MA4H9 Modular Forms - Lecture Notes

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1 Introduction

These are lecture notes for the 2013 Modular forms course, taught at the University of Warwick by Peter Bruin.

The recommended books are:

- F. Diamond & J. Sherman - A first course in Modular Forms
- J. S. Milne - Modular functions and Modular forms

- T. Miyake - Modular Forms
- J-P. Serre - Cours d'Arithmetique

2 Motivation

Lecture 1

The big picture:

2.1 Historical prologue: Sums of squares

Famous question: given integers $n \geq 0,\ k \geq 0$, in how many ways can n be written as a sum of k squares (of integers).

In other words, what is

$$r_k(n) = \#\{(x_1, \dots, x_n) \in \mathbb{Z}^k \mid x_1^2 + \cdots + x_k^2 = n\}?$$

Note that sign changes and permutations of $(x_1, \dots, x_k)$ count separately.

Example. $k = 5,\ n = 2$

$5 = 1^2 + 2^2 = (-1)^2 + 2^2 = 1^2 + (-2)^2 = (-1)^2 + (-2)^2 = 2^2 + 1^2 = \cdots$

So $r_2(5) = 8$.

We can reinterpret this geometrically (using Pythagoras) as the statement that there are 8 points in $\mathbb{Z}^2$ of distance $\sqrt{5}$ from the origin.

Trivial cases: $k = 0$, $k = 1$.

$$r_0(n) = \begin{cases} 1, & n = 0 \\ 0, & \text{otw} \end{cases} \quad , \ r_1(n) = \begin{cases} 1, & n = 0 \\ 2, & n \text{ is a square} \\ 0, & \text{otw} \end{cases}$$

The case: $k = 2$.

Diophantus in the 3^{rd} century showed that if $m,\ n$ are numbers each of which is the sum of two squares then mn is also a sum of two squares. In our language this says that if $r_2(m) > 0$ and $r_2(n) > 0$ then $r_2(mn) > 0$.

Exercise. Prove this!

Fermat (1637): If n is a positive odd integer then n is a sum of two squares $\iff$ every prime number $p \mid n$ with $p \equiv 3 \pmod 4$ occurs an even number of times in the factorisation of n.

In particular a prime p is a sum of two squares $\iff$ $p = 2$ or $p \equiv 1 \pmod 4$.

2.1.1 Formulas for $r_k(n)$

Jacobi (1829): $r_2(n) = 4 \sum_{d \mid n} \chi(d)$, for $n > 0$. Where

$$\chi(d) = \begin{cases} 0, & \text{if } 2 \mid d \\ 1, & \text{if } d \equiv 1 \pmod 4 \\ -1, & \text{if } d \equiv 3 \pmod 4 \end{cases}$$

Gauss (1801) found a formula for $r_3(n)$, but it is much more complicated.

Jacobi (1829): $r_4(n) = 8\sum_{d|n,\ 4\nmid d} d$ for all $n > 0$. In particular the positivity of this sum implies Lagrange's Theorem.

Jacobi, F. G. Eisenstein, H. J. Smith found that

$$r_6(n) = \sum_{d|n} (16\chi\left(\frac{n}{d}\right) - 4\chi(d))d^2 \text{ and } r_8(n) = \sum_{d|n} (-1)^{n-d} d^3.$$

J. Liouville (1864/65) found

$$r_{10}(n) = \frac{4}{5}\sum_{d|n} (16\chi\left(\frac{n}{d}\right) + \chi(d))d^2 + \frac{8}{5}\sum_{z\in\mathbb{Z}[i]} .$$

For even n the formula

$$r_{12}(n) = 8\sum_{d|n} d^5 - 512\sum_{d|\frac{n}{4}} d^5$$

holds, the second sum is omitted if $4 \nmid n$.

Remark. We note the following about the above formulae:

- Many are sums over the positive divisors of n.
- The function $\chi(d)$ appears in the formulas for $k \equiv 2 \pmod 4$.
- When $k = 10$ there is an unexpected term involving $\mathbb{Z}[i]$.
- Formulas exist only for $k \leq 12$.

Amazingly, all these facts can be explained using modular forms.

2.1.2 Generating series for $r_k(n)$

The generating series of $r_k(n)$ is

$$\sum_{n=0}^{\infty} r_k(n) q^n \in \mathbb{Z}[[q]]$$

When $k = 1$ this is $1 + 2q + 2q^4 + 2q^9 + \ldots = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \sum_{n\in\mathbb{Z}} q^{n^2}$. This series is known as Jacobi's ϑ series, and is an example of a modular form.

Exercise. Try to show that for all $k \geq 0$ we have $\sum_{n=0}^{\infty} r_k(n)q^n = \vartheta(q)^k$.

One can show that these power series for k even are examples of "q-expansions of modular forms".

2.2 $\mathbb{H}$ and the group $SL_2(\mathbb{R})$

Lecture 2

We now introduce some of the basic objects of use when studying modular forms.

The complex upper half plane $\mathbb{H}$ is $\{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$.

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R},\ ad - bc = 1 \right\}.$$

Notation. For $\gamma \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{H}$, write

$$\gamma z = \frac{az + b}{cz + d} \text{ noting that } cz + d \neq 0.$$

Proposition. *This formula defines a map*

$$\mathrm{SL}_2(\mathbb{R}) \times \mathbb{H} \to \mathbb{H},\ (\gamma, z) \mapsto \gamma z$$

which is a group action.

Proof. First note that for all $\gamma \in \mathrm{SL}_2(\mathbb{R})$ and all $z \in \mathbb{H}$

$$\operatorname{Im}(\gamma z) = \operatorname{Im}\left(\frac{az + b}{cz + d}\right) = \operatorname{Im}\left(\frac{(az + b)(c\bar{z} + d)}{|cz + d|^2}\right) = \frac{\operatorname{Im}((az + b)(c\bar{z} + d))}{|az + d|^2}.$$

Now $\operatorname{Im}((az + b)(c\bar{z} + d)) = \operatorname{Im}(ac|z|^2 + bd + adz + bc\bar{z})$, so letting $z = x + iy$ we see this equals

$$\operatorname{Im}(adz + bc\bar{z}) + \operatorname{Im}((ad - bc)iy) = \operatorname{Im}(iy) = y = \operatorname{Im}(z).$$

Therefore

$$\operatorname{Im}(\gamma z) = \frac{\operatorname{Im}(z)}{|cz + d|^2}$$

This shows the action is well defined. It is a group action as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z = \frac{z+0}{0+1} = z$, and $\gamma(\gamma' z) = (\gamma\gamma')z$. $\square$

2.3 The modular group

Definition. The modular group is the group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\} \subset \mathrm{SL}_2(\mathbb{R}).$$

As a subgroup of $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$.

Remark. Apart from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially on $\mathbb{H}$.

Since $N = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{R})$ (and $\mathrm{SL}_2(\mathbb{Z})$), the quotient $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/N$ (resp. $\mathrm{PSL}_2(\mathbb{Z}) = \dots$) is again a group with an action on $\mathbb{H}$.

Sometimes it is convenient to work with PSL rather than SL.

We now give names to some useful elements of the modular group, let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{then } ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \text{ etc.}$$

We can see for all $z \in \mathbb{H}$ we have $Sz = \frac{-1}{z}$ and $Tz = z + 1$.

2.4 A fundamental domain

Let $D = \{z \in \mathbb{H} \colon |z| \geq 1 \text{ and } \frac{-1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}\}$ and let $\rho = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, then $\rho + 1 = e^{2\pi i/3} + 1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Theorem (Properties of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{H}$)**.**

1. *For all $z \in \mathbb{H}$ there exists some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ s.t. $\gamma z \in D$.*
2. *$z, z' \in D$ are in the same $\mathrm{SL}_2(\mathbb{Z})$ orbit $\iff$ either $z = z'$ or $\mathrm{Re}(z) \pm \frac{1}{2}$ and $z = z' \pm 1$ or $|z| = 1$ and $z' = -\frac{1}{z}$.*
3. *Let $z \in D$ and let $H_z = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \colon \gamma z = z\}$ be that stabilisser of z under the action of $\mathrm{SL}_2(\mathbb{Z})$, then*

$$H_z \text{ is } \begin{cases} \textit{cyclic of order 6, generated by } ST \textit{ when } z = \rho. \\ \textit{cyclic of order 6, generated by ? when } z = \rho + 1. \\ \textit{cyclic of order 4, generated by ? when } z = i. \\ \textit{cyclic of order 2, generated by ? otherwise.} \end{cases}$$

4. *The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by the two elements S and T.*

Lecture 3

Proof. 1. Let $z \in \mathbb{H}$. For $c, d \in \mathbb{Z}$ we have $|cz + d|^2 = |(cx + d) + (cy)i|^2 = (cx + d)^2 + (cy)^2$, so there are only finitely many c, d s.t. $|cz + d| < 1$. I.e. only finitely many s.t. $\mathrm{Im}(\gamma z) > \mathrm{Im}(z)$ this implies that there is some $\gamma \in \langle S, T\rangle$ s.t. $\mathrm{Im}(\gamma z) \geq \mathrm{Im}(\gamma' z)$ for all $\gamma' \in \langle S, T\rangle$. By multiplying on the left by an appropriate power of T (i.e. by translating γz by some integer) we may assume that γ is chosen s.t. $|\mathrm{Re}(\gamma z)| \leq \frac{1}{2}$.

Claim With this γ we have $|\gamma z| \geq 1$, and hence $\gamma z \in D$, this proves part 1, with moreover the result that γ can be chosen in $\langle S, T\rangle$.

Proof of the claim. By the choice of γ we have $\mathrm{Im}(\gamma z) \geq \mathrm{Im}(S\gamma z) = \mathrm{Im}(\frac{-1}{\gamma z}) = \frac{\mathrm{Im}(\gamma z)}{|\gamma z|^2}$ this implies that $|\gamma z| \geq 1$ which proves the claim. □

2. We now let $z, z' \in D$ be in the same $\mathrm{SL}_2(\mathbb{Z})$ orbit. We may assume $\mathrm{Im}(z') \geq \mathrm{Im}(z)$. Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ be such that $z' = \gamma z$ in particular,

$$\mathrm{Im}\, z' = \frac{\mathrm{Im}\, z}{|cz + d|^2} \leq \frac{\mathrm{Im}\, z'}{|cz + d|^2}$$

so $|cz+d|^2 \leq 1$. Since $|cz+d|^2 = |cx+d|^2 + |cy|^2$ and $y \geq \frac{\sqrt{3}}{2} > \frac{1}{2}$ this gives $|c| \leq 1$, so we deal with the three possible cases for $c \in \{-1, 0, 1\}$ separately.

Case $c = 0$

$$\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \implies ad = 1 \implies a = d = \pm 1 \implies z' = \gamma z = \frac{\pm z + b}{\pm 1} = z \pm b.$$

This is only possible if $b \in \{0, \pm 1\}$, and in non-zero cases we must have $|\operatorname{Re} z| = |\operatorname{Re} z'| = \frac{1}{2}$.

Case $c = 1$ $\quad \gamma = \begin{pmatrix} a & b \\ 1 & d \end{pmatrix}$,

$$1 \geq |cz+d|^2 = (x+d)^2 + y^2 = x^2 + y^2 + 2xd + d^2 = |z|^2 + 2xd + d^2 \geq 1 + 2xd + d^2$$

So we have $|cz+d| = 1$ and $2xd + d^2 = 0$, hence either $d = 0$ or $d = -2x$ which gives $d = \pm 1$, $x = \pm \frac{1}{2}$. If $d = -1$, $x = -\frac{1}{2}$ then $z = \rho$, $z' = \frac{a\rho + b}{\rho + 1}$.

Exercise. Show this only lies in D if $(a, b) = (0, -1)$ or $(1, 0)$.

The cases $d = 0$, $d = -1$ are similar, every time we get a small finite number of γ and the corresponding possibilities for $z, z' \in D$ s.t. $z' = \gamma z$.

Case $c = -1$ is completely analogous since γ and $-\gamma$ act in the same way on $\mathbb{H}$.

So we are left with a few different possibilities which are summarised in the table.

γ	z	z'	Fixed points of γ
$\pm\mathrm{Id}$	any $z \in D$	z	all of D
$\pm T$	i	i	i

Table 1: Pairs (γ, z) with z and $z' = \gamma z$ both in D

Parts 2. and 3. of the theorem can be read off of this table, it remains to show 4.

4. Choose any z in the interior of D (e.g. $z = 2i$) and $\gamma \in \mathrm{SL}_2(\digamma)$. There exists $\gamma_0 \in \langle S, T \rangle$ s.t. $\gamma_0(\gamma z) \in D$ this means that both z and $(\gamma_0 \gamma) z$ are in D, and z is not on the boundary, so $\gamma_0 \gamma = \pm I$ and $\gamma \in \langle S, T \rangle$.

$\square$

3 Modular forms

Definition. Let f be a meromorphic function on $\mathbb{H}$ and let k be an integer.

We say that f is *weakly modular of weight k* if

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $z \in \mathbb{H}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_z(\mathbb{Z})$.

Notation.

$$(f|_k\gamma)(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \text{ for } \gamma \in \mathrm{SL}_2(\mathbb{R}).$$

Exercise. Show that if $\mathcal{F}$ is the set of all meromorphic functions on $\mathbb{H}$, then the map $\mathcal{F} \times \mathrm{SL}_2(\mathbb{R}) \to \mathcal{F}$ sending (f, γ) to $f|_k\gamma$ is a right action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{F}$.

Remark. Note that f is weakly modular of weight $k \iff f$ is $\mathrm{SL}_2(\mathbb{Z})$ invariant with respect to the $|_k$ action.

Since $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T this is equivalent to saying that $f|_k S = f$ and $f|_k T = f$ i.e. $f(z+1) = f(z)$ and $f(\frac{-1}{z}) = z^k f(z)$.

Lecture 4

In particular f is weakly modular of weight $k \implies f$ is periodic of period 1. Consider the exponential map $z \mapsto e^{2\pi i z}$, this is also periodic of period 1.

Consider the diagram

There is a *unique* function $\tilde{f} \colon \mathbb{D}^* \to \mathbb{C} \cup \{\infty\}$ s.t. $f(z) = \tilde{f}(e^{2\pi i z})$. $\tilde{f}(q) = f(\frac{\log q}{2\pi i})$ is well defined as f is periodic.

Definition. Let f be meromorphic on $\mathbb{H}$ and weakly modular of weight k. We say that f is *meromorphic at infinity* if $\tilde{f}(q)$ can be continued to a meromorphic function on the whole open unit disc $\mathbb{D} = \{q \in \mathbb{C} \colon |q| < 1\}$. Equivalently, as a Laurent series

$$\tilde{f}(q) = \sum_{n=-\infty}^{\infty} a_n q^n, \ a_n \in \mathbb{C} \text{ with } a_n = 0 \text{ for all } n \text{ sufficiently negative,}$$

converges in some open neighbourhood of $0 \in \mathbb{D}$ (but potentially having a pole at 0.

We say that f is *holomorphic at infinity* if this Laurent series is a power series, i.e.

$$\tilde{f}(z) = \sum_{n=0}^{\infty} a_n q^n.$$

In this case we define $f(\infty) = \tilde{f}(0) = a_0$.

Definition. A *modular form of weight k* is a holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ which is weakly modular of weight k and holomorphic at ∞.

A *cusp form* of weight k is a modular form of weight k which vanishes at infinity.

The reason for this terminology is if we construct the quotient space $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ we can do this by gluing the edges of D. Upon doing this we observe a surface that looks like a raindrop, the squeezed point at infinity is called the cusp.

By the above definition a modular form of weight k is a holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ that can be expressed as a convergent series

$$f(z) = \sum_{n=0}^{\infty} a_n q^n = \sum_{n=0}^{\infty} a_n e^{2\pi i z},$$

with $f\left(\frac{-1}{z}\right) = z^k f(z)$.

It is a cusp form if $a_0 = 0$.

The series $\sum_{n=0}^{\infty} a_n q^n$ is called the q-expansion (or Fourier expansion) of f.

4 Eisenstein series

Let k be an even integer with $k \geq 4$. Consider the infinite sum

$$G_k(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(mz+n)^k} = \sum_{m,n}{}' \frac{1}{(mz+n)^k}, \ z \in \mathbb{H}.$$

Proposition. *This series converges absolutely and uniformly on subsets of $\mathbb{H}$ of the form*

$$R_{r,s} = \{x + iy\colon |x| \leq r,\ y \geq s\},\ r, s > 0$$

Proof. (Sketch) Given $r, s > 0$ one first shows that there exists $c > 0$ (depending on r, s) such that $|mz+n|^2 \geq c(m^2+n^2)$ for all $m, n \in \mathbb{Z}$, $z \in R_{r,s}$. This means

$$\begin{aligned}
|G_k(z)| \leq c^{-k/2} \sum_{m,n}{}' \frac{1}{(mz+n)^{k/2}} &= c^{-k/2} \sum_{j=1}^{\infty} \sum_{m,n \text{ in } j\text{th } \square}{}' \frac{1}{(mz+n)^{k/2}} \\
\leq c^{-k/2} \sum_{j=1}^{\infty} 8j \frac{1}{j^k} = \frac{8}{c^{k/2}} \sum_{j=1}^{\infty} \frac{1}{j^{k-1}} &\leq \frac{8}{c^{k/2}} \left(1 + \int_{j=1}^{\infty} \frac{1}{t^{k-1}} dt\right) \\
&= \frac{8}{c^{k/2}} \left(1 + \frac{1}{k-2}\right) < \infty
\end{aligned}$$

$\square$

In fact this proposition implies that $G_k(z)$ is a holomorphic function on $\mathbb{H}$.

Theorem. *For all even integers ≥ 4 the function G_k is a modular form of weight k.*

Proof. It is holomorphic on $\mathbb{H}$ by the proposition. We need to check its invariance under $\mathrm{SL}_2(\mathbb{Z})$, i.e. that

$$G_k(z+1) = G_k(z) \text{ and } G_k\left(\frac{-1}{z}\right) = z^k G_k(z).$$

Proof. of $G_k(\frac{-1}{z}) = z^k G_k(z)$

$$G_k\left(\frac{-1}{z}\right) = \sum_{m,n}{}' \frac{1}{(m \cdot \frac{-1}{z} + n)^k} = z^k \sum_{m,n}{}' \frac{1}{(-m+nz)^k} = G_k(z).$$

The last step is by a couple of variable changes, we can see that the sum will always run over the same values. □

It just remains to check that G_k is holomorphic at infinity, we will do this by calculating the q-expansion at infinity, however this is long and requires its own section. □

5 q-expansions of Eisenstein series

Notation.

$$\sigma_t(n) = \sum_{d|n,\ d>0} d^t \text{ for } n \geq 1,\ t \geq 0.$$

The Riemann zeta function is

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},\ s \in \mathbb{C},\ \mathrm{Re}\, s > 1.$$

We will only need the zeta function for $s \in \mathbb{N}$ even.

These values can be expressed in terms of the Bernoulli numbers $B_k \in \mathbb{Q}$, defined by the identity

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k \text{ in } \mathbb{Q}[[t]].$$

We have $B_k \neq 0 \iff k = 1$ or $k \geq 0$ even. The first few non-zero terms are

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42}, \ldots.$$

Fact. For any even positive integer k

$$\zeta(k) = -\frac{(2\pi i)^k B_k}{2k!} \text{ for example } \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

We can rewrite the series defining G_k as follows

$$G_k(z) = \sum_{n\neq 0} \frac{1}{n^k} + \sum_{m\neq 0}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k} = 2\sum_{n=1}^{\infty} \frac{1}{n^k} + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k}$$
$$= 2\zeta(k) + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k}.$$

Fact. For $k \geq 2$ we have

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{d=1}^{\infty} d^{k-1}q^d \text{ where } q = e^{2\pi i z}.$$

Proof. sketch. Start with the following product formula for the sine function

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

Take logarithmic derivatives ($\frac{d}{dz}\log f(z) = \frac{f'(z)}{f(z)}$) gives

$$\pi\frac{\cos(\pi z)}{\sin(\pi z)} = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right).$$

On the other hand, using$e^{\pm\pi i z} = \cos\pi z \pm i\sin\pi z$ and $1/1-q = \sum_{d=0}^{\infty}$, one can prove that

$$\pi\frac{\cos(\pi z)}{\sin(\pi z)} = i\pi i - 2\pi i\sum_{d=1}^{\infty} e^{2\pi i d z}.$$

Comparing gives

$$\frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right) = -\pi i - 2\pi i\sum_{d=1}^{\infty} e^{2\pi i d z},$$

then taking derivatives gives

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^2} = (2\pi i)^2\sum_{d=1}^{\infty} de^{2\pi i d z}$$

as required for $k = 1$, the general case follows by induction upon taking successive derivatives. □

Applying this fact gives a series for G_k, we obtain

$$G_k(Z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sum_{d=1}^{\infty} d^{k-1}e^{2\pi i d m z}$$
$$= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sum_{d|n} d^{k-1}e^{2\pi i n z} = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n.$$

This gives the q-expansion of G_k and shows at the same time that G_k is holomorphic at infinity. Indeed we have now shown that G_k is a modular form of weight k.

It is useful to introduce a rescaled version of G_k

$$E_k(z) = \frac{(k-1)!}{2(2\pi i)^k} G_k(z) = \frac{-B_k}{2_k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

In particular, note that all q-expansion coefficients of E_k are rational numbers.

Remark. Another common normalisation of G_k is s.t. the constant coefficient becomes 1.

Example. $k = 4$:

$$E_4(z) = \frac{1}{240} + \sum_{n=1}^{\infty} \sigma_3(n) q^n = \frac{1}{240} + q + 9q^2 + \dots.$$

6 Motivation: Lattices in $\mathbb{C}$

Definition. A *lattice* in the complex plane $\mathbb{C}$ is a subgroup $L \subset \mathbb{C}$ of the form

$$L = \mathbb{Z}\omega_1 + \mathbb{X}\omega_2,$$

where $\omega_1, \omega_2 \in \mathbb{C}$ are $\mathbb{R}$ linearly independent.

Example. $\omega_1 = z \in \mathbb{H}$, $\omega_2 = 1$ then we let $L_z = \mathbb{Z}z + \mathbb{Z} \cdot 1$ and we see

$$G_k(z) = \sum_{m,n}{}' \frac{1}{(mz+n)^k} = \sum_{\omega \in L_z \setminus 0} \frac{1}{\omega^k}.$$

For an arbitrary lattice L one can similarly define

$$\mathcal{G}_k(L) = \sum_{\omega \in L \setminus 0} \frac{1}{\omega^k}.$$

Lattices can be scaled by complex numbers $\lambda \in \mathbb{C}^\times$, if L is a lattice we put $\lambda L = \{\lambda\omega \mid \omega \in L\}$. This is again a lattice. Scaling defines an equivalence relation on the set $\mathcal{L}$ of all lattices in $\mathbb{C}$, two lattices L, L' are *homothetic* if $L' = \lambda L$ for some $\lambda \in \mathbb{C}^\times$.

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ then another basis of L is $(a\omega_1 + b\omega_2, c\omega_1 + d\omega_2)$. We observe that

$$L_z = \mathbb{Z}z + \mathbb{Z} \cdot 1 = \mathbb{Z}(az+b) + \mathbb{Z}(cz+d),$$

this hints at a connection between lattices and modular forms.

Consider functions $\mathcal{F}\colon \{\text{lattices in } \mathbb{C}\} \to \mathbb{C}$ with the property that $\mathcal{F}(\lambda L) = \lambda^{-k}\mathcal{F}(L)$. Given such an $\mathcal{F}$ we put $f(z) = \mathcal{F}(L_z)$. Then we have

$$\begin{aligned}
f(z) &= \mathcal{F}(\mathbb{Z}z + \mathbb{Z}\cdot 1) = \mathcal{F}(\mathbb{Z}(az+b) + \mathbb{Z}(cz+d)) \\
&= \mathcal{F}\left((cz+d)\left(\mathbb{Z}\left(\frac{az+b}{cz+d}\right) + \mathbb{Z}\cdot 1\right)\right) = (cz+d)^{-k}\mathcal{F}\left(\mathbb{Z}\frac{az+b}{cz+d} + \mathbb{Z}\cdot 1\right) \\
&= \lambda^{-k}\mathcal{F}\left(L_{\frac{az+b}{cz+d}}\right) = \lambda^{-k} f\left(\frac{az+b}{cz+d}\right) = \left(f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z).
\end{aligned}$$

So there is an interpretation of modular forms as functions of *lattices* instead of functions on $\mathbb{H}$. However in the setting of functions on $\mathbb{H}$ with the property of being holomorphic (both on $\mathbb{H}$ and at infinity) many statements are easier to formulate.

7 More examples: the modular form Δ and the modular function j

Recall that

$$E_4 = \frac{1}{240} + q + \ldots, \; E_6 = \frac{1}{540} + q + \ldots.$$

Definition. We define

$$\Delta = \frac{(240E_4)^3 - (-504E_6)^2}{1728}.$$

Working out the q-expansion we see that

$$\Delta = q - 24q^2 + 252q^3 - 1472q^4 + \ldots$$

is a cusp form of weight 12. One can in fact show that all the q-expansion coefficients are integral.

Moreover we define the function $\tau\colon \mathbb{Z} \to \mathbb{Z}$ through the following equality

$$\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n.$$

τ is known as Ramanujan's τ function.

τ has many interesting number-theoretic properties e.g.

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$

Definition. The j-function is defined as

$$j(z) = \frac{(240E_4)^3}{\Delta} = q^{-1} + 744 + 196884q + \ldots.$$

This is not a modular form as it is not holomorphic at infinity, it is however an example of a *modular function*.

Definition. A *modular function* is a function f satisfying $f(\gamma z) = f(z)$ for all $z \in \mathbb{H}$, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ that is also meromorphic on $\mathbb{H}$ and at infinity.

8 The valence formula

The valence formula gives us information about the zeroes and poles of modular and weakly modular functions.

Let f be a meromorphic function on an open subset $U \subseteq \mathbb{C}$, and $w \in U$. Then f can be expanded in a Laurent series around w.

$$f(z) = c_n(z-w)^n + c_{n+1}(z-w)^{n+1} + \dots \ (n \in \mathbb{Z},\ c_j \in \mathbb{C},\ c_n \neq 0).$$

The *order* or *valuation* of f at w (denoted $\operatorname{ord}_w(f)$ or $\nu_w(f)$) is the n in the above expansion. The *residue* of f at w is c_{-1} ($= 0$ if $n \geq 0$). We denote this by $\operatorname{Res}_w(f)$

From this expression we can deduce that

$$\frac{f'}{f}(z) = \frac{n}{z-w} + b_0 + b_1(z-w) + \dots.$$

In particular $\frac{f'}{f}$ has a simple zero precisely at the zeroes and poles of f and

$$\operatorname{Res}_w\left(\frac{f'}{f}\right) = n = \operatorname{ord}_w(f).$$

Theorem (Cauchy's integral formula)**.** *Let g be holomorphic on an open subset $U \subseteq \mathbb{C}$. Let $\mathcal{C}$ be a contour (simple closed curve) in U, and let $w \in U$. Then we have*

$$\oint_{\mathcal{C}} \frac{g(z)}{z-w} dz = 2\pi i g(w).$$

Theorem (Argument principle)**.** *Let f be meromorphic on an open subset $U \in \mathbb{C}$, let $\mathcal{C}$ be a contour in U. Assume f has no zeroes or poles on $\mathcal{C}$. Then*

$$\oint_{\mathcal{C}} \frac{f'(z)}{f(z)} \mathrm{d}z = 2\pi i \sum_{z \in \operatorname{interior}(\mathcal{C})} \operatorname{ord}_z(f)$$

$$(= 2\pi i \cdot (\textit{number of zeroes} \ - \ \textit{number of poles, with multiplicity})).$$

There are variants of this statement, such as:

Let $\mathcal{C}$ be an arc around $w \in U$ with angle α and radius r. If g is holomorphic at w we have

$$\lim_{r \to 0} \oint_{\mathcal{C}} \frac{g(z)}{z-w} \mathrm{d}z = \alpha i g(w).$$

If f is meromorphic at w we have

$$\lim_{r \to 0} \oint_{\mathcal{C}} \frac{f'(z)}{z-w} \mathrm{d}z = \alpha i \operatorname{ord}_i f.$$

Let f be meromorphic on $\mathbb{H}$ and weakly modular of some weight k. Let $z \in \mathbb{H}$, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Using the transformation formula it is not hard to check that $\operatorname{ord}_{\gamma z} f = \operatorname{ord}_z f$. Finally if in addition f is meromorphic at infinity and $\tilde{f}(q)$ is defined by

$$f(z) = \tilde{f}(e^{2\pi i z} = \tilde{f}(q)$$

we define $\operatorname{ord}_{z=\infty} f = \operatorname{ord}_{q=0} \tilde{f}$.

Theorem (Valence formula)**.** *Let f be meromorphic on $\mathbb{H}$, weakly modular of weight k and meromorphic at infinity. Then we have*

$$\operatorname{ord}_\infty f + \frac{1}{2}\operatorname{ord}_i f + \frac{1}{3}\operatorname{ord}_\rho f + \sum_{w \in W} \operatorname{ord}_w f = \frac{k}{12}.$$

Where W is the set of $\mathrm{SL}_2(\mathbb{Z})$-orbits in $\mathbb{H}$ with the orbits of i and ρ omitted.

Proof. We may take all orbit representatives in the set D. We assume for simplicity that f has no poles or zeroes on the boundary except for possibly at i, ρ or $\rho + 1$. Let $\mathcal{C}$ be the following contour

By the argument principle

$$\oint_{\mathcal{C}} \frac{f'(z)}{f(z)} \mathrm{d}z = 2\pi i \sum_{w \in W} \operatorname{ord}_w f.$$

On the other hand we can compute the same integral by cutting $\mathcal{C}$ into 8 pieces and using the transformation formula. We consider the pieces as follows

- We have

  $$\int_{D'}^{E} \frac{f'(z)}{f(z)} \mathrm{d}z = \int_B^A \frac{f'}{f}(z'+1)\mathrm{d}z' = \int_B^A \frac{f'}{f}(z)\mathrm{d}z = \int_A^B \frac{f'}{f}(z)\mathrm{d}z.$$

  I.e. the integrals $\int_B^A$ and $\int_E^{D'}$ cancel.

- From the equation $f(-1/z) = z^k f(z)$ we deduce

  $$z^{-2} f'\left(\frac{-1}{z}\right) = kz^{k-1} f(z) + z^k f'(z) \implies z^{-2}\frac{f'}{f}\left(\frac{-1}{z}\right) = \frac{k}{2} + \frac{f'}{f}(z).$$

  So

  $$\begin{aligned} \int_{C'}^{D} \frac{f'}{f}(z)\mathrm{d}z &= \int_C^{B'} \frac{f'}{f}\left(\frac{-1}{z}\right)(z')^{-2}\mathrm{d}z' \text{ where } z' = \frac{-1}{z} \\ &= \int_C^{B'} \frac{k}{z'} + \frac{f'}{f}(z)\mathrm{d}z' \\ &= k\int_C^{B'} \frac{1}{z}\mathrm{d}z - \int_{B'}^{C} \frac{f'}{f}(z)\mathrm{d}z. \end{aligned}$$

  This implies

  $$\int_{B'}^{C} \frac{f'}{f}(z)\mathrm{d}z + \int_{C'}^{D} \to k^{\frac{\pi i}{6}} \text{ as } r \to \infty.$$

- As $r \to 0$, we have (by a result quoted above).

$$\int_B^{B'} \frac{f'}{f}(z)\mathrm{d}z \to -\frac{\pi i}{3}\operatorname{ord}_\rho f,$$
$$\int_C^{C'} \frac{f'}{f}(z)\mathrm{d}z \to -\pi i \operatorname{ord}_i f,$$
$$\int_D^{D'} \frac{f'}{f}(z)\mathrm{d}z \to -\frac{\pi i}{3}\operatorname{ord}_{\rho+1} f = -\frac{\pi i}{3}\operatorname{ord}_\rho f.$$

- To calculate the integral from E to A we make the change of variables $q = e^{2\pi i z}$. Recall that $\tilde{f}$ is defined by

$$f(z) = \tilde{f}(e^{2\pi i z}.$$

This gives

$$\tilde{f}' = 2\pi i e^{2\pi i z} \text{ and so } \frac{f'}{f} = 2\pi i e^{2\pi i z}\frac{\tilde{f}'}{\tilde{f}}(e^{2\pi i z}.$$

Furthermore $\mathrm{d}q/\mathrm{d}z = 2\pi i e^{2\pi i z}$. Hence we get

$$\int_E^A \frac{f'}{f}\mathrm{d}z = -\oint_{|q|=e^{-2\pi R}} \frac{\tilde{f}'}{\tilde{f}}(q)\mathrm{d}q = -2\pi i \operatorname{ord}_{q=0} \tilde{f} = -2\pi i \operatorname{ord}_{z=\infty} f.$$

Summing the eight contributions we get

$$\oint_{C'} \frac{f'}{f}(z)\mathrm{d}z = k\frac{\pi i}{6} - \pi i \operatorname{ord}_i f - \frac{2\pi i}{3}\operatorname{ord}_\rho f - 2\pi i \operatorname{ord}_\infty f.$$

Combining the expressions for $\oint f'/f(z)$ found in these two ways we obtain

$$2\sum_{w\in W}\operatorname{ord}_w f = \frac{k\pi i}{6} - \pi i \operatorname{ord}_i f - \frac{2\pi i}{3}\operatorname{ord}_\rho f - 2\pi i \operatorname{ord}_\infty f.$$

Rearranging gives the result. □

We will use this theorem to prove a finiteness theorem about the space of modular forms. But first we define the following corollary of the theorem.

Theorem. *1. The Eisenstein series E_4 has a simple zero at ρ and no other zeroes in $\mathbb{H}$ or at infinity.*

2. The Eisenstein series E_6 has a simple zero at i and no other zeroes in $\mathbb{H}$ or at infinity.

3. The modular form Δ of weight 12 has a simple zero at infinity and no other zeroes in $\mathbb{H}$.

Proof. Using the valence formula

$$\operatorname{ord}_\infty f + \frac{1}{2}\operatorname{ord}_i f + \frac{1}{3}\operatorname{ord}_\rho f + \sum_{w \in W} \operatorname{ord}_w f = \begin{cases} \frac{1}{3} \text{ in case 1,} \\ \frac{1}{2} \text{ in case 2,} \\ 1 \text{ in case 3.} \end{cases}$$

all terms on the left are non-negative, in case 3 we know moreover $\operatorname{ord}_\infty \Delta \geq 1$. The only way to satisfy the formula is if the location of the zeroes is as claimed. $\square$

Notation. $M_k =$ the set of all modular forms of weight k.
$S_k =$ the subset of M_k consisting of cusp forms.

Corollary. *Multiplication by Δ is an isomorphism*

$$M_k \tilde{\to} S_{k+12}, \ f \mapsto \Delta f, \ \frac{g}{\Delta} \leftarrow g.$$

Note that M_k and S_k are $\mathbb{C}$ vector spaces (a $\mathbb{C}$ linear combination of modular forms of weight k (resp. cusp forms) is again such a form).

Theorem. *The spaces M_k and S_k are finite dimensional for every $bk \in \mathbb{Z}$. Furthermore $M_k = \{0\}$ is $k < 0$ or k is odd, and the dimension of M_k for $k \geq 0$ even is*

$$\dim_{\mathbb{C}} M_k = \begin{cases} \lfloor \frac{k}{12} \rfloor \ \textit{if } k \equiv 2 \pmod{12}, \\ \lfloor \frac{k}{12} \rfloor + 1 \ \textit{if } k \not\equiv 2 \pmod{12}. \end{cases}$$

Note that this also gives the dimension of $S_k \cong M_{k-12}$ for all $k \in \mathbb{Z}$.
In particular we can compute

k	$\dim M_k$	$\dim S_k$
0	1	0
2	0	0
4	1	0
6	1	0
8	1	0
10	1	0
12	2	1
14	1	0
16	2	1

Table 2: Dimensions of the spaces of modular and cusp form of weight k for even $k \leq 16$

Proof. The claim that $M_k = \{0\}$ if $k < 0$ follows from the valence formula.
For k odd, note that if f is a modular form of weight k, then applying the transformation formula

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \text{ with } a = d = -1, \ b = c = 0$$

shows $f(z) = -f(z)$ so $f = 0$.

$\square$