

## Part III Algebraic Geometry 2014



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# Chapter 1

## Sheaf Theory

### 1.1 Introduction

These are lecture notes for the 2014 Part III Algebraic Geometry course taught by Dr. P.M.H. Wilson.

The recommended books are:

- Algebraic Geometry - ??

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### 1.2 Sheaves

Let  $X$  be a topological space.

**Definition 1.2.1** (Presheaves). A **presheaf**  $\mathcal{F}$  of abelian groups (resp. rings) on  $X$  consists of data:

1. For every open  $U \subset X$  an abelian group (resp. ring)  $\mathcal{F}(U)$ .
2. For every inclusion of open sets  $V \subset U$  a homomorphism called **restriction**, denoted  $\rho_V^U: \mathcal{F}(U) \rightarrow \mathcal{F}(V)$  such that
  - (a)  $\mathcal{F}(\emptyset) = 0$ .
  - (b)  $\rho_U^U: \mathcal{F}(U) \rightarrow \mathcal{F}(U)$  is  $id_{\mathcal{F}(U)}$ .
  - (c) If  $W \subset V \subset U$  open then  $\rho_W^U = \rho_W^V \circ \rho_V^U$ .

**Remark 1.2.2.** If  $\mathcal{U}$  denotes the category of open sets in  $X$  (the morphisms are inclusions) then a presheaf of abelian groups over  $X$  is a contravariant functor  $\mathcal{F}: \mathcal{U} \rightarrow \text{Abgp}$  i.e. an element of  $\text{Abgp}^{\mathcal{U}^{op}}$ .

An element  $s \in \mathcal{F}(U)$  is called a **section** of  $\mathcal{F}$  over  $U$ . For  $s \in \mathcal{F}(U)$  we denote  $\rho_V^U(s)$  by  $s|_V$ .

**Definition 1.2.3** (Sheaves). A presheaf  $\mathcal{F}$  on  $X$  is a **sheaf** if it satisfies two further conditions:

1. If  $U$  is open and has  $U = \bigcup_i U_i$  an open cover and if  $s \in \mathcal{F}(U)$  is such that  $s|_{V_i} = 0$  for all  $i$  then  $s = 0$ . (A presheaf satisfying this condition is called a **monopresheaf**).
2. If  $U = \bigcup_i V_i$  is an open cover and we have  $s_i \in \mathcal{F}(V_i)$  such that  $\forall i, j$   $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$  then there exists  $s \in \mathcal{F}(U)$  with  $s|_{V_i} = s_i$  for all  $i$ .

**Example 1.2.4.**  $X$  a topological space,  $A$  any abelian group (resp. ring). The constant sheaf  $\mathcal{A}$  determined by  $A$  is defined as follows:  $\mathcal{A}(\emptyset) = \{0\}$ , for  $U \neq \emptyset$  open in  $X$

$$\mathcal{A}(U) = \{\text{locally constant maps } U \rightarrow A\},$$

this is an abelian group (resp. ring) under pointwise operations. With obvious restriction maps we obtain a sheaf. If  $U \neq \emptyset$  is open and connected, then  $\mathcal{A}(U) = A$ . If  $U$  an open set whose connected components are open (e.g. in a locally connected topological space  $X$ ) then the section  $\mathcal{A}(U)$  is a direct product of copies of  $A$ .

**Example 1.2.5.** If  $X$  is a differentiable ( $C^\infty$ ) manifold, we can define the sheaf of  $C^\infty$ -functions ( $\mathbf{R}$  or  $\mathbf{C}$  valued) on  $X$ . Which is a sheaf of rings. Similarly if  $X$  is a complex manifold, we can define a sheaf of holomorphic functions on  $X$ . In both cases, the sheaf is called the **structure sheaf** of  $X$ , sometimes denoted  $\mathcal{O}_X$ .

**Example 1.2.6.** For  $V$  an (irreducible) variety (affine, projective, quasi-projective). We can consider  $V$  as a topological space with the Zariski topology. For  $U$  open in  $V$  set

$$\mathcal{O}_V(U) = \{\text{regular functions on } U\} = \{f \in k(V) \text{ s.t. } f \text{ regular on } U\}.$$

This is a sheaf of rings with respect to the Zariski topology, and is known as the **structure sheaf** for varieties. If  $V$  is affine we have that  $\mathcal{O}_V(V) = k[V]$ .

**Definition 1.2.7** (Stalks). If  $\mathcal{F}$  is a presheaf on a topological space  $X$  and  $P \in X$  we define the stalk  $\mathcal{F}_P$  of  $\mathcal{F}$  at  $P$  to be  $\varinjlim_{U \ni P} \mathcal{F}(U)$  i.e. an element of  $\mathcal{F}_P$  is represented by a pair  $(U, s)$  where  $U$  is an open neighbourhood of  $P$  and  $s \in \mathcal{F}(U)$ , where  $(U, s)$  and  $(V, t)$  define the same element of  $\mathcal{F}_P$  if there exists an open neighbourhood  $W \ni P$  with  $W \subset V \cap U$  such that  $s|_W = t|_W$  the elements  $s_P$  of  $\mathcal{F}_P$  are called **germs**. If  $\mathcal{F}$  is a sheaf of abelian groups or rings then  $\mathcal{F}_P$  is an abelian group, ring, etc.

**Example 1.2.8.** For the constant sheaf  $\mathcal{A}$  associated to  $A$  we have  $\mathcal{A}_P = A$ .

**Example 1.2.9.** For  $X$  a  $C^\infty$  manifold (resp. complex) with structure sheaf  $\mathcal{O}_X$ , the stalk  $\mathcal{O}_{X,P}$  of  $\mathcal{O}_X$  at  $P \in V$  consists of germs of  $C^\infty$  (resp. holomorphic) functions.

**Example 1.2.10.** For  $V$  (irreducible) affine, projective or quasi-projective variety with structure sheaf  $\mathcal{O}_V$  the stalk at  $P \in V$   $\mathcal{O}_{V,P} =$  local ring at  $P$  (defined before).

**Definition 1.2.11** (Morphisms of (pre)sheaves). If  $\mathcal{F}$  and  $\mathcal{G}$  are presheaves (resp. sheaves) on  $X$  a morphism  $\Phi: \mathcal{F} \rightarrow \mathcal{G}$  consists of homomorphisms  $\mathcal{F}(U) \xrightarrow{\Phi} \mathcal{G}(U)$  for all open  $U$  such that for  $V \subseteq U$  open

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \rho_V^U \downarrow & & \downarrow \rho_V'^U \\ \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{G}(V) \end{array}$$

commutes.

A morphism  $\phi: \mathcal{F} \rightarrow \mathcal{G}$  induces a homomorphism  $\phi_P: \mathcal{F}_P \rightarrow \mathcal{G}_P$  for each  $P$ , namely  $\phi_P[(U, s)] = [(U, \phi(U)(s))]$ , which is well defined.



**Definition 1.2.12** (Injective and isomorphic sheaf morphisms). A morphism  $\phi: \mathcal{F} \rightarrow \mathcal{G}$  of (pre)sheaves is **injective** if  $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is injective for all open  $U$ . e.g. sheaves of subgroups or subrings where  $\mathcal{F}(U) \subseteq \mathcal{G}(U)$  for all  $U$ . In this case  $\mathcal{F}$  is called a **subsheaf** of  $\mathcal{G}$ .

A morphism  $\phi: \mathcal{F} \rightarrow \mathcal{G}$  is called an isomorphism if there exists an inverse morphism  $\chi: \mathcal{G} \rightarrow \mathcal{F}$ . This is equivalent to  $\phi(U): \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  being bijective for all  $U$  since we can define  $\chi(U) = \phi(U)^{-1}$  as the inverse.

**Lemma 1.2.13.** *Let  $\phi: \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of sheaves then*

1.  $\phi$  is injective iff  $\phi_P: \mathcal{F}_P \rightarrow \mathcal{G}_P$  is injective for all  $P \in X$ ,
2.  $\phi$  is an isomorphism iff  $\phi_P: \mathcal{F}_P \rightarrow \mathcal{G}_P$  is an isomorphism for all  $P \in X$ .

*Proof.* ( $\Rightarrow$ ) (true for presheaves too).

1. Suppose there exists a germ  $s_P \in \mathcal{F}_P$  such that  $\phi_P(s_P) = 0 \in \mathcal{G}_P$ . i.e. there exists an open neighbourhood  $W \subset U$  with  $P \in W$  such that  $\phi(U)(s|_W) = 0$ . So by commutativity  $\phi(W)(s|_W) = 0$  but  $\phi$  injective implies  $s|_W = 0$ .
2. Clear.

( $\Leftarrow$ )

1. Needs first sheaf condition on  $\mathcal{F}$ . If  $\phi_P$  injective for all  $P$  and  $U$  is open in  $X$  it remains to prove that  $\phi(U): \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is injective. Suppose not and there exists  $0 \neq s \in \mathcal{F}(U)$  such that  $\phi(U)(s) = 0 \in \mathcal{G}(U)$ . Let  $s_P$  denote the germ of  $s$  at  $P \in U$ , then  $0 = \phi(U)(s)_P = \phi_P(s_P)$  for all  $P \in U$ . So  $s_P$  in  $\mathcal{F}_P$  for all  $P \in U$ . Hence for all  $P \in U$  we have an open neighbourhood  $U \supset W \ni P$  such that  $s|_W = 0$ . So  $U$  is covered by open sets  $U_\alpha$  such that  $s|_{U_\alpha} = 0$  for all  $\alpha$ , which implies that  $s = 0$ .
2.  $\phi(U): \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  is an injection for all open  $U$  by the first part, so it remains to prove that it is surjective also. Suppose  $t \in \mathcal{G}(U)$  and let  $t_P \in \mathcal{G}_P$  be its germ at  $P \in U$ . Since  $\phi_P$  is surjective we have some  $s_P \in \mathcal{F}_P$  such that  $\phi_P(s_P) = t_P$ . Now suppose that  $s_P$  is represented by a pair  $(V, s)$  with  $P \in V \subseteq U$  and  $s \in \mathcal{F}(V)$ . We then have that  $t_P$  is represented by  $\phi(V)(s)$ , i.e.  $(U, t) \sim (V, \phi(V)(s))$ . Shrinking  $V$  we may assume that we have an open neighbourhood  $V_P \ni P$  such that  $\phi(V)(s)|_{V_P} = t|_{V_P}$ . In this way we cover  $U$  by open sets giving  $U = \bigcup_\alpha U_\alpha$  to obtain sections  $s_\alpha \in \mathcal{F}_\alpha$  such that  $\phi(U_\alpha)(s_\alpha) = t|_{U_\alpha}$ . On the overlaps  $U_{\alpha\beta} = U_\alpha \cap U_\beta$  we have  $\phi(U_{\alpha\beta})(s_\alpha|_{U_{\alpha\beta}}) = t|_{U_{\alpha\beta}} = \phi(U_{\alpha\beta})(s_\beta|_{U_{\alpha\beta}})$ , therefore the injectivity of  $\phi(U_{\alpha\beta})$  gives that  $s_\alpha|_{U_{\alpha\beta}} = s_\beta|_{U_{\alpha\beta}}$ . Since  $\mathcal{F}$  is a sheaf the  $s_\alpha$  patch together to give a section  $s \in \mathcal{F}(U)$  such that  $s|_{U_\alpha} = s_\alpha$  (using the second sheaf condition for  $\mathcal{F}$ ). But then  $\phi(U)(s)$  and  $t$  are sections of  $\mathcal{G}(U)$  such that  $\phi(U)(s)|_{U_\alpha} = \phi(U_\alpha)(s_\alpha) = t|_{U_\alpha}$  for all  $\alpha$ . The first sheaf condition for  $\mathcal{G}$  now gives that  $\phi(U)(s) = t$  as required.

□

**Definition 1.2.14** (Surjective sheaf morphisms). A morphism of sheaves  $\phi: \mathcal{F} \rightarrow \mathcal{G}$  is called **surjective** if  $\phi_P: \mathcal{F}_P \rightarrow \mathcal{G}_P$  is surjective for all  $P \in X$ .

**Definition 1.2.15** (Induced (pre)sheaves). Given a (pre)sheaf  $\mathcal{F}$  on a space  $X$  and a continuous map  $f: X \rightarrow Y$  we have an **induced (pre)sheaf**, denoted  $f_*\mathcal{F}$  on  $Y$  defined by

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$$

for  $U$  open in  $Y$ . With the restriction maps coming from  $\mathcal{F}$  as  $V \subset U$  implies  $f^{-1}(V) \subset f^{-1}(U)$ . It should be checked that indeed  $\mathcal{F}$  being a (pre)sheaf implies  $f_*\mathcal{F}$  is a (pre)sheaf.

**Definition 1.2.16** (Ringed spaces). A **ringed space** is a pair  $(X, \mathcal{O}_X)$  where  $X$  is a topological space and  $\mathcal{O}_X$  is a sheaf of rings on  $X$ .

**Definition 1.2.17** (Morphisms of ringed spaces). Given two ringed spaces  $(X, \mathcal{O}_X)$  and  $(Y, \mathcal{O}_Y)$  a **morphism of ringed spaces**  $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  is a pair  $(f, f^\#)$  where  $f: X \rightarrow Y$  is continuous and  $f^\#: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$  is a morphism of sheaves of rings. So  $f^\#$  defines homomorphisms  $\mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(f^{-1}(U))$  for all  $U$  open in  $Y$ , compatible with restrictions. Hence we have homomorphisms on stalks too  $\mathcal{O}_{Y, f(P)} \rightarrow \mathcal{O}_{X, P}$ .

**Definition 1.2.18** (Ringed spaces over a ring). If  $R$  is a commutative ring (e.g. a field), a **ringed space over  $R$**  is a ringed space with  $\mathcal{O}_X$  a sheaf of  $R$ -algebras. (Therefore the restriction maps are homomorphisms of  $R$ -algebras.) A morphism of ringed spaces over  $R$  is defined in the obvious way.

**Definition 1.2.19** (Locally ringed spaces). A ringed space  $(X, \mathcal{O}_X)$  is a **locally ringed space** (also known as a geometric space) if all  $\mathcal{O}_{X, P}$  are local rings.

**Definition 1.2.20** (Morphisms of locally ringed spaces). A **morphism of locally ringed spaces** is a morphism of ringed spaces as above where all the induced maps  $f_P^\#: \mathcal{O}_{Y, f(P)} \rightarrow \mathcal{O}_{X, P}$  are local homomorphisms of local rings.

**Example 1.2.21.**  $(X, \mathbf{Z})$  is a ringed space but not locally ringed.

**Example 1.2.22.** If  $X$  is a  $C^\infty$  (resp. complex) manifold with structure sheaf  $\mathcal{O}_X$  then  $(X, \mathcal{O}_X)$  is a locally ringed space over  $\mathbf{R}$  (resp. over  $\mathbf{C}$ ). A smooth (resp. holomorphic) map of manifolds  $f: X \rightarrow Y$  yields a morphism of  $\mathbf{R}$  (resp.  $\mathbf{C}$ )-algebras  $f^\#: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$  namely  $f^\#: \mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(f^{-1}(U))$  given by  $g \mapsto g \circ f$  (smooth (resp. holomorphic) functions on  $Y$  pullback to ones on  $X$ ). Clearly  $g(f(P)) = 0 \iff f^\#(g)(P) = 0$  and so  $f^\#(m_{Y, f(P)}) \subseteq m_{X, P}$ . So  $(f, f^\#)$  is a morphism of locally ringed spaces over  $\mathbf{R}$  (resp.  $\mathbf{C}$ ).

**Example 1.2.23.**  $(V, \mathcal{O}_V)$  for  $V$  an irreducible affine variety  $V$  is a ringed space via its structure sheaf, it is locally ringed over the base field  $k$ . If  $\Phi: V \rightarrow W$  is a morphism of affine varieties then there exists a morphism of locally ringed spaces over  $k$   $(\phi, \phi^\#): (V, \mathcal{O}_V) \rightarrow (W, \mathcal{O}_W)$  given by  $\phi^\#(g) = g \circ \phi \in \mathcal{O}_V(\phi^{-1}(U))$  for  $g \in \mathcal{O}_W(U)$ .

**Lemma 1.2.24.** *If  $V, W$  are (irreducible) affine varieties and  $(f, f^\#): (V, \mathcal{O}_V) \rightarrow (W, \mathcal{O}_W)$  is a morphism of locally ringed spaces over  $k$  then  $f$  is induced from a morphism of varieties  $\phi: V \rightarrow W$  with  $f^\# = \phi^\#$  defined as above.*

*Proof.* Suppose  $V \subseteq \mathbf{A}^n$ ,  $W \subseteq \mathbf{A}^m$  let  $y_j$  be the  $j$ th coordinate function on  $W$  and define  $g_j = f^\#(y_j) \in \mathcal{O}_V(V) = k[V]$ . Let  $\phi = (g_1, \dots, g_m)$ , this is a morphism  $V \rightarrow \mathbf{A}^m$ . Suppose that  $f(P) = (b_1, \dots, b_m)$  for  $P \in V$  then  $y_j - b_j \in m_{W, f(P)}$  for all  $j$  which implies that  $g_j(P) = b_j$  for all  $j$ . Since  $f^\#$  is local we have that  $\phi(P) = f(P)$  and so  $\phi: V \rightarrow W$  is the same map as  $f$  on topological spaces. Moreover  $y_1, \dots, y_m$  generate  $k[W]$  as a  $k$ -algebra and also generate  $k(W)$  as a field over  $k$ .  $f^\#(y_j) = g_j = y_j \circ \phi = \phi^\#(y_j)$  and it follows that  $f^\# = \phi^\#$  on any  $\mathcal{O}_W(U)$ .  $k[W] \subset \mathcal{O}_W(U) \subset k(W)$  for  $U$  open in  $W$ .  $\square$

**Definition 1.2.25** (Morphisms of varieties). Given  $V$  and  $W$  (irreducible) quasi-projective varieties, we define a **morphism** of varieties  $V \rightarrow W$  to be a morphism of the corresponding locally ringed spaces over  $k$   $(V, \mathcal{O}_V) \rightarrow (W, \mathcal{O}_W)$ .

### 1.2.1 $\mathcal{O}_X$ -modules

**Definition 1.2.26** ( $\mathcal{O}_X$ -modules). Let  $M$  be a sheaf of abelian groups on a ringed space  $(X, \mathcal{O}_X)$ ,  $M$  is said to be an  $\mathcal{O}_X$ -**module** if for every open set  $U \subset X$   $M(U)$  is an  $\mathcal{O}_X(U)$ -module and for any  $W \subseteq U$  open  $\alpha \in \mathcal{O}_X(U)$ ,  $m \in M(U)$ , we have  $(\alpha m)|_W = (\alpha|_W)(m|_W)$ . Similarly we have the obvious definition for morphisms of  $\mathcal{O}_X$ -modules  $\phi: M \rightarrow N$  (all maps respect the  $\mathcal{O}_X$ -module structure).

**Example 1.2.27.** For  $V$  a (irreducible) quasi-projective variety with structure sheaf  $\mathcal{O}_V$  and  $W \subset V$  a closed subvariety we have the **sheaf of ideals**  $\mathcal{I}_W \subset \mathcal{O}_V$  a subsheaf of  $\mathcal{O}_V$  given by

$$\mathcal{I}_W(U) = \{f \in \mathcal{O}_V(U) : f|_{W \cap U} \equiv 0\}.$$

This is clearly an  $\mathcal{O}_V$ -module.

Most things go through unchanged e.g. if  $M$  is an  $\mathcal{O}_X$ -module then any stalk  $M_P$  is an  $\mathcal{O}_{X,P}$ -module etc.

However there is the following technicality: If we take the pushforward of an  $\mathcal{O}_X$ -module  $M$  under a morphism of ringed spaces  $(X, \mathcal{O}_X) \xrightarrow{\phi} (Y, \mathcal{O}_Y)$ . The sheaf  $f_*M$  is naturally an  $f_*\mathcal{O}_X$ -module via the morphism  $f^\#: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$  we can consider  $f_*M$  as an  $\mathcal{O}_Y$ -module which we denote by  $\phi_*M$ . Explicitly for  $U$  open in  $Y$  we let

### 1.2.2 Sheafification

**Definition 1.2.28** (Sheafification). Given a preheaf  $\mathcal{F}$  on  $X$  there exists an associated sheaf  $\mathcal{F}^+$  and a morphism  $\Theta: \mathcal{F} \rightarrow \mathcal{F}^+$  with the universal property that for any sheaf  $\mathcal{G}$  on  $X$  and morphism  $\phi: \mathcal{F} \rightarrow \mathcal{G}$  there is a unique morphism  $\psi: \mathcal{F}^+ \rightarrow \mathcal{G}$  such that  $\phi = \psi\Theta$ . We construct  $\mathcal{F}^+$  as follows: For  $U$  open in  $X$  we let

$$\mathcal{F}^+(U) = \left\{ \text{functions } s: U \rightarrow \coprod \mathcal{F}_p : \forall p \in U s(p) \in \mathcal{F}_p, \forall p \in U \exists \text{ open nbhd } W, t \in \mathcal{F}(W) \text{ with } s(q) = t_q \forall q \in W \right\}$$

It is clear that this is a sheaf since sections are given in terms of functions.

There exists a morphism  $\Theta: \mathcal{F} \rightarrow \mathcal{F}^+$  where  $\Theta(U): \mathcal{F}(U) \rightarrow \mathcal{F}^+(U)$  is given by  $\sigma \mapsto s$  with  $s(p) = \sigma_p$  for all  $p \in U$ .



## Chapter 2

# Locally free and coherent $\mathcal{O}_X$ -modules

### 2.1 Locally free and coherent $\mathcal{O}_X$ -modules

**Definition 2.1.1** (Locally free  $\mathcal{O}_X$ -modules). An  $\mathcal{O}_X$ -module  $M$  is locally free of rank  $r$  if for each  $x \in X$  there exists an open neighbourhood  $U \ni x$  s.t.  $M|_U$  is isomorphic to  $\mathcal{O}_U^r = \bigoplus_{i=1}^r \mathcal{O}_U$

**Example 2.1.2.** If  $(X, \mathcal{O}_X)$  is a  $C^\infty$  (resp. complex) manifold and  $E \rightarrow X$  a rank  $r$   $C^\infty$  (resp. holomorphic) vector bundle over  $X$  we can define a corresponding  $\mathcal{O}_X$ -module  $\mathcal{E}$  by letting  $\mathcal{E}(U)$  be the  $C_*^\infty$  (resp. holomorphic) sections  $\sigma$  of  $\mathcal{E}$  over  $U$ . Since  $\mathcal{E}$  is locally trivial it's clear that  $\mathcal{E}$  is locally free of rank  $r$ .

**Definition 2.1.3** (Transition functions). For  $M$  a locally free  $\mathcal{O}_X$ -module of rank  $r$  we have an open cover  $\{U_i\}$  of  $X$  (when  $X$  is a variety we may in fact take a finite open cover) and trivialisations  $M|_{U_i} \cong \mathcal{O}_{U_i}^r$  this then gives rise to isomorphisms on the overlaps  $U_{ij} = U_i \cap U_j$ . We write  $\psi_{ji}$  for the map from the  $i$ th trivialisation to the  $j$ th. These functions are called the **transition functions**. They satisfy some compatibility conditions:  $\psi_{ii} = id$ ,  $\psi_{ij} = \psi_{ji}^{-1}$  and  $\psi_{kj}\psi_{ji} = \psi_{ki}$ .

