Singular Moduli

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In this talk:

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&\approx 12^3(9^2-1)^3 + 744 - 10^{-4} \cdot 2.225\ldots \\
e^{\pi\sqrt{67}} &\approx 147197952743.999998662454 \\
&\approx 12^3(21^2-1)^3 + 744 - 10^{-6} \cdot 1.337\ldots \\
e^{\pi\sqrt{163}} &\approx 262537412640768743.99999999999925007 \\
&\approx 12^3(231^2-1)^3 + 744 - 10^{-13} \cdot 7.499\ldots
\end{aligned}$$

Some definitions

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Non-examples:

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$\mathrm{cl}(\mathbf{Z}_K)$ **measures** how far $\mathbf{Z}_K$ is from having unique factorisation.

The Hilbert class field (of an imaginary quadratic field)

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$$\mathfrak{p}\,\mathbf{Z}_L = \mathfrak{P}_1\,\mathfrak{P}_2 \cdots \mathfrak{P}_n$$

into **distinct** prime ideals $\mathfrak{P}_i$ of $\mathbf{Z}_L$.

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$\mathbf{Q}(\sqrt{-163})$	$\mathbf{Q}(\sqrt{-163})$	1

The Artin reciprocity theorem for the Hilbert class field

Theorem

If K is a number field and L is its Hilbert class field then

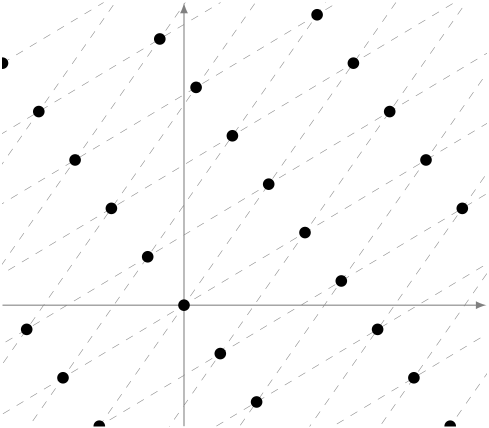
$$\mathrm{cl}(\mathbf{Z}_K) \cong \mathrm{Gal}(L|K).$$

$$\begin{tikzcd}
L \arrow[d, dash, "\mathrm{Gal}(L|K)"] \\
K \arrow[d, dash] \\
\mathbf{Q}
\end{tikzcd}$$

Lattices

Definition

A **lattice** is an additive subgroup of $\mathbf{C}$ that is isomorphic to $\mathbf{Z}^2$.



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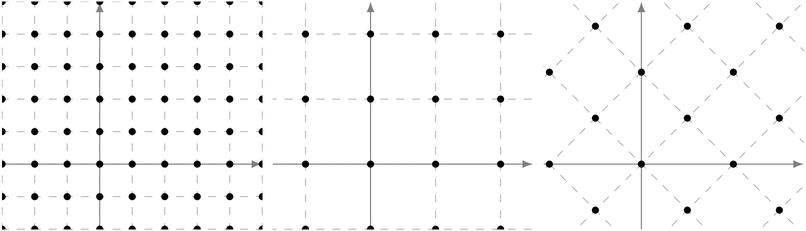

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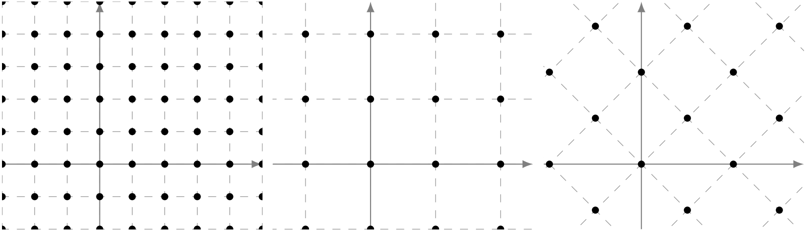

Every lattice is homothetic to one of the form $\mathbf{Z} + \mathbf{Z}\tau$ for some $\tau \in \mathbf{C}$ with positive imaginary part.

The j-invariant

The j-invariant is a function

$$j\colon \{\text{lattices}\} \to \mathbf{C}$$

such that $j(L) = j(L') \iff L$ and L' are homothetic.

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We can define j on the upper half plane by $j(\tau) = j(\mathbf{Z} + \mathbf{Z}\,\tau)$.
Letting $q = e^{2\pi i\tau}$ it turns out that

$$\begin{aligned} j(\tau) = q^{-1} &+ 744 + 196884q + 21493760q^2 \\ &+ 864299970q^3 + 20245856256q^4 + \cdots . \end{aligned}$$

The j-invariant

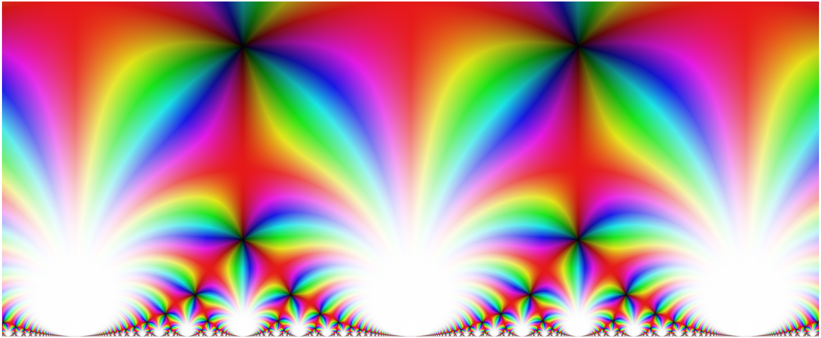

Figure : The j-invariant, picture by Fredrik Johansson

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Examples

$$j(i) = 1728,$$
$$j\left(\frac{1+\sqrt{-3}}{2}\right) = 0,$$
$$j\left(\frac{1+\sqrt{-15}}{2}\right) = \frac{-191025-85995\sqrt{5}}{2}.$$

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Theorem

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A (kind of) converse (Schneider)

If τ is an algebraic number that is not imaginary quadratic then $j(\tau)$ is transcendental.

Explaining Hermite's observations

$$K = \mathbf{Q}(\sqrt{-d}) \text{ with } \mathrm{cl}(\mathbf{Z}_K) = 1, \ \mathbf{Z}_K = \mathbf{Z} + \mathbf{Z}\,\tau.$$

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$$e^{-2\pi i\tau} + 744 + 196884e^{2\pi i\tau} + \ldots \in \mathbf{Z}_K \cap \mathbf{R}$$

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is an integer.
The trailing terms are tiny (of order 10^{-13}) here giving

$$e^{\pi\sqrt{163}} \approx -j(\tau) + 744.$$

The class number 1 problem

Theorem (Stark-Heegner)

The only imaginary quadratic number fields with trivial class group are $\mathbf{Q}(\sqrt{-d})$ for

$$d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}.$$

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So we might expect $e^{\pi\sqrt{19}}$ to be close to an integer too, however

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isn't really. The value is not as close to the corresponding singular modulus as $e^{-\pi\sqrt{d}}$ has larger absolute value for smaller d.

A formula of Gross-Zagier

We have that $j((1+\sqrt{-67})/2) = 12^3(21^2-1)^3$ and $j((1+\sqrt{-163})/2) = 12^3(231^2-1)^3$ and so

$$j\left(\frac{1+\sqrt{-163}}{2}\right) - j\left(\frac{1+\sqrt{-67}}{2}\right) = 2^{15}\cdot 3^7\cdot 5^3\cdot 7^2\cdot 13\cdot 139\cdot 331.$$

A formula of Gross-Zagier

Theorem (Gross-Zagier, '84)

Letting $\epsilon(n) =$ we get

$$N(j(\tau_1) - j(\tau_2))^2 = \prod_{x^2 \equiv D \pmod 4} F\left(\frac{D - x^2}{4}\right)$$

where

$$F(m) = \prod_{\substack{n,n'>0 \\ nn'=m}} n^{-\epsilon(n)}.$$

Closing remarks

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- Singular moduli are not particularly complex objects in and of themselves.
- But their relation between different areas of mathematics ensures that they are still a research topic to this day.

Sources

I used some of the following when preparing this talk, and so they are probably good places to look to learn more about the topic:

- "Primes of the form $x^2 + ny^2$" – David A. Cox