

Motivation for p -adic modular forms

Alex J. Best

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STAGE

Overview

These slides are available online (in handout form) at

[https://alexjbest.github.io/talks/
motivation-for-p-adic-modular-forms/slides_h.pdf](https://alexjbest.github.io/talks/motivation-for-p-adic-modular-forms/slides_h.pdf)



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Goal: Introduce, post hoc, motivation for Katz's definition of p -adic modular forms, especially to motivate Serre's ∂ operator.



Recall

A *modular form* of weight k is a function

$$f: \{(E \xrightarrow{\pi} R \text{ an ell. curve}, \omega \in \Gamma(E, \Omega_{E/R}^1) \text{ nowhere vanishing})\} \rightarrow R$$



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s.t.

1. $\forall \lambda \in R^\times, f(E, \lambda\omega) = \lambda^{-k} f(E, \omega)$
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we then have

$$f(E, \omega) \cdot \omega^{\otimes k} \in \Gamma(R, \underbrace{\pi_*(\Omega_{E/R}^1)}_{\underline{\omega}_{E/R}}^{\otimes k})$$



De Rham cohomology

The sheaf of values $\underline{\omega}_{E/R}$ is a subsheaf of the de Rham cohomology of E/R :

$$H_{\mathrm{dR}}^1(E/R) := \mathbb{H}^1(E, \Omega_{E/R}^\bullet)$$
$$0 \rightarrow \underline{\omega}_{E/R} \rightarrow H_{\mathrm{dR}}^1(E/R) \rightarrow \underbrace{H^1(E, \mathcal{O}_E)}_{=\underline{\omega}_{E/R}} \rightarrow 0$$



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assuming $1/6 \in R$ we can canonically split this sequence:

Fixing $(E, \omega)/R$ we have a unique pair of meromorphic functions with poles only at ∞ , of orders 2 and 3 resp., denoted by X, Y so that

$$\omega = \frac{dX}{Y} \text{ and } E: Y^2 = 4X^3 - g_2X - g_3, g_i \in R$$



Then we have an inclusion of 2-term complexes

$$(\mathcal{O}_E \rightarrow \Omega_{E/R}^1) \subseteq (\mathcal{O}_E(\infty) \rightarrow \Omega_{E/R}^1(2\infty))$$

which induces an isomorphism on \mathbb{H}^1 . Moreover for $i > 0$,

$$H^i(E, \mathcal{O}_E(\infty)) = 0$$

$$H^i(E, \Omega_{E/R}^1(2\infty)) = 0$$

giving

$$\begin{aligned} H_{\mathrm{dR}}^1(E/R) &\cong \mathbb{H}^1(E, \mathcal{O}_E(\infty) \rightarrow \Omega_{E/R}^1(2\infty)) \\ &= \mathrm{coker}(H^0(E, \mathcal{O}_E(\infty)) \rightarrow H^0(E, \Omega_{E/R}^1(2\infty))) \\ &= \mathrm{coker}(R \xrightarrow{0} H^0(E, \Omega_{E/R}^1(2\infty))) \\ &= H^0(E, \Omega_{E/R}^1(2\infty)) \end{aligned}$$







Theorem (Katz, Manin-Vishik)

