Raynaud's proof III

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BUNTES

Overview

These slides are available online (in handout form) at

`https://alexjbest.github.io/talks/raynaud2/slides_h.pdf`

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Goal: To explain the proof of case 3) of Raynaud's proof of Abhyankar.

Recall

A group is **rev-**p if it appears as an unramified cover of $\mathbf{A}^1_{\mathbf{F}_p}$.

We are proving Abhyankar's conjecture via the following:

Theorem

Let G be a quasi-p-group and S a p-Sylow subgroup of G then we let $G(S)$ be the subgroup of G generated by all strict quasi-p-subgroups of G which have a p-Sylow subgroup contained in S.

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2. If the strict quasi-p subgroups of G are all rev-p then $G(S)$ is rev-p.
3. If $G(S) \neq G$ and if G does not contain a non-trivial normal p-subgroup, then G is rev-p.

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What about $D_{2\ell}$ for prime ℓ, this is quasi-2 (and not quasi-ℓ), the only normal subgroup is C_ℓ which is not a 2-group.

As $D_{2\ell}$ has ℓ distinct subgroups that are isomorphic to C_2, each of which is a 2-Sylow, we have that fixing only one 2-Sylow S constrains $G(S)$ to be simply S again.

So $D_{2\ell}$ is an example of case 3) for a quasi-2-group.

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3. (The combinatorial step, next week) Show that a graph with a group action satisfying additional properties must contain a vertex on which the group acts in a specific way.

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2. Associate a graph with a group action to this situation, where irreducible components of $Y''_k \leftrightarrow$ vertices.
3. (The combinatorial step, next week) Show that a graph with a group action satisfying additional properties must contain a vertex on which the group acts in a specific way.
4. This vertex corresponds to a component C of Y''_k covering P in X''_k in such a way that the restriction of C to $P - \{\infty\}$ is etale and Galois of group G.

Semi-stable curves

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Definition

Let Y be a smooth projective curve over K with $H^0(Y, \mathcal{O}_Y) = K$. Then Y is said to be semi-stable if there exists a proper model of Y which is at-worst-nodal of relative dimension 1 over R (i.e. all closed points of X_k are either in the smooth locus of the structure morphism $X \to \mathrm{Spec}(k)$ or are ordinary double points). We call this a semi-stable model.

Inertia graphs

Given a semistable model, the special fibre consists of a set of irreducible components linked by double points.

We can take the dual graph of this set-up, i.e. vertices for irreducible components, with edges connecting the vertices corresponding to a pair of components that meet (this could include self-loops).

If G acts on a semistable model then we get an action on the corresponding graph.

Theorem (Semi-stable reduction theorem)

Let X be a proper R-curve with geometrically connected generic fibre. Then there exists a finite extension R' of R, such that there exists a birational and proper R'-morphism $\pi\colon \widetilde{X} \to X \times \operatorname{Spec} R'$ where $\widetilde{X}$ is semi-stable.

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Example

Consider the nodal cubic

$$y^2 = x^3 + p/\mathbf{Z}_p$$

this does not have semi-stable reduction as on the special fibre the singularity is not an ordinary double point.

However upon base-extension to $\mathbf{Z}_p[\sqrt[6]{p}]$ we can change the model to get

$$y^2 = x^3 + 1/\mathbf{Z}_p$$

which in fact has good reduction (for $p \neq 2, 3$).

Let X be an R-curve and x a closed point of X such that X_k is reduced at x. Then let

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Invariants

Then we set

$$\mu_x = 2\delta_x - m_x + 1 \in \mathbf{Z}_{\geq 0}$$

which has the property that:
$\mu_x = 0 \iff x$ smooth and
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Proposition

Let $f\colon Y \to X$ be a covering of R-curves with X_k, Y_k both reduced. Let y be a closed point of Y with $x = f(y)$. Then

$$\mu_y \geq \mu_x.$$

Local Riemann-Hurwitz

Proposition (Kato)

Let $f\colon Y \to X$ be a covering of R-curves X_k, Y_k both reduced. Let y be a closed point of Y with $x = f(y)$. And $(x_j)_{j\in J}$ the points of the normalization $\widetilde{X}_k$ over x. Likewise let $(y_{i,j})_{j\in J, i\in I_j}$ be the points of the normalization of Y_k.

Assume $f_k\colon Y_k \to X_k$ is generically etale. Then

$$\mu_y - 1 = n(\mu_x - 1) + d_K - d_k^w$$

where $n = \deg(\operatorname{Spec}\hat{\mathcal{O}}_{Y,y} \to \operatorname{Spec}\hat{\mathcal{O}}_{X,x})$, the value d_K the degree of the ramification divisor of

$$\operatorname{Spec}\left(\hat{\mathcal{O}}_{Y,y} \otimes_R K\right) \to \operatorname{Spec}\left(\hat{\mathcal{O}}_{X,x} \otimes_R K\right)$$

and $d_k^w := \sum d_{i,j}^w, \quad d_{i,j}^w := v_{x_j}\left(\delta_{y_{i,j},x_j}\right) - e_{i,j} + 1$

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where $\delta_{y_{i,j},x_j}$ is the discriminant ideal of the extension $\hat{\mathcal{O}}_{\tilde{X}_k,x_j} \to \hat{\mathcal{O}}_{\tilde{Y}_k,y_{i,j}}$ of complete DVRs, and $e_{i,j}$ its ramification index. The integer d_k^w equals 0 if and only if the morphism $\tilde{Y}_k \to \tilde{X}_k$ between the normalisations of X_k and Y_k is tamely ramified.

Proposition

Let Y be an R-curve. Let G be a finite group acting by automorphisms on Y. Then the quotient $X := Y/G$ of Y by G exists.

Proof.

A quotient exists if and only if every orbit of G is contained in an open affine of Y, but as G is finite and every finite set of points in a quasi-projective space (such as Y) is contained in an affine. $\square$

Proposition

Let $f\colon Y \to X$ be a covering. If Y is semi-stable then X is semi-stable.

Proof.

Let y be a closed point of Y, and let x be its image in X. We have $\mu_y \geq \mu_x$ and $\mu_y \in \{0, 1\}$, hence $\mu_x \in \{0, 1\}$. Thus if y is a smooth point then x is smooth, and if y is an ordinary double point, x is either a double point or a smooth point depending on the number of branches passing through x. $\square$

Proposition

Let $X := \operatorname{Spec} \mathcal{O}_{X',x}$ be the localisation of an R -curve X' at a smooth closed point x, and let $s : S \to X$ be an S point of X. Let $f : Y \to X$ be a Galois covering, and let e be its ramification index above the point $\tilde{x} := s(\operatorname{Spec} K)$. Assume that f is étale outside $s(S)$, and that e is prime to $\operatorname{char}(K)$. Then Y is smooth, and the morphism $f_k : Y_k \to X_k$ is tamely ramified at x with ramification index e. In particular the inertia subgroup at a point y of Y above x is cyclic of order e.

Proof

Let y be a closed point of Y above the point x. After étale localisation at y and x we can assume that y is the unique closed point of Y which is above x. Use local Riemann-Hurwitz.

Proof

$$\mu_y - 1 = n\left(\mu_x - 1\right) + d_K - d_k^w$$

We have $\mu_x = 0$. We compute d_K. Let $\{\tilde{y}_i\}_{i=1}^r$ be the points of Y_K above $\tilde{x}$ and let f be the residual degree at these points, then $d_K = r(ef - 1)$. Hence
$\mu_y = 1 - n + n - r - d_w = 1 - r - d_w$. The only possibility is that $r = 1, d_w = 0$ and $\mu_y = 0$ as claimed. The inertia subgroup at y is then the same as the inertia of the extension $\mathcal{O}_{X_k,x} \to \mathcal{O}_{Y_k,y}$ which is cyclic of order e .

Proposition

Assume $\mathrm{char}(K) = 0$ now. Let $\tilde{y}$ be a rational point of Y_K which specialises to a point y of Y_k, and let η be the generic point of the irreducible component of Y_k containing y. Let $\tilde{x}$ be the image of $\tilde{y}$ in X_K and assume that $f_K : Y_K \to X_K$ is étale outside $\tilde{x}$. Let $e'p^a$ be the ramification index at $\tilde{y}$, with e' prime to p. (Note that $I(\tilde{y})$ and $I(\eta)$ are subgroups of $I(y)$.) Then:

1. $I(\eta)$ is a p-group.
2. $I(\eta)$ is invariant in $I(y)$, and the quotient $I(y)/I(\eta)$ is cyclic of order e'

In particular if $e' = 1$, then $I(\tilde{y}) \subset I(\eta) = I(y)$, and moreover if $a \geq 1$ then $f : Y \to X$ is ramified along the irreducible component containing y.

Proof

Omitted

Proposition

Let $f : Y \to X$ be a Galois covering of group G where Y and X are semi-stable. Assume that $f_K : Y_K \to X_K$ is étale. Let y be an ordinary double point of Y, whose image in X is a double point x. Let C_1 and C_2 be the two irreducible components of Y_k passing through y (which may be equal), and let η_1 and η_2 be the corresponding generic points of Y_k. Then:

1. $I(\eta_1)$ and $I(\eta_2)$ are normal p-subgroups of $I(y)$, and they generate the (normal) p-sylow subgroup of $I(y)$
2. the quotient $I(y)/\langle I(\eta_1), I(\eta_2)\rangle$ is a cyclic group of order prime to p

Proof

1. Etale localize to assume y is the unique point of Y above x so that $G = I(y)$ and $C_1 \neq C_2$. Then $D(\eta_1) = D(\eta_2) = G$, and $I(\eta_1)$, $I(\eta_2)$ are p-groups.

2. We can pass to the quotient curve $Y' = Y / \langle I(\eta_1), I(\eta_2)\rangle$ to trivialise this subgroup, then use local Riemann-Hurwitz. If y' is the image of y we have

$$\mu_{y'} - 1 = n(\mu_x - 1) + d_K - d_k^w$$

with $\mu_x = 1$ and $d_K = 0$ so $\mu_{y'} = 0$. So the only possibility is $\mu_{y'} = 0$ and $d_k^w = 0$ so the cover $Y'_k \to X'_k$ is tamely ramified. So the original quotient $I(y)/\langle I(\eta_1), I(\eta_2)\rangle$ is cyclic.

A nicer cover

Let X be a smooth proper R-curve with geom. conn. X_K and $\{a_i : \operatorname{Spec} R \to X\}_{i=1,\dots r}$ all R-points of X s.t. they have disjoint support (distinct on the special fibre).

Let $f\colon Y \to X$ be a galois cover with group G s.t. $f_K\colon Y_K \to X_K$ is etale away from the points x_i, Y not necessarily smooth.

After extending R we can find $Y' \to Y$ proper birational with Y' semi-stable and in such a way that the G-action extends (this follows from choosing a minimal one).

We can quotient to get $X' = Y'/G$.

A nicer cover

The points x_i induce points x_i' on X', which remain disjoint but may have support on a double point, to fix this blow up X' and Y' to get a semi-stable model Y'' with a G-action and hence $X'' = Y''/G$ in such a way that

- The irreducible components of the special fibre of Y'' are smooth.
- The integral points $\{x_i\}_{i=1}^r$ extend to points $\{x_i''\}_{i=1}^r$ of $X''(R)$ which have disjoint support and are contained in the smooth locus of X''

as X was smooth originally and X'' is a semi-stable model of X_K the special fibre of X'' is a tree with a bunch of $\mathbf{P}^1$'s added to the original special fibre at double points.

Back to Abhyankar

If G is a quasi-p-group then it is generated by a family $(\alpha_1, \ldots, \alpha_m)$ of elements of p-power order, by adding new generators if needed we can assume $\alpha_1 \cdots \alpha_m = 1$.

Consider a complete DVR R with algebraically closed residue field k of characteristic p and fraction field K of characteristic 0, π a uniformizer.

Choose m distinct R-points $x_1, \ldots, x_m$ of the projective line $\mathbf{P}^1_R$, which have disjoint support (i.e. do not reduce to the same point on the special fibre). Write

$$U = \mathbf{P}^1_R \smallsetminus \{x_1, \ldots, x_m\}.$$

We consider the fundamental group of $U_{\overline{K}}$, the geometric generic fibre, as K was assumed to be of characteristic 0 the Lefschetz principle tells us that the answer agrees with the usual topological one (after profinite completion).

The fundamental group is a free profinite group where we have m topological generators $(\sigma_1, \dots, \sigma_m)$ satisfying $\sigma_1 \cdots \sigma_m = 1$, so that G is a quotient of π_1. And we have a connected Galois cover $Y_{\overline{K}} \to \mathbf{P}^1_{\overline{K}}$ with group G etale away from $\{x_i\}_i$.

The inertia subgroups above these points are cyclic p-groups. We can choose σ_i to generate inertia above x_i.

Enlarging K everything is defined over K and we can take integral versions of everything.

We can now apply the theory we developed to obtain semi-stable models Y'' and X'' for this setup where

- The irreducible components of the special fibre of Y'' are smooth.
- The rational points $\{x_i\}_{i=1}^m$ extend to integral points $\{x_i''\}_{i=1}^m$ of $X''(R)$ which have disjoint support and are contained in the smooth locus of X''

and where the special fibre of X'' is a tree of projective lines.

Proving 3)

Assume we are in the case of 3); $G(S) \neq G$ and G does not contain a non-trivial normal p-subgroup, we will apply the theory of semi-stable curves to prove that G is rev-p.

Combinatorial step shows that: the graph of the special fibre of Y'' with action of G is a **graph with inertia** (satisfies 8 conditions see next week).

It then says that there is a vertex s of the graph of Y''_k for which the decomposition group for that component is all of G, and whose image in the quotient tree is a leaf. As we have no non-trivial normal p-subgroup by assumption we have $I_s = 1$.

Restricting to this component covering a single $\mathbf{P}^1$ in the tree below as the vertex is a leaf it meets the rest of the special fibre only once, we have only one bad point in the component?, which when removed gives us an etale G-galois cover.