Computing Coleman integrals on superelliptic curves

Arithmetic of low dimensional abelian varieties – ICERM

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Background

Coleman integration is a p-adic integration theory that may be applied to integrate 1-forms on curves.

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Fix a curve $C/\mathbf{Z}_{p^n}$ with good reduction, a point $b \in C$, and A the ring of (overconvergent) functions on C, defines $\int_b \colon \Omega^1_A \to A$, satisfying the usual properties (fundamental theorem of calculus, linearity, additivity in endpoints).

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This algorithm took time proportional to p, as have extensions.

Theorem
Let

$$C/\mathbf{Z}_{p^n} : y^a = h(x)$$

with $\gcd(a, \deg(h)) = 1$, $p \nmid a$, *Let* M *be the matrix of Frobenius, acting on* $H^1_{\mathrm{dR}}(C)$, *basis* $\{\omega_{i,j} = x^i \,\mathrm{d}x/y^j\}_{i=0,\dots,b-2, j=1,\dots,a}$, *and points* $P, Q \in C(\mathbf{Q}_{p^n})$ *known to precision* p^N, *if* $p > (aN-1)b$, *the vector of Coleman integrals* $\left(\int_P^Q \omega_{i,j}\right)_{i,j}$ *can be computed in time*

$$\widetilde{O}\left(g^3\sqrt{p}nN^{5/2} + N^4 g^4 n^2 \log p\right)$$

to absolute precision $N - v_p(\det(M - I))$.

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Speed of this algorithm may lend itself to answering distributional questions?