Part III Algebraic Topology 2014

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Chapter 1

Homology

1.1 Introduction

These are lecture notes for the 2014 Part III Algebraic Topology course taught by Dr. Jacob Rasmussen.

The recommended books are:

- Algebraic Topology - Allen Hatcher,
- Homology Theory - James W. Vick,
- Differential Forms in Algebraic Topology - Raoul Bott and Loring W. Tu.

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1.2 Homotopy

1.2.1 Homotopies

Definition 1.2.1 (Homotopic maps)**.** Maps $f_0, f_1 \colon X \to Y$ are said to be **homotopic** if there is a continuous map $F \colon X \times I \to Y$ such that

$$F(x,0) = f_0(x) \text{ and } F(x,1) = f_1(x)\ \forall x \in X.$$

We let $\mathrm{Map}(X,Y) = \{f \colon X \to Y \text{ continuous}\}$. Then letting $f_t(x) = F(x,t)$ in the above definition we see that f_t is a path from f_0 to f_1 in $\mathrm{Map}(X,Y)$.

Example 1.2.2.

1. $X = Y = \mathbf{R}^n$, $f_0(\overline{x}) = \overline{0}$ and $f_1(\overline{x}) = \overline{x}$ are homotopic via $f_t(\overline{x}) = t\overline{x}$.
2. $S^1 = \{z \in \mathbf{C} : |z| = 1\}$ then
3. $S^n = \{\overline{x} \in \mathbf{R}^n : |\overline{x}| = 1\}$

Lemma 1.2.3. *Homotopy is an equivalence relation on* $\mathrm{Map}(X,Y)$.

Lemma 1.2.4. *If $f_0 \sim f_1 \colon X \to Y$ and $g_0 \sim g_1 \colon Y \to Z$ then $g_0 \circ f_0 \sim g_1 \circ f_1$.*

Corollary 1.2.5. *For any space X the set $[X, \mathbf{R}^n]$ has one element.*

Proof. Define $0_X \colon X \to \mathbf{R}^n$ by $0_X(x) = 0 \in \mathbf{R}^n$ for any $x \in X$. $\square$

Definition 1.2.6 (Contractible space)**.** X is **contractible** if 1_X is homotopic to a constant map.

Proposition 1.2.7. *Y is contractible $\iff [X, Y]$ has one element for any space X.*

Proof. ($\Rightarrow$) as in corollary. ($\Leftarrow$) $[X, Y]$ has one element so $1_Y \sim$ a constant map. $\square$

Given a space X how can we tell if X is contractible? If X is contractible then it must be path connected for one.

Proof. Contractible implies that $[S^0, X]$ has one element and so $f\colon S^0 \to X$ extends to D^1, and therefore X is path connected. $\square$

Similarly if $[S^1, X]$ has more than one element then X is not contractible.

Definition 1.2.8 (Simply connected)**.** We say X is **simply connected** if $[S^1, X]$ has only one element.

We say two space X and Y are *homotopy equivalent* if there exists $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f \sim 1_X$ and $f \circ g \sim 1_Y$.

Example 1.2.9. X is contractible if and only if $X \sim \{p\}$.

Proof. X contractible $\implies 1_X \sim c$, a constant map. Choose $f\colon X \to \{p\}$, $f(x) = p$ and $g\colon \{p\} \to X$, $g(p) = c$. Then $g \circ f = c \sim 1$ and $f \circ g = 1_{\{p\}}$. Converse: exercise. $\square$

Exercise 1.2.10.

Given X and Y how can we determine if $X \sim Y$? How do we determine $[X, Y]$? For example is $S^n \sim S^m$.

1.2.2 Homotopy groups

Definition 1.2.11 (Map of pairs)**.** A **map of pairs**$f\colon (X, A) \to (Y, B)$ is a map $f\colon X \to Y$ with sets $A \subset X$ and $B \subset Y$ such that $f(A) \subset B$.

If we have maps of pairs $f_0, f_1\colon (X, A) \to (Y, B)$ then we write $f_0 \sim f_1$ if there exists $F\colon (X \times I, A \times I) \to (Y, B)$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$.

Definition 1.2.12 (Homotopy groups)**.** If $* \in X$ then the n**th homotopy group** is

$$\pi_n(X, *) = [(D^n, S^{n-1}) \to (X, \{*\})].$$

We now note several properties of this definition:

1. $\pi_0(X, *) =$ set of path components of X.
2. $\pi_1(X, *)$ is a group.$\pi_n(X, *)$ is an abelian group.
3. π_n is a functor

$$\begin{Bmatrix}\text{pointed spaces}\\ \text{pointed maps}\end{Bmatrix} \to \begin{Bmatrix}\text{groups}\\ \text{group homomorphisms}\end{Bmatrix}.$$

So given

$$f\colon (X, p) \to (Y, q)$$

we get

$$f_*\colon \pi_n(X, p) \to \pi_n(y, q)$$

defined by

$$f_*(\gamma) = f \circ \gamma.$$

n	1	2	3	4	5	6	7
$\pi_n(S^2)$	0	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/12$	$\mathbf{Z}/15$

Example 1.2.13 (Homotopy groups of S^2)**.**

1.3 Homology

Our goal is to construct a functor H_* from the category of topological spaces and continuous maps to the category of $\mathbf{Z}$-modules and $\mathbf{Z}$-linear maps. This means to each space X we associate an abelian group $H_*(X) = \bigoplus_{n\geq 0} H_n(X)$, and to each map $f\colon X \to Y$ a function $f_*\colon H_n(X) \to H_n(Y)$ satisfying $(1_X)_* = 1_{H_n(X)}$ and $(f \circ g)_* = f_* \circ g_*$.

Some properties we would like to have for our construction are:

1. Homotopy invariance, if $f \sim g\colon X \to Y$ then $f_* = g_*$.
2. The dimension axiom, $H_n(X) = 0$ for any $n > \dim X$.

1.3.1 Chain complexes

Definition 1.3.1 (Chain complex)**.** If R is a commutative ring then a **chain complex** over R is a pair (C, d) satisfying:

1. $C = \bigoplus_{n\in\mathbf{Z}} C_n$ for R-modules C_n.
2. $d\colon C \to C$ where $d = \bigoplus d_n$ for R-linear maps d_n.
3. $d \circ d = 0$.

The indexing by n is called a **grading**. Usually we take $C_n = 0$ for $n < 0$. An element of $\ker d_n$ is called **closed** or a **cycle**. An element of $\operatorname{im} d_n$ is called a **boundary**. d is the **boundary map** or **differential**.

Definition 1.3.2 (Homology groups)**.** If (C, d) is a chain complex, its n**th homology group** is

$$H_n(C, d) = \ker d_n / \operatorname{im} d_{n+1}.$$

If $x \in \ker d_n$ we write $[x]$ for its image in $H_n(C)$.

Example 1.3.3.

1. $C_0 = C_1 = \mathbf{Z}$, $C_i = 0$ otherwise,
$$0 \to \mathbf{Z} \xrightarrow{\cdot 3} \mathbf{Z} \to 0.$$
Then $H_1 = 0$, $H_0 = \mathbf{Z}/3$.
2. $$\mathbf{Z} = \langle e\rangle \to \mathbf{Z}^2 = \langle f_1, f_2\rangle \to \mathbf{Z} = \langle g\rangle \to 0$$
with $d(e) = f_1 - f_2$, $d(f_1) = d(f_2) = g$, then $H_*(C) = 0$(exercise).

1.3.2 The chain complex of a simplex

Definition 1.3.4 (n-simplex)**.** The n**-dimensional simplex**Δ^n is

$$\Delta^n = \left\{ (x_0, \ldots, x_n) \in \mathbf{R}^n : \sum_i x_i = 1,\, x_i \geq 0 \forall i \right\}.$$

Δ^n has **vertices**$v_0, \ldots, v_n$ which are the intersections with the coordinate axes. The k-dimensional **faces** are in bijection with the $k+1$element subsets of $\{0, \ldots, n\}$.

Definition 1.3.5 (Simplicial chain complex)**.** $S_*(\Delta^n)$ is the chain complex with $S_k(\Delta^n)$ the free $\mathbf{Z}$-module generated by the k-dimensional faces of Δ^n. So

$$S_k(\Delta^n) = \langle e_I : I = \{i_0, \ldots, i_k : 0 \leq i_0 \leq \cdots \leq i_k \leq n\}\rangle.$$

To define d it suffices to define $d(e_I)$, we let

$$d(e_I) = \sum_{j=0}^{k} (-1)^j e_{i_0,\ldots,i_{j-1},i_{j+1},\ldots,i_k} \in S_{k-1}(\Delta^n).$$

Example 1.3.6. 1. For Δ^1 we have $d(e_{0,1}) = e_1 - e_0$.

2. For Δ^2 we have $d(e_{0,1,2}) = e_{12} - e_{02} + e_{01}$ and so $d^2(e_I = (e_2 - e_1) - (e_2 - e_0) + (e_1 - e_0) = 0$.

Lemma 1.3.7. $d^2 = 0$

Proof. It suffices to show $d^2(e_I) = 0$ for all I.

$$\begin{aligned}
d^2(e_I) &= d\left(\sum_{j=0}^{k}(-1)^j e_{i_0,\ldots,\hat{i}_j,\ldots,i_k}\right) \\
&= \sum_{j=0}^{k}(-1)^j d\left(e_{i_0,\ldots,\hat{i}_j,\ldots,i_k}\right) \\
&= \sum_{j=0}^{k}(-1)^j\left(\sum_{l<j}(-1)^l e_{i_0,\ldots,\hat{i}_l,\ldots,\hat{i}_j,\ldots,i_k} + \sum_{l>j}(-1)^{l-1} e_{i_0,\ldots,\hat{i}_j,\ldots,\hat{i}_l,\ldots,i_k}\right) \\
&= \sum_{j=0}^{k}\left(\sum_{l<j}(-1)^{l+j} e_{I-i_l-i_j} - \sum_{l>j}(-1)^{l+j} e_{I-i_j-i_l}\right) = 0.
\end{aligned}$$

□

Example 1.3.8 (Computing $H_*(S_*(\Delta^2))$)**.**

$$0 \to \mathbf{Z} = \langle e_{012}\rangle \to \mathbf{Z}^3 = \langle e_{01}, e_{02}, e_{12}\rangle \to \mathbf{Z}^3 = \langle e_0, e_1, e_2\rangle \to 0$$

with $d(e_{012}) = e_{12} - e_{02} + e_{01}$. So $\ker d_2 = 0 \implies H_2 = 0$.

$$d(ae_{01} + be_{02} + ce_{12}) = a(e_1 - e_0) + b(e_2 - e_1) + c(e_2 - e_0)$$

so $ae_{01} + be_{02} + ce_{12} \in \ker d_1 \iff -a - b = 0,\ a - c = 0,\ c + b = 0 \iff a = -b = c$ hence $\ker d_1 = \langle e_{01} - e_{02} + e_{12} = \operatorname{im} d_2$ and so $H_1 = 0$. $\ker(d_0) = 0$, $\operatorname{im} d_1 = \langle e_1 - e_0, e_2 - e_1, e_2 - e_0\rangle$ so $H_0 = \mathbf{Z} = \langle [e_0]\rangle$.

Exercise 1.3.9. Show that $H_*(S_*(\Delta^n)) = 0$ if $k \neq 0$ and $\mathbf{Z}$ if $k = 0$.

1.3.3 The singular chain complex

Definition 1.3.10 (Singular chain complex)**.** If X is a space, the **singular chain complex** of X, $C_*(X)$ is defined by

$$C_n(x) = \langle e_\sigma : \sigma\colon \Delta^n \to X \text{ any continuous map}\rangle.$$

Where

$$d(e_\sigma) = \sum_{j=0}^{n}(-1)^j e_{\sigma\circ F_j} \in C_{n-1}(X)$$

where $F_j\colon \Delta^{n-1} \to \Delta^n$ is given by $F_j(x_0, \ldots, x_{n_1} = (x_0, \ldots, 0, \ldots, x_{n-1})$ with the 0 in the jth place.

1.3.3.1 Homotopy invariance

Definition 1.3.11 (Chain homotopic maps)**.** Suppose $\phi, \psi\colon C_* \to C'_*$ are chain maps, we say that ϕ are **chain homotopic** if there exists an R-linear map $h\colon C_* \to C'_{*+1}$ such that $d' \circ h + h \circ d' = \phi - \psi$. We denote this relation by $\phi \sim \psi$.

Lemma 1.3.12. *If $\phi \sim \psi$ then $\phi_* = \psi_*$.*

Proof.

$$\begin{aligned}\phi_*([x]) - \psi_*([x]) &= [\phi(x) - \psi(x)] \\ &= [d'hx + hdx] = [d'hx] = 0 \in H_*(C').\end{aligned}$$

□

Theorem 1.3.13. *Suppose $f \sim g\colon X \to Y$ via H then $f_\# \sim g_\# \implies f_* = g_*$.*

Proof. □

Corollary 1.3.14. *If $X \sim Y$ then $H_*(X) \cong H_*(Y)$.*

Corollary 1.3.15. *If X is contractible then $H_*(X) \cong H_*(\{p\}) \cong \mathbf{Z}$ if $* = 0$, 0 otherwise.*

1.4 Homology of a pair

1.4.1 Exact sequences

Definition 1.4.1 (Exact sequence)**.** A sequence

$$\cdots \to A_{n+1} \to A_n \to A_{n-1} \to \cdots$$

of R-modules and R-linear maps is **exact at** A_n if $\ker f_n = \operatorname{im} f_{n+1}$.
A sequence is **exact** if it is exact at all A_n, then (A_*, f) is known as a **acyclic** chain complex (the homology is zero).

Example 1.4.2.

1. $0 \to A \xrightarrow{f} B$ is exact if and only if f is surjective.
2. $B \xrightarrow{g} C \to 0$ is exact if and only if g is injective.
3. $0 \to A \xrightarrow{f} A' \to 0$ is exact if and only if f is an isomorphism.
4. $0 \to A \to B \to C \to 0$ is exact if and only if $A \subset B$ and $C \cong B/A$.

Definition 1.4.3 (Short exact sequence)**.** A sequence

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{\pi} C_* \to 0$$

is a **short exact sequence** of chain complexes if

1. A_*, B_*, C_* are chain complexes.
2. i, π are chain maps.
3. $$0 \to A_n \xrightarrow{i} B_n \xrightarrow{\pi} C_n \to 0$$
   is exact for all n.

Lemma 1.4.4 (Snake lemma)**.** *If*

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{\pi} C_* \to 0$$

*is a short exact sequence of chain complexes then there is an associated **long exact sequence** of homology groups*

$$
\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] & H_n(B) \arrow[r] & H_n(C) \arrow[dll, rounded corners, "\partial_n"] \\
& H_{n-1}(A) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(C) \arrow[r] & \dots
\end{tikzcd}
$$

*The map ∂ is called the **boundary map** in the exact sequence.*

Proof. We first define ∂, given $[c] \in H_n(C)$ pick $b \in B_n$ such that $\pi(b) = c$. Now $\pi db = d\pi b = dc = 0$ and so $db = ia$ for some $a \in A_{n-1}$. We have that $ida = dia = ddb = 0$ and so $da = 0$ and we can define $\partial[c] = [a] \in H_{n-1}(A)$.
We must check that the definition of ∂ does not depend on the choice of b or c and also check exactness at each term. Here we prove exactness at $H_n(C)$. With notation as above suppose $[c] = \pi_*([x])$ then we can take $b = x$, $db = 0$ implying that $a = 0$ and so $\partial[c] = [a] = 0$. We have then that $\operatorname{im}\pi_* \subset \ker\partial$, so it remains to show that $\ker\partial \subset \operatorname{im}\pi_*$. If $[a] = da'$, $a' \in A_n$ we let $b' = b - i(a')$ then $d(b') = db - d(ia') = db - i(da') = db - db = 0$. Then $[b'] \in H_n(B)$ so $\pi_*([b']) = [\pi(b')] = [\pi(b) - \pi(i(a'))] = [\pi(b)] = [c]$. $\square$

Exercise 1.4.6. Suppose we have a map of short exact sequences commuting, then show there is a map of long exact sequences of homology commuting .

Example 1.4.7 (Reduced homology)**.** If X is a space let

$$\tilde{C}_*(X) = \begin{cases} C_*(X), & * \neq -1, \\ \mathbf{Z}, & * = -1, \end{cases}$$

then define $d(e_p) = e$ for $p \in X$. We have $d^2 e_\gamma = d(e_{\gamma(1)} - e_{\gamma(0)}) = e - e = 0$. So $\tilde{C}_*$ is a chain complex.
This construction can be motivated by thinking of $\Delta^{-1} = \{\}$ and then considering $\operatorname{Map}(\{\}, X) = \{e\}$. If

$$A_* = \begin{cases} \mathbf{Z} = \langle e \rangle, & * = -1, \\ 0, & * = -1. \end{cases}$$

Then we have a short exact sequence

$$0 \to A_* \to \tilde{C}_*(X) \to C_*(X) \to 0.$$

We have $H_*(A) = A_*$ and so the long exact sequence of homology then says that for $n \geq 0$

$$\cdots \to H_n(A) = 0 \to H_n(\tilde{C}(X)) \to H_n(C(X)) \to H_{n-1}(A) = 0 \to \cdots$$

giving that $H_n(\tilde{C}_*(X)) \cong H_n(C_*(X))$ for $n > 0$. We write $\tilde{H}_*(X)$ for $H_*(\tilde{C}_*(X))$. At $n = 0$ we get

$$\cdots \to 0 \to \tilde{H}_0(X) \to H_0(X) \to \mathbf{Z} \to 0 \to \cdots$$

giving that $\tilde{H}_0(X)$ has one fewer copies of $\mathbf{Z}$ than $H_0(X)$.

Example 1.4.8 (Homology of a pair)**.** Suppose $A \subset X$, then $C_*(A) \subset C_*(X)$ is a subcomplex. Define $C_*(X, A) = C_*(X)/C_*(A)$ and so we have a short exact sequence

$$0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0.$$

1.4.2 Homology of a pair

Definition 1.4.9 (Homology of a pair)**.** $H_*(X, A) = H_*(C_*(X, A))$ is the **homology of the pair** (X, A).

From this we obtain:

Definition 1.4.10 (Long exact sequence of a pair)**.** The **long exact sequence of the pair** (X, A) is

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots .$$

If $f\colon (X, A) \to (Y, B)$ is a map of pairs then we get an **induced map** $f_\#\colon C_*(X) \to C_*(Y)$ defined by

$$(\sigma\colon \Delta^n \to A) \mapsto f \circ \sigma.$$

Observe that $f_\#(C(A)) \subset C_*(B)$ and so $f_\#$ descends to a map

$$f_\#\colon C_*(X)/C_*(A) \to C_*(Y)/C_*(B)$$

or equivalently $f_\#\colon C_*(X, A) \to C_*(Y, B)$ this then induces $f_*\colon H_*(X, A) \to H_*(Y, B)$.

Proposition 1.4.11 (Homotopy invariance)**.** *If $f, g\colon (X, A) \to (Y, B)$ are homotopic as maps of pairs then $f_* = g_*\colon H_*(X, A) \to H_*(Y, B)$.*

Proof. Let $H\colon (X \times I, A \times I) \to (Y, B)$ be the homotopy, H induces a chain homotopy $h\colon C_*(X) \to C_{*+1}(Y)$ where $dh + hd = f_\# - g_\#$. $H(A \times I) \subset B$ so $h(C_*(A)) \subset C_{*+1}(B)$ this implies that h descends to a map

$$h\colon C_*(X)/C_*(A) \to C_{*+1}(Y)/C_{*+1}(B)$$

with $hd+dh = f_\# - g_\#$ as any relation satisfied will remain satisfied in the quotient. So we have $h\colon C_*(X, A) \to C_{*+1}(Y, A)$ and hence $f_\#, g_\#\colon C_*(X, A) \to C_*(Y, B)$ are chain homotopic. □

1.4.2.1 Visualising relative homology classes

If W^{n+1} is a connected oriented compact manifold then we'll show that $H_{n+1}(W, \partial W) \cong \mathbf{Z} = \langle [W, \partial] \rangle$ (where the ∂ notation means relative to boundary). So given $f\colon (W, \partial) \to (X, A)$ we get $f_*([W, \partial W]) \in H_{n+1}(X, A)$.

Example 1.4.12. Let $X = \mathbf{R}^3$ and $A = S^1$ then $W = D^2$ defines a class in $H_2(\mathbf{R}^3, S^1)$ (the boundary of W lies inside of A).

1.4.3 Good pairs

Definition 1.4.13 (Good pair)**.** (X, A) is a **good pair** if

1. $\exists U \subset X$ open and $A \subset U$
2. $\exists \pi\colon U \to A$ with $\pi|_A = \mathrm{id}_A$.
3. $\pi \sim 1_{(U,A)}$ as maps of pairs from (U, A) to itself.

(i.e. A is a deformation retract of U).

Example 1.4.14 (Good pairs)**.** 1. (smooth manifold, closed submanifold).

2. (simplicial complex, subcomplex).

Example 1.4.15 (Not good pairs)**.** 1. $(\mathbf{R}, \mathbf{Q})$.

2. Letting $H \subset \mathbf{R}^2$ be the Hawaiian earring then $(\mathbf{R}^2, H)$ is not a good pair.

Theorem 1.4.16. *Take $A \subset X$ and let π be the natural map $\pi\colon (X, A) \to (X/A, A/A) \cong (X/A, *)$. Then if (X, A) is a good pair the induced map $\pi_*\colon H_*(X, A) \to H_*(X/A, *)$ is an isomorphism.*

Proof. Postponed. □

Exercise 1.4.17. The composite map ϕ in

$$\tilde{H}_*(X) \to H_*(X) \to H_*(X, *)$$

is an isomorphism. i.e.

$$H_*(X, *) = \begin{cases} H_*(X), & * > 0, \\ H_0(X)/\mathbf{Z}, & * \le 0. \end{cases}$$

Proposition 1.4.18.

$$\tilde{H}_*(S^n) = \begin{cases} \mathbf{Z}, & * = n, \\ 0, & * \ne n. \end{cases}$$

Proof. We proceed by induction on n. For $n = 0$ we have $S^0 = \{p_+, p_-\}$ implying

$$H_*(S^0) = H_*(p_+) \oplus H_*(p_-) = \begin{cases} \mathbf{Z}^2, & * = 0, \\ 0, & * \ne 0, \end{cases}$$

giving

$$\tilde{H}_*(S^0) = \begin{cases} \mathbf{Z}, & * = 0, \\ 0, & * \ne 0. \end{cases}$$

Now consider the long exact sequence of the pair (D^n, S^{n-1})

$$H_*(D^n) \to H_*(D^n, S^{n-1}) \to H_{*-1}(S^{n-1}) \xrightarrow{\phi} H_{*-1}(D^n),$$

we can break this up using the kernel and cokernel to get

$$0 \to \operatorname{coker} \phi_m \to H_m(D^n, S^{n-1}) \to \ker \phi_{m-1} \to 0. \tag{1.4.1}$$

D^n is contractible and so we get

$$H_*(D^n) = \begin{cases} \mathbf{Z}, & * = 0, \\ 0, & * \ne 0. \end{cases}$$

We have $\phi\colon H_0(S^{n-1}) \to H_0(D^n) \cong \mathbf{Z} = \langle e_p \rangle$ given by $\phi(e_p) = e_p$. We see that $\ker \phi = \tilde{H}_*(S^{n-1})$ and $\operatorname{coker} \phi = 0$. Now by looking at (1.4.1) we see that

$$0 \to \tilde{H}_*(S^n) = H_*(S^n, *) = H_*(D^n, S^{n-1}) \to \ker \phi = \tilde{H}_*(S^{n-1}) \to 0$$

is exact giving $\tilde{H}_*(S^n) \cong \tilde{H}_*(S^{n-1})$. The claim then follows by induction. □

Corollary 1.4.19.

$$S^n \sim S^m \implies m = n.$$

Example 1.4.20 (Chains generating $H_n(S^n, *)$)**.** 1. Choose $f\colon \Delta^n \to \Delta^n$ a homeomorphism and let $e\colon \Delta^n \to \Delta^n$ be the identity map. Then $a_{n-1} = f_\#(de) \in C_*(S^{n-1})$ and we have $da_{n-1} = df_\#(de) = f_\#(d^2 e) = 0$ and so a_{n-1} is closed.

2. Choose $g\colon (D^n/S^{n-1}, S^{n-1}/S^{n-1}) \to (S^n, *)$. Now let $b_n = g_\#(f_\#(e)) \in C_n(S^n, *)$. Then $db_n = g_\#(a_{n-1}) \in C_*(*)$ implies b_n is closed in $C_*(S^n, *)$. For example if $n = 2$ we are crushing the boundary of the 2-simplex to a point.

Proposition 1.4.21. $[a_n]$ *generates* $\tilde{H}_n(S^n) = \mathbf{Z}$ *(statement* (A_n)*).*
$[b_n]$ *generates* $\tilde{H}_n(S^n, *) = \mathbf{Z}$ *(statement* (B_n)*).*

Proof. Statement (A_0): $S^0 = \{p_+, p_-\}$, $de = e_{p_+} - e_{p_-}$ so de generates $\tilde{H}_0(S^0)$. We'll show that $A_{n-1} \implies B_n$. Here $\partial\colon H_n(S^n, *) = H_n(D^n, S^{n-1}) \xrightarrow{\sim} \tilde{H}_{n-1}(S^{n-1})$ (by 1.4.18). So it suffices to check that $\partial[b_n] = [a_{n-1}]$ □

Definition 1.4.22 (Wedge product)**.** If (X_i, p_i) are pointed spaces the **wedge product**

$$\bigvee_{i\in I}(X_i, p_i) \text{ is } \coprod_{i\in I} X_i \Big/ \{p_i : i \in I\}.$$

If X_i is such that for any $p, q \in X_i$ there exists a homeomorphism $f\colon X_i \to X_i$ with $f(p) = q$ we can drop p_i from the notation (for example in the case of X_i a connected manifold we can do this).

Example 1.4.23.

Corollary 1.4.24. *If* (X_i, p_i) *are good pairs then*

$$\tilde{H}_*\left(\bigvee_i (X_i, p_i)\right) = \bigoplus_i \tilde{H}_*(X_i).$$

Proof.

$$\begin{aligned}\tilde{H}_*\left(\bigvee_{i\in I} X_i\right) &\cong H_*\left(\bigvee_{i\in I} X_i, p\right) \cong H_*\left(\coprod_{i\in I} X_i, \{p_i : i \in I\}\right)\\ &\cong \bigoplus_{i\in I} H_*\left(X_i, p_i\right) \cong \bigoplus_{i\in I} \tilde{H}_*(X_i).\end{aligned}$$

□

Example 1.4.25.

$$H_*(S^1 \vee S^2) = \begin{cases}\mathbf{Z}, & * = 0, 1, 2\\ 0, & \text{otherwise.}\end{cases}$$

1.5 Subdivision and Excision

Definition 1.5.1. If $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of X, let

$$C_n^{\mathcal{U}}(X) = \langle e_\sigma : \sigma\colon \Delta^n \to X, \operatorname{im}\sigma \subset U_i \text{ for some } i\rangle \subset C_n(X).$$

Observe that $\operatorname{im}\sigma \subset U_i$ implies $\operatorname{im}\sigma \circ F_j \subset U_i$ and so $C_*^{\mathcal{U}}$ is a subcomplex of C_*. Let $H_*^{\mathcal{U}}$ be the homology of this complex, then we have a map

$$i\colon C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X).$$

Lemma 1.5.2 (Subdivision)**.** $C_*\colon H_*^{\mathcal{U}}(X) \to H_*(X)$ *is an isomorphism.*

1.6 Degree and Orientations

1.7 Cell Complexes

Definition 1.7.1 (Attaching of cells)**.** If $f\colon S^{n-1} \to X$ then

$$X \cup_f D^n = X \amalg D^n / \sim$$

is the space obtained by **attaching** an n-dimensional cell to X via the map f.

Example 1.7.2. If $X = \{p\}$ and $f\colon S^{n-1} \to X$ then

$$X \cup_f D^n \cong D^n / S^{n-1} \cong S^n.$$

Definition 1.7.3 (Finite cell complex)**.** A 0-dimensional **finite cell complex** is a finite disjoint union of points.
A k-dimensional finite cell complex is a space obtained by attaching finitely many k-cells to a $(k-1)$-dimensional finite cell complex.

Example 1.7.4.

Example 1.7.5. If a finite cell complex X has one 0-cell and one n-cell then $X \cong S^n$. Similarly if X has one 0-cell and kn-cells then $X \cong \bigvee_{i=1}^k S^n$.

Example 1.7.6. T^2 is a finite cell complex with one 0-cell, two 1-cells and one 2-cell.

Example 1.7.7. Any simplicial complex is a finite cell complex and any closed manifold can be given the structure of a finite cell complex.

Chapter 2

Cohomology and Products

2.1 Homology with Coefficients and Cohomology

2.1.1 Hom and $\otimes$ for modules

Definition 2.1.1 (Tensor product of R-modules)**.** Let M, N be R-modules. Then the **tensor product** $M \otimes N$ is the R-modules generated by all pairs $m \otimes n$ for $m \in M$, $n \in N$ modulo the relations:

1. $(m_1 + m_2) \otimes (n_1 + n_2) = \sum m_i \otimes n_j$.
2. $r(m \otimes n) = (rm) \otimes n = m \otimes (rn)$.

We have the following properties of this definition:

1. $(M_1 \oplus M_2) \otimes (N_1 \oplus N_2) = \bigoplus M_i \otimes N_j$.
2. $M \otimes N \cong N \otimes M$.
3. $M \otimes R = M$

Example 2.1.2 (Tensor products)**.** 1. $R^n \otimes R^m \cong R^{mn}$.

2. Letting $R = \mathbf{Z}$, $\mathbf{Q} \otimes \mathbf{Z}/a \cong 0$.
3. $\mathbf{Z}/a \otimes \mathbf{Z}/b \cong \mathbf{Z}/(a, b)$.

If $f\colon M_1 \to M_2$ and $g\colon N_1 \to N_2$ then there is a map

$$\begin{aligned} f \otimes g\colon M_1 \otimes N_1 &\to M_2 \otimes N_2, \\ m \otimes n &\mapsto f(m) \otimes g(n). \end{aligned}$$

Example 2.1.3. If $f\colon R^n \to R^m$ is given by multiplication by $A \in \mathrm{Mat}_{n\times m}(R)$ then $f \otimes 1_M\colon M^n \to M^m$ is given by multiplication by A.

Definition 2.1.4 (Hom)**.**

$$\mathrm{Hom}(M, N) = \{f\colon M \to N : f \text{ is } R\text{-linear}\}$$

is an R-module, via $(f + rg)(m) = f(m) + rg(m)$.

From this definition we see that

1. $\mathrm{Hom}(\bigoplus M_i, \bigoplus N_j) \cong \bigoplus_{i,j} \mathrm{Hom}(M_i, N_j)$.
2. $\mathrm{Hom}(R, M) \cong M$.

Note however that we do not have $\mathrm{Hom}(M, N) = \mathrm{Hom}(N, M)$ as for example $\mathrm{Hom}(\mathbf{Z}/2, \mathbf{Z}) = 0$ but $\mathrm{Hom}(\mathbf{Z}, \mathbf{Z}/2) = \mathbf{Z}/2$.

Definition 2.1.5 (Dual module)**.** Given an R-module M the **dual** of M is $M^* = \mathrm{Hom}(M, R)$.

Now if we have $f\colon M \to N$ we get a map

$$f^*\colon \mathrm{Hom}(N, O) \to \mathrm{Hom}(M, O)$$

given by $f^* g = g \circ f$.

Example 2.1.6. If $f\colon R^n \to R^m$ is multiplication by A then

$$f^*\colon \mathrm{Hom}(R^m, O) \cong O^m \to \mathrm{Hom}(R^n, O) \cong O^n$$

is multiplication by $A^\top$.

2.1.2 Hom and $\otimes$ for chain complexes

If (C, d) is a chain complex defined over R then so are $(C_* \otimes M, d \otimes 1_M) = C_* \otimes M$ and $(\mathrm{Hom}(C_*, M), d^*) = \mathrm{Hom}(C_*, M)$.

2.2 Universal Coefficient Theorems

2.3 Products

Definition 2.3.1 (Product of chain complexes)**.** If A, B are chain complexes the $C = A \otimes B$ is the chain complex with

$$C_i = \bigoplus_{j+k=i} A_j \otimes B_k$$

and

$$d(a \otimes b) = (da) \otimes b + (-1)^{|a|} a \otimes (db)$$

where $|a| = j$ if $a \in A_j$.

The following theorem is true but we will not prove it.

Theorem 2.3.2 (Eilenberg-Zilber)**.**

$$C_*(X \times Y) \sim C_*(X) \otimes C_*(Y).$$

Given $H_*(A)$ and $H_*(B)$ how can we compute $H_*(A \otimes B)$.

Lemma 2.3.3. *Suppose $A \sim A'$. Then $A \otimes B \sim A' \otimes B$.*

Proof. Easy. ∎

Corollary 2.3.4. *Suppose A, B are finitely generated chain complexes, defined over a field. Then*

$$H_*(A \otimes B) = \bigoplus_{j+k=i} H_j(A) \otimes H_k(B)$$

i.e.

$$H_*(A \otimes B) \cong H_*(A) \otimes H_*(B).$$

Proof. A is a chain complex over a field, so $A \sim A'$ where $A'_i = H_i(A)$ and $d \equiv 0$ on A'. Similarly for B. Then

$$H_*(A \otimes B) \cong H_*(A' \otimes B') \cong A' \otimes B' = H_*(A) \otimes H_*(B).$$

□

Corollary 2.3.5. *If k is a field and X, Y are finite chain complexes then*

$$H_*(X \times Y; k) \cong H_*(X; k) \otimes_k H_*(Y; k).$$

Definition 2.3.6 (Poincare polynomial)**.** The **Poincare polynomial** of X with coefficients in k is

$$P_k(X) = \sum_{i \geq 0} \dim H_i(X; k) t^i.$$

Example 2.3.7.

$$P_{\mathbf{Z}/2}(\mathbf{RP}^2) = 1 + t + t^2.$$

2.3.5 tells us that

$$P_k(X \times Y) = P_k(X) P_k(Y).$$

Example 2.3.8.

$$P_{\mathbf{Z}/2}(\mathbf{RP}^3 \times \mathbf{RP}^2) = (1 + t + t^2)(1 + t + t^2 + t^3).$$

For now we will suppose that A, B are finitely generated chain complexes over a PID R.

Lemma 2.3.9. *If $H_*(A)$ is a free R-module then $A \sim A'$ where $A'_i = H_i(A)$ and $d \equiv 0$ on A'.*

Proof. A is the direct sum of short and stupid complexes. $H_*(A)$ being torsion free implies that every short complex is of the form

$$C\colon R \underset{\cdot a^{-1}}{\overset{\cdot a}{\rightleftarrows}} R$$

with $a \in R^\times$ such a $C \sim 0$ (same proof as for a field). □

Corollary 2.3.10. *If $H_*(A)$ is free then $H_*(A \otimes B) \cong H_*(A) \otimes H_*(B)$.*

Proof. $A \otimes B \cong A' \otimes B$ where A' is as in 2.3.9. $A' \otimes B$ is a direct sum of copies of B, one for each generator of A'. This gives that $H_*(A' \otimes B)$ is a direct sum of copies of $H_*(B)$, one for each generator of $H_*(A)$ which implies that

$$H_*(A \otimes B) \cong H_*(A) \otimes H_*(B)$$

(since $H_*(A)$ is free).

$$R^n \otimes H_*(B) \cong (H_*(B))^n.$$

□

Corollary 2.3.11. *If X and Y are finite cell complexes and $H_*(X)$ is free over $\mathbf{Z}$, then $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$.*

Fact: If C_1 is a free resolution of M and C_2 is a free resolution of N. Then

$$\mathrm{Tor}_i(M, N) \cong H_i(C_1 \otimes C_2) \cong H_i(C_1 \otimes N) \cong H_i(M \otimes C_2)$$

.

Theorem 2.3.12 (Kunneth formula)**.** *If A, B are free (finitely generated) chain complexes over a PID R then*

$$H_i(A \otimes B) \cong \bigoplus_{j+k=i} \mathrm{Tor}_0(H_j(A), H_k(B)) \oplus \bigoplus_{j+k=i-1} \mathrm{Tor}_1(H_j(A), H_k(B)).$$

Proof. It suffices to check the statement where A, B are short and/or stupid. We just did the case where both are short. The case where one or more is stupid was covered in the statement that

$$H_*(A \otimes B) \cong H_*(A) \otimes H_*(B)$$

if $H_*(A)$ is free. □

Corollary 2.3.13. *We can compute $H_*(X \times Y)$ from $H_*(X)$ and $H_*(Y)$.*

2.4 Cup product

Motivation: if k is a field

$$\begin{aligned} H^*(X \times X; k) &\cong (H_*(X \times X; k))^* \\ &\cong (H_*(X; k) \otimes H_*(X; k))^* \\ &\cong H^*(X; k) \otimes H^*(X; k) \end{aligned}$$

Definition 2.4.1 (Cup product)**.** Suppose $a \in C^i(X)$ and $b \in C^j(X)$. If $\sigma \colon \Delta^{i+j} \to X$ then we define $a \smile b \in C^{i+j}(X)$ by

$$(a \smile b)(\sigma) = a(\sigma \circ F_i^+) b(\sigma \circ F_j^-)$$

where

$$\begin{aligned} F_i^+ \colon \Delta^i &\to \Delta^{i+j} \\ (x_0, \ldots, x_i) &\mapsto (x_0, \ldots, x_i, 0, \ldots, 0) \end{aligned}$$

and

$$\begin{aligned} F_j^- \colon \Delta^j &\to \Delta^{i+j} \\ (x_0, \ldots, x_j) &\mapsto (0, \ldots, 0, x_0, \ldots, x_j). \end{aligned}$$

Lemma 2.4.2.

$$d(a \smile b) = da \smile b + (-1)^{|a|} a \smile db.$$

Proof. Denote by $[v_0, \ldots, v_k]$ the singular k-simplex that sends

$$\Delta^k \to \Delta^{i+j} \xrightarrow{\sigma} X$$

$$\text{vertices of } \Delta^k \to \text{vertices } v_0, \ldots, v_k \text{ of } \Delta^{i+j}.$$

Then taking $\sigma\colon \Delta^{i+j+1} \to X$

$$\begin{aligned}
d(a \smile b)(\sigma) &= (a \smile b)(d\sigma) \\
&= \sum_{k=0}^{i+j+1} (-1)^k a \smile b([v_0, \ldots, \hat{v}_k, \ldots, v_n]) \\
&= \sum_{k \leq i} (-1)^k a([v_0, \ldots, \hat{v}_k, \ldots, v_{i+1}]) b([v_{i+1}, \ldots, v_n]) \\
&+ \sum_{k > i} (-1)^k a([v_0, \ldots, v_i]) b([v_i, \ldots, \hat{v}_k, \ldots, v_n]) \\
&= \sum_{k \leq i+1} (-1)^k a([v_0, \ldots, \hat{v}_k, \ldots, v_{i+1}]) b([v_{i+1}, \ldots, v_n]) \\
&+ \sum_{k \geq i} (-1)^k a([v_0, \ldots, v_i]) b([v_i, \ldots, \hat{v}_k, \ldots, v_n]) \\
&= a(d([v_0, \ldots, v_{i+1}])) b([v_{i+1}, \ldots, v_n]) \\
&+ (-1)^i a([v_0, \ldots, v_i]) b(d([v_i, \ldots, v_n])) \\
&= da \smile b([v_0, \ldots, v_n]) + (-1)^i a \smile db([v_0, \ldots, v_n]).
\end{aligned}$$

□

Corollary 2.4.3. *There is a well-defined cup product*

$$\begin{gathered}
H^i(X) \times H^j(X) \to H^{i+j}(X), \\
[a] \times [b] \mapsto [a \smile b].
\end{gathered}$$

Proof. Must check

1. $da = c$, $db = 0 \implies d(a \smile b) = da \smile b \pm a \smile db = 0$ so ok.

2. $(a + dc) \smile b = a \smile b + dc \smile b = a \smile b + d(c \smile b)$ implies $[(a + dc) \smile b] = [a \smile b]$ and so $\smile$ is well defined.

□

Proposition 2.4.4 (Properties of $\smile$)**.** *Let $\alpha, \beta, \gamma \in H^*(X)$ then*

1. *(Associative) $(\alpha \smile \beta) \smile \gamma = \alpha \smile (\beta \smile \gamma)$.*

2. *(Graded commutative) $a \smile b = (-1)^{|\alpha||\beta|} \beta \smile \alpha$.*

3. *(Identity) There is a class $1 \in H^0(X)$ such that $1 \smile \alpha = \alpha \smile 1 = \alpha$ for all $\alpha \in H^*(X)$.*

4. *(Functorial) If $f\colon X \to Y$, $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$.*

Proof.

1. If $a \in C^i(X)$, $b \in C^j(X)$, $c \in C^k(X)$, $\sigma\colon \Delta^{i+j+k} \to X$ then

$$\begin{aligned}
((a \smile b) \smile c)(\sigma) &= a(\sigma \circ F_i^+) b(\sigma \circ F_j^{\mathrm{mid}}) c(\sigma \circ F_k^-) \\
&= (a \smile (b \smile c))(\sigma).
\end{aligned}$$

   Where $F_i^+(X) = (X^i\ 0)$, $F_k^-(X) = (0^{i+j}\ X^{i+j+k})$, $F_j^{\mathrm{mid}}(X) = (0^i\ X^{i+j}\ 0)$.

2. We'll do later.

3. Let $e \in C^0(X)$ be given by

$$e(e_p) = 1 \text{ for all } p \in X$$

If $\gamma\colon [0,1] \to X$ is a 1-chain then

$$de(e_\gamma) = e(de_\gamma) = e(e_{\gamma(1)} - e_{\gamma(0)}) = 1 - 1 = 0.$$

Easy to see $e \smile a = a \smile e = a$ for all $a \in C^*(X)$ then $1 = [e] \in H^0(X)$.

4.

$$\begin{aligned} f^\#(a \smile b)(\sigma) &= (a \smile b)(f \circ \sigma) \\ &= a(f \circ \sigma \circ F_i^+)b(f \circ \sigma \circ F_j^-) \\ &= f^\#(a)(\sigma \circ F_i^+)f^\#(b)(\sigma \circ F_j^-) \\ &= [f^\#(a) \smile f^\#(b)](\sigma). \end{aligned}$$

□

2.4.1 Digression: De Rahm Cohomology

Let M be a smooth manifold and $\omega \in \Omega^k(M)$. If $\sigma\colon \Delta^k \to M$ is a smooth we define $\omega(\sigma) = \int_\sigma \omega$. Stokes' theorem then gives us that $d\omega(\sigma) = \omega(d\sigma)$. Where on the left $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ is the differential map and on the right d is the boundary map in the sense of (smooth) singualr chains. I.e. the two d's are dual to each other on $C_*^{\text{smooth}}(M)$.

Theorem 2.4.5 (de Rahm)**.**

$$H_k(\Omega^*(M), d) \cong H^k(M; \mathbf{R}).$$

Under this correspondence

$$[\omega] \smile [\nu] = [\omega \wedge \nu].$$

Example 2.4.6. 1. $H^*(S^n) = \langle 1, X_n \rangle$ where $\langle X_n \rangle = H^n(S^n)$ and we have

$$\begin{aligned} 1 \smile X_n &= X_n, \\ X_n \smile 1 &= X_n, \\ 1 \smile 1 &= 1, \\ X_n \smile X_n &= 0 \in H^{2n}(S^n) = 0. \end{aligned}$$

2. For arbitrary X, Y

$$C_*(X \amalg Y) \cong C_*(X) \oplus C_*(Y)$$

which implies

$$\begin{aligned} &C^*(X \amalg Y) \cong C^*(X) \oplus C_*(Y) \\ \implies\ &H^*(X \amalg Y) \cong H^*(X) \oplus H_*(Y) \text{ as rings} \end{aligned}$$

with

$$(\alpha_1 \oplus \alpha_2) \smile (\beta_1 \oplus \beta 2) = (\alpha_1 \smile \beta_1) \oplus (\alpha_2 \smile \beta_2).$$

3. Considering $X \vee Y$ we have a map $p\colon X \amalg Y \to X \vee Y$ and so Mayer-Vietoris for $X \vee Y$ gives

$$\cdots \to H^*(X \vee Y) \xrightarrow{p^*} H^*(X) \oplus H^*(Y) \xrightarrow{i^*} H^*(\text{pt}) \to \cdots$$

Now assuming X and Y are path connected p^* is injective and an isomorphism in grading > 0. In grading 0 we have

$$H^0(X \vee Y) \to H^0(X) \oplus H^0(Y) \to H^0(\mathrm{pt})$$

with $1 \mapsto 1 \oplus 1$ and $a \oplus b \mapsto a - b$. And so as a ring we have

$$H^*(X \vee Y) \cong \ker i^* \subset H^*(X) \oplus H^*(Y)$$

where elements in grading > 0 look like (α, β) and in grading 0 look like $(n, n) \in \mathbf{Z} \times \mathbf{Z}$.

4. $S^2 \vee S^4$, $H^*(S^2 \vee S^4) = \langle 1, X_2, X_4 \rangle$ with $X_2^2 = 0$ since $X_2^2 = 0$ in $H^*(S^2)$.

2.4.2 Exterior Product

Let $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$.

Definition 2.4.7 (Exterior product)**.** If $\alpha \in H^i(X)$, $\beta \in H^j(X)$ then their **exterior product** is $\alpha \times \beta = \pi_1^*(\alpha) \smile \pi_2^*(\beta) \in H^{i+j}(X \times Y)$.

Proposition 2.4.8. *1. $\alpha \smile \beta = \Delta^*(\alpha \times \beta)$ where $\Delta \colon X \to X \times X$ is the diagonal map.*

2. $(\alpha \times \beta) \smile (\alpha' \times \beta') = (-1)^{|\beta||\alpha'|}(\alpha \smile \alpha') \times (\beta \smile \beta')$.

Proof. 1.

$$\Delta^*(\alpha \times \beta) = \Delta^*(\pi_1^*(\alpha) \smile \pi_2^*(\beta))$$

□

2.5 Notation

Symbol	Description	Page
		1