

Coleman Integration in Larger Characteristic

ANTS XIII — University of Wisconsin, Madison

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Boston University

Kedlaya

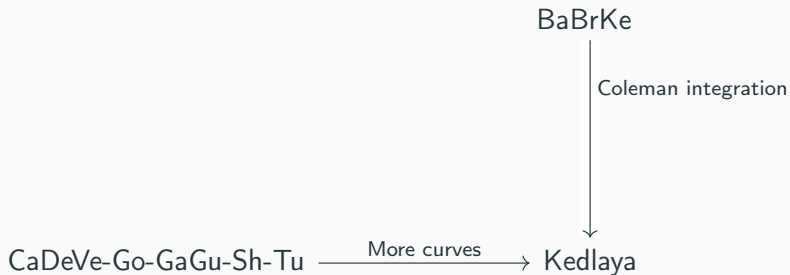
Arul, Balakrishnan, Best, Bradshaw, Castryk, Costa, Denef, Gaudry, Gurel, Harvey, Kedlaya, Magner, Minzlaff, Shieh, Triantafillou, Tuitman, Vercauteren, and more. . .

The big picture

CaDeVe-Go-GaGu-Sh-Tu $\xrightarrow{\text{More curves}}$ Kedlaya

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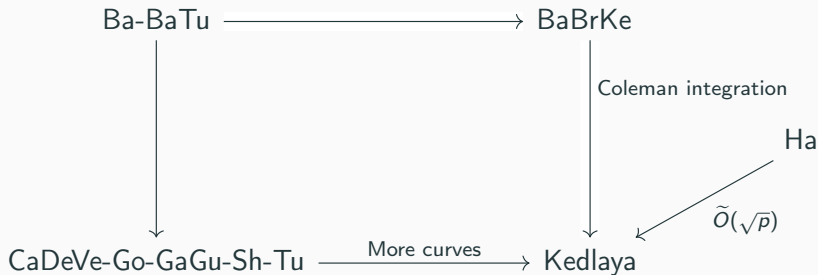
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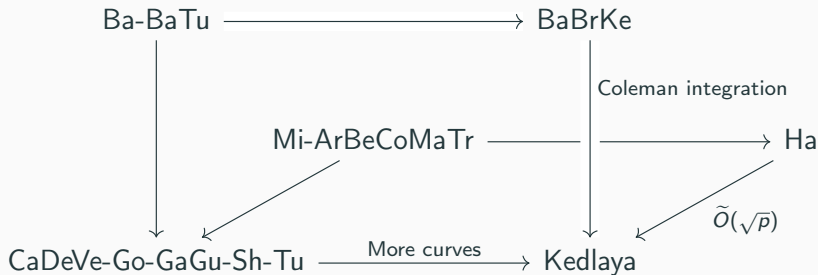
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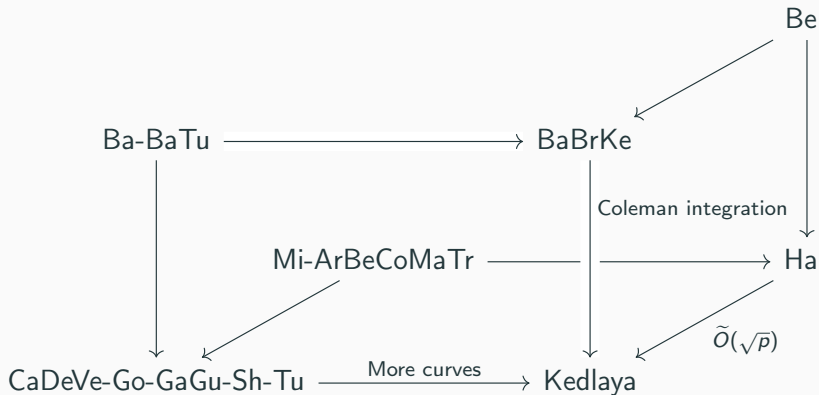
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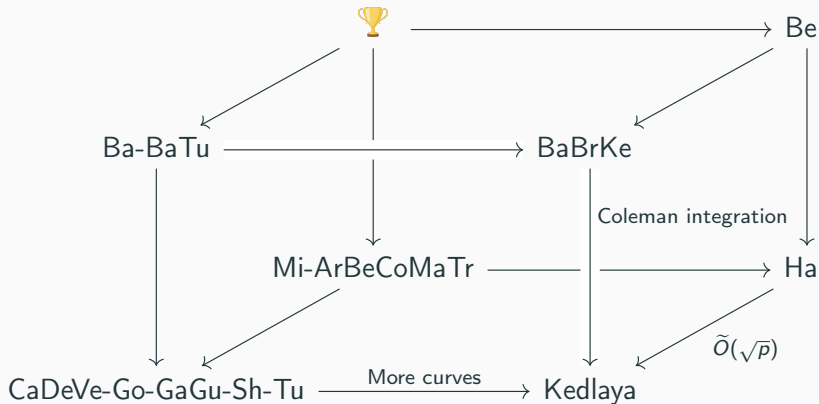
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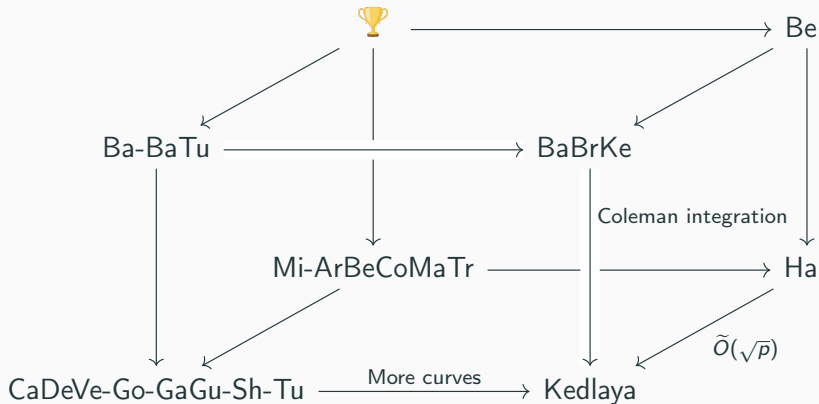
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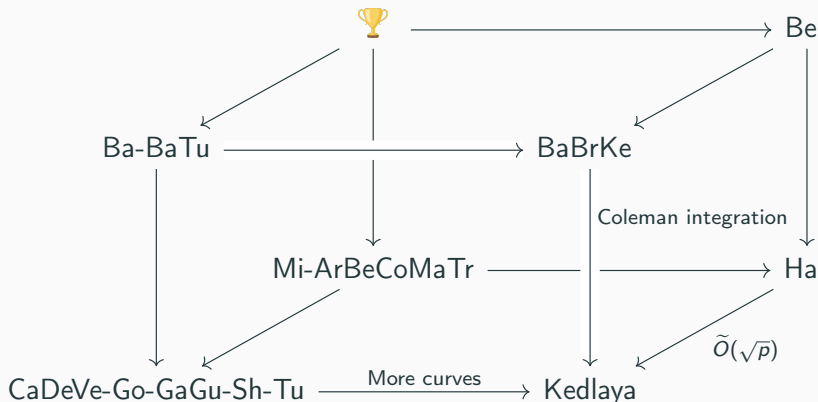
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There is (at least) one dimension missing: Small p !

Motivation

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- Applications to rational points, combining congruence information for many primes, 1-step (Mordell-Weil) sieving.

Coleman integration

Throughout we take X/\mathbf{Q}_p a genus g odd degree hyperelliptic curve, and p an odd prime. We pick a lift of the Frobenius map, $\phi^*: X \rightarrow X$.

Theorem (Coleman)

There is a \mathbf{Q}_p -linear map $\int_b^x: \Omega_{A^\dagger}^1 \otimes \mathbf{Q}_p \rightarrow A_{\text{loc}}(X)$ satisfying:

1.

$$d \circ \int_b^x = \text{id}: \Omega_{A^\dagger}^1 \otimes \mathbf{Q}_p \rightarrow \Omega_{A^\dagger}^1 \otimes \mathbf{Q}_p \leftarrow (FTC)$$

2.

$$\int_b^x \circ d: A^\dagger \rightarrow A_{\text{loc}}$$

3.

$$\int_b^x \phi^* \omega = \phi^* \int_b^x \omega \leftarrow (\text{Frobenius equivariance})$$

Reduction to reduction

Balakrishnan-Bradshaw-Kedlaya reduce the problem of computing all Coleman integrals of basis differentials ω_i of H^i between $\infty \in X$ and a point $x \in X(\mathbf{Q}_p)$, to:

1. Finding “tiny integrals” between nearby points,
2. Writing $\phi^*\omega_i - df_i = \sum_j a_j\omega_j$ and evaluating the primitive f_i for a point P near x , for each i .

Kedlaya's algorithm

Theorem (Kedlaya)

The action of ϕ^ on $H^1(X)$ (and hence the zeta function of X) can be computed in time*

$$\tilde{O}(p).$$

Theorem (Harvey)

If $p > (2g + 1)(2N - 1)$ the action of ϕ^ on $H^1(X)$ can be computed in time*

$$\tilde{O}(\sqrt{p}).$$

How do these work?

So we need to compute f along with $\omega - df$.

For vanilla Kedlaya this is “easy”, the reduction procedure is transparent, whenever we subtract dg to reduce, add g onto f .

For faster variants, this is not so simple!

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Spaces of differentials W_t , indexed by degree, each of dimension $2g$.

Goal

Reduce all differentials from W_t to a cohomologous one in W_0 , write in terms of fixed basis of W_0 .

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Key fact

Entries of $R(t)$ are fractions of *linear* functions of t , with \mathbf{Z}_p coefficients; work of Bostan-Gaudry-Schost (& Harvey) \implies products can be interpolated

$$R(a, b) = R(a+1) \cdots R(b) \rightsquigarrow R(a+1+t, b+t)$$

This interpolation is what gives us a $\tilde{O}(\sqrt{p})$ algorithm.

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Vital remark

We must use evaluations of primitives here, instead of trying to compute f as a power series.

Stumbling block

This is no longer linear in the index! You cannot apply BGS to this recurrence.

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Horner to the rescue

Instead of computing a series $\sum_{i=0}^N a_i x^i$ by computing sequentially

$$\left(\sum_{i=t}^N a_i x^i \right)_{t=N, N-1, \dots, 0}$$

We instead compute

$$((\cdots ((a_N)x + a_{N-1})x + \cdots)x + a_0)$$

from the inside to the out. This *is* an iterated composition of linear functions, each of which is linear in the index t .

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The evaluation is only correct at the end!

In matrix form we are augmenting the reduction matrices in the following sort of way:

$$\begin{array}{c}
 x^{t-2g} dx/y \\
 \vdots \\
 x^{t-2g} dx/y \\
 f(P)
 \end{array}
 \left(
 \begin{array}{ccc|c}
 x^{t-2g} dx/y & \dots & x^t dx/y & f(P) \\
 (2t-1)r_{0,0} + 2s'_{0,0} & \cdots & (2t-1)r_{2g-1,0} + 2s'_{2g-1,0} & \\
 \vdots & \ddots & \vdots & \\
 (2t-1)r_{0,2g-1} + 2s'_{0,2g-1} & \cdots & (2t-1)r_{2g-1,2g-1} + 2s'_{2g-1,2g-1} & \\
 \hline
 -S_0(x) & \cdots & -S_{2g-1}(x) & y^{-2}D_V(t)
 \end{array}
 \right)$$

so that we keep in memory a vector $v \in W_t \times \mathbf{Q}_p$.

Many integrals simultaneously

We may wish to do this with multiple points in several residue disks. Instead of repeating the whole procedure (repeating computing the Frobenius matrix), augment with many points.

$$\begin{array}{c}
x^{t-2g} dx/y \quad \dots \quad x^t dx/y \quad \quad \quad f(P_1) \quad \dots \quad f(P_L) \\
x^{t-2g} dx/y \left(\begin{array}{ccc} (2t-1)r_{0,0} + 2s'_{0,0} & \dots & (2t-1)r_{2g-1,0} + 2s'_{2g-1,0} \\ \vdots & \ddots & \vdots \\ (2t-1)r_{0,2g-1} + 2s'_{0,2g-1} & \dots & (2t-1)r_{2g-1,2g-1} + 2s'_{2g-1,2g-1} \end{array} \right. \\
\left. \begin{array}{ccc} f(P_1) & -S_0(x(P_1)) & \dots & -S_{2g-1}(x(P_1)) \\ \vdots & \vdots & \ddots & \vdots \\ f(P_L) & -S_0(x(P_L)) & \dots & -S_{2g-1}(x(P_L)) \end{array} \right| \begin{array}{ccc} y^{-2}(P_1)D_V(t) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & y(P_L)^{-2}D_V(t) \end{array} \right)
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\hline
f(P_1) \quad \quad \quad -S_0(x(P_1)) \quad \dots \quad -S_{2g-1}(x(P_1)) \quad \quad \quad y^{-2}(P_1)D_V(t) \quad \dots \quad 0 \\
\vdots \quad \quad \quad \vdots \quad \ddots \quad \vdots \quad \quad \quad \vdots \quad \ddots \quad \vdots \\
f(P_L) \quad \quad \quad -S_0(x(P_L)) \quad \dots \quad -S_{2g-1}(x(P_L)) \quad \quad \quad 0 \quad \dots \quad y(P_L)^{-2}D_V(t)
\end{array}$$

Note

This matrix and products of it have the same fixed form, when running BGS we don't try and interpolate entries that are always 0 \rightsquigarrow better run time.

Core algorithm is potentially more general? Evaluation of primitives is “integration”, but maybe of a different type?

Some amusing factoids

It is faster to evaluate the power series this way than to evaluate the power series you get from vanilla Kedlya, even when multiple points are needed computing the power series is no use.

Thanks for listening!

Questions/comments?