Szemerédi's regularity lemma

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Recap: Studying Manin's conjecture and (equi)-distribution of rational points on a Fano variety. Another instance of pseudo-randomness emerging with scale is in extremal combinatorics / graph theory.

Tao (paraphrased): "The various proofs of Szemerédi's theorem and related theorems and proofs using measure theory, ergodic theory, graph theory, hypergraph theory, probability theory, information theory, and Fourier analysis share a number of common features and serve as a "Rosetta stone" for connecting these fields, notably they often use dichotomy between randomness and structure".

This time: Give more detail on Szemerédi's regularity lemma (SRL) and its proof, and the reduction to Roth's theorem / Szemerédi's theorem.

The statement

Slogan: The vertices of a sufficiently large graph can be partitioned into a fixed number of subsets in a way that the interactions between each behave pseudorandomly.

There are several variants of SRL, some place more restrictions on the partition (and so appear at first sight stronger).

Definition 1. *Let $G = (V, E)$ be a graph and $A, B \subseteq V$ two disjoint sets. The density of edges between A and B is*

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

We call a pair (A, B) ϵ-regular (or uniform) for a given $\epsilon > 0$, if for all $X \subseteq A, Y \subseteq B$ where $|X| \geq \epsilon|A|$ and $|Y| \geq \epsilon|B|$, we have:

$$|d(X, Y) - d(A, B)| \leq \epsilon$$

Definition 2. *A partition $V_1, \dots, V_k$ of the vertices of a graph is said to be an equipartition if it is as balanced as possible, i.e.*

$$\max\{||V_k| - |V_j||\} \leq 1 \text{ or } \left\lfloor \frac{|V|}{k} \right\rfloor \leq |V_j| \leq \left\lceil \frac{|V|}{k} \right\rceil$$

Definition 3. *A partition $V_1, \dots, V_k$ of the vertices of a graph is said to be ϵ-regular if most pairs (V_i, V_j) are ϵ-regular, in the sense that at most*

$$\epsilon \binom{k}{2} \text{ are not } \epsilon\text{-regular}$$

Theorem 4 (Szemerédi)**.** *Let $\varepsilon > 0$, and let l be a natural number. Then there exists an integer L such that every graph G with $|G| \geq l$ has an ε-uniform equipartition into m parts for some m such that $l \leq m \leq L$.*

Note that L does not depend on G or even the size of G, so that for large enough graphs we can consider this a partition into a small number of pieces relative to the size of the graph.

Applications

Theorem 5 (Roth)**.** *A subset of the natural numbers with positive upper density contains a 3-term arithmetic progression.*

Szemerédi was motivated by generalizing this result and proved:

Theorem 6 (Szemerédi)**.** *A subset of the natural numbers with positive upper density contains a k-term arithmetic progression for any k.*

Method

Theorem 7 (Roth')**.** *For every $\delta > 0$, there exists an n_0 such that for any $n \geq n_0$ and any subset $A \subseteq \{1, \dots, n\}$ satisfying $|A| \geq \delta n$, there are distinct elements $a, b, c \in A$ such that $a + c = 2b$.*

To prove this we instead let

$$B = \{(x, y) : x - y \in A\} \subseteq \{1, \dots, 2n\}^2$$

Definition 8 (Corners)**.** *A* corner *in a set $B \subseteq \{1, \dots, n\}^2$ is a triple of the form $(x, y), (x+h, y), (x, y+h) \in B$, $0 < h$ (anticorner if $h < 0$).*

So, given a corner in B we get a 3 term AP (a, b, c) in A from $(x - y, x + h - y, x - y - h)$.

Theorem 9 (Corners theorem)**.** *For every $\delta > 0$, there exists an n_0 such that for any $n \geq n_0$ and any subset $B \subseteq \{1, \dots, n\}^2$ satisfying $|B| \geq \delta n^2$, there is a corner in B.*

Reduce to the weak corners theorem (as above but allowing anti-corners).

Then construct a tripartite graph where the triangles correspond to (anti or trivial)-corners of B and so that all triangles are edge disjoint.

The construction is to have vertices for horizontal, vertical and diagonal lines in $\{1, \dots, 2n\}$ and put an edge when two such lines meet at a point of B.

There are at least δn^2 triangles in this graph from trivials alone so to remove all of them we must remove at least δn^2 edges.

Theorem 10 (Triangle Removal Lemma)**.** *For all $1 \geq \delta > 0$, there exists $\epsilon > 0$ such that any graph on n vertices with less than or equal to ϵn^3 triangles can be made triangle-free by removing at most δn^2 edges.*

(If there are not too many triangles you can remove a small number of edges to remove all triangles.)

(this is a special case of the more general *Graph removal lemma* and the *hypergraph removal lemma* (used for the full Szemerédi lemma on k-term APs).)

Now the above graph must have at least ϵn^3 triangles as the triangle removal lemma does not apply.

So we have at least

$$\epsilon n^3 - \delta n^2$$

nontrivial triangles, so pick n such that $\epsilon n > \delta$ and we are done.

Roth's theorem can be proved directly from the TRL, but the *Corners theorem* is in some sense a stronger version of Roth.

We can prove the TRL from the so called triangle counting lemma, from SRL (though other proofs are available).

Proof of SRL

The proof idea is fairly straightforward in the outline:

- We define a (bounded above) quantity called *energy* of an equipartition, that unless the equipartition is ϵ-regular can be increased by at least a fixed positive amount (via some modification of the equipartition), without adding too many sets to the equipartition.

- We start with a trivial equipartition and may inductively apply this process which must eventually stop (after a number of steps bounded independently of the input graph) at which point we have proved the lemma.

The details of this are quite involved however!

Definition 11. *The* energy *of a partition $P = \{V_1, \dots, V_k\}$ is*

$$0 \leq q_G(P) = \frac{1}{k^2} \sum_{1 \leq i < j \leq k} d_G\left(V_i, V_j\right)^2 \leq 1$$

Lemma 12. *Let G be a graph of order n with an equipartition $V = \bigcup_{i=0}^k C_i$*

$$|C_1| = |C_2| = \cdots = |C_k| = c \geq 2^{3k+1}.$$

Suppose that the partition $\mathcal{P} = (C_i)_{i=0}^k$ is not ε-uniform, where $0 < \varepsilon < \frac{1}{2}$ and $2^{-k} \leq \varepsilon^5/8$. Then there is an equitable partition $\mathcal{P}' = \left(C'_i\right)_{i=0}^{\ell}$ with $\ell = k(4^k - 2^{k-1})$ such that

$$q\left(\mathcal{P}'\right) \geq q(\mathcal{P}) + \frac{\varepsilon^5}{2}$$

To prove this we take, for each (C_i, C_j) non-uniform, some sets $C_{ij} \subset C_i$, $C_{ji} \subset C_j$ witnessing this so that

$$\left|C_{ij}\right| \geq \varepsilon\left|C_i\right| = \varepsilon c, \left|C_{ji}\right| \geq \varepsilon\left|C_j\right| = \varepsilon c$$

$$\left|d\left(C_{ij}, C_{ji}\right) - d\left(C_i, C_j\right)\right| \geq \varepsilon$$

.

We would like to partition simultaneously all possible C_{ij} into new sets C_h so that each C_{ij} is a union of a bunch of C_h's, which would have larger energy, this isn't quite possible, but we can "atomise" each C_i by making equivalent points not distinguishable by being in different C_{ij}s. We then pick $H = 4^k - 2^{k-1}$ different $\lfloor c/4^k \rfloor$-subsets, all contained in some atom of C_i.

This new partition can be shown to have an energy of at least $\epsilon^5/4$ more.