The (inescapable) p-adics

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VU Master Seminar - Algebra

Linear recurrence sequences

Definition: Linear recurrence sequence

A **linear recurrence sequence**, is a sequence whose nth term is a linear combination of the previous k terms (for all $n \geq k$).

We often consider sequences of integers, but we will consider these over fields like $\mathbb{Q}, \mathbb{C}$ also.

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Example: Fibonacci

Let $a_0 = 0, a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq k = 2$:

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 41$

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a_n grows exponentially.

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Example: A periodic sequence

$a_0 = 1, a_1 = 0$ with $a_n = -a_{n-1} - a_{n-2}$

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a_n is periodic now.

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$a_0 = 1, a_1 = 0, a_2 = 2, a_3 = 0$ with $a_n = 2a_{n-2} - a_{n-4}$

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not periodic but the zeroes *do* have a regular repeating pattern, being zero half the time.

The ultimate question

Can we describe the set $\{n : a_n = 0\}$ for a general linear recurrence?

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To answer this we will use power series:

Observation

A linear recurrence sequence is the Taylor expansion around 0 of a rational function

$$\frac{a_0 + a_1 x + \cdots + a_\ell x^\ell}{b_0 + b_1 x + \cdots + b_k x^k}$$

with $b_0 \neq 0$ (so that the expansion makes sense).

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$$\frac{(1+x)^3 - x^3}{(1+x)^5 - x^5} \leftrightarrow 1, -2, 3, -5, 10, -20, 35, -50, 50, 0, -175, 625,$$

$$-1625, 3625, -7250, 13125, -21250, 29375, -29375$$

$$0, 106250, -384375, 1006250, -2250000, 4500000,$$

$$-8140625, 13171875, -18203125, 18203125, 0, -658$$

Consequences

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The set of all linear recurrence sequences over a field is a vector space! Hard to tell how the rule changes.

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The set of all linear recurrence sequences over a field is a vector space! Hard to tell how the rule changes.

As we can add any finite sequence can always mess up the behaviour of finitely many initial terms. So we want to know what is the structure of the zero set asymptotically?We can also multiply rational functions, which convolutes the sequences!

Example

$$\frac{1}{(1-x^2)^2}-(1-x+2x^2+3x^4+4x^6) \leftrightarrow 0,1,0,0,0,0,0,0,5,0,6,0,7,0,8$$

Interlacing with 0 and *shifting* correspond to plugging in x^2 and multiplying by x respectively in the rational functions

$$\frac{1}{(1-x)^2} \leftrightarrow 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,$$

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$$\frac{1+2x}{(1-x^4)^2} \leftrightarrow 1, 2, 0, 0, 2, 4, 0, 0, 3, 6, 0, 0, 4, 8, 0, 0, 5, 10, 0, 0, 6, 12, 0, 0, 7, 1$$

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Still has periodic zero set, all n congruent to $2, 3$ modulo 4.

Approach

Expand into partial fractions

$$\frac{p(x)}{q(x)} = \sum_{i=1}^{m} \sum_{j=1}^{n_j} \frac{r_{ij}}{(1 - \alpha_i x)^j}$$

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do some rearranging:

$$\sum a_n x^n = \sum_{n=0}^{\infty} \left(\sum_{i=1}^{m} \sum_{j=1}^{n_j} r_{ij} \binom{n+j-1}{j-1} \alpha_i^n \right) x^n$$

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Upshot: there are polynomials $A_i(n)$ over $\mathbb{C}$ such that

$$a_n = \sum_{i=1}^{m} A_i(n) \alpha_i^n.$$

Example: Binet's formula for Fibonacci

$F_n = \frac{1}{\sqrt{5}} \phi^n - \frac{1}{\sqrt{5}} (-\phi^{-1})^n$

Approach

So a_n is an analytic function of n which has zeroes for infinitely many integer values.

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Ridiculous suggestion
What if the integers were bounded? If that was true, infinitely many zeroes $\implies$ the function is zero!

What is bounded?

What if the integers were bounded?

How do we define boundedness?

Definition: Absolute Values

Let C be a commutative ring, an **absolute value** on C, is a function $|\cdot|\colon R \to \mathbb{R}_{\geq 0}$ satisfying for all $x, y \in C$

$$|x| = 0 \iff x = 0$$

$$|xy| = |x||y|$$

$$|x + y| \leq |x| + |y|$$

Are there other absolute values for the integers?

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$$|x + y| \leq |x| + |y|$$

Property 2 implies that $|1| = 1$ and $|-1|^2 = 1$ so $|-1| = 1$ also. So it remains to decide what happens for all primes $p \in \mathbb{Z}$.

We could set $|x| = 1$ for all $x \neq 0$, this is the **trivial absolute value**.

Or $|x| = x$ for all positive x, this gives the usual absolute value.

A strange absolute value

We can in fact define another absolute value $|\cdot|_p$ for each prime p.

Pick a value $\alpha = |p|_p < 1$, and let $|q|_p = 1$ for all other primes q.

Now we have that

$$|x+y| \leq \max(|x|, |y|) \leq |x| + |y|$$

Theorem: Ostrowski

The only nontrivial absolute values on $\mathbb{Q}$ are

$$x \mapsto \operatorname{sgn}(x)x \;\; \textit{and} \;\; |\cdot|_p \textit{ for some prime } p$$

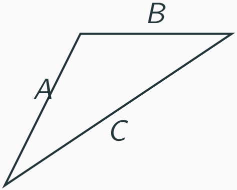
Why is it strange?

The inequality

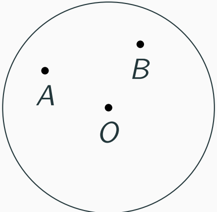
$$|x + y| \leq \max(|x|, |y|)$$

is called **ultrametric** and has some unusual properties:

Every triangle is isosceles.



Every point of a ball is its center



Completion

We can define the real numbers as equivalence classes of Cauchy sequences of rationals, (with respect to the normal absolute value).

We can do the same thing with $|\cdot|_p$ to get the *p*-adics, a topologically complete ring $\mathbb{Q}_p$, which is analogous to $\mathbb{R}$ but very different.

$$\{(x_n)_n : x_n \in \mathbb{Q}, \forall \epsilon > 0, \exists N, n, m > N \implies |x_n - x_m| < \epsilon\}/\sim$$

The absolute value $|\cdot|_p$ extends to give one on $\mathbb{Q}_p$ which is still ultrametric.

How to think about p-adic numbers

For real numbers we use a decimal expansion, to write a number with the most significant digits first. This works as $\frac{1}{10^n}$ gets smaller ($|\cdot| \to 0$) as $n \to \infty$.

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$$-\frac{1}{} = \frac{1}{} = 1 + 5 + 5^2 + 5^3 + \cdots$$

With $|\cdot|_p$ the integers are bounded!

Back to recurrences

With $|\cdot|_p$ the integers are bounded! Going back to the problem of finding the zero's of

$$a_n = \sum_{i=1}^{m} A_i(n)\alpha_i^n$$

as a function of n. We can think of this as a function $\mathbb{N} \to \overline{\mathrm{Q}}_p$. And ask, are the α_i^n p-adic analytic functions of n?

Do they have an expression as a p-adic power series?

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The fix

Choose p so that $|\alpha_i|_p = 1$ for all i, then $\alpha_i^{p-1} = 1 + \lambda_i$ with $|\lambda_i|_p \leq \frac{1}{p}$. Now $(\alpha_i^{p-1})^n$ is analytic!

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for each fixed r this function of n' is analytic. Infinitely many zeroes for integer n means $\exists r$ with infinitely many zeroes of the form $r + (p-1)n'$. So the function

$$\sum_{i=1}^{m} A_i(r + (p-1)n')\alpha_i^r(\alpha_i^{(p-1)})^{n'}$$

is identically zero, and all these $a_n = 0$ when $n \equiv r \pmod{p-1}$.

Finale

Theorem: Skolem $\rightsquigarrow$ Mahler $\rightsquigarrow$ Lech

Let x_n be a linear recurrence sequence in a field of characteristic 0 (i.e. containing $\mathbb{Q}$). Then the n for which $a_n = 0$ lie in a union of finitely many arithmetic progressions, i.e. they are all of the form $tM + b$ for some fixed M and $b \in B \subset \{0, \ldots, M-1\}$, as $t \in \mathbb{N}$ varies.

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What about for fields of positive characteristic? For finite fields everything is periodic by default. But for fields like $\mathbb{F}_p(x)$ the theorem is false!

(Example from Harm Derksen) Consider the sequence $a_n \in \mathbb{F}_4(x)$ defined by

$$a(n) = x^n + (x+1)^n + (x+\alpha)^n + (x+1+\alpha)^n,$$

where $\alpha \in \mathbb{F}_4 \backslash \mathbb{F}_2$. We can compute $a(0) = a(1) = a(2) = 0$ and $a(3) = 1$. This sequence satisfies a recurrence relation of order 4

$$a(n+4) = a(n+1) + \left(x^4 + x\right) a(n).$$

the set of n for which $a_n = 0$ is

$$\left\{4^a + 4^b | a, b \in \mathbb{N}\right\} \cup \left\{2 \cdot \left(4^a + 4^b\right) | a, b \in \mathbb{N}\right\} \cup \{0, 1\}$$