

Coleman-de Shalit's p -adic regulator

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Overview/History/Philosophy

Goal: Introduce Coleman-de Shalit's regulator and show a relation to p -adic L -functions.

Big picture: Regulators are maps from K -groups / motivic cohomology to absolute Hodge cohomology (Deligne-Beilinson / syntomic). They relate to special values of L -functions.

History:

- Beilinson - Define regulators (+Bloch and many more), Deligne-Beilinson cohomology is absolute Hodge.
- Coleman-de Shalit - Construct a p -adic analogue
- Fontaine-Messing - Syntomic cohomology
- Gros - *Rigid* syntomic cohomology
- Besser - Coleman integrals compute regulators from K -theory to rigid syntomic cohomology
- Bannai - Rigid syntomic cohomology is absolute Hodge coh.

Beilinson regulators (Complex theory)

Let C/\mathbf{C} be a smooth complete curve, $f, g \in \mathbf{C}(C)^\times$. Beilinson defines

$$r_{\infty, C}(f, g)(\omega) = \frac{1}{2\pi i} \int_{C(\mathbf{C})} \log |g|^2 \overline{d \log f} \wedge \omega$$

the relation to K -groups comes via

$$K_2(\mathbf{C}(C)) = \mathbf{C}(C)^\times \otimes \mathbf{C}(C)^\times / \langle f \otimes 1 - f \rangle.$$

and $r_{\infty, C}$ satisfies this relation.

Relation to L -values

Fix E/κ be an elliptic curve with CM by \mathcal{O}_κ , κ a CM field of class number 1. Let $\Psi = \Psi_{E/\kappa}$ be the associated Grossencharacter, p be a prime that splits in κ , $p = \mathfrak{p}\bar{\mathfrak{p}}$. ω an invariant differential.

Proposition (Bloch, Rohrlich, Deninger-Wingberg)

$$r_{\infty,E}(f,g)(\omega) = c_{f,g} \underbrace{\Omega L_{\infty}(E,0)}_{=L_{\infty}(\Psi,0)}, \quad c_{f,g} \in \mathbb{Q}$$

(L_{∞} includes Gamma factors), and there exists f, g with $c_{f,g} \neq 0$.

We can associate a canonical *period-pair-class* to κ :

$\langle \Omega, \Omega_p \rangle \in (\mathbf{C}^\times \times \mathbf{C}_p^\times) / \overline{\mathbf{Q}}^\times$ so that:

Theorem (Katz, Manin-Vishik)

Let $1 \neq \mathfrak{g}$ an ideal of κ relatively prime to \mathfrak{p} . Then $\exists!$,

$W(\overline{\mathbf{F}}_p)$ -valued measure μ on $\mathcal{G}(\mathfrak{g}) = \text{Gal}(\kappa(\mathfrak{gp}^\infty)/\kappa)$ so that $\forall \epsilon$
Grossencharacter of conductor dividing \mathfrak{g} with infinity type $(k, 0)$
 $k \geq 1$, if

$$L_{\infty, \mathfrak{g}}(\epsilon^{-1}, s) = \Gamma(s + k) \prod_{\mathfrak{l} \nmid \mathfrak{g}} (1 - \epsilon^{-1}(\mathfrak{l}) \text{Nm } \mathfrak{l}^{-s})^{-1}$$

$$L_{p, \mathfrak{g}}(\epsilon^{-1}) = \int_{\mathcal{G}(\mathfrak{g})} \epsilon(\sigma) d\mu(\sigma)$$

we have

$$\Omega_p^{-k} L_{p, \mathfrak{g}}(\epsilon^{-1}) = \Omega^{-k} (1 - p^{-1} \epsilon(\mathfrak{p})) L_{\infty, \mathfrak{g}}(\epsilon^{-1}, 0) \in \overline{\mathbf{Q}}.$$

p -adic regulators?

Can rewrite $r_{\infty, \mathbb{C}}$ as

$$r_{\infty, \mathbb{C}}(f, g) = \sum_{b \in C(\mathbb{C})} \text{ord}_b(g) F_{f, \omega}(b)$$

where $F_{f, \omega}$ satisfies

$$\bar{\partial}(\mathrm{d}F) = \bar{\partial}(\log |f|^2 \omega)$$

Even without p -adic $\bar{\partial}$ we can just try to find $F_{f, \omega}$ satisfying

$$\mathrm{d}F = \log f \cdot \omega$$

and define

$$''r_{p, \mathbb{C}}(f, g) = \sum_{b \in C(\mathbb{C}_p)} \text{ord}_b(g) F_{f, \omega}(b)''$$

p -adic tools (Coleman integration)

Let $K = \mathbf{C}_p = \widehat{\mathbf{Q}_p}$, $R = \mathcal{O}_K$, $k = R/\mathfrak{m}$. We will work with 1-dimensional rigid spaces (curves) over K . We fix a branch of the p -adic logarithm $\log: K^\times \rightarrow K$.

It is always possible to integrate rigid 1-forms locally on a disk: given ω we have a local expression in terms of a convergent power series

$$\omega|_D = \sum_i a_i t^i dt$$

which can be integrated formally (up to a constant).

Let X/\mathcal{O}_K be a smooth projective curve, if $Y \subseteq X$ smooth affine open, then in the special fibre

$$X_k \setminus Y_k = \{e_1, \dots, e_n\}.$$

What is hard is to integrate globally, iteratively and include $\int \frac{dz}{z}!$

p -adic tools (Coleman integration)

We then remove rigid disks around e_i . Y_k is locally given by \bar{h} so we can take the rigid subspace

$$U_r \text{ locally defined by } |h| > r$$

and the underlying affinoid is $X_K - \bigcup_i B_{<}(e_i, 1)$.

We have

$$U = \varprojlim_{r \rightarrow 1} U_r$$

and spaces of overconvergent functions and 1-forms

$$A(U) = \varinjlim_{r \rightarrow 1} A(U_r)$$

Let Y be an affinoid with good reduction then Y_k finite type, and we have $F: Y_k \rightarrow Y_k$ the q -power frobenius.

p -adic tools (Coleman integration)

Proposition

There exists

$$\phi: U \rightarrow U, \tilde{\phi} = F$$

a lift of frobenius or frobenius morphism of X , of degree q .

Note: Whatever we choice of frobenius we make should not matter!

Example

Have $X = \mathbf{P}_{\mathcal{O}_{\mathbf{C}_p}}^1 \supseteq Y = \mathbf{G}_m = \mathbf{P}^1 \setminus \{0, \infty\}$ then

$$U_r = \{r < |z| < 1/r\}$$

$\phi(z) = z^q: U_r \rightarrow U_{rq}$. (But we could add some other p -junk!)

$$\Omega^1(U) = \varinjlim_{r \rightarrow 1} \Omega^1(U_r)$$

$$A(U_x) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ converging for } |z| < 1 \right\}$$

$$A(U_{e_i}) = \left\{ f(z_{e_i}) = \sum_{n=-\infty}^{\infty} a_n z_{e_i}^n \text{ converging for } r < |z_{e_i}| < 1, r < 1 \right\}.$$

$$A_{\log}(U_x) = A(U_x), \, A_{\log}(U_{e_i}) = A(U_{e_i})[\log(z_{e_i})]$$

$$\Omega^1_{\log}(U_?) = A_{\log}(U_?) \, dz_?$$

$$A_{\mathrm{loc}}(U) = \prod_x A_{\mathrm{log}}(U_x)$$

Theorem (Coleman integration)

There is a subspace $M(U)$ of $A_{\text{loc}}(U)$, which we call the space of Coleman functions, and linear map (integration), which we denote by \int or by $\omega \mapsto F_\omega$, from $M(U) \otimes_{A(U)} \Omega(U)$ to $M(U)/\mathbb{C}_p$.

The map f is characterized by three properties:

- 1. It is a primitive for the differential in the sense that $dF_\omega = \omega$.*
- 2. It is Frobenius equivariant $F_{\phi^*\omega} = \phi^*F_\omega$.*
- 3. If $g \in A(U)$, then $F_{dg} = g + \mathbb{C}_p$.*

We also have properties such as:

$$f \in M(U)$$

vanishes on one residue disk, then f is identically zero.

The space $M(U)$ is constructed iteratively $M(U) = \bigcup_n A_n(U)$ with each step being obtained as functions you get by integration from the previous.

The p -adic regulator

We can now define a p -adic version of the above regulator.

(Let C to be a complete non-singular curve whose jacobian has good reduction.)

If $f \in K(C)^\times$, $U = C \setminus |\operatorname{div}(f)|$ we can take a global 1-form $\omega \in H^0(C, \Omega_{C/K}^1)$ and the function

$$\log(f) = \int \frac{df}{f} \in A_1(U)$$

and obtain

$$\log(f)\omega \in \Omega_1(U).$$

Integration gives

$$F_{f,\omega} \in A_2(U) \text{ with } dF_{f,\omega} = \log(f)\omega \in \Omega_1(U),$$

unique up to a constant.

The regulator

Definition (The p -adic regulator)

Take f, g, ω as before defined over \bar{k} , then define

$$r(f, g)(\omega) = - \int_{(g)} \log(f) \omega = - \sum_{b \in C(K)} \text{ord}_b(g) F_{f, \omega}(b) \in \bar{k}$$

Theorem (Coleman-de Shalit)

$r_C(f, g)$ is a skew-symmetric bilinear pairing on $\bar{k}(C)^\times$ that

1. *factors through $K_2(\bar{k}(C))$*
2. *depends only on $\text{div}(f), \text{div}(g)$*
3. *is $\text{Gal}(\bar{k}/k)$ equivariant*
4. *for finite morphisms of complete non-singular curves $u: C' \rightarrow C$ we get $r_{C'}(u^*f, u^*g) = u^*r_C(f, g)$.*

Comparison of the p -adic and C theories

We now move to a very special situation, where the above regulators can be shown to be related to L -values.

$C = E/\kappa$ will be an elliptic curve with CM by \mathcal{O}_κ . $\Psi = \Psi_{E/\kappa}$ the corresponding Grossencharacter with conductor \mathfrak{f} and assume

$$w_{\mathfrak{f}} = \#\{\zeta \in \mu(K) : \zeta \equiv 1 \pmod{\mathfrak{f}}\} = 1.$$

let ω be a κ -rational invariant differential, \mathcal{L} the period lattice of (E, ω) .

The theorem

Theorem (Rohrlich, others?)

$$r_{\infty}(f, g) = \underbrace{12(Nm \mathfrak{a} - \Psi^{-1}(\mathfrak{a})) \sum_{\text{orbits } \langle Q \rangle} \text{ord}_Q g \cdot \Omega(Q) \prod_{\mathfrak{l} | \mathfrak{g}_Q} (1 - \Psi(\mathfrak{l})) L_{\infty}(\Psi, 0)}_{c_{f,g}}$$

\mathfrak{g}_Q ideal of annihilators of Q .

Theorem (Coleman-de Shalit)

We have the formula

$$r_{p,E}(f, g)(\omega) = c_{f,g} \Omega_p(1 - (\Psi(\mathfrak{p})p)^{-1})^{-1} L_p(\Psi).$$

Where do these terms come from? (in the p -adic case)

The rest of the talk: proof overview, see where the terms come in.

Proof

We use a specific class of f 's (for $(\mathfrak{a}, \mathfrak{f}\mathfrak{p}) = 1$), the functions

$$f(P) = \Theta_{\mathfrak{a}}(P) = \Delta(\mathcal{L})\Delta(\mathfrak{a}^{-1}\mathcal{L})^{-1} \prod'_{R \in E[\mathfrak{a}]} \frac{\Delta(\mathcal{L})}{(x(P) - x(R))^6} \in \kappa(E)^{\times}$$

whose values are **elliptic units**, the divisor of $\Theta_{\mathfrak{a}}$ is

$$12 \left((\mathrm{Nm} \mathfrak{a} - 1) \cdot (0) - \sum'_{R \in E[\mathfrak{a}]} (R) \right)$$

and we have the **distribution relation**

$$f(\pi P) = \prod_{v \in E[\mathfrak{p}]} f(P + v)$$

These functions generate the set of all functions with divisors supported on torsion.

We also take $g \in \kappa(E)^{\times}$ with divisor supported on torsion and $Q \in |\mathrm{div} g| \implies \mathfrak{f}|g_Q, (g_Q, \mathfrak{a}\mathfrak{p}) = 1$.

Take E with the \mathfrak{a} -torsion points removed,

$$X(\mathfrak{a}) = E \setminus \bigcup_{P \in E[\mathfrak{a}]} B(P, 1) \subseteq U_r(\mathfrak{a}) = E \setminus \bigcup_{P \in E[\mathfrak{a}]} B(P, r).$$

Take D to be a derivation that is dual to ω (so $DF_\omega = 1$). Then

$$F_{f, \omega}$$

is the unique (up to constant) $F \in A_2(U_r(\mathfrak{a}))$ for which

$$DF = \log f$$

Then we have

$$D(F(\pi P)) = \pi \cdot (DF)(\pi P)$$

and the distribution relation gives

$$D(F(\pi P)) = \pi \sum_{v \in E[\pi]} (DF)(P + v)$$

By definition $\pi = \psi(\mathfrak{p})$ is a lift of frobenius (which is algebraic!). As $F \in A_2(U_r(\mathfrak{a}))$, for some (possibly different) r close to 1 we have

$$F(\pi P) - \pi \sum_{v \in E[\pi]} F(P + v) \in A_2(U_r(\mathfrak{a}))$$

the above implies this is locally constant, hence constant! So we change F to get that

$$F(\pi P) - \pi \sum_{v \in E[\pi]} F(P + v) = 0.$$

$$F(\pi P) - \pi \sum_{v \in E[\pi]} F(P + v) = 0.$$

Now define

$$F^\#(P) = F(P) - p^{-1} \sum_{v \in E[\pi]} F(P + v)$$

so that as $Q \in |\operatorname{div} g|$ is Galois conjugate to πQ over κ :

$$r_p(f, g) = - \sum_Q \operatorname{ord}_Q gF(Q) = - \sum_Q \operatorname{ord}_Q gF(\pi Q)$$

giving

$$\left(1 - \frac{1}{\pi p}\right) r_p(f, g) = - \sum_Q \operatorname{ord}_Q gF^\#(Q).$$

We also have

$$\log(f)^\#(P) = \log f(P) - p^{-1} \sum_{v \in E[\pi]} \log f(P + v).$$

If Q is a torsion point in $X(\mathfrak{a})$ relatively prime to \mathfrak{p} order, then de Shalit has associated a

$$\eta_Q: \widehat{\mathbf{G}}_m \xrightarrow{\sim} \widehat{E}$$

so $Q + \eta_Q(S)$ parameterises the residue disk of Q and a $W = W(\overline{\mathbf{F}}_p)$ valued measure μ_Q on \mathbf{Z}_p^\times s.t.

$$\log(f)^\#(Q + \eta_Q(S)) = \int_{\mathbf{Z}_p^\times} (1 + S)^x d\mu_Q(x) \in W[[S]]$$

Then work of de Shalit shows that

$$F^\#(Q + \eta_Q(S)) = \underbrace{\eta'_Q(0)}_{\Omega_p(Q)} \int_{\mathbf{Z}_p^\times} (1+S)^x x^{-1} d\mu_Q(x) + c$$

for some constant c , and that $F^\#(P)$ is rigid analytic on $X(\mathfrak{a})$.

So we get

$$\left(1 - \frac{1}{\pi p}\right) r_p(f, g) = - \sum_Q \text{ord}_Q g \Omega_p(Q) \int_{\mathbf{Z}_p^\times} x^{-1} d\mu_Q(x).$$

We need to move to the correct group and remove the dependence on Q , by identifying $G = \text{Gal}(\kappa(\mathfrak{gp}^\infty)/\kappa(\mathfrak{g})) \cong \mathbf{Z}_p^\times$ so that

$$\begin{aligned} &= - \sum_{\langle Q \rangle} \text{ord}_Q g \Omega_p(Q) \sum_{\tau \in \mathcal{G}/G} \int_G \Psi^{-1}(\sigma) d\mu_{\tau(Q)}(\sigma) \\ &= - \sum_{\langle Q \rangle} \text{ord}_Q g \Omega_p(Q) \int_{\mathcal{G}(\mathfrak{g}_Q)} \Psi^{-1}(\sigma) d\mu_e(\sigma) \end{aligned}$$

Theorem (Coates-Wiles)

$$\mu_e = 12(\sigma_a - Nm\, a)\mu(\mathfrak{g}_Q)$$

where $\mu(\mathfrak{g}_Q)$ is the measure which defines the p -adic L -function of conductor \mathfrak{g}_Q , so removing those factors we reach

$$\overbrace{(1 - (\pi p)^{-1})}^{\text{Distribution relations}} r_p(f, g) = \underbrace{12(Nm\, a - \Psi^{-1}(a))}_{\text{Coates-Wiles}} \sum_{\text{orbits } \langle Q \rangle} \text{ord}_Q g \Omega_p(Q) \underbrace{\prod_{l|\mathfrak{g}_Q} (1 - \Psi(l)) L_p(\Psi)}_{L_{p, \mathfrak{g}_Q} \rightsquigarrow L_p},$$

Fin.