Computations with p-adic polylogarithms in Sage

– Global Virtual SageDays 109

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Overview

These slides are available online (in handout form) at

`https://alexjbest.github.io/talks/sage-computations-polylogs/slides_h.pdf`

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Goal: Introduce you to (p-adic polylogarithms) in Sage and explain some applications of these computations to solving S-unit equations.

What are polylogarithms?

Polylogarithms are special functions of a complex variable z, obtained by iteratively dividing by z and taking antiderivatives, starting with $\mathrm{Li}_0 = \frac{z}{1-z}$:

$$\mathrm{Li}_1(z) = \int_0^z \frac{t}{(1-t)t}\,\mathrm{d}t = -\log(1-z),$$

$$\mathrm{Li}_2(z) = \int_0^z \frac{-\log(1-t)}{t}\,\mathrm{d}t \qquad \text{the } \textit{dilogarithm},$$

$$\mathrm{Li}_3(z) = \int_0^z \frac{\mathrm{Li}_2(t)}{t}\,\mathrm{d}t$$

$$\vdots$$

What are polylogarithms?

Their power series expansions around zero are rather nice:

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} = z + \frac{z^2}{2^n} + \frac{z^3}{3^n} + \cdots$$

(for $|z| < 1$) and we can see that

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indeed Sage knows this symbolically

```
sage: polylog(3, 1)
zeta(3)
sage: polylog(2, 1)
1/6*pi^2
```

```
sage: polylog(2, 1/2)
1/12*pi^2 - 1/2*log(2)^2
sage: polylog(2, 7.0)
1.24827318209942 -
    6.11325702881799*I
```

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Properties of polylogarithms

These functions satisfy many interesting functional equations:

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \mathrm{Li}_2(1) - \log(x)\log(1-x)$$

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Powering:

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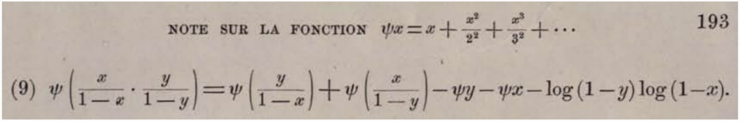
Spence (1809), **Abel (1827)**, Hill (1828), Kummer (1840), Schaeffer (1846):


NOTE SUR LA FONCTION $\psi x = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots$ 193

(9) $\psi\left(\frac{x}{1-x}\cdot\frac{y}{1-y}\right) = \psi\left(\frac{y}{1-x}\right) + \psi\left(\frac{x}{1-y}\right) - \psi y - \psi x - \log(1-y)\log(1-x).$

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What are the p-adics?

Parallel with the real/complex numbers. They are defined by:

1. Fixing a norm on $\mathbf{Q}$, defined by

$$|x|_p = p^{-\overbrace{\max\{i \in \mathbf{Z} : p^i | x\}}^{=\nu_p(x)}}$$

2. Completing $\mathbf{Q}$ with respect to $|\cdot|_p$, to get a complete normed field.

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2. Completing $\mathbf{Q}$ with respect to $|\cdot|_p$, to get a complete normed field.

Upshot: p is now *small* ($|p|_p = p^{-1}$), so instead of decimal expansions for elements of $\mathbf{R}$:
$$\frac{1}{3} = 3 \cdot \frac{1}{10} + 3 \cdot \frac{1}{10^2} + 3 \cdot \frac{1}{10^3} + 3 \cdot \frac{1}{10^4} + \text{smaller terms}$$
we have p-adic expansions for elements of $\mathbf{Q}_p$:
$$\frac{1}{3} = 5 + 4 \cdot 7 + 4 \cdot 7^2 + 4 \cdot 7^3 + 4 \cdot 7^4 + \text{smaller terms}$$

p-adics in Sage

There is now good support for p-adics in Sage, thanks to many people, but in particular Xavier Caruso, David Roe and Julian Rüth are regularly working on this (on Zulip).

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Support includes:

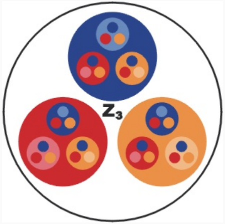
- Basic arithmetic
- Many different precision tracking modes (absolute / relative, fixed / capped precision)
- Hensel lifting (Newton's method)
- $\exp$ and $\log$
- Frobenius, and Teichmüller representatives
- Extensions
- Li_n?
- Much more!

To define Li_n, p-adically, we must define antiderivatives of p-adic functions.

Easy! Just write out power series locally and take the antiderivative termwise!

Problem: we can work out local antiderivatives here, and calculate integrals between nearby points, but we can't analytically continue. Distinct disks don't overlap in the p-adic topology.

A different constant of integration to be chosen on each p-adic disk.



Bad topology!

p-adic polylogarithms

Assume more of the integral, to pin down the function defined: assume Frobenius equivariance:

$$\int_{x^p}^{y^p} f(t)\,\mathrm{d}t = \int_x^y f(t^p)\,\mathrm{d}(t^p)$$

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For example: If $f(t) = 1/t$ we can define

$$\log(z) := \int_1^z \frac{\mathrm{d}t}{t}$$

and find values for z (p-adically) near 1 by integrating a power series,

$$\log(z^p) = \int_1^{z^p} \frac{\mathrm{d}t}{t} = \int_1^z \frac{pt^{p-1}\,\mathrm{d}t}{t^p} = p\int_1^z \frac{\mathrm{d}t}{t} = p\log(z)$$

so for a $p^k - 1$st root of unity ζ we have

$$\log(\zeta) = \log(\zeta^{p^k}) = p^k \log(\zeta) \implies \log(\zeta) = 0.$$

p-adic polylogarithms in Sage

Some initial cases (but with restrictions on p, n, z) implemented by Jennifer Balakrishnan (at a Sage days).

Sage Days 87: p-adics in Sage and the LMFDB (2017), I wrote a complete implementation and #20260 was merged.

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```
sage: K = Qp(5, prec=7);
```

```
sage: K(1 + 5).polylog(2)
5 + 5^2 + 5^3 + O(5^4)
sage: K(1 + 5^2).polylog(2)
5^2 + 5^4 + O(5^5)
sage: K(1 + 5^3).polylog(2)
5^3 + O(5^5)
sage: K(1 + 5^4).polylog(2)
5^4 + O(5^6)
```

```
sage: K(1 + 5^5).polylog(2)
5^5 + O(5^6)
sage: K(1/2).polylog(2)
3*5^2 + 3*5^3 + O(5^4)
sage: -K(1/2).log()^2/2
3*5^2 + 3*5^3 + 2*5^4 + 5^5 +
    2*5^7 + O(5^8)
sage: K(7).polylog(3)
3*5^3 + O(5^4)
```

How does it work?

Besser – de Jeu: "Li^p-Service? An Algorithm for Computing p-Adic Polylogarithms." Math. Comp. 77, no. 262 (2008).

- Near 0: use the power series $\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$
- Near ∞: use the relation

  $$\mathrm{Li}_n(z) + (-1)^n \mathrm{Li}_n\left(z^{-1}\right) = -\frac{1}{n!}\log^n(z)$$

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  to reduce to the first case.
- Else: Must be near a $(p^k - 1)$st root of unity for some k (except near 1). Letting

$$\mathrm{Li}_n^{(p)}(z) = \mathrm{Li}_n(z) - \frac{1}{p^n} \mathrm{Li}_n\left(z^p\right)$$

  Reduces to computing $\mathrm{Li}_m^{(p)}\left(\zeta^{p^j}\right)$ for $m \leq n$ and $j < k$.
- Near 1: see the paper!

Application: The S-unit equation

One classic diophantine equation is the S-unit equation: for a fixed finite set of primes S

$$u + v = 1,\ u, v \in \mathbf{Q}^\times$$

where we ask that the only primes present in the factorization of u, v are those in S.

So

$$\frac{4}{3} - \frac{1}{3} = 1$$

is a solution of the $\{2, 3\}$-unit equation, but not of the $\{2\}$-unit equation or $\{3\}$-unit equation alone.

The most difficult cases of this equation are when S is large, or over number fields instead.

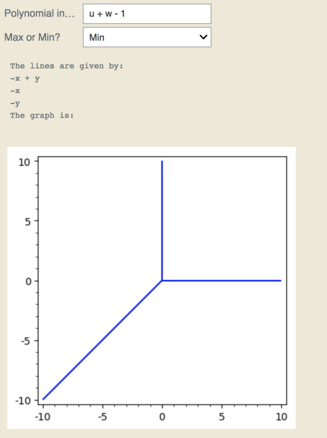
In a joint project with Theresa Kumpitsch, Martin Lüdtke, Angus McAndrew, Lie Qian, Elie Studnia, and Yujie Xu, we have been using p-adic polylogarithms (in Sage) to provably determine the full set of solutions to these equations.

For now only for small S, over $\mathbf{Q}$.

The S-unit equation

Note that for any prime p, either $p|u$, $p|v$, or both $p|u^{-1}$ and $p|v^{-1}$.

Plotting $\nu_p(u)$ against $\nu_p(v)$ we get a diagonal Y shape:



Interact by Wang Weikun

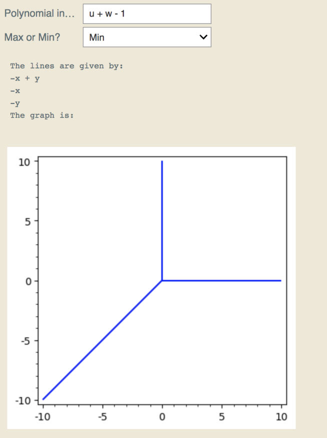
The S-unit equation

Note that for any prime p, either $p|u$, $p|v$, or both $p|u^{-1}$ and $p|v^{-1}$.

Plotting $\nu_p(u)$ against $\nu_p(v)$ we get a diagonal Y shape:

This is an instance of a more general phenomenon:
the valuations lie on
the tropical curve associated with
the defining equation $u + v = 1$.

Note: If $p \notin S$ then $u \pmod p$ cannot be any of $0, 1, \infty$.



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Applications

Minhyong Kim has developed a programme of *non-abelian Chabauty*.

Extends classical Chabauty's method for finding rational and integral points on curves as zeroes of abelian integrals on the curve.

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One specific consequence of this theory due to Dan-Cohen–Wewers: there exists a commutative diagram for any fixed prime p not in S.

$$\begin{tikzcd}
\mathbf{P}^1 \smallsetminus \{0,1,\infty\}(\mathbf{Z}[\frac{1}{S}]) \arrow[r] \arrow[d, "(\nu_\ell(z),\nu_\ell(1-z))_{\ell\in S}"'] & \mathbf{P}^1 \smallsetminus \{0,1,\infty\}(\mathbf{Z}_p) \arrow[d, "(\log(z),\log(1-z),-\mathrm{Li}_2(z))"] \\
\mathbf{A}^{2|S|}_{\mathbf{Q}_p} \arrow[r, "\left(\sum_{\ell\in S} x_\ell \log(\ell),\sum_{\ell\in S} y_\ell \log(\ell),h(\underline{x},\underline{y})\right)"'] & \mathbf{A}^3_{\mathbf{Q}_p}
\end{tikzcd}$$

Applications

In this diagram everything is defined, except h, it is a bilinear form in the x_ℓ and y_ℓ.

Strategy:

- Given enough points in the top $\mathbf{P}^1 \smallsetminus \{0, 1, \infty\}(\mathbf{Z}[\frac{1}{S}])$, we can find their image in the $\mathbf{A}^3_{\mathbf{Q}_p}$ going round the diagram both ways, commutativity then determines h.
- Given a subvariety V of $\mathbf{A}^3_{\mathbf{Q}_p}$ we can find all S-units that land in V by pulling back along the right vertical arrow.

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- Given a subvariety V of $\mathbf{A}^3_{\mathbf{Q}_p}$ we can find all S-units that land in V by pulling back along the right vertical arrow.
- Have to solve a polynomial (with $\mathbf{Q}_p$ coefficients) combinations of $\log(z), \log(1-z), \mathrm{Li}_2(z)$.
- To find a useful collection of Vs covering all possible S-units when $|S| = 2$ we use the tropical picture, we have 3 components $\{-, |, /\}$ for each prime $\ell \in S$ (Betts–Dogra).

Example

When $S = \{2, 3\}$ we have many solutions

$$\left\{2, \frac{1}{2}, -1\right\} \cup \left\{3, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, -\frac{1}{2}, -2\right\}$$

$$\cup \left\{4, \frac{1}{4}, \frac{4}{3}, \frac{3}{4}, -\frac{1}{3}, -3\right\} \cup \left\{-\frac{1}{8}, \frac{1}{9}, \frac{9}{8}, \frac{8}{9}, 9, -8\right\}$$

from which we can determine that:

$$h = \tfrac{1}{2}\log(2)^2 x_2 y_2 - \mathrm{Li}_2(-2) x_2 y_3 - \mathrm{Li}_2(3) x_3 y_2 + \tfrac{1}{2}\log(3)^2 x_3 y_3$$

Example

For one choice of V we get that for any S-unit z with $2|z$ and $3|(1-z)$ we have $\mathrm{Li}_2(-2)\,\mathrm{Li}_2(z) = \mathrm{Li}_2(3)\,\mathrm{Li}_2(1-z)$ which we can solve

```
sage: allr = allroots(K(1-3).log(p_branch)*K(3).log(p_branch)*Li2z -
      K(3).polylog(2)*logz*logone_z,p)
sage: for r in allr:
....:     print("root: ",r)
....:     print(algdep(r, 2))

root:  2*5^-1 + 1 + 5^2 + 5^5 + 5^6 + 5^8 + 5^9 + 3*5^11 + 3*5^12 + 4*5^13 +
      4*5^14 + 2*5^15 + 4*5^16 + 3*5^17 + 4*5^18 + 2*5^19 + O(5^20)
11775*x^2 - 119800*x - 28359
root:  2 + O(5^24)
x - 2
root:  2 + 4*5 + 4*5^2 + 4*5^3 + 4*5^4 + 4*5^5 + 4*5^6 + 4*5^7 + 4*5^8 +
      4*5^9 + 4*5^10 + 4*5^11 + 4*5^12 + 4*5^13 + 4*5^14 + 4*5^15 + 4*5^16 +
      4*5^17 + 4*5^18 + 4*5^19 + 4*5^20 + 4*5^21 + 4*5^22 + 4*5^23 + O(5^24)
x + 3
root:  3 + 4*5^23 + O(5^24)
x - 3
root:  3 + 5^2 + 2*5^3 + 5^4 + 3*5^5 + 5^6 + 5^7 + 5^9 + 2*5^10 + 3*5^11 +
      2*5^12 + 3*5^13 + 3*5^14 + 4*5^15 + 5^16 + 4*5^17 + 3*5^18 + 2*5^22 +
      5^23 + O(5^24)
128901*x^2 - 49672*x - 62943
root:  4 + 5 + O(5^24)
x - 9
```