Riemann Hypotheses

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WMS Talks

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In this talk:

The Riemann zeta function: Euler's work

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- He also discovered more general formulae for $\sum_{n=1}^{\infty} n^{-2k}$ in terms of the Bernoulli numbers B_{2k} for all natural k.
- In fact, a nice form for

$$\sum_{n=1}^{\infty} n^{-2k-1},$$

is still unknown today.

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VII.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859.)

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$$\Pi \frac{1}{1-\frac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch $\zeta(s)$. Beide convergiren nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^\infty e^{-nx} x^{s-1}\, dx = \frac{\Pi(s-1)}{n^s}$$

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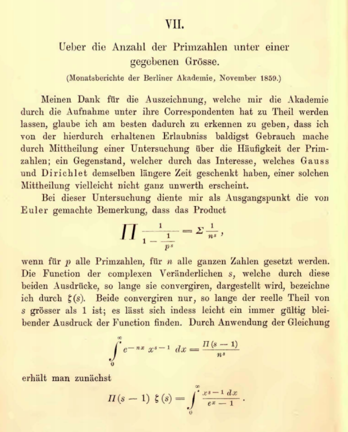

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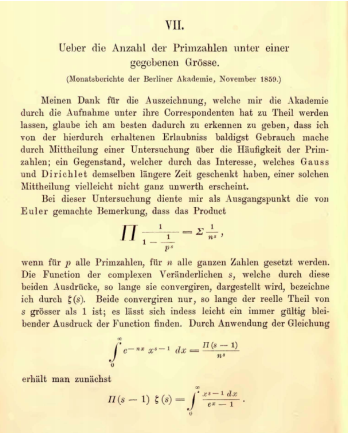

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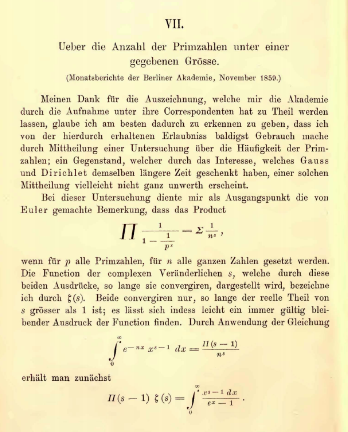
- Here the notation ζ for this function is used for the first time.
- Along the way he (essentially) makes four hypotheses.



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The Riemann zeta function: What Riemann did

In his work Euler had (more or less) found that

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$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

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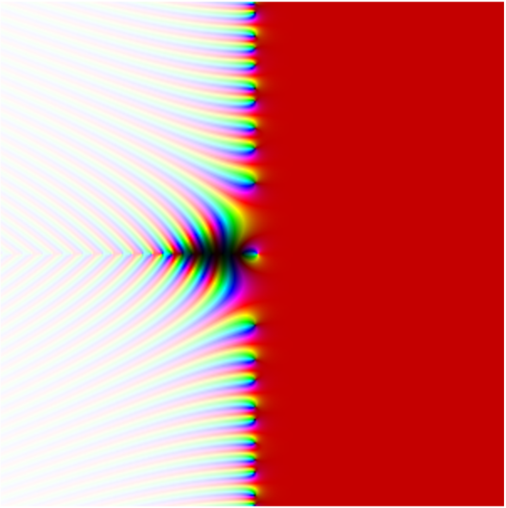

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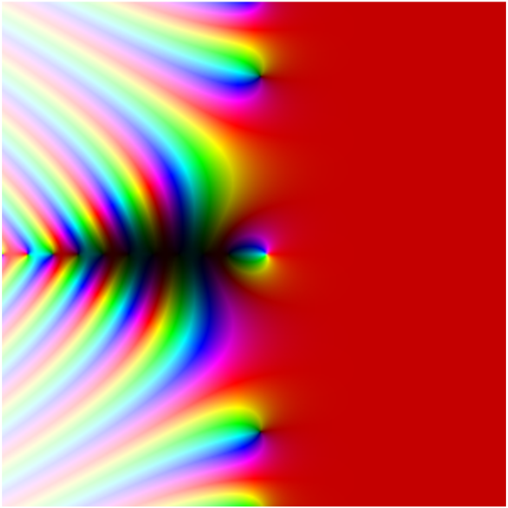

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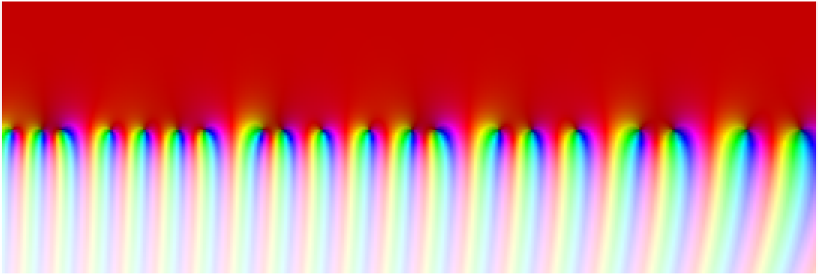

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All non-trivial zeroes of the Riemann zeta function lie on the **critical line**

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- The location of the zeroes of $\zeta(s)$ relates in a strong way to the distribution of the primes.

The Riemann zeta function: Why number theorists care

The prime counting function $\pi(x)$

We define

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The Riemann hypothesis is actually equivalent to the statement that there exists $c_2 > c_1 > 0$ such that

$$\mathrm{Li}(x) + c_1\sqrt{x}\log(x) \leq \pi(x) \leq \mathrm{Li}(x) + c_2\sqrt{x}\log(x)$$

eventually. These are the best possible bounds!

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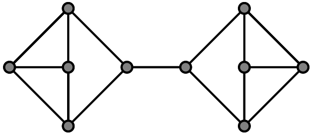

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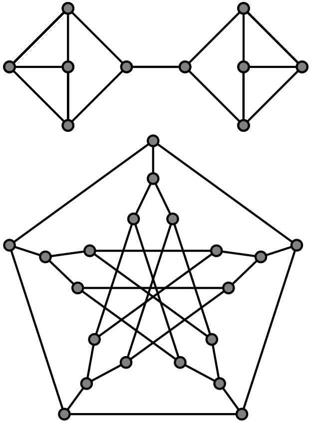

The spectrum

Given such a graph G we can fix a numbering of the vertices and form:

The adjacency matrix

Is an $n \times n$ matrix $A_G = (a_{i,j})$ of 1's and 0's given by

$$a_{i,j} = 1 \iff \text{there is an edge between vertex } i \text{ and vertex } j.$$

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The gap between this value and k is very important for optimisation problems of this nature. The larger the better!

Ramanujan graphs

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Only in 2013 were Ramanujan graphs shown to exist for $k \neq p^n + 1$ (Marcus, Speilman, Srivastava).

The Ihara zeta function

Take a graph G as described above (k-regular for some k, finite and connected). We define:

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The Ihara zeta function of G

$$\zeta_G(u) = \prod_{P \text{ a prime of } G} \left(1 - u^{\text{length}(p)}\right)^{-1}.$$

The Ihara determinant formula

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In fact we always have the following expression for ζ_G:

The Ihara determinant formula

$$\zeta_G(u)^{-1} = (1-u^2)^{|E|-|V|}\det((1-(k-1)u^2)I - Au).$$

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In the strip $0 < \mathrm{Re}(u) < 1$ the only poles of $\zeta_G(u)$ are on the line $\mathrm{Re}(u) = \frac{1}{2}$.

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Theorem

A graph G satisfies the RH for graphs $\iff$ it is Ramanujan.

The sketchiest rough idea of a proof you will ever see

Proof: We are looking for zeroes of $\zeta_G(u)^{-1}$ which is given by the Ihara determinant formula as

$$(1-u^2)^{|E|-|V|}\det(I(1-(k-1)u^2)-Au)$$

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Then check some cases using the fact that $\lambda(G)\leq 2\sqrt{k-1}$ to see when this is zero.

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- Curves over finite fields.
- Fractal strings.
- Schemes (over finite type over $\mathbb{Z}$).

The Dedekind zeta function

Richard Dedekind (1831–1916) wanted to use analysis to study more general fields than just $\mathbb{Q}$, specifically he was interested in number fields.

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So we should deal with ideals instead of elements!

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So Dedekind defined for a number field K

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A proof of these hypotheses for all number fields (known as the **extended** Riemann hypothesis) would give approximations for the number of prime ideals of bounded norm, exactly the same as for the original hypothesis.

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- Zeta functions can be used to pack up lots of useful information into one big package (a complex function).

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- Zeta functions can be used to pack up lots of useful information into one big package (a complex function).
- The properties of this package can tell us about the objects we started with.
- We can also see links between different objects via their zeta functions.
- Due to the abundant computational evidence (over ten trillion non-trivial zeroes found so far, all on the critical line) a huge number of papers have been written that assume the Riemann hypothesis is true. So a proof of the (generalised) hypothesis would imply hundreds of other results true also.

Sources used

I used some of the following when preparing this talk, and so they are possibly good places to look to learn more about the topic:

1. "What is... an expander?" – Peter Sarnak
2. "Problems of the Millennium: The Riemann Hypothesis" – Peter Sarnak
3. "Problems of the Millennium: The Riemann Hypothesis" (Official Millennium prize problem description) – Enrico Bombieri
4. "Zeta Functions of Graphs: A Stroll through the Garden" – Audrey Terras
5. Wikipedia – Enough said
6. http://graphtheoryinlatex.blogspot.com/ – Pretty pictures
7. "Fractal Geometry, Complex Dimensions and Zeta Functions" – Lapidus and van Frankenhuijsen (not used in talk but still cool)