Coleman-de Shalit's p-adic regulator

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Overview/History/Philosophy

Goal: Introduce Coleman-de Shalit's regulator and show a relation to p-adic L-functions.

Big picture: Regulators are maps from K-groups / motivic cohomology to absolute Hodge cohomology (Deligne-Beilinson / syntomic). They relate to special values of L-functions.

History:

- Beilinson - Define regulators (+Bloch and many more), Deligne-Beilinson cohomology is absolute Hodge.
- Coleman-de Shalit - Construct a p-adic analogue
- Fontaine-Messing - Syntomic cohomology
- Gros - *Rigid* syntomic cohomology
- Besser - Coleman integrals compute regulators from K-theory to rigid syntomic cohomology
- Bannai - Rigid syntomic cohomology is absolute Hodge coh.

Beilinson regulators (Complex theory)

Let $C/\mathbb{C}$ be a smooth complete curve, $f, g \in \mathbb{C}(C)^\times$. Beilinson defines

$$r_{\infty,C}(f,g)(\omega) = \frac{1}{2\pi i}\int_{C(\mathbb{C})} \log|g|^2 \overline{\mathrm{d}\log f} \wedge \omega$$

the relation to K-groups comes via

$$K_2(\mathbb{C}(C)) = \mathbb{C}(C)^\times \otimes \mathbb{C}(C)^\times / \langle f \otimes 1 - f \rangle.$$

and $r_{\infty,C}$ satisfies this relation.

Relation to L-values

Fix E/κ be an elliptic curve with CM by $\mathcal{O}_\kappa$, κ a CM field *of class number 1*. Let $\Psi = \Psi_{E/\kappa}$ be the associated Grossencharacter, p be a prime that splits in κ, $p = \mathfrak{p}\overline{\mathfrak{p}}$. ω an invariant differential.

Proposition (Bloch, Rohrlich, Deninger-Wingberg)

$$r_{\infty,E}(f,g)(\omega) = c_{f,g}\Omega \underbrace{L_\infty(E,0)}_{=L_\infty(\Psi,0)},\ c_{f,g} \in \mathbb{Q}$$

(L_∞ includes Gamma factors), and there exists f, g with $c_{f,g} \neq 0$.

p-adic version

We can associate a canonical *period-pair-class* to κ: $\langle \Omega, \Omega_p \rangle \in (\mathsf{C}^\times \times \mathsf{C}_p^\times)/\overline{\mathsf{Q}}^\times$ so that:

Theorem (Katz, Manin-Vishik)
Let $1 \neq \mathfrak{g}$ an ideal of κ relatively prime to $\mathfrak{p}$. Then $\exists!$, $W(\overline{\mathsf{F}}_p)$-valued measure μ on $\mathscr{G}(\mathfrak{g}) = \mathrm{Gal}(\kappa(\mathfrak{g}\mathfrak{p}^\infty)/\kappa)$ so that $\forall \epsilon$ Grossencharacter of conductor dividing $\mathfrak{g}$ with infinity type $(k, 0)$ $k \geq 1$, if

$$L_{\infty,\mathfrak{g}}(\epsilon^{-1}, s) = \Gamma(s+k) \prod_{\mathfrak{l} \nmid \mathfrak{g}} (1 - \epsilon^{-1}(\mathfrak{l}) \operatorname{Nm} \mathfrak{l}^{-s})^{-1}$$

$$L_{p,\mathfrak{g}}(\epsilon^{-1}) = \int_{\mathscr{G}(\mathfrak{g})} \epsilon(\sigma) \, \mathrm{d}\mu(\sigma)$$

we have

$$\Omega_p^{-k} L_{p,\mathfrak{g}}(\epsilon^{-1}) = \Omega^{-k} (1 - p^{-1} \epsilon(\mathfrak{p})) L_{\infty,\mathfrak{g}}(\epsilon^{-1}, 0) \in \overline{\mathsf{Q}}.$$

p-adic regulators?

Can rewrite $r_{\infty,C}$ as

$$r_{\infty,C}(f,g) = \sum_{b \in C(\mathbb{C})} \mathrm{ord}_b(g) F_{f,\omega}(b)$$

where $F_{f,\omega}$ satisfies

$$\bar{\partial}(\mathrm{d}F) = \bar{\partial}(\log|f|^2\omega)$$

Even without p-adic $\bar{\partial}$ we can just try to find $F_{f,\omega}$ satisfying

$$\mathrm{d}F = \log f \cdot \omega$$

and define

$$"r_{p,C}(f,g) = \sum_{b \in C(\mathbb{C}_p)} \mathrm{ord}_b(g) F_{f,\omega}(b)"$$

p-adic tools (Coleman integration)

Let $K = \mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$, $R = \mathcal{O}_K$, $k = R/\mathfrak{m}$. We will work with 1-dimensional rigid spaces (curves) over K. We fix a branch of the p-adic logarithm $\log\colon K^\times \to K$.

It is always possible to integrate rigid 1-forms locally on a disk: given ω we have a local expression in terms of a convergent power series

$$\omega|_D = \sum_i a_i t^i \,\mathrm{d}t$$

which can be integrated formally (up to a constant).

Let $X/\mathcal{O}_K$ be a smooth projective curve, if $Y \subseteq X$ smooth affine open, then in the special fibre

$$X_k \smallsetminus Y_k = \{e_1, \dots, e_n\}.$$

What is hard is to integrate globally, iteratively and include $\int \frac{\mathrm{d}z}{z}$!

p-adic tools (Coleman integration)

We then remove rigid disks around e_i. Y_k is locally given by $\bar{h}$ so we can take the rigid subspace

$$U_r \text{ locally defined by } |h| > r$$

and the underlying affinoid is $X_K - \bigcup_i B_<(e_i, 1)$.

We have

$$U = \varprojlim_{r \to 1} U_r$$

and spaces of overconvergent functions and 1-forms

$$A(U) = \varinjlim_{r \to 1} A(U_r)$$

Let Y be an affinoid with good reduction then Y_k finite type, and we have $F\colon Y_k \to Y_k$ the q-power frobenius.

Proposition
There exists

$$\phi\colon U \to U, \widetilde{\phi} = F$$

*a **lift of frobenius** or frobenius morphism of X, of degree q.*

Note: Whatever we choice of frobenius we make should not matter!

Example
Have $X = \mathsf{P}^1_{\mathcal{O}_{\mathsf{C}_p}} \supseteq Y = \mathsf{G}_m = \mathsf{P}^1 \smallsetminus \{0, \infty\}$ then

$$U_r = \{r < |z| < 1/r\}$$

$\phi(z) = z^q\colon U_r \to U_{r^q}$. (But we could add some other $p \cdot$ junk!)

$$\Omega^1(U) = \varinjlim_{r \to 1} \Omega^1(U_r)$$

$$A(U_x) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ converging for } |z| < 1 \right\}$$

$$A(U_{e_i}) = \left\{ f(z_{e_i}) = \sum_{n=-\infty}^{\infty} a_n z_{e_i}^n \text{ converging for } r < |z_{e_i}| < 1, r < 1 \right\}.$$

$$A_{\log}(U_x) = A(U_x),\ A_{\log}(U_{e_i}) = A(U_{e_i})[\log(z_{e_i})]$$

$$\Omega^1_{\log}(U_?) = A_{\log}(U_?)\, \mathrm{d}z_?$$

$$A_{\mathrm{loc}}(U) = \prod_x A_{\log}(U_x)$$

Theorem (Coleman integration)
There is a subspace $M(U)$ of $A_{\mathrm{loc}}(U)$, which we call the space of Coleman functions, and linear map (integration), which we denote by $\int$ or by $\omega \mapsto F_\omega$, from $M(U) \otimes_{A(U)} \Omega(U)$ to $M(U)/\mathbb{C}_p$.

The map f is characterized by three properties:

1. *It is a primitive for the differential in the sense that $\mathrm{d}F_\omega = \omega$.*
2. *It is Frobenius equivariant $F_{\phi^*\omega} = \phi^* F_\omega$.*
3. *If $g \in A(U)$, then $F_{\mathrm{d}g} = g + \mathbb{C}_p$.*

We also have properties such as:

$$f \in M(U)$$

vanishes on one residue disk, then f is identically zero.

The space $M(U)$ is constructed iteratively $M(U) = \bigcup_n A_n(U)$ with each step being obtained as functions you get by integration from the previous.

The p-adic regulator

We can now define a p-adic version of the above regulator.

(Let C to be a complete non-singular curve whose jacobian has good reduction.)

If $f \in K(C)^\times$, $U = C \smallsetminus |\operatorname{div}(f)|$ we can take a global 1-form $\omega \in H^0(C, \Omega^1_{C/K})$ and the function

$$\log(f) = \int \frac{\mathrm{d}f}{f} \in A_1(U)$$

and obtain

$$\log(f)\omega \in \Omega_1(U).$$

Integration gives

$$F_{f,\omega} \in A_2(U) \text{ with } \mathrm{d}F_{f,\omega} = \log(f)\omega \in \Omega_1(U),$$

unique up to a constant.

Definition (The p-adic regulator)
Take f, g, ω as before defined over $\overline{k}$, then define

$$r(f,g)(\omega) = -\int_{(g)} \log(f)\omega = -\sum_{b \in C(K)} \mathrm{ord}_b(g) F_{f,\omega}(b) \in \overline{k}$$

Theorem (Coleman-de Shalit)
$r_C(f,g)$ is a skew-symmetric bilinear pairing on $\overline{k}(C)^\times$ that

1. *factors through $K_2(\overline{k}(C))$*
2. *depends only on $\mathrm{div}(f), \mathrm{div}(g)$*
3. *is $\mathrm{Gal}(\overline{k}/k)$ equivariant*
4. *for finite morphisms of complete non-singular curves $/k$ $u \colon C' \to C$ we get $r_{C'}(u^*f, u^*g) = u^* r_C(f,g)$.*

Comparison of the p-adic and $\mathbb{C}$ theories

We now move to a very special situation, where the above regulators can be shown to be related to L-values.

$C = E/\kappa$ will be an elliptic curve with CM by $\mathcal{O}_\kappa$. $\Psi = \Psi_{E/\kappa}$ the corresponding Grossencharacter with conductor $\mathfrak{f}$ and assume

$$w_{\mathfrak{f}} = \#\{\zeta \in \mu(K) : \zeta \equiv 1 \pmod{\mathfrak{f}}\} = 1.$$

let ω be a κ-rational invariant differential, $\mathscr{L}$ the period lattice of (E, ω).

Theorem (Rohrlich, others?)

$$r_\infty(f,g) = \underbrace{12(\mathrm{Nm}\,\mathfrak{a} - \Psi^{-1}(\mathfrak{a})) \sum_{\textit{orbits } \langle Q\rangle} \mathrm{ord}_Q\, g \cdot \Omega(Q) \prod_{\mathfrak{l}|\mathfrak{g}_Q} (1-\Psi(\mathfrak{l}))}_{c_{f,g}} L_\infty(\Psi, 0)$$

$\mathfrak{g}_Q$ *ideal of annihilators of* Q*.*

Theorem (Coleman-de Shalit)
We have the formula

$$r_{p,E}(f,g)(\omega) = c_{f,g}\Omega_p(1-(\Psi(\mathfrak{p})p)^{-1})^{-1}L_p(\Psi).$$

Where do these terms come from? (in the p-adic case)

The rest of the talk: proof overview, see where the terms come in.

Proof

We use a specific class of f's (for $(\mathfrak{a}, \mathfrak{f}\mathfrak{p}) = 1$), the functions

$$f(P) = \Theta_{\mathfrak{a}}(P) = \Delta(\mathcal{L})\Delta(\mathfrak{a}^{-1}\mathcal{L})^{-1} \prod_{R \in E[\mathfrak{a}]}{}' \frac{\Delta(\mathcal{L})}{(x(P) - x(R))^6} \in \kappa(E)^\times$$

whose values are **elliptic units**, the divisor of $\Theta_{\mathfrak{a}}$ is

$$12\left((\mathrm{Nm}\,\mathfrak{a} - 1)\cdot(0) - \sum_{R \in E[\mathfrak{a}]}{}'(R)\right)$$

and we have the **distribution relation**

$$f(\pi P) = \prod_{v \in E[\mathfrak{p}]} f(P + v)$$

These functions generate the set of all functions with divisors supported on torsion.

We also take $g \in \kappa(E)^\times$ with divisor supported on torsion and $Q \in |\operatorname{div} g| \implies \mathfrak{f} | \mathfrak{g}_Q$, $(\mathfrak{g}_Q, \mathfrak{a}\mathfrak{p}) = 1$.

Take E with the $\mathfrak{a}$-torsion points removed,

$$X(\mathfrak{a}) = E \smallsetminus \bigcup_{P \in E[\mathfrak{a}]} B(P,1) \subseteq U_r(\mathfrak{a}) = E \smallsetminus \bigcup_{P \in E[\mathfrak{a}]} B(P,r).$$

Take D to be a derivation that is dual to ω (so $DF_\omega = 1$). Then

$$F_{f,\omega}$$

is the unique (up to constant) $F \in A_2(U_r(\mathfrak{a}))$ for which

$$DF = \log f$$

Then we have

$$D(F(\pi P)) = \pi \cdot (DF)(\pi P)$$

and the distribution relation gives

$$D(F(\pi P)) = \pi \sum_{v \in E[\pi]} (DF)(P+v)$$

By definition $\pi = \psi(\mathfrak{p})$ is a lift of frobenius (which is algebraic!). As $F \in A_2(U_r(\mathfrak{a}))$, for some (possibly different) r close to 1 we have

$$F(\pi P) - \pi \sum_{v \in E[\pi]} F(P+v) \in A_2(U_r(\mathfrak{a}))$$

the above implies this is locally constant, hence constant! So we change F to get that

$$F(\pi P) - \pi \sum_{v \in E[\pi]} F(P+v) = 0.$$

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Now define

$$F^{\#}(P) = F(P) - p^{-1} \sum_{v \in E[\pi]} F(P+v)$$

so that as $Q \in |\operatorname{div} g|$ is Galois conjugate to πQ over κ:

$$r_p(f,g) = -\sum_Q \operatorname{ord}_Q gF(Q) = -\sum_Q \operatorname{ord}_Q gF(\pi Q)$$

giving

$$\left(1 - \frac{1}{\pi p}\right) r_p(f,g) = -\sum_Q \operatorname{ord}_Q gF^{\#}(Q).$$

We also have

$$\log(f)^{\#}(P) = \log f(P) - p^{-1} \sum_{v \in E[\pi]} \log f(P+v).$$

If Q is a torsion point in $X(\mathfrak{a})$ relatively prime to $\mathfrak{p}$ order, then de Shalit has associated a

$$\eta_Q \colon \widehat{\mathsf{G}}_m \xrightarrow{\sim} \widehat{E}$$

so $Q + \eta_Q(S)$ parameterises the residue disk of Q and a $W = W(\overline{\mathsf{F}}_p)$ valued measure μ_Q on $\mathsf{Z}_p^\times$ s.t.

$$\log(f)^{\#}(Q + \eta_Q(S)) = \int_{\mathsf{Z}_p^\times} (1+S)^x \, \mathrm{d}\mu_Q(x) \in W[[S]]$$

Then work of de Shalit shows that

$$F^{\#}(Q+\eta_Q(S)) = \underbrace{\eta_Q'(0)}_{\Omega_p(Q)} \int_{\mathbb{Z}_p^\times} (1+S)^x x^{-1}\,\mathrm{d}\mu_Q(x) + c$$

for some constant c, and that $F^{\#}(P)$ is rigid analytic on $X(\mathfrak{a})$.

So we get

$$\left(1-\frac{1}{\pi p}\right) r_p(f,g) = -\sum_{Q} \mathrm{ord}_Q\, g\Omega_p(Q) \int_{\mathbb{Z}_p^\times} x^{-1}\,\mathrm{d}\mu_Q(x).$$

We need to move to the correct group and remove the dependence on Q, by identifying $G = \mathrm{Gal}(\kappa(\mathfrak{g}\mathfrak{p}^\infty)/\kappa(\mathfrak{g})) \cong \mathbb{Z}_p^\times$ so that

$$= -\sum_{\langle Q\rangle} \mathrm{ord}_Q\, g\Omega_p(Q) \sum_{\tau\in\mathscr{G}/G} \int_G \Psi^{-1}(\sigma)\,\mathrm{d}\mu_{\tau(Q)}(\sigma)$$

$$= -\sum_{\langle Q\rangle} \mathrm{ord}_Q\, g\Omega_p(Q) \int_{\mathscr{G}(\mathfrak{g}_Q)} \Psi^{-1}(\sigma)\,\mathrm{d}\mu_e(\sigma)$$

Theorem (Coates-Wiles)

$$\mu_e = 12(\sigma_{\mathfrak{a}} - \operatorname{Nm}\mathfrak{a})\mu(\mathfrak{g}_Q)$$

where $\mu(\mathfrak{g}_Q)$ is the measure which defines the p-adic L-function of conductor $\mathfrak{g}_Q$,so removing those factors we reach

$$\overbrace{\left(1-(\pi p)^{-1}\right)}^{\text{Distribution relations}} \; r_p(f,g) =$$

$$\underbrace{12(\operatorname{Nm}\mathfrak{a} - \Psi^{-1}(\mathfrak{a}))}_{\text{Coates-Wiles}} \sum_{\text{orbits } \langle Q\rangle} \operatorname{ord}_Q g\Omega_p(Q) \underbrace{\prod_{\mathfrak{l}|\mathfrak{g}_Q}(1-\Psi(\mathfrak{l}))\, L_p(\Psi)}_{L_{p,\mathfrak{g}_Q} \rightsquigarrow L_p},$$

Fin.