Zeta functions and p-adic integrals; computations and applications

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AMS Graduate Student Conference, Brown University

Where are we going?

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Example
If $C = \mathbf{P}^1/\mathbf{F}_p$ then we have $C(\mathbf{F}_q) = \mathbf{F}_q \cup \{\infty\}$ so

$$\#C(\mathbf{F}_{p^n}) = p^n + 1.$$

An elliptic curve

Example
If $E\colon y^2 = x^3 - 1/\mathbf{F}_5$ then

n	1	2	3	4	5	6	7	8
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It initially seemed like we had an infinite amount of data here: $\#E(\mathbf{F}_{5^n})$ for all $n \in \mathbf{N}$. But we don't!

The Weil polynomial

Rephrased: we have a polynomial

$$L_E = t^2 + 5$$

so that

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Theorem (Schmidt?, Weil?)
Let $C/\mathbf{F}_q$ be a curve, there exists a monic $L_C(t) \in \mathbf{Z}[t]$ of degree $2 \cdot \mathrm{genus}(C)$. Whose roots α_i come in complex conjugate pairs with $|\alpha_i| = q^{1/2}$ and

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The zeta function

The condition on the roots means $\alpha_i \overline{\alpha}_i = q$ so we may write $L_C(t) = q^g \prod_i (1 - \frac{\alpha_i}{q} t)$ then

$$\log(L_C(t)/q^g) = -\sum_i \sum_{n=1}^{\infty} \frac{\alpha_i^n t^n}{q^n n} = \sum_{n=1}^{\infty} -\left(\sum_i \alpha_i^n\right) \frac{t^n}{q^n n}$$

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The (Hasse-Weil) zeta function of $C/\mathbf{F}_q$ is

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And we have that

$$Z(C,t) = \frac{q^{-g} L_C(qt)}{(1-t)(1-qt)}.$$

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Example (A completely random example, I promise)

$$C\colon y^2 = x^5 + 6x^2 + x + 3/\mathbf{F}_{43}$$

$$L_C(t) = t^4 + 9t^3 + 64t^2 + 387t + 1849$$

$$\implies \#J(\mathbf{F}_{43}) = L_C(1) = 2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$$

so $J(\mathbf{F}_{43}) = C_{2\cdot 3\cdot 5\cdot 7\cdot 11}$.

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so $J(\mathbf{F}_{43}) = C_{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11}$. So never use this curve for cryptography!!

Distributional questions - Sato-Tate

Let $C/\mathbf{Q}$ be a genus g curve. We can reduce mod p for all primes p of good reduction, get a polynomial $L_{C_{\mathbf{F}_p}}(t)$ for all these p.

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a unitary symplectic polynomial, i.e. the characteristic polynomial of a unitary symplectic matrix.

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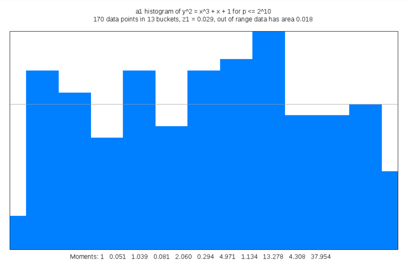
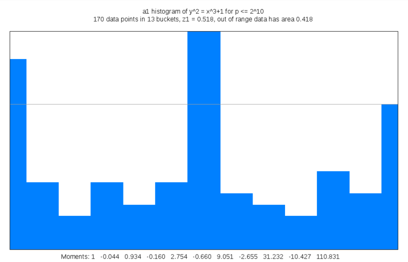
How is the image distributed as $p \to \infty$?

Genus 1

$$y^2 = x^3 + x + 1 \qquad y^2 = x^3 + 1$$





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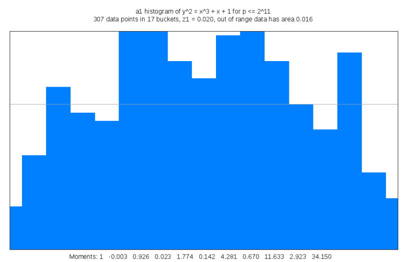

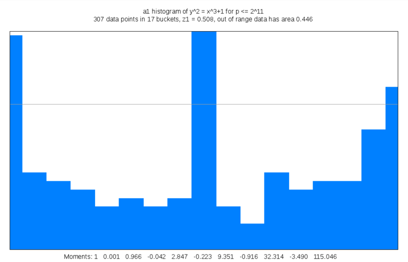

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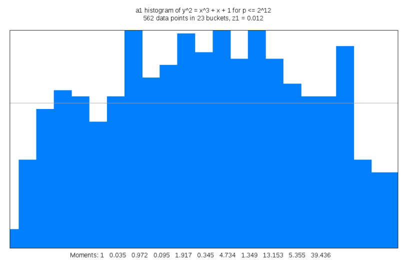

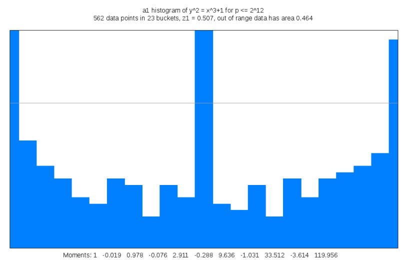

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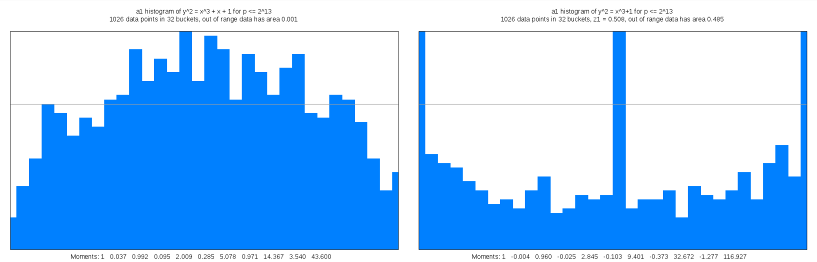

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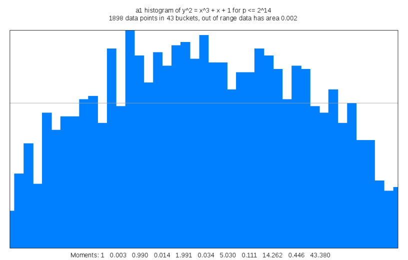

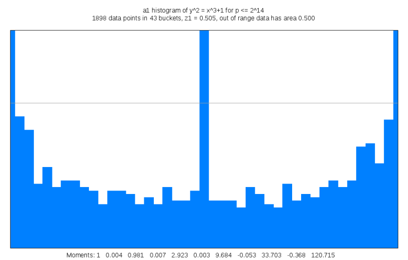

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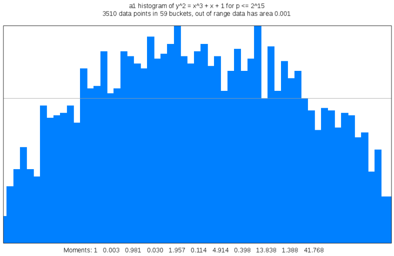

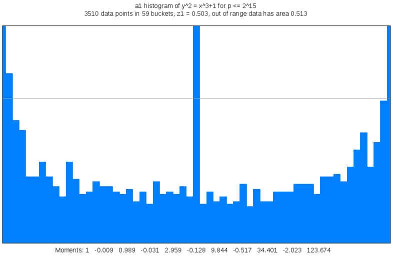

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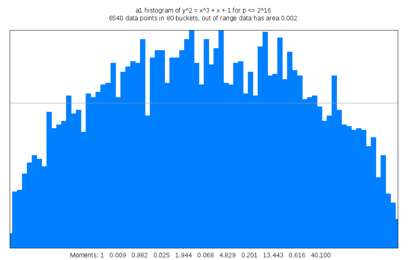

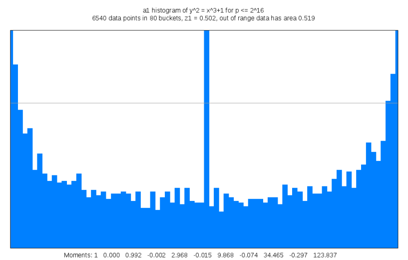

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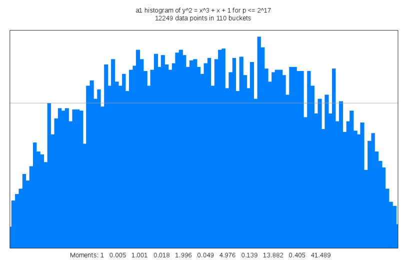

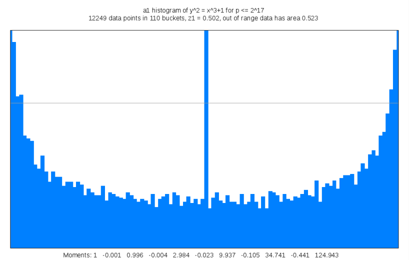

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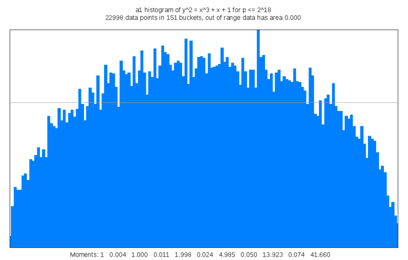

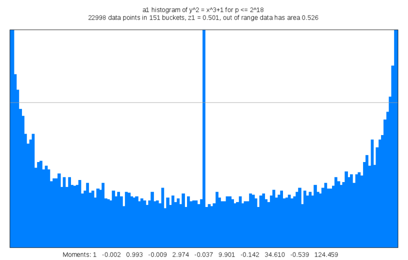

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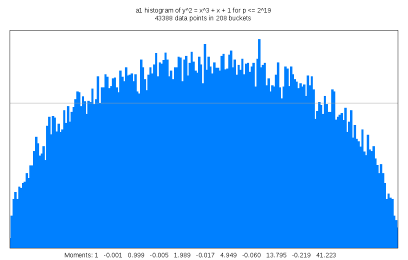

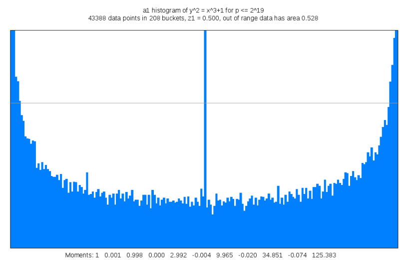

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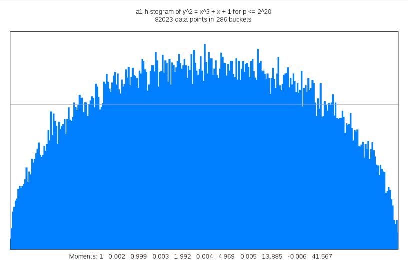

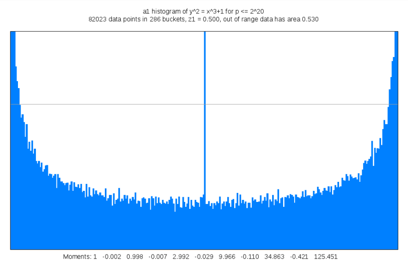

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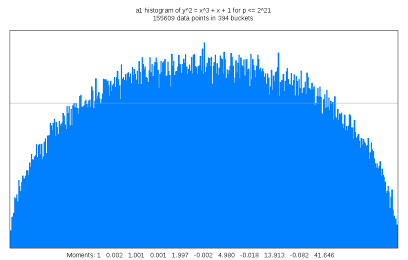

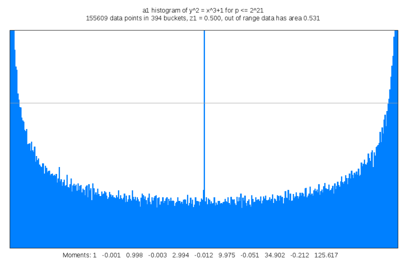

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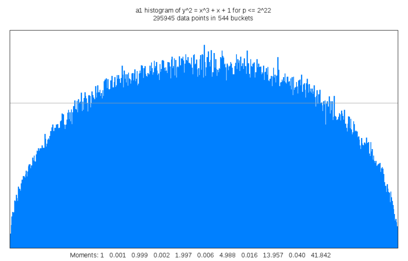

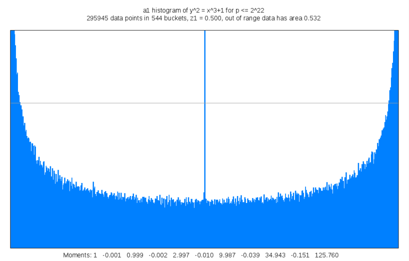

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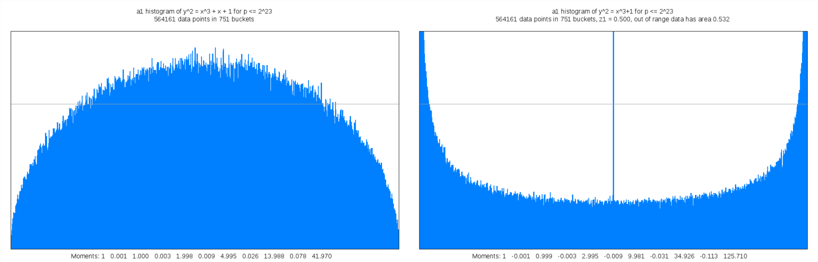

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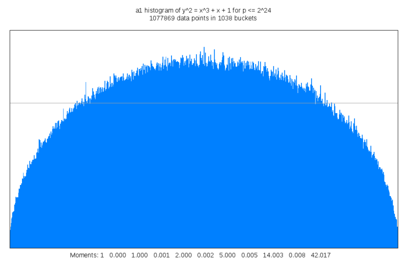

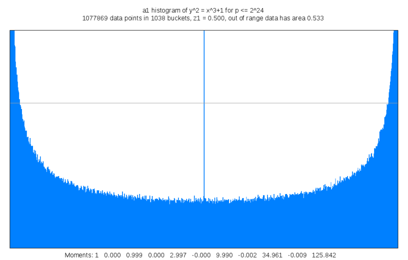

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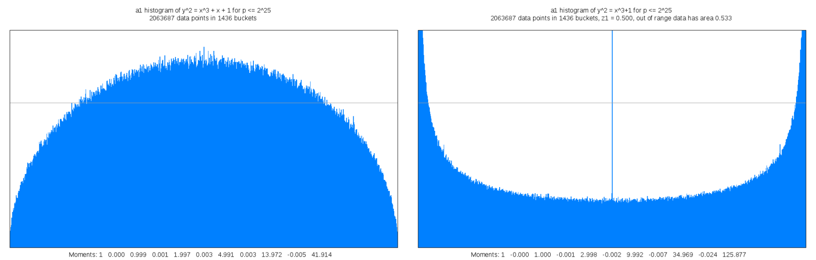

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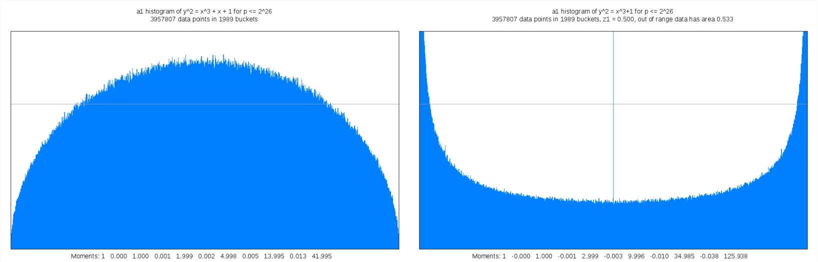

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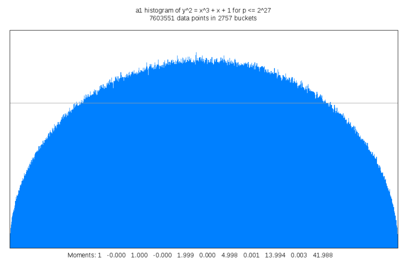

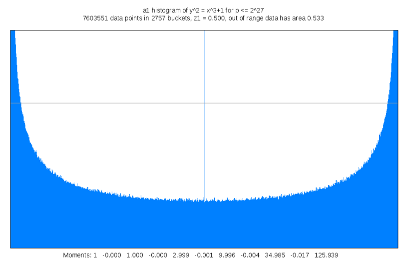

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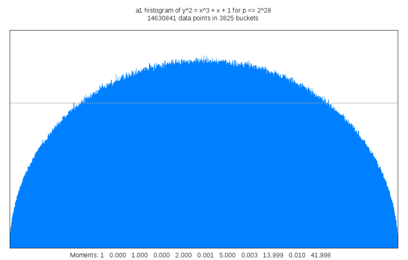

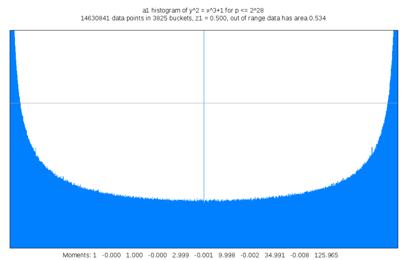

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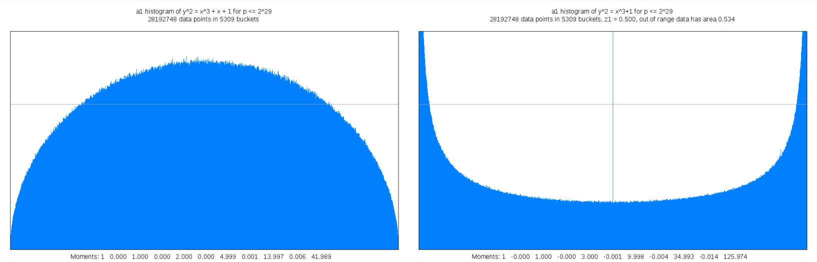

Pictures due to Drew Sutherland.

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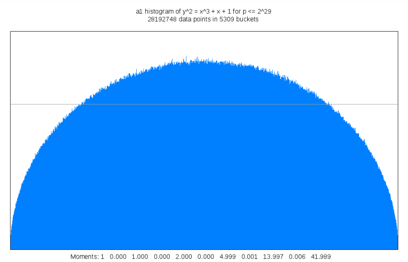

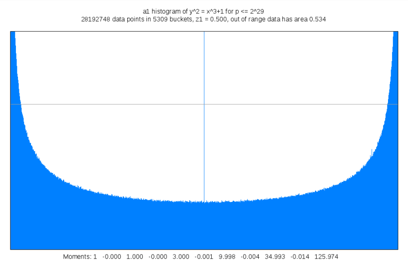

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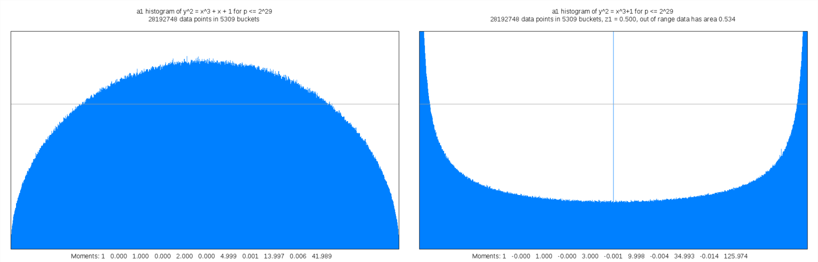

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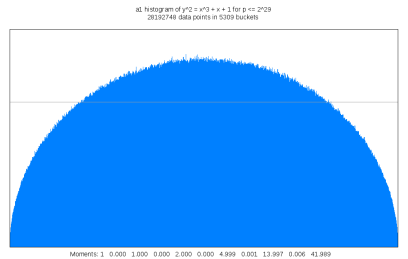

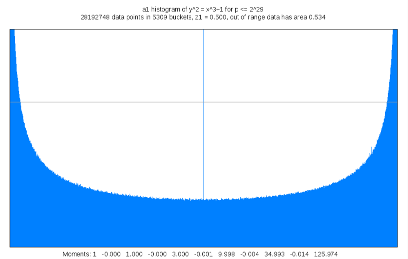

Pictures due to Drew Sutherland. Left is a generic elliptic curve, the right has CM (over $\mathbf{Q}$). By computing enough zeta functions we can *see* the endomorphism algebra of our curve.

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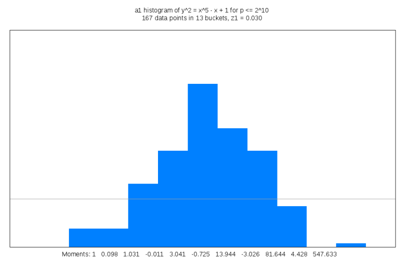

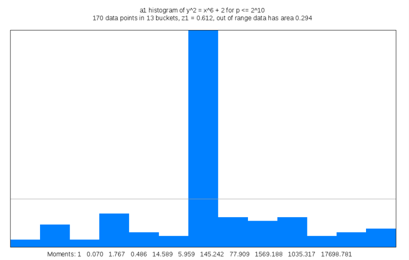

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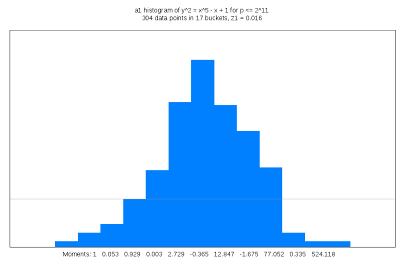

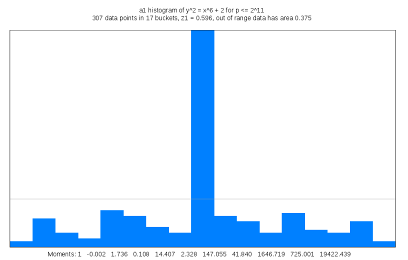

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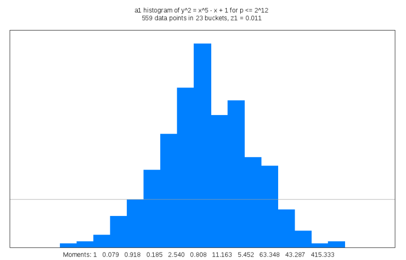

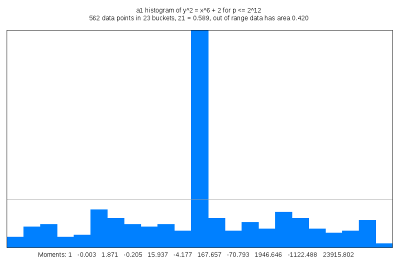

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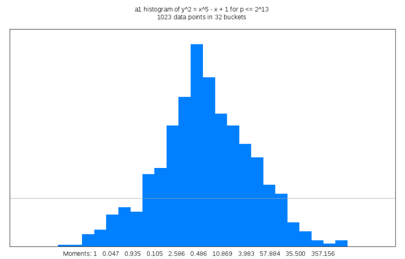

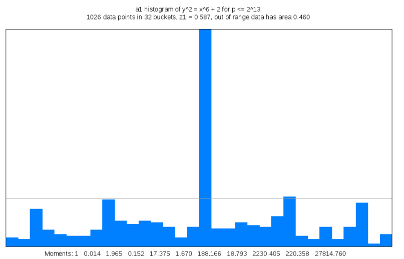

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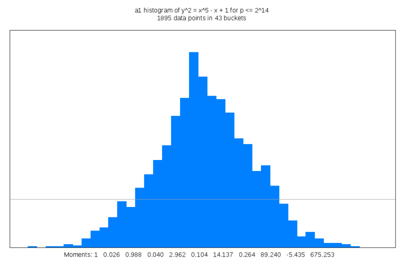

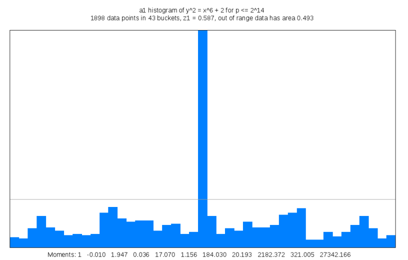

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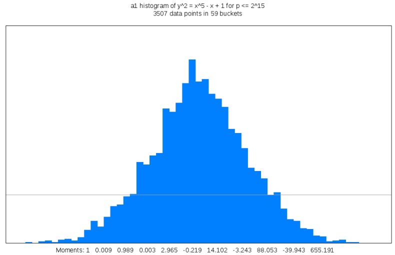

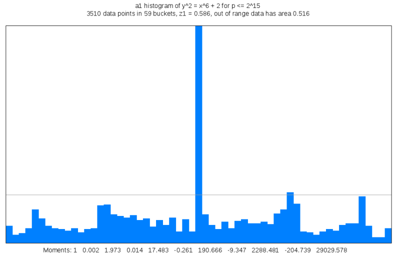

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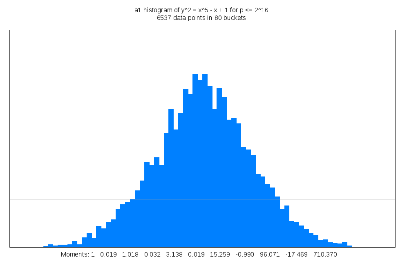

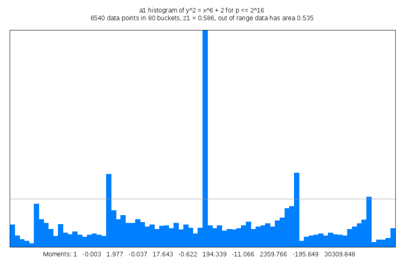

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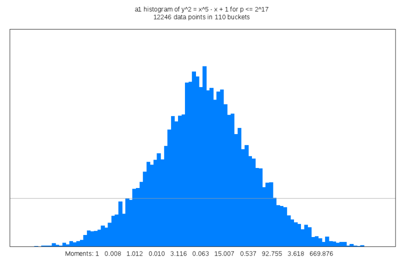

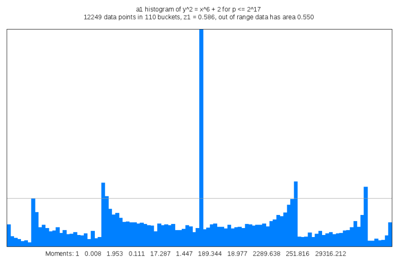

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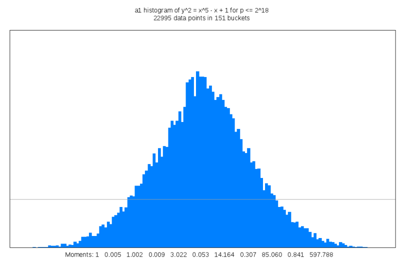

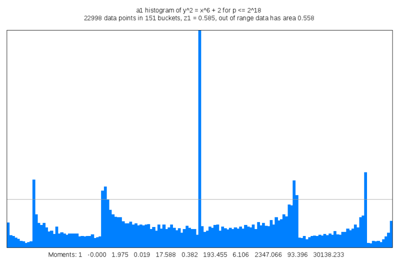

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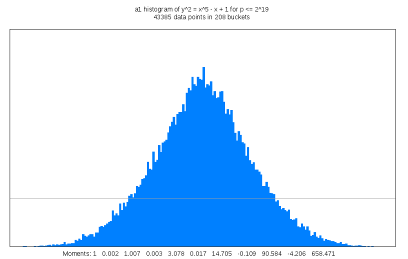

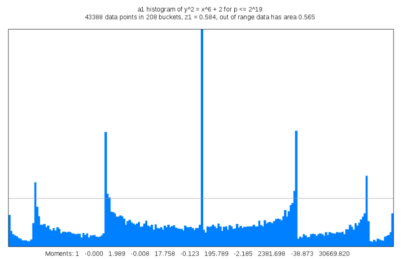

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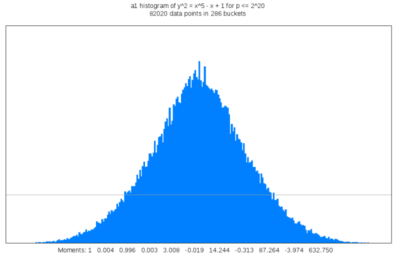

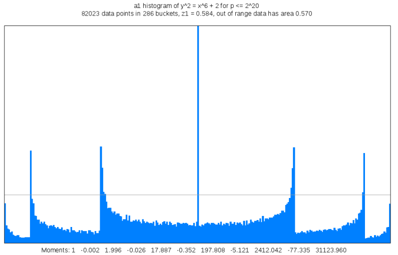

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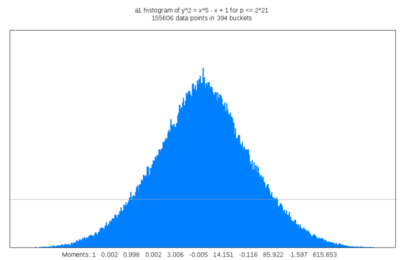

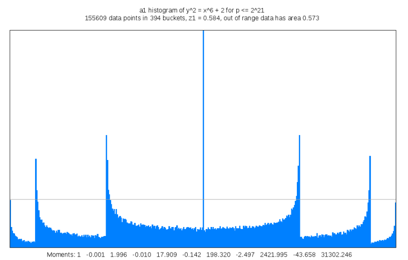

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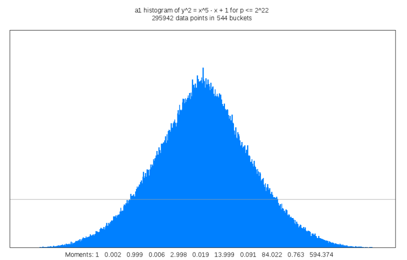

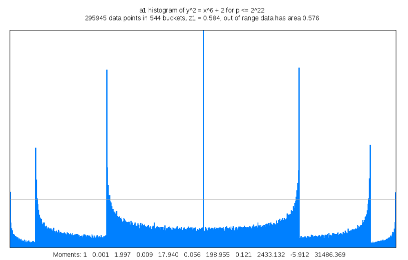

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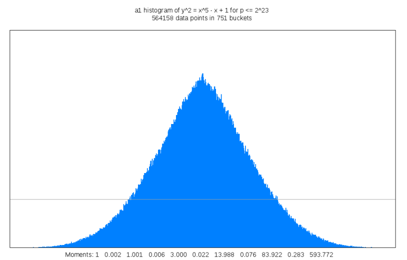

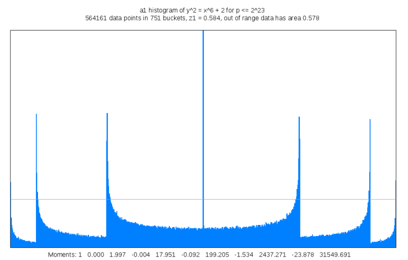

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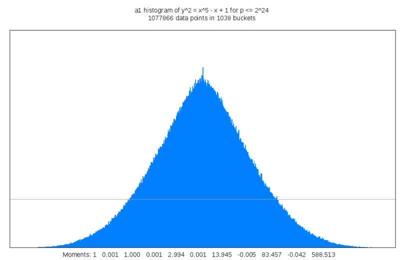

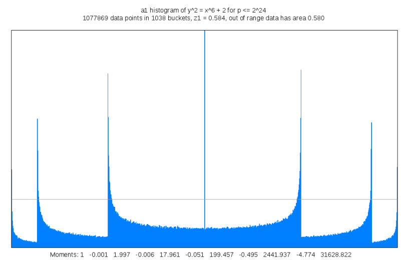

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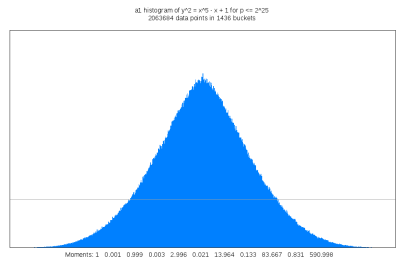

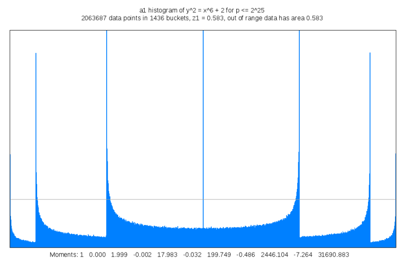

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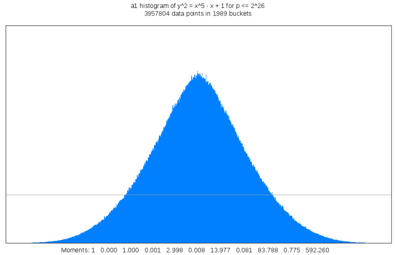

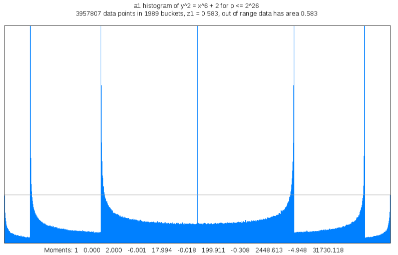

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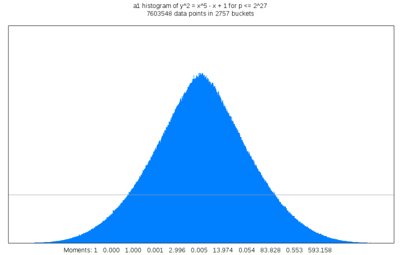

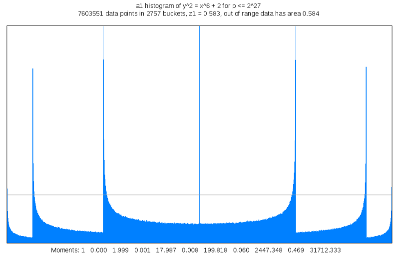

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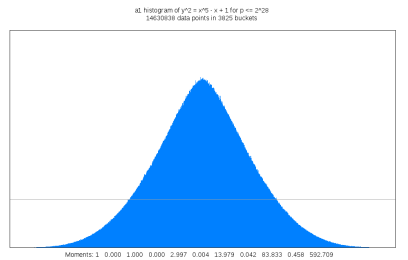

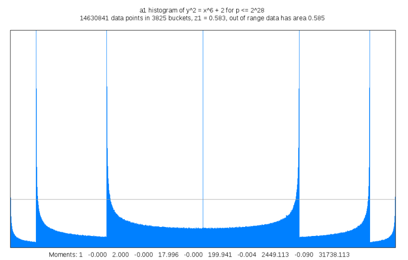

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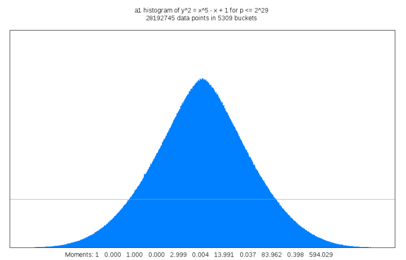

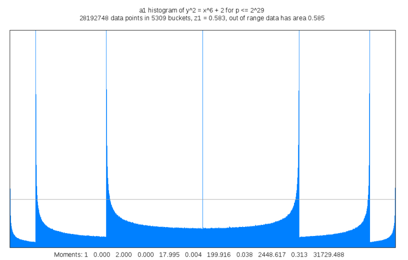

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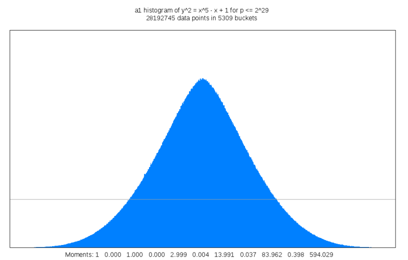

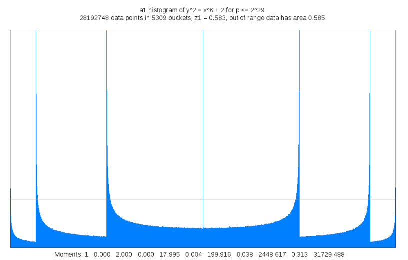

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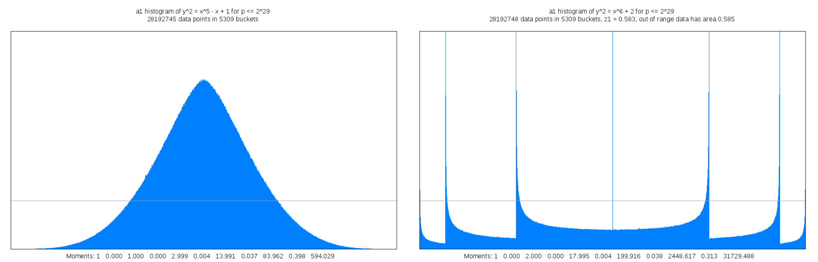

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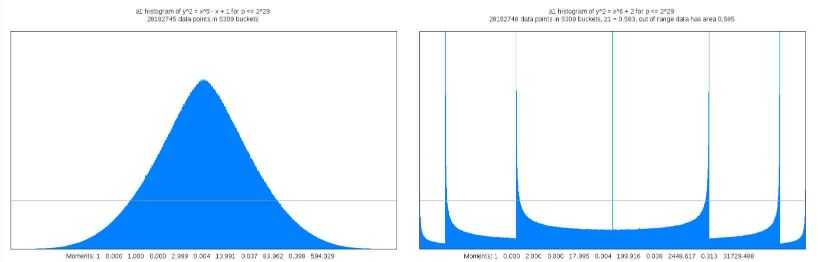

Pictures due to Drew Sutherland. Left is a generic genus 2 curve, the right has $\mathrm{End}(\mathrm{Jac}(C)_{\overline{\mathbf{Q}}}) \otimes \mathbf{R} = \mathrm{Mat}_2(\mathbf{C})$. After the work of Fité-Kedlaya-Rotger-Sutherland we can recognise these distributions and guess the structure of the Jacobian, the right one should be square of a CM elliptic curve.

Relations

Let

$$C_1 \colon y^2 + y = x^3 + x/\mathbf{F}_2, \; C_2 \colon y^2 + y = x^5 + x/\mathbf{F}_2$$

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In our example we have a map

$$(x, y) \mapsto (x^2 + x, y + x^3 + x^2).$$

Relations again

The converse is false!

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where

$$L_{D_1}(t) = t^4 + t^3 + 2t + 4,\ L_{D_2} = (t^4 + t^3 + 2t + 4)(t^2 + 2)$$

but no map exists!

How to compute?

Reverse reverse engineering: Count points for a few n ($n \leq g$ is sufficient), recover $L_C(t)$.

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p**-adic cohomology:** A method due to Kedlaya relates $L_C(t)$ to p-adic cohomology. $L_C(t)$ is the characteristic polynomial of "Frobenius" acting on "$H^1_{MW}(\tilde{C})$". If we can compute this action (as a matrix) we win!

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Average time: Harvey-Sutherland have an approach to compute $L_{C_{\mathbf{F}_p}}(t)$ for a curve over $\mathbf{Q}$ for all $p < N$ at once! This works out faster on average.

Monsky-Washnitzer cohomology in general

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First choose a lift $\tilde{C}/\mathbf{Z}_q$ and an affine open $U = \operatorname{Spec}(A) \subseteq C$.
And a lift of the q-power Frobenius on $\overline{A} = A/pA$ to $\phi \colon A^\dagger \to A^\dagger$.

Now the weak completion $A^\dagger$ is the set of p-adic power series on U that p-adically overconverge.

We have differentials $\Omega^1_{A^\dagger}$ and a derivative $\mathrm{id} \colon A^\dagger \to \Omega^1_{A^\dagger}$

$$H^1_{\mathrm{MW}}(\overline{A}) = \Omega^1_{A^\dagger} \otimes \mathbf{Q}_p / \mathrm{d}(A^\dagger \otimes \mathbf{Q}_p)$$

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Let $C\colon y^2 = \overline{Q}(x)/\mathbf{F}_q$ be an (odd) hyperelliptic curve.

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The affine coordinate ring of the punctured curve is

$$A = \mathbf{Z}_p[x, y, y^{-1}]/(y^2 - Q(x))$$

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First choose a lift $\tilde{C}\colon y^2 = Q(x)/\mathbf{Z}_q$.

The affine coordinate ring of the punctured curve is

$$A = \mathbf{Z}_p[x, y, y^{-1}]/(y^2 - Q(x))$$

$$A^\dagger = \left\{ \sum_{i=-\infty}^{\infty} R_i(x) y^{-i} : R_i \in \mathbf{Z}_p[x]_{\deg \leq 2g} \text{ where } \liminf_{|i| \to \infty} v_p(R_i)/|i| > 0 \right\}$$

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The q-power Frobenius on A/pA can be lifted to $\phi\colon A^\dagger \to A^\dagger$

$$x \mapsto x^p$$

$$y \mapsto y^{-p} \sum_{k=0}^{\infty} \binom{-1/2}{k} (\phi(Q(x)) - Q(x)^p)^k / y^{2pk}.$$

Monsky-Washnitzer cohomology (for hyperelliptic curves)

$$\Omega_{A^\dagger} = A^\dagger \,\mathrm{d}x \oplus A^\dagger \,\mathrm{d}y/(2y\,\mathrm{d}y - Q'(x)\,\mathrm{d}x))$$

$$\begin{aligned} \mathrm{d}\colon A^\dagger &\to \Omega^1_{A^\dagger} \\ \sum_{i=-\infty}^{\infty} \frac{R_i(x)}{y^i} &\mapsto \sum_{i=-\infty}^{\infty} R_i'(x)y^{-i}\,\mathrm{d}x - R_i(x)iy^{-i-1}\,\mathrm{d}y. \end{aligned}$$

Reductions in cohomology

$\{\omega_i = x^i \, \mathrm{d}x/y\}_{i=1,\dots,2g}$ are a basis for $H^1_{MW}(C)$ and for each i we get an expansion

$$\phi^*\omega_i \equiv \sum_{j=0}^{N-1} \sum_{r=0}^{(2g+1)j} B_{j,r} x^{p(i+r+1)-1} y^{-p(2j+1)+1} \frac{\mathrm{d}x}{2y} \pmod{p^N}$$

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to do this we iteratively use relations like

$$\begin{aligned}\mathrm{d}(x^s y^{-2t+1}) = &(2s - (2t-1)(2g+1))\, x^{2g+1} x^{s-1} y^{-2t} \frac{\mathrm{d}x}{2y} \\ &+ \left(2sP(x) - (2t-1)xP'(x)\right) x^{s-1} y^{-2t} \frac{\mathrm{d}x}{2y}.\end{aligned}$$

to reduce the exponents of monomials appearing in the expansion.

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The L-polynomial is then the characteristic polynomial of the matrix $F = (a_{ij})_{i,j}$.

Interlude: Computing things quickly - a silly example

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$$N! = P(0) \cdot P(\sqrt{N}) \cdot P(2\sqrt{N}) \cdots P((\sqrt{N} - 1)\sqrt{N})$$

where

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once we compute $\sqrt[4]{N}$ of these $P(i)$ (for $i = 0, \ldots, \sqrt[4]{N}$) in $\sqrt{N}$ steps we have a degree $\sqrt[4]{N}$ polynomial evaluated at $\sqrt[4]{N}$ points. If you know a (monic) degree n polynomial at n points, you know the polynomial! $\rightsquigarrow$ interpolate to find other values

Interlude: Computing things quickly - Fancy version

In general if we have $M(t) \in \mathrm{Mat}_{n\times n}(R[t])$ a matrix with linear polynomials as coefficients. We can evaluate lots of products

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\begin{aligned}
&M(0)M(1)\cdots M(k-1),\\
&M(k)M(k+1)\cdots M(2k-1),\\
&\vdots\\
&M((m-1)k)M((m-1)k+1)\cdots M(mk-1)
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Using this we can reduce quickly and compute $L_C(t)$ in time roughly $\sqrt{p}$ (Harvey).

With Arul, Costa, Magner, Triantafillou we can do this for general cyclic covers $y^a = f(x)$.

Take $C/\mathbf{Z}_p$ a genus g curve and p an odd prime.

Theorem (Coleman)
There is a $\mathbf{Q}_p$-linear map $\int_b^x \colon \Omega^1_{A^\dagger} \otimes \mathbf{Q}_p \to A_{\mathrm{loc}}(X)$ for which:

$$\mathrm{d} \circ \int_b^x = (\mathrm{id}\colon \Omega^1_{A^\dagger} \otimes \mathbf{Q}_p \to \Omega^1_{loc}) \quad \text{“FTC”}$$

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Locally we can integrate power series formally.

To integrate between far away points we use Frobenius equivariance.

Switch to an odd hyperelliptic curve now, some manipulation with the set of all $\int_P^\infty \omega_i$ gives:

$$\begin{pmatrix} \vdots \\ \int_P^\infty \omega_i \\ \vdots \end{pmatrix} = (F - I)^{-1} \begin{pmatrix} \vdots \\ f_i(P) \\ \vdots \end{pmatrix}$$

where from earlier

$$\phi^* \omega_i \equiv \sum_{j=1}^{2g} M_{ij} \omega_j - \mathrm{d} f_i$$

One consequence of Coleman's work we saw earlier is

Theorem (Coleman's effective Chabauty)
Let $C/\mathbf{Q}$ be a curve of genus g. If $\operatorname{rank} J(C)(\mathbf{Q}) < g$ and $p > 2$ is a prime of good reduction for C then

$$\#C(\mathbf{Q}) \leq \#C_p(\mathbf{F}_p) + 2g - 2.$$

Explicit Chabauty

Given an individual curve we can often compute $X(\mathbf{Q})$ by explicitly evaluating enough of these integrals.

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$$X_s(13)$$

or

$$X_0(67)^+ \text{ and friends?}$$

A fun converse

Thinking about effective Chabauty backwards: if we have a lot of $\mathbf{Q}$-points and few $\mathbf{F}_p$ points, the Jacobian must have large rank!

Example
To force a curve to have many $\mathbf{Q}$ points and few $\mathbf{F}_7$ points, let

$$C\colon y^2 = x(x-7)(x-14)(x+7)(x+14)+1$$

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To force a curve to have many $\mathbf{Q}$ points and few $\mathbf{F}_7$ points, let

$$C\colon y^2 = x(x-7)(x-14)(x+7)(x+14)+1$$

this has a bunch of rational points $(7n, \pm 1)$ for $n = -2, -1, 0, 1, 2$ (and ∞ so ≥ 11 in all), but these give the same $\mathbf{F}_7$ points $(0, \pm 1)$. In fact $\#C(\mathbf{F}_7) = 8$ so we fail the Coleman bound as

$$11 \leq 8 + 2g - 2 = 10$$

so we must have $\operatorname{rank} \operatorname{Jac}(C)(\mathbf{Q}) \geq g = 2$. (Magma tells me $\operatorname{rank} \operatorname{Jac}(C)(\mathbf{Q}) = 5$ in fact!)

More generally

In fact Coleman showed:

Corollary (Coleman)
Let $k \in \mathbf{Z}$, $p \nmid k$ *prime and* $f(x)/\mathbf{Z}$ *monic with* $f(x) \equiv x^k \pmod{p}$ *and* $\lfloor (k+1)/2 \rfloor$ *roots over* $\mathbf{Z}$ *then the rank of the Jacobian of*

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The proof is a little more serious than our example above, it shows that the points $(\alpha_i, 1)$ where α_i are roots of f are actually linearly independent in the Jacobian.

Fast Coleman

Recall to compute a Coleman integral we need to find

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we can coerce the evaluation of $f_i(P)$ into a linear recurrence and apply Bostan-Gaudry-Schost!

Generalities

Coleman integration can be more general, Coleman de Shallit define:

$$r_C \colon K_2(\overline{k}(C)) \to \mathrm{Hom}(H^0(C, \Omega^1_{C/\overline{k}}), \overline{k}).$$

$$r(f, g)(\omega) = -\int_{(g)} \log(f) \in \overline{k}$$

Where next?

- Coleman integration quickly on general curves
- Coleman integration for many primes at once?
- Distribution?