# Motivation for $p$-adic modular forms

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STAGE

## Overview

These slides are available online (in handout form) at

`https://alexjbest.github.io/talks/motivation-for-p-adic-modular-forms/slides_h.pdf`

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**Goal:** Introduce, post hoc, motivation for Katz's definition of $p$-adic modular forms, especially to motivate Serre's $\partial$ operator.

# Recall

A *modular form* of weight $k$ is a function

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s.t.

1. $\forall \lambda \in R^\times, f(E, \lambda\omega) = \lambda^{-k} f(E, \omega)$g
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we then have

$$f(E, \omega) \cdot \omega^{\otimes k} \in \Gamma(R, \underbrace{\pi_*(\Omega^1_{E/R})}_{\underline{\omega}_{E/R}}{}^{\otimes k})$$

# De Rham cohomology

The sheaf of values $\underline{\omega}_{E/R}$ is a subsheaf of the de Rham cohomology of $E/R$:

$$H^1_{\mathrm{dR}}(E/R) := \mathbb{H}^1(E, \Omega^\bullet_{E/R})$$

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Fixing $(E, \omega)/R$ we have a unique pair of meromorphic functions with poles only at $\infty$, of orders 2 and 3 resp., denoted by $X$, $Y$ so that

$$\omega = \frac{\mathrm{d}X}{Y} \text{ and } E\colon\ Y^2 = 4X^3 - g_2X - g_3,\ g_i \in R$$

Then we have an inclusion of 2-term complexes

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this induces an isomorphism on $\mathbb{H}^1$. Moreover for $i > 0$,

$$H^i(E, \mathcal{O}_E(\infty)) = 0$$

$$H^i(E, \Omega^1_{E/R}(2\infty)) = 0$$

giving

$$\begin{aligned} H^1_{\mathrm{dR}}(E/R) &\cong \mathbb{H}^1(E, \mathcal{O}_E(\infty) \to \Omega^1_{E/R}(2\infty)) \\ &= \operatorname{coker}(H^0(E, \mathcal{O}_E(\infty)) \to H^0(E, \Omega^1_{E/R}(2\infty))) \\ &= \operatorname{coker}(R \xrightarrow{0} H^0(E, \Omega^1_{E/R}(2\infty))) \\ &= H^0(E, \Omega^1_{E/R}(2\infty)) \end{aligned}$$

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Theorem (Katz, Manin-Vishik)