Riemann Hypotheses

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WMS Talks

4/2/2014

In this talk:

The Riemann zeta function: Euler's work

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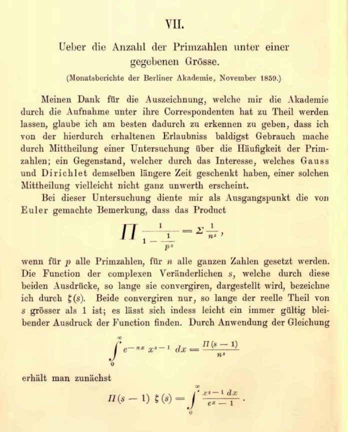
- He also discovered more general formulae for $\sum_{n=1}^{\infty} n^{-2k}$ in terms of the Bernoulli numbers B_{2k} for all natural k.
- In fact, a nice form for

$$\sum_{n=1}^{\infty} n^{-2k-1},$$

is still unknown today.

The Riemann zeta function: Along comes Riemann

- In 1859 Bernhard Riemann, a well known analyst, publishes a paper on his work counting the primes using analysis.



VII.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859.)

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Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\Pi \frac{1}{1-\frac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch $\zeta(s)$. Beide convergiren nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

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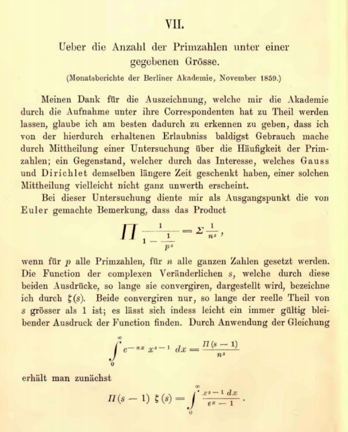

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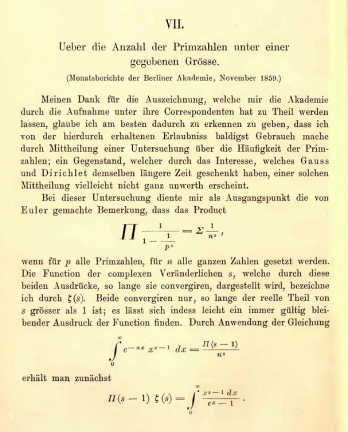

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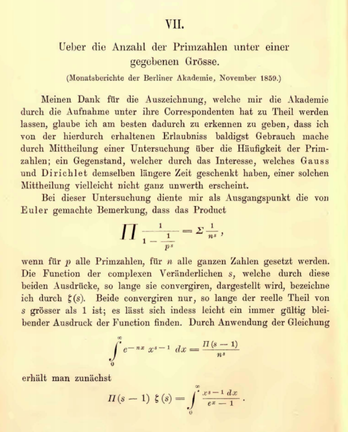
- Along the way he (essentially) makes four hypotheses.



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$$\frac{1}{\pi^{s/2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{\pi^{(1-s)/2}}\Gamma\left(\frac{s}{2}\right)\zeta(1-s).$$

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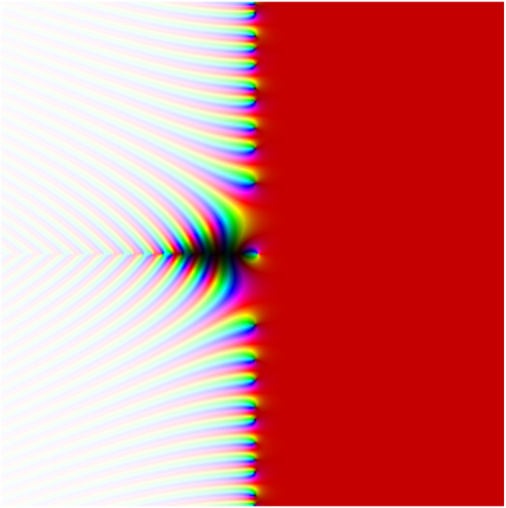

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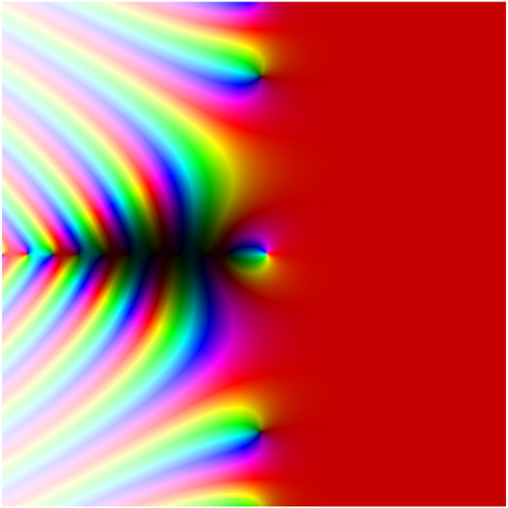

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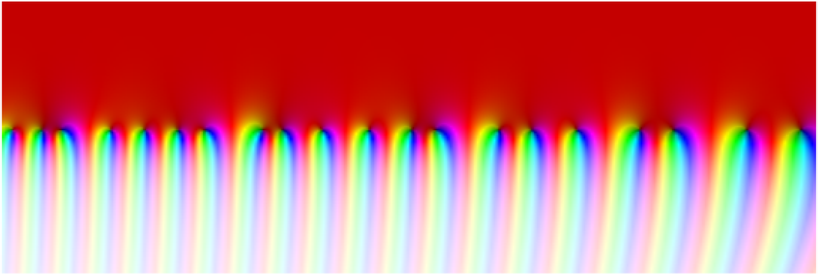

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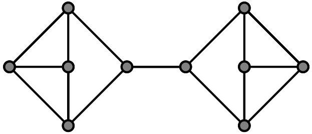

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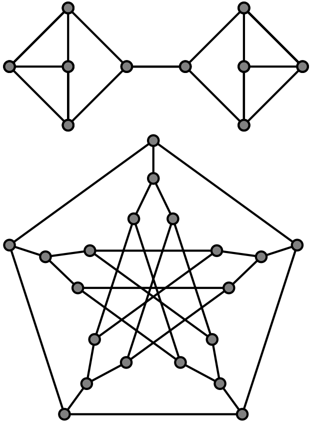

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The zeta function of a scheme

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In a general number field the idea of a prime element doesn't work out so well, however if we consider nice subgroups of the field (ideals) then everything works out well, so we should deal with ideals instead of elements! For example we have unique factorisation of ideals into prime ideals.

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- The properties of this package can tell us about the objects we started with.
- We can also see links between different objects via their zeta functions.
- Due to the abundant computational evidence (over ten trillion non-trivial zeroes found so far, all on the critical line) a huge number of papers have been written that assume the Riemann hypothesis is true. So a proof of the (generalised) hypothesis would imply hundreds of other results true also.

Sources used

I used some of the following when preparing this talk, and so they are possibly good places to look to learn more about the topic:

1. "What is... an expander?" – Peter Sarnak
2. "Problems of the Millennium: The Riemann Hypothesis" – Peter Sarnak
3. "Problems of the Millennium: The Riemann Hypothesis" (Official Millennium prize problem description) – Enrico Bombieri
4. Wikipedia – Enough said
5. http://graphtheoryinlatex.blogspot.com/ – Pretty pictures