Explicit computation with Coleman integrals

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1. Thank the audience for being awake.

Explicit computation with Coleman integrals

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Why do we integrate things? Logarithms

Take $\frac{dx}{x}$, as a differential on the group $\mathbf{R}^{\times}$,

there are many answers to this question

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Why do we integrate things? Logarithms

Take $\frac{\mathrm{d}x}{x}$, as a differential on the group $\mathbf{R}^\times$, this is translation invariant, i.e. $(a \cdot -)^*(\mathrm{d}x/x) = \mathrm{d}(ax)/ax = \mathrm{d}x/x$, hence

$$\int_1^t \frac{\mathrm{d}x}{x} = \log|t| \colon \mathbf{R}^\times \to \mathbf{R}$$

has the property that

$$\int_1^{ab} \frac{\mathrm{d}x}{x} = \int_a^{ab} \frac{\mathrm{d}x}{x} + \int_1^{a} \frac{\mathrm{d}x}{x} = \int_1^{b} \frac{\mathrm{d}x}{x} + \int_1^{a} \frac{\mathrm{d}x}{x}$$

Integration can define logarithm maps between groups and their tangent spaces.

How do we calculate $\log|t|$? Power series on $\mathbf{R}_{>0}$ and use the relation $\log|t| = \frac{1}{2}\log t^2$

there are many answers to this question

Coleman integration

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For instance near a point α:

$$\omega = \frac{\mathrm{d}(\alpha + x)}{\alpha + x} = \frac{\mathrm{d}x}{\alpha + x} = \frac{1}{\alpha}\sum\left(\frac{-x}{\alpha}\right)^n \mathrm{d}x$$

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so that

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Bad topology!

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But we cannot find C! There is a different choice in each disk.

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Applications: Chabauty-Kim

Minhyong Kim has vastly generalised the above to cases where

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Theorem (Balakrishnan-Dogra-Muller-Tuitman-Vonk)
The (cursed) modular curve $X_{split}(13)$ (of genus 3 and jacobian rank 3), has 7 rational points: one cusp and 6 points that correspond to CM elliptic curves whose mod-13 Galois representations land in normalizers of split Cartan subgroups.

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Their method can also be applied to other interesting curves:

Theorem (WIP B.-Bianchi-Triantafillou-Vonk)
The modular curve $X_0(67)^+$ (of genus 2 and jacobian rank 2), has rational points contained in an explicitly computable finite set of 7-adic points.

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