cyclic groups $H_n(S^n)$.

We fix homeomorphisms

$$\sigma^{n+1}\colon (\Delta^{n+1}, \partial\Delta^{n+1}) \to (B^{n+1}, S^n),$$

for example σ^n could be a radial projection.