Ranks and Parity of Ranks of Curves and Abelian Surfaces

MA842 at BU Spring 2019

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1 Mordell-Weil groups

These are notes for Céline Maistret's course MA842 at BU Spring 2019.

The course webpage is https://sites.google.com/view/cmaistret/teaching#h.p_BYGoPzU848FJ.

Course overview:

Main references that we will be following:

1. Vladimir Dokchitser - Notes?
2. Silverman - Arithmetic of Elliptic Curves
3. Milne - Abelian Varieties?

1.1 Mordell-Weil

Lecture 4 5/2/2018

Remark 1.1 A homomorphism $\phi\colon \mathrm{Gal}(\overline{K}/K) \to G$ for a finite group G is continuous if it comes from a finite Galois extension, i.e.

$$\exists F/K \text{ finite Galois }, \tilde{\phi}\colon \mathrm{Gal}(F/K) \to G$$

s.t. ϕ is the composition $\mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(F/K) \xrightarrow{\tilde{\phi}} G$. So $\phi(g)$ only cares about what g does to F.

Proposition 1.2 *Let E/K be an elliptic curve*

$$y^2 = (x-\alpha)(x-\beta)(x-\gamma)$$

for $P \in E(K)$ have $\frac{1}{2}P \in E(\overline{K})$ s.t. $\frac{1}{2}P \oplus \frac{1}{2}P = P$.

1. *$K(\frac{1}{2}P)/K$ is a Galois extension and $\mathrm{Gal}(K(\frac{1}{2}P)/K) = C_2 \times C_2$ from Lemma 1.*
2. $$\phi_P\colon \mathrm{Gal}(\overline{K}/K) \to E(K)[2]$$ $$g \mapsto Q^\sigma - Q = g(\frac{1}{2}P) - \frac{1}{2}P$$ *is well defined and has kernel $\mathrm{Gal}(K/K(\frac{1}{2}P))$.*

3.
$$\phi\colon E(K)/2E(K) \to \mathrm{Hom}_{cts}(\mathrm{Gal}(\overline{K}/K), E(K)[2])$$
$$P \mapsto \phi_P$$

is well defined and injective. Now ϕ_P *is continuous by* 2. *and so*

$$\phi_{P\oplus Q}(g) = g(\frac{1}{2}(P \oplus Q)) - (\frac{1}{2}P \oplus \frac{1}{2}Q)$$
$$= g(\frac{1}{2}P) \oplus g(\frac{1}{2}Q) - \frac{1}{2}P \ominus \frac{1}{2}Q$$
$$= \phi_P(g) \oplus \phi_Q(g)$$

a homomorphism.

$$\phi_{2Q}(g) = g(\frac{1}{2}2Q)) - \frac{1}{2}2(Q) = g(Q) - Q = 0$$

for all $g \in \mathrm{Gal}(\overline{K}/K)$ *if* $Q \in E(K)$ *so this is well defined. For injectivity:*

$$\phi_P(g) = 0 \implies g(\frac{1}{2}P) = \frac{1}{2}P \forall g \in \mathrm{Gal}(\overline{K}/K)$$
$$\implies \frac{1}{2}P \in E(K) \implies P \in 2E(K)$$

which gives injectivity.

4.
$$\eta\colon \mathrm{Hom}_{cts}(\mathrm{Gal}(\overline{K}/K), E(K)[2]) \to K^\times/K^{\times 2} \times K^\times/K^{\times 2} \times K^\times/K^{\times 2}$$
$$\psi \mapsto \psi_\alpha, \psi_\beta, \psi_\gamma$$
$$\psi(g) \in \{0, (\alpha, 0)\} \subseteq E(K) \iff g \in \mathrm{Gal}(\overline{K}/K(\sqrt{\psi_\alpha}))$$

then η *is an injective homomorphism. It is an isomorphism to the subgroup of triples* a, b, c *s.t.* $abc \in K^{\times 2}$. *Proof:*

$$\mathrm{Hom}_{cts}(\mathrm{Gal}(\overline{K}/K), C_2) \simeq K^\times/K^{\times 2}$$

with ψ *s.t.* $\ker\psi = \mathrm{Gal}(\overline{K}/K\sqrt{d}) \leftrightarrow d$. *It is an isomorphism:*

$$\ker\psi_i = \mathrm{Gal}(\overline{K}/K(\sqrt{d_i})),\ i = 1, 2$$
$$\ker\psi_1\psi_2 = \mathrm{Gal}(\overline{K}/K(\sqrt{d_1 d_2}))$$

Now apply this to $E(K)[2] = C_2 \times C_2$ *to get an isomorphism to* $K^\times/K^{\times 2} \times K^\times/K^{\times 2}$. *Record this third homomorphism to get* η.

5. *If* $P = (x_0, y_0) \in E(K)$ *then*

$$\eta(\phi_P) = (x_0 - \alpha, x_0 - \beta, x_0 - \gamma).$$

Proof sketch: If
$$E\colon y^2 = x^3 + Ax^2 + Bx$$
then for $Q = (x_0, y_0) \in E(K)$.

$$2Q = \left(\left(\frac{x_0 - B}{2y_0}\right)^2, \dots\right)$$

Hence if $2Q = P = (x_1, y_1)$ *then* $\sqrt{x_1} \in K(\frac{1}{2}P)$. *So if*

$$E\colon y^2 = (x-\alpha)(x-\beta)(x-\gamma)$$

then

$$P = (x_2, y_2)$$

then

$$\sqrt{x_2-\alpha}, \sqrt{x_2-\beta}, \sqrt{x_2-\gamma} \in K(\frac{1}{2}P)$$

$$K(\sqrt{x_2-\alpha}), K(\sqrt{x_2-\beta}), K(\sqrt{x_2-\gamma}) \subseteq K(\frac{1}{2}P)$$

$$\implies K(\frac{1}{2}P) = K(\sqrt{x_2-\alpha}, \sqrt{x_2-\beta}, \sqrt{x_2-\gamma})$$

Example 1.3 Let

$$E\colon y^2 = x(x-1)(x+1)$$

for $P \in E(\mathbf{Q})$, $\mathbf{Q}(\frac{1}{2}P)/\mathbf{Q}$ can only ramify at 2.

$$\mathbf{Q}(\frac{1}{2}P) \subseteq \mathbf{Q}(i, \sqrt{2})$$

$$P = (x_0, y_0) \mapsto x_0, x_0 - 1, x_0 + 1 \in \mathbf{Q}^\times/\mathbf{Q}^{\times 2}$$

is a homomorphism so x_0, x_0-1, x_0+1 are $\pm 1, \pm 2$ up to square.

x_0	$x_0 - 1$	$x_0 + 1$	rat?
1	1	1	1) rat
1	-1	-1	2) non-rat
1	2	2	1) rat
1	-2	-2	2) non-rat
-1	1	-1	2) non-rat
-1	-1	1	1) rat
-1	2	-1	2) non-rat
-1	-2	2	1) rat
2	1	2	3) non-rat
2	-1	-2	2) non-rat
2	2	1	4) rat
2	-2	-1	2) non-rat
-2	1	-2	?
-2	-1	2	?
-2	2	-1	?
-2	-2	1	?

Table 1.4: Images

1) The 2-torsion points $P = 0, (0,0), (1,0), (-1,0) \in E(\mathbf{Q})$ give us some rows. 2) As we have $x_0 > -1$ we get $x_0 + 1 > 0$ so $x_0(x_0-1) > 0$ for the product to be a square (and hence > 0). 3) $x_0 = 2A^2$, $x_0 - 1 = B^2$, $x_0 + 1 = 2C^2$ with $A, B, C \in \mathbf{Q} \smallsetminus \{0\}$. Let $A = m/n$ so $2m^2/n^2 - 1 = B^2$

$$2m^2 - n^2 = (Bn)^2$$

and

$$2m^2 + n^2 = 2(Cn)^2$$

if $m \equiv 0(2) \implies -1 = \square \pmod 8$ a contradiction.

$$m \equiv 1 \pmod 2 \implies m^2 \equiv 1 \pmod 8.$$

So $2 - n^2 = \square \pmod 8 \implies n^2 \equiv 1 \pmod 8$

$$2 + n^2 = 2\square \pmod 8 \implies n^2 \equiv 0 \pmod 8$$

$$|E(\mathbf{Q})/2E(\mathbf{Q})| = 4$$
$$|E(\mathbf{Q})[2]| = 4 \implies \text{rk} = 0$$
$$E(\mathbf{Q}) \cong E(\mathbf{Q})[2].$$

4) Use the group structure! $\square$

Theorem 1.5 Complete 2-decent. *Let K be a field of characteristic 0 and*

$$E\colon y^2 = (x-\alpha)(x-\beta)(x-\gamma),\ \alpha, \beta, \gamma \text{ distinct.}$$

The map

$$P \mapsto (x_0 - \alpha, x_0 - \beta, x_0 - \gamma)$$

replacing $x_0 - \alpha$ with $(x_0 - \beta)(x_0 - \gamma)$ if 0.

$$E(K)/2E(K) \to (K^\times/K^{\times 2})^3$$

Triples (a, b, c) that lie in the image satisfy $abc \in K^{\times 2}$. A triple a, b, c with $abc \in K^{\times 2}$ lies in the image iff it is in the image of $E(K)[2]$ or

$$cz_3^2 - \alpha + \gamma = az_1^2$$

$$cz_3^2 - \beta + \gamma = bz_1^2$$

is soluble with $z_i \in K^\times$. In which case

$$P = (az_1^2 + \alpha, \sqrt{abc}, z_1 z_2 z_3) \mapsto (a, b, c)$$

iii) If K is a number field and (a, b, c) is in the image then

$$K(\sqrt{a}, \sqrt{b}, \sqrt{c})/K$$

only ramifies at primes dividing $2(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)$.

Exercise 1.6

$$E\colon y^2 = x(x-5)(x+5).$$

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Recall:

$$\phi\colon E(K)/2E(K) \to \mathrm{Hom}_{cts}(G_K, E(K)[2])$$
$$P \mapsto \phi_P$$

where $\phi_P\colon \sigma \mapsto Q^\sigma - Q$ where $Q = 2P$. Which is well-defined and injective.

Elements of

$$\mathrm{Hom}_{cts}(G_K, E[2]) \leftrightarrow a, b, c \in (K^\times/K^{\times 2}) \text{ s.t. } abc \in K^{\times 2}$$

$$(x_0, y_0) \mapsto (x_0 - \alpha, x_0 - \beta, x_0 - \gamma).$$

Lemma 1.7 *Let $n \geq 1$*

1.
$$\psi\colon E(K)/nE(K) \longrightarrow \{K \subseteq F \subseteq \overline{K}\}$$
$$P \mapsto K(\frac{1}{n}P, E[n])$$
is well defined.

2. *$K(\frac{1}{n}P, E[n])/K$ only ramifies at $\mathfrak{p}|n\Delta_E$.*

3.
$$\mathrm{Gal}(K(\frac{1}{n}P, E[n])/K) \leq \mathbf{Z}/n \times \mathbf{Z}/n$$

4. *There are only finitely many fields satisfying 2. and 3. so $\operatorname{im}\psi$ is finite.*

To do descent, need more than ψ (i.e. injection).

Definition 1.8 Let G be a group and M a G-module then let

$$H^0(G, M) = M^G = \{m \in M : gm = m \forall g \in G\}$$

$H^1(G, M) = \{\text{skew homs } G \to M\}/\{\text{skew homs } G \to M \text{ of the form } g \mapsto g(t) - t,\ t \in M\}$.

◊

Remark 1.9 If G acts trivially on M then

$$H^0(G, M) = M$$
$$H^1(G, M) = \mathrm{Hom}(G, M).$$

When G is profinite then we want that the skew homomorphisms factor through finite Galois groups. We will prove that

$$E(K)/nE(K) \hookrightarrow H^1(G_K, E[n]).$$

Theorem 1.10 *If*
$$0 \to A \to B \to C \to 0$$
is an exact sequence of G-modules then

$$0 \to H^0(G, A) \to H^0(G, B) \to H^0(G, C) \to H^1(G, A) \to H^1(G, B) \to H^1(G, C).$$

Lemma 1.11

1. *ψ is finite-to-one (gives Mordell-Weil)*

2. *Let*
$$\phi_P\colon G_K \to E[n]$$
$$\phi_P(gh) = \phi_P(g) + g\phi_P(h)$$
is a skew (or crossed) homomorphism. If $(\frac{1}{n}P)'$ is another choice of $\frac{1}{n}P$ and ϕ'_P is the corresponding skew homomorphism, then
$$\phi_P - \phi'_P$$
is of the form
$$g \mapsto T \ominus gT$$
where $T \in E[n]$.

3. *ϕ_P factors through*

$$\mathrm{Gal}(K(\frac{1}{n}P, E[n])/K).$$

4.

$$\phi\colon E(K)/nE(K) \to Z/B$$
$$P \mapsto \phi_P$$

is an injective homomorphism. Where

$$Z = \{\textit{skew homs } G_K \to E[n]\}$$
$$B = \{\textit{skew homs } G_K \to E[n] \textit{ of the form } g \mapsto T \ominus gT,\ T \in E[n]\}.$$

Proof.

1. There are finitely many skew homomorphisms

$$\mathrm{Gal}(K(\frac{1}{n}P, E[n])/K) \to E[n]$$

and by 4.

$$P \mapsto \{\phi_P, K(\frac{1}{n}P, E[n])\}$$

is injective. So $\psi\colon P \mapsto K(\frac{1}{n}P, E[n])$ is finite to one by 3.

2.

$$\phi_P(gh) = \frac{1}{n}P \ominus gh\frac{1}{n}P$$
$$= \left((\frac{1}{n}P) \ominus g(\frac{1}{n}P)\right) \oplus \left(g(\frac{1}{n}P) \ominus g(h(\frac{1}{n}P))\right)$$
$$= \phi_P \oplus g(\phi_P(h)).$$

Remark: If $E[n] \subseteq E(K)$ then ϕ_P is a homomorphism. Recall for $n = 2$

$$\phi_P(gh) = \frac{1}{2}P \ominus gh(\frac{1}{2}P)$$
$$= \frac{1}{2}P \ominus h(\frac{1}{2}P) \oplus h(\frac{1}{2}P) \ominus g(h(\frac{1}{2}P))$$
$$= \phi_P(h) \oplus \phi_P(g)$$

since $2h(\frac{1}{2}P) = h(P) = P$. Consider now

$$\frac{1}{n}P = \frac{1}{n}P' \oplus T$$

for some $T \in E[n]$

$$(\phi_P \ominus \phi'_P)(g) = \phi_P(g) - \phi'_P(g) = \frac{1}{n}P \ominus g(\frac{1}{n}P) - [(\frac{1}{n}P) \oplus T \ominus g(\frac{1}{n}P) \oplus gT]$$
$$= T \ominus gT.$$

■

Take $G = G_K$

$$B = E(\overline{K}), A = E[n], C = E(\overline{K})$$

to get

$$0 \to E[n] \to E(\overline{K}) \xrightarrow{\cdot n} E(\overline{K}) \to 0$$

which gives the long exact sequence

$$0 \to E(K)[n] \to E(K) \xrightarrow{\cdot n} E(K) \xrightarrow{\delta} H^1(G_K, E[n]) \to H^1(G_K, E(\overline{K})) \to$$

$$\implies E(K)/nE(K) \hookrightarrow H^1(G_K, E[n]).$$

Problem:

$$H^1(G_K, E[n])$$

is infinite. What subgroup of

$$H^1(G_K, E[n])$$

do we land in?

Notation: When v is a place of K we have $G_{K_v} \subseteq G_K$, for any module M have $M^{G_K} \le M^{G_{K_v}}$ and

$$\operatorname{Res}\colon H^1(G_K, E[n]) \to H^1(G_{K_v}, E[n]).$$

We have from the theorem

$$\begin{tikzcd}
0 \arrow[r] & E(K)/nE(K) \arrow[r, "\delta"] \arrow[d] & H^1(G_K, E[n]) \arrow[r] \arrow[d, "\mathrm{Res}"] & H^1(G_K, E(\overline{K}))[n] \arrow[r] \arrow[d, "\mathrm{Res}"] & 0 \\
0 \arrow[r] & \prod_v E(K_v)/nE(K_v) \arrow[r, "\delta"] & \prod_v H^1(G_{K_v}, E[n]) \arrow[r] & \prod_v H^1(G_{K_v}, E(\overline{K}))[n] \arrow[r] & 0
\end{tikzcd}$$

we want to understand $\operatorname{im}\delta$ i.e. the subgroup

$$\ker\{H^1(G_K, E[n]) \to H^1(G_K, E(\overline{K}))\}$$

this is as hard as finding $E(K)$, here is why:

Claim 1.12

$$H^1(G_K, E(\overline{K}))$$

corresponding to principal homogeneous spaces for E (genus 1 curves whose jacobian is E)

Finding

$$\ker\{H^1(G_K, E[n]) \to H^1(G_K, E(\overline{K}))\}$$

is equivalent to finding which PHS coming from H^1 have a rational point. ??? Hensel's lemma.

Let C be a curve

$$\operatorname{Isom}(C) \leftrightarrow C(\overline{K}) \times \operatorname{Aut}(C)$$

$$\tau_p \circ \alpha \leftrightarrow (P, \sigma)$$

$$\operatorname{Twist}(E/K) \leftrightarrow H^1(G_K, \operatorname{Isom}(C))$$

$$C \simeq_{\overline{K}} E$$

$$PHS \leftrightarrow H^1(G_K, E(\overline{K}))$$

C is a PHS for E iff E is the jacobian of C.

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$$\begin{tikzcd}
0 \arrow[r] & E(K)/nE(K) \arrow[r, "\delta"] \arrow[d] & H^1(G_K, E[n]) \arrow[r] \arrow[d, "\mathrm{Res}"] & H^1(G_K, E(\overline{K}))[n] \arrow[r] \arrow[d, "\mathrm{Res}"] & 0 \\
0 \arrow[r] & \prod_v E(K_v)/nE(K_v) \arrow[r, "\delta"] & \prod_v H^1(G_{K_v}, E[n]) \arrow[r] & \prod_v H^1(G_{K_v}, E(\overline{K}))[n] \arrow[r] & 0
\end{tikzcd}$$

Definition 1.13 Twists of curves. A **twist** of C/K is a smooth curve C'/K that is isomorphic to C over $\overline{K}$. ◊

If C_1, C_2 are twists of C/K and $C_1 \simeq_K C_2$ then we say that C_1 and C_2 are equivalent modulo K-isomorphism.

We denote Twist(C/K) - the set of twists of C/K modulo K-isomorphism.

Theorem 1.14 *The twists of C/K up to K-isomorphism are in 1-1 correspondence with elements of*

$$H^1(G_K, \mathrm{Isom}(C))$$

where

$$\mathrm{Isom}(C) = \{\overline{K}\text{-isomorphisms } C \to C\}.$$

Proof. Let C'/K be a twist of C/K then there exists an isomorphism $/\overline{K}$

$$\phi\colon C' \to C$$

associate the following map

$$\xi\colon G_K \to \mathrm{Isom}(C)$$

$$\sigma \mapsto \phi^\sigma \phi^{-1}.$$

Check that ξ is a cocycle

$$\xi_{\sigma\tau} = (\xi_\sigma)^\tau \xi_\tau$$

for all $\sigma, \tau \in G_K$. Denote $\{\xi\}$ the associated class in H^1. $\{\xi\}$ is determined by the K-isomorphism class of C' independent of the choice ϕ.

The map

$$\mathrm{Twist}(C/K) \leftrightarrow H^1(G_K, \mathrm{Isom}(C))$$

$$C' \mapsto \{\xi\}$$

is a bijection.

Injective, trace through.

Surjectivity, define the function field using the curve. ■

Remark 1.15 If C is an elliptic curve then Isom(C) is generated by

$$\mathrm{Aut}(C)(\text{fixing } 0)$$

and translations

$$\tau_P\colon C \to C$$

$$Q \mapsto Q + P.$$

Example 1.16 E/K elliptic, consider

$$K(\sqrt{d})$$

a quadratic extension and χ the associated character

$$\chi\colon G_K \to \{\pm 1\}$$
$$\sigma \mapsto \sigma(\sqrt{d})/\sqrt{d}.$$

The group ± 1 can be viewed as automorphisms of C. So use χ to define the cocycle

$$\xi\colon G_K \to \mathrm{Isom}(C)$$
$$\sigma \mapsto [\chi(\sigma)].$$

Let C/K be the corresponding twist of E/K, we find an equation for C/K. Choose

$$y^2 = f(x) \text{ for } E/K$$

and write

$$\overline{K}(E) = \overline{K}(x, y)$$
$$\overline{K}(C) = \overline{K}(x, y)_\xi$$

since $[-1](x, y) = (x, -y)$ the action of $\sigma \in G_K$ on

$$\overline{K}(x, y)_\xi \text{ is given by } \sqrt{d}^\sigma = \chi(\sigma)\sqrt{d}$$
$$x^\sigma = x,\ y = \chi(\sigma)y$$

note that the function $x' = x$ and $y' = y/\sqrt{d}$ are in $\overline{K}(x, y)_\xi$ and are fixed by G_K. Now x', y' satisfy

$$dy'^2 = f(x')/K$$

is defined over K and defines an elliptic curve. Moreover

$$(x, y) \mapsto (x', y'\sqrt{d})$$

is an isomorphism over $K(\sqrt{d})$. □

Note C/K is not a principal homogeneous space for E/K.

Definition 1.17 Homogenous spaces. Let E/K be an elliptic curve, a principal homogeneous space for E/K is a smooth curve C/K together with a simply transitive algebraic group action of E on C defined over K.

$$\mu\colon C \times E \to C$$

morphism defined over K satisfying

1. $$\mu(P, 0) = P\ \forall P \in C$$

2. $$\mu(\mu(p, P), Q) = \mu(p, P + Q)\ \forall P \in C$$

3. $$\forall p, q \in C,\ \exists! P \in E \text{ s.t.}$$
$$\mu(p, P) = q$$

   so we may define a subtraction map

$$\nu\colon C \times C \to E$$
$$p, q \mapsto P$$

   as above.

$\lozenge$

Proposition 1.18 *Let E/K and C/K be a principal homogeneous space for E/K. Fix a point $p_0 \in C$ and define a map*

$$\theta \colon E \to C$$

$$P \mapsto \underbrace{p_0 + P}_{\mu(p_0,P)} .$$

1. *θ is an isomorphism over $K(p_0)$. In particular C/K is a twist of E/K.*
2. *$\forall p, q \in C$*
$$q - p = \theta^{-1}(q) - \theta^{-1}(p).$$
3. *θ is a morphism over K.*

Definition 1.19 Two homogeneous space C/K and C'/K for E/K are equivalent if there is an isomorphism

$$\phi \colon C \to C'$$

defined over K and is compatible with the action of E on C and C'.

$$\begin{tikzcd} C \arrow[r, "\theta"] \arrow[d, "\phi"'] & E \arrow[d] \\ C' \arrow[r] & E' \end{tikzcd}$$

$\lozenge$

The equivalence class of PHS for E/K containing E/K acting on itself via translation is called the trivial class.

The collection of equivalence classes of PHS for E/K is called the Weil-Châtelet group, denoted

$$WC(E/K).$$

Proposition 1.20 *Let C/K be a PHS for E/K then C/K is in the trivial class $\iff$ $C(K) \neq \emptyset$.*

Theorem 1.21 *Let E/K then there is a natural bijection after fixing $p_0 \in C$*

$$WC(E/K) \to H^1(G_K, \underbrace{E(\overline{K})}_{\subseteq \mathrm{Isom}(E)})$$

$$\{C/K\} \mapsto \{\sigma \mapsto p_0^\sigma - p_0\}$$

Proof. Well-definedness:

$$\sigma \mapsto p_0^\sigma - p_0$$

is a cocycle. Suppose that C'/K and C/K are two equivalent PHS then

$$p_0^\sigma - p_0$$

and

$$p_0'^\sigma - p_0'$$

are cohomologous.

Injective, suppose that $p_0^\sigma - p_0$ and $p_0'^\sigma - p_0'$ corresponding to C/K and C'/K that are cohomologous and prove that $C \simeq_K C'$.

Surjective: let $\xi \colon G_K \to E(\overline{K})$ be a cocycle representing an element in

$H^1(G_K, E)$. Embed

$$E(\overline{K}) \hookrightarrow \mathrm{Isom}(E)$$
$$P \mapsto \tau_P$$

and view

$$\xi \in H^1(G_K, \mathrm{Isom}\, E).$$

From the theorem on

$$\mathrm{Twist}(E/K) \leftrightarrow H^1(G_K, \mathrm{Isom}(E))$$

there exists a curve C/K and a $\overline{K}$-isomorphism

$$\phi\colon C \to E$$

s.t.

$$\forall \sigma \in G_K : \phi^\sigma \phi^{-1} = \text{translation by } -\xi_\sigma.$$

Define a map $\mu\colon C \times E \to C$

$$(p, Q) \mapsto \phi^{-1}(\phi(p) + Q).$$

Show that μ is simply transitive.

Show μ defined over K. Compute the cohomology class associated to C/K and show it is ξ. ■

Remark 1.22 For a given C/K of genus 1 one can define several structures of PHS.

$$\{C/K, \mu\}^\alpha = \{C/K, \mu \circ (1 \times \alpha)\}$$
$$\mu^\alpha(p, Q) = \mu(p, \alpha Q)$$

for $\alpha \in \mathrm{Aut}(E)$.

$$\begin{tikzcd} C \arrow[r, "\mu"] & E \arrow[d, "p"'] \\ C' \arrow[r, "\mu^\alpha"] & E' \end{tikzcd}$$

Lecture 7 21/2/2018

Example 1.23 E/K and $K(\sqrt{d})/K$ a quadratic extension. Let $T \in E(K)$ be a non-trivial point of order 2. Then $\xi\colon G_K \to E$

$$\sigma \mapsto \begin{cases} 0 & \text{if } (\sqrt{d})^\sigma = \sqrt{d}, \\ T & \text{if } (\sqrt{d})^\sigma = -\sqrt{d}. \end{cases}.$$

We construct the PHS corresponding to $\{\xi\} \in H^1(G_K, E(\overline{K}))$. Since $T \in E(K)$ can choose a Weierstraß equation for E/K

$$E\colon y^2 = x^3 + ax^2 + bx \text{ with } T = (0, 0)$$

then the translation by T map is given by

$$\tau_T(P) = (x, y) + (0, 0) = \left(\frac{b}{x}, -\frac{by}{x^2}\right)$$

for

$$P = (x, y).$$

Thus if $\sigma \in G_K$ is non-trivial, σ acts on $\overline{K}(E)_\xi$, which is isomorphic to $\overline{K}(E)$ but $\mathrm{Gal}(\overline{K}/K)$ action is twisted by ξ, i.e. $x^{\mathrm{id}} \mapsto (x^{\mathrm{id}})^\sigma$.

$$(\sqrt{d})^\sigma = -\sqrt{d}$$

$$x^\sigma = \frac{b}{x},\ y^\sigma = -\frac{by}{x^2}$$

need to find the subfield of $K(\sqrt{d})(x,y)_\xi$ fixed by σ. Note:

$$\frac{\sqrt{d}x}{y},\ \sqrt{d}\left(x - \frac{b}{x}\right)$$

are invariant, take

$$z = \frac{\sqrt{d}x}{y},\ w = \sqrt{d}\left(x - \frac{b}{x}\right)\left(\frac{x}{y}\right)^2$$

and find relations between z and w to get

$$C\colon dw^2 = d^2 - 2adz^2 + (a^2 - 4b)z^4.$$

Claim: C/K is the PHS of E/K corresponding to $\{\xi\}$. There is a natural map

$$\phi\colon E \to C$$

$$(x,y) \mapsto (z,w)$$

$$(x,y) \mapsto \left(\frac{\sqrt{d}y}{x^2+ax+b}, \frac{\sqrt{d}(x^2-b)}{x^2+ax+b}\right)$$

so that

$$\phi(0,0) = (0, -\sqrt{d})$$

$$\phi(0) = (0, \sqrt{d})$$

- Prove that ϕ is an isomorphism so C is a twist.
- C is the PHS corresponding to $\{\xi\}$. Take $p \in C$ and compute

  $$\sigma \mapsto p^\sigma - p = \phi^{-1}(p^\sigma) - \phi^{-1}(p)$$

  for example let $p = (0, \sqrt{d}) \in C$, if $\sigma = \mathrm{id}$ then $p^\sigma - p = 0 - 0 = 0$. If $\sigma = -\mathrm{id}$ then $p^\sigma - p = T - 0 = T$.

$\square$

Back to Selmer, we want to have the image of our weak Mordell-Weil land in something finite.

$$\begin{tikzcd}
0 \arrow[r] & E(K)/mE(K) \arrow[r, "\delta"] \arrow[d] & H^1(G_K, E[m]) \arrow[r] \arrow[d, "\mathrm{Res}"] & WC(E/K)[m] \arrow[r] \arrow[d, "\mathrm{Res}"] & 0 \\
0 \arrow[r] & \prod_v E(K_v)/nE(K_v) \arrow[r, "\delta"] & \prod_v H^1(G_{K_v}, E[n]) \arrow[r] & \prod_v WC(E/K_v)[m] \arrow[r] & 0
\end{tikzcd}$$

Definition 1.24 ***m*-Selmer groups.** The m-Selmer group of E/K is the subgroup of

$$H^1(G_K, E[m])$$

defined by

$$\mathrm{Sel}^m(E/K) = \ker\left\{H^1(G_K, E[m]) \to \prod_v WC(E/K_v)\right\}.$$

$\Diamond$

Definition 1.25 The Shafarevich-Tate group. The **Shafarevich-Tate** group of E/K is the subgroup of

$$WC(E/K)$$

defined by

$$\text{Ш}(E/K) = \ker\left\{WC(E/K) \to \prod_v WC(E/K_v)\right\}.$$

$\Diamond$

Theorem 1.26 *There is an exact sequence*

1. $$0 \to E(K)/mE(K) \to \mathrm{Sel}^m(E/K) \to \text{Ш}(E/K)[m] \to 0$$

2. $\mathrm{Sel}^m(E/K)$ *is finite.*

1.2 p^∞-Selmer and the structure of Ш

$H^1(G_K, E(\overline{K}))$ is torsion for general galois cohomological reasons. So

$$\text{Ш}(E/K) \subseteq H^1(G_K, E(\overline{K}))$$

is torsion.

So we may write

$$\text{Ш}(E/K) = \bigoplus_p \text{Ш}_{p^\infty}(E/K)$$

where for each prime p

$$\text{Ш}_{p^\infty}(E/K)$$

denotes the p-primary part of Ш(E/K). (i.e. the subgroup of elements whose order is a power of p.) By descent

$$\text{Ш}(E/K)[m] \text{ is finite for all } m \geq 1.$$

So

$$\text{Ш}_{p^\infty}(E/K) \cong (\mathbf{Q}_p/\mathbf{Z}_p)^{\delta_p} \oplus T_p,\ \delta_p \in \mathbf{Z}_{\geq 0}$$

where T_p is a finite abelian p-group.

$$T_p \cong \mathbf{Z}/p^{s_1}\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/p^{s_l}\mathbf{Z},\ s_i \in \mathbf{Z}_{\geq 0}.$$

The group

$$\bigoplus_p (\mathbf{Q}_p/\mathbf{Z}_p)^{\delta_p} \subseteq \text{Ш}(E/K)$$

is called the infinitely divisible subgroup of Ш denoted Ш_{div}.

The conjecture that Ш is finite implies $\delta_p = 0$ for all p. And $T_p \neq 0$ for only finitely many p.

There is a pairing called the Cassels-Tate pairing

$$\text{Ш}(E/K) \times \text{Ш}(E/K) \to \mathbf{Q}/\mathbf{Z}$$

which is bilinear and alternating, and the kernel on either side is the infinitely divisible group. If $\text{Ш}(E/K)$ is finite then the pairing is non-degenerate and hence

$$|\text{Ш}(E/K)| = \square \in \mathbf{Z}.$$

Definition 1.27 p^∞**-Selmer group.** Consider $\mathrm{Sel}_{p^n}(E/K)$ and take the direct limit

$$\varinjlim_n \mathrm{Sel}_{p^n}(E/K)$$

to define the p^∞-Selmer group. ◊

One shows that

$$X_p(E/K) = \mathrm{Hom}_{\mathbf{Z}_p}(\varinjlim_n \mathrm{Sel}_{p^n}(E/K), \mathbf{Q}_p/\mathbf{Z}_p)$$

called the Pontyragin dual of the p^∞ Selmer group is a finitely generated $\mathbf{Z}_p$-module. The associated $\mathbf{Q}_p$-vector space, denoted $\mathcal{X}_p(E/K) = X_p(E/K) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ has dimension rk_p.

Definition 1.28 rk_p is called the p^∞-Selmer rank of E/K and satisfies

$$\mathrm{rk}_p = \mathrm{rk}(E/K) + \delta_p.$$

◊

So if Ш is finite then $\delta_p = 0$ for all p. Use BSD to compute parity of rk_p.