

# Riemann Hypotheses

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WMS Talks

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- 1 Introduction
- 2 The original hypothesis
- 3 Zeta functions for graphs
- 4 More assorted zetas
- 5 Back to number theory
- 6 Conclusion

# The Riemann zeta function: Euler's work

A brief history of  $\zeta$ :

- In 1735 Euler solves the **Basel problem** by finding that

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- He also discovered more general formulae for  $\sum_{n=1}^{\infty} n^{-2k}$  in terms of the Bernoulli numbers  $B_{2k}$  for all natural  $k$ .
- In fact, a nice form for

$$\sum_{n=1}^{\infty} n^{-2k-1},$$

is still unknown today.

# The Riemann zeta function: Along comes Riemann

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- In 1859 Bernhard Riemann, a well known analyst, publishes a paper on his work counting the primes using analysis.

VII.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859.)

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diene mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1 - \frac{1}{p^s}} = \sum \frac{1}{n^s},$$

wenn für  $p$  alle Primzahlen, für  $n$  alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen  $s$ , welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch  $\zeta(s)$ . Beide convergiren nur, so lange der reelle Theil von  $s$  grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^{\infty} e^{-sx} x^{s-1} dx = \frac{\Gamma(s-1)}{s^s}$$

erhält man zunächst

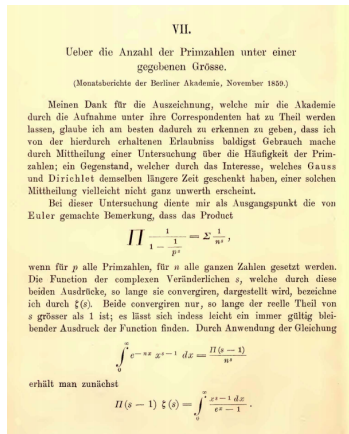
$$\Gamma(s-1) \zeta(s) = \int_0^{\infty} \frac{x^{s-1} dx}{e^x - 1}.$$

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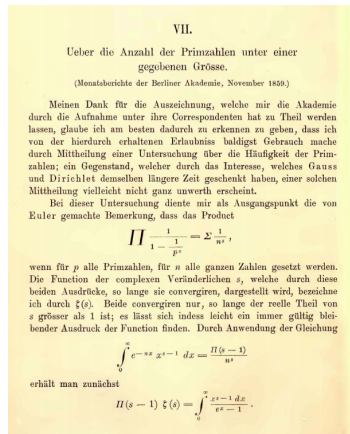
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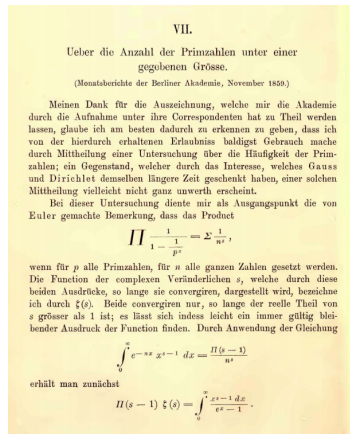
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- Along the way he (essentially) makes four hypotheses.



# The Riemann zeta function: What Riemann did

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Riemann also discovers a **functional equation** for the zeta function by showing that

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

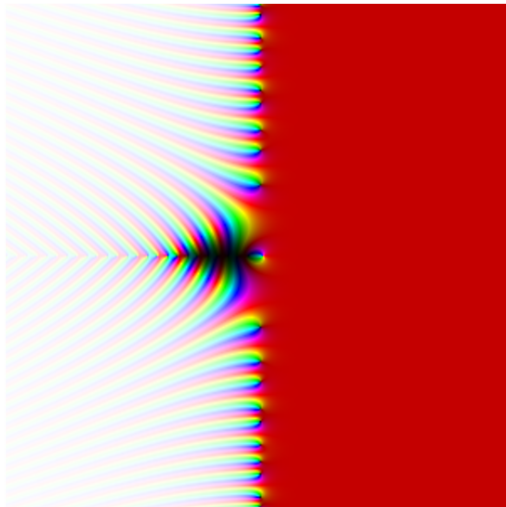
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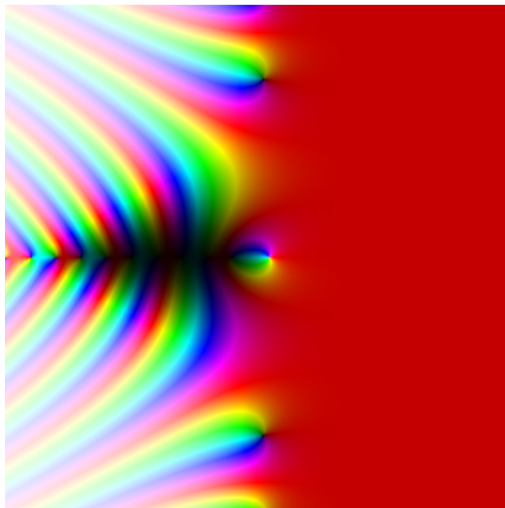
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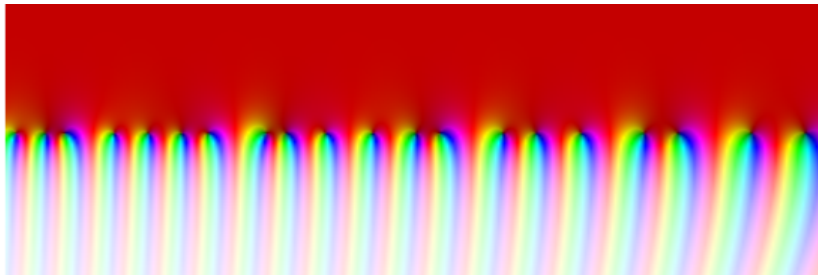
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# The Riemann zeta function: The hypothesis

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All non-trivial zeroes of the Riemann zeta function lie on the **critical line**

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- It is a natural function to consider, and knowing the zeroes of a complex function are key to understanding it.
- The location of the zeroes of  $\zeta(s)$  relates in a strong way to the distribution of the primes.

# The Riemann zeta function: Why number theorists care

## The prime counting function $\pi(x)$

We define

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The Riemann hypothesis is actually equivalent to the statement that there exists  $c_2 > c_1 > 0$  such that

$$\text{Li}(x) + c_1 \sqrt{x} \log(x) \leq \pi(x) \leq \text{Li}(x) + c_2 \sqrt{x} \log(x)$$

eventually. These are the best possible bounds!

# Expander graphs

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If we split our network into two nonempty parts (a partition) there should be lots of edges between the two halves.

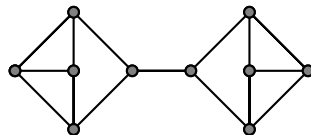
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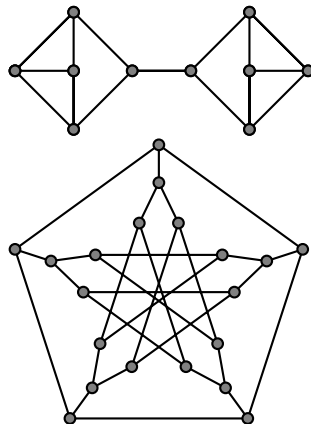
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# The spectrum

Given such a graph  $G$  we can fix a numbering of the vertices and form:

## The adjacency matrix

Is an  $n \times n$  matrix  $A_G = (a_{ij})$  of 1's and 0's given by

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What are the eigenvalues of this matrix? So we define  $\lambda(G)$  to be the next largest eigenvalue after  $d$ , i.e.

$$\lambda(G) = \max_{|\lambda_i| < d} |\lambda_i|.$$

# Ramanujan graphs

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These are highly optimal from our point of view! Ramanujan graphs are only known to exist for  $k = p^n + 1$ .

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The Ihara zeta function of  $G$

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Is this true?

## Theorem

A graph  $G$  satisfies the RH for graphs  $\iff$  it is Ramanujan.

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- Fractal strings.
- Schemes (over finite type over  $\mathbb{Z}$ ).

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In a general number field the idea of a prime element doesn't work out so well, however if we consider nice subgroups of the field (ideals of the ring of integers) then everything works out well, so we should deal with ideals instead of elements! For example we have unique factorisation of ideals into prime ideals.

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A proof of these hypotheses for all number fields (known as the **extended** Riemann hypothesis) would give approximations for the number of prime ideals of bounded norm, exactly the same as for the original hypothesis.

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- The properties of this package can tell us about the objects we started with.
- We can also see links between different objects via their zeta functions.
- Due to the abundant computational evidence (over ten trillion non-trivial zeroes found so far, all on the critical line) a huge number of papers have been written that assume the Riemann hypothesis is true. So a proof of the (generalised) hypothesis would imply hundreds of other results true also.

## Sources used

I used some of the following when preparing this talk, and so they are possibly good places to look to learn more about the topic:

- ① “What is... an expander?” – Peter Sarnak
- ② “Problems of the Millennium: The Riemann Hypothesis” – Peter Sarnak
- ③ “Problems of the Millennium: The Riemann Hypothesis” (Official Millennium prize problem description) – Enrico Bombieri
- ④ “Zeta Functions of Graphs: A Stroll through the Garden” – Audrey Terras
- ⑤ Wikipedia – Enough said
- ⑥ <http://graphtheoryinlatex.blogspot.com/> – Pretty pictures
- ⑦ “Fractal Geometry, Complex Dimensions and Zeta Functions” – Lapidus and van Frankenhuysen (not used in talk but still cool)