# Motivation for $p$-adic modular forms

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STAGE

## Overview

These slides are available online (in handout form) at

https://alexjbest.github.io/talks/
motivation-for-p-adic-modular-forms/slides_h.pdf

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**Goal:** Introduce, post hoc, motivation for Katz's definition of $p$-adic modular forms, especially to motivate Serre's $\partial$ operator.

## Recall

A *modular form* of weight $k$ is a function

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s.t.

1. $\forall \lambda \in R^\times, f(E, \lambda\omega) = \lambda^{-k} f(E, \omega)$
2. $f(E, \omega)$ is isomorphism invariant.
3. $f$ is functorial w.r.t. $R$.

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3. $f$ is functorial w.r.t. $R$.

we then have

$$f(E, \omega) \cdot \omega^{\otimes k} \in \Gamma(R, \underbrace{\pi_*(\Omega^1_{E/R})^{\otimes k}}_{\underline{\omega}_{E/R}})$$

## De Rham cohomology

The sheaf of values $\underline{\omega}_{E/R}$ is a subsheaf of the de Rham cohomology of $E/R$:

$$0 \to \underline{\omega}_{E/R} \to \overbrace{H^1_{\mathrm{dR}}(E/R)}^{:=\mathbb{H}^1(E,\Omega^\bullet_{E/R})} \to \underbrace{H^1(E,\mathcal{O}_E)}_{=\underline{\omega}^{\otimes -1}_{E/R}} \to 0$$

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assuming $1/6 \in R$ we can canonically split this sequence:

Fixing $(E, \omega)/R$ we have a unique pair of meromorphic functions with poles only at $\infty$, of orders 2 and 3 resp., denoted by $X, Y$ so that

$$\omega = \frac{\mathrm{d}X}{Y} \text{ and } E\colon Y^2 = 4X^3 - g_2 X - g_3, \ g_i \in R$$

Then we have an inclusion of 2-term complexes

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this induces an isomorphism on $\mathbb{H}^1$. Moreover for $i > 0$,

$$H^i(E, \mathcal{O}_E(\infty)) = 0$$
$$H^i(E, \Omega^1_{E/R}(2\infty)) = 0$$

giving

$$\begin{aligned}
H^1_{\mathrm{dR}}(E/R) &\cong \mathbb{H}^1(E, \mathcal{O}_E(\infty) \to \Omega^1_{E/R}(2\infty)) \\
&= \mathrm{coker}(H^0(E, \mathcal{O}_E(\infty)) \to H^0(E, \Omega^1_{E/R}(2\infty))) \\
&= \mathrm{coker}(R \xrightarrow{0} H^0(E, \Omega^1_{E/R}(2\infty))) \\
&= H^0(E, \Omega^1_{E/R}(2\infty)) \\
&\ni \underbrace{\frac{\mathrm{d}X(E,\omega)}{Y(E,\omega)}}_{=\omega}, \underbrace{X(E,\omega) \cdot \omega}_{=\eta}
\end{aligned}$$

## How does $R^\times$ act?

By uniqueness

$$X(E, \lambda\omega) = \lambda^{-2} \cdot Y(E, \omega)$$
$$Y(E, \lambda\omega) = \lambda^{-3} \cdot Y(E, \omega)$$
$$g_2(E, \lambda\omega) = \lambda^{-4} g_2(E, \omega)$$
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hence

$$\lambda\omega = \frac{\mathrm{d}X(E, \lambda\omega)}{Y(E, \lambda\omega)} \quad \text{and} \quad \lambda^{-1}\eta = \frac{X(E, \lambda\omega)\mathrm{d}X(E, \lambda\omega)}{Y(E, \lambda\omega)}$$

$$H^1_{\mathrm{dR}}(E/R) \simeq \underline{\omega}_{E/R} \oplus \underline{\omega}_{E/R}^{-1}$$

$$\mathrm{Symm}^k\left(H^1_{\mathrm{dR}}(E/R)\right) \simeq \left(\underline{\omega}_{E/R}\right)^{\otimes k} \oplus \left(\underline{\omega}_{E/R}\right)^{\otimes k-2} \oplus \cdots \oplus \left(\underline{\omega}_{E/R}\right)^{\otimes -k}$$

## Connections

Let $f : S \to T$ be a smooth $T$-scheme, $\mathcal{E}$ a quasi-coherent sheaf of $\mathcal{O}_S$-modules. A *connection* on $\mathcal{E}$ is a homomorphism

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{S/T}$$

of abelian sheaves satisfying the "Leibniz rule"

$$\nabla(ge) = g\nabla(e) + \mathrm{d}g \otimes e$$

where $g$ and $e$ are sections of $\mathcal{O}_S$ and $\mathcal{E}$, respectively, over an open subset of $S$ and $\mathrm{d} : \mathcal{O}_S \to \Omega^1_{S/T}$ the exterior derivative.

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Given an element of the tangent space $t \in (\Omega^1_{S/T})^*$ we can define

$$\nabla_t : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{S/T} \to \mathcal{E}$$

by "contraction".

## The Gauss-Manin connection – complex case

Let $R$ be the ring of holomorphic functions of $\tau$, and $E$ be the relative elliptic curve

$$\mathbf{C}/\mathbf{Z} + \mathbf{Z}\tau$$

which can be expressed as $y^2 = 4x^3 - \frac{E_4}{12}x + \frac{E_6}{216}$, $E_i \in R$.

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The de Rham homology $H_1^{\mathrm{dR}}(E/R)$ is then free on $\gamma_1, \gamma_2$, the paths $\overrightarrow{0\tau}, \overrightarrow{01}$ respectively.

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To give the Gauss-Manin connection in this context we need only define $\nabla_\tau$, we do so via the dual connection on $H_1^{\mathrm{dR}}(E/R)$ as

$$\int_{\gamma_1} \nabla_\tau(\xi) = \frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\gamma_1} \xi \quad \text{for} \quad \xi \in H_{\mathrm{dR}}^1(E/R), \text{ and } 1 = 1, 2$$

## Computation of the connection matrix

Let

$$\omega = \frac{\mathrm{d}x}{y}, \ \eta = \frac{x\,\mathrm{d}x}{y}$$

Poincaré duality gives elements $\gamma_i \in H^1_{\mathrm{dR}}(E/R)$ also, satisfying

$$\langle \gamma_2, \gamma_1 \rangle = 1 = - \langle \gamma_1, \gamma_2 \rangle$$

$$\langle \gamma_1, \gamma_1 \rangle = 0 = \langle \gamma_2, \gamma_2 \rangle$$

we then define

$$\omega_i = \int_{\gamma_i} \omega = \langle \omega, \gamma_i \rangle, \ \eta_i = \int_{\gamma_i} \eta = \langle \eta, \gamma_i \rangle \in R$$

so that we have

$$\begin{pmatrix} \omega_1 & -\omega_2 \\ \eta_1 & -\eta_2 \end{pmatrix} \begin{pmatrix} \gamma_2 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

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to invert this we note that

$$\eta_1 \omega_2 - \eta_1 \omega_2 = 2\pi i$$

so

$$2\pi i \begin{pmatrix} \gamma_2 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} -\eta_2 & \omega_2 \\ -\eta_1 & \omega_1 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

to which we want to apply $\nabla_\tau$

$$\int_{\gamma_i} \nabla_\tau(\gamma_j) = \frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\gamma_i} \gamma_j = 0$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \nabla_\tau \left( \begin{pmatrix} -\eta_2 & \omega_2 \\ -\eta_1 & \omega_1 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix} \right)$$

$$= \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}\tau}\eta_2 & \frac{\mathrm{d}}{\mathrm{d}\tau}\omega_2 \\ -\frac{\mathrm{d}}{\mathrm{d}\tau}\eta_1 & \frac{\mathrm{d}}{\mathrm{d}\tau}\omega_1 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix} + \begin{pmatrix} -\eta_2 & \omega_2 \\ -\eta_1 & \omega_1 \end{pmatrix} \begin{pmatrix} \nabla_\tau(\omega) \\ \nabla_\tau(\eta) \end{pmatrix}$$

$$\left( \begin{array}{c} 0 \\ 0 \end{array} \right) = \nabla_\tau \left( \left( \begin{array}{cc} -\eta_2 & \omega_2 \\ -\eta_1 & \omega_1 \end{array} \right) \left( \begin{array}{c} \omega \\ \eta \end{array} \right) \right)$$

$$= \left( \begin{array}{cc} -\frac{\mathrm{d}}{\mathrm{d}\tau}\eta_2 & \frac{\mathrm{d}}{\mathrm{d}\tau}\omega_2 \\ -\frac{\mathrm{d}}{\mathrm{d}\tau}\eta_1 & \frac{\mathrm{d}}{\mathrm{d}\tau}\omega_1 \end{array} \right) \left( \begin{array}{c} \omega \\ \eta \end{array} \right) + \left( \begin{array}{cc} -\eta_2 & \omega_2 \\ -\eta_1 & \omega_1 \end{array} \right) \left( \begin{array}{c} \nabla_\tau(\omega) \\ \nabla_\tau(\eta) \end{array} \right)$$

so we get

$$\left( \begin{array}{c} \nabla_\tau(\omega) \\ \nabla_\tau(\eta) \end{array} \right) = \frac{-1}{2\pi i} \left( \begin{array}{cc} \omega_1 & -\omega_2 \\ \eta_1 & -\eta_2 \end{array} \right) \left( \begin{array}{cc} -\eta_2' & \omega_2' \\ -\eta_1' & \omega_1' \end{array} \right) \left( \begin{array}{c} \omega \\ \eta \end{array} \right)$$

$$= \frac{-1}{2\pi i} \left( \begin{array}{cc} \eta_1'\omega_2 - \eta_2'\omega_1 & \omega_1\omega_2' - \omega_2\omega_1' \\ \eta_2\eta_1' - \eta_1\eta_2' & \eta_1\omega_2' - \eta_2\omega_1' \end{array} \right) \left( \begin{array}{c} \omega \\ \eta \end{array} \right)$$

In fact $\omega_1 = \tau$ and $\omega_2 = 1$ so that $\omega_1' = 1, \omega_2' = 0$ and $\eta_1 - \tau\eta_2 = 2\pi i$, giving $\eta_1' - \tau\eta_2' = \eta_2$ and we simplify

$$\begin{pmatrix} \nabla_\tau(\omega) \\ \nabla_\tau(\eta) \end{pmatrix} = \frac{-1}{2\pi i} \begin{pmatrix} \eta_1'\omega_2 - \eta_2'\omega_1 & \omega_1\omega_2' - \omega_2\omega_1' \\ \eta_2\eta_1' - \eta_1\eta_2' & \eta_1\omega_2' - \eta_2\omega_1' \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

$$= \frac{-1}{2\pi i} \begin{pmatrix} \eta_1' - \eta_2'\tau & -1 \\ \eta_2\eta_1' - \eta_1\eta_2' & -\eta_2 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

$$= \frac{-1}{2\pi i} \begin{pmatrix} \eta_2 & -1 \\ (\eta_2)^2 - 2\pi i\eta_2' & -\eta_2 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

purely in terms of $\eta_2$.

Let $q = e^{2\pi i \tau}$, and

$$P(q) = E_2(q) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n, \text{ where } \sigma_1(n) = \sum_{d \geq 1, d \mid n} d$$

then

**Lemma**

$$\eta_2 = -\sum_m \sum_n {}' \frac{1}{(m\tau + n)^2} = \frac{-\pi^2}{3} P$$

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$$\eta = X \, dX/Y = \wp(z) \, dz = -\, d\zeta$$

where

$$\zeta = \frac{1}{z} + \sum_m \sum_n {}' \frac{1}{z - m\tau - n} + \frac{1}{m+n} + \frac{z}{(m\tau + n)^2}$$

□

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$$\eta_2 = \int_{\gamma_2} \eta = \int_0^1 (-\, d\zeta(z)) = \int_z^{z+1} (-\, d\zeta(z)) = \zeta(z) - \zeta(z+1)$$

$$= \frac{1}{2} - \frac{1}{z+1} + \sum_m \sum_n {}' \left\{ \frac{1}{z - m\tau - n} - \frac{1}{z - m\tau - n + 1} - \frac{1}{(m\tau + n)^2} \right.$$

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$\square$

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$$= \sum_m \sum_n {}' \frac{1}{(m\tau + n)^2}$$

$\square$

## Aside on $\eta_1$

Similarly

$$\eta_1 = \zeta(z) - \zeta(z + \tau) = -\sum_n \sum_m{}' \frac{\tau}{(m\tau + n)^2}$$

giving

$$\eta_2(-1/\tau) = \tau_{\eta_1}(\tau)$$

so

$$\frac{\eta_2(-1/\tau)}{\tau} - \tau\eta_2(\tau) = 2\pi i$$

$$\implies \eta_2(-1/\tau) = \tau^2\eta_2(\tau) + 2\pi i\tau$$

$$P(-1/\tau) = \tau^2 P(\tau) - \frac{6i\tau}{\pi}$$

In conclusion we have

$$
\left( \begin{array}{c} \nabla_\tau(\omega) \\ \nabla_\tau(\eta) \end{array} \right) = \frac{1}{2\pi i} \left( \begin{array}{cc} \frac{\pi^2 P}{3} & 1 \\ \frac{\pi^4}{9} P^2 - \frac{12}{2\pi i} P' & -\frac{\pi^2}{3} P \end{array} \right) \left( \begin{array}{c} \omega \\ \eta \end{array} \right)
$$

We can consider this in terms of $\omega_{\mathrm{can}} = 2\pi i \omega$, $\eta_{\mathrm{can}} = \frac{1}{2\pi i} \eta$ and $\theta = \frac{1}{2\pi i}$, $\frac{\mathrm{d}}{\mathrm{d}\tau} = q \frac{\mathrm{d}}{\mathrm{d}q}$ For the Tate curve over $\boldsymbol{C}((q))$, we have $\omega_{can} = \mathrm{d}t/t$.

$$
\nabla(\theta) \left( \begin{array}{c} \omega \\ \eta \end{array} \right) = \left( \begin{array}{cc} \frac{-P}{12} & \frac{-1}{4\pi^2} \\ \frac{\pi^2}{36} \left( P^2 - 12\theta P \right) & \frac{P}{12} \end{array} \right) \left( \begin{array}{c} \omega \\ \eta \end{array} \right)
$$

so

$$
\nabla(\theta) \left( \begin{array}{c} \omega_{\mathrm{can}} \\ \eta_{\mathrm{can}} \end{array} \right) = \left( \begin{array}{cc} \frac{-P}{12} & 1 \\ \frac{P^2 - 12\theta P}{144} & \frac{P}{12} \end{array} \right) \left( \begin{array}{c} \omega_{\mathrm{can}} \\ \eta_{\mathrm{can}} \end{array} \right)
$$

## Kodaira-Spencer

As before let $f \colon S \to T$ be a smooth $T$-scheme, $E/S$ an elliptic curve.

We can take a nowhere vanishing invariant differential $\omega \in \Omega^1_{E/S}$ and a derivation $D \in Der(S/T)$ and form the cup product

$$\langle \omega, \nabla(D)\omega \rangle \in \mathcal{O}_S$$

as we have $\omega$ in both sides this defines a pairing between $Der(S/T)$ and $\underline{\omega}^2$ for

$$\underline{\omega} = f_* \Omega^1_{E/S}$$

this gives a map

$$\underline{\omega}^{\otimes 2} \to \Omega^1_{S/T}$$

**Lemma**
*On the Tate curve over $\mathbf{Z}((q))$*

$$\omega_{can}^{\otimes 2} \mapsto \mathrm{d}q/q$$

**Proof.**
We must check that

$$\langle \omega_{can}, \nabla(\theta)\omega_{can} \rangle = 1$$

but we already found

$$\nabla(\theta)\left(\omega_{\mathrm{can}}\right) = \frac{-P}{12}\omega_{\mathrm{can}} + \eta_{\mathrm{can}}$$

$\square$

### The Gauss-Manin connection

To determine if a $q$-expansion $f(q) \in \boldsymbol{C}[[q]]$ is a modular form of weight $k$ we must check if

$$f(q)(\omega_{can})^{\otimes k}$$

extends to all of $\underline{\omega}^{\otimes k}$. Viewing this inside of $H^1_{\mathrm{dR}}$ we ask instead that there exist $a, b \in \boldsymbol{N}$ with $a - b = k$ such that

$$f(q)(\omega_{can})^{\otimes a}(\eta_{can})^{\otimes b}$$

extends to

$$\mathrm{Symm}^{a+b}\left(H^1_{\mathrm{dR}}(E/S)\right)$$

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The Gauss-Manin connection is

$$\nabla : H^1_{\mathrm{dR}}(E/S) \longrightarrow H^1_{\mathrm{dR}}(E/S) \otimes \Omega^1_{S/T}$$

We can tensor to get

$$\mathsf{Symm}^k\left(\mathrm{H}^1\right) \longrightarrow \mathsf{Symm}^k\left(\mathrm{H}^1\right) \otimes \Omega^1_\mathrm{S}/\mathrm{T}$$

if

$$\Omega^1_{S/T} \simeq \underline{\omega}^{\otimes 2}$$

we can view this as

$$\sum_{j=0}^{k} \underline{\omega}^{\otimes k-2j} \longrightarrow \sum_{j=0}^{k} \underline{\omega}^{\otimes k-2j} \otimes \omega^{\otimes 2} = \sum_{j=0}^{k} \underline{\omega}^{\otimes k+2-2j}$$

## The image of $f(q)$

Under the Gauss-Manin connection

$$f \mapsto \theta(f) \cdot (\omega_{can})^{\otimes 2} \cdot (\omega_{can})^{\otimes a} \cdot (\eta_{can})^{\otimes b}$$
$$+ f \cdot a \cdot (\omega_{can})^{\otimes a-1} \left( \frac{-P}{12} \omega_{can} + \eta_{can} \right) \otimes (\omega_{can})^{\otimes 2} \otimes (\eta_{can})^b$$
$$+ f \cdot (\omega_{can})^{\otimes a} \cdot b \cdot (\eta_{can})^{\otimes b-1} \left( \frac{P^2 - 12\theta P}{144} \omega_{can} + \frac{P}{12} \eta_{can} \right) \cdot (\omega_{can})^{\otimes}$$

$$= \left( \theta(f) - (a-b)f\frac{P}{12} \right) (\omega_{can})^{\otimes a+2} (\eta_{can})^{\otimes b}$$
$$+ (af) (\omega_{can})^{\otimes a+1} (\eta_{can})^{\otimes b+1}$$
$$+ \left( bf \frac{P^2 - 12\theta P}{144} \right) (\omega_{can})^{\otimes a+3} (\eta_{can})^{\otimes b-1}$$

So if $f$ is modular of weight $k = a - b$ then

$$\begin{cases} \theta(f) - (a-b)f\frac{P}{12}, & \text{is modular of weight } a+2-b \\ af, & \text{is modular of weight } a-b \\ bf\frac{P^2-12\theta P}{144}, & \text{is modular of weight } a+3-b+1 \end{cases}$$

the operator

$$\partial(F) = 12\theta(f) - kPf$$

due to Serre therefore raises the weight by 2.

**Corollary**

$$P^2 - 12\theta P$$

*is modular of weight 4 and hence*

$$P^2 - 12\theta P = E_4 = Q$$

**Proof.**
Start with $f = 1$ modular of weight $1 - 1 = 0$ to see $P^2 - 12\theta P$
has weight 4. $\square$

**Corollary (Deligne)**

$$P = \frac{\theta\Delta}{\Delta}$$

**Proof.**

$$\theta(\Delta) - \Delta P = 0$$

as it lies in weight 14 and level 1.

$\square$