

# Motivation for $p$ -adic modular forms

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STAGE

# Overview

These slides are available online (in handout form) at

[https://alexjbest.github.io/talks/  
motivation-for-p-adic-modular-forms/slides\\_h.pdf](https://alexjbest.github.io/talks/motivation-for-p-adic-modular-forms/slides_h.pdf)



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**Goal:** Introduce, post hoc, motivation for Katz's definition of  $p$ -adic modular forms, especially to motivate Serre's  $\partial$  operator.



# Recall

A *modular form* of weight  $k$  is a function

$$f: \{(E \xrightarrow{\pi} R \text{ an ell. curve}, \omega \in \Gamma(E, \Omega_{E/R}^1) \text{ nowhere vanishing})\} \rightarrow R$$



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s.t.

1.  $\forall \lambda \in R^\times, f(E, \lambda\omega) = \lambda^{-k} f(E, \omega)$
2.  $f(E, \omega)$  is isomorphism invariant.
3.  $f$  is functorial w.r.t.  $R$ .



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we then have

$$f(E, \omega) \cdot \omega^{\otimes k} \in \Gamma(R, \underbrace{\pi_*(\Omega_{E/R}^1)}_{\underline{\omega}_{E/R}}^{\otimes k})$$



# De Rham cohomology

The sheaf of values  $\underline{\omega}_{E/R}$  is a subsheaf of the de Rham cohomology of  $E/R$ :

$$0 \rightarrow \underline{\omega}_{E/R} \rightarrow \overbrace{H_{\mathrm{dR}}^1(E/R)}^{:= \mathbb{H}^1(E, \Omega_{E/R}^\bullet)} \rightarrow \underbrace{H^1(E, \mathcal{O}_E)}_{= \underline{\omega}_{E/R}^{\otimes -1}} \rightarrow 0$$



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assuming  $1/6 \in R$  we can canonically split this sequence:

Fixing  $(E, \omega)/R$  we have a unique pair of meromorphic functions with poles only at  $\infty$ , of orders 2 and 3 resp., denoted by  $X, Y$  so that

$$\omega = \frac{dX}{Y} \text{ and } E: Y^2 = 4X^3 - g_2X - g_3, g_i \in R$$





Then we have an inclusion of 2-term complexes

$$(\mathcal{O}_E \rightarrow \Omega_{E/R}^1) \subseteq (\mathcal{O}_E(\infty) \rightarrow \Omega_{E/R}^1(2\infty)),$$



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this induces an isomorphism on  $\mathbb{H}^1$ . Moreover for  $i > 0$ ,

$$H^i(E, \mathcal{O}_E(\infty)) = 0$$

$$H^i(E, \Omega_{E/R}^1(2\infty)) = 0$$

giving

$$\begin{aligned} H_{\mathrm{dR}}^1(E/R) &\cong \mathbb{H}^1(E, \mathcal{O}_E(\infty) \rightarrow \Omega_{E/R}^1(2\infty)) \\ &= \operatorname{coker}(H^0(E, \mathcal{O}_E(\infty)) \rightarrow H^0(E, \Omega_{E/R}^1(2\infty))) \\ &= \operatorname{coker}(R \xrightarrow{0} H^0(E, \Omega_{E/R}^1(2\infty))) \\ &= H^0(E, \Omega_{E/R}^1(2\infty)) \\ &\ni \underbrace{\frac{dX(E, \omega)}{Y(E, \omega)}}_{=\omega}, \underbrace{X(E, \omega) \cdot \omega}_{=\eta} \end{aligned}$$



# How does $R^\times$ act?

By uniqueness

$$X(E, \lambda\omega) = \lambda^{-2} \cdot Y(E, \omega)$$

$$Y(E, \lambda\omega) = \lambda^{-3} \cdot Y(E, \omega)$$

$$g_2(E, \lambda\omega) = \lambda^{-4} g_2(E, \omega)$$

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hence

$$\lambda\omega = \frac{dX(E, \lambda\omega)}{Y(E, \lambda\omega)} \quad \text{and} \quad \lambda^{-1}\eta = \frac{X(E, \lambda\omega)dX(E, \lambda\omega)}{Y(E, \lambda\omega)}$$

$$H_{\text{dR}}^1(E/R) \simeq \underline{\omega}_{E/R} \oplus \underline{\omega}_{E/R}^{-1}$$

$$\text{Sym}^k(H_{\text{dR}}^1(E/R)) \simeq \left(\underline{\omega}_{E/R}\right)^{\otimes k} \oplus \left(\underline{\omega}_{E/R}\right)^{\otimes k-2} \oplus \cdots \oplus \left(\underline{\omega}_{E/R}\right)^{\otimes -k}$$



# Connections

Let  $f : S \rightarrow T$  be a smooth  $T$ -scheme,  $\mathcal{E}$  a quasi-coherent sheaf of  $\mathcal{O}_S$ -modules. A *connection* on  $\mathcal{E}$  is a homomorphism

$$\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_S} \Omega_{S/T}^1$$

of abelian sheaves satisfying the "Leibniz rule"

$$\nabla(g e) = g \nabla(e) + dg \otimes e$$

where  $g$  and  $e$  are sections of  $\mathcal{O}_S$  and  $\mathcal{E}$ , respectively, over an open subset of  $S$  and  $d : \mathcal{O}_S \rightarrow \Omega_{S/T}^1$  the exterior derivative.

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Given an element of the tangent space  $t \in (\Omega_{S/T}^1)^*$  we can define

$$\nabla_t : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{S/T}^1 \rightarrow \mathcal{E}$$

by "contraction".

# The Gauss-Manin connection – complex case

Let  $R$  be the ring of holomorphic functions of  $\tau$ , and  $E$  be the relative elliptic curve

$$\mathcal{C}/\mathcal{Z} + \mathcal{Z}\tau$$

which can be expressed as  $y^2 = 4x^3 - \frac{E_4}{12}x + \frac{E_6}{216}$ ,  $E_i \in R$ .



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The de Rham homology  $H_1^{\text{dR}}(E/R)$  is then free on  $\gamma_1, \gamma_2$ , the paths  $\overset{\rightarrow}{0\tau}, \overset{\rightarrow}{01}$  respectively.





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To give the Gauss-Manin connection in this context we need only define  $\nabla_\tau$ , we do so via the dual connection on  $H_1^{\text{dR}}(E/R)$  as

$$\int_{\gamma_1} \nabla_\tau(\xi) = \frac{d}{d\tau} \int_{\gamma_1} \xi \quad \text{for} \quad \xi \in H_1^{\text{dR}}(E/R), \text{ and } 1 = 1, 2$$



# Computation of the connection matrix

Let

$$\omega = \frac{dx}{y}, \eta = \frac{x dx}{y}$$

Poincaré duality gives elements  $\gamma_i \in H_{\text{dR}}^1(E/R)$  also, satisfying

$$\langle \gamma_2, \gamma_1 \rangle = 1 = -\langle \gamma_1, \gamma_2 \rangle$$

$$\langle \gamma_1, \gamma_1 \rangle = 0 = \langle \gamma_2, \gamma_2 \rangle$$

we then define

$$\omega_i = \int_{\gamma_i} \omega = \langle \omega, \gamma_i \rangle, \eta_i = \int_{\gamma_i} \eta = \langle \eta, \gamma_i \rangle \in R$$

so that we have

$$\begin{pmatrix} \omega_1 & -\omega_2 \\ \eta_1 & -\eta_2 \end{pmatrix} \begin{pmatrix} \gamma_2 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$



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to invert this we note that

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$$

so

$$2\pi i \begin{pmatrix} \gamma_2 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} -\eta_2 & \omega_2 \\ -\eta_1 & \omega_1 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

to which we want to apply  $\nabla_\tau$

$$\int_{\gamma_i} \nabla_\tau(\gamma_j) = \frac{d}{d\tau} \int_{\gamma_i} \gamma_j = 0$$



$$\begin{aligned}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \nabla_{\tau} \left( \begin{pmatrix} -\eta_2 & \omega_2 \\ -\eta_1 & \omega_1 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix} \right) \\
&= \begin{pmatrix} -\frac{d}{d\tau}\eta_2 & \frac{d}{d\tau}\omega_2 \\ -\frac{d}{d\tau}\eta_1 & \frac{d}{d\tau}\omega_1 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix} + \begin{pmatrix} -\eta_2 & \omega_2 \\ -\eta_1 & \omega_1 \end{pmatrix} \begin{pmatrix} \nabla_{\tau}(\omega) \\ \nabla_{\tau}(\eta) \end{pmatrix}
\end{aligned}$$



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\end{aligned}$$

so we get

$$\begin{aligned}
\begin{pmatrix} \nabla_{\tau}(\omega) \\ \nabla_{\tau}(\eta) \end{pmatrix} &= \frac{-1}{2\pi i} \begin{pmatrix} \omega_1 & -\omega_2 \\ \eta_1 & -\eta_2 \end{pmatrix} \begin{pmatrix} -\eta'_2 & \omega'_2 \\ -\eta'_1 & \omega'_1 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix} \\
&= \frac{-1}{2\pi i} \begin{pmatrix} \eta'_1\omega_2 - \eta'_2\omega_1 & \omega_1\omega'_2 - \omega_2\omega'_1 \\ \eta_2\eta'_1 - \eta_1\eta'_2 & \eta_1\omega'_2 - \eta_2\omega'_1 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}
\end{aligned}$$



In fact  $\omega_1 = \tau$  and  $\omega_2 = 1$  so that  $\omega'_1 = 1, \omega'_2 = 0$  and  $\eta_1 - \tau\eta_2 = 2\pi i$ , giving  $\eta'_1 - \tau\eta'_2 = \eta_2$  and we simplify

$$\begin{aligned} \begin{pmatrix} \nabla_\tau(\omega) \\ \nabla_\tau(\eta) \end{pmatrix} &= \frac{-1}{2\pi i} \begin{pmatrix} \eta'_1\omega_2 - \eta'_2\omega_1 & \omega_1\omega'_2 - \omega_2\omega'_1 \\ \eta_2\eta'_1 - \eta_1\eta'_2 & \eta_1\omega'_2 - \eta_2\omega'_1 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix} \\ &= \frac{-1}{2\pi i} \begin{pmatrix} \eta'_1 - \eta'_2\tau & -1 \\ \eta_2\eta'_1 - \eta_1\eta'_2 & -\eta_2 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix} \\ &= \frac{-1}{2\pi i} \begin{pmatrix} \eta_2 & -1 \\ (\eta_2)^2 - 2\pi i\eta'_2 & -\eta_2 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix} \end{aligned}$$

purely in terms of  $\eta_2$ .



# Determining $\eta_2$

Let  $q = e^{2\pi i\tau}$ , and

$$P(q) = E_2(q) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n, \text{ where } \sigma_1(n) = \sum_{d \geq 1, d|n} d$$

then

**Lemma**

$$\eta_2 = - \sum_m \sum_n ' \frac{1}{(m\tau + n)^2} = \frac{-\pi^2}{3} P$$



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$$\begin{aligned} \eta_2 &= \int_{\gamma_2} \eta = \int_0^1 (-d\zeta(z)) = \int_z^{z+1} (-d\zeta(z)) = \zeta(z) - \zeta(z+1) \\ &= \frac{1}{2} - \frac{1}{z+1} + \sum_m \sum_n' \left\{ \frac{1}{z - m\tau - n} - \frac{1}{z - m\tau - n + 1} - \frac{1}{(m\tau + n)} \right\} \end{aligned}$$

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where

$$\zeta = \frac{1}{z} + \sum_m \sum_n' \frac{1}{z - m\tau - n} + \frac{1}{m + n} + \frac{z}{(m\tau + n)^2}$$

$$\eta_2 = \frac{1}{z} - \frac{1}{z+1} + \sum_{m \neq 0} \sum_n \frac{-1}{(m\tau + n)^2} + \sum_{n \neq 0} \left\{ \frac{-1}{n^2} + \frac{1}{z - n} - \frac{1}{z + 1 - n} \right\}$$

□

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□

## Aside on $\eta_1$

Similarly

$$\eta_1 = \zeta(z) - \zeta(z + \tau) = - \sum_n \sum'_m \frac{\tau}{(m\tau + n)^2}$$

giving

$$\eta_2(-1/\tau) = \tau \eta_1(\tau)$$

so

$$\begin{aligned} \frac{\eta_2(-1/\tau)}{\tau} - \tau \eta_2(\tau) &= 2\pi i \\ \implies \eta_2(-1/\tau) &= \tau^2 \eta_2(\tau) + 2\pi i \tau \\ P(-1/\tau) &= \tau^2 P(\tau) - \frac{6i\tau}{\pi} \end{aligned}$$



In conclusion we have

$$\begin{pmatrix} \nabla_{\tau}(\omega) \\ \nabla_{\tau}(\eta) \end{pmatrix} = \frac{1}{2\pi i} \begin{pmatrix} \frac{\pi^2 P}{3} & 1 \\ \frac{\pi^4}{9} P^2 - \frac{12}{2\pi i} P' & -\frac{\pi^2}{3} P \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

We can consider this in terms of  $\omega_{\text{can}} = 2\pi i \omega$ ,  $\eta_{\text{can}} = \frac{1}{2\pi i} \eta$  and  $\theta = \frac{1}{2\pi i}$ ,  $\frac{d}{d\tau} = q \frac{d}{dq}$ . For the Tate curve over  $\mathbf{C}((q))$ , we have  $\omega_{\text{can}} = dt/t$ .

$$\nabla(\theta) \begin{pmatrix} \omega \\ \eta \end{pmatrix} = \begin{pmatrix} \frac{-P}{12} & \frac{-1}{4\pi^2} \\ \frac{\pi^2}{36} (P^2 - 12\theta P) & \frac{P}{12} \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

so

$$\nabla(\theta) \begin{pmatrix} \omega_{\text{can}} \\ \eta_{\text{can}} \end{pmatrix} = \begin{pmatrix} \frac{-P}{12} & 1 \\ \frac{P^2 - 12\theta P}{144} & \frac{P}{12} \end{pmatrix} \begin{pmatrix} \omega_{\text{can}} \\ \eta_{\text{can}} \end{pmatrix}$$





As before let  $f: S \rightarrow T$  be a smooth  $T$ -scheme,  $E/S$  an elliptic curve.

We can take a nowhere vanishing invariant differential  $\omega \in \Omega_{E/S}^1$  and a derivation  $D \in \text{Der}(S/T)$  and form the cup product

$$\langle \omega, \nabla(D)\omega \rangle \in \mathcal{O}_S$$

as we have  $\omega$  in both sides this defines a pairing between  $\text{Der}(S/T)$  and  $\underline{\omega}^2$  for

$$\underline{\omega} = f_* \Omega_{E/S}^1$$

this gives a map

$$\underline{\omega}^{\otimes 2} \rightarrow \Omega_{S/T}^1$$



## Lemma

On the Tate curve over  $\mathbb{Z}((q))$

$$\omega_{can}^{\otimes 2} \mapsto dq/q$$

## Proof.

We must check that

$$\langle \omega_{can}, \nabla(\theta) \omega_{can} \rangle = 1$$

but we already found

$$\nabla(\theta)(\omega_{can}) = \frac{-P}{12} \omega_{can} + \eta_{can}$$



# The Gauss-Manin connection

To determine if a  $q$ -expansion  $f(q) \in \mathbf{C}[[q]]$  is a modular form of weight  $k$  we must check if

$$f(q)(\omega_{can})^{\otimes k}$$

extends to all of  $\underline{\omega}^{\otimes k}$ . Viewing this inside of  $H_{dR}^1$  we ask instead that there exist  $a, b \in \mathbf{N}$  with  $a - b = k$  such that

$$f(q)(\omega_{can})^{\otimes a}(\eta_{can})^{\otimes b}$$

extends to

$$\mathrm{Sym}^{a+b}(H_{dR}^1(E/S))$$

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The Gauss-Manin connection is

$$\nabla : H_{dR}^1(E/S) \longrightarrow H_{dR}^1(E/S) \otimes \Omega_{S/T}^1$$

We can tensor to get

$$\mathrm{Sym}^k(H^1) \longrightarrow \mathrm{Sym}^k(H^1) \otimes \Omega_S^1/T$$

if

$$\Omega_{S/T}^1 \simeq \underline{\omega}^{\otimes 2}$$

we can view this as

$$\sum_{j=0}^k \underline{\omega}^{\otimes k-2j} \longrightarrow \sum_{j=0}^k \underline{\omega}^{\otimes k-2j} \otimes \omega^{\otimes 2} = \sum_{j=0}^k \underline{\omega}^{\otimes k+2-2j}$$



# The image of $f(q)$

Under the Gauss-Manin connection

$$\begin{aligned} f &\mapsto \theta(f) \cdot (\omega_{can})^{\otimes 2} \cdot (\omega_{can})^{\otimes a} \cdot (\eta_{can})^{\otimes b} \\ &+ f \cdot a \cdot (\omega_{can})^{\otimes a-1} \left( \frac{-P}{12} \omega_{can} + \eta_{can} \right) \otimes (\omega_{can})^{\otimes 2} \otimes (\eta_{can})^{\otimes b} \\ &+ f \cdot (\omega_{can})^{\otimes a} \cdot b \cdot (\eta_{can})^{\otimes b-1} \left( \frac{P^2 - 12\theta P}{144} \omega_{can} + \frac{P}{12} \eta_{can} \right) \cdot (\omega_{can})^{\otimes} \\ &= \left( \theta(f) - (a-b)f \frac{P}{12} \right) (\omega_{can})^{\otimes a+2} (\eta_{can})^{\otimes b} \\ &+ (af) (\omega_{can})^{\otimes a+1} (\eta_{can})^{\otimes b+1} \\ &+ \left( bf \frac{P^2 - 12\theta P}{144} \right) (\omega_{can})^{\otimes a+3} (\eta_{can})^{\otimes b-1} \end{aligned}$$



So if  $f$  is modular of weight  $k = a - b$  then

$$\begin{cases} \theta(f) - (a - b)f \frac{P}{12}, & \text{is modular of weight } a + 2 - b \\ af, & \text{is modular of weight } a - b \\ bf \frac{P^2 - 12\theta P}{144}, & \text{is modular of weight } a + 3 - b + 1 \end{cases}$$

the operator

$$\partial(F) = 12\theta(f) - kPf$$

due to Serre therefore raises the weight by 2.

## Corollary

$$P^2 - 12\theta P$$

*is modular of weight 4 and hence*

$$P^2 - 12\theta P = E_4 = Q$$

## Proof.

Start with  $f = 1$  modular of weight  $1 - 1 = 0$  to see  $P^2 - 12\theta P$  has weight 4. □



Corollary (Deligne)

$$P = \frac{\theta\Delta}{\Delta}$$

Proof.

$$\theta(\Delta) - \Delta P = 0$$

as it lies in weight 14 and level 1.

