Part III Homotopy Theory 2014

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Chapter 1

Homotopy Theory

1.1 Introduction

These are lecture notes for the 2014 Part III Homotopy Theory course taught by Dr. Oscar Randal-Williams, these notes are part of MJOLNIR.

The recommended books are:

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1.2 Homotopy groups

The goal of the course is to introduce tools to compute higher homotopy groups, though it will turn out that the tools developed are more interesting than the groups themselves.

Let (X, x_0) be a based space and write $I = [0, 1]$, I^n for the n-cube. We then have $\partial I^n = \{(x_1, \dots, x_n) \in I^n : \text{some } x_i \in \{0, 1\}\}$.

Definition 1.2.1 (Homotopy groups)**.** $\pi_n(X, x_0)$ is the set of homotopy classes of maps $f\colon I^n \to X$ such that $f(\partial I^n) = \{x_0\}$ and homotopies are taken through such maps.

If A is a subspace we write (X, A). Then a map $f\colon (X, A) \to (Y, B)$ is a map $f\colon X \to Y$ such that $f(A) \subset B$. A homotopy of maps of pairs is a homotopy $H\colon X \times I \to Y$ such that $H(A \times I) \subset B$.

Thus $\pi_n(X, x_0)$ is the set of homotopy classes of maps $f\colon (I^n, \partial I^n) \to (X, \{x_0\})$.

For $n = 1$ this is the usual fundamental ggroup of (X, x_0), for $n = 0$ let $I^n = \{*\}$, $\partial I^n = \emptyset$ and so $\pi_0(X, x_0)$ is the set of path components of X.

We may define a composition law on $\pi_n(X, x_0)$ by

$$f \cdot g(x_1, \dots, x_n) = \begin{cases} f(2x_1, \dots, x_n) & 0 \leq x_1 \leq \frac{1}{2} \\ g(2x - 1, \dots, x_n) & \frac{1}{2} \leq x_1 \leq 1. \end{cases}$$

Just as for $\pi_1(X, x_0)$ this composition law makes $\pi_n(X, x_0)$ into a group for $n \geq 1$. For $n \geq 2$ this group is abelian.

If u is a path from x_0 to x_1 in X we obtain a map $u_\#\colon \pi_n(X, x_1) \to \pi_n(X, x_0)$ given by using the path to bridge the outer boundary of the cube to an inner cube of half the size. This map satisfies

1. If $u \simeq u'$ as paths then $u_\# = u'_\#$.

2. $(C_{x_0})_\# = \mathrm{id}$.
3. $u_\#$ is a homomorphism.
4. $v_\#(u_\#(f)) = (v \cdot u)_\#(f)$.

We have the following consequences of this definition.

1. If x_0, x_1 are in the smae path component then $\pi_n(X, x_0) \cong \pi_n(X, x_1)$, but not canonically so.
2. Taking $x_1 = x_0$ in the above we get a left action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ (i.e. for $n \geq 2$ $pi_n(X, x_0)$ is a $\mathbf{Z}\,\pi_1(X, x_0)$-module.
3. $\pi_n(-)$ is a functor
$$\text{based spaces} \to \begin{cases} \text{groups} & n = 1, \\ \text{abelian groups} & n \geq 2. \end{cases}$$
and if $f\colon (X, x_0) \to (Y, y_0)$ then we have $f_*([\gamma] \cdot x) = f_*([\gamma]) \cdot f_*(x)$.
4. If $f\colon (X, x_0) \to (Y, y_0)$ is a based homotopy equivalence then f_* is an isomorphism on all π_n. (This is still true if f is just a naked homotopy equivalence.)

1.3 Relative homotopy groups

Relative homotopy groups are defined for a space X a subspace $A \subset X$ and a point $x_0 \in A$.

Let $\sqcap^{n-1} \subset \partial I^n \subset I^n$ be the closure of the complement of $I^{n-1} \times \{0\}$.

Definition 1.3.1 (Relative homotopy groups)**.** The **relative homotopy group**$\pi_n(X, A, x_0)$ is the set of homotopy classes of maps $f\colon I^n \to X$ such that

1. $f(\partial I^n) \subset A$.
2. $f(\sqcap^{n-1}) = \{x_0\}$.

For $n \geq 2$ the usual formula defines a composition law on $\pi_n(X, A, x_0)$. For $n \geq 3$ the usual argument shows $\pi_n(X, A, x_0)$ is an abelian group.

Observe that as $(I^n/\sqcap^{n-1}, \partial I^n/\sqcap^{n-1}, \sqcap^{n-1}/\sqcap^{n-1}) \cong (D^n, \partial D^n, *)$ we can define $\pi_n(X, A, x_0) = \{\text{homotopy classes of maps } f\colon (D^n, \partial D^n, *) \to (X, A, x_0)\}$.

If we have $f\colon (X, A, x_0) \to (Y, B, y_0)$ then f induces a map $f_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ and if $f \simeq g$ through such maps then $f_* = g_*$.

Proposition 1.3.2 (Compression criterion)**.** *A map $f\colon (D^n, \partial D^n, *) \to (X, A, x_0)$ is trivial in $\pi_n(X, A, x_0)$ $\iff$ f is homotopic relative to ∂D^n to a map into A.*

Proof. ($\Rightarrow$) Let f be homotopic relative to ∂D^n to a map into A, call the homotopy $H\colon D^n \times I \to X$. So $H(D^n \times \{1\} \subset A$ and $H(\partial D^n \times I \subset A$, projection from the point $(0, \ldots, 0, -1)$ then gives a deformation retract of $D^n \times I$ to $B = (\partial D^n \times I) \cup (D^n \times \{0\})$ so it gives a homotopy from H to H' relative to B such that H' lands in A. Restriction this homotopy to $D^n \times \{0\}$ gives a homotopy from f to a $f'\colon D^n \to A$ relative to ∂D^n.
($\Leftarrow$) If $f \simeq g\colon (D^n, \partial D^n, *) \to (X, A, x_0)$ and $g(D^n) \subset A$ then $[f] = [g]$, now consider $g\colon (D^n, *) \to (A, x_0)$. The deformation of D^n to $*$ gives a based homotopy from g to C_{x_0}, i.e. letting $r\colon D^n \times I \to D^n$ be the linear deformation retract then $g \circ r\colon D^n \times I \to A$ is a based homotopy which at $t = 1$ is C_{x_0}. $\square$