

Part III Homological and Homotopical Algebra 2014

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Chapter 1

Elements of Homological Algebra

1.1 Introduction

These are lecture notes for the 2014 Part III Homological and Homotopical Algebra course taught by Dr. Julian Holstein, these notes are part of [Mjolnir](#).

The recommended books are:

- W. G. Dwyer and J. Spalinski, Homotopy theories and model categories
- S. I. Gelfand and Yu. I. Manin, Methods of Homological Algebra
- C. Weibel, An introduction to homological algebra

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1.2 Motivation

Start with a graded ring $\mathbf{C}[x_0, \dots, x_n]$ with $\deg x_i = 1$. Consider a graded module $M = \bigoplus_d M_d$ over R . Hilbert looked at the map $d \mapsto H_M(d) = \dim_{\mathbf{C}} M_d$. For example we can take R to be the homogeneous coordinate ring of \mathbf{P}^n and $V(I) \subset \mathbf{P}^n$ a subvariety where I is a homogeneous ideal. We then take $M = R/I$, if V is a curve C then $H_{R/I}(d) = \deg(V) \cdot d + (1 - g(C))$. Hilbert showed that the function $H_M(d)$ is eventually polynomial. We can compute this function easily if M is free so we try to replace M by free modules. First we take

$$K_0 \rightarrow F_0 \rightarrow M$$

where K_0 is the kernel of the surjective map from F_0 to M . We can continue this getting

$$\begin{aligned} K_1 &\rightarrow F_1 \rightarrow K_0 \\ K_2 &\rightarrow F_2 \rightarrow K_1 \\ &\vdots \end{aligned}$$

we can then write

$$\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0,$$

this is a free resolution of M . We also have the following.

Theorem 1.2.1 (Hilbert). $F_{n+1} = 0$.

Corollary 1.2.2. $H_M(d) = \sum_i (-1)^i H_{F_i}(d)$.

1.3 Categorical notions

1.3.1 Abelian Categories

Example 1.3.1. \mathbf{Rmod} - the category of left R -modules for R an associative ring is an abelian category.

Example 1.3.2. The categories of sheaves of abelian groups on a topological space, sheaves of \mathcal{O} -modules on a scheme and (quasi-)coherent sheaves on a scheme are all abelian.

Definition 1.3.3 (Additive categories). An **additive category** is a category in which:

1. Every hom-space has the structure of an abelian group.
2. There exists a 0-object (one with exactly one map to and from every other object).
3. Finite products exist (these are automatically equal to sums $A \times B = A \oplus B = A \amalg B$).

In such a category we let

$$\ker(f) = \text{eq}(A \xrightarrow[f]{\quad} B)$$

and

$$\text{coker}(f) = \text{coeq}(A \xrightarrow[f]{\quad} B).$$

Definition 1.3.4 (Abelian categories). An **abelian category** \mathcal{A} is an additive category in which:

1. Every map f has a kernel and cokernel.
2. For all f we have $\text{coker}(\ker(f)) = \text{im}(f) = \text{coim}(f) = \ker(\text{coker}(f))$.

Example 1.3.5. Let \mathcal{B} be the category of pairs of vector spaces $V \subset W$, with morphisms the compatible linear maps. Consider the natural map $f: 0 \subset V \rightarrow V \subset V$, we then have $\text{im } f \cong 0 \subset V$ but $\text{coim } f \cong V \subset V$. So this category is not abelian.

From now on we take \mathcal{A} to be any abelian category.

1.3.2 Exactness

Definition 1.3.6 (Exact sequences). A sequence of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in \mathcal{A} is **exact at** B if $\text{im } f = \ker g$. A sequence is then exact if it is exact everywhere. An exact sequence of the form

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is called a **short exact sequence**.

Definition 1.3.7 (Mono and epi morphisms). A morphism f is a **monomorphism** if $fg = fh \implies g = h$ and it is an **epimorphism** if $gf = hf \implies g = h$.

Proof. Exercise. □

Naturality here means given two short exact sequences and compatible chain maps the induced maps on homology are compatible with ∂_n . (The obvious diagram commutes.)

Recall that f is a chain map if $\partial f - f \partial = 0$.

Definition 1.4.8. Let $\underline{\text{Hom}}_n(A, B)$ consist of functions $\{f_i: A_i \rightarrow B_{i+n}\}$ and define $df = d \cdot f - (-1)^n f d$ if $f \in \underline{\text{Hom}}_n$. Check that

$$d^2 f = d \cdot (d \cdot f - (-1)^n f d) - (-1)^n (d \cdot f - (-1)^n f d) \cdot d = 0.$$

We use the “Sign rule” to help with definitions, this states that if a moves past b we pick a sign $(-1)^{\deg a \deg b}$.

$\text{Ch}(\mathcal{A})$ can be enriched over $\text{Ch}(\mathbf{Z})$.

Definition 1.4.9 (Shifted complexes). The **shifted complex** $C[n]$ for $C \in \text{Ch}(\mathcal{A})$ is defined by $C[n]_i = C_{n+i}$ and $d_i^{C[n]} = (-1)^n d_{n+i}^C$.

Note that $H_i(C) = H_0(C[i])$.

So a chain map $f: A \rightarrow B[n]$ is exactly a cycle in $\underline{\text{Hom}}_n(A, B)$.

Now $\text{Hom}(A, B) = Z_0(\underline{\text{Hom}}(A, B))$, so what is $H_0(\underline{\text{Hom}}(A, B))$?

Definition 1.4.10 (Chain homotopies). A **chain homotopy** S between chain maps $f, g: A \rightarrow B$ is a collection $S_i: A_i \rightarrow B_i$ such that $\partial S + S \partial = f - g$. Equivalently we could say a map $A \rightarrow B[1]$ such that $dS = g - f$ (note: not a chain map). We write $f \simeq g$ to denote the fact that f is chain homotopic to g .

Definition 1.4.11 (Chain homotopy equivalences). Two chain complexes A and B are said to be **chain homotopy equivalent** if there are some $f: A \rightarrow B$, $g: B \rightarrow A$ such that $gf \simeq 1_A$ and $fg \simeq 1_B$.

Lemma 1.4.12. If $f \simeq g$ then $f_* = g_*$ on homology.

Proof. Check. □

Definition 1.4.13 (Quasi-isomorphisms). A chain map f inducing isomorphisms on homology is called a **quasi-isomorphism**. Two chains A, B are quasi-isomorphic if there is a quasi-isomorphism $A \rightarrow B$ and $B \rightarrow A$.

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow n & & \downarrow & & \\ \dots & \longrightarrow & \mathbf{Z} & \xrightarrow{\text{pr}} & \mathbf{Z} & \longrightarrow & \dots \end{array}$$

Example 1.4.14. is a quasi-isomorphism.

Any chain homotopy equivalence is a quasi-isomorphism, the converse is false however.

Definition 1.4.16 (Cones). Given $f: A \rightarrow B$ we define a chain complex called the **cone** of f by $\text{cone}(f)_n = A_{n-1} \oplus B_n$ with maps

$$d = \begin{pmatrix} -d_A & 0 \\ -f & d_B \end{pmatrix}.$$

Note that there exists a short exact sequence

$$B \xrightarrow[b \mapsto (0,b)]{\quad} \text{cone}(f) \xrightarrow[\{a,b\} \mapsto -a]{\quad} A[-1].$$

Doing the diagram chase of the [Snake lemma 1.4.6](#) we see that the boundary map is induced by f on homology i.e.

$$f_*: H_{n-1}A \rightarrow H_{n-1}B.$$

This proves the following.

Lemma 1.4.17. *f is a quasi-isomorphism if and only if $\text{cone}(f)$ is exact.*

Proof. Look at the long exact sequence of $B \rightarrow \text{cone}(f) \rightarrow A[-1]$

$$H_n(\text{cone}(f)) \rightarrow H_n(A) \xrightarrow{f_*} H_{n-1}(B) \rightarrow H_{n-1}(\text{cone}(f)).$$

□

1.5 Exact Functors

Definition 1.5.1 (Exact functors). An additive functor F is **exact** if it preserves short exact sequences. It is **left exact** if it sends a short exact sequence of the form

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

to an exact sequence

$$0 \rightarrow FA \rightarrow FB \rightarrow FC.$$

We have a similar definition for **right exact**.

Example 1.5.2. The functor $\text{Hom}_{\mathcal{A}}(M, -)$ is left exact from \mathcal{A} to $\text{Ab} = \mathbf{Z} \text{ mod}$. The functor $\text{Hom}_{\mathcal{A}}(-, M): \mathcal{A}^{\text{op}} \rightarrow \text{Ab}$ is left exact.

Note that left adjoint functors are right exact as they preserve colimits.

Example 1.5.3. Let M be an R, S -bimodule (i.e. a left R -module and a right S -module). Then for $A \in S\text{mod}$, $B \in R\text{mod}$

$$\text{Hom}_R(M \otimes_S A, B) \cong \text{Hom}_S(M, \text{Hom}_R(A, B))$$

Clearly not all functors are exact. However they all preserve split exact sequences, i.e. those of the form

$$0 \rightarrow A \rightarrow A \oplus C \rightarrow C.$$

Because they preserve finite direct sums

$$\begin{array}{ccccc} A & \xleftarrow{r} & B & \xleftarrow{s} & C \\ & \searrow f & \uparrow (f,g) & \searrow g & \\ & & A & & C \\ & \swarrow & \downarrow (r,s) & \swarrow & \\ A & \longrightarrow & A \oplus C & \longrightarrow & C \end{array}$$

$(r, g), (f, s)$ are inverse isomorphisms if and only if $B = fr + sg$.

1.6 Derived Functors, Introduction

We fix \mathcal{A} , and $\text{Ch}(\mathcal{A})$. If we have some right exact functor F we obtain exact sequences of the form

$$FA \rightarrow FB \rightarrow FC \rightarrow 0$$

and the question arises, can we extend this exact sequence by placing objects to the left of it?

If F is exact on short exact sequences of complexes we get a long exact sequence of homology $H_i FA$. F is exact on complexes if it is level wise exact, but F is exact if it is level wise exact. We know F is exact on split exact sequences. So we can try to force a short exact sequence to be exact by replacing objects by complexes.

Definition 1.6.1 (Projective and injective objects). An object M is **projective** if for all epimorphisms q and maps $M \xrightarrow{f} B$ there exists a lift making

$$\begin{array}{ccc} & M & \\ & \downarrow f & \\ A & \xrightarrow{q} & B \longrightarrow 0 \end{array}$$

commute. The dual notion is called **injective**

$$\begin{array}{ccc} & I & \\ & \uparrow & \\ 0 & \longrightarrow & B \longrightarrow A \end{array}$$

Example 1.6.2. Free modules in Rmod are projective. In $\text{Mat}_n(R)\text{-mod}$ the column vectors R^n form a projective object. \mathbf{Q} is injective in Ab .

Lemma 1.6.3. If C is projective or A is injective then

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is split.

Proof. (We prove the C projective case) Consider

$$\begin{array}{ccc} & C & \\ & \downarrow f & \\ B & \xrightarrow{q} & C \longrightarrow 0 \end{array}$$

then $gs = 1$. Now produce r such that $rf = 1_A$, $fr + sg = 1_B$ and $rs = 0$. Let $h = 1 - sg$. Now $gh = 0$ giving that $h = fr$ by the properties of the kernel.

$$\begin{array}{ccccc} B & \xrightarrow{h} & B & \xrightarrow{g} & C \\ & \searrow \exists! r & \nearrow & & \\ & A & & & \end{array}$$

Now check $rf = 1_A$ and $rs = 0$. □

Note that in Rmod this shows projectives are exactly summands of free modules.

Definition 1.6.4 (Projective resolutions). A **projective resolution** $P_\bullet \xrightarrow{\epsilon} A$ of A is a non-negative chain complex such that all P_i are projective and ϵ is a quasi-isomorphism. So $H_i P = 0$ if $i > 0$ and $H_0 P = A$.

Definition 1.6.5 (Derived functors). The i th **left derived functor** $L_i F(A)$ of a right exact functor F is defined as $H_i F(P)$ for some projective resolution P of A .

Dually we may define injective resolutions $B \xrightarrow{\sim} I^\bullet$ with $I \in \text{Ch}^{\geq 0}(\mathcal{A})$ and we get **right derived functors** of a left exact functor,

$$R^i F(B) = H^i(FI).$$

Note $L_{<0} F(A) = 0$ and $L_0 = fP_0/FP_1 = F(P_0/P_1) = F(A)$.

Example 1.6.6 (Tor). Define $\text{Tor}_i^R(A, B)$ to be $L_i(- \otimes_R B)(A)$. Let $\mathcal{A} = \text{Ab}$. What is $\text{Tor}_i(\mathbf{Z}/p, B)$?

$$\begin{array}{c} \mathbf{Z} \\ \downarrow p \\ \mathbf{Z} \xrightarrow{\sim} \mathbf{Z}_p \end{array}$$

is a projective resolution. So $\text{Tor}_* = H_*(B \xrightarrow{p} B)$ and we have $\text{Tor}_0^{\mathbf{Z}}(\mathbf{Z}/p, B) = B/pB$ and $\text{Tor}_1^{\mathbf{Z}}(\mathbf{Z}/p, B) = {}_p B = \{b : pb = 0\}$.

Example 1.6.7 (Ext). Define $\text{Ext}_R^i(A, B)$ to be $R^i \text{Hom}_R(-, B)(A)$. Injective in Rmod^{op} correspond to projectives in Rmod . So $\text{Ext}_{\mathbf{Z}}^* i(\mathbf{Z}/p, B) = H_*(B \xrightarrow{p} B)$ hence $\text{Ext}^0(\mathbf{Z}/p, B) = {}_p B$ and $\text{Ext}^1(\mathbf{Z}/p, B) = B/pB$.

1.7 Derived Functors, Proofs

Definition 1.7.1. \mathcal{A} has **enough projectives** if for all $M \in \mathcal{A}$ there exists a projective P such that $P \rightarrow M \rightarrow 0$.

Example 1.7.2. Rmod has enough projectives.

Warning: The category of abelian sheaves on a topological space does not have enough projectives in general.

Lemma 1.7.3. *Projective resolutions exist in \mathcal{A} if \mathcal{A} has enough projectives.*

Proof. Let $A \in \mathcal{A}$, then there exists

$$0 \rightarrow K_0 \rightarrow P_0 \rightarrow A \rightarrow 0$$

and inductively

$$0 \rightarrow K_{n+1} \rightarrow P_{n+1} \rightarrow K_n \rightarrow 0$$

with P_i projective. We can splice these together to get

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0.$$

□

Theorem 1.7.4 (Comparison Theorem). *Let $\epsilon: P \rightarrow M$ and $\eta: Q \rightarrow N$ be two projective resolutions and let $f: M \rightarrow N$ then there exists a lift $\tilde{f}: P \rightarrow Q$ (a chain map) unique up to chain homotopy.*

$$\begin{array}{ccc} P_1 & & Q_1 \\ \downarrow & & \downarrow \\ P_0 & & Q_0 \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

Proof. Exercise. □

Corollary 1.7.5. *Projective resolutions are well defined up to chain homotopy equivalence and so derived functors are well defined.*

Proof. Lift the identity to get chain maps in both directions. Uniqueness implies that they are inverse up to homotopy. □

Corollary 1.7.6. *$L_i F$ are functors.*

Lemma 1.7.7 (Horseshoe Lemma). *Given a short exact sequence*

$$A^1 \rightarrow A^2 \rightarrow A^3$$

and projective resolutions $P^1 \rightarrow A^1$ and $P^3 \rightarrow A^3$ there exists a projective resolution P^2 of A^2 with $P_i^2 = P_i^1 \oplus P_i^3$ and the inclusion and projection maps lift. So we have the following situation

$$\begin{array}{ccccccc} & P_1^1 & & P_1^2 & & P_1^3 & \\ & \downarrow & & \downarrow & & \downarrow & \\ & P_0^1 & & P_0^2 & & P_0^3 & \\ \epsilon^1 \downarrow & & \epsilon^2 \downarrow & & \epsilon^3 \downarrow & & \\ 0 \longrightarrow & A^1 & \xrightarrow{i} & A^2 & \xrightarrow{p} & A^3 & \longrightarrow 0 \end{array}$$

Proof. By induction: To get $\epsilon^2: P_0^2 \rightarrow A^2$ we use i and ϵ^1 and a lift of p . Now the [Snake lemma 1.4.6](#) shows that $\text{coker } \epsilon^i$ and $\ker \epsilon^i$ fit into a long exact sequence and hence $\text{coker } \epsilon^3 = 0$. Now apply the induction assumption to the short exact sequence of kernels

$$0 \rightarrow \ker \epsilon^1 \rightarrow \ker \epsilon^2 \rightarrow \ker \epsilon^3 \rightarrow \text{coker } \epsilon^1 = 0.$$

□

Corollary 1.7.8. *A short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathcal{A} gives a long exact sequence of left derived functors*

$$\rightarrow L_2 FC \rightarrow L_1 FA \rightarrow L_1 FB \rightarrow L_1 FC \rightarrow FA \rightarrow FB \rightarrow FC \rightarrow 0.$$

Proof. Combine [Horseshoe Lemma 1.7.7](#), [1.6.3](#) and [Snake lemma 1.4.6](#). □

Proposition 1.7.9. *The boundary map ∂ is natural, i.e. given*

$$\begin{array}{ccccc} A^1 & \longrightarrow & A^2 & \longrightarrow & A^3 \\ \downarrow f^1 & & \downarrow f^2 & & \downarrow f^3 \\ B^1 & \longrightarrow & B^2 & \longrightarrow & B^3 \end{array}$$

we have lifts $\partial \circ L_i f_3 = L_{i-1} f_1 \circ \partial$.

Proof. See Weibel theorem 2.4.6 □

Note that there is no extra work needed to do all of this for right derived functors.

1.8 The Derived Category

Idea: We want to talk about complexes up to quasi-isomorphism. We will reinterpret derived functors as ways of lifting functors to derived categories.

Remark 1.8.1. If we add simply inverses of quasi-isomorphisms we get nasty stuff!

Definition 1.8.2 (Homotopy categories). Let the homotopy category $K(\mathcal{A})$ of \mathcal{A} have objects the objects of $\text{Ch}(\mathcal{A})$ and morphism the chain homotopy classes of chain maps. We can add boundedness conditions to our categories. So we let $\text{Ch}_+(\mathcal{A})$ be only those chain complexes A with $A_n = 0$ when $n \ll 0$, these are **bounded below** chain complexes. Similarly we define $\text{Ch}_-(\mathcal{A})$ and $\text{Ch}_b(\mathcal{A}) = \text{Ch}_-(\mathcal{A}) \cap \text{Ch}_+(\mathcal{A})$. We also define $\text{Ch}^+(\mathcal{A})$ etc. for cochain complexes. Finally we define $K^+(\mathcal{A})$ etc. in the obvious way.

Definition 1.8.3 (Localisations of categories). Given a category \mathcal{C} and a class of morphisms S define the localisation of \mathcal{C} at S to be a category $\mathcal{C}[S^{-1}]$ with a functor $\mathcal{C} \xrightarrow{Q} \mathcal{C}[S^{-1}]$ such that Q sends any $s \in S$ to an isomorphism, and also such that Q is universal with respect to having this property. If $\mathcal{C} \xrightarrow{R} \mathcal{B}$ sends S to isomorphisms then there exists some P so that we have

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{R} & \mathcal{B} \\ & \searrow Q \quad \nearrow P & \\ & \mathcal{C}[S^{-1}] & \end{array}$$

Definition 1.8.4. Let $D(\mathcal{A})$, the **derived category** of \mathcal{A} , be the localisation of $K(\mathcal{A})$ at the quasi-isomorphisms. Similarly define the usual suspects $D^b = K^b(\mathcal{A})[\text{quasi-isomorphisms}]$, etc.

Theorem 1.8.5. $D(\mathcal{A})$ exists.

Proof. See Weibel 10.3, Gelfand-Manin III 2. □

Although we didn't prove this we should note that we can write morphisms in $D(\mathcal{A})$ as

$$A \xleftarrow{\sim} A' \xrightarrow{f} B$$

with $f \in \text{Hom}_{K(\mathcal{A})}(A', B)$ and $q \in \text{Hom}_{K(\mathcal{A})}(A', A)$.

Remark 1.8.6. $D^b(\mathcal{A})$ is equivalent to the subcategory of $D(\mathcal{A})$ with cohomology in bounded degrees.

Example 1.8.7. Let X be a scheme, $\text{Coh}(X)$ the abelian category of coherent sheaves on X . Then the derived category of X is defined to be $D^b(X) = D^b(\text{Coh}(X))$.

Note that $D(\mathcal{A})$ is an additive, but not necessarily abelian category.

Theorem 1.8.8. Given a complex $I^\bullet \in K^+(\mathcal{A})$ of injective objects and any chain complex A^\bullet then

$$\text{Hom}_{D(\mathcal{A})}(A^\bullet, I^\bullet) \cong \text{Hom}_{K(\mathcal{A})}(A^\bullet, I^\bullet).$$

Proof. (Sketch) Crucial ingredient: $\text{Hom}_{K(\mathcal{A})}(-, I^\bullet)$ sends quasi-isomorphisms to quasi-isomorphisms. So we can replace

$$A \xleftarrow{\sim} A' \rightarrow I$$

by $A \rightarrow I$. By considering cones it suffices to check that $\underline{\text{Hom}}_{K(\mathcal{A})}(-, I)$ sends acyclics to complexes homotopy equivalent to 0. One can construct the homotopy equivalence by hand, using injectivity. □

Corollary 1.8.9.

$$\mathrm{Hom}_{D(\mathcal{A})}(A, B[i]) = \mathrm{Ext}_{\mathcal{A}}^i(A, B).$$

Proof. Let $B \rightarrow I^\bullet$ be an injective resolution. Then both sides are isomorphic to

$$\mathrm{Hom}_{K(\mathcal{A})}(A, I[i]) = H^0 \underline{\mathrm{Hom}}_{K(\mathcal{A})}(A, I[i]).$$

□

Corollary 1.8.10. *Assume \mathcal{A} has enough injectives and write $\mathrm{inj} \mathcal{A} \subset \mathcal{A}$ for the full subcategory of injective objects. Then*

$$K^+(\mathrm{inj} \mathcal{A}) \cong D^+(\mathcal{A}).$$

Proof. We have fully faithfulness by 1.8.8. To see that it is essentially surjective write down injective resolutions for complexes (see later). □

1.9 Total derived functors

We now interpret/redefine derived functors as lifts to the derived category.

Definition 1.9.1. Let F :

Chapter 2

Applications

Chapter 3

Spectral Sequences

Chapter 4

Model Categories

