

# Riemann Hypotheses

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WMS Talks

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- 1 Introduction
- 2 The original hypothesis
- 3 Zeta functions for graphs
- 4 More assorted zetas
- 5 Back to number theory
- 6 Conclusion

# The Riemann zeta function: Euler's work

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- He also discovered more general formulae for  $\sum_{n=1}^{\infty} n^{-2k}$  in terms of the Bernoulli numbers  $B_{2k}$  for all natural  $k$ .
- In fact, a nice form for

$$\sum_{n=1}^{\infty} n^{-2k-1},$$

is still unknown today.

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VII.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859.)

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hiedurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1 - \frac{1}{p^s}} = \sum \frac{1}{n^s},$$

wenn für  $p$  alle Primzahlen, für  $n$  alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen  $s$ , welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch  $\xi(s)$ . Beide convergiren nur, so lange der reelle Theil von  $s$  grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^{\infty} e^{-sx} x^{s-1} dx = \frac{\Gamma(s-1)}{s^s}$$

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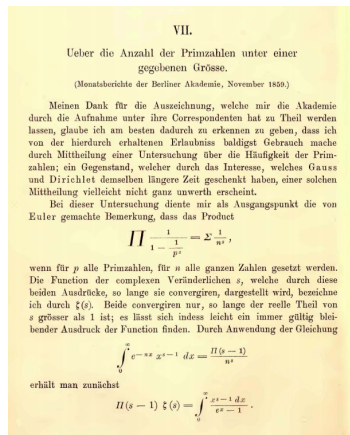
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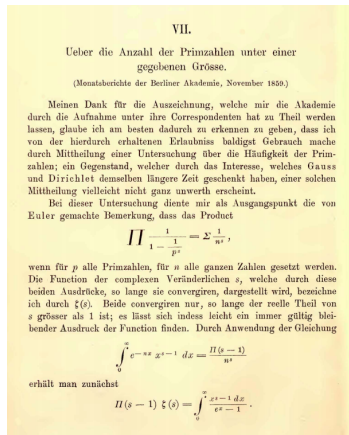


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- Along the way he (essentially) makes four hypotheses.



# The Riemann zeta function: What Riemann did

In his paper Riemann takes the function

$\zeta: \{\sigma + it \in \mathbb{C} \mid \sigma > 1\} \rightarrow \mathbb{C}$  and extends it to all of  $\mathbb{C}$ .

# Ramanujan graphs

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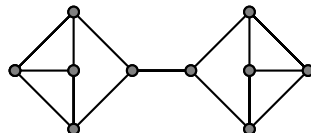
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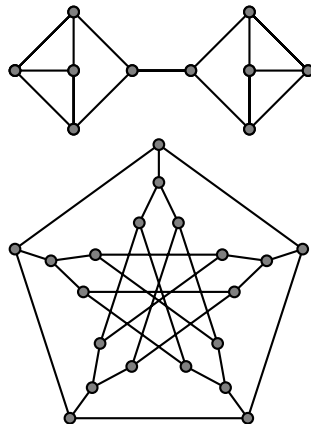
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# The Ihara zeta function

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$\mathbb{Q}$  is a number field too, and it turns out that the ideals of  $\mathbb{Q}$  correspond one to one with the set of natural numbers! We also have that the norm for these ideals is simply the natural number they correspond to. Therefore

$$\zeta(s) = \zeta_{\mathbb{Q}}(s)$$

and the Dedekind zeta is a direct generalisation of our original zeta.

## Closing remarks

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- Zeta functions can be used to pack up lots of useful information into one big package (a complex function).
- The properties of this package can tell us about the objects we started with.
- We can also see links between different objects via their zeta functions.
- Due to the abundant computational evidence (over ten trillion non-trivial zeroes found so far, all on the critical line) a huge number of papers have been written that assume the Riemann hypothesis is true. So a proof of the (generalised) hypothesis would imply hundreds of other results true also.

# Sources used

I used some of the following when preparing this talk, and so they are possibly good places to look to learn more about the topic:

- 1 Sarnak “What is... an expander?”
- 2 Wikipedia
- 3 <http://graphtheoryinlatex.blogspot.com/> – pretty pictures