Part III Algebraic Topology 2014

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Abstract

These are lecture notes for the 2014 Part III Algebraic Topology course taught by Dr. Jacob Rasmussen.

Contents

1 Introduction

The recommended books are:

- Algebraic Topology - Allen Hatcher,
- Homology Theory - James W. Vick,
- Differential Forms in Algebraic Topology - Raoul Bott and Loring W. Tu.

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2 Homotopy

2.1 Homotopies

Definition 2.1 (Homotopic maps)**.** Maps $f_0, f_1 \colon X \to Y$ are said to be **homotopic** if there is a continuous map $F \colon X \times I \to Y$ such that

$$F(x,0) = f_0(x) \text{ and } F(x,1) = f_1(x) \ \forall x \in X.$$

We let $\mathrm{Map}(X,Y) = \{f \colon X \to Y \text{ continuous}\}$. Then letting $f_t(x) = F(x,t)$ in the above definition we see that f_t is a path from f_0 to f_1 in $\mathrm{Map}(X,Y)$.

Example 2.2.

1. $X = Y = \mathbf{R}^n$, $f_0(\overline{x}) = \overline{0}$ and $f_1(\overline{x}) = \overline{x}$ are homotopic via $f_t(\overline{x}) = t\overline{x}$.
2. $S^1 = \{z \in \mathbf{C} : |z| = 1\}$ then
3. $S^n = \{\overline{x} \in \mathbf{R}^n : |\overline{x}| = 1\}$

Lemma 2.3. *Homotopy is an equivalence relation on* $\mathrm{Map}(X,Y)$*.*

Lemma 2.4. *If $f_0 \sim f_1 \colon X \to Y$ and $g_0 \sim g_1 \colon Y \to Z$ then $g_0 \circ f_0 \sim g_1 \circ f_1$.*

Corollary 2.5. *For any space X the set $[X, \mathbf{R}^n]$ has one element.*

Proof. Define $0_X \colon X \to \mathbf{R}^n$ by $0_X(x) = 0 \in \mathbf{R}^n$ for any $x \in X$. □

Definition 2.6 (Contractible space)**.** X is **contractible** if 1_X is homotopic to a constant map.

Proposition 2.7. *Y is contractible $\iff$ $[X,Y]$ has one element for any space X.*

Proof. ($\Rightarrow$) as in corollary. ($\Leftarrow$) $[X,Y]$ has one element so $1_Y \sim$ a constant map. □

Given a space X how can we tell if X is contractible? If X is contractible then it must be path connected for one.

Proof. Contractible implies that $[S^0, X]$ has one element and so $f \colon S^0 \to X$ extends to D^1, and therefore X is path connected. □

Similarly if $[S^1, X]$ has more than one element then X is not contractible.

Definition 2.8 (Simply connected)**.** We say X is **simply connected** if $[S^1, X]$ has only one element.

We say two space X and Y are *homotopy equivalent* if there exists $f \colon X \to Y$ and $g \colon Y \to X$ such that $g \circ f \sim 1_X$ and $f \circ g \sim 1_Y$.

Example 2.9. X is contractible if and only if $X \sim \{p\}$.

Proof. X contractible $\implies 1_X \sim c$, a constant map. Choose $f\colon X \to \{p\}$, $f(x) = p$ and $g\colon \{p\} \to X$, $g(p) = c$. Then $g \circ f = c \sim 1$ and $f \circ g = 1_{\{p\}}$. Converse: exercise. $\square$

Exercise 2.10.

Given X and Y how can we determine if $X \sim Y$? How do we determine $[X, Y]$? For example is $S^n \sim S^m$.

2.2 Homotopy groups

Definition 2.11 (Map of pairs)**.** A **map of pairs**$f\colon (X, A) \to (Y, B)$ is a map $f\colon X \to Y$ with sets $A \subset X$ and $B \subset Y$ such that $f(A) \subset B$.

If we have maps of pairs $f_0, f_1\colon (X, A) \to (Y, B)$ then we write $f_0 \sim f_1$ if there exists $F\colon (X \times I, A \times I) \to (Y, B)$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$.

Definition 2.12 (Homotopy groups)**.** If $* \in X$ then the n**th homotopy group** is

$$\pi_n(X, *) = [(D^n, S^{n-1}) \to (X, \{*\})].$$

We now note several properties of this definition:

1. $\pi_0(X, *) =$ set of path components of X.
2. $\pi_1(X, *)$ is a group.$\pi_n(X, *)$ is an abelian group.
3. π_n is a functor

$$\left\{\begin{smallmatrix}\text{pointed spaces}\\ \text{pointed maps}\end{smallmatrix}\right\} \to \left\{\begin{smallmatrix}\text{groups}\\ \text{group homomorphisms}\end{smallmatrix}\right\}.$$

So given

$$f\colon (X, p) \to (Y, q)$$

we get

$$f_*\colon \pi_n(X, p) \to \pi_n(y, q)$$

defined by

$$f_*(\gamma) = f \circ \gamma.$$

n	1	2	3	4	5	6	7
$\pi_n(S^2)$	0	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/12$	$\mathbf{Z}/15$

Example 2.13 (Homotopy groups of S^2)**.**

3 Homology

Our goal is to construct a functor ″ from the category of topological spaces and continuous maps to the category of $\mathbf{Z}$-modules and $\mathbf{Z}$-linear maps. This means to each space X we associate an abelian group ″$(X) = \bigoplus_{n \geq 0} H_n(X)$, and to each map $f\colon X \to Y$ a function $f_*\colon H_n(X) \to H_n(Y)$ satisfying $(1_X)_* = 1_{H_n(X)}$ and $(f \circ g)_* = f_* \circ g_*$.

Some properties we would like to have for our construction are:

1. Homotopy invariance, if $f \sim g\colon X \to Y$ then $f_* = g_*$.
2. The dimension axiom, $H_n(X) = 0$ for any $n > \dim X$.

3.1 Chain complexes

Definition 3.1 (Chain complex)**.** If R is a commutative ring then a **chain complex** over R is a pair (C, d) satisfying:

1. $C = \bigoplus_{n \in \mathbf{Z}} C_n$ for R-modules C_n.
2. $d \colon C \to C$ where $d = \bigoplus d_n$ for R-linear maps d_n.
3. $d \circ d = 0$.

The indexing by n is called a **grading**. Usually we take $C_n = 0$ for $n < 0$. An element of $\ker d_n$ is called **closed** or a **cycle**. An element of $\operatorname{im} d_n$ is called a **boundary**. d is the **boundary map** or **differential**.

Definition 3.2 (Homology groups)**.** If (C, d) is a chain complex, its n**th homology group** is

$$H_n(C, d) = \ker d_n / \operatorname{im} d_{n+1}.$$

If $x \in \ker d_n$ we write $[x]$ for its image in $H_n(C)$.

Example 3.3.

1. $C_0 = C_1 = \mathbf{Z}$, $C_i = 0$ otherwise,

$$0 \to \mathbf{Z} \xrightarrow{\cdot 3} \mathbf{Z} \to 0.$$

Then $H_1 = 0$, $H_0 = \mathbf{Z}/3$.

2.

$$\mathbf{Z}$$

Notation

Symbol	Description	Page
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