Something to Lean on; fun with interactive theorem provers

Alex J. Best

BU Math Retreat 2019

The problem

Mathematicians make mistakes.

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However this uncertainty takes up time and energy, what if referees only needed to judge the importance, novelity and quality of exposition, not check the arguments.

What is proof

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> *This story got me scared. Starting from 1993 multiple groups of mathematicians studied the "Cohomological Theory" paper at seminars and used it in their work and none of them noticed the mistake.*
>
> *And it clearly was not an accident. A technical argument by a trusted author, which is hard to check and looks similar to arguments known to be correct, is hardly ever checked in detail.*
>
> *— Vladimir Voevodsky*

Some examples: Grunwald(-Wang) and K-theory

Some days later I was with Artin in his office when Wang appeared. He said he had a counterexample to a lemma which had been used in the proof. An hour or two later, he produced a counterexample to the theorem itself... Of course he [Artin] was astonished, as were all of us students, that a famous theorem with two published proofs, one of which we had all heard in the seminar without our noticing anything, could be wrong.

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The groundbreaking 1986 paper "Algebraic Cycles and Higher K-theory" by Spencer Bloch was soon after publication found by Andrei Suslin to contain a mistake in the proof of Lemma 1.1. The proof could not be fixed.

— Voevodsky

The solution

Work at an extremely high level of rigour, with every statement logical step spelled out in extreme detail and checked by multiple independent people.

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This is really boring!

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This is really boring!

> *But to do the work at the level of rigor and precision I felt was necessary would take an enormous amount of effort and would produce a text that would be very difficult to read. And who would ensure that I did not forget something and did not make a mistake, if even the mistakes in much more simple arguments take years to uncover?*
>
> — *Voevodsky*

The solution: Take 2

What if computers could do the boring work for us?

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Computers are:

- Capable of checking basic logical statements,

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What if computers could do the boring work for us?

Computers are:

- Capable of checking basic logical statements,
- Fast,
- Never complain.

The new problem

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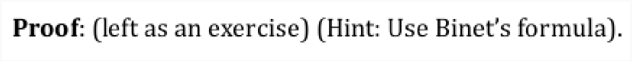

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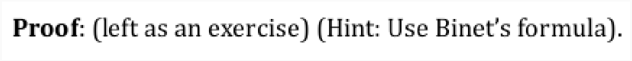
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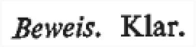
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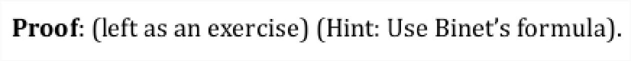
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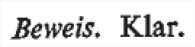
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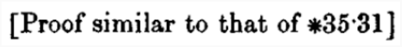
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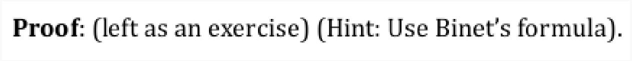
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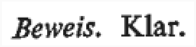
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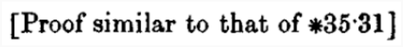
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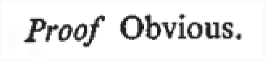
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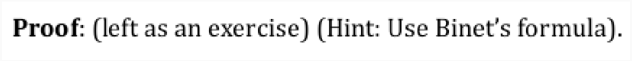
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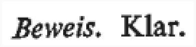
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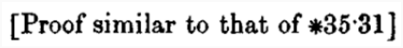
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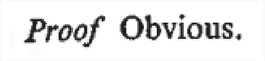
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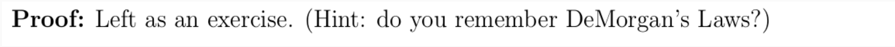
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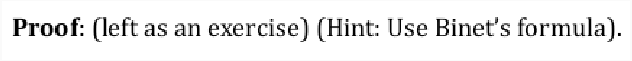
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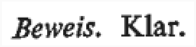
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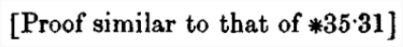
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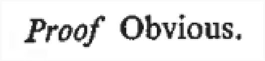
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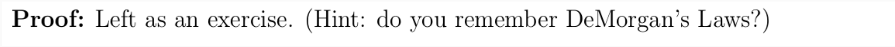
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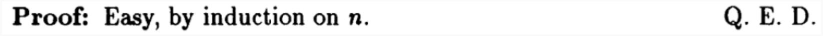
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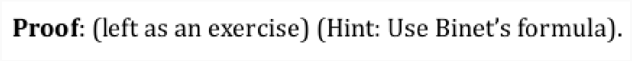
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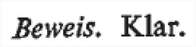
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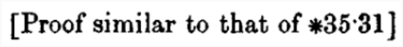
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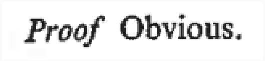
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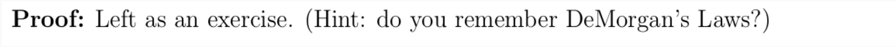
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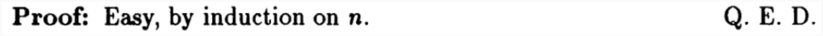
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The computer will probably not understand these, but in order to stay sane we must strike a balance between detail and verbosity.

Enter Lean

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Recently an interactive proof assistant called Lean has been under heavy development. And I've been playing with it.

Let me show you some lean code:

```
lemma fact_rec (n : ℕ) :
factorial (n + 1) = factorial n * (n+1) :=
begin
-- write out the definition of factorial
unfold factorial,
-- remember {1,...,n+1} = {1,...,n} ∪ {n+1}
rewrite list.range'_concat 1 n,
-- the product of two sequences joined together is
    just the product of the products of each sequence
rewrite list.prod_append,
-- I'm bored already are we done here?
simp,
-- YES!
end
```

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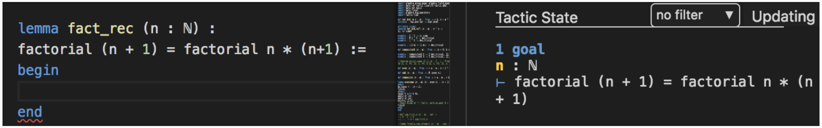
We can replace all of the above with: `by unfold factorial; simp [list.range'_concat, list.prod_append]`

(not so?) Live demo!



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begin

end
```

```
Tactic State
1 goal
n : ℕ
⊢ factorial (n + 1) = factorial n * (n
+ 1)
```

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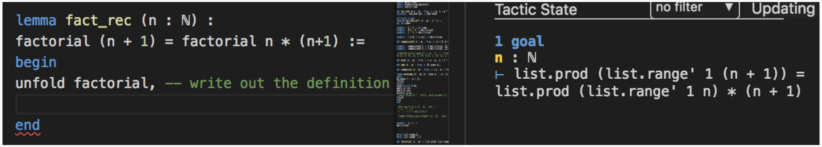

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```
1 goal
n : ℕ
⊢ list.prod (list.range' 1 (n + 1)) =
list.prod (list.range' 1 n) * (n + 1)
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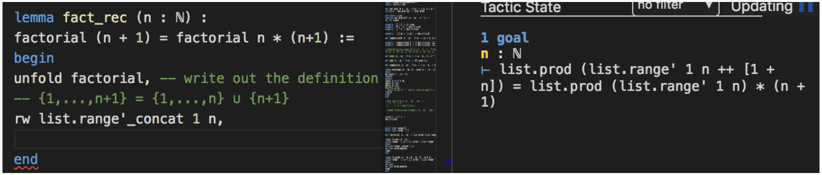

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rw list.range'_concat 1 n,

end
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⊢ list.prod (list.range' 1 n ++ [1 +
n]) = list.prod (list.range' 1 n) * (n +
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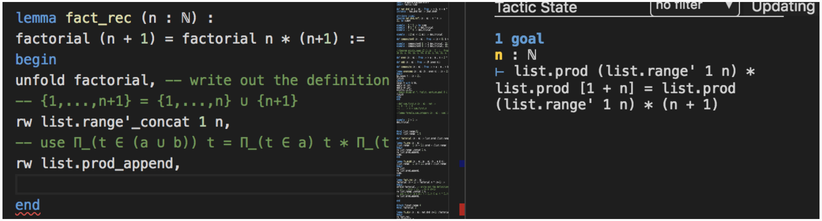

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rw list.prod_append,

end
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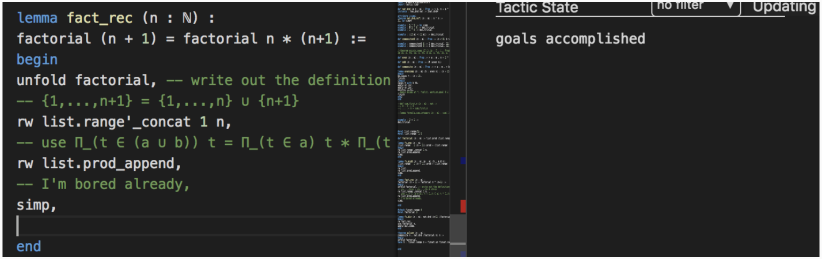

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```
goals accomplished
```

But can it do research?

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Research level mathematics requires a vast body of knowledge to even think about.

As humans we forget this and also gloss over things we (think we) know well.

For instance one can forget what a Dedekind cut is, or Peano arithmetic and still think about real and natural numbers without an issue. Our intuition allows us to abstract these concepts so far away that we don't have to work from the ground up when approaching a problem.

Perfectoids

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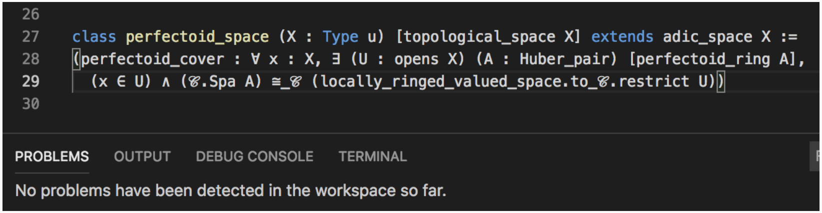
Last week Kevin Buzzard, Johan Commelin, Patrick Massot (and others) completed a long term project to define a perfectoid space formally in Lean.

Perfectoids



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class perfectoid_space (X : Type u) [topological_space X] extends adic_space X :=
(perfectoid_cover : ∀ x : X, ∃ (U : opens X) (A : Huber_pair) [perfectoid_ring A],
  (x ∈ U) ∧ (𝒞.Spa A) ≅_𝒞 (locally_ringed_valued_space.to_𝒞.restrict U))
```

```
No problems have been detected in the workspace so far.
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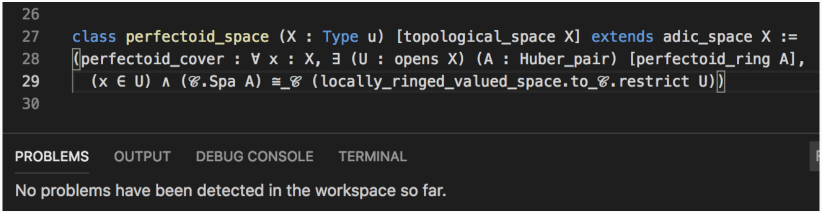
Lean has accepted the chain of definitions that lead to this are all valid, topological spaces, sheaves, valuations, adic spaces, perfectoid rings,...

Perfectoids



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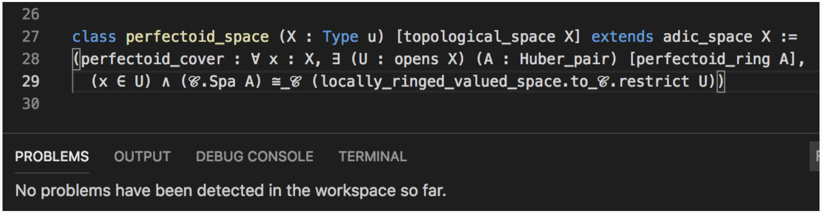

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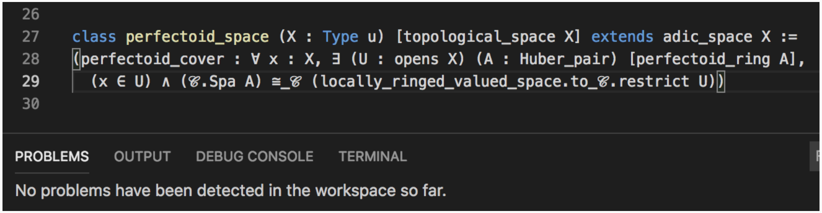

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PROBLEMS OUTPUT DEBUG CONSOLE TERMINAL

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However, I would guess it compares favourably to the length of time needed to teach a human with zero mathematical background

A call to arms

I want to learn more about this!

A call to arms

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If you think this sounds fun and/or interesting let me know, I'd love to organise a casual semi-regular meetup, to play with this together and collaborate!