Ranks and Parity of Ranks of Curves and Abelian Surfaces

MA842 at BU Spring 2019

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1 Mordell-Weil groups

These are notes for Céline Maistret's course MA842 at BU Spring 2019.

The course webpage is https://sites.google.com/view/cmaistret/teaching#h.p_BYGoPzU848FJ.

Lecture 1 22/1/2018

Outline

1. Elliptic curves and their ranks
   (a) Background
      i. Mordell Weil theorem (state and prove) (ANT and cohomological proof)
      ii. Non-effectivity
      iii. Computing the rank (descent)
   (b) The Birch and Swinnerton-Dyer conjecture
      i. Heuristic via counting points omn the reduced curve
      ii. L-functions
      iii. BSD-1
      iv. Local arithmetic invariants and BSD-2
   (c) Parity of ranks
      i. Isogeny invariants of BSD 2
      ii. Galois representations and local root numbers
      iii. The parity conjecture
2. Abelian surfaces
   (a) Background on genus 2 curves and their Jacobians
   (b) BSD in this case
   (c) Computability of local arithmetic invariants
   (d) Parity conjecture

Evaluation, none, when not around will give exercise/project, if you come regularly and do a computation you pass.

Main references that we will be following:

1. Vladimir Dokchitser - Lecture course
2. Silverman - Arithmetic of Elliptic Curves
3. Milne - Abelian Varieties?

1.1 Elliptic curves and their ranks

Sources: Silverman I, V. Dokchitser's lectures.

1.1.1 Mordell-Weil

Let K be a number field and let E/K be an elliptic curve. The group $E(K)$ is finitely generated.

$$E(K) \simeq E(K)_{\text{tors}} \oplus \mathbf{Z}^r.$$

Where $E(K)_{\text{tors}}$ is a finite subgroup and r is the rank, a non-negative integer.

Assuming that we can compute the torsion subgroup, computing the rank would completely determine $E(K)$ and hence solve the associated diophantine problem.

Plan

1. Understand the proof of Mordell-Weil
2. See where it is non-effective.
3. From the proof, extract a strategy to sometimes compute the rank (define Selmer groups, Shafarevich-Tate group).

Outline proof of Mordell-Weil. Part 1: Prove that

$$E(K)/mE(K)$$

is finite for some $m \geq 2$.

Part 2: use a descent argument with heights of points. ∎

Of these two parts of the proof, part 1 is the challenging/interesting one.

For part 2: Assuming that

$$E(K)/mE(K)$$

is finite and that E has a "height function" then $E(K)$ is finitely generated.

Theorem 1.1 Descent theorem (see Thm. VIII 3.1). *Let A be an abelian group, suppose that there exists a function*

$$h : A \to \mathbf{R}$$

with the following properties:

1. *Let $Q \in A$ then there is a constant c_1 depending on Q and A such that*
$$h(P + Q) = 2h(P) + c_1, \ \forall P \in A.$$
2. *There is an integer $m \geq 2$ and a constant c_2 depending on A s.t.*
$$h(mP) \geq m^2 h(P) - c_2, \ \forall P \in A.$$
3. *For every constant c_3, the set*
$$\{P \in A : h(P) \leq c_3\}$$

is finite.

suppose further that for the m in 2. we have A/mA is finite. Then A is finitely generated.

Proof. Choose elements $Q_1, \ldots, Q_r \in A$ to represent the finitely many cosets in A/mA. Let P be a point in A. We show that P can be generated by $Q_1, \ldots, Q_r$ plus a set of finitely many points of bounded height.

First write

$$P = mP_1 + Q_{i_1}$$

for some $1 \leq i \leq r$. Repeat this for

$$\begin{aligned} P_1 &= mP_2 + Q_{i_2} \\ P_2 &= mP_3 + Q_{i_3} \\ &\vdots \\ P_{n-1} &= mP_n + Q_{i_n} \end{aligned}$$

by property 2. of h we have

$$\begin{aligned} h(P_j) &\leq \frac{1}{m^2}(h(mP_j) + c_2) \\ &\frac{1}{m^2}(h(P_{j-1}) - Q_{i_j}) + c_2) \\ &\leq \frac{1}{m^2}(2h(P_{j-1}) + c_1' + c_2) \end{aligned}$$

by 1. Where c_1' is the maximum of the constants from i for Q in $\{-Q_1, \ldots, -Q_r\}$. Note that c_1' and c_2 do not depend on P and that $h(P) \geq 0$. We repeat this inequality starting from P_n and working back to P.

$$\begin{aligned} h(P_n) &\leq \left(\frac{2}{m^2}\right)^n h(P) + \frac{1}{m^2}\left(1 + \frac{2}{m^2} + \left(\frac{2}{m^2}\right)^2 + \cdots + \left(\frac{2}{m^2}\right)^{n-1}\right)(c_1' + c_2) \\ &= \left(\frac{2}{m^2}\right)^n h(P) + \frac{1}{m^2}\left(1 + \frac{2}{m^2} + \left(\frac{2}{m^2}\right)^2 + \cdots + \left(\frac{2}{m^2}\right)^{n-1}\right)(c_1' + c_2) \\ &< \left(\frac{2}{m^2}\right)^n h(P) + \frac{c_1' + c_2}{m^2 - 2} \\ &\leq \frac{1}{2^n} h(P) + \frac{c_1' + c_2}{2}, \end{aligned}$$

since $m \geq 2$. Hence for n sufficiently large (to make $\frac{1}{2^n} h(P) \leq 1$) we have

$$h(P_n) \leq 1 + \frac{1}{2}(c_1' + c_2).$$

Since P is a linear combination of P_n and Q_i

$$P = m^n P_n + \sum_{j=1}^{n} m^{j-1} Q_{i_j},$$

it follows that every $P \in A$ is a linear combination of points in

$$\{Q_1, \ldots, Q_r\} \cup \{Q \in A : h(Q) \leq 1 + \frac{1}{2}(c_1' + c_2)\}.$$

■

Remark 1.2 On $E/\mathbf{Q}$ the height function

$$h\colon E(\mathbf{Q}) \to \mathbf{Q}$$

$$P \mapsto \begin{cases} \log(\max\{|p|,|q|\}),\ x(P) = \frac{p}{q}, & P \neq 0, \\ 0, & P = 0. \end{cases}$$

satisfies the conditions of Theorem 1.1.

Remark 1.3 The above proof is effective. To find generators of $E(\mathbf{Q})$ first compute $c_1 = c_1(Q_i)$ for each i, then compute c_2. Find points of bounded height. Note that we need $Q_1, \ldots, Q_r$ to start with.

It remains to show part 1:

Theorem 1.4 Weak Mordell-Weil. *Let K be a number field E/K an elliptic curve, $m \geq 2$ then*

$$\#E(K)/mE(K) < \infty.$$

We will prove this under the assumption that $E[m] \subseteq E(K)$. This is WLOG since:

Lemma 1.5 *Let L/K be a finite Galois extension, if*

$$E(L)/mE(L)$$

is finite then so is

$$E(K)/mE(K).$$

Proof.

$$0 \to \phi \to E(K)/mE(K) \xrightarrow{\varphi} E(L)/mE(L) \to 0$$

induced by

$$E(K) \subseteq E(L),$$

and prove that ϕ is finite. Kernel ϕ is given by

$$\frac{E(K) \cap mE(L)}{mE(K)},$$

take $P \in \phi$. We can choose $Q_P \in E(L)$ such that $Q_P = P$. Define a map of sets

$$\lambda_P\colon G_{L/K} \to E[m]$$

$$\sigma \mapsto Q_P^\sigma - Q_P.$$

Note that

$$[m](Q_P^\sigma - Q_P) = ([m]Q_P)^\sigma - [m]Q_P = 0.$$

Now we show that the association

$$\phi \to \mathrm{Map}(G_{L/K}, E[m])$$

$$P \mapsto \lambda_P$$

is 1 to 1.

Suppose that $P, P' \in E(K) \cap mE(L)$ satisfying $\lambda_P = \lambda_{P'}$ then

$$(Q_P - Q_{P'})^\sigma = Q_P - Q_{P'}$$

for all $\sigma \in G_{L/K}$ so $Q_P - Q_{P'} \in E(K)$ and hence

$$P - P' = [m]Q_P - [m]Q_{P'} \in mE(K)$$

hence

$$P = P' \pmod{mE(K)}.$$

$G_{L/K}$ and $E[m]$ are both finite, hence so is ϕ. ∎

Lecture 2 29/1/2018

Now we will prove the weak Mordell-Weil theorem. Using the above lemma we can reduce to the case where $E[m] \subseteq E(K)$, so we assume this going forwards.

Definition 1.6 The Kummer pairing. The **Kummer pairing** is

$$\kappa \colon E(K) \times G_{\overline{K}/K} \to E[m]$$

$$P, \sigma \mapsto Q^\sigma - Q$$

where Q is a choice of point in $E(\overline{K})$ such that $mQ = P$. ◊

Proposition 1.7 *κ is well defined, bilinear, the kernel in the first argument is $mE(K)$ and in the second argument is $G_{\overline{K}/L}$ where $L = K([m]^{-1}E(K))$ is the compositum of all fields $\kappa(x(Q), y(Q))$ as Q ranges over all the points of $E(\overline{K})$ s.t. $mQ \in E(K)$.*

Hence the Kummer pairing induces a perfect bilinear pairing

$$E(K)/mE(K) \times G_{L/K} \to E[m]$$

i.e. the map

$$E(K)/mE(K) \to \mathrm{Hom}_K(G_{L/K}, E[m])$$
$$P \mapsto (\sigma \mapsto Q^\sigma - Q)$$

is an isomorphism.

Lecture 3 31/1/2018

Lecture 4 5/2/2018

Remark 1.8 A homomorphism $\phi \colon \mathrm{Gal}(\overline{K}/K) \to G$ for a finite group G is continuous if it comes from a finite Galois extension, i.e.

$$\exists F/K \text{ finite Galois }, \ \tilde{\phi} \colon \mathrm{Gal}(F/K) \to G$$

s.t. ϕ is the composition $\mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(F/K) \xrightarrow{\tilde{\phi}} G$. So $\phi(g)$ only cares about what g does to F.

Proposition 1.9 *Let E/K be an elliptic curve*

$$y^2 = (x - \alpha)(x - \beta)(x - \gamma)$$

for $P \in E(K)$ have $\frac{1}{2}P \in E(\overline{K})$ s.t. $\frac{1}{2}P \oplus \frac{1}{2}P = P$.

1. *$K(\frac{1}{2}P)/K$ is a Galois extension and $\mathrm{Gal}(K(\frac{1}{2}P)/K) = C_2 \times C_2$ from Lemma 1.*
2. $$\phi_P \colon \mathrm{Gal}(\overline{K}/K) \to E(K)[2]$$
   $$g \mapsto Q^\sigma - Q = g(\frac{1}{2}P) - \frac{1}{2}P$$
   is well defined and has kernel $\mathrm{Gal}(K/K(\frac{1}{2}P))$.
3. $$\phi \colon E(K)/2E(K) \to \mathrm{Hom}_{cts}(\mathrm{Gal}(\overline{K}/K), E(K)[2])$$
   $$P \mapsto \phi_P$$

is well defined and injective. Now ϕ_P is continuous by 2. *and so*

$$\phi_{P\oplus Q}(g) = g(\frac{1}{2}(P \oplus Q)) - (\frac{1}{2}P \oplus \frac{1}{2}Q)$$
$$= g(\frac{1}{2}P) \oplus g(\frac{1}{2}Q) - \frac{1}{2}P \ominus \frac{1}{2}Q$$
$$= \phi_P(g) \oplus \phi_Q(g)$$

a homomorphism.

$$\phi_{2Q}(g) = g(\frac{1}{2}2Q)) - \frac{1}{2}2(Q) = g(Q) - Q = 0$$

for all $g \in \mathrm{Gal}(\overline{K}/K)$ if $Q \in E(K)$ so this is well defined. For injectivity:

$$\phi_P(g) = 0 \implies g(\frac{1}{2}P) = \frac{1}{2}P \forall g \in \mathrm{Gal}(\overline{K}/K)$$

$$\implies \frac{1}{2}P \in E(K) \implies P \in 2E(K)$$

which gives injectivity.

4.

$$\eta\colon \mathrm{Hom}_{cts}(\mathrm{Gal}(\overline{K}/K), E(K)[2]) \to K^\times/K^{\times 2} \times K^\times/K^{\times 2} \times K^\times/K^{\times 2}$$

$$\psi \mapsto \psi_\alpha, \psi_\beta, \psi_\gamma$$

$$\psi(g) \in \{0, (\alpha, 0)\} \subseteq E(K) \iff g \in \mathrm{Gal}(\overline{K}/K(\sqrt{\psi_\alpha}))$$

then η is an injective homomorphism. It is an isomorphism to the subgroup of triples a, b, c s.t. $abc \in K^{\times 2}$. Proof:

$$\mathrm{Hom}_{cts}(\mathrm{Gal}(\overline{K}/K), C_2) \simeq K^\times/K^{\times 2}$$

with ψ s.t. $\ker\psi = \mathrm{Gal}(\overline{K}/K\sqrt{d}) \leftrightarrow d$. It is an isomorphism:

$$\ker\psi_i = \mathrm{Gal}(\overline{K}/K(\sqrt{d_i})),\ i = 1, 2$$

$$\ker\psi_1\psi_2 = \mathrm{Gal}(\overline{K}/K(\sqrt{d_1 d_2}))$$

Now apply this to $E(K)[2] = C_2 \times C_2$ to get an isomorphism to $K^\times/K^{\times 2} \times K^\times/K^{\times 2}$. Record this third homomorphism to get η.

5. *If $P = (x_0, y_0) \in E(K)$ then*

$$\eta(\phi_P) = (x_0 - \alpha, x_0 - \beta, x_0 - \gamma).$$

Proof sketch: If

$$E\colon y^2 = x^3 + Ax^2 + Bx$$

then for $Q = (x_0, y_0) \in E(K)$.

$$2Q = \left(\left(\frac{x_0 - B}{2y_0}\right)^2, \dots\right)$$

Hence if $2Q = P = (x_1, y_1)$ then $\sqrt{x_1} \in K(\frac{1}{2}P)$. So if

$$E\colon y^2 = (x - \alpha)(x - \beta)(x - \gamma)$$

then

$$P = (x_2, y_2)$$

then

$$\sqrt{x_2 - \alpha}, \sqrt{x_2 - \beta}, \sqrt{x_2 - \gamma} \in K(\frac{1}{2}P)$$

$$K(\sqrt{x_2 - \alpha}), K(\sqrt{x_2 - \beta}), K(\sqrt{x_2 - \gamma}) \subseteq K(\frac{1}{2}P)$$

$$\implies K(\frac{1}{2}P) = K(\sqrt{x_2 - \alpha}, \sqrt{x_2 - \beta}, \sqrt{x_2 - \gamma})$$

Example 1.10 Let

$$E\colon y^2 = x(x-1)(x+1)$$

for $P \in E(\mathbf{Q})$, $\mathbf{Q}(\frac{1}{2}P)/\mathbf{Q}$ can only ramify at 2.

$$\mathbf{Q}(\frac{1}{2}P) \subseteq \mathbf{Q}(i, \sqrt{2})$$

$$P = (x_0, y_0) \mapsto x_0, x_0 - 1, x_0 + 1 \in \mathbf{Q}^\times/\mathbf{Q}^{\times 2}$$

is a homomorphism so $x_0, x_0 - 1, x_0 + 1$ are $\pm 1, \pm 2$ up to square.

x_0	$x_0 - 1$	$x_0 + 1$	rat?
1	1	1	1) rat
1	-1	-1	2) non-rat
1	2	2	1) rat
1	-2	-2	2) non-rat
-1	1	-1	2) non-rat
-1	-1	1	1) rat
-1	2	-1	2) non-rat
-1	-2	2	1) rat
2	1	2	3) non-rat
2	-1	-2	2) non-rat
2	2	1	4) rat
2	-2	-1	2) non-rat
-2	1	-2	?
-2	-1	2	?
-2	2	-1	?
-2	-2	1	?

Table 1.11: Images

1) The 2-torsion points $P = 0, (0,0), (1,0), (-1,0) \in E(\mathbf{Q})$ give us some rows. 2) As we have $x_0 > -1$ we get $x_0 + 1 > 0$ so $x_0(x_0 - 1) > 0$ for the product to be a square (and hence > 0). 3) $x_0 = 2A^2$, $x_0 - 1 = B^2$, $x_0 + 1 = 2C^2$ with $A, B, C \in \mathbf{Q} \setminus \{0\}$. Let $A = m/n$ so $2m^2/n^2 - 1 = B^2$

$$2m^2 - n^2 = (Bn)^2$$

and

$$2m^2 + n^2 = 2(Cn)^2$$

if $m \equiv 0(2) \implies -1 = \square \pmod 8$ a contradiction.

$$m \equiv 1 \pmod 2 \implies m^2 \equiv 1 \pmod 8.$$

So $2 - n^2 = \square \pmod 8 \implies n^2 \equiv 1 \pmod 8$

$$2 + n^2 = 2\square \pmod 8 \implies n^2 \equiv 0 \pmod 8$$

$$|E(\mathbf{Q})/2E(\mathbf{Q})| = 4$$

$$|E(\mathbf{Q})[2]| = 4 \implies \text{rk} = 0$$

$$E(\mathbf{Q}) \cong E(\mathbf{Q})[2].$$

4) Use the group structure! □

Theorem 1.12 Complete 2-decent. *Let K be a field of characteristic 0 and*

$$E\colon y^2 = (x-\alpha)(x-\beta)(x-\gamma),\ \alpha,\beta,\gamma \text{ distinct.}$$

The map

$$P \mapsto (x_0 - \alpha, x_0 - \beta, x_0 - \gamma)$$

replacing $x_0 - \alpha$ with $(x_0 - \beta)(x_0 - \gamma)$ if 0.

$$E(K)/2E(K) \to (K^\times/K^{\times 2})^3$$

Triples (a,b,c) that lie in the image satisfy $abc \in K^{\times 2}$. A triple a,b,c with $abc \in K^{\times 2}$ lies in the image iff it is in the image of $E(K)[2]$ or

$$cz_3^2 - \alpha + \gamma = az_1^2$$

$$cz_3^2 - \beta + \gamma = bz_1^2$$

is soluble with $z_i \in K^\times$. In which case

$$P = (az_1^2 + \alpha, \sqrt{abc}, z_1z_2z_3) \mapsto (a,b,c)$$

iii) If K is a number field and (a,b,c) is in the image then

$$K(\sqrt{a}, \sqrt{b}, \sqrt{c})/K$$

only ramifies at primes dividing $2(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)$.

Exercise 1.13

$$E\colon y^2 = x(x-5)(x+5).$$

Lecture 5 7/2/2018

Recall:

$$\phi\colon E(K)/2E(K) \to \text{Hom}_{cts}(G_K, E(K)[2])$$

$$P \mapsto \phi_P$$

where $\phi_P\colon \sigma \mapsto Q^\sigma - Q$ where $Q = 2P$. Which is well-defined and injective.

Elements of

$$\text{Hom}_{cts}(G_K, E[2]) \leftrightarrow a,b,c \in (K^\times/K^{\times 2}) \text{ s.t. } abc \in K^{\times 2}$$

$$(x_0, y_0) \mapsto (x_0 - \alpha, x_0 - \beta, x_0 - \gamma).$$

Lemma 1.14 *Let $n \geq 1$*

1.

$$\psi\colon E(K)/nE(K) \to \{K \subseteq F \subseteq \overline{K}\}$$

$$P \mapsto K(\frac{1}{n}P, E[n])$$

is well defined.

2. *$K(\frac{1}{n}P, E[n])/K$ only ramifies at $\mathfrak{p}|n\Delta_E$.*

3. $$\mathrm{Gal}(K(\frac{1}{n}P, E[n])/K) \leq \mathbf{Z}/n \times \mathbf{Z}/n$$

4. *There are only finitely many fields satisfying 2. and 3. so* $\mathrm{im}\ \psi$ *is finite.*

To do descent, need more than ψ (i.e. injection).

Definition 1.15 Let G be a group and M a G-module then let

$$H^0(G, M) = M^G = \{m \in M : gm = m \forall g \in G\}$$

$H^1(G, M) = \{\text{skew homs } G \to M\}/\{\text{skew homs } G \to M \text{ of the form } g \mapsto g(t) - t,\ t \in M\}.$

◊

Remark 1.16 If G acts trivially on M then

$$H^0(G, M) = M$$

$$H^1(G, M) = \mathrm{Hom}(G, M).$$

When G is profinite then we want that the skew homomorphisms factor through finite Galois groups. We will prove that

$$E(K)/nE(K) \hookrightarrow H^1(G_K, E[n]).$$

Theorem 1.17 *If*

$$0 \to A \to B \to C \to 0$$

is an exact sequence of G-modules then

$$0 \to H^0(G, A) \to H^0(G, B) \to H^0(G, C) \to H^1(G, A) \to H^1(G, B) \to H^1(G, C).$$

Lemma 1.18

1. *ψ is finite-to-one (gives Mordell-Weil)*

2. *Let*
$$\phi_P \colon G_K \to E[n]$$
$$\phi_P(gh) = \phi_P(g) + g\phi_P(h)$$
is a skew (or crossed) homomorphism. If $(\frac{1}{n}P)'$ is another choice of $\frac{1}{n}P$ and ϕ'_P is the corresponding skew homomorphism, then
$$\phi_P - \phi'_P$$
is of the form
$$g \mapsto T \ominus gT$$
where $T \in E[n]$.

3. *ϕ_P factors through*
$$\mathrm{Gal}(K(\frac{1}{n}P, E[n])/K).$$

4. $$\phi \colon E(K)/nE(K) \to Z/B$$
$$P \mapsto \phi_P$$

is an injective homomorphism. Where

$$Z = \{\textit{skew homs } G_K \to E[n]\}$$

$$B = \{\textit{skew homs } G_K \to E[n] \textit{ of the form } g \mapsto T \ominus gT,\ T \in E[n]\}.$$

Proof.

1. There are finitely many skew homomorphisms

$$\mathrm{Gal}(K(\frac{1}{n}P, E[n])/K) \to E[n]$$

and by 4.

$$P \mapsto \{\phi_P, K(\frac{1}{n}P, E[n])\}$$

is injective. So $\psi\colon P \mapsto K(\frac{1}{n}P, E[n])$ is finite to one by 3.

2.

$$\phi_P(gh) = \frac{1}{n}P \ominus gh\frac{1}{n}P$$

$$= \left((\frac{1}{n}P) \ominus g(\frac{1}{n}P)\right) \oplus \left(g(\frac{1}{n}P) \ominus g(h(\frac{1}{n}P))\right)$$

$$= \phi_P \oplus g(\phi_P(h)).$$

Remark: If $E[n] \subseteq E(K)$ then ϕ_P is a homomorphism. Recall for $n = 2$

$$\phi_P(gh) = \frac{1}{2}P \ominus gh(\frac{1}{2}P)$$

$$= \frac{1}{2}P \ominus h(\frac{1}{2}P) \oplus h(\frac{1}{2}P) \ominus g(h(\frac{1}{2}P))$$

$$= \phi_P(h) \oplus \phi_P(g)$$

since $2h(\frac{1}{2}P) = h(P) = P$. Consider now

$$\frac{1}{n}P = \frac{1}{n}P' \oplus T$$

for some $T \in E[n]$

$$(\phi_P \ominus \phi'_P)(g) = \phi_P(g) - \phi'_P(g) = \frac{1}{n}P \ominus g(\frac{1}{n}P) - [(\frac{1}{n}P) \oplus T \ominus g(\frac{1}{n}P) \oplus gT]$$

$$= T \ominus gT.$$

■

Take $G = G_K$

$$B = E(\overline{K}), A = E[n], C = E(\overline{K})$$

to get

$$0 \to E[n] \to E(\overline{K}) \xrightarrow{\cdot n} E(\overline{K}) \to 0$$

which gives the long exact sequence

$$0 \to E(K)[n] \to E(K) \xrightarrow{\cdot n} E(K) \xrightarrow{\delta} H^1(G_K, E[n]) \to H^1(G_K, E(\overline{K})) \to$$

$$\implies E(K)/nE(K) \hookrightarrow H^1(G_K, E[n]).$$

Problem:

$$H^1(G_K, E[n])$$

is infinite. What subgroup of

$$H^1(G_K, E[n])$$

do we land in?

Notation: When v is a place of K we have $G_{K_v} \subseteq G_K$, for any module M have $M^{G_K} \leq M^{G_{K_v}}$ and

$$\text{Res}\colon H^1(G_K, E[n]) \to H^1(G_{K_v}, E[n]).$$

We have from the theorem

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E(K)/nE(K) & \xrightarrow{\delta} & H^1(G_K, E[n]) & \longrightarrow & H^1(G_K, E(\overline{K}))[n] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \text{Res}} & & \downarrow{\scriptstyle \text{Res}} & & \\ 0 & \longrightarrow & \prod_v E(K_v)/nE(K_v) & \xrightarrow{\delta} & \prod_v H^1(G_{K_v}, E[n]) & \longrightarrow & \prod_v H^1(G_{K_v}, E(\overline{K}))[n] & \longrightarrow & 0 \end{array}$$

we want to understand im δ i.e. the subgroup

$$\ker\{H^1(G_K, E[n]) \to H^1(G_K, E(\overline{K}))\}$$

this is as hard as finding $E(K)$, here is why:

Claim 1.19

$$H^1(G_K, E(\overline{K}))$$

corresponding to principal homogeneous spaces for E (genus 1 curves whose jacobian is E)

Finding

$$\ker\{H^1(G_K, E[n]) \to H^1(G_K, E(\overline{K}))\}$$

is equivalent to finding which PHS coming from H^1 have a rational point. ??? Hensel's lemma.

Let C be a curve

$$\text{Isom}(C) \leftrightarrow C(\overline{K}) \times \text{Aut}(C)$$

$$\tau_p \circ \alpha \leftrightarrow (P, \sigma)$$

$$\text{Twist}(E/K) \leftrightarrow H^1(G_K, \text{Isom}(C))$$

$$C \simeq_{\overline{K}} E$$

$$PHS \leftrightarrow H^1(G_K, E(\overline{K}))$$

C is a PHS for E iff E is the jacobian of C.

Lecture 6 14/2/2018

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E(K)/nE(K) & \xrightarrow{\delta} & H^1(G_K, E[n]) & \longrightarrow & H^1(G_K, E(\overline{K}))[n] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \text{Res}} & & \downarrow{\scriptstyle \text{Res}} & & \\ 0 & \longrightarrow & \prod_v E(K_v)/nE(K_v) & \xrightarrow{\delta} & \prod_v H^1(G_{K_v}, E[n]) & \longrightarrow & \prod_v H^1(G_{K_v}, E(\overline{K}))[n] & \longrightarrow & 0 \end{array}$$

Definition 1.20 Twists of curves. A **twist** of C/K is a smooth curve C'/K that is isomorphic to C over $\overline{K}$. ◊

If C_1, C_2 are twists of C/K and $C_1 \simeq_K C_2$ then we say that C_1 and C_2 are equivalent modulo K-isomorphism.

We denote Twist(C/K) - the set of twists of C/K modulo K-isomorphism.

Theorem 1.21 *The twists of C/K up to K-isomorphism are in 1-1 correspondence with elements of*

$$H^1(G_K, \mathrm{Isom}(C))$$

where

$$\mathrm{Isom}(C) = \{\overline{K}\text{-isomorphisms } C \to C\}.$$

Proof. Let C'/K be a twist of C/K then there exists an isomorphism $/\overline{K}$

$$\phi \colon C' \to C$$

associate the following map

$$\xi \colon G_K \to \mathrm{Isom}(C)$$

$$\sigma \mapsto \phi^\sigma \phi^{-1}.$$

Check that ξ is a cocycle

$$\xi_{\sigma\tau} = (\xi_\sigma)^\tau \xi_\tau$$

for all $\sigma, \tau \in G_K$. Denote $\{\xi\}$ the associated class in H^1. $\{\xi\}$ is determined by the K-isomorphism class of C' independent of the choice ϕ.

The map

$$\mathrm{Twist}(C/K) \leftrightarrow H^1(G_K, \mathrm{Isom}(C))$$

$$C' \mapsto \{\xi\}$$

is a bijection.

Injective, trace through.

Surjectivity, define the function field using the curve. ■

Remark 1.22 If C is an elliptic curve then $\mathrm{Isom}(C)$ is generated by

$$\mathrm{Aut}(C)(\text{fixing } 0)$$

and translations

$$\tau_P \colon C \to C$$

$$Q \mapsto Q + P.$$

Example 1.23 E/K elliptic, consider

$$K(\sqrt{d})$$

a quadratic extension and χ the associated character

$$\chi \colon G_K \to \{\pm 1\}$$

$$\sigma \mapsto \sigma(\sqrt{d})/\sqrt{d}.$$

The group ± 1 can be viewed as automorphisms of C. So use χ to define the cocycle

$$\xi \colon G_K \to \mathrm{Isom}(C)$$

$$\sigma \mapsto [\chi(\sigma)].$$

Let C/K be the corresponding twist of E/K, we find an equation for C/K. Choose

$$y^2 = f(x) \text{ for } E/K$$

and write

$$\overline{K}(E) = \overline{K}(x, y)$$

$$\overline{K}(C) = \overline{K}(x, y)_\xi$$

since $[-1](x, y) = (x, -y)$ the action of $\sigma \in G_K$ on

$$\overline{K}(x, y)_\xi \text{ is given by } \sqrt{d}^\sigma = \chi(\sigma)\sqrt{d}$$

$$x^\sigma = x,\ y = \chi(\sigma)y$$

note that the function $x' = x$ and $y' = y/\sqrt{d}$ are in $\overline{K}(x, y)_\xi$ and are fixed by G_K. Now x', y' satisfy

$$dy'^2 = f(x')/K$$

is defined over K and defines an elliptic curve. Moreover

$$(x, y) \mapsto (x', y'\sqrt{d})$$

is an isomorphism over $K(\sqrt{d})$. □

Note C/K is not a principal homogeneous space for E/K.

Definition 1.24 Homogenous spaces. Let E/K be an elliptic curve, a principal homogeneous space for E/K is a smooth curve C/K together with a simply transitive algebraic group action of E on C defined over K.

$$\mu : C \times E \to C$$

morphism defined over K satisfying

1.
$$\mu(P, 0) = P\ \forall P \in C$$

2.
$$\mu(\mu(p, P), Q) = \mu(p, P + Q)\ \forall P \in C$$

3.
$$\forall p, q \in C,\ \exists! P \in E \text{ s.t.}$$
$$\mu(p, P) = q$$

so we may define a subtraction map

$$\nu : C \times C \to E$$

$$p, q \mapsto P$$

as above.

◊

Proposition 1.25 *Let E/K and C/K be a principal homogeneous space for E/K. Fix a point $p_0 \in C$ and define a map*

$$\theta : E \to C$$

$$P \mapsto \underbrace{p_0 + P}_{\mu(p_0,P)}.$$

1. *θ is an isomorphism over $K(p_0)$. In particular C/K is a twist of E/K.*
2. $\forall p, q \in C$
$$q - p = \theta^{-1}(q) - \theta^{-1}(p).$$
3. *θ is a morphism over K.*

Definition 1.26 Two homogeneous space C/K and C'/K for E/K are equivalent if there is an isomorphism
$$\phi \colon C \to C'$$
defined over K and is compatible with the action of E on C and C'.

$$\begin{array}{ccc} C & \xrightarrow{\theta} & E \\ {\scriptstyle\phi}\downarrow & & \downarrow \\ C' & \longrightarrow & E' \end{array}$$

◊

The equivalence class of PHS for E/K containing E/K acting on itself via translation is called the trivial class.

The collection of equivalence classes of PHS for E/K is called the Weil-Châtelet group, denoted
$$WC(E/K).$$

Proposition 1.27 *Let C/K be a PHS for E/K then C/K is in the trivial class $\iff C(K) \neq \emptyset$.*

Theorem 1.28 *Let E/K then there is a natural bijection after fixing $p_0 \in C$*
$$WC(E/K) \to H^1(G_K, \underbrace{E(\overline{K})}_{\subseteq \mathrm{Isom}(E)})$$
$$\{C/K\} \mapsto \{\sigma \mapsto p_0^\sigma - p_0\}$$

Proof. Well-definedness:
$$\sigma \mapsto p_0^\sigma - p_0$$
is a cocycle. Suppose that C'/K and C/K are two equivalent PHS then
$$p_0^\sigma - p_0$$
and
$$p_0'^\sigma - p_0'$$
are cohomologous.

Injective, suppose that $p_0^\sigma - p_0$ and $p_0'^\sigma - p_0'$ corresponding to C/K and C'/K that are cohomologous and prove that $C \simeq_K C'$.

Surjective: let $\xi \colon G_K \to E(\overline{K})$ be a cocycle representing an element in $H^1(G_K, E)$. Embed
$$E(\overline{K}) \hookrightarrow \mathrm{Isom}(E)$$
$$P \mapsto \tau_P$$
and view
$$\xi \in H^1(G_K, \mathrm{Isom}\, E).$$

From the theorem on

$$\mathrm{Twist}(E/K) \leftrightarrow H^1(G_K, \mathrm{Isom}(E))$$

there exists a curve C/K and a $\overline{K}$-isomorphism

$$\phi\colon C \to E$$

s.t.

$$\forall \sigma \in G_K : \phi^\sigma \phi^{-1} = \text{translation by } -\xi_\sigma.$$

Define a map $\mu\colon C \times E \to C$

$$(p, Q) \mapsto \phi^{-1}(\phi(p) + Q).$$

Show that μ is simply transitive.

Show μ defined over K. Compute the cohomology class associated to C/K and show it is ξ. ■

Remark 1.29 For a given C/K of genus 1 one can define several structures of PHS.

$$\{C/K, \mu\}^\alpha = \{C/K, \mu \circ (1 \times \alpha)\}$$

$$\mu^\alpha(p, Q) = \mu(p, \alpha Q)$$

for $\alpha \in \mathrm{Aut}(E)$.

$$\begin{array}{ccc} C & \xrightarrow{\mu} & E \\ & & \downarrow{\scriptstyle P} \\ C' & \xrightarrow{\mu^\alpha} & E' \end{array}$$

Lecture 7 21/2/2018

Example 1.30 E/K and $K(\sqrt{d})/K$ a quadratic extension. Let $T \in E(K)$ be a non-trivial point of order 2. Then $\xi\colon G_K \to E$

$$\sigma \mapsto \begin{cases} 0 & \text{if } (\sqrt{d})^\sigma = \sqrt{d}, \\ T & \text{if } (\sqrt{d})^\sigma = -\sqrt{d}. \end{cases}.$$

We construct the PHS corresponding to $\{\xi\} \in H^1(G_K, E(\overline{K}))$. Since $T \in E(K)$ can choose a Weierstraß equation for E/K

$$E\colon y^2 = x^3 + ax^2 + bx \text{ with } T = (0, 0)$$

then the translation by T map is given by

$$\tau_T(P) = (x, y) + (0, 0) = \left(\frac{b}{x}, -\frac{by}{x^2}\right)$$

for

$$P = (x, y).$$

Thus if $\sigma \in G_K$ is non-trivial, σ acts on $\overline{K}(E)_\xi$, which is isomorphic to $\overline{K}(E)$ but $\mathrm{Gal}(\overline{K}/K)$ action is twisted by ξ, i.e. $x^{\mathrm{id}} \mapsto (x^{\mathrm{id}})^\sigma$.

$$(\sqrt{d})^\sigma = -\sqrt{d}$$

$$x^\sigma = \frac{b}{x},\ y^\sigma = -\frac{by}{x^2}$$

need to find the subfield of $K(\sqrt{d})(x,y)_\xi$ fixed by σ. Note:

$$\frac{\sqrt{d}x}{y},\ \sqrt{d}\left(x-\frac{b}{x}\right)$$

are invariant, take

$$z=\frac{\sqrt{d}x}{y},\ w=\sqrt{d}\left(x-\frac{b}{x}\right)\left(\frac{x}{y}\right)^2$$

and find relations between z and w to get

$$C\colon dw^2=d^2-2adz^2+(a^2-4b)z^4.$$

Claim: C/K is the PHS of E/K corresponding to $\{\xi\}$. There is a natural map

$$\phi\colon E\to C$$

$$(x,y)\mapsto(z,w)$$

$$(x,y)\mapsto\left(\frac{\sqrt{d}y}{x^2+ax+b},\frac{\sqrt{d}(x^2-b)}{x^2+ax+b}\right)$$

so that

$$\phi(0,0)=(0,-\sqrt{d})$$

$$\phi(0)=(0,\sqrt{d})$$

- Prove that ϕ is an isomorphism so C is a twist.
- C is the PHS corresponding to $\{\xi\}$. Take $p\in C$ and compute

  $$\sigma\mapsto p^\sigma-p=\phi^{-1}(p^\sigma)-\phi^{-1}(p)$$

  for example let $p=(0,\sqrt{d})\in C$, if $\sigma=\mathrm{id}$ then $p^\sigma-p=0-0=0$. If $\sigma=-\mathrm{id}$ then $p^\sigma-p=T-0=T$.

□

Back to Selmer, we want to have the image of our weak Mordell-Weil land in something finite.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E(K)/mE(K) & \xrightarrow{\delta} & H^1(G_K,E[m]) & \longrightarrow & WC(E/K)[m] & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} & & \\
0 & \longrightarrow & \prod_v E(K_v)/nE(K_v) & \xrightarrow{\delta} & \prod_v H^1(G_{K_v},E[n]) & \longrightarrow & \prod_v WC(E/K_v)[m] & \longrightarrow & 0
\end{array}$$

Definition 1.31 **m-Selmer groups.** The m-Selmer group of E/K is the subgroup of

$$H^1(G_K,E[m])$$

defined by

$$\mathrm{Sel}^m(E/K)=\ker\left\{H^1(G_K,E[m])\to\prod_v WC(E/K_v)\right\}.$$

◊

Definition 1.32 The Shafarevich-Tate group. The **Shafarevich-Tate** group of E/K is the subgroup of

$$WC(E/K)$$

defined by

$$Ш(E/K) = \ker\left\{WC(E/K) \to \prod_v WC(E/K_v)\right\}.$$

◊

Theorem 1.33 *There is an exact sequence*

1.
$$0 \to E(K)/mE(K) \to \mathrm{Sel}^m(E/K) \to Ш(E/K)[m] \to 0$$

2. $\mathrm{Sel}^m(E/K)$ *is finite.*

1.1.2 p^∞-Selmer and the structure of Ш

$H^1(G_K, E(\overline{K}))$ is torsion for general galois cohomological reasons. So

$$Ш(E/K) \subseteq H^1(G_K, E(\overline{K}))$$

is torsion.

So we may write

$$Ш(E/K) = \bigoplus_p Ш_{p^\infty}(E/K)$$

where for each prime p

$$Ш_{p^\infty}(E/K)$$

denotes the p-primary part of $Ш(E/K)$. (i.e. the subgroup of elements whose order is a power of p.) By descent

$$Ш(E/K)[m] \text{ is finite for all } m \geq 1.$$

So

$$Ш_{p^\infty}(E/K) \cong (\mathbf{Q}_p/\mathbf{Z}_p)^{\delta_p} \oplus T_p,\ \delta_p \in \mathbf{Z}_{\geq 0}$$

where T_p is a finite abelian p-group.

$$T_p \cong \mathbf{Z}/p^{s_1}\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/p^{s_l}\mathbf{Z},\ s_i \in \mathbf{Z}_{\geq 0}.$$

The group

$$\bigoplus_p (\mathbf{Q}_p/\mathbf{Z}_p)^{\delta_p} \subseteq Ш(E/K)$$

is called the infinitely divisible subgroup of Ш denoted $Ш_{div}$.

The conjecture that Ш is finite implies $\delta_p = 0$ for all p. And $T_p \neq 0$ for only finitely many p.

There is a pairing called the Cassels-Tate pairing

$$Ш(E/K) \times Ш(E/K) \to \mathbf{Q}/\mathbf{Z}$$

which is bilinear and alternating, and the kernel on either side is the infinitely divisible group. If $Ш(E/K)$ is finite then the pairing is non-degenerate and hence

$$|Ш(E/K)| = \square \in \mathbf{Z}.$$

Definition 1.34 **p^∞-Selmer group.** Consider $\mathrm{Sel}_{p^n}(E/K)$ and take the direct limit

$$\varinjlim_n \mathrm{Sel}_{p^n}(E/K)$$

to define the p^∞-Selmer group. ◊

One shows that

$$X_p(E/K) = \mathrm{Hom}_{\mathbf{Z}_p}(\varinjlim_n \mathrm{Sel}_{p^n}(E/K), \mathbf{Q}_p/\mathbf{Z}_p)$$

called the Pontyragin dual of the p^∞ Selmer group is a finitely generated $\mathbf{Z}_p$-module. The associated $\mathbf{Q}_p$-vector space, denoted $\mathcal{X}_p(E/K) = X_p(E/K) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ has dimension rk_p.

Definition 1.35 rk_p is called the p^∞-Selmer rank of E/K and satisfies

$$\mathrm{rk}_p = \mathrm{rk}(E/K) + \delta_p.$$

◊

So if Ш is finite then $\delta_p = 0$ for all p. Use BSD to compute parity of rk_p.

Lecture ? 19/3/2018

1.1.3 Consequences of BSD

Consider $E/\mathbf{Q}$: Mordell-Weil implies that

$$E(\mathbf{Q}) \simeq \mathbf{Z}^{\mathrm{rk}} \oplus \text{torsion}$$

then BSD 1 says that

$$\underbrace{\mathrm{ord}_{s=1} L(E,s)}_{\mathrm{rk}_{\mathrm{an}}} = \mathrm{rk}\,,$$

functional equation for $L(E,s)$.

$$L^*(E,s) = wL^*(E,2-s)$$

with $w \in \{\pm 1\}$ the sign of the functional equation. If $w = 1$ then $L(E,s)$ is (essentially) symmetric at $s = 1$. So $\mathrm{ord}_{s=1} L(E,s)$ is even. If $w = -1$ then $\mathrm{ord}_{s=1} L(E,s)$ is odd.

We get BSD mod 2:

$$(-1)^{\mathrm{rk}} = w(\text{sign of f.e.})$$

a conjecture based on conjecture is bad so we go one step further.

Theorem 1.36 *The sign in the functional equation of $L(E,s)$ is equal to the global root number of E.*

This is defined by

$$w_\infty \prod_p w_p,$$

the local root numbers defined in terms of the local galois representations. Non-trivial to understand, but manageable.

Conjecture 1.37 Parity conjecture.

$$(-1)^{\mathrm{rk}} = \prod_v w_v = w.$$

Example 1.38

$$E/\mathbf{Q}\colon\ y^2 + y = x^3 + x^2 - 7x + 5$$

$$\Delta_E = -7 \cdot 13$$

$w_v = 1$ if $v \nmid \infty 7 \cdot 13$

$$w_\infty = -1$$

(in general -1^g where g is dimension of the abelian variety).

$$w_7 = -1$$

$$w_{13} = -1$$

so $w = -1$ and the rank is odd, hence there is a point of infinite order on this curve. □

Problem. On the one hand $\prod_v w_v$ is computable. On the other hand $(-1)^{\mathrm{rk}}$ is precisely unknown.

$$(-1)^{\mathrm{rk}} = \prod_v w_v.$$

Theorem 1.39 *Assume* Ш *is finite, let* $\phi\colon E \to E'$ *be an isogeny whose degree is not divisible by* $\mathrm{char}(K)$*, then*

$$\frac{|\text{Ш}_E|\,\mathrm{Reg}_E \prod_p c_p \Omega_E}{|E_{\mathrm{tors}}|^2} = \frac{|\text{Ш}_{E'}|\,\mathrm{Reg}_{E'} \prod_p c'_p \Omega_{E'}}{|E'_{\mathrm{tors}}|^2}.$$

Remark 1.40 In fact this is true for all abelian varieties over K.

Example 1.41 Let

$$E/\mathbf{Q}\colon\ y^2 + xy = x^3 - x$$

http://www.lmfdb.org/EllipticCurve/Q/65/a/1. $\Delta_E = 5 \cdot 13$, it has a 2-isogenous curve E'.

Compute

$$c_5 = c_{13} = 1$$

$$c'_5 = c'_{13} = 2$$

$$\Omega_E = 2\Omega_{E'}$$

then

$$\frac{\mathrm{Reg}_{E'}}{\mathrm{Reg}_E} = \frac{|\text{Ш}_E|\,|E'_{\mathrm{tors}}|^2 \prod_p c_p \Omega_E}{|\text{Ш}_{E'}|\,|E_{\mathrm{tors}}|^2 \prod_p c'_p \Omega_{E'}} \equiv \square\frac{2}{4} \not\equiv 1\square.$$

So $\mathrm{Reg}_E \neq 1$, $\mathrm{Reg}_{E'} \neq 1$ so E has at least one rational point of infinite order, so $\mathrm{rk} \geq 1$. □

Lemma 1.42 *Assume* Ш *is finite, let*

$$\phi\colon E/K \to E'/K$$

be a K*-rational isogeny of degree* d*.*

Write $n = \mathrm{rk}_E = \mathrm{rk}_{E'}$*. Pick a basis* $\Lambda = \langle P_1, \ldots, P_n \rangle$ *for*

$$E(K)/\mathrm{tors}$$

write Λ' *for a basis of* $E'(K)/\mathrm{tors}$*. Write* $\phi^\vee\colon E' \to E$ *for the dual isogeny s.t.* $\phi\phi^\vee = [d]$*.*

using the following fact

$$\langle \phi(P), Q \rangle_{E'} = \langle P, \phi^\vee(Q) \rangle_E$$

Then

$$d^n \operatorname{Reg}_E = \det(\langle dP_i, P_j \rangle_E)_{i,j}$$
$$= \det(\langle \phi^\vee \phi P_i, P_j \rangle_E) = \det(\langle \phi P_i, \phi P_j \rangle_{E'})$$
$$= \operatorname{Reg}_{E'} [\Lambda' : \phi(\Lambda)]^2.$$

Back to the example

$$\frac{\operatorname{Reg}_E}{\operatorname{Reg}_{E'}} \equiv \frac{1}{2} \square$$

so by the lemma rk is odd. Here we assumed that Ш is finite for elliptic curves, one can drop the assumption of finiteness of Ш to get unconditional results on the parity of rk_p for all p.

Conjecture 1.43 ***p*-parity.**

$$(-1)^{\operatorname{rk}_p} = w.$$

This is known over $\mathbf{Q}$ and totally real fields.

How to compute the parity of $\operatorname{rk}_p(E/K)$? Need BSD-invariance for Selmer groups. (Details T. and V. Dokchitser "On the BSD quotients modulo squares", and Milne "Arithmetic duality theorems")

Definition 1.44 For an isogeny

$$\Psi \colon A \to B$$

of abelian varieties over K. Let

$$Q(\Psi) = |\operatorname{coker}(\Psi \colon A(K)/A(K)_{\text{tors}} \to B(K)/B(K)_{\text{tors}})| \cdot |\ker(\psi \colon Ш(A)_{\text{div}} \to Ш(B)_{\text{div}})|.$$

◊

Recall $\operatorname{rk}_p = \operatorname{rk} + \delta_p$ where

$$Ш = \bigoplus Ш_{p^\infty}$$

and

$$Ш_{p^\infty} \simeq (\mathbf{Q}_p/\mathbf{Z}_p)_p^\delta \oplus T_p$$
$$Ш_{\text{div}} = \bigoplus (\mathbf{Q}_p/\mathbf{Z}_p)^{\delta_p}.$$

Strategy, we show that for Ψ an isogeny s.t. $\Psi\Psi^\vee = [p]$. Then

$$o^{\operatorname{rk}_p(E/K)} \equiv \frac{Q(\Psi^\vee)}{Q(\Psi)} \equiv \frac{\prod_v c_p}{\prod_v c'_v} \frac{\Omega_E}{\Omega_{E'}} \pmod{K^{\times 2}}.$$

Remark 1.45 Let $A^\vee$ be the dual of A. $A^\vee = \operatorname{Pic}^0(A)$.

So

$$(-1)^{\operatorname{rk}_p(E/K)} = (-1)^{\operatorname{ord}_p\left(\frac{\prod_v c_v \Omega_E}{\prod_v c'_v \Omega_{E'}}\right)}$$

the parity of $\operatorname{rk}_p(E/K)$ is computable from local invariants of E and E'.

To prove the p-parity conjecture it remains to prove

$$(-1)^{\operatorname{ord}_p\left(\frac{\prod_v c_v \Omega_E}{\prod_v c'_v \Omega_{E'}}\right)} = \prod_v w_v.$$