

# Raynaud's proof III

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BUNTES

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**Goal:** To explain the proof of case 3) of Raynaud's proof of Abhyankar.



## Recall

A group is **rev- $p$**  if it appears as an unramified cover of  $\mathbf{A}_{\mathbf{F}_p}^1$ .

We are proving Abhyankar's conjecture via the following:

### Theorem

Let  $G$  be a quasi- $p$ -group and  $S$  a  $p$ -Sylow subgroup of  $G$  then we let  $G(S)$  be the subgroup of  $G$  generated by all strict quasi- $p$ -subgroups of  $G$  which have a  $p$ -Sylow subgroup contained in  $S$ .

1. If  $H$  is a normal  $p$ -subgroup of  $G$  and  $G/H$  is rev- $p$  then  $G$  is rev- $p$ .

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## Example of case 3)

Pop-quiz: what is an example of case 3)?



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If we consider simple groups that are not  $p$ -groups we rule out case 1).

To find a quasi- $p$ -group whose strict quasi- $p$ -subgroups do not generate it, we could take one with no strict quasi- $p$ -subgroups.

What about  $A_4$  of order 12, it's simple, quasi  $p$  for  $p = 2, 3$ , not a  $p$ -group. The only strict quasi-3 subgroup is the 3-Sylow (the 3-cycle), so it is not generated by strict quasi-3-subgroups.

1. Construct a curve  $Y''$  over a DVR  $R$  (with residue field  $k$ ) with an action of  $G$  on  $Y''$ , such that  $Y''/G$  has special fibre a tree of  $\mathbf{P}^1$ 's.

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3. (The combinatorial step, next week) Show that a graph with a group action satisfying additional properties must contain a vertex on which the group acts in a specific way.
4. This vertex corresponds to a component  $C$  of  $Y''_k$  covering  $P$  in  $X''_k$  in such a way that the restriction of  $C$  to  $P - \{\infty\}$  is etale and Galois of group  $G$ .

# Semi-stable curves

Let  $R$  be a discrete valuation ring with fraction field  $K$ .

We will outline a general construction that takes  $G$ -covers between smooth proper  $R$ -curves  $Y \rightarrow X$  and creates combinatorially simpler models that can be analysed explicitly.

# Semi-stable curves

Let  $R$  be a discrete valuation ring with fraction field  $K$ .

We will outline a general construction that takes  $G$ -covers between smooth proper  $R$ -curves  $Y \rightarrow X$  and creates combinatorially simpler models that can be analysed explicitly.

## Definition

Let  $Y$  be a smooth projective curve over  $K$  with  $H^0(Y, \mathcal{O}_Y) = K$ . Then  $Y$  is said to be semi-stable if there exists a proper model of  $Y$  which is at-worst-nodal of relative dimension 1 over  $R$  (i.e. all closed points of  $X_k$  are either in the smooth locus of the structure morphism  $X \rightarrow \operatorname{Spec}(k)$  or are ordinary double points). We call this a semi-stable model.

Given a semistable model, the special fibre consists of a set of irreducible components linked by double points.

We can take the dual graph of this set-up, i.e. vertices for irreducible components, with edges connecting the vertices corresponding to a pair of components that meet (this could include self-loops).

If  $G$  acts on a semistable model then we get an action on the corresponding graph.



## Theorem (Semi-stable reduction theorem)

Let  $X$  be a proper  $R$ -curve with geometrically connected generic fibre. Then there exists a finite extension  $R'$  of  $R$ , such that there exists a birational and proper  $R'$ -morphism  $\pi: \tilde{X} \rightarrow X \times \operatorname{Spec} R'$  where  $\tilde{X}$  is semi-stable.

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## Example

Consider the nodal cubic

$$y^2 = x^3 + p/\mathbf{Z}_p$$

this does not have semi-stable reduction as on the special fibre the singularity is not an ordinary double point.

However upon base-extension to  $\mathbf{Z}_p[\sqrt[6]{p}]$  we can change the model to get

$$y^2 = x^3 + 1/\mathbf{Z}_p$$

which in fact has good reduction (for  $p \neq 2, 3$ ).

Let  $X$  be an  $R$ -curve and  $x$  a closed point of  $X$  such that  $X_k$  is reduced at  $x$ . Then let

$$\delta_x = \dim_k \tilde{\mathcal{O}}_x / \mathcal{O}_x$$

(the normalization of localization of the local ring at  $x$  inside its fraction ring).

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Let  $m_x$  be the number of maximal ideals of  $\mathcal{O}_x$ .

Then we set

$$\mu_x = 2\delta_x - m_x + 1 \in \mathbf{Z}_{\geq 0}$$

which has the property that:

$$\mu_x = 0 \iff x \text{ smooth and}$$

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## Proposition

Let  $f: Y \rightarrow X$  be a covering of  $R$ -curves with  $X_k, Y_k$  both reduced. Let  $y$  be a closed point of  $Y$  with  $x = f(y)$ . Then

$$\mu_y \geq \mu_x.$$

## Proposition (Kato)

Let  $f: Y \rightarrow X$  be a covering of  $R$ -curves  $X_k, Y_k$  both reduced. Let  $y$  be a closed point of  $Y$  with  $x = f(y)$ . And  $(x_j)_{j \in J}$  the points of the normalization  $\tilde{X}_k$  over  $x$ . Likewise let  $(y_{i,j})_{j \in J, i \in I_j}$  be the points of the normalization of  $Y_k$ .

Assume  $f_k: Y_k \rightarrow X_k$  is generically etale. Then

$$\mu_y - 1 = n(\mu_x - 1) + d_K - d_k^w$$

where  $n = \deg(\mathrm{Spec} \hat{\mathcal{O}}_{Y,y} \rightarrow \mathrm{Spec} \hat{\mathcal{O}}_{X,x})$ , the value  $d_K$  the degree of the ramification divisor of

$$\mathrm{Spec} \left( \hat{\mathcal{O}}_{Y,y} \otimes_R K \right) \rightarrow \mathrm{Spec} \left( \hat{\mathcal{O}}_{X,x} \otimes_R K \right)$$

and  $d_k^w := \sum d_{i,j}^w$ ,  $d_{i,j}^w := v_{x_j}(\delta_{y_{i,j}, x_j}) - e_{i,j} + 1$



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where  $\delta_{y_{i,j}, x_j}$  is the discriminant ideal of the extension  $\hat{\mathcal{O}}_{\tilde{X}_k, x_j} \rightarrow \hat{\mathcal{O}}_{\tilde{Y}_k, y_{i,j}}$  of complete DVRs, and  $e_{i,j}$  its ramification index. The integer  $d_K^w$  equals 0 if and only if the morphism  $\tilde{Y}_k \rightarrow \tilde{X}_k$  between the normalisations of  $X_k$  and  $Y_k$  is tamely ramified.

### Proposition

Let  $Y$  be an  $R$ -curve. Let  $G$  be a finite group acting by automorphisms on  $Y$ . Then the quotient  $X := Y/G$  of  $Y$  by  $G$  exists.

### Proof.

A quotient exists if and only if every orbit of  $G$  is contained in an open affine of  $Y$ , but as  $G$  is finite and every finite set of points in a quasi-projective space (such as  $Y$ ) is contained in an affine. □

### Proposition

Let  $f: Y \rightarrow X$  be a covering. If  $Y$  is semi-stable then  $X$  is semi-stable.

### Proof.

Let  $y$  be a closed point of  $Y$ , and let  $x$  be its image in  $X$ . We have  $\mu_y \geq \mu_x$  and  $\mu_y \in \{0, 1\}$ , hence  $\mu_x \in \{0, 1\}$ . Thus if  $y$  is a smooth point then  $x$  is smooth, and if  $y$  is an ordinary double point,  $x$  is either a double point or a smooth point depending on the number of branches passing through  $x$ .  $\square$

## Proposition

Let  $X := \operatorname{Spec} \mathcal{O}_{X',x}$  be the localisation of an  $R$ -curve  $X'$  at a smooth closed point  $x$ , and let  $s : S \rightarrow X$  be an  $S$  point of  $X$ . Let  $f : Y \rightarrow X$  be a Galois covering, and let  $e$  be its ramification index above the point  $\tilde{x} := s(\operatorname{Spec} K)$ . Assume that  $f$  is étale outside  $s(S)$ , and that  $e$  is prime to  $\operatorname{char}(K)$ . Then  $Y$  is smooth, and the morphism  $f_k : Y_k \rightarrow X_k$  is tamely ramified at  $x$  with ramification index  $e$ . In particular the inertia subgroup at a point  $y$  of  $Y$  above  $x$  is cyclic of order  $e$ .

## Proof

Let  $y$  be a closed point of  $Y$  above the point  $x$ . After étale localisation at  $y$  and  $x$  we can assume that  $y$  is the unique closed point of  $Y$  which is above  $x$ . Use local Riemann-Hurwitz.

## Proof

$$\mu_y - 1 = n(\mu_x - 1) + d_K - d_k^w$$

We have  $\mu_x = 0$ . We compute  $d_K$ . Let  $\{\tilde{y}_i\}_{i=1}^r$  be the points of  $Y_K$  above  $\tilde{x}$  and let  $f$  be the residual degree at these points, then  $d_K = r(e f - 1)$ . Hence

$\mu_y = 1 - n + n - r - d_w = 1 - r - d_w$ . The only possibility is that  $r = 1$ ,  $d_w = 0$  and  $\mu_y = 0$  as claimed. The inertia subgroup at  $y$  is then the same as the inertia of the extension  $\mathcal{O}_{X_k, x} \rightarrow \mathcal{O}_{Y_k, y}$  which is cyclic of order  $e$ .

## Proposition

Assume  $\text{char}(K) = 0$  now. Let  $\tilde{y}$  be a rational point of  $Y_K$  which specialises to a point  $y$  of  $Y_k$ , and let  $\eta$  be the generic point of the irreducible component of  $Y_k$  containing  $y$ . Let  $\tilde{x}$  be the image of  $\tilde{y}$  in  $X_K$  and assume that  $f_K : Y_K \rightarrow X_K$  is étale outside  $\tilde{x}$ . Let  $e'p^a$  be the ramification index at  $\tilde{y}$ , with  $e'$  prime to  $p$ . (Note that  $I(\tilde{y})$  and  $I(\eta)$  are subgroups of  $I(y)$ .) Then:

1.  $I(\eta)$  is a  $p$ -group.
2.  $I(\eta)$  is invariant in  $I(y)$ , and the quotient  $I(y)/I(\eta)$  is cyclic of order  $e'$

In particular if  $e' = 1$ , then  $I(\tilde{y}) \subset I(\eta) = I(y)$ , and moreover if  $a \geq 1$  then  $f : Y \rightarrow X$  is ramified along the irreducible component containing  $y$ .

## Proof

Omitted

## Proposition

Let  $f : Y \rightarrow X$  be a Galois covering of group  $G$  where  $Y$  and  $X$  are semi-stable. Assume that  $f_K : Y_K \rightarrow X_K$  is étale. Let  $y$  be an ordinary double point of  $Y$ , whose image in  $X$  is a double point  $x$ . Let  $C_1$  and  $C_2$  be the two irreducible components of  $Y_k$  passing through  $y$  (which may be equal), and let  $\eta_1$  and  $\eta_2$  be the corresponding generic points of  $Y_k$ . Then:

1.  $I(\eta_1)$  and  $I(\eta_2)$  are normal  $p$ -subgroups of  $I(y)$ , and they generate the (normal)  $p$ -syllow subgroup of  $I(y)$
2. the quotient  $I(y) / \langle I(\eta_1), I(\eta_2) \rangle$  is a cyclic group of order prime to  $p$

## Proof

1. Etale localize to assume  $y$  is the unique point of  $Y$  above  $x$  so that  $G = I(y)$  and  $C_1 \neq C_2$ . Then  $D(\eta_1) = D(\eta_2) = G$ , and  $I(\eta_1), I(\eta_2)$  are  $p$ -groups.
2. We can pass to the quotient curve  $Y' = Y / \langle I(\eta_1), I(\eta_2) \rangle$  to trivialise this subgroup, then use local Riemann-Hurwitz. If  $y'$  is the image of  $y$  we have

$$\mu_{y'} - 1 = n(\mu_x - 1) + d_K - d_K^w$$

with  $\mu_x = 1$  and  $d_K = 0$  so  $\mu_{y'} = 0$ . So the only possibility is  $\mu_{y'} = 0$  and  $d_K^w = 0$  so the cover  $Y'_k \rightarrow X'_k$  is tamely ramified. So the original quotient  $I(y) / \langle I(\eta_1), I(\eta_2) \rangle$  is cyclic.



## A nicer cover

Let  $X$  be a smooth proper  $R$ -curve with geom. conn.  $X_K$  and  $\{a_i : \operatorname{Spec} R \rightarrow X\}_{i=1, \dots, r}$  all  $R$ -points of  $X$  s.t. they have disjoint support (distinct on the special fibre).

Let  $f: Y \rightarrow X$  be a galois cover with group  $G$  s.t.  $f_K: Y_K \rightarrow X_K$  is etale away from the points  $x_i$ ,  $Y$  not necessarily smooth.

After extending  $R$  we can find  $Y' \rightarrow Y$  proper birational with  $Y'$  semi-stable and in such a way that the  $G$ -action extends (this follows from choosing a minimal one).

We can quotient to get  $X' = Y'/G$ .

## A nicer cover

The points  $x_i$  induce points  $x'_i$  on  $X'$ , which remain disjoint but may have support on a double point, to fix this blow up  $X'$  and  $Y'$  to get a semi-stable model  $Y''$  with a  $G$ -action and hence  $X'' = Y''/G$  in such a way that

- The irreducible components of the special fibre of  $Y''$  are smooth.
- The integral points  $\{x_i\}_{i=1}^r$  extend to points  $\{x''_i\}_{i=1}^r$  of  $X''(R)$  which have disjoint support and are contained in the smooth locus of  $X''$

as  $X$  was smooth originally and  $X''$  is a semi-stable model of  $X_K$  the special fibre of  $X''$  is a tree with a bunch of  $\mathbf{P}^1$ 's added to the original special fibre at double points.

If  $G$  is a quasi- $p$ -group then it is generated by a family  $(\alpha_1, \dots, \alpha_m)$  of elements of  $p$ -power order, by adding new generators if needed we can assume  $\alpha_1 \cdots \alpha_m = 1$ .

Consider a complete DVR  $R$  with algebraically closed residue field  $k$  of characteristic  $p$  and fraction field  $K$  of characteristic 0,  $\pi$  a uniformizer.

Choose  $m$  distinct  $R$ -points  $x_1, \dots, x_m$  of the projective line  $\mathbf{P}_R^1$ , which have disjoint support (i.e. do not reduce to the same point on the special fibre). Write

$$U = \mathbf{P}_R^1 \setminus \{x_1, \dots, x_m\}.$$

We consider the fundamental group of  $U_{\overline{K}}$ , the geometric generic fibre, as  $K$  was assumed to be of characteristic 0 the Lefschetz principle tells us that the answer agrees with the usual topological one (after profinite completion).

The fundamental group is a free profinite group where we have  $m$  topological generators  $(\sigma_1, \dots, \sigma_m)$  satisfying  $\sigma_1 \cdots \sigma_m = 1$ , so that  $G$  is a quotient of  $\pi_1$ . And we have a connected Galois cover  $Y_{\overline{K}} \rightarrow \mathbf{P}_{\overline{K}}^1$  with group  $G$  etale away from  $\{x_i\}_i$ .

The inertia subgroups above these points are cyclic  $p$ -groups. We can choose  $\sigma_i$  to generate inertia above  $x_i$ .

Enlarging  $K$  everything is defined over  $K$  and we can take integral versions of everything.

We can now apply the theory we developed to obtain semi-stable models  $Y''$  and  $X''$  for this setup where

- The irreducible components of the special fibre of  $Y''$  are smooth.
- The rational points  $\{x_i\}_{i=1}^m$  extend to integral points  $\{x''_i\}_{i=1}^m$  of  $X''(R)$  which have disjoint support and are contained in the smooth locus of  $X''$

and where the special fibre of  $X''$  is a tree of projective lines.

## Proving 3)

Assume we are in the case of 3);  $G(S) \neq G$  and  $G$  does not contain a non-trivial normal  $p$ -subgroup, we will apply the theory of semi-stable curves to prove that  $G$  is rev- $p$ .

Combinatorial step shows that: the graph of the special fibre of  $Y''$  with action of  $G$  is a **graph with inertia** (satisfies 8 conditions see next week).

It then says that there is a vertex  $s$  of the graph of  $Y''_k$  for which the decomposition group for that component is all of  $G$ , and whose image in the quotient tree is a leaf. As we have no non-trivial normal  $p$ -subgroup by assumption we have  $I_s = 1$ .

Restricting to this component covering a single  $\mathbf{P}^1$  in the tree below as the vertex is a leaf it meets the rest of the special fibre only once, giving us only one double point in that component, which when removed gives us an étale  $G$ -galois cover.