Raynaud's proof – II: Extension

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Last time: Angus gave an overview and some cases of Raynaud's proof of Abhyankar's conjecture.

This time: Give more detail on the extension steps of Raynaud's proof.

Extension

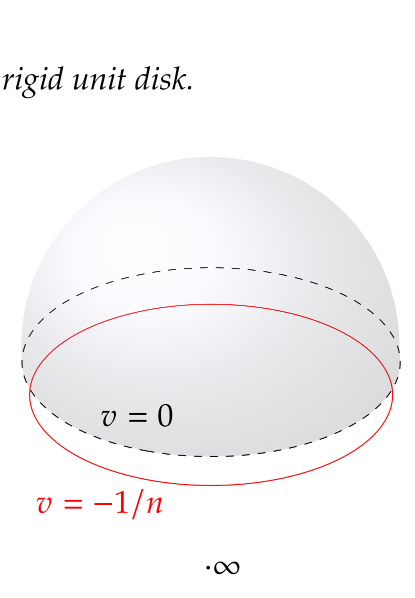
Let k be an algebraically closed field of characteristic p. R be a complete DVR with residue field k and fraction field K, π a normalized uniformizer.

Theorem 1. *Consider a connected etale cover $X \to \mathbf{A}^1_k$, which is Galois with Galois group G, and let $Q \subseteq G$ be the inertia subgroup at ∞.*

Write $\mathscr{X} \to \mathscr{D}$ for the corresponding connected cover of the rigid unit disk.

Then for n sufficiently large this cover extends to a connected Galois cover $\mathscr{X}' \to \mathscr{D}'$ = the disk defined by $v(T) \geq -1/n$.

Additionally for any n sufficiently large, the connected components of $\mathscr{X}'$ above the boundary annulus $\mathscr{A}'$ defined by $v(T) = -1/n$ are in a natural bijection with the points at infinity of X. In particular if $\mathscr{U}' \to \mathscr{A}'$ is such a component then it is a connected Galois cover with Galois group Q.



Theorem 2 (3.4.1)**.** *Let $\mathscr{D}$ be the closed unit rigid disk ($|T| \leq 1$). And $\mathscr{U} \to \mathscr{D}$ a finite etale cover.*

Then there exists $\epsilon > 0$ and a finite etale cover of $\mathscr{D}'$ defined by $|T| < 1 + \epsilon$, which extends $\mathscr{U} \to \mathscr{D}$.

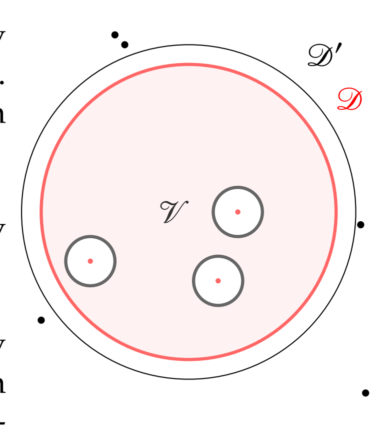
This cover is unique, any two such are isomorphic via an isomorphism which is the identity on $\mathscr{U}$.

Proof. If $A = K\{T\}$ is the Tate algebra of $\mathscr{D}$ and B is the finite etale A algebra corresponding to $\mathscr{U}$.

As the residue fields at points of $\operatorname{Spec} A$ are infinite B is locally over $\operatorname{Spec} A$ a monogenic A-algebra. I.e. there exists for each $x \in \mathscr{D}$ an element $w \in B$ such that w generates $B \otimes_A K(x)$.

Then the subalgebra $B' = A[w] \subseteq B$ equals B except in a finite set of points $S \subseteq \mathscr{D}$. We write $B' = A[W]/f$ with $f = \sum a_i W^i$ for $a_i \in A$.

For sufficiently small $\delta > 0$ the disks $\Delta_{s,\delta}$ defined by $|T - s| \leq \delta$ for $s \in S$ are disjoint and contained in $\mathscr{D}$. Taking the affinoid $\mathscr{V}$ to be the complement of the union of these disks inside of $\mathscr{D}$.



On the space $\mathscr{V}$ the series a_i are approximated arbitrarily well by polynomials $b_i \in K[T]$.

We can therefore let $g = \sum b_i W^i \in K[T][W]$ in such a way that $C = K[T][W]/(g)$ is a finite $K[T]$-algebra. Which above $\mathscr{V}$ is etale and isomorphic to the algebra corresponding to $\mathscr{U}$.

C will be ramified over the affine line over K at finitely many points, so there exists some $\epsilon > 0$ such that C is etale over the annulus $1 < |T| \leq 1 + \epsilon$.

Therefore C defines for us such an etale cover.

To prove uniqueness we must first introduce some more results on models. □

Let n be an integer > 0. Consider the closed rigid disk $v(T) \geq -1/n$, as the union of the unit disk $v(T) \leq 0$ and the closed annulus $0 \geq v(T) \geq -1/n$. The formal model $D^{(n)}$ is no longer affine.

$D^{(n)}$ can be obtained by gluing the standard disc $\mathscr{D}$, with coordinate x, having coordinate ring $R\{x\}$, with the annulus $C^{(n)}$ with coordinate ring $R\{x, y\}/(x^n y - \pi)$. Gluing is done on the annulus of zero thickness, which has coordinate ring $R\{x, x^{-1}\}$. The special fiber $D^{(n)}_k$ has two irreducible components, a projective line with multiplicity 1 and an affine line A_k with multiplicity n.

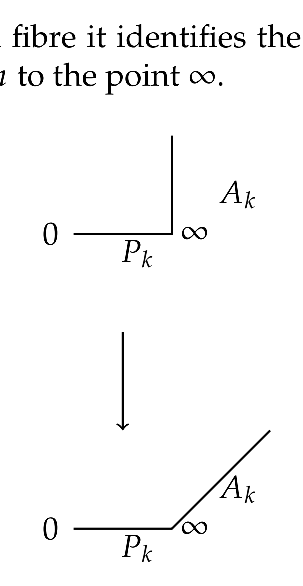
The two components intersect transversely at a point ∞.

On P_k is a point called 0 that all points with $v(T) > 0$ specialise to.

If $n' > n$ is an integer, there is a natural inclusion of disks as $v(T) \geq -1/n'$ implies $v(T) \geq -1/n$. This induces a morphism of the formal models

$$D^{(n')} \to D^{(n)}$$

when we reduce to the special fibre it identifies the P_k components and sends the component A_k of multiplicity n to the point ∞.



Let $\mathscr{Y}^{(n)}$ be a rigid rigid finite covering of $\mathscr{D}^{(n)}$. It extended to a unique normal formal scheme $Y^{(n)}$, finite above $D^{(n)}$.

Let us consider the morphism $Y^{(n)}_k \to D^{(n)}_k$ of the special fibers, and let Q_k be the reduced inverse image of P_k. Then $Q_k \to P_k$ is finite.

Theorem 3 (3.2.6)**.** *Let X be a normal formal R-curve with generic fiber $\mathscr{X}$ and let x be a closed point of the special fiber of X.*

Then the set of points z in $\mathscr{X}$ which reduce to x is an open rigid connected subspace of $\mathscr{X}$, it is a union of a increasing set of connected quasi-compact opens.

Theorem 4 (3.4.2)**.** *For $n \gg 0$ the fibre Q_k is normal at all points above ∞.*

Proof. Consider the normalization of Q_k call it Q'_k, the morphism $Q'_k \to Q_k$ is a blow up at points above infinity, and so we can obtain it inside of a blow up $Y'^{(n)} \to Y^{(n)}$ above infinity, with Q'_k being the strict transform of Q_k.

This blow up introduces on the special fiber of Y, irreducible components above the point ∞. Let E be the open subscheme of the formal scheme $Y'^{(n)}$ which, on the special fiber, is the complement of the union of the irreducible components which are finite on $D^{(n)}_k$ and let $\mathscr{E}$ be its generic fiber. Then E lies above the open annulus $0 > v(T) > -1/n$ and by the maximum modulus principle, there exists an integer $n' > n$ such that on $\mathscr{E}$, we have $v(T) < -1/n'$.

So let's make the base change $D^{(n')} \to D^{(n)}$ and remove the π-torsion from the fiber product $Y'^{(n)} \times_{D^{(n)}} D^{(n')}$. We then obtain a formal R-scheme Z which is now finite over $D^{(n')}$ as it is disjoint from $E \times_{D^{(n)}} D^{(n')}$. Consequently, if we normalize Z, we find the model $Y^{(n')}$. Then, like $Y^{(n)}$ dominates $Y^{(n')}$, the closed sub-scheme of its reduced special fiber located above the component P_k is still normal at infinity. □

We now assume n is chosen such that this holds.

Theorem 5 (3.4.6)**.** *For each integer $n > 0$ consider the formal R-model $D^{(n)}$ of the rigid disk $\mathscr{D}^{(n)}$ defined by $v(T) \geq -1/n$.*

Write P_k (resp. A_k) for the irreducible component of the projective (resp. affine) line of the special fibre of $D^{(n)}$ and write ∞ for the point of intersection of these fibres.

Let $\mathscr{U}^{(n)} \to \mathscr{D}^{(n)}$ be a rigid finite etale cover and $U^{(n)} \to D^{(n)}$ the finite morphism of normal formal schemes which extends it.

We write Q_k for the reduced subscheme of the special fibre of $U^{(n)}$ which lies above P_k and y_i for $i \in I$ for the points above ∞. Then for n sufficiently large Q_k is normal above the point ∞.

Letting $\mathscr{U}^{(n)}_i$ be the set of points z of $\mathscr{U}^{(n)}$ that reduce to y_i the map $y_i \mapsto \mathscr{U}^{(n)}_i$ induces a natural bijection between the points of Q_k lying above infinity and the connected components of the cover $\mathscr{U}^{(n)} \to \mathscr{D}^{(n)}$ above the open annulus $0 > v(T) > -1/n$.

These components remain connected when restricted to $0 > v(T) > -1/n'$ for any $n' > n$.

Proof. The normality of Q_k is proved in 3.4.2.

The points z of $\mathscr{U}^{(n)}$ above the annulus specialise in $U^{(n)}_k$ to the points y_i for $i \in I$.

On the other hand $U^{(n)}$ is normal so by 3.2.6 the rigid open subspaces given by taking those that reduce to a single point are connected, hence they are the connected components of the cover.

If $n' > n$ the canonical morphism $U^{(n')} \to U^{(n)}$ induces the identity on Q_k and is normal at infinity, hence the restriction of $\mathscr{U}^{(n)}_i$ to the smaller annulus remains connected. □

Interlude: The complex picture. Consider the extension of function fields

$$\mathbf{C}(\sqrt{x}, \sqrt{x+1})/\mathbf{C}(x)$$

this is a V_4 extension with $\sqrt{x} \mapsto \pm\sqrt{x}$, $\sqrt{x+1} \mapsto \pm\sqrt{x+1}$.

We can take a primitive element to be $y = \sqrt{x} + \sqrt{x+1}$ so that

$$y^2 = x + x + 1 + 2\sqrt{x}\sqrt{x+1}$$

and

$$(y^2 - 2x - 1)^2 - 4x(x+1) = 0$$

is an equation for the curve C giving the cover $C \xrightarrow{x} \mathbf{P}^1$.

(Ignoring infinity for this story).

This cover is ramified at the locus where $4y(y^2 - 2x - 1) = 0$, i.e. when $y = 0$ or $\pm\sqrt{2x+1}$. In the $y = 0$ case we have

$$(2x+1)^2 - 4x(x+1) = 0$$

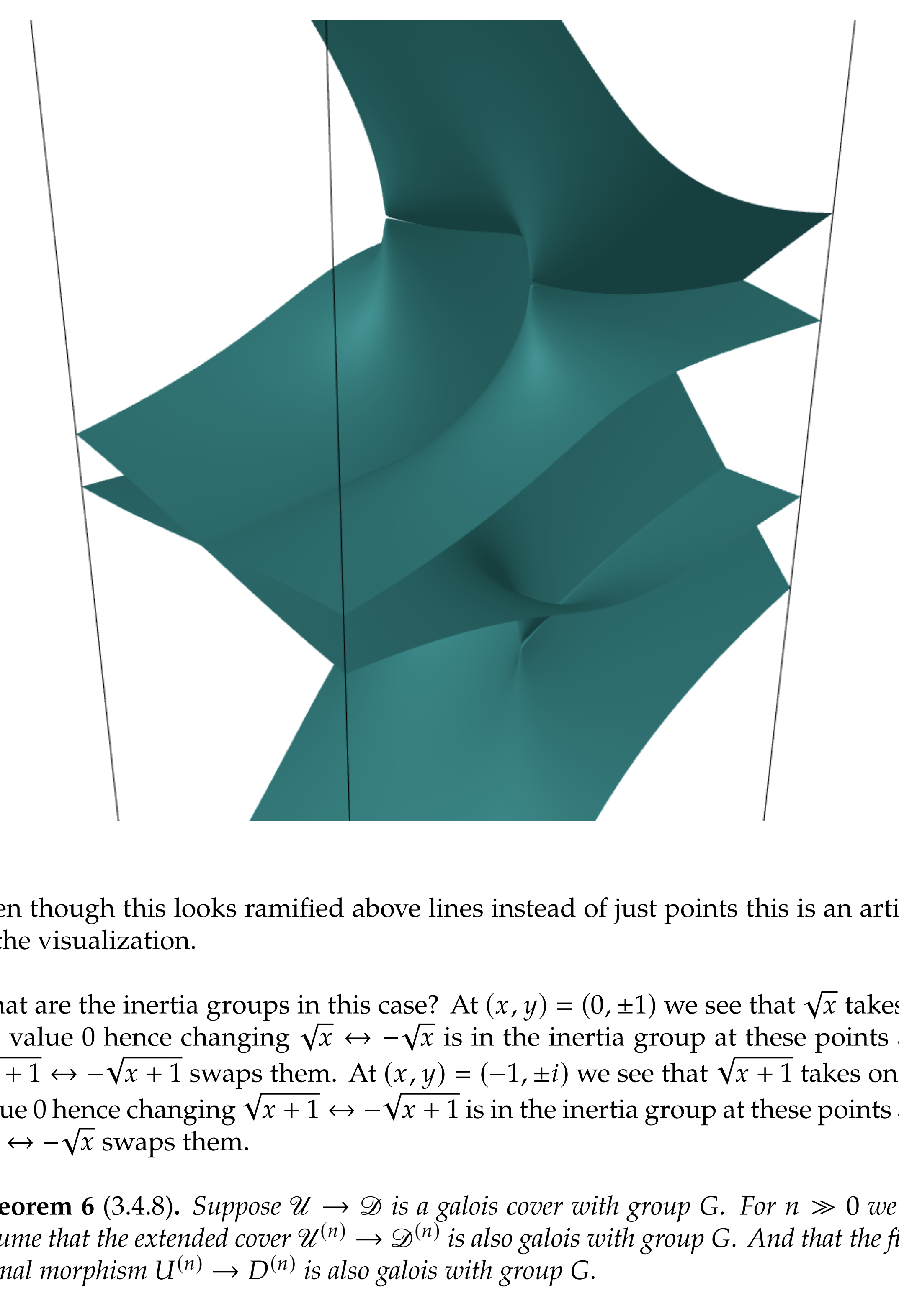
i.e.

$$1 = 0$$

so this point does not lie on the curve.

For $y = \pm\sqrt{2x+1}$ we have $4x(x+1) = 0$ so the ramification points are $x = 0, -1$. Above $x = 0$ we have two points $y = \pm 1$ each with ramification index 2. Above $x = -1$ we have two points $y = \pm i$ each with ramification index 2.

The cover can be visualised as follows



Even though this looks ramified above lines instead of just points this is an artifact of the visualization.

What are the inertia groups in this case? At $(x, y) = (0, \pm 1)$ we see that $\sqrt{x}$ takes on the value 0 hence changing $\sqrt{x} \leftrightarrow -\sqrt{x}$ is in the inertia group at these points and $\sqrt{x+1} \leftrightarrow -\sqrt{x+1}$ swaps them. At $(x, y) = (-1, \pm i)$ we see that $\sqrt{x+1}$ takes on the value 0 hence changing $\sqrt{x+1} \leftrightarrow -\sqrt{x+1}$ is in the inertia group at these points and $\sqrt{x} \leftrightarrow -\sqrt{x}$ swaps them.

Theorem 6 (3.4.8)**.** *Suppose $\mathscr{U} \to \mathscr{D}$ is a galois cover with group G. For $n \gg 0$ we can assume that the extended cover $\mathscr{U}^{(n)} \to \mathscr{D}^{(n)}$ is also galois with group G. And that the finite formal morphism $U^{(n)} \to D^{(n)}$ is also galois with group G.*

If H_i is the decomposition group at the point y_i then $\mathscr{U}^{(n)}$ is an etale galois cover of the annulus $0 > v(T) > -1/n$ with galois group H_i.

In addition H_i is a solvable group.

Proof. For the last part. Note that if y_i belongs to the irreducible component of Q_k, whose generic point is η_i we can consider the decomposition group D_i and the inertia group I_i of η_i, and $G_i = D_i/I_i$ acts faithfully on the component of Q_k, passing through y_i.

Then H_i, is extension of the inertia subgroup of G_i at y_i by I_i.

An inertia group of a local field is solvable, as it is an extension of a cyclic group of order coprime to the residue characteristic by a p group.

□