Singular Moduli

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In this talk:

Observations (Hermite, 1859):

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$$\begin{aligned}
e^{\pi\sqrt{43}} &\approx 884736743.999777466 \\
&\approx 12^3(9^2-1)^3 + 744 - 10^{-4}\cdot 2.225\ldots \\
e^{\pi\sqrt{67}} &\approx 147197952743.999998662454 \\
&\approx 12^3(21^2-1)^3 + 744 - 10^{-6}\cdot 1.337\ldots \\
e^{\pi\sqrt{163}} &\approx 262537412640768743.99999999999925007 \\
&\approx 12^3(231^2-1)^3 + 744 - 10^{-13}\cdot 7.499\ldots
\end{aligned}$$

Some definitions

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Non-examples:

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$$\mathrm{cl}(\mathbf{Z}_K) = I(\mathbf{Z}_K)/P(\mathbf{Z}_K).$$

$\mathrm{cl}(\mathbf{Z}_L)$ **measures** how far $\mathbf{Z}_K$ is from having unique factorisation.

The Hilbert class field (of an imaginary quadratic field)

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$$\mathfrak{p}\,\mathbf{Z}_L = \mathfrak{P}_1\,\mathfrak{P}_2\cdots\mathfrak{P}_n$$

into **distinct** prime ideals $\mathfrak{P}_i$ of $\mathbf{Z}_L$.

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Examples

The Artin reciprocity theorem for the Hilbert class field

Theorem

If K is a number field and L is its Hilbert class field then

$$\mathrm{cl}(\mathbf{Z}_K) \cong \mathrm{Gal}(L|K).$$

The j-invariant

Letting $q = e^{2\pi i z}$ we have

$$j(z) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots .$$

Closing remarks

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- Singular moduli are not particularly complex objects in and of themselves.
- But their relation between different areas of mathematics ensures that they are still a research topic to this day.

Sources

I used some of the following when preparing this talk, and so they are probably good places to look to learn more about the topic:

- "Primes of the form $x^2 + ny^2$" – David A. Cox