Part III Topics in Algebraic Geometry 2014

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Chapter 1

Schemes

1.1 Introduction

These are lecture notes for the 2014 Part III Topics in Algebraic Geometry course taught by Dr. Mark Gross, these notes are part of Mjolnir.

Some reference for commutative algebra:

- Atiyah-Macdonald
- Matsumura - Commutative Algebra
- Matsumura - Commutative Ring Theory

The general course reference is Hartshorne.

Generated: April 10, 2015, 14:26:52 (Z)

1.2 Motivation

Take $I \subset k[x_1, \dots, x_n]$ then $X = V(I) \subset \mathbf{A}^n$ and $(X, \mathcal{O}_X)$ is a ringed space. $A(X) = k[x_1, \dots, x_n]/\sqrt{I}$. A point of X is given by a map of k-algebras $\phi\colon A(X) \to k$ $x_i \mapsto a_i$, giving the point $(a_1, \dots, a_n) \in X$. The kernel of ϕ is a maximal ideal and conversely given a maximal ideal $m \subset A(C)$ we get a map $A(X) \to A(X)/m = k$ if $k = \bar{k}$ by the nullstellensatz. Similarly if l is a field extension of k then a k-algebra homomorphism $A(X) \to l$ can be viewed as giving a solution to the system of equations with values in l. Note that the group $\mathrm{Gal}(l|k)$ acts on the set of solutions over l by postcomposition. We might as well consider all possible field extensions $l|k$, then $\ker(A(X) \to l)$ might not be maximal.

Example 1.2.1. $X = \mathbf{A}^1 k[X] \hookrightarrow k(X)$. This is a $k(X)$-valued point on $\mathbf{A}^1$.

More generally if R is a k-algebra an R-valued point of X is given by a k-algebra homomorphism $\phi\colon A(X) \to R$.

Example 1.2.2. $R = k[t]/(t^2)$ and the k-algebra homomorphism $\phi\colon A(X) \to R$ induces by composition with $t \mapsto 0$ a k-valued point $x \in X$. (Assuming now $k = \bar{k}$) x corresponds to a maximal ideal $m_x = \ker(A(X) \to k)$ and $\phi(m_x) \subseteq (t)$. So $\phi(m_x^2) = 0$. Thus we get a map $\phi\colon m_x/m_x^2 \to (t) = k$.

Exercise 1.2.3. Check that giving $x \in X$ and a map $m_x/m_x^2 \to k$ is equivalent to giving a map $\phi\colon A(X) \to R$.

A map $m_x/m_x^2 \to k$ is an element of $(m_x/m_x^2)^*$.

Recall that if X, Y are affine varieties, giving a morphism $X \to Y$ is equivalent to giving a k-algebra homomorphism $A(Y) \to A(X)$. This suggests that we should allow any k-algebra to be a coordinate ring and if A, B are k-algebras then a map of k-algebras $A \to B$ should be equivalent to giving a morphism of the corresponding "varieties" . More generally we could work over a ring R, rather than a field k. A and B could then be R-algebras. This includes the case where $R = \mathbf{Z}$ and A and B are just rings.

Definition 1.2.4. The category of affine schemes is the opposite category of the category of commutative rings.

Definition 1.2.5. A **scheme** is a geometric object covered by affine schemes.

In general we tend to work with schemes over a base scheme S (e.g. $S \leftrightarrow k$) and consider morphisms defined over S i.e. diagrams .

For T, X schemes over S a T-valued point of X is a diagram .

Definition 1.2.6. If A is a commutative ring

$$\operatorname{Spec} A = \{p \subset A : p \text{ is a prime ideal}\}.$$

If $I \subset A$ is an ideal (or set) we define $V(I) = \{p \in \operatorname{Spec} A : p \supseteq I\}$.

Note that if $k = \bar{k}$ and $A = k[x_1, \dots, x_n]$, $m = (x_1 - a_1, \dots, x_n - a_n)$ contains I if and only if $(a_1, \dots, a_n) \in V(I)$.

Exercise 1.2.7. Show that the sets $V(I)$ form the closed sets of a topology on $\operatorname{Spec} A$, the **Zariski topology**.

$\Gamma(X, \mathcal{O}_X) = A(X)$. Our goal is to construct a sheaf $\mathcal{O}_{\operatorname{Spec} A}$ with stalks $\mathcal{O}_{\operatorname{Spec} A, p} = A_p$ and $\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A$.

Definition 1.2.8 (Structure sheaf on Spec)**.** Let $\mathcal{O}_{\operatorname{Spec} A}$ be the sheaf on $\operatorname{Spec} A$ whose sections over U open are functions

$$s \colon U \to \coprod_{p \in U} A_p$$

such that

1. $s(p) \in A_p$
2. for $p \in U$ there exists some open $V \subset U$ with $p \in V$ and $f, g \in A$ such that for all $q \in V$ we have $g \in q$ and $s(q) = f/g \in A_q$

Definition 1.2.9 (Spectrums of rings)**.** The spectrum of A is the locally ringed space $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.

Here locally ringed space means $\mathcal{O}_{\operatorname{Spec} A, p}$ is local for all $p \in \operatorname{Spec} A$.

Proposition 1.2.10. *1. For any* $p \in \operatorname{Spec} A$ $\mathcal{O}_p = A_p$.

2. For any $f \in A$ *let* $D(f) = \{p \in \operatorname{Spec} A : f \notin p\} = \operatorname{Spec} A \smallsetminus V(f)$.

3. $\Gamma(\operatorname{Spec} A, \mathcal{O}) = A$.

Exercise 1.2.11. Show that sets of the form $D(f)$ form a basis of the topology on $\operatorname{Spec} A$.

Proof. 1. Define a map $\mathcal{O}_p \to A_p$ by $(U, s) \mapsto s(p)$. To see this is surjective we take $f/g \in A_p$ ($g \in p$) then $(D(g), f/g) \in \mathcal{O}_p$ which maps to f/g. To see injectivity we let (U, s), $(V, t) \in \mathcal{O}_p$ with $s(p) = t(p)$. By shrinking U and V we can assume $U = V$ and s is given by f/g with t given by f'/g' where $g, g' \notin p$. Thus $f/g = f'/g' \in A_p$ and hence there exists some $h \notin p$ with $(f'g - g'f)h = 0$. Then for and

□

Definition 1.2.12 (Morphisms of locally ringed spaces)**.** A morphism $(X, \mathcal{O}_x) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces is a continuous map $f\colon X \to Y$ along with a morphism of sheaves of rings

$$f^{\#}\colon \mathcal{O}_Y \to \mathcal{O}_X$$

such that the induced maps $f^{\#}_p\colon \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ are local homomorphisms for all p. The maps $f^{\#}_p$ are induced by $(U, s) \mapsto (U, f^{\#}(s))$.

Proposition 1.2.13. *1. If $\phi\colon A \to B$ is a ring homomorphism then ϕ induces a morphism of locally ringed spaces*

$$(f, f^{\#})\colon (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}).$$

2. Any morphism of locally ringed spaces is induced in this way.

Proof. 1. Define $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ by $f(p) = \phi^{-1}(p) \in \operatorname{Spec} A$. This is continuous as

$$\begin{aligned} f^{-1}(V(I)) &= \{p \in \operatorname{Spec} B : \phi^{-1}(p) \supseteq I\} \\ &= \{p \in \operatorname{Spec} B : p \supseteq \phi(I)\} \\ &= V(\phi(I)). \end{aligned}$$

For any $p \in \operatorname{Spec} B$ we have a ring map

$$\begin{aligned} \phi_p\colon A_{\phi^{-1}(p)} &\to B_p \\ \frac{a}{s} &\mapsto \frac{\phi(a)}{\phi(s)}. \end{aligned}$$

Define

$$\begin{gathered} f^{\#}\colon \mathcal{O}_{\operatorname{Spec} A}(V) \to \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(V) = (f_* \mathcal{O}_{\operatorname{Spec} B})(V) \\ \left(s\colon V \to \coprod_{p \in V} A_p\right) \mapsto \left(f^{\#}(s)\colon f^{-1}(V) \to \coprod_{q \in f^{-1}(V)} B_q\right) \end{gathered}$$

with $(f^{\#}(s))(q) = \phi_{f(q)}(s(f(q)))$, $s(f(q)) \in A_{f(q)} = A_{\phi^{-1}(q)}$. Note that $f^{\#}$ induces the map ϕ_q on stalks and ϕ_q is a local homomorphism.

2. Suppose we are given $(f, f^{\#})\colon (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, then we can get $\phi\colon \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A \xrightarrow{f^{\#}} \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) = B$. For $p \in \operatorname{Spec} B$ we get

$$\begin{aligned} f^{\#}_p\colon \mathcal{O}_{\operatorname{Spec} A, f(p)} &\to \mathcal{O}_{\operatorname{Spec} B, p} \\ A_{f(p)} &\to B_p \end{aligned}$$

a local homomorphism. We also have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\substack{a \\ \downarrow \\ a/1}"'] & B \arrow[d, "\substack{b \\ b/1}"] \\ A_{f(p)} \arrow[r, "f_p^{\#}"'] & B_p \end{tikzcd}$$

$\square$

1.3 Properties of Schemes

Definition 1.3.1 (Irreducible schemes)**.** A scheme $(X, \mathcal{O}_X)$ is said to be **irreducible** if X is irreducible as a topological space.

Definition 1.3.2 (Reduced schemes)**.** A scheme is **reduced** if $\mathcal{O}_X(U)$ is an integral domain for any $U \subset X$ open.

Proposition 1.3.3. *A scheme is integral if and only if it is reduced and irreducible.*

Proof. Integral implies reduced is clear.
Suppose $X = Y_1 \cup Y_2$ with $Y_1, Y_2 \subset X$ closed $Y_1, Y_2 \neq X$. So we find that $U_1 = Y_1 \setminus Y_2 = X_1 \setminus Y_2$ and $U_2 = Y_2 \setminus Y_1 = X_2 \setminus Y_1$ are open disjoint sets.

Then $\mathcal{O}_X(U_1 \cup U_2) = \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$ by the sheaf axioms ⟨⟨Unresolved xref, ref="defn-sheaf"; check spelling or use "provisional" attribute⟩⟩. But this is not an integral domain. Conversely suppose X is reduced and irreducible. If $U \subset X$ open, $f, g \in \mathcal{O}_X(U)$ with $fg = 0$ we want to show that either $f = 0$ or $g = 0$. Let $Y = \{x \in U : f_x \in \mathfrak{m}_x\}$ where f_x is the germ of f in $\mathcal{O}_{X,x}$ and $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the maximal ideal. Let $Z = \{x \in U : g_x \in \mathfrak{m}_x\}$ then Y and Z are closed subsets of U (its enough to check this on an open cover of U which we can assume to be affine, but if $U = \operatorname{Spec} A$ is affine $Y = V(f)$, which is closed). Since $fg = 0$ we have $f_x g_x = 0$ for all x and so $U = Z \cup Y$. U is an open subset of X which is irreducible so U is irreducible (exercise!). So $U = Y$ or $U = Z$. Assume $U = Y$, we will show $f = 0$. We can restrict to affine open subsets of U and hence $U = \operatorname{Spec} A$ is affine. Thus $\emptyset = U \setminus Y = D(f)$. Thus $f = \bigcap_{p \in \operatorname{Spec} A} p = \sqrt{(0)}$. Thus f is nilpotent so $f = 0$ since X is reduced, therefore $\mathcal{O}_X(U)$ is an integral domain. $\square$

Example 1.3.4. $\operatorname{Spec} k[x, y]/(xy)$ is the two axes, and not irreducible.
$\operatorname{Spec} k[t]/(t^2)$ is a point, with global sections $k[t]/(t^2)$.

Definition 1.3.5 ((Locally) Noetherian schemes)**.** A scheme X is said to be **locally Noetherian** if it can be covered by open affines of the form $\operatorname{Spec} A$ with A a Noetherian ring. It is said to be **Noetherian** if there is a finite such cover.

Proposition 1.3.6. *X is locally Noetherian is and only if for every affine open subset* $\operatorname{Spec} A \subset X$ *we have A Noetherian. In particular* $\operatorname{Spec} A$ *is noetherian if and only if A is Noetherian.*

Proof. $\Leftarrow$ clear, $\Rightarrow$ let $U \subset X$ be open affine, $U = \operatorname{Spec} A$. In general if B is Notherian and $f \in B$ then B_f is Noetherian and $D(f) \cong \operatorname{Spec} B_p$ as schemes. Also the $D(f)$s form a basis for the topology on $\operatorname{Spec} B$. Thus any open set of $\operatorname{Spec} B$ can be covered by open affines of the form $\operatorname{Spec} B_f$ with B_f Noetherian. $U \cap \operatorname{Spec} B$ can be covered by affine schemes of the form $\operatorname{Spec} B_f$ with B_f Noetherian. We need to show that if $\operatorname{Spec} A$ can be covered by sets of the form $\operatorname{Spec} B$ with B Noetherian, then A is Noetherian. $\square$