# Explicit computation with Coleman integrals

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1. Thank the audience for being awake.

# Explicit computation with Coleman integrals

## Why do we integrate things? Logarithms

there are many answers to this question

### Why do we integrate things? Logarithms

Take $\frac{dx}{x}$, as a differential on the group $\mathbf{R}^\times$, this is translation invariant, i.e. $(a \cdot -)^*(dx/x) = d(ax)/ax = dx/x$, hence

$$\int_1^t \frac{dx}{x} = \log|t| \colon \mathbf{R}^\times \to \mathbf{R}$$

has the property that

$$\int_1^{ab} \frac{dx}{x} = \int_a^{ab} \frac{dx}{x} + \int_1^a \frac{dx}{x} = \int_1^b \frac{dx}{x} + \int_1^a \frac{dx}{x}$$

Integration can define logarithm maps between groups and their tangent spaces.

How do we calculate $\log|t|$? Power series on $\mathbf{R}_{>0}$ and use the relation $\log|t| = \frac{1}{2}\log t^2$

### Coleman integration

Is there $p$-adic analogue of this? Given a $p$-adic space, (as $p$-adic solutions to some equations) we can locally write down convergent power series for a 1-form and integrate.

For instance near a point $\alpha$:

$$\omega = \frac{\mathrm{d}(\alpha + x)}{\alpha + x} = \frac{\mathrm{d}x}{\alpha + x} = \frac{1}{\alpha}\sum\left(\frac{-x}{\alpha}\right)^n \mathrm{d}x$$

Bad topology!

so that

$$\int_{\alpha+x}\omega = -\sum\frac{1}{n+1}\left(\frac{-x}{\alpha}\right)^{n+1} + C$$

But we cannot find $C$! There is a different choice in each disk.

As number theorists it is natural to ask,

### Applications: Chabauty-Kim

Minhyong Kim has vastly generalised the above to cases where

$$\mathrm{rank}(\mathrm{Jac}(X))(\mathbf{Q}) \geq \mathrm{genus}(X)$$

This can be made effective, and computable

**Theorem (Balakrishnan-Dogra-Muller-Tuitman-Vonk)**
*The (cursed) modular curve $X_{split}(13)$ (of genus 3 and jacobian rank 3), has 7 rational points: one cusp and 6 points that correspond to CM elliptic curves whose mod-13 Galois representations land in normalizers of split Cartan subgroups.*

Their method can also be applied to other interesting curves:

**Theorem (WIP B.-Bianchi-Triantafillou-Vonk)**
*The modular curve $X_0(67)^+$ (of genus 2 and jacobian rank 2), has rational points contained in an explicitly computable finite set of 7-adic points.*

many authors have tried to make this explicit and useful in examples culminating in