Coleman Integration in Larger Characteristic

ANTS XIII — University of Wisconsin, Madison

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Boston University

The big picture

Kedlaya

Arul, Balakrishnan, Best, Bradshaw, Castryk, Costa, Denef, Gaudry, Gurel, Harvey, Kedlaya, Magner,

Minzlaff, Shieh, Triantafillou, Tuitman, Vercauteren, and more. . .

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$$\text{CaDeVe-Go-GaGu-Sh-Tu} \xrightarrow{\text{More curves}} \text{Kedlaya}$$

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There is (at least) one dimension missing: Small p!

Motivation

(Explicit) Coleman integration is a central tool in (non-abelian) Chabauty and can be applied to find: heights, torsion points and regulators also, but, algorithms lag behind the related ones for zeta functions.

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- Applications to rational points, combining congruence information for many primes, 1-step (Mordell-Weil) sieving.

Coleman integration

Throughout we take $X/\mathbf{Z}_p$ a genus g odd degree hyperelliptic curve, and p an odd prime. We pick a lift of the Frobenius map, $\phi^*\colon X \to X$, and write $A^\dagger$ (resp. $A_{\mathrm{loc}}(X)$) for overconvergent (resp. locally analytic) functions on X.

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Theorem (Coleman)

There is a $\mathbf{Q}_p$-linear map $\int_b^x \colon \Omega^1_{A^\dagger} \otimes \mathbf{Q}_p \to A_{\mathrm{loc}}(X)$ for which:

$$\mathrm{d} \circ \int_b^x = \mathrm{id}\colon \Omega^1_{A^\dagger} \otimes \mathbf{Q}_p \to \Omega^1_{loc} \quad \text{“FTC”}$$

$$\int_b^x \circ\, \mathrm{d} = \mathrm{id}\colon A^\dagger \hookrightarrow A_{\mathrm{loc}}$$

$$\int_b^x \phi^*\omega = \phi^* \int_b^x \omega \quad \text{“Frobenius equivariance”}$$

Reduction to reduction

Balakrishnan-Bradshaw-Kedlaya reduce the problem of computing all Coleman integrals of basis differentials ω_i of $H^1_{\mathrm{dR}}(X)$ between $\infty \in X$ and a point $x \in X(\mathbf{Q}_p)$, to:

1. Finding "tiny integrals" between nearby points,
2. Writing $\phi^*\omega_i - \mathrm{d}f_i = \sum_j a_{ij}\omega_j$ and evaluating the primitive f_i for a point P near x, for each i.

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Applying ϕ^* to the basis $x^i\,\mathrm{d}x/2y$ for $i = 0, \ldots, 2g-1$ gives

$$\phi^*\omega_i \equiv \sum_{j=0}^{N-1} \sum_{r=0}^{(2g+1)j} B_{j,r} x^{p(i+r+1)-1} y^{-p(2j+1)+1} \frac{\mathrm{d}x}{2y} \pmod{p^N}$$

$B_{j,r} \in \mathbf{Z}_p$ are in terms of coefficients of the curve and binomial coefficients.

Kedlaya's algorithm

Theorem (Kedlaya)

The action of ϕ^* on $H^1_{\mathrm{MW}}(X)$ (which determines the zeta function of X) can be computed in time

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The problem solved here is almost the same: determining a_{ij} s.t.

$$\phi^* \omega_i - \sum_j a_{ij} \omega_j \in \mathrm{image}(\mathrm{d}).$$

Primitive technology

Revised problem

Computing f along with $\omega - \mathrm{d}f$ when reducing degree.

For vanilla Kedlaya this is "easy", the reduction procedure is transparent, whenever we subtract $\mathrm{d}g$ to reduce, add g onto f.

For faster variants, this is not so simple!

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Spaces of differentials W_t, indexed by degree, each of dimension $2g$.

Goal

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Relations in the de Rham cohomology $\rightsquigarrow$ linear maps $R(t)\colon W_t \to W_{t-1}\,\forall t$, with $R(t)\omega \sim \omega$. Want to evaluate

$$W_t \ni \omega \mapsto R(1)R(2)\cdots R(t-1)R(t)\omega \in W_0$$

$O(\sqrt{p})$

Key fact

Entries of $R(t)$ are fractions of *linear* functions of t, with $\mathbf{Z}_p$ coefficients; work of Bostan-Gaudry-Schost (& Harvey) $\implies$ products can be interpolated

$R(a, b) = R(a+1) \cdots R(b) \rightsquigarrow R(a+1+t, b+t)$

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Vital remark

We must use evaluations of primitives here, instead of trying to compute f as a power series.

A problem and a solution

Stumbling block

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Horner to the rescue!

Instead of computing a series $\sum_{i=0}^{N} a_i x^i$ by computing sequentially

$$\left(\sum_{i=t}^{N} a_i x^i\right)_{t=N,N-1,\dots,0}$$

we can instead compute

$$((\cdots((a_N)x + a_{N-1})x + \cdots)x + a_0)$$

from the inside to the out. This *is* an iterated composition of linear functions, each of which is linear in the index t.

Explicit recurrence

In matrix form we augment the (numerators of) the reduction matrices:

$$
\begin{array}{c|c}
 & \begin{array}{cccc} y^{-2t}\,\mathrm{d}x/2y & \cdots & x^{2g-1}y^{-2t}\,\mathrm{d}x/2y & f(P) \end{array} \\
\begin{array}{c} y^{-2(t-1)}\,\mathrm{d}x/2y \\ \vdots \\ x^{2g-1}y^{-2(t-1)}\,\mathrm{d}x/2y \\ f(P) \end{array} &
\left(\begin{array}{ccc|c}
(2t-1)r_{0,0}+2s'_{0,0} & \cdots & (2t-1)r_{2g-1,0}+2s'_{2g-1,0} & \\
\vdots & \ddots & \vdots & \\
(2t-1)r_{0,2g-1}+2s'_{0,2g-1} & \cdots & (2t-1)r_{2g-1,2g-1}+2s'_{2g-1,2g-1} & \\
\hline
-S_0(x) & \cdots & -S_{2g-1}(x) & y^{-2}D_V(t)
\end{array}\right)
\end{array}
$$

so that we keep in memory a vector $v \in W_t \times \mathbf{Q}_p$ which gives the evaluation at the end.

Many integrals simultaneously

We may wish to do this with multiple points in several residue disks. Instead of repeating the whole procedure (repeating computing the Frobenius matrix), augment with many points.

$$\begin{array}{cc}
 & \begin{array}{cccccc} y^{-2t}\,\mathrm{d}x/2y & \cdots & x^{2g-1}y^{-2t}\,\mathrm{d}x/2y & f(P_1) & \cdots & f(P_L) \end{array} \\
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(2t-1)r_{0,2g-1}+2s'_{0,2g-1} & \cdots & (2t-1)r_{2g-1,2g-1}+2s'_{2g-1,2g-1} & & & \\
\hline
-S_0(x(P_1)) & \cdots & -S_{2g-1}(x(P_1)) & y^{-2}(P_1)D_V(t) & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-S_0(x(P_L)) & \cdots & -S_{2g-1}(x(P_L)) & 0 & \cdots & y(P_L)^{-2}D_V(t)
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Note

This matrix and its iterates have the same fixed form, when running BGS don't try and interpolate entries that are always 0 $\rightsquigarrow$ better run time.

Thanks for listening!

Questions/comments?