2019-07-04

Explicit computation with Coleman integrals

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Journées Arithmétiques XXXI – Istanbul University

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1. Thank the audience for being awake.

Coleman integration

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near a point $\alpha \in \mathbf{G}_m(\mathbf{Q}_p) = \mathbf{Q}_p^\times$:

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}(\alpha + t)}{\alpha + t} = \frac{\mathrm{d}t}{\alpha + t} = \frac{1}{\alpha}\sum\left(\frac{-t}{\alpha}\right)^n \mathrm{d}t$$

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so that

$$\int_\alpha^{\alpha+t} \frac{\mathrm{d}x}{x} = -\sum \frac{1}{n+1}\left(\frac{-t}{\alpha}\right)^{n+1} + C$$

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Bad topology!

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Applications: Chabauty-Kim

Minhyong Kim has vastly generalised the above to cases where

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Theorem (Balakrishnan-Dogra-Muller-Tuitman-Vonk)
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Their method can also be applied to other interesting curves:

Theorem (WIP B.-Bianchi-Triantafillou-Vonk)
The modular curve $X_0(67)^+$ (of genus 2 and jacobian rank 2), has rational points contained in an explicitly computable set of 7-adic points of cardinality 16.

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