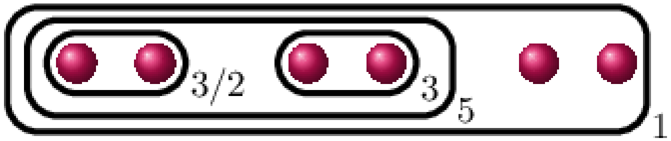
Cluster pictures & BY-trees in SageMath

[https://github.com/alexjbest/cluster-pictures (https://github.com/alexjbest/cluster-pictures)](https://github.com/alexjbest/cluster-pictures)

implementation by Alex J. Best and Raymond van Bommel

In [50]:
```
from sage_cluster_pictures.cluster_pictures import *
K = Qp(5)
x = polygen(K)
H = HyperellipticCurve((x^2 + 5^2)*(x^2 - 5^15)*(x - 5^6)*(x - 5^6 - 5^9))
R = Cluster.from_curve(H)
view(R)
```

Out[50]:

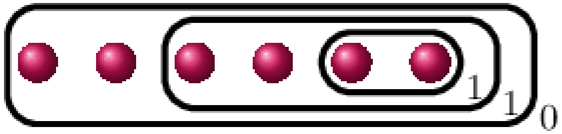


Creating clusters

From roots:

In [3]:
```
view(Cluster.from_roots([K(1), K(2), K(5), K(10), K(25), K(50)]))
```

Out[3]:

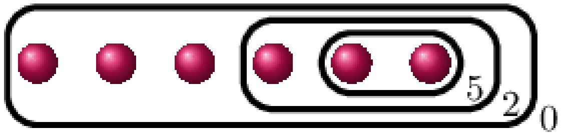


From valuations:

$$y^2 = (x-1)(x-2)(x-3)(x-p^2)(x-p^7)(x+p^7)$$

In [4]:
```
view(Cluster([[oo, 0,  0,  0,  0,  0],
              [0,  oo, 0,  0,  0,  0],
              [0,  0,  oo, 0,  0,  0],
              [0,  0,  0 , oo, 2,  2],
              [0,  0,  0,  2,  oo, 7],
              [0,  0,  0,  2,  7,  oo]]))
```

Out[4]:



Basic properties

In [5]:
```
print(R.children())
```

```
[Cluster with 4 roots and 2 children, Cluster with 1 roots and 0 children, Clu
ster with 1 roots and 0 children]
```

In [6]:
```
[unicode_art(D) for D in R.all_descendents()]
```

Out[6]:
```
['(((● ●)_3/2 (● ●)_3)_5 ● ●)_1',
 '((● ●)_3/2 (● ●)_3)_5',
 '(● ●)_3/2',
 '●',
 '●',
 '(● ●)_3',
 '●',
 '●',
 '●',
 '●']
```

In [7]:
```
R.is_semistable(K)
```

Out[7]:
```
True
```

In [51]:
```
(R.jacobian_has_potentially_good_reduction(), R.potential_toric_rank())
```

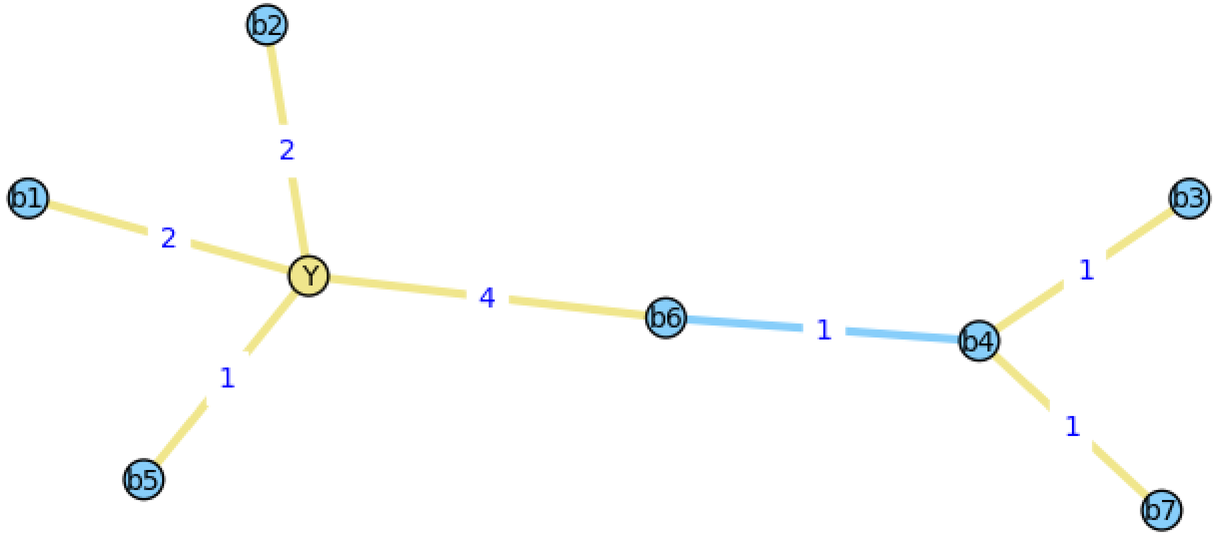
Out[51]: (False, 2)

BY-trees:

In [9]:

```
T = BYTree(name="Stick person")
T.add_blue_vertices(['b1', 'b2', 'b3', 'b4', 'b5', 'b6', 'b7'])
T.add_yellow_vertex('Y')
T.add_yellow_edges([('b1', 'Y', 2), ('b2', 'Y', 2), ('b5', 'Y', 1), ('b6', 'Y', 4
), ('b3', 'b4', 1), ('b7', 'b4', 1)])
T.add_blue_edge(('b6', 'b4', 1))
plot(T,vertex_labels=True)
```

Out[9]:



In [10]:

```
T.validate()
```

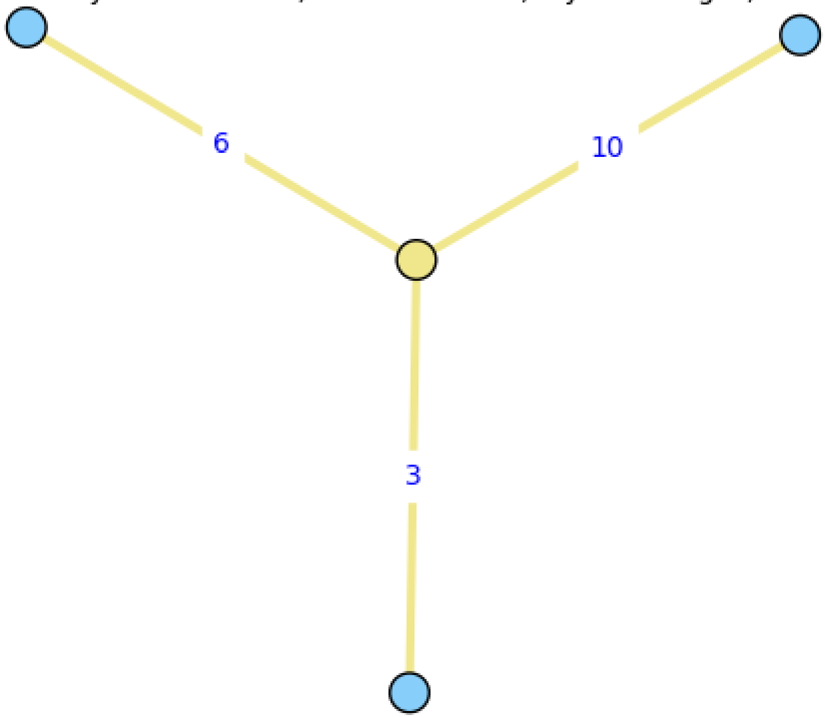
Out[10]:

```
True
```

In [11]:
```
T = R.BY_tree(); T
```

Out[11]:

BY-tree with 1 yellow vertices, 3 blue vertices, 3 yellow edges, 0 blue edges



In [12]:
```
T.vertices()
```

Out[12]:
```
[Cluster with 6 roots and 3 children,
 Cluster with 4 roots and 2 children,
 Cluster with 2 roots and 2 children,
 Cluster with 2 roots and 2 children]
```

Via the cluster picture and the homology of the dual graph of the special fibre:

In [13]:
```
R.root_number()
```
Out[13]:
```
1
```

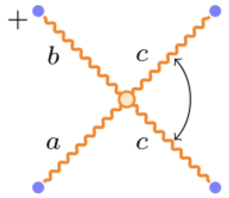
Via the associated BY-tree:

In [14]:
```
R.tamagawa_number()
```
Out[14]:
```
108
```

In [15]:
```
T, F = R.BY_tree(with_frob=True)
T.tamagawa_number(F)
```
Out[15]:
```
108
```

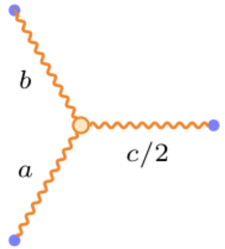
Out[21]:

FIGURE 1. The BY tree associated to X_f



In this diagram, the whole graph represents the tree T, while the blue/solid vertices represent the vertices of S (which has no edges in this example) – by contrast, the vertices of T not in S are represented by yellow/open circles and the edges of T not in S are represented by yellow/squiggly lines. The lengths of the edges are indicated by the parameters a, b and c, while the signed automorphism is indicated both with double-headed arrows for the underlying unsigned automorphism of (T, S) (which here has order 2) and with $\pm$ signs next to each connected component[10] of $T \setminus S$ (so here the sign is $+$).

Since $T \setminus S$ is connected, there is only one term in the formula from theorem 3.0.1, and since $\epsilon(C_i) = +1$ for the unique component C_i, we have $(T_i', S_i') = (T, S)$ and $\tilde{c}_{1,i} = 1$. By inspection, $Q_i = 2$ and the quotient tree T_i'', along with its subgraph S_i'', are given by the following diagram



where again the blue/solid vertices indicate the subset $S' \subseteq T'$ and the labels indicate edge-lengths. The removal of any two of the three edges of this graph disconnects the three points of S' from one another, and hence the formula in theorem 3.0.1 provides that the Tamagawa number of X_f is

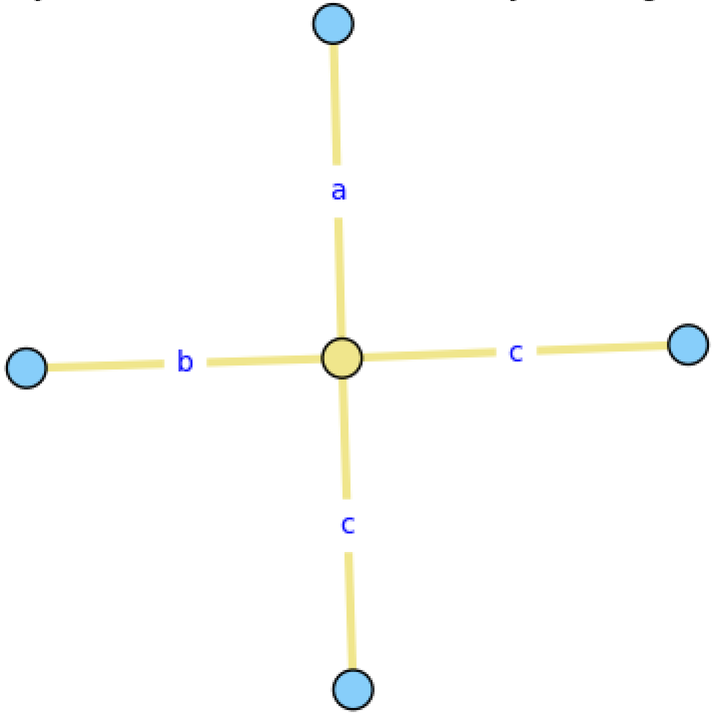
$$c_{X_f} = 2 \cdot \left(ab + b\frac{c}{2} + \frac{c}{2}a\right) = 2ab + bc + ca.$$

In [44]:

```
a, b, c = var('a b c', domain="positive")
R = Cluster([[oo, a/2, 0,   0,   0,   0,    0,   0],
             [a/2, oo, 0,   0,   0,   0,    0,   0],
             [0,   0,   oo, b/2, 0,   0,    0,   0],
             [0,   0,   b/2, oo, 0,   0,    0,   0],
             [0,   0,   0,   0,   oo, c/2,  0,   0],
             [0,   0,   0,   0,   c/2, oo,  0,   0],
             [0,   0,   0,   0,   0,   0,   oo, c/2],
             [0,   0,   0,   0,   0,   0,   c/2, oo], ])
A,B,C1,C2 = R.children()
T = R.BY_tree(); T
```

Out[44]:

BY-tree with 1 yellow vertices, 4 blue vertices, 4 yellow edges, 0 blue edges



In [46]:
```
T.yellow_vertices()
```
Out[46]:
```
[Cluster with 8 roots and 4 children]
```
In [48]:
```
F = BYTreeIsomorphism(T, T,
                      lambda x: {A:A, B:B, C1:C2, C2:C1, R:R}[x],
                      lambda Y: 1)
T.tamagawa_number(F)
```
Out[48]:

$$2\,ab + ac + bc$$

Thank you! And thanks to the organisers!