

# Ranks and Parity of Ranks of Curves and Abelian Surfaces

MA842 at BU Spring 2019

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These are notes for Céline Maistret's course MA842 at BU Spring 2019.

The course webpage is [https://sites.google.com/view/cmaistret/teaching#h.p\\_BYGoPzU848FJ](https://sites.google.com/view/cmaistret/teaching#h.p_BYGoPzU848FJ).

Lecture 1 22/1/2018

Outline

1. Elliptic curves and their ranks
  - (a) Background
    - i. Mordell Weil theorem (state and prove) (ANT and cohomological proof)
    - ii. Non-effectivity
    - iii. Computing the rank (descent)
  - (b) The Birch and Swinnerton-Dyer conjecture
    - i. Heuristic via counting points on the reduced curve
    - ii.  $L$ -functions
    - iii. BSD-1
    - iv. Local arithmetic invariants and BSD-2
  - (c) Parity of ranks
    - i. Isogeny invariants of BSD 2
    - ii. Galois representations and local root numbers
    - iii. The parity conjecture
2. Abelian surfaces
  - (a) Background on genus 2 curves and their Jacobians
  - (b) BSD in this case
  - (c) Computability of local arithmetic invariants
  - (d) Parity conjecture

Evaluation, none, when not around will give exercise/project, if you come regularly and do a computation you pass.

Main references that we will be following:

1. Vladimir Dokchitser - Lecture course
2. Silverman - Arithmetic of Elliptic Curves
3. Milne - Abelian Varieties?

# 1 Elliptic curves and their ranks

Sources: Silverman I, V. Dokchitser's lectures.

## 1.1 Mordell-Weil

Let  $K$  be a number field and let  $E/K$  be an elliptic curve. The group  $E(K)$  is finitely generated.

$$E(K) \simeq E(K)_{\text{tors}} \oplus \mathbf{Z}^r.$$

Where  $E(K)_{\text{tors}}$  is a finite subgroup and  $r$  is the rank, a non-negative integer.

Assuming that we can compute the torsion subgroup, computing the rank would completely determine  $E(K)$  and hence solve the associated diophantine problem.

Plan

1. Understand the proof of Mordell-Weil
2. See where it is non-effective.
3. From the proof, extract a strategy to sometimes compute the rank (define Selmer groups, Shafarevich-Tate group).

*Outline proof of Mordell-Weil.* Part 1: Prove that

$$E(K)/mE(K)$$

is finite for some  $m \geq 2$ .

Part 2: use a descent argument with heights of points. ■

Of these two parts of the proof, part 1 is the challenging/interesting one.

For part 2: Assuming that

$$E(K)/mE(K)$$

is finite and that  $E$  has a "height function" then  $E(K)$  is finitely generated.

**Theorem 1.1 Descent theorem (see Thm. VIII 3.1).** *Let  $A$  be an abelian group, suppose that there exists a function*

$$h: A \rightarrow \mathbf{R}$$

*with the following properties:*

1. *Let  $Q \in A$  then there is a constant  $c_1$  depending on  $Q$  and  $A$  such that*

$$h(P + Q) = 2h(P) + c_1, \forall P \in A.$$

2. *There is an integer  $m \geq 2$  and a constant  $c_2$  depending on  $A$  s.t.*

$$h(mP) \geq m^2h(P) - c_2, \forall P \in A.$$

3. *For every constant  $c_3$ , the set*

$$\{P \in A : h(P) \leq c_3\}$$

*is finite.*

*suppose further that for the  $m$  in 2. we have  $A/mA$  is finite. Then  $A$  is finitely generated.*

*Proof.* Choose elements  $Q_1, \dots, Q_r \in A$  to represent the finitely many cosets in  $A/mA$ . Let  $P$  be a point in  $A$ . We show that  $P$  can be generated by  $Q_1, \dots, Q_r$  plus a set of finitely many points of bounded height.

First write

$$P = mP_1 + Q_{i_1}$$

for some  $1 \leq i \leq r$ . Repeat this for

$$P_1 = mP_2 + Q_{i_2}$$

$$P_2 = mP_3 + Q_{i_3}$$

$$\vdots$$

$$P_{n-1} = mP_n + Q_{i_n}$$

by property 2. of  $h$  we have

$$h(P_j) \leq \frac{1}{m^2}(h(mP_j) + c_2)$$

$$\frac{1}{m^2}(h(P_{j-1}) - Q_{i_j}) + c_2)$$

$$\leq \frac{1}{m^2}(2h(P_{j-1}) + c'_1 + c_2)$$

by 1. Where  $c'_1$  is the maximum of the constants from  $i$  for  $Q$  in  $\{-Q_1, \dots, -Q_r\}$ . Note that  $c'_1$  and  $c_2$  do not depend on  $P$  and that  $h(P) \geq 0$ . We repeat this inequality starting from  $P_n$  and working back to  $P$ .

$$\begin{aligned} h(P_n) &\leq \left(\frac{2}{m^2}\right)^n h(P) + \frac{1}{m^2} \left(1 + \frac{2}{m^2} + \left(\frac{2}{m^2}\right)^2 + \dots + \left(\frac{2}{m^2}\right)^{n-1}\right) (c'_1 + c_2) \\ &= \left(\frac{2}{m^2}\right)^n h(P) + \frac{1}{m^2} \left(1 + \frac{2}{m^2} + \left(\frac{2}{m^2}\right)^2 + \dots + \left(\frac{2}{m^2}\right)^{n-1}\right) (c'_1 + c_2) \\ &< \left(\frac{2}{m^2}\right)^n h(P) + \frac{c'_1 + c_2}{m^2 - 2} \\ &\leq \frac{1}{2^n} h(P) + \frac{c'_1 + c_2}{2}, \end{aligned}$$

since  $m \geq 2$ . Hence for  $n$  sufficiently large (to make  $\frac{1}{2^n} h(P) \leq 1$ ) we have

$$h(P_n) \leq 1 + \frac{1}{2}(c'_1 + c_2).$$

Since  $P$  is a linear combination of  $P_n$  and  $Q_i$

$$P = m^n P_n + \sum_{j=1}^n m^{j-1} Q_{i_j},$$

it follows that every  $P \in A$  is a linear combination of points in

$$\{Q_1, \dots, Q_r\} \cup \{Q \in A : h(Q) \leq 1 + \frac{1}{2}(c'_1 + c_2)\}.$$

■

**Remark 1.2** On  $E/\mathbb{Q}$  the height function

$$h: E(\mathbb{Q}) \rightarrow \mathbb{Q}$$

$$P \mapsto \begin{cases} \log(\max\{|p|, |q|\}), & x(P) = \frac{p}{q}, \quad P \neq 0, \\ 0, & P = 0. \end{cases}$$

satisfies the conditions of [Theorem 1.1](#).

**Remark 1.3** The above proof is effective. To find generators of  $E(\mathbb{Q})$  first compute  $c_1 = c_1(Q_i)$  for each  $i$ , then compute  $c_2$ . Find points of bounded height. Note that we need  $Q_1, \dots, Q_r$  to start with.

It remains to show part 1:

**Theorem 1.4 Weak Mordell-Weil.** *Let  $K$  be a number field  $E/K$  an elliptic curve,  $m \geq 2$  then*

$$\#E(K)/mE(K) < \infty.$$

We will prove this under the assumption that  $E[m] \subseteq E(K)$ . This is WLOG since:

**Lemma 1.5** *Let  $L/K$  be a finite Galois extension, if*

$$E(L)/mE(L)$$

*is finite then so is*

$$E(K)/mE(K).$$

*Proof.*

$$0 \rightarrow \phi \rightarrow E(K)/mE(K) \xrightarrow{\varphi} E(L)/mE(L) \rightarrow 0$$

induced by

$$E(K) \subseteq E(L),$$

and prove that  $\phi$  is finite. Kernel  $\phi$  is given by

$$\frac{E(K) \cap mE(L)}{mE(K)},$$

take  $P \in \phi$ . We can choose  $Q_P \in E(L)$  such that  $Q_P = P$ . Define a map of sets

$$\lambda_P: G_{L/K} \rightarrow E[m]$$

$$\sigma \mapsto Q_P^\sigma - Q_P.$$

Note that

$$[m](Q_P^\sigma - Q_P) = ([m]Q_P)^\sigma - [m]Q_P = 0.$$

Now we show that the association

$$\phi \rightarrow \text{Map}(G_{L/K}, E[m])$$

$$P \mapsto \lambda_P$$

is 1 to 1.

Suppose that  $P, P' \in E(K) \cap mE(L)$  satisfying  $\lambda_P = \lambda_{P'}$  then

$$(Q_P - Q_{P'})^\sigma = Q_P - Q_{P'}$$

for all  $\sigma \in G_{L/K}$  so  $Q_P - Q_{P'} \in E(K)$  and hence

$$P - P' = [m]Q_P - [m]Q_{P'} \in mE(K)$$

hence

$$P = P' \pmod{mE(K)}.$$

$G_{L/K}$  and  $E[m]$  are both finite, hence so is  $\phi$ . ■

Lecture 2 29/1/2018

Now we will prove the weak Mordell-Weil theorem. Using the above lemma we can reduce to the case where  $E[m] \subseteq E(K)$ , so we assume this going forwards.

**Definition 1.6 The Kummer pairing.** The Kummer pairing is

$$\kappa: E(K) \times G_{\bar{K}/K} \rightarrow E[m]$$

$$P, \sigma \mapsto Q^\sigma - Q$$

where  $Q$  is a choice of point in  $E(\bar{K})$  such that  $mQ = P$ .  $\diamond$

**Proposition 1.7**  $\kappa$  is well defined, bilinear, the kernel in the first argument is  $mE(K)$  and in the second argument is  $G_{\bar{K}/L}$  where  $L = K([m]^{-1}E(K))$  is the compositum of all fields  $\kappa(x(Q), y(Q))$  as  $Q$  ranges over all the points of  $E(\bar{K})$  s.t.  $mQ \in E(K)$ .

Hence the Kummer pairing induces a perfect bilinear pairing

$$E(K)/mE(K) \times G_{L/K} \rightarrow E[m]$$

i.e. the map

$$E(K)/mE(K) \rightarrow \text{Hom}_K(G_{L/K}, E[m])$$

$$P \mapsto (\sigma \mapsto Q^\sigma - Q)$$

is an isomorphism.

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**Remark 1.8** A homomorphism  $\phi: \text{Gal}(\bar{K}/K) \rightarrow G$  for a finite group  $G$  is continuous if it comes from a finite Galois extension, i.e.

$$\exists F/K \text{ finite Galois, } \tilde{\phi}: \text{Gal}(F/K) \rightarrow G$$

s.t.  $\phi$  is the composition  $\text{Gal}(\bar{K}/K) \rightarrow \text{Gal}(F/K) \xrightarrow{\tilde{\phi}} G$ . So  $\phi(g)$  only cares about what  $g$  does to  $F$ .

**Proposition 1.9** Let  $E/K$  be an elliptic curve

$$y^2 = (x - \alpha)(x - \beta)(x - \gamma)$$

for  $P \in E(K)$  have  $\frac{1}{2}P \in E(\bar{K})$  s.t.  $\frac{1}{2}P \oplus \frac{1}{2}P = P$ .

1.  $K(\frac{1}{2}P)/K$  is a Galois extension and  $\text{Gal}(K(\frac{1}{2}P)/K) = C_2 \times C_2$  from Lemma 1.

2.

$$\phi_P: \text{Gal}(\bar{K}/K) \rightarrow E(K)[2]$$

$$g \mapsto Q^\sigma - Q = g(\frac{1}{2}P) - \frac{1}{2}P$$

is well defined and has kernel  $\text{Gal}(K/K(\frac{1}{2}P))$ .

3.

$$\phi: E(K)/2E(K) \rightarrow \text{Hom}_{cts}(\text{Gal}(\bar{K}/K), E(K)[2])$$

$$P \mapsto \phi_P$$

is well defined and injective. Now  $\phi_P$  is continuous by 2. and so

$$\begin{aligned}\phi_{P \oplus Q}(g) &= g\left(\frac{1}{2}(P \oplus Q)\right) - \left(\frac{1}{2}P \oplus \frac{1}{2}Q\right) \\ &= g\left(\frac{1}{2}P\right) \oplus g\left(\frac{1}{2}Q\right) - \frac{1}{2}P \oplus \frac{1}{2}Q \\ &= \phi_P(g) \oplus \phi_Q(g)\end{aligned}$$

a homomorphism.

$$\phi_{2Q}(g) = g\left(\frac{1}{2}2Q\right) - \frac{1}{2}2(Q) = g(Q) - Q = 0$$

for all  $g \in \text{Gal}(\bar{K}/K)$  if  $Q \in E(K)$  so this is well defined. For injectivity:

$$\begin{aligned}\phi_P(g) = 0 &\implies g\left(\frac{1}{2}P\right) = \frac{1}{2}P \forall g \in \text{Gal}(\bar{K}/K) \\ &\implies \frac{1}{2}P \in E(K) \implies P \in 2E(K)\end{aligned}$$

which gives injectivity.

4.

$$\eta: \text{Hom}_{cts}(\text{Gal}(\bar{K}/K), E(K)[2]) \rightarrow K^\times/K^{\times 2} \times K^\times/K^{\times 2} \times K^\times/K^{\times 2}$$

$$\psi \mapsto \psi_\alpha, \psi_\beta, \psi_\gamma$$

$$\psi(g) \in \{0, (\alpha, 0)\} \subseteq E(K) \iff g \in \text{Gal}(\bar{K}/K(\sqrt{\psi_\alpha}))$$

then  $\eta$  is an injective homomorphism. It is an isomorphism to the subgroup of triples  $a, b, c$  s.t.  $abc \in K^{\times 2}$ . Proof:

$$\text{Hom}_{cts}(\text{Gal}(\bar{K}/K), C_2) \simeq K^\times/K^{\times 2}$$

with  $\psi$  s.t.  $\ker \psi = \text{Gal}(\bar{K}/K\sqrt{d}) \leftrightarrow d$ . It is an isomorphism:

$$\ker \psi_i = \text{Gal}(\bar{K}/K(\sqrt{d_i})), i = 1, 2$$

$$\ker \psi_1 \psi_2 = \text{Gal}(\bar{K}/K(\sqrt{d_1 d_2}))$$

Now apply this to  $E(K)[2] = C_2 \times C_2$  to get an isomorphism to  $K^\times/K^{\times 2} \times K^\times/K^{\times 2}$ . Record this third homomorphism to get  $\eta$ .

5. If  $P = (x_0, y_0) \in E(K)$  then

$$\eta(\phi_P) = (x_0 - \alpha, x_0 - \beta, x_0 - \gamma).$$

Proof sketch: If

$$E: y^2 = x^3 + Ax^2 + Bx$$

then for  $Q = (x_0, y_0) \in E(K)$ .

$$2Q = \left( \left( \frac{x_0 - B}{2y_0} \right)^2, \dots \right)$$

Hence if  $2Q = P = (x_1, y_1)$  then  $\sqrt{x_1} \in K(\frac{1}{2}P)$ . So if

$$E: y^2 = (x - \alpha)(x - \beta)(x - \gamma)$$

then

$$P = (x_2, y_2)$$

then

$$\begin{aligned} \sqrt{x_2 - \alpha}, \sqrt{x_2 - \beta}, \sqrt{x_2 - \gamma} &\in K(\tfrac{1}{2}P) \\ K(\sqrt{x_2 - \alpha}), K(\sqrt{x_2 - \beta}), K(\sqrt{x_2 - \gamma}) &\subseteq K(\tfrac{1}{2}P) \\ \implies K(\tfrac{1}{2}P) &= K(\sqrt{x_2 - \alpha}, \sqrt{x_2 - \beta}, \sqrt{x_2 - \gamma}) \end{aligned}$$

**Example 1.10** Let

$$E: y^2 = x(x-1)(x+1)$$

for  $P \in E(\mathbf{Q})$ ,  $\mathbf{Q}(\frac{1}{2}P)/\mathbf{Q}$  can only ramify at 2.

$$\mathbf{Q}(\tfrac{1}{2}P) \subseteq \mathbf{Q}(i, \sqrt{2})$$

$$P = (x_0, y_0) \mapsto x_0, x_0 - 1, x_0 + 1 \in \mathbf{Q}^\times / \mathbf{Q}^{\times 2}$$

is a homomorphism so  $x_0, x_0 - 1, x_0 + 1$  are  $\pm 1, \pm 2$  up to square.

$x_0$	$x_0 - 1$	$x_0 + 1$	rat?
1	1	1	1) rat
1	-1	-1	2) non-rat
1	2	2	1) rat
1	-2	-2	2) non-rat
-1	1	-1	2) non-rat
-1	-1	1	1) rat
-1	2	-1	2) non-rat
-1	-2	2	1) rat
2	1	2	3) non-rat
2	-1	-2	2) non-rat
2	2	1	4) rat
2	-2	-1	2) non-rat
-2	1	-2	?
-2	-1	2	?
-2	2	-1	?
-2	-2	1	?

**Table 1.11:** Images

1) The 2-torsion points  $P = 0, (0, 0), (1, 0), (-1, 0) \in E(\mathbf{Q})$  give us some rows.  
2) As we have  $x_0 > -1$  we get  $x_0 + 1 > 0$  so  $x_0(x_0 - 1) > 0$  for the product to be a square (and hence  $> 0$ ). 3)  $x_0 = 2A^2$ ,  $x_0 - 1 = B^2$ ,  $x_0 + 1 = 2C^2$  with  $A, B, C \in \mathbf{Q} \setminus \{0\}$ . Let  $A = m/n$  so  $2m^2/n^2 - 1 = B^2$

$$2m^2 - n^2 = (Bn)^2$$

and

$$2m^2 + n^2 = 2(Cn)^2$$

if  $m \equiv 0(2) \implies -1 \equiv \square \pmod{8}$  a contradiction.

$$m \equiv 1 \pmod{2} \implies m^2 \equiv 1 \pmod{8}.$$

$$\text{So } 2 - n^2 = \square \pmod{8} \implies n^2 \equiv 1 \pmod{8}$$

$$2 + n^2 = 2\square \pmod{8} \implies n^2 \equiv 0 \pmod{8}$$

$$|E(\mathbf{Q})/2E(\mathbf{Q})| = 4$$

$$|E(\mathbf{Q})[2]| = 4 \implies \text{rk} = 0$$

$$E(\mathbf{Q}) \cong E(\mathbf{Q})[2].$$

4) Use the group structure! □

**Theorem 1.12 Complete 2-decent.** *Let  $K$  be a field of characteristic 0 and*

$$E: y^2 = (x - \alpha)(x - \beta)(x - \gamma), \alpha, \beta, \gamma \text{ distinct.}$$

*The map*

$$P \mapsto (x_0 - \alpha, x_0 - \beta, x_0 - \gamma)$$

*replacing  $x_0 - \alpha$  with  $(x_0 - \beta)(x_0 - \gamma)$  if 0.*

$$E(K)/2E(K) \rightarrow (K^\times/K^{\times 2})^3$$

*Triples  $(a, b, c)$  that lie in the image satisfy  $abc \in K^{\times 2}$ . A triple  $a, b, c$  with  $abc \in K^{\times 2}$  lies in the image iff it is in the image of  $E(K)[2]$  or*

$$cz_3^2 - \alpha + \gamma = az_1^2$$

$$cz_3^2 - \beta + \gamma = bz_1^2$$

*is soluble with  $z_i \in K^\times$ . In which case*

$$P = (az_1^2 + \alpha, \sqrt{abc}, z_1z_2z_3) \mapsto (a, b, c)$$

*iii) If  $K$  is a number field and  $(a, b, c)$  is in the image then*

$$K(\sqrt{a}, \sqrt{b}, \sqrt{c})/K$$

*only ramifies at primes dividing  $2(\alpha - \beta)(\alpha - \gamma)(\beta - \gamma)$ .*

**Exercise 1.13**

$$E: y^2 = x(x - 5)(x + 5).$$

Lecture 5 7/2/2018

Recall:

$$\phi: E(K)/2E(K) \rightarrow \text{Hom}_{cts}(G_K, E(K)[2])$$

$$P \mapsto \phi_P$$

where  $\phi_P: \sigma \mapsto Q^\sigma - Q$  where  $Q = 2P$ . Which is well-defined and injective.

Elements of

$$\text{Hom}_{cts}(G_K, E[2]) \leftrightarrow a, b, c \in (K^\times/K^{\times 2}) \text{ s.t. } abc \in K^{\times 2}$$

$$(x_0, y_0) \mapsto (x_0 - \alpha, x_0 - \beta, x_0 - \gamma).$$

**Lemma 1.14** *Let  $n \geq 1$*

*1.*

$$\psi: E(K)/nE(K) \rightarrow \{K \subseteq F \subseteq \bar{K}\}$$

$$P \mapsto K(\frac{1}{n}P, E[n])$$



is well defined.

2.  $K(\frac{1}{n}P, E[n])/K$  only ramifies at  $\mathfrak{p} | n\Delta_E$ .

3.

$$\text{Gal}(K(\frac{1}{n}P, E[n])/K) \leq \mathbf{Z}/n \times \mathbf{Z}/n$$

4. There are only finitely many fields satisfying 2. and 3. so  $\text{im } \psi$  is finite.

To do descent, need more than  $\psi$  (i.e. injection).

**Definition 1.15** Let  $G$  be a group and  $M$  a  $G$ -module then let

$$H^0(G, M) = M^G = \{m \in M : gm = m \forall g \in G\}$$

$$H^1(G, M) = \{\text{skew homs } G \rightarrow M\} / \{\text{skew homs } G \rightarrow M \text{ of the form } g \mapsto g(t) - t, t \in M\}.$$

◇

**Remark 1.16** If  $G$  acts trivially on  $M$  then

$$H^0(G, M) = M$$

$$H^1(G, M) = \text{Hom}(G, M).$$

When  $G$  is profinite then we want that the skew homomorphisms factor through finite Galois groups. We will prove that

$$E(K)/nE(K) \hookrightarrow H^1(G_K, E[n]).$$

**Theorem 1.17** If

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is an exact sequence of  $G$ -modules then

$$0 \rightarrow H^0(G, A) \rightarrow H^0(G, B) \rightarrow H^0(G, C) \rightarrow H^1(G, A) \rightarrow H^1(G, B) \rightarrow H^1(G, C).$$

**Lemma 1.18**

1.  $\psi$  is finite-to-one (gives Mordell-Weil)

2. Let

$$\phi_P : G_K \rightarrow E[n]$$

$$\phi_P(gh) = \phi_P(g) + g\phi_P(h)$$

is a skew (or crossed) homomorphism. If  $(\frac{1}{n}P)'$  is another choice of  $\frac{1}{n}P$  and  $\phi'_P$  is the corresponding skew homomorphism, then

$$\phi_P - \phi'_P$$

is of the form

$$g \mapsto T \ominus gT$$

where  $T \in E[n]$ .

3.  $\phi_P$  factors through

$$\text{Gal}(K(\frac{1}{n}P, E[n])/K).$$

4.

$$\phi : E(K)/nE(K) \rightarrow \mathbf{Z}/B$$

$$P \mapsto \phi_P$$

is an injective homomorphism. Where

$$Z = \{\text{skew homs } G_K \rightarrow E[n]\}$$

$$B = \{\text{skew homs } G_K \rightarrow E[n] \text{ of the form } g \mapsto T \ominus gT, T \in E[n]\}.$$

*Proof.*

1. There are finitely many skew homomorphisms

$$\text{Gal}(K(\frac{1}{n}P, E[n])/K) \rightarrow E[n]$$

and by 4.

$$P \mapsto \{\phi_P, K(\frac{1}{n}P, E[n])\}$$

is injective. So  $\psi: P \mapsto K(\frac{1}{n}P, E[n])$  is finite to one by 3.

- 2.

$$\begin{aligned} \phi_P(gh) &= \frac{1}{n}P \ominus gh \frac{1}{n}P \\ &= \left( \left( \frac{1}{n}P \right) \ominus g \left( \frac{1}{n}P \right) \right) \oplus \left( g \left( \frac{1}{n}P \right) \ominus g \left( h \left( \frac{1}{n}P \right) \right) \right) \\ &= \phi_P \oplus g(\phi_P(h)). \end{aligned}$$

Remark: If  $E[n] \subseteq E(K)$  then  $\phi_P$  is a homomorphism. Recall for  $n = 2$

$$\begin{aligned} \phi_P(gh) &= \frac{1}{2}P \ominus gh \left( \frac{1}{2}P \right) \\ &= \frac{1}{2}P \ominus h \left( \frac{1}{2}P \right) \oplus h \left( \frac{1}{2}P \right) \ominus g \left( h \left( \frac{1}{2}P \right) \right) \\ &= \phi_P(h) \oplus \phi_P(g) \end{aligned}$$

since  $2h(\frac{1}{2}P) = h(P) = P$ . Consider now

$$\frac{1}{n}P = \frac{1}{n}P' \oplus T$$

for some  $T \in E[n]$

$$\begin{aligned} (\phi_P \ominus \phi'_P)(g) &= \phi_P(g) - \phi'_P(g) = \frac{1}{n}P \ominus g \left( \frac{1}{n}P \right) - \left[ \left( \frac{1}{n}P \right) \oplus T \ominus g \left( \frac{1}{n}P \right) \oplus gT \right] \\ &= T \ominus gT. \end{aligned}$$

■

Take  $G = G_K$

$$B = E(\bar{K}), A = E[n], C = E(\bar{K})$$

to get

$$0 \rightarrow E[n] \rightarrow E(\bar{K}) \xrightarrow{\cdot n} E(\bar{K}) \rightarrow 0$$

which gives the long exact sequence

$$\begin{aligned} 0 \rightarrow E(K)[n] \rightarrow E(K) \xrightarrow{\cdot n} E(K) \xrightarrow{\delta} H^1(G_K, E[n]) \rightarrow H^1(G_K, E(\bar{K})) \rightarrow \\ \implies E(K)/nE(K) \hookrightarrow H^1(G_K, E[n]). \end{aligned}$$

Problem:

$$H^1(G_K, E[n])$$

is infinite. What subgroup of

$$H^1(G_K, E[n])$$

do we land in?

Notation: When  $v$  is a place of  $K$  we have  $G_{K_v} \subseteq G_K$ , for any module  $M$  have  $M^{G_K} \leq M^{G_{K_v}}$  and

$$\text{Res}: H^1(G_K, E[n]) \rightarrow H^1(G_{K_v}, E[n]).$$

We have from the theorem

$$\begin{array}{ccccccc} 0 & \longrightarrow & E(K)/nE(K) & \xrightarrow{\delta} & H^1(G_K, E[n]) & \longrightarrow & H^1(G_K, E(\bar{K}))[n] \longrightarrow 0 \\ & & \downarrow & & \downarrow \text{Res} & & \downarrow \text{Res} \\ 0 & \longrightarrow & \prod_v E(K_v)/nE(K_v) & \xrightarrow{\delta} & \prod_v H^1(G_{K_v}, E[n]) & \longrightarrow & \prod_v H^1(G_{K_v}, E(\bar{K}))[n] \longrightarrow 0 \end{array}$$

we want to understand  $\text{im } \delta$  i.e. the subgroup

$$\ker\{H^1(G_K, E[n]) \rightarrow H^1(G_K, E(\bar{K}))\}$$

this is as hard as finding  $E(K)$ , here is why:

**Claim 1.19**

$$H^1(G_K, E(\bar{K}))$$

corresponding to principal homogeneous spaces for  $E$  (genus 1 curves whose jacobian is  $E$ )

Finding

$$\ker\{H^1(G_K, E[n]) \rightarrow H^1(G_K, E(\bar{K}))\}$$

is equivalent to finding which PHS coming from  $H^1$  have a rational point. ???  
Hensel's lemma.

Let  $C$  be a curve

$$\text{Isom}(C) \leftrightarrow C(\bar{K}) \times \text{Aut}(C)$$

$$\tau_p \circ \alpha \leftrightarrow (P, \sigma)$$

$$\text{Twist}(E/K) \leftrightarrow H^1(G_K, \text{Isom}(C))$$

$$C \simeq_{\bar{K}} E$$

$$\text{PHS} \leftrightarrow H^1(G_K, E(\bar{K}))$$

$C$  is a PHS for  $E$  iff  $E$  is the jacobian of  $C$ .

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$$\begin{array}{ccccccc} 0 & \longrightarrow & E(K)/nE(K) & \xrightarrow{\delta} & H^1(G_K, E[n]) & \longrightarrow & H^1(G_K, E(\bar{K}))[n] \longrightarrow 0 \\ & & \downarrow & & \downarrow \text{Res} & & \downarrow \text{Res} \\ 0 & \longrightarrow & \prod_v E(K_v)/nE(K_v) & \xrightarrow{\delta} & \prod_v H^1(G_{K_v}, E[n]) & \longrightarrow & \prod_v H^1(G_{K_v}, E(\bar{K}))[n] \longrightarrow 0 \end{array}$$

**Definition 1.20 Twists of curves.** A **twist** of  $C/K$  is a smooth curve  $C'/K$  that is isomorphic to  $C$  over  $\bar{K}$ .  $\diamond$

If  $C_1, C_2$  are twists of  $C/K$  and  $C_1 \simeq_K C_2$  then we say that  $C_1$  and  $C_2$  are equivalent modulo  $K$ -isomorphism.

We denote  $\text{Twist}(C/K)$  - the set of twists of  $C/K$  modulo  $K$ -isomorphism.

**Theorem 1.21** *The twists of  $C/K$  up to  $K$ -isomorphism are in 1-1 correspondence with elements of*

$$H^1(G_K, \text{Isom}(C))$$

where

$$\text{Isom}(C) = \{\bar{K}\text{-isomorphisms } C \rightarrow C\}.$$

*Proof.* Let  $C'/K$  be a twist of  $C/K$  then there exists an isomorphism  $\phi: C' \rightarrow C$  over  $\bar{K}$ .

$$\phi: C' \rightarrow C$$

associate the following map

$$\xi: G_K \rightarrow \text{Isom}(C)$$

$$\sigma \mapsto \phi^\sigma \phi^{-1}.$$

Check that  $\xi$  is a cocycle

$$\xi_{\sigma\tau} = (\xi_\sigma)^\tau \xi_\tau$$

for all  $\sigma, \tau \in G_K$ . Denote  $\{\xi\}$  the associated class in  $H^1$ .  $\{\xi\}$  is determined by the  $K$ -isomorphism class of  $C'$  independent of the choice  $\phi$ .

The map

$$\text{Twist}(C/K) \leftrightarrow H^1(G_K, \text{Isom}(C))$$

$$C' \mapsto \{\xi\}$$

is a bijection.

Injective, trace through.

Surjectivity, define the function field using the curve.  $\blacksquare$

**Remark 1.22** If  $C$  is an elliptic curve then  $\text{Isom}(C)$  is generated by

$$\text{Aut}(C)(\text{fixing } 0)$$

and translations

$$\tau_P: C \rightarrow C$$

$$Q \mapsto Q + P.$$

**Example 1.23**  $E/K$  elliptic, consider

$$K(\sqrt{d})$$

a quadratic extension and  $\chi$  the associated character

$$\chi: G_K \rightarrow \{\pm 1\}$$

$$\sigma \mapsto \sigma(\sqrt{d})/\sqrt{d}.$$

The group  $\pm 1$  can be viewed as automorphisms of  $C$ . So use  $\chi$  to define the cocycle

$$\xi: G_K \rightarrow \text{Isom}(C)$$

$$\sigma \mapsto [\chi(\sigma)].$$

Let  $C/K$  be the corresponding twist of  $E/K$ , we find an equation for  $C/K$ . Choose

$$y^2 = f(x) \text{ for } E/K$$

and write

$$\bar{K}(E) = \bar{K}(x, y)$$

$$\bar{K}(C) = \bar{K}(x, y)_\xi$$

since  $[-1](x, y) = (x, -y)$  the action of  $\sigma \in G_K$  on

$$\bar{K}(x, y)_\xi \text{ is given by } \sqrt{d}^\sigma = \chi(\sigma)\sqrt{d}$$

$$x^\sigma = x, y = \chi(\sigma)y$$

note that the function  $x' = x$  and  $y' = y/\sqrt{d}$  are in  $\bar{K}(x, y)_\xi$  and are fixed by  $G_K$ . Now  $x', y'$  satisfy

$$dy'^2 = f(x')/K$$

is defined over  $K$  and defines an elliptic curve. Moreover

$$(x, y) \mapsto (x', y'\sqrt{d})$$

is an isomorphism over  $K(\sqrt{d})$ . □

Note  $C/K$  is not a principal homogeneous space for  $E/K$ .

**Definition 1.24 Homogenous spaces.** Let  $E/K$  be an elliptic curve, a principal homogeneous space for  $E/K$  is a smooth curve  $C/K$  together with a simply transitive algebraic group action of  $E$  on  $C$  defined over  $K$ .

$$\mu: C \times E \rightarrow C$$

morphism defined over  $K$  satisfying

1.

$$\mu(P, 0) = P \forall P \in C$$

2.

$$\mu(\mu(p, P), Q) = \mu(p, P + Q) \forall P \in C$$

3.

$$\forall p, q \in C, \exists! P \in E \text{ s.t.}$$

$$\mu(p, P) = q$$

so we may define a subtraction map

$$\nu: C \times C \rightarrow E$$

$$p, q \mapsto P$$

as above. ◇

**Proposition 1.25** Let  $E/K$  and  $C/K$  be a principal homogeneous space for  $E/K$ . Fix a point  $p_0 \in C$  and define a map

$$\theta: E \rightarrow C$$

$$P \mapsto p_0 + \underbrace{P}_{\mu(p_0, P)}.$$

1.  $\theta$  is an isomorphism over  $K(p_0)$ . In particular  $C/K$  is a twist of  $E/K$ .

2.  $\forall p, q \in C$

$$q - p = \theta^{-1}(q) - \theta^{-1}(p).$$

3.  $\theta$  is a morphism over  $K$ .

**Definition 1.26** Two homogeneous space  $C/K$  and  $C'/K$  for  $E/K$  are equivalent if there is an isomorphism

$$\phi: C \rightarrow C'$$

defined over  $K$  and is compatible with the action of  $E$  on  $C$  and  $C'$ .

$$\begin{array}{ccc} C & \xrightarrow{\theta} & E \\ \phi \downarrow & & \downarrow \\ C' & \longrightarrow & E' \end{array}$$

◇

The equivalence class of PHS for  $E/K$  containing  $E/K$  acting on itself via translation is called the trivial class.

The collection of equivalence classes of PHS for  $E/K$  is called the Weil-Châtelet group, denoted

$$WC(E/K).$$

**Proposition 1.27** Let  $C/K$  be a PHS for  $E/K$  then  $C/K$  is in the trivial class  $\iff C(K) \neq \emptyset$ .

**Theorem 1.28** Let  $E/K$  then there is a natural bijection after fixing  $p_0 \in C$

$$WC(E/K) \rightarrow H^1(G_K, \underbrace{E(\bar{K})}_{\subseteq \text{Isom}(E)})$$

$$\{C/K\} \mapsto \{\sigma \mapsto p_0^\sigma - p_0\}$$

*Proof.* Well-definedness:

$$\sigma \mapsto p_0^\sigma - p_0$$

is a cocycle. Suppose that  $C'/K$  and  $C/K$  are two equivalent PHS then

$$p_0^\sigma - p_0$$

and

$$p_0'^\sigma - p_0'$$

are cohomologous.

Injective, suppose that  $p_0^\sigma - p_0$  and  $p_0'^\sigma - p_0'$  corresponding to  $C/K$  and  $C'/K$  that are cohomologous and prove that  $C \simeq_K C'$ .

Surjective: let  $\xi: G_K \rightarrow E(\bar{K})$  be a cocycle representing an element in  $H^1(G_K, E)$ . Embed

$$E(\bar{K}) \hookrightarrow \text{Isom}(E)$$

$$P \mapsto \tau_P$$

and view

$$\xi \in H^1(G_K, \text{Isom } E).$$

From the theorem on

$$\text{Twist}(E/K) \leftrightarrow H^1(G_K, \text{Isom}(E))$$

there exists a curve  $C/K$  and a  $\bar{K}$ -isomorphism

$$\phi: C \rightarrow E$$

s.t.

$$\forall \sigma \in G_K : \phi^\sigma \phi^{-1} = \text{translation by } -\xi_\sigma.$$

Define a map  $\mu: C \times E \rightarrow C$

$$(p, Q) \mapsto \phi^{-1}(\phi(p) + Q).$$

Show that  $\mu$  is simply transitive.

Show  $\mu$  defined over  $K$ . Compute the cohomology class associated to  $C/K$  and show it is  $\xi$ . ■

**Remark 1.29** For a given  $C/K$  of genus 1 one can define several structures of PHS.

$$\{C/K, \mu\}^\alpha = \{C/K, \mu \circ (1 \times \alpha)\}$$

$$\mu^\alpha(p, Q) = \mu(p, \alpha Q)$$

for  $\alpha \in \text{Aut}(E)$ .

$$\begin{array}{ccc} C & \xrightarrow{\mu} & E \\ & & \downarrow P \\ C' & \xrightarrow{\mu^\alpha} & E' \end{array}$$

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**Example 1.30**  $E/K$  and  $K(\sqrt{d})/K$  a quadratic extension. Let  $T \in E(K)$  be a non-trivial point of order 2. Then  $\xi: G_K \rightarrow E$

$$\sigma \mapsto \begin{cases} 0 & \text{if } (\sqrt{d})^\sigma = \sqrt{d}, \\ T & \text{if } (\sqrt{d})^\sigma = -\sqrt{d}. \end{cases}$$

We construct the PHS corresponding to  $\{\xi\} \in H^1(G_K, E(\bar{K}))$ . Since  $T \in E(K)$  can choose a Weierstraß equation for  $E/K$

$$E: y^2 = x^3 + ax^2 + bx \text{ with } T = (0, 0)$$

then the translation by  $T$  map is given by

$$\tau_T(P) = (x, y) + (0, 0) = \left( \frac{b}{x}, -\frac{by}{x^2} \right)$$

for

$$P = (x, y).$$

Thus if  $\sigma \in G_K$  is non-trivial,  $\sigma$  acts on  $\bar{K}(E)_\xi$ , which is isomorphic to  $\bar{K}(E)$  but  $\text{Gal}(\bar{K}/K)$  action is twisted by  $\xi$ , i.e.  $x^{\text{id}} \mapsto (x^{\text{id}})^\sigma$ .

$$(\sqrt{d})^\sigma = -\sqrt{d}$$

$$x^\sigma = \frac{b}{x}, y^\sigma = -\frac{by}{x^2}$$

need to find the subfield of  $K(\sqrt{d})(x, y)_\xi$  fixed by  $\sigma$ . Note:

$$\frac{\sqrt{d}x}{y}, \sqrt{d}\left(x - \frac{b}{x}\right)$$

are invariant, take

$$z = \frac{\sqrt{d}x}{y}, w = \sqrt{d}\left(x - \frac{b}{x}\right)\left(\frac{x}{y}\right)^2$$

and find relations between  $z$  and  $w$  to get

$$C: dw^2 = d^2 - 2adz^2 + (a^2 - 4b)z^4.$$

Claim:  $C/K$  is the PHS of  $E/K$  corresponding to  $\{\xi\}$ . There is a natural map

$$\phi: E \rightarrow C$$

$$(x, y) \mapsto (z, w)$$

$$(x, y) \mapsto \left( \frac{\sqrt{d}y}{x^2 + ax + b}, \frac{\sqrt{d}(x^2 - b)}{x^2 + ax + b} \right)$$

so that

$$\phi(0, 0) = (0, -\sqrt{d})$$

$$\phi(0) = (0, \sqrt{d})$$

- Prove that  $\phi$  is an isomorphism so  $C$  is a twist.
- $C$  is the PHS corresponding to  $\{\xi\}$ . Take  $p \in C$  and compute

$$\sigma \mapsto p^\sigma - p = \phi^{-1}(p^\sigma) - \phi^{-1}(p)$$

for example let  $p = (0, \sqrt{d}) \in C$ , if  $\sigma = \text{id}$  then  $p^\sigma - p = 0 - 0 = 0$ . If  $\sigma = -\text{id}$  then  $p^\sigma - p = T - 0 = T$ .

□

Back to Selmer, we want to have the image of our weak Mordell-Weil land in something finite.

$$\begin{array}{ccccccc} 0 & \longrightarrow & E(K)/mE(K) & \xrightarrow{\delta} & H^1(G_K, E[m]) & \longrightarrow & WC(E/K)[m] \longrightarrow 0 \\ & & \downarrow & & \downarrow \text{Res} & & \downarrow \text{Res} \\ 0 & \longrightarrow & \prod_v E(K_v)/nE(K_v) & \xrightarrow{\delta} & \prod_v H^1(G_{K_v}, E[n]) & \longrightarrow & \prod_v WC(E/K_v)[m] \longrightarrow 0 \end{array}$$

**Definition 1.31** *m-Selmer groups.* The  $m$ -Selmer group of  $E/K$  is the subgroup of

$$H^1(G_K, E[m])$$

defined by

$$\text{Sel}^m(E/K) = \ker \left\{ H^1(G_K, E[m]) \rightarrow \prod_v WC(E/K_v) \right\}.$$

◇



**Definition 1.32 The Shafarevich-Tate group.** The **Shafarevich-Tate** group of  $E/K$  is the subgroup of

$$WC(E/K)$$

defined by

$$\text{III}(E/K) = \ker \left\{ WC(E/K) \rightarrow \prod_v WC(E/K_v) \right\}.$$

◇

**Theorem 1.33** *There is an exact sequence*

1.

$$0 \rightarrow E(K)/mE(K) \rightarrow \text{Sel}^m(E/K) \rightarrow \text{III}(E/K)[m] \rightarrow 0$$

2.  $\text{Sel}^m(E/K)$  is finite.

## 1.2 $p^\infty$ -Selmer and the structure of III

$H^1(G_K, E(\bar{K}))$  is torsion for general galois cohomological reasons. So

$$\text{III}(E/K) \subseteq H^1(G_K, E(\bar{K}))$$

is torsion.

So we may write

$$\text{III}(E/K) = \bigoplus_p \text{III}_{p^\infty}(E/K)$$

where for each prime  $p$

$$\text{III}_{p^\infty}(E/K)$$

denotes the  $p$ -primary part of  $\text{III}(E/K)$ . (i.e. the subgroup of elements whose order is a power of  $p$ .) By descent

$$\text{III}(E/K)[m] \text{ is finite for all } m \geq 1.$$

So

$$\text{III}_{p^\infty}(E/K) \cong (\mathbf{Q}_p/\mathbf{Z}_p)^{\delta_p} \oplus T_p, \quad \delta_p \in \mathbf{Z}_{\geq 0}$$

where  $T_p$  is a finite abelian  $p$ -group.

$$T_p \cong \mathbf{Z}/p^{s_1}\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/p^{s_l}\mathbf{Z}, \quad s_i \in \mathbf{Z}_{\geq 0}.$$

The group

$$\bigoplus_p (\mathbf{Q}_p/\mathbf{Z}_p)^{\delta_p} \subseteq \text{III}(E/K)$$

is called the infinitely divisible subgroup of III denoted  $\text{III}_{div}$ .

The conjecture that III is finite implies  $\delta_p = 0$  for all  $p$ . And  $T_p \neq 0$  for only finitely many  $p$ .

There is a pairing called the Cassels-Tate pairing

$$\text{III}(E/K) \times \text{III}(E/K) \rightarrow \mathbf{Q}/\mathbf{Z}$$

which is bilinear and alternating, and the kernel on either side is the infinitely divisible group. If  $\text{III}(E/K)$  is finite then the pairing is non-degenerate and hence

$$|\text{III}(E/K)| = \square \in \mathbf{Z}.$$

**Definition 1.34**  $p^\infty$ -Selmer group. Consider  $\text{Sel}_{p^n}(E/K)$  and take the direct limit

$$\varinjlim_n \text{Sel}_{p^n}(E/K)$$

to define the  $p^\infty$ -Selmer group. ◇

One shows that

$$X_p(E/K) = \text{Hom}_{\mathbf{Z}_p}(\varinjlim_n \text{Sel}_{p^n}(E/K), \mathbf{Q}_p/\mathbf{Z}_p)$$

called the Pontryagin dual of the  $p^\infty$  Selmer group is a finitely generated  $\mathbf{Z}_p$ -module. The associated  $\mathbf{Q}_p$ -vector space, denoted  $X_p(E/K) = X_p(E/K) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$  has dimension  $\text{rk}_p$ .

**Definition 1.35**  $\text{rk}_p$  is called the  $p^\infty$ -Selmer rank of  $E/K$  and satisfies

$$\text{rk}_p = \text{rk}(E/K) + \delta_p.$$

◇

So if III is finite then  $\delta_p = 0$  for all  $p$ . Use BSD to compute parity of  $\text{rk}_p$ .

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### 1.3 Consequences of BSD

Consider  $E/\mathbf{Q}$ : Mordell-Weil implies that

$$E(\mathbf{Q}) \simeq \mathbf{Z}^{\text{rk}} \oplus \text{torsion}$$

then BSD 1 says that

$$\underbrace{\text{ord}_{s=1} L(E, s)}_{\text{rk}_{\text{an}}} = \text{rk},$$

functional equation for  $L(E, s)$ .

$$L^*(E, s) = w L^*(E, 2 - s)$$

with  $w \in \{\pm 1\}$  the sign of the functional equation. If  $w = 1$  then  $L(E, s)$  is (essentially) symmetric at  $s = 1$ . So  $\text{ord}_{s=1} L(E, s)$  is even. If  $w = -1$  then  $\text{ord}_{s=1} L(E, s)$  is odd.

We get BSD mod 2:

$$(-1)^{\text{rk}} = w(\text{sign of f.e.})$$

a conjecture based on conjecture is bad so we go one step further.

**Theorem 1.36** *The sign in the functional equation of  $L(E, s)$  is equal to the global root number of  $E$ .*

*This is defined by*

$$w_\infty \prod_p w_p,$$

*the local root numbers defined in terms of the local galois representations. Non-trivial to understand, but manageable.*

**Conjecture 1.37** Parity conjecture.

$$(-1)^{\text{rk}} = \prod_v w_v = w.$$

**Example 1.38**

$$E/\mathbf{Q}: y^2 + y = x^3 + x^2 - 7x + 5$$

$$\Delta_E = -7 \cdot 13$$

$$w_v = 1 \text{ if } v \nmid \infty 7 \cdot 13$$

$$w_\infty = -1$$

(in general  $-1^g$  where  $g$  is dimension of the abelian variety).

$$w_7 = -1$$

$$w_{13} = -1$$

so  $w = -1$  and the rank is odd, hence there is a point of infinite order on this curve.  $\square$

**Problem.** On the one hand  $\prod_v w_v$  is computable. On the other hand  $(-1)^{\text{rk}}$  is precisely unknown.

$$(-1)^{\text{rk}} = \prod_v w_v.$$

**Theorem 1.39** Assume III is finite, let  $\phi: E \rightarrow E'$  be an isogeny whose degree is not divisible by  $\text{char}(K)$ , then

$$\frac{|\text{III}_E| \text{Reg}_E \prod_p c_p \Omega_E}{|E_{\text{tors}}|^2} = \frac{|\text{III}_{E'}| \text{Reg}_{E'} \prod_p c'_p \Omega_{E'}}{|E'_{\text{tors}}|^2}.$$

**Remark 1.40** In fact this is true for all abelian varieties over  $K$ .

**Example 1.41** Let

$$E/\mathbf{Q}: y^2 + xy = x^3 - x$$

<http://www.lmfdb.org/EllipticCurve/Q/65/a/1>.  $\Delta_E = 5 \cdot 13$ , it has a 2-isogenous curve  $E'$ .

Compute

$$c_5 = c_{13} = 1$$

$$c'_5 = c'_{13} = 2$$

$$\Omega_E = 2\Omega_{E'}$$

then

$$\frac{\text{Reg}_{E'}}{\text{Reg}_E} = \frac{|\text{III}_E| |E'_{\text{tors}}|^2 \prod_p c_p \Omega_E}{|\text{III}_{E'}| |E_{\text{tors}}|^2 \prod_p c'_p \Omega_{E'}} \equiv \square \frac{2}{4} \not\equiv 1 \square.$$

So  $\text{Reg}_E \neq 1$ ,  $\text{Reg}_{E'} \neq 1$  so  $E$  has at least one rational point of infinite order, so  $\text{rk} \geq 1$ .  $\square$

**Lemma 1.42** Assume III is finite, let

$$\phi: E/K \rightarrow E'/K$$

be a  $K$ -rational isogeny of degree  $d$ .

Write  $n = \text{rk}_E = \text{rk}_{E'}$ . Pick a basis  $\Lambda = \langle P_1, \dots, P_n \rangle$  for

$$E(K)/\text{tors}$$

write  $\Lambda'$  for a basis of  $E'(K)/\text{tors}$ . Write  $\phi^\vee: E' \rightarrow E$  for the dual isogeny s.t.  $\phi\phi^\vee = [d]$ .

using the following fact

$$\langle \phi(P), Q \rangle_{E'} = \langle P, \phi^\vee(Q) \rangle_E$$

Then

$$\begin{aligned} d^n \operatorname{Reg}_E &= \det(\langle dP_i, P_j \rangle_E)_{i,j} \\ &= \det(\langle \phi^\vee \phi P_i, P_j \rangle_E) = \det(\langle \phi P_i, \phi P_j \rangle_{E'}) \\ &= \operatorname{Reg}_{E'}[\Lambda' : \phi(\Lambda)]^2. \end{aligned}$$

Back to the example

$$\frac{\operatorname{Reg}_E}{\operatorname{Reg}_{E'}} \equiv \frac{1}{2} \square$$

so by the lemma  $\operatorname{rk}$  is odd. Here we assumed that  $\operatorname{III}$  is finite for elliptic curves, one can drop the assumption of finiteness of  $\operatorname{III}$  to get unconditional results on the parity of  $\operatorname{rk}_p$  for all  $p$ .

**Conjecture 1.43**  $p$ -parity.

$$(-1)^{\operatorname{rk}_p} = w.$$

This is known over  $\mathbf{Q}$  and totally real fields.

How to compute the parity of  $\operatorname{rk}_p(E/K)$ ? Need BSD-invariance for Selmer groups. (Details T. and V. Dokchitser "On the BSD quotients modulo squares", and Milne "Arithmetic duality theorems")

**Definition 1.44** For an isogeny

$$\Psi: A \rightarrow B$$

of abelian varieties over  $K$ . Let

$$Q(\Psi) = |\operatorname{coker}(\Psi: A(K)/A(K)_{\operatorname{tors}} \rightarrow B(K)/B(K)_{\operatorname{tors}})| \cdot |\ker(\psi: \operatorname{III}(A)_{\operatorname{div}} \rightarrow \operatorname{III}(B)_{\operatorname{div}})|.$$

◇

Recall  $\operatorname{rk}_p = \operatorname{rk} + \delta_p$  where

$$\operatorname{III} = \bigoplus \operatorname{III}_{p^\infty}$$

and

$$\operatorname{III}_{p^\infty} \simeq (\mathbf{Q}_p/\mathbf{Z}_p)_p^\delta \oplus T_p$$

$$\operatorname{III}_{\operatorname{div}} = \bigoplus (\mathbf{Q}_p/\mathbf{Z}_p)^{\delta_p}.$$

Strategy, we show that for  $\Psi$  an isogeny s.t.  $\Psi\Psi^\vee = [p]$ . Then

$$p^{\operatorname{rk}_p(E/K)} \equiv \frac{Q(\Psi^\vee)}{Q(\Psi)} \equiv \frac{\prod_v c_p \Omega_E}{\prod_v c'_v \Omega_{E'}} \pmod{K^{\times 2}}.$$

**Remark 1.45** Let  $A^\vee$  be the dual of  $A$ .  $A^\vee = \operatorname{Pic}^0(A)$ .

So

$$(-1)^{\operatorname{rk}_p(E/K)} = (-1)^{\operatorname{ord}_p \left( \frac{\prod_v c_p \Omega_E}{\prod_v c'_v \Omega_{E'}} \right)}$$

the parity of  $\operatorname{rk}_p(E/K)$  is computable from local invariants of  $E$  and  $E'$ .

To prove the  $p$ -parity conjecture it remains to prove

$$(-1)^{\operatorname{ord}_p \left( \frac{\prod_v c_p \Omega_E}{\prod_v c'_v \Omega_{E'}} \right)} = \prod_v w_v.$$

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**Aside: Generalisation of the definition of  $\text{Sel}^n(E/\mathbf{Q})$ .** Consider

$$\Psi: A \rightarrow B$$

an isogeny of abelian varieties. We have

$$0 \rightarrow A(K)[\Psi] \rightarrow A(K) \xrightarrow{\Psi} B(K) \xrightarrow{\delta} H^1(G_K, A[\Psi]) \rightarrow H^1(G_K, A) \xrightarrow{\Psi} H^1(G_K, B)$$

from which we extract

$$\begin{aligned} 0 \rightarrow B(K)/\Psi(A(K)) &\xrightarrow{\delta} H^1(G_K, A[\Psi]) \rightarrow H^1(G_K, A)[\Psi] \rightarrow 0 \\ 0 \rightarrow \prod_v B(K_v)/\Psi(A(K_v)) &\xrightarrow{\delta} H^1(G_{K_v}, A[\Psi]) \rightarrow \prod_v H^1(G_{K_v}, A)[\Psi] \rightarrow 0 \end{aligned}$$

we then define

$$\begin{aligned} \text{Sel}^{(\Psi)}(A/K) &= \ker \left\{ H^1(G_K, A[\Psi]) \rightarrow \prod_v H^1(G_{K_v}, A) \right\} \\ \text{III}(A/K) &= \ker \left\{ H^1(G_K, A) \rightarrow \prod_v H^1(G_{K_v}, A) \right\} \end{aligned}$$

so

$$0 \rightarrow \underbrace{B(K)/\Psi(A(K))}_{\text{coker}(\Psi: A(K) \xrightarrow{\Psi} B(K))} \rightarrow \text{Sel}^{(\Psi)}(A/K) \rightarrow \text{III}(A/K) \rightarrow 0.$$

We want to show:

**Theorem 1.46** *Let  $E/K$  be an elliptic curve,  $K$  a number field, if  $\Psi$  is s.t.  $\Psi\Psi^\vee = [p]$  then*

$$p^{\text{rk}_p(E/K)} \equiv \frac{Q(\Psi)}{Q(\Psi^\vee)} \equiv \frac{\prod_v c_p \Omega_E}{\prod_v c'_p \Omega_{E'}} \pmod{K^{\times 2}}$$

We will show this in 3 parts, first the left, then the right, then the equality with the global root number.

**Step 1. Proposition 1.47**

$$p^{\text{rk}_p(E/K)} \equiv \frac{Q(\Psi)}{Q(\Psi^\vee)} \pmod{K^{\times 2}}$$

*Proof.* Note that

$$Q(\Psi \circ \Psi^\vee) = Q(\Psi)Q(\Psi^\vee)$$

hence

$$\frac{Q(\Psi)}{Q(\Psi^\vee)} \equiv \underbrace{Q(\Psi)Q(\Psi^\vee)}_{=Q([p])} \pmod{K^{\times 2}}$$

now

$$|\text{coker}([p]: E(K)/E(K)_{\text{tors}} \rightarrow E'(K)/E'(K)_{\text{tors}})| = p^{\text{rk}(E/K)}$$

Proof of this: For each generator  $R$  of  $E(K)/E(K)_{\text{tors}}$  then

$$\frac{1}{p}R, \frac{2}{p}R, \dots, \frac{p-1}{p}R, R$$

are not in the image of  $[p]$  which implies the size is  $p^{\text{rk}(E/K)}$ . Also

$$|\ker([p]: \text{III}(E/K)_{\text{div}} \rightarrow \text{III}(E'/K)_{\text{div}})| = p^{\delta_p}$$

since

$$\text{III}(E/K)_{\text{div}} = \bigoplus_p (\mathbf{Q}_p/\mathbf{Z}_p)^{\delta_p}$$

and since  $[p]$  is trivial on all

$$(\mathbf{Q}_l/\mathbf{Z}_l)^{\delta_l}, l \neq p$$

then look at  $[p]: (\mathbf{Q}_p/\mathbf{Z}_p)^{\delta_p} \rightarrow (\mathbf{Q}_p/\mathbf{Z}_p)^{\delta_p}$  if  $x \in \mathbf{Q}_p/\mathbf{Z}_p$  and  $\ker[p]$  then  $px \in \mathbf{Z}_p \implies x = a/p$  for  $a \in \mathbf{F}_p$ . so

$$p^{\delta_p}.$$

■

**Step 2.** We show that

$$\frac{Q(\Psi)}{Q(\Psi^\vee)} \equiv \frac{\prod_v c_p}{\prod_v c'_v} \frac{\Omega_E}{\Omega_{E'}} \pmod{K^{\times 2}}.$$

**Theorem 1.48** Let  $A, B/K$  be abelian varieties given with a non-zero global exterior form  $\omega_A, \omega_B$ . Suppose

$$\Psi: A \rightarrow B$$

is an isogeny and

$$\Psi^\vee: B^\vee \rightarrow A^\vee$$

its dual.

Let  $\text{III}_0(A/K)$  denote  $\text{III}(A/K)$  mod its divisible part. And

$$\Omega_A = \prod_{v|\infty, \text{real}} \int_{A(K_v)} |\omega_A| \prod_{v|\infty, \text{complex}} 2^{\dim A} \int_{A(K_v)} \omega_A \wedge \overline{\omega_A}.$$

Then

$$\frac{Q(\Psi^\vee)}{Q(\Psi)} = \frac{|B(K)_{\text{tors}}| |B^\vee(K)_{\text{tors}}|}{|A(K)_{\text{tors}}| |A^\vee(K)_{\text{tors}}|} \frac{\prod_v c_p(A/K)}{\prod_v c_v(B/K)} \frac{\Omega_A}{\Omega_B} \prod_{p|\deg \Psi} \frac{|\text{III}_0(A)[p^\infty]|}{|\text{III}_0(B)[p^\infty]|}.$$

**Remark 1.49** If  $A = E, B = E'$  with  $\Psi$  s.t.  $\Psi\Psi^\vee = [p]$  then

$$E \simeq E^\vee, E' \simeq E'^\vee$$

and  $|\text{III}_0| = \square$ .

$$\frac{Q(\Psi^\vee)}{Q(\Psi)} \equiv \frac{\prod_v c_p}{\prod_v c'_v} \frac{\Omega_E}{\Omega_{E'}} \pmod{K^{\times 2}}.$$

*Sketch proof of theorem.* We show how to obtain the quotient of Tamagawa numbers, for a sufficiently large set of places  $S$  of  $K$

$$\frac{Q(\Psi^\vee)}{Q(\Psi)} \frac{|\text{III}[\Psi^\vee]|}{|\text{III}[\Psi]|} = \prod_{v \in S} \frac{|\ker \Psi_v|}{|\ker \Psi_v^\vee|}$$

where  $\Psi_v$  is the induced map on  $E(K_v) \rightarrow E'(K_v)$ . If  $v \nmid \infty$  and  $v \in S$  what is

$$\frac{|\ker \Psi_v|}{|\text{coker } \Psi_v|}?$$

$$\begin{array}{ccccccc}
& & & 0 & & & \\
& & & \downarrow & & & \\
& & 0 & \downarrow & \ker \Psi_v & \downarrow & H_1 \\
0 & \longrightarrow & E_1(K_v) & \longrightarrow & E(K_v) & \longrightarrow & E(K_v)/E_1(K_v) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & E'_1(K_v) & \longrightarrow & E'(K_v) & \longrightarrow & E'(K_v)/E'_1(K_v) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & \text{coker } \Psi_v & & H_2
\end{array}$$

Snake lemma gives

$$\begin{aligned}
0 \rightarrow \ker \Psi_v \rightarrow H_1 \rightarrow 0 \rightarrow \text{coker } \Psi_v \rightarrow H_2 \rightarrow 0 \\
\implies |\ker \Psi_v| = |H_1|
\end{aligned}$$

and

$$|\text{coker } \Psi_v| = |H_2|.$$

Also

$$\left| \frac{E(K_v)/E_1(K_v)}{H_1} \right| = \left| \frac{E'(K_v)/E'_1(K_v)}{H_2} \right|.$$

Moreover since  $E, E'$  are isogenous we have

$$|\tilde{E}_{\text{ns}}(\bar{k})| = |\tilde{E}'_{\text{ns}}(\bar{k})|$$

hence since

$$0 \rightarrow E_1(K_v) \rightarrow E_0(K_v) \rightarrow \tilde{E}_{\text{ns}}(\bar{k}) \rightarrow 0$$

similarly for  $E'$ . We have

$$\begin{aligned}
|E_0(K_v)/E_1(K_v)| &= |E'_0(K_v)/E'_1(K_v)| \\
\implies \left| \frac{E'(K_v)/E'_1(K_v)}{E(K_v)/E_1(K_v)} \right| &= \left| \frac{E'(K_v)/E'_0(K_v)}{E(K_v)/E_0(K_v)} \right| = \frac{c_v}{c'_v}.
\end{aligned}$$

■

Hence

$$\begin{aligned}
(-1)^{\text{rk}_p(E/K)} &= (-1)^{\text{ord}_p \left( \prod_v \left| \frac{\text{coker } \Psi_v}{\ker \Psi_v} \right| \right)} \\
&= (-1)^{\text{ord}_p \left( \frac{\prod_v c'_v}{\prod_v c_v} \underbrace{\prod_{v|\infty} \left| \frac{\text{coker } \Psi_v}{\ker \Psi_v} \right|}_{\Omega_E/\Omega_{E'}} \right)}.
\end{aligned}$$

**Step 3.** We need to show that

$$(-1)^{\text{rk}_p(E/K)} = w_E \text{ (p-parity)}$$

i.e. we need to show that

$$(-1)^{\text{ord}_p\left(\frac{\prod_v c'_v}{\prod_v c_v} \frac{\Omega_E}{\Omega_{E'}}\right)} = w_E$$

Strategy:

$$(-1)^{\text{ord}_p\left(\frac{\prod_v c'_v}{\prod_v c_v} \frac{\Omega_E}{\Omega_{E'}}\right)} = \prod_{v \nmid \infty} (-1)^{\text{ord}_p \frac{c'_v}{c_v}} \prod_{v \mid \infty} (-1)^{\text{ord}_p \left| \frac{\ker \Psi_v}{\text{coker } \Psi_v} \right|}$$

and relate

$$(-1)^{\text{ord}_p \frac{c'_v}{c_v}}$$

to  $w_v$  for  $v \nmid \infty$  and

$$(-1)^{\text{ord}_p \left| \frac{\ker \Psi_v}{\text{coker } \Psi_v} \right|}$$

to  $w_v$  for  $v \mid \infty$ .

Then take product over all places.

Lecture ? 26/3/2018

Let  $E/K$  be an elliptic curve admitting an isogeny  $\Psi$  of degree  $p$  (defined over  $K$ ). Recall that we proved

$$p^{\text{rk}_p(E/K)} = \prod_v \frac{c_v}{c'_v} \frac{\Omega_E}{\Omega_{E'}}$$

$v$  missing  $p$ . More precisely

$$p^{\text{rk}_p(E/K)} \equiv \prod_{v \nmid p\infty} \frac{c_v}{c'_v} \prod_{v \mid \infty} \left| \frac{\ker \psi_v}{\text{coker } \psi_v} \right|$$

where  $\psi_v$  is the map induced by  $\psi$  on  $E(K_v)$ .

What about  $v \mid p$  to extract

$$\frac{c_v}{c'_v}$$

from

$$\left| \frac{\ker \psi_v}{\text{coker } \psi_v} \right|$$

at finite places we can use a diagram involving

$$0 \rightarrow E_1(K_v) \rightarrow E'_1(K_v) \rightarrow \text{coker} \rightarrow 0.$$

If  $v \nmid p$  then  $|\text{coker}| = 1$  since then on the level of the formal group  $\psi$  induces a map

$$\begin{aligned} \hat{\psi}: \hat{E}(\mathfrak{m}_K) &\rightarrow \hat{E}'(\mathfrak{m}_K) \\ T &\mapsto aT + \dots \end{aligned}$$

power series rep of  $\psi$   $\psi(x, y) = (x', y')$  Silverman IV cor 4.3/  $\omega' \circ \psi = \psi' \circ \omega$ . with leading  $a = \psi^* \omega' / \omega \times \text{unit} \in \mathcal{O}_K$ .

$$\implies aa' = p \in \mathcal{O}_K^\times \implies \hat{\psi} \text{ isom.}$$

If  $v \mid p$  then coker contributes to the snake lemma and at that place

$$\frac{c_v}{c'_v} \left| \frac{\psi^* \omega'}{\omega} \right|_v = \frac{c_E}{c'_E} \left| \frac{\omega}{\omega_v^0} \right|_v$$

for a particular choice of  $\omega$ .



**Proving  $p$ -parity.** To prove the  $p$ -parity conjecture

$$(-1)^{\text{rk}_p(E/K)} = w_E.$$

We will show that

$$(-1)^{\text{ord}_p \Pi_v \frac{c_v}{c_v'} \frac{\Omega}{\Omega_{E'}}} = w_E$$

by relating

$$(-1)^{\text{ord}_p \frac{c_v}{c_v'}}$$

and  $w_v$  at some place  $v \nmid p\infty$

$$(-1)^{\text{ord}_p \frac{\Omega_E}{\Omega_{E'}}} = (-1)^{\text{ord}_p \left| \frac{\ker \psi_v}{\text{coker } \psi_v} \right|}$$

and  $w_v$  at  $v|\infty$ .

We only sketch these steps for  $v \nmid p$  and  $E$  is semistable at  $v$ .

The proofs of  $p$ -parity for  $p$  odd and  $p = 2$  are different.

**$p$  odd.** The  $p$ -parity conjecture is proven for principally polarized abelian varieties with a  $p$ -cyclic isogeny with  $p \geq 2g + 2$  or  $p \geq 2$  and semistable reduction and some local constraints at  $v|p$ . see Root numbers selmer groups and non-commutative Iwasawa theory, Coates, Fukaya, Kato, Sujatha

Sketch, for an elliptic curve with a  $p$ -isogeny  $\psi$  we look at  $v|\infty$  where  $w_v = -1$ , and

$$(-1)^{\text{ord}_p \left| \frac{\ker \psi_v}{\text{coker } \psi_v} \right|}$$

if  $v$  is complex  $|\ker \psi_v| = p \mid \text{coker } \psi_v| = 1$ . so

$$(-1)^{\text{ord}_p \left| \frac{\ker \psi_v}{\text{coker } \psi_v} \right|} = -1 = w_v.$$

If  $v|\infty$  is real what does  $E(\mathbf{R})$  look like? Either there is a real period and so two real components, and all real  $p$ -torsion (if any) is on the identity component. Or there is no real period and only 1 real component that contains all real  $p$ -tors if any.

1.  $|\ker \psi_v| = p$  (the  $p$ -tors in  $\ker \psi$  are real)
2.  $|\ker \psi_v| = 1$  (the  $p$ -tors in  $\ker \psi$  are not real)

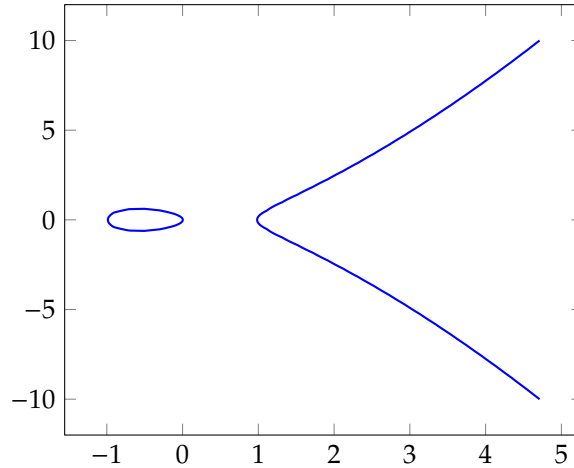


Figure 1.50

Moreover  $|\operatorname{coker} \psi| = 1$  always,  $\operatorname{sgn}(\Delta_E) = \operatorname{sgn}(\Delta_{E'})$

More generally if  $\deg \Psi$  is odd then

$$E'(\mathbf{R})/\psi(E(\mathbf{R})) \hookrightarrow H^1(\operatorname{Gal}(\mathbf{C}/\mathbf{R}), E[\psi]) = 0$$

since  $[\mathbf{C} : \mathbf{R}] = 2$  is coprime to  $E[\psi]$  (see Atiyah's book).

In the first case

$$(-1)^{\operatorname{ord}_p \left| \frac{\ker \psi_v}{\operatorname{coker} \psi_v} \right|} = -1 = w_v$$

In the second case

$$(-1)^{\operatorname{ord}_p \left| \frac{\ker \psi_v}{\operatorname{coker} \psi_v} \right|} = 1 \neq w_v$$

For  $K$  a local field let  $F = K(\ker \psi_v)$  noting that

$$\operatorname{Gal}(F/K) \hookrightarrow (\mathbf{Z}/p\mathbf{Z})^\times$$

from its action on points in  $\ker \psi = F/K$  is cyclic.

Consider the composition

$$F^\times \xrightarrow{\text{local rec.}} \operatorname{Gal}(F/K) \hookrightarrow (\mathbf{Z}/p\mathbf{Z})^\times.$$

and denote

$$(-1, F/K)$$

the image of  $-1$  under the above map.

$$(-1, F/K) = \begin{cases} 1 & \text{if } -1 \text{ is a norm from } F \text{ to } K, \\ -1 & \text{otw} \end{cases}$$

this is the Artin symbol.

This is perfect as they cancel out globally.

If  $v$  is complex then  $F = \mathbf{C}$ ,  $K = \mathbf{C}$  and  $(-1, F/K) = 1$

If  $v$  is real and  $|\ker \psi_v| = p$  then  $F = \mathbf{R}$ ,  $K = \mathbf{R}$  and  $(-1, F/K) = 1$

If  $v$  is real and  $|\ker \psi_v| = 1$  then  $F = \mathbf{R}$ ,  $K = \mathbf{R}$  and  $(-1, F/K) = -1$

$p = 2$ . Note that  $(-1, F/K) = 1$  for all places of  $K$  since if  $E$  admits a 2-isogeny  $\psi/K$  then it admits a 2-torsion point over  $K$ .

Hence  $F = K(\ker \psi_v) = K$

set-up

$$E/K$$

with a 2-isogeny  $\psi/K$

$$E: y^2 = x(x + ax + b)$$

by translating 2-torsion to  $(0,0)$

$$\psi: E \rightarrow E': y^2 = x(x^2 - 2ax + \delta)$$

where  $\delta = a^2 - 4b = \text{disc}(x^2 + ax + b)$  if  $\delta > 0$  then  $E(\mathbf{R})$  has two connected components.  $\delta < 0$  only 1. Have  $16b = \text{disc}(x^2 - 2ax + \delta)$  likewise for  $E'$

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & \ker \psi_v^0 & & \ker \psi_v & & \ker \psi_{/} & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \longrightarrow & E^0(\mathbf{R}) & \longrightarrow & E(\mathbf{R}) & \longrightarrow & E(\mathbf{R})/E^0(\mathbf{R}) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \longrightarrow & E'^0(\mathbf{R}) & \longrightarrow & E'(\mathbf{R}) & \longrightarrow & E'(\mathbf{R})/E'^0(\mathbf{R}) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \text{coker } \psi_v^0 & & \text{coker } \Psi_v & & \text{coker } \psi_{/} & 
 \end{array}$$

by snakey

$$\begin{aligned}
 & \frac{|\ker \psi_v^0| |\ker \psi_{/}| |\ker \psi|}{|\ker \psi_v| |\text{coker } \psi_v^0| |\text{coker } \psi_{/}|} = 1 \\
 \implies & \left| \frac{\text{coker } \psi_v}{\ker \psi_v} \right| = \frac{|\text{coker } \psi_v^0| |\text{coker } \psi_{/}|}{|\ker \psi_v^0| |\ker \psi_{/}|}
 \end{aligned}$$

let  $n(E), n(E')$  be the number of real connected components  $n = E(\mathbf{R})/E^0(\mathbf{R})$

By the third column

$$\frac{n(E')}{n(E)} \frac{|\ker \psi_{/}|}{|\text{coker } \psi_{/}|} = 1$$

now  $|\text{coker } \psi_v^0| = 1$  as the map on identity component is surjective. hence

$$\left| \frac{\text{coker } \psi_v}{\ker \psi_v} \right| = \frac{n(E')}{n(E) |\ker \psi_v^0|}$$

Lecture ? 28/3/2018

Recall: to prove the 2-parity conjecture for  $E/K$

$$(-1)^{\text{rk}_2(E)} = w????????????????$$

missed

Notation

$$E: y^2 = x(x^2 + ax + b) = xq_1(x)$$

$$E': y^2 = x(x^2 - 2ax + \delta) = xq_2(x), \delta = a^2 - 4b$$

$$\text{disc}(q_1(x)) = \delta \text{disc}(q_2(x)) = 16b$$

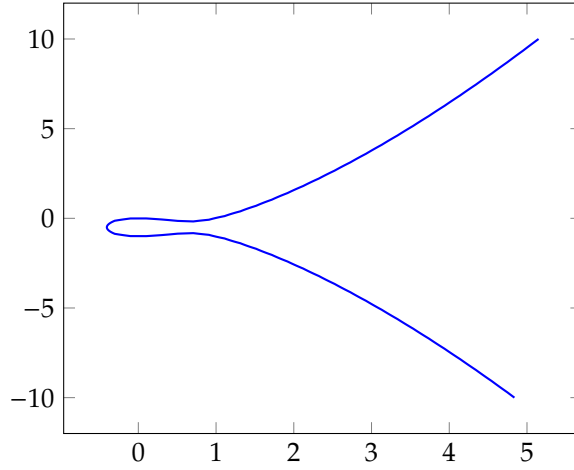


Figure 1.51

a) If  $\delta > 0, b > 0$  then  $E, E'$  both have two real components,  $n(E) = n(E') = 2$ .

$$|\ker \psi_v^0| = \begin{cases} 1 & \text{if } (0, 0) \text{ is not on } E^0(\mathbf{R}) \\ 2 & \text{if } (0, 0) \text{ is on } E^0(\mathbf{R}) \end{cases} = \begin{cases} 1 & \text{if } a < 0 \\ 2 & \text{if } a > 0 \end{cases}$$

write  $q_1(x) = x^2 + ax + b = (x - \alpha)(x - \beta)$  then if  $(0, 0) \in E^0(\mathbf{R})$ ,  $\alpha, \beta < 0$  but  $a = -\alpha - \beta$  hence in this case  $a > 0$ .

$$(-1)^{\text{ord}_2 \left| \frac{\ker \psi_v}{\text{coker } \psi_v} \right|} = \begin{cases} 1 & \text{if } a < 0 \\ -1 & \text{if } a > 0 \end{cases}$$

so we need some correction if  $\delta > 0, b > 0, a < 0$ .

b) If  $\delta > 0, b < 0$   $E$  has two real components and  $E'$  only 1  $n(E) = 2, n(E') = 1$ .

$$|\ker \psi_v^0| = 1$$

since  $b < 0$  and  $b = \alpha\beta$ .

$$(-1)^{\text{ord}_2 \left| \frac{\ker \psi_v}{\text{coker } \psi_v} \right|} = -1$$

so no correction if  $\delta > 0, b < 0$ .

c) If  $\delta < 0, b > 0, n(E) = 1, n(E') = 2$ .

$$|\ker \psi_v^0| = 2$$

and

$$(-1)^{\text{ord}_2 \left| \frac{\ker \psi_v}{\text{coker } \psi_v} \right|} = 1$$

need correction if  $\delta < 0, b > 0$ .

d)  $b < 0, \delta < 0$  contradiction,  $\delta = a^2 - 4b$ .

So in summary if  $\delta > 0, b > 0, a < 0$  or  $\delta < 0, b > 0$  need a correction, if  $\delta > 0, b > 0, a > 0$  or  $\delta > 0, b < 0$  no correction.

$$(-1)^{\text{ord}_2 \left| \frac{\ker \psi_v}{\text{coker } \psi_v} \right|} = ? w_v$$

First guess

$$(a, -b)(-a, \delta)$$

Recall: let  $K$  be a local field

$$K^\times \times K^\times \rightarrow \{\pm 1\}$$

$$(a, b) \mapsto \begin{cases} 1 & \text{if } a \text{ is a norm from } K(\sqrt{b}) \rightarrow K, \\ -1 & \text{otw} \end{cases}$$

If  $K$  is archimidean  $(a, b) = -1 \iff a < 0, b < 0$ . If  $K$  is non-archimidean with odd residue characteristic then

$$(\text{unit}, \text{unit}) = 1$$

$$(\text{unit}, \pi^n) = -1$$

if  $n$  odd and unit is not a square.

$$(a, bc) = (a, b)(a, c).$$

So guess

$$(a, -b)(-a, \delta)$$

works over  $\mathbf{R}$ .

$v \nmid 2\infty$  need to show that

$$(-1)^{\text{ord}_2 \frac{c_v}{c'_v}} = (a, -b)(-a, \delta)w_v$$

if  $E$  has good reduction at  $v$ .

$$c_v = c'_v = 1.$$

Need to show that

$$(a, -b)(-a, \delta) = 1.$$

Since  $E, E'$  have good reduction at  $v$ . then  $b, \delta$  are units in  $K$ . If  $a \in \mathcal{O}_K^\times$  then  $(a, -b)(-a, b) = 1$  if  $a \equiv 0 \pmod{\pi_K}$  then since  $a^2 - 4b = \delta$  then  $\delta \equiv -4b \pmod{\pi_K}$ .

If  $E$  has split multiplicative reduction, (multiplicative reduction is when  $y^2 = f(x)$  and  $f(x)$  has a double root mod  $\pi_K$ , any two distinct tangents at the node, both defined over  $k$  (fixed by frob)). so  $E'$  also has split multiplicative reduction as  $\psi$  commutes with frobenius.

Need to compute

$$\frac{c_v}{c'_v}$$

by Tates algorithm

$$c_E = v(\Delta_E) = n$$

we show that

$$c_{E'} = v(\Delta_{E'}) = \begin{cases} 2n, \\ \frac{1}{2}n \end{cases}$$

Recall

$$E: y^2 = x \overbrace{(x^2 + ax + b)}^{f_E(x)} = x(x - \alpha)(x - \beta) = xq_1(x)$$

$$\Delta_{f_E} = \alpha^2 \beta^2 (\alpha - \beta)^2 = b^2 (\alpha - \beta)^2 = b^2 \delta$$

$$E': y^2 = x \overbrace{(x^2 - 2ax + \delta)}^{f_{E'}(x)} = x(x - A)(x - B) = xq_2(x)$$

$$\Delta_{f_{E'}} = A^2 B^2 (A - B)^2 = \delta^2 (A - A)^2 = \delta^2 16b$$

if  $v(\delta) = n$  then  $v(\Delta_{f_E}) = n$  so  $c_E = -n$  and  $v(\Delta_{f_{E'}}) = 2n$  so  $c_{E'} = 2n$  in general if  $E$  admits a  $p$ -isogeny and  $E$  has split multiplicative reduction then

$$\frac{c_E}{c_{E'}} = p^{\pm 1}.$$

here  $w_v = -1$  and

$$(-1)^{\text{ord}_2 \frac{c_E}{c_{E'}}} = -1$$

need to show that

$$(a, -b)(-a, \delta) = 1$$

if  $E$  has a double root at  $(0, 0)$  wlog  $\alpha \equiv 0 \pmod{\pi_K}$  then  $v(\delta) = 0$ ,  $v(b) > 0$  and both slopes of tangent at  $(0, 0)$  are defined over  $k$ .

Taylor expansion at  $(0, 0)$

$$\begin{aligned} f(x, y) &= y^2 - x^3 - ax^2 - bx \\ &= (y - s_1 x)(y - s_2 x) + h.o.t. \\ &= y^2 - xy(s_1 + s_2) + s_1 s_2 x^2 + h.o.t. \end{aligned}$$

so  $s_1 = -s_2$  and  $s_1 s_2 = -a$  implies  $s_1^2 = a$ .

so  $s_1 \in k^\times$  then  $a \in k^{\times 2}$

$$(a, -b) = 1 \implies (-a, \delta) = 1$$

as both are units.

Now  $b = \alpha\beta \equiv 0 \pmod{\pi_K}$  so

$$x^2 - 2ax + \delta \equiv (a - A)^2 \pmod{\pi_K}$$

same Taylor expansion gives

$$\begin{aligned} f(x, y) &= y^2 - x^3 + 2ax^2 - \delta x \\ &= f(x, y) - f(A, 0) = (y - s_3(x - A))(y - s_4(x - A)) + h.o.t. \end{aligned}$$

so  $s_3 = -s_4$  and  $s_3 s_4 = 2a$ ,  $s_3^2 = -2a$  hence

$$(a, -b)(-2a, \delta)$$

split multiplicative

$$-2a \in K^{\times 2},$$

So we should use this Hilbert symbol instead, it doesn't change the real case.

If  $E$  has non-split multiplicative reduction

$$\frac{c_E}{c_{E'}} = \begin{cases} 1, & \text{if } v(\Delta_E), v(\Delta_{E'}) \text{ even} \\ 2, & \text{if } v(\Delta_{E'}) \text{ odd} \\ \frac{1}{2}, & \text{if } v(\Delta_E) \text{ odd} \end{cases}$$

$$\implies (-1)^{\text{ord}_2 \frac{c_E}{c_{E'}}} = \begin{cases} 1, \\ -1, \\ -1, \end{cases}$$

done since  $a, -2a$  precisely not squares.

What are these invariants purely in theory?

## 2 Abelian varieties

Lecture ? 2/4/2018

What about generalising this method to abelian varieties?

For  $p$  odd Coates et. al. (ppav with  $p$ -cyclic isogenies and local constraints)

For  $p = 2$ .

Recall let  $X, Y/K$  be abelian varieties over a number field and suppose that  $\Psi: X \rightarrow Y$  is an isogeny, then  $\Psi^\vee: Y^\vee \rightarrow X^\vee$  its dual. Then

$$\frac{Q(\Psi^\vee)}{Q(\Psi)} = \frac{|Y(K)_{\text{tors}}|}{|X(K)_{\text{tors}}|} \frac{|Y^\vee(K)_{\text{tors}}|}{|X^\vee(K)_{\text{tors}}|} \frac{\prod_v c(X/K_v) \Omega_X}{\prod_v c(Y/K_v) \Omega_Y} \prod_{p|\deg \Psi} \frac{|\text{III}_0(X)[p^\infty]|}{|\text{III}_0(Y)[p^\infty]|} \quad (2.1)$$

on the other hand we showed that if  $\Psi\Psi^\vee = [p]$  then

$$\frac{Q(\Psi^\vee)}{Q(\Psi)} \equiv p^{\text{rk}_p(X/K)} \pmod{K^{\times 2}}$$

note that in this case  $\deg \psi = p^{\dim(X)}$ .

To be able to use the same method we need to compute the RHS of (2.1).

For  $E$  since  $E \simeq E^\vee$  and  $|\text{III}_0(E)| = \square$ , this only meant computing

$$\prod_v \frac{c(E/k) \Omega_E}{c(E'/k) \Omega_{E'}}.$$

First consider a ppav  $X/K$  s.t.

$$(2.1) \equiv \frac{\prod_v c(X/K_v) \Omega_X}{\prod_v c(Y/K_v) \Omega_Y} \frac{|\text{III}_0(X)[p^\infty]|}{|\text{III}_0(Y)[p^\infty]|} \pmod{K^{\times \vee}} \quad (2.2)$$

1. Can we compute

$$\frac{\prod_v c(X/K_v) \Omega_X}{\prod_v c(Y/K_v) \Omega_Y} ? \quad (2.3)$$

Leads us to Jacobians of hyperelliptic curves of genus  $g$

2. Can we compute

$$\frac{|\text{III}_0(X)[p^\infty]|}{|\text{III}_0(Y)[p^\infty]|} ? \quad (2.4)$$

Leads us to Jacobians of hyperelliptic curves of genus  $g$

3. Need an isogeny  $\Psi$  of degree  $2^g$  s.t.

$$\Psi: J \rightarrow J'$$

i.e. the codomain must be a Jacobian of a hyperelliptic curve otherwise we cannot compute 1. or 2.

To satisfy 1., 2. and 3. we take  $g = 2$  because of the following:

**Theorem 2.1 González, Josep, Jordi Guardia, and Victor Rotger. Abelian surfaces of GL2-type as Jacobians of curves. arXiv preprint math/0409352 (2004).** *Let  $A/K$  be a principally polarized abelian surface defined over a number field. Then  $A$  is one of the following types*

•

$$A/K \simeq_K J(C)$$

where  $C/K$  is a smooth genus 2 curve.

•

$$A/K \simeq_K C_1 \times C_2$$

where  $C_1, C_2/K$  are elliptic curves defined over  $K$ .

•

$$A/K \simeq_K \text{Res}_{F/K} C$$

where  $\text{Res}_{F/K} C$  is the Weil restriction of an elliptic curve defined over a quadratic extension  $F/K$ .

**Remark 2.2** The parity of the rank of  $A/K$  in the last two cases can be computed from that of the underlying elliptic curves.

We will concentrate on  $A \simeq_K J(C)$ ,

$$C: y^2 = f(x)$$

for  $\deg(f) = 6$ .

The generalisation of a 2-isogeny is called a Richelot isogeny.

Plan:

1. Review of hyperelliptic curves and their Jacobians.
2. Richelot isogeny
3. Compute contribution of the real places
4. Compute Tamagawa numbers/local root numbers
5. Compute  $|\text{III}_0(J)[2^\infty]|$  up to squares
6. Find and prove the right error term

## 2.1 Review of hyperelliptic curves and Jacobians

See Stoll's notes.

By a hyperelliptic curve  $C$  over a number field  $K$  given by

$$C/K: y^2 = f(x)$$

of genus  $g$  where  $f(x) \in K[x]$  of degree  $2g+1$  or  $2g+2$  with no multiple roots, we mean the pair of affine patches

$$U_x: y^2 = f(x)$$

$$U_t: v^2 = t^{2g+2} f\left(\frac{1}{t}\right)$$

glued together along the maps

$$x = \frac{1}{t}, y = \frac{v}{t^{g+1}}.$$

We refer to as the points at  $\infty$  (i.e.  $C \setminus U_x$ ) the points with  $t = 0$  on  $U_t$ .

Explicitly denote by  $c$  the leading term of  $f(x)$ .

If  $f(x)$  is of degree  $2g+1$  then

$$U_x: y^2 = c \prod_{i=1}^{2g+1} (x - r_i)$$



$$U_t: v^2 = tc \prod_{i=1}^{2g+1} (tr_i - 1)$$

we denote  $P_\infty = (0, 1)$  the only point at infinity with  $t = 0$ .

Otherwise if  $f(x)$  is of degree  $2g + 2$  then

$$U_x: y^2 = c \prod_{i=1}^{2g+2} (x - r_i)$$

$$U_t: v^2 = c \prod_{i=1}^{2g+2} (tr_i - 1)$$

we denote  $P_\infty^\pm = (0, \pm\sqrt{c})$  the two points on  $U_t$  with  $t = 0$ .

**Divisors and the picard group.** Let  $G_K$  be the absolute galois group of  $K$ , recall that  $G_K$  acts on

$$C(K^{\text{sep}})$$

via its action on coordinates.

**Definition 2.3** A divisor  $D$  on  $C$  is a formal sum

$$\sum_{P \in C(K^{\text{sep}})} n_P P$$

where  $n_P \in \mathbf{Z}$  and  $n_P = 0$  for all but finitely many  $P \in C(K^{\text{sep}})$ . The integer  $n_P$  is called the multiplicity of  $P$  in  $D$  and  $\deg(D) = \sum_P n_P$  is the degree of  $D$ .

Divisors on  $C$  are elements of the free abelian group on the set of points  $P \in C(K^{\text{sep}})$ . Denote by  $\text{Div}(C)$  the group of divisors on  $C$ .  $\diamond$

**Definition 2.4** A divisor

$$D = \sum_{P \in C(F)} n_P P$$

for some Galois extension  $F|K$ . Is  $K$ -rational or defined over  $K$  if

$$D^\sigma = D \quad \forall \sigma \in \text{Gal}(F/K).$$

$\diamond$

**Example 2.5**

$$C: y^2 = f(x)$$

$$\alpha \in K$$

$$P = (\alpha, \sqrt{f(\alpha)})$$

$$\bar{P} = (\alpha, -\sqrt{f(\alpha)})$$

then

$$D = P + \bar{P}$$

is a  $K$ -rational divisor.  $\square$

**Definition 2.6** Let  $f$  be a non-zero rational function on  $C$ . Define

$$[f] = \sum_{P \in C} \text{ord}_P(f) P$$

where the multiplicity of  $P$  in  $[f]$  is given by the order of vanishing of  $f$  at

$P$ . These divisors are called principal divisors, the group of such is denote  $\text{Princ}(P)$ . Note that these are all of degree 0.  $\diamond$

**Definition 2.7** The picard group of  $C$  is defined to be

$$\text{Pic}(C) = \text{Div}(C)/\text{Princ}(P).$$

Note that this inherits a notion of degree from  $\text{Div}(C)$ .  $\diamond$

**Theorem 2.8** *Let  $C$  be a smooth, projective, absolutely irreducible curve of genus  $g$  over some field  $K$ . Then there exists an abelian variety  $J$  of dimension  $g$  over  $K$  s.t. for each field*

$$K \subseteq L \subseteq K^{\text{sep}}$$

$$J(L) = \text{Pic}_C^0(L)$$

**Definition 2.9**  $J$  is called the Jacobian variety of  $C$ .  $\diamond$

**Remark 2.10**  $J$  is a projective variety (abelian), thus it can be embedded in some projective space  $\mathbf{P}^N$  over  $K$ . One can show that

$$N = 4g - 1$$

always works for hyperelliptic curves.

This is too large to work with an explicit model for  $J$  instead we will work with the curve  $C$ .

Lecture ? 4/4/2018

**Jacobians of genus 2 curves.** Let  $C$  be a hyperelliptic curve of genus 2 defined over  $K$ .

$$C: y^2 = f(x)$$

with  $f(x) \in K[x]$  of degree 6.

Points on  $C(\bar{K})$  and  $J(\bar{K})$ :

A point  $D$  on  $J(\bar{K})$  is given by a divisor on  $C$  of the form

$$D = P + Q - P_{\infty}^+ - P_{\infty}^-$$

for some  $P, Q \in C(\bar{K})$ . For  $D$  to be defined over  $K$  either  $P, Q \in C(K)$  or  $P = Q^{\sigma}$  for  $\sigma \in \text{Gal}(F/K)$  where  $[F : K] = 2$ .

**Remark 2.11** If  $P = (x, y)$  and  $P' = (x, -y)$  then

$$D = P + Q - P_{\infty}^+ - P_{\infty}^-$$

is zero in  $J(\bar{K})$ .

Addition:

Choose 4 points  $P, P', Q, Q' \in C(\bar{K})$  (in general position to make it easier).

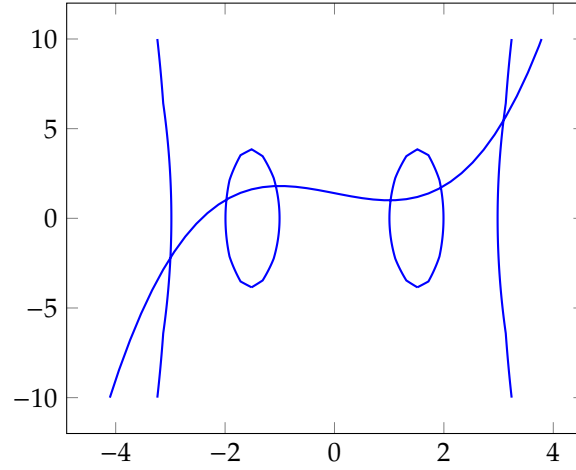


Figure 2.12

We can find a cubic polynomial  $y = p(x)$  through the four points. It also intersects at two additional points  $S, S'$  so that

$$\begin{aligned} [y - p(x)] &= P + P' + Q + Q' + S + S' - 3P_{\infty}^+ - 3P_{\infty}^- \\ (P + P' - P_{\infty}^+ - P_{\infty}^-) + (Q + Q' - P_{\infty}^+ - P_{\infty}^-) &= -(S + S' - P_{\infty}^+ - P_{\infty}^-) \end{aligned}$$

hence

$$\underbrace{[P, P']}_{=P+P'-P_{\infty}^+-P_{\infty}^-} + [Q, Q'] = [R, R']$$

where  $[R, R'] = -[S, S']$ . Where negation is taking negative of all  $y$ -coordinates.

So what is 2-torsion?

**Lemma 2.13** *Each non-zero element of  $J(\bar{K})[2]$  may be uniquely represented by the following pairs of points on  $C(\bar{K})$ , let  $x_1, \dots, x_6$  be the roots of  $f(x)$  then*

$$J(\bar{K})[2] = \{[T_i, T_k], i \neq k\}, T_i = (x_i, 0) \in C(\bar{K}).$$

**Remark 2.14** For the Richelot isogeny  $\phi$ :

$$\begin{array}{ccccc} J & \xrightarrow{\phi} & J' & \xrightarrow{\phi^\vee} & J \\ \uparrow & & \uparrow & & \\ C & \xrightarrow{\Gamma} & C' & & \end{array}$$

where  $\phi^\vee \circ \phi = [2]$  and  $\Gamma$  is a correspondence.

## 2.2 Richelot isogenies and the Richelot construction

Richelot isogenies are defined for Jacobians of genus 2 curves, they split multiplication by 2. Their codomain is the Jacobian of a curve, a model of which is explicitly given by the Richelot construction.

**Definition 2.15 The Richelot operator.** Given two polynomials  $P(x), Q(x) \in K[x]$  of degree at most 2 we define the **Richelot operator**  $[-, -]$  by

$$[P(x), Q(x)] = P'(x)Q(x) - Q'(x)P(x).$$

◇

**Definition 2.16 Richelot polynomials.** We say that a polynomial  $G(x) \in K[x]$  of degree 5 or 6 is a **Richelot polynomial** over  $K$  if we can fix a factorisation

$$G(x) = G_0(x)G_1(x)G_2(x)$$

where each  $G_i$  is of degree at most 2, defined over  $\bar{K}$  and defined over  $K$  as a set.

Write

$$G_i(x) = g_{i2}x^2 + g_{i1}x + g_{i0} = g_i(x - \alpha_i)(x - \beta_i)$$

for its factorisation over  $\bar{K}$  and define

$$\Delta_G = \det((g_{ij})_{0 \leq i, j \leq 2}).$$

◇

**Definition 2.17 Richelot dual polynomials.** To a Richelot polynomial  $G(x)$  with a fixed factorisation

$$G(x) = G_0(x)G_1(x)G_2(x)$$

such that  $\Delta_G \neq 0$ . We associate its **Richelot dual polynomial**  $F(x)$  given by

$$F(x) = \prod_{i=1}^3 F_i(x), \quad F_i(x) = \frac{1}{\Delta_G} [G_{i+1}(x), G_{i+2}(x)]$$

where we take indices mod 3. Write  $F_i(x) = f_i(x - A_i)(x - B_i)$

◇

$\Delta_G$  may not be defined over  $K$  but  $\Delta_G^2$  is.

**Definition 2.18 Richelot (dual) curves.** We say that a hyperelliptic curve  $C/K$  of genus 2 is a **Richelot curve** over  $K$  if it is given by  $y^2 = G(x)$  together with the factorisation

$$G(x) = G_0(x)G_1(x)G_2(x)$$

as a Richelot polynomial over  $K$  such that  $\Delta_G \neq 0$ .

To a Richelot curve  $C/K$  we associate its **Richelot dual curve**  $\widehat{C}$  given by

$$\widehat{C}: y^2 = F(x)$$

where  $F(x)$  is the Richelot dual polynomial of  $G(x)$  with respect to the given factorisation.

◇

**Remark 2.19** Let  $G(x) \in K[x]$  be a polynomial of degree 5 or 6. Denote by  $K_G$  its splitting field. Then the conditions for  $G(x)$  to be a Richelot polynomial can be rephrased as

$$\text{Gal}(K_G/K) \subseteq C_2^3 \rtimes S_3 \subseteq S_6$$

$$G(x) = G_0(x)G_1(x)G_2(x)$$

**Richelot isogenies.** **Definition 2.20 Richelot isogenies.** Let  $C/K$  be a Richelot curve with fixed factorisation

$$G(x) = G_0(x)G_1(x)G_2(x).$$

Let  $J$  be its Jacobian, consider the 2-torsion points of  $J(\bar{K})$  defined by the quadratic factorisation of  $G(x)$ .

$$D_i = [P_i, Q_i]$$

where  $P_i = (\alpha_i, 0)$ ,  $Q_i = (\beta_i, 0)$ . Then the isogeny over  $K$  for  $J$  whose kernel is  $\{0, D_1, D_2, D_3\}$  is called a **Richelot isogeny**.  $\diamond$

We say that a Jacobian admits a Richelot isogeny over  $K$  if its underlying curve is a Richelot curve  $/K$ .

**Theorem 2.21** Let  $C/K$  be a Richelot curve with fixed factorisation

$$G(x) = G_0(x)G_1(x)G_2(x).$$

Let  $\widehat{C}/K$  be its Richelot dual curve and let  $\phi$  denote the associated Richelot isogeny on  $J$ . Then  $\phi: J \rightarrow \widehat{J}$  where  $\widehat{J}$  is the Jacobian of  $\widehat{C}$  and moreover  $\hat{\phi}\phi = [2]$ .

Lecture ? 9/4/2018

**Brauer groups Galois cohomology and local invariants (Angus).** Reference Milne's CFT.

Central simple algebras:

We will consider finite dimensional  $k$ -algebra for  $k$  a field.

**Definition 2.22** A  $k$ -algebra  $A$  is central if the center  $Z(A) = k$ . A  $k$ -algebra is simple if the only two sided ideals are  $A$  and  $(0)$ .  $\diamond$

**Example 2.23** The matrix algebra  $M_n(k)$  is central simple for  $k$ .  $\square$

**Example 2.24** A quaternion algebra like  $\mathbf{H} = \mathbf{R}\{i, j, k\}$  is central simple for  $k$ .  $\square$

**Example 2.25** A division algebra is simple.  $\square$

**Definition 2.26** Two central simple  $k$ -algebras  $A, B$  are similar, if there exists  $m, n \in \mathbf{Z}_{>0}$  s.t.  $A \otimes_k M_m(k) \simeq B \otimes_k M_n(k)$ . Denote this by  $A \sim B$ .  $\diamond$

**Definition 2.27 Brauer groups.** The **Brauer group** of a field  $k$  denoted  $\text{Br}(k)$  is the set of similarity classes of central simple algebras  $[A]$  with operation

$$[A][B] = [A \otimes B].$$

$\diamond$

**Remark 2.28**

1. The class  $[M_n(k)]$  is the identity for all  $n$ .
2. The operation is well defined.
3. Given  $A$  let  $A^{\text{op}}$  be the algebra with order of multiplication reversed. Then

$$A \otimes_k A^{\text{op}} \xrightarrow{\sim} \text{End}_k(A) \simeq M_{\dim_k(A)}(k)$$

$$(a \otimes a') \mapsto (v \mapsto av a').$$

So

$$[A]^{-1} = [A^{\text{op}}].$$

Galois cohomology:

**Theorem 2.29 Noether-Skolem.** *Let  $A, B$  be central simple  $k$ -algebras and  $f, g: A \rightarrow B$  a  $k$ -algebra morphism. Then there exists*

$$b \in B^\times$$

such that

$$f(a) = bg(a)b^{-1}, \forall a \in A.$$

Let  $A$  be a central simple  $k$ -algebra with maximal subfield  $L/k$ .

Let  $\sigma \in \text{Gal}(\bar{k}/k)$ , it induces a map

$$\sigma: A \rightarrow A,$$

comparing this to the identity Noether-Skolem gives an element

$$e_\sigma \text{ s.t. } \sigma a = e_\sigma a e_\sigma^{-1}, \forall a \in L$$

defined up to multiplication by  $L^\times$ .

Given another  $\tau \in \text{Gal}(\bar{k}/k)$  I have

$$e_{\sigma\tau} a e_{\sigma\tau}^{-1} = \sigma(\tau a) = e_\sigma e_\tau a e_\tau^{-1} e_\sigma^{-1}$$

thus there exists

$$\phi(\sigma, \tau) \in L^\times$$

s.t.

$$e_{\sigma\tau} = \phi(\sigma, \tau) e_\sigma e_\tau$$

this gives a map

$$\{\text{central simple algebras}/k\} \rightarrow H^2(\text{Gal}(\bar{k}/k), \bar{k}^\times).$$

**Theorem 2.30** *This descends to*

$$\text{Br}(k) \simeq H^2(\text{Gal}(\bar{k}/k), \bar{k}^\times).$$

Some special  $k$ .

**Theorem 2.31 Wedderburn.** *Every central simple  $k$ -algebra is isomorphic to  $M_n(D)$  for  $D$  a division  $k$ -algebra.*

**Proposition 2.32** *If  $k = \bar{k}$  then any division  $k$ -algebra  $D$  is isomorphic to  $k$ . Thus  $\text{Br}(k) = 0$ .*

**Theorem 2.33 Wedderburn.** *Every finite division ring is a field. So if  $k$  is a finite field then  $\text{Br}(k) = 0$ .*

**Theorem 2.34 Frobenius.** *Every central division  $\mathbf{R}$ -algebra is isomorphic to either  $\mathbf{R}$  or  $\mathbf{H}$ . Thus  $\text{Br}(\mathbf{R}) \simeq \mathbf{Z}/2$ .*

Let  $k$  be a non-archimidean local field with valuation

$$v: k^\times \rightarrow \mathbf{Z}$$

for a central division algebra  $D$  there exists  $n \in \mathbf{Z}$  s.t.

$$v: D^\times \rightarrow \frac{1}{n}\mathbf{Z}.$$

Consider a maximal unramified subfield

$$K \subseteq L \subseteq D$$

with  $\sigma \in \text{Gal}(L/K)$  lifting Frobenius.

Noether-Skolem gives  $\alpha \in D^\times$  s.t.

$$\sigma x = \alpha x \alpha^{-1}, \forall x \in L$$

up to  $L^\times$ .

If we take  $\alpha' = c\alpha$  for  $c \in L^\times$  we can compute

$$v(\alpha') = v(c) + v(\alpha) \equiv v(\alpha) \pmod{\mathbf{Z}}.$$

We get a map

$$\{\text{central division algebras}/k\} \rightarrow \mathbf{Q}/\mathbf{Z}.$$

**Theorem 2.35** *This descends to an isomorphism*

$$\text{Br}(k) \simeq \mathbf{Q}/\mathbf{Z}.$$

If  $F$  is a number field with a place  $v \in |F|$  get a map

$$\iota_v^{-1}: \text{Br}(F) \rightarrow \text{Br}(F_v) \simeq \begin{cases} 0, & F_v = \mathbf{C}, \\ \mathbf{Z}/2, & F_v = \mathbf{R}, \\ \mathbf{Q}/\mathbf{Z}, & F_v \text{ nonarch.} \end{cases}.$$

Global CFT gives an exact seq

$$0 \rightarrow \text{Br}(F) \rightarrow \bigoplus_v \text{Br}(F_v) \rightarrow \mathbf{Q}/\mathbf{Z} \rightarrow 0.$$

**Root numbers of elliptic curves (Ricky).** Based on Rohrlich's article elliptic curves and the Weil-Deligne group

$K$  non-archimidean local field,  $\bar{K}$  is its separable closure.

$$\phi = (x \mapsto x^q)^{-1} \in \text{Gal}(\bar{k}/k), \quad q = |k|$$

$\Phi$  some lift of  $\phi$  in  $\text{Gal}(\bar{K}/K)$ .

$W(\bar{K}/K)$  = Weil group, the preimage of  $\langle \phi \rangle$  in  $\text{Gal}(\bar{k}/k)$  under  $G_K \twoheadrightarrow G_k$ .

We consider  $\sigma: W(\bar{K}/K) \rightarrow \text{GL}(v)$ , representations over  $V/\mathbf{C}$  (always cts.)

Say  $\sigma$  is of Galois type if it factors through a finite quotient.

Another source of examples is

$$\omega: W \rightarrow \mathbf{C}^\times$$

given by

$$\omega(I) = \{1\}$$

where

$$I = \ker(G_K \rightarrow G_k)$$

and  $\omega(\Phi) = q^{-1}$ .

Fact, all irreducible  $\sigma \cong \rho \otimes \omega^s$  for some  $s \in \mathbf{C}$  and  $\rho$  of Galois type.

**Definition 2.36 The Weil-Deligne group.** The Weil-Deligne group is

$$W'(\overline{K}/K) = W(\overline{K}/K) \ltimes \mathbb{C}$$

where  $W$  acts on  $\mathbb{C}$  via  $\omega$

$$gzg^{-1} = \omega(g)z, \quad g \in W(\overline{K}/K), z \in \mathbb{C}.$$

◇

Upshot: Representations  $\sigma'$  of  $W'$  are the same as  $(\sigma, N)$  where

$$\sigma: W \rightarrow \mathrm{GL}(V)$$

a representation and  $N$  is a nilpotent linear operator on  $V$ . Satisfying

$$\sigma(g)N\sigma(g)^{-1} = \omega(g)N.$$

One motivation for studying those is a general construction of Grothendieck and Deligne which turn an  $l$ -adic representation of  $G_K$  into a representation of  $W'$  (given  $i: \mathbb{Q}_l \hookrightarrow \mathbb{C}$ ).

**Example 2.37**

$$\mathrm{sp}(n) = \mathbb{C}^n$$

with action of  $W'$  given by

$$\sigma(g)e_j = \omega(g)^j e_j, \quad \forall g \in W$$

$$Ne_j = e_{j+1}, \quad Ne_n = 0$$

check relation  $\sigma N \sigma^{-1} = \omega N$ . □

We want to define  $\epsilon$ -factors for representations of  $W'$ . We need two choices:

$$\psi: K \rightarrow \mathbb{C}^\times$$

an additive character of  $K$ . And

$$dx$$

a Haar measure on  $K$ .

Then

$$\epsilon(\sigma', \psi, dx) = \epsilon(\sigma, \psi, dx) \delta(\sigma')$$

where

$$\delta(\sigma') = \det(-N|V^I/V_N^I)$$

and  $\epsilon(\sigma, \psi, dx)$  is defined by the following proposition.

**Proposition 2.38 Deligne-Langlands.** *There exists a unique function  $\epsilon(\sigma, \psi, dx)$  satisfying*

1.  $\epsilon(*, \psi, dx)$  is multiplicative in short exact sequences.
2. If  $L/K$  is finite then

$$\epsilon(\mathrm{Ind}_{L/K} \rho, \psi, dx) = \epsilon(\rho, \psi \circ \mathrm{Tr}_{L/K}, dx_L) \cdot \left( \epsilon(\mathrm{Ind}_{L/K} 1_L, \psi, dx) / \epsilon(1_L, \psi \circ \mathrm{Tr}_{L/K}, dx_L) \right)^{\dim \rho}$$

3. For  $\chi$  a character

$$\epsilon(\chi, \psi, dx)$$

agrees with the ones defined in Tate's thesis. They're both given by an integral formula.



**Definition 2.39 Root numbers.** The **root number** of  $\sigma'$  is defined to be

$$w(\sigma', \psi) = \frac{\epsilon(\sigma', y, dx)}{|\epsilon(\sigma', y, dx)|}.$$

◇

For  $E/K$  an elliptic curve we have a representation on  $V_l^*$  ( $l \neq p$ ).

Using the Grothendieck-Deligne construction, let  $\sigma_{E/K}$  be a representation of  $W'$  it has the following property

- $E$  pot. good reduction then

$$N_{E/K} = 0$$

and  $\sigma_{E/K}$  is semisimple.  $E$  has good reduction iff  $\sigma_{E/K}$  is unramified.

- $E$  has potential multiplicative reduction implies that we can take  $\chi$  a character of  $W$  with  $\chi^2 = 1$ , so that

$$E^\chi$$

has split multiplicative reduction. Then

$$\sigma'_{E/K} \simeq \chi \omega^{-1} \otimes \text{sp}(2)$$

$\chi$  is trivial / unramified and non-trivial / ramified according to  $E$  having split / non-split / additive reduction.

- $\sigma'_{E/K}$  is essentially symplectic.  $W(E/K) = W(\sigma'_{E/K})$  is independent of  $\psi$  and must be  $\pm 1$ .

**Proposition 2.40**

1.  $E$  has good reduction implies  $W(E/K) = 1$ .
2.  $E$  potentially multiplicative reduction implies

$$W(E/K) = \begin{cases} -1 & \text{split} \\ 1 & \text{nonsplit} \end{cases}.$$

If additive reduction take  $\xi$  quadratic character s.t.

$$E^\xi$$

has split multiplicative reduction and  $W(E/K) = \xi(-1)$ .