The (inescapable) p-adics

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VU Master Seminar - Algebra

Linear recurrence sequences

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Example: Fibonacci

$a_0 = 0, a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq k = 2$:

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 41$

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a_n grows exponentially.

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$a_0 = 1, a_1 = 0$ with $a_n = -a_{n-1} - a_{n-2}$

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a_n is periodic now.

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Example: Natural numbers interlaced with zeroes

$a_0 = 1, a_1 = 0, a_2 = 2, a_3 = 0$ with $a_n = 2a_{n-2} - a_{n-4}$

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not periodic but the zeroes *do* have a regular repeating pattern.

The ultimate question

Question

What possible patterns are there for the zeroes of a linear recurrence sequence?

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Observation

A linear recurrence sequence is the Taylor expansion around 0 of a rational function

$$\frac{a_1 + a_2 x + \cdots + a_\ell x^\ell}{b_1 + b_2 x \cdots + b_k x^k}$$

with $b_1 \neq 0$ (so that the expansion makes sense).

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$$\frac{(1+x)^3-x^3}{(1+x)^5-x^5} \leftrightarrow 1,-2,3,-5,10,-20,35,-50,50,0,-175,625,$$
$$-1625,3625,-7250,13125,-21250,29375,-29375$$
$$0,106250,-384375,1006250,-2250000,4500000,$$
$$-8140625,13171875,-18203125,18203125,0,-658$$

Consequences

Observation

The set of all linear recurrence sequences is a vector space!
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The set of all linear recurrence sequences is a vector space! Hard to tell how the rule changes.

We can always mess up a finite amount of behaviour. So assume a_n has infinitely many zeroes, what is the structure of the zero set?

Linear recurrence sequences

Example

$$\frac{1}{(1-x^2)^2}-(1-x+2x^2+3x^4+4x^6) \leftrightarrow 0, 1, 0, 0, 0, 0, 0, 0, 5, 0, 6, 0, 7, 0, 8$$

Interlacing with 0 and *shifting* correspond to plugging in x^2 and multiplying by x respectively in the rational functions

$$\frac{1}{(1-x)^2} \leftrightarrow 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21,$$

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$$\frac{1+2x}{(1-x^4)^2} \leftrightarrow 1, 2, 0, 0, 2, 4, 0, 0, 3, 6, 0, 0, 4, 8, 0, 0, 5, 10, 0, 0, 6, 12, 0, 0, 7, 1$$

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Still has periodic zero set, all n congruent to $2, 3$ modulo 4.

Approach

Expand into partial fractions

$$\frac{p(x)}{q(x)} = \sum_{i=1}^{m} \sum_{j=1}^{n_j} \frac{r_{ij}}{(1 - \alpha_i x)^j}$$

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do some math:

$$\sum_{n=0}^{\infty} \left(\sum_{i=1}^{m} \sum_{j=1}^{n_j} r_{ij} \binom{n+j-1}{j-1} \alpha_i^n \right) x^n$$

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Upshot: there are polynomials $A_i(n)$ such that

$$a_n = \sum_{i=1}^{m} A_i(n) \alpha_i^n.$$

Like that formula for Fibonacci with the golden ratio in.

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Ridiculous suggestion

What if the integers were bounded? In that case infinitely many zeroes $\implies$ the function is zero!

What is bounded?

What if the integers were bounded?

How do we define boundedness?

Definition: Absolute Values

Let C be a commutative ring, an **absolute value** on C, is a function $|\cdot|\colon R \to \mathbb{R}_{\geq 0}$ satisfying for all $x, y \in C$

$$|x| = 0 \iff x = 0$$

$$|xy| = |x||y|$$

$$|x + y| \leq |x| + |y|$$

Are there other absolute values for the integers?

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$$|x + y| \leq |x| + |y|$$

Property 2 implies that $|1| = 1$ and $|-1|^2 = 1$ so $|-1| = 1$ also. So it remains to decide what happens for all primes $p \in \mathbb{Z}$.

We could set $|x| = 1$ for all $x \neq 0$, this is the **trivial absolute value**.

Or $|x| = x$ for all positive x, this gives the usual absolute value.

A strange absolute value

We can in fact define another absolute value $|\cdot|_p$ for each prime p.

Pick a value $\alpha = |p|_p < 1$, and let $|q|_p = 1$ for all other primes q.

Now we have that

$$|x+y| \leq \max(|x|, |y|) \leq |x| + |y|$$

Theorem: Ostrowski

The only nontrivial absolute values on $\mathbb{Q}$ are

$$x \mapsto \operatorname{sgn}(x)x \ \textit{ and } \ |\cdot|_p \textit{ for some prime } p$$

Result

With $|\cdot|_p$ the integers are bounded!

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The p-adic exponential function has finite radius of convergence.

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The p-adic exponential function has finite radius of convergence.

The fix
Choose p so that $|\alpha_i|_p = 1$ for all i, then $\alpha_i^{p-1} = 1 + \lambda_i$ with $|\lambda_i|_p \leq \frac{1}{p}$. Now $(\alpha_i^{p-1})^n$ is analytic!

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$$= \sum_{i=1}^{m} A_i(r + (p-1)n')\alpha_i^r(\alpha_i^{(p-1)})^{n'}$$

for each fixed r this function of n' is analytic.

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for each fixed r this function of n' is analytic. Infinitely many zeroes for integer n means $\exists r$ with infinitely many zeroes of the form $r + (p-1)n'$. So the function

$$\sum_{i=1}^{m} A_i(r + (p-1)n')\alpha_i^r(\alpha_i^{(p-1)})^{n'}$$

is identically zero, and all these $a_n = 0$ when $n \equiv r \pmod{p-1}$.

Finale

Theorem: Skolem $\rightsquigarrow$ Mahler $\rightsquigarrow$ Lech

All except finitely many indices of the zeroes of a linear recurrence lie in a finite union of arithmetic progressions, i.e. they are all of the form $nM + b$ for some $b \in B \subset \{0, \ldots, M-1\}$, $n \in \mathbb{N}$.

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