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Coleman-de Shalit's p-adic regulator

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1. Thank the audience for being awake.

The paper is a little ad hoc, so it is interesting to note that subsequent work has placed their regulator in a broader framework.absolute hodge means derived hom in derived cat of mhs

of the above in an ad hoc wayso *a posteriori* everything we do here is "right".

Beilinson regulators (Complex theory)

Let $C/\mathbf{C}$ be a smooth complete curve, $f, g \in \mathbf{C}(C)^\times$. Beilinson defines

$$r_{\infty,C}(f,g)(\omega) = \frac{1}{2\pi i}\int_{C(\mathbf{C})} \log|g|^2 \overline{\mathrm{d}\log f} \wedge \omega$$

the relation to K-groups comes via

$$K_2(\mathbf{C}(C)) = \mathbf{C}(C)^\times \otimes \mathbf{C}(C)^\times / \langle f \otimes 1 - f \rangle.$$

and $r_{\infty,C}$ satisfies this relation.

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Relation to L-values

Fix E/κ be an elliptic curve with CM by $\mathcal{O}_\kappa$, κ a CM field *of class number 1*. Let $\Psi = \Psi_{E/\kappa}$ be the associated Grossencharacter, p be a prime that splits in κ, $p = \mathfrak{p}\bar{\mathfrak{p}}$. ω an invariant differential.

Proposition (Bloch, Rohrlich, Deninger-Wingberg)

$$r_{\infty,E}(f,g)(\omega) = c_{f,g}\Omega \underbrace{L_\infty(E,0)}_{=L_\infty(\Psi,0)},\ c_{f,g} \in \mathbf{Q}$$

(L_∞ includes Gamma factors), and there exists f,g with $c_{f,g} \neq 0$.

One very interesting aspect of this definition is the relation to the L-function of an elliptic curve E.

p-adic version

We can associate a canonical *period-pair-class* to κ: $\langle \Omega, \Omega_p \rangle \in (\mathbf{C}^\times \times \mathbf{C}_p^\times)/\overline{\mathbf{Q}}^\times$ so that:

Theorem (Katz, Manin-Vishik)
Let $1 \neq \mathfrak{g}$ an ideal of κ relatively prime to $\mathfrak{p}$. Then $\exists!$, $W(\overline{\mathbf{F}}_p)$-valued measure μ on $\mathscr{G}(\mathfrak{g}) = \mathrm{Gal}(\kappa(\mathfrak{g}\mathfrak{p}^\infty)/\kappa)$ so that $\forall \epsilon$ Grossencharacter of conductor dividing $\mathfrak{g}$ with infinity type $(k, 0)$ $k \geq 1$, if

$$L_{\infty,\mathfrak{g}}(\epsilon^{-1}, s) = \Gamma(s+k) \prod_{\mathfrak{l} \nmid \mathfrak{g}} (1 - \epsilon^{-1}(\mathfrak{l}) \operatorname{Nm} \mathfrak{l}^{-s})^{-1}$$

$$L_{p,\mathfrak{g}}(\epsilon^{-1}) = \int_{\mathscr{G}(\mathfrak{g})} \epsilon(\sigma)\, \mathrm{d}\mu(\sigma)$$

we have

$$\Omega_p^{-k} L_{p,\mathfrak{g}}(\epsilon^{-1}) = \Omega^{-k}(1 - p^{-1}\epsilon(\mathfrak{p})) L_{\infty,\mathfrak{g}}(\epsilon^{-1}, 0) \in \overline{\mathbf{Q}}.$$

There is a p-adic analogue of the right hand side:

p-adic regulators?

Can rewrite $r_{\infty,C}$ as

$$r_{\infty,C}(f,g) = \sum_{b\in C(\mathbf{C})} \mathrm{ord}_b(g) F_{f,\omega}(b)$$

where $F_{f,\omega}$ satisfies

$$\bar\partial(\mathrm{d}F) = \bar\partial(\log|f|^2\omega)$$

Even without p-adic $\bar\partial$ we can just try to find $F_{f,\omega}$ satisfying

$$\mathrm{d}F = \log f \cdot \omega$$

and define

$$''r_{p,C}(f,g) = \sum_{b\in C(\mathbf{C}_p)} \mathrm{ord}_b(g) F_{f,\omega}(b)''$$

So we have a p-adic L-function and p-adic period, can we define a p-adic regulator and obtain a similar theorem with $L_p(\Psi)$?The first step in this process is to rephrase the regulator pairing asfor which

$$F(P) = \log|f(P)|^2 \int^P \omega + \text{smooth}$$

near $P_0 \in |\operatorname{div} f|$.We will see that even without a p-adic ∂ we can solve

$$\mathrm{d}F_{f,\omega} = \log f \cdot \omega$$

to get a candidate analogue of r_∞, the proof is in the pudding though, some relation to the L-value.

Besser does have a padic partial bar though?!

p-adic tools (Coleman integration)

Let $K = \mathbf{C}_p = \widehat{\overline{\mathbf{Q}}}_p$, $R = \mathcal{O}_K$, $k = R/\mathfrak{m}$. We will work with 1-dimensional rigid spaces (curves) over K. We fix a branch of the p-adic logarithm $\log\colon K^\times \to K$.

It is always possible to integrate rigid 1-forms locally on a disk: given ω we have a local expression in terms of a convergent power series

$$\omega|_D = \sum_i a_i t^i \, \mathrm{d}t$$

which can be integrated formally (up to a constant).

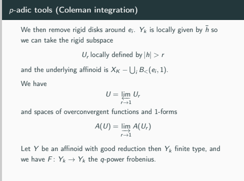
Let $X/\mathcal{O}_K$ be a smooth projective curve, if $Y \subseteq X$ smooth affine open, then in the special fibre

$$X_k \smallsetminus Y_k = \{e_1, \ldots, e_n\}.$$

What is hard is to integrate globally, iteratively and include $\int \frac{\mathrm{d}z}{z}$!

If we want to find an analogue of the above picture we need a p-adic definition of integrals such as We are trying to define a p-adic

$$\int \log(f) \cdot \omega$$


p-adic tools (Coleman integration)

We then remove rigid disks around e_i. Y_k is locally given by $\bar{h}$ so we can take the rigid subspace

U_r locally defined by $|h| > r$

and the underlying affinoid is $X_K - \bigcup_i B_<(e_i, 1)$.

We have

$$U = \varprojlim_{r \to 1} U_r$$

and spaces of overconvergent functions and 1-forms

$$A(U) = \varinjlim_{r \to 1} A(U_r)$$

Let Y be an affinoid with good reduction then Y_k finite type, and we have $F\colon Y_k \to Y_k$ the q-power frobenius.

The problem comes when trying to piece these together, the discs are disconnected, there are no overlaps where we can match up values of our integral, we need more structure to find an integral unique up to a single global (additive) constant.

The additional structure we will use is the Frobenius coming from the reduction mod p.

p-adic tools (Coleman integration)

Proposition
There exists

$$\phi\colon U \to U,\ \widetilde{\phi} = F$$

*a **lift of frobenius** or frobenius morphism of X, of degree q.*

Note: Whatever we choice of frobenius we make should not matter!

Example
Have $X = \mathbf{P}^1_{\mathcal{O}_{\mathbf{C}_p}} \supseteq Y = \mathbf{G}_m = \mathbf{P}^1 \smallsetminus \{0, \infty\}$ then

$$U_r = \{r < |z| < 1/r\}$$

$\phi(z) = z^q\colon U_r \to U_{r^q}$. (But we could add some other $p \cdot$ junk!)

Here's a theorem which is needed in general, but technically unnecessary: One can imagine two different ways of computing a Coleman integral, picking a frobenius lift in a versatile but "arbitrary" way, using existance in theory or making some simple choice in practice. In some situations there are canonical Frobenius lifts, perhaps an algebraic lift of Frobenius.

Our final theory should be invariant under our choice, so we should be able to use a widely applicable computational approach à la Balakrishnan-Bradshaw-Kedlaya. Or use a lift coming from some specific structure and get the same answer.

Coleman-de Shalit's p-adic regulator

Theorem (Coleman integration)
There is a subspace $M(U)$ of $A_{\mathrm{loc}}(U)$, which we call the space of Coleman functions, and linear map (integration), which we denote by $\int$ or by $\omega \mapsto F_\omega$, from $M(U) \otimes_{A(U)} \Omega(U)$ to $M(U)/\mathbf{C}_p$.

The map f is characterized by three properties:

1. *It is a primitive for the differential in the sense that $\mathrm{d}F_\omega = \omega$.*
2. *It is Frobenius equivariant $F_{\phi^*\omega} = \phi^* F_\omega$.*
3. *If $g \in A(U)$, then $F_{\mathrm{d}g} = g + \mathbf{C}_p$.*

We also have properties such as:

$$f \in M(U)$$

vanishes on one residue disk, then f is identically zero.

The space $M(U)$ is constructed iteratively $M(U) = \bigcup_n A_n(U)$ with each step being obtained as functions you get by integration from the previous.

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The p-adic regulator

We can now define a p-adic version of the above regulator.

(Let C to be a complete non-singular curve whose jacobian has good reduction.)

If $f \in K(C)^\times$, $U = C \smallsetminus |\operatorname{div}(f)|$ we can take a global 1-form $\omega \in H^0(C, \Omega^1_{C/K})$ and the function

$$\log(f) = \int \frac{\mathrm{d}f}{f} \in A_1(U)$$

and obtain

$$\log(f)\omega \in \Omega_1(U).$$

Integration gives

$$F_{f,\omega} \in A_2(U) \text{ with } \mathrm{d}F_{f,\omega} = \log(f)\omega \in \Omega_1(U),$$

unique up to a constant.

. If $a \in |\operatorname{div}(f)| = C \smallsetminus U$ then we can fix R_a a rigid disc around a, and $V_a = R_a \smallsetminus \{a\}$. On V_a we have

$$\int \log(f)\omega = \log(f)\int \omega - \int \left(\frac{\mathrm{d}f}{f} \int \omega \right)$$

we choose $\int \omega$ to vanish at a, so this is a function which differs from $F_{f,\omega}$ by a constant.

Doing this extends $F_{f,\omega}$ to a function on $C(K)$ rather than just in $A_2(U)$.

The regulator

Definition (The p-adic regulator)
Take f, g, ω as before defined over $\overline{k}$, then define

$$r(f,g)(\omega) = -\int_{(g)} \log(f)\omega = -\sum_{b \in C(K)} \mathrm{ord}_b(g) F_{f,\omega}(b) \in \overline{k}$$

Theorem (Coleman-de Shalit)
$r_C(f,g)$ is a skew-symmetric bilinear pairing on $\overline{k}(C)^\times$ that

1. *factors through $K_2(\overline{k}(C))$*
2. *depends only on $\mathrm{div}(f), \mathrm{div}(g)$*
3. *is $\mathrm{Gal}(\overline{k}/k)$ equivariant*
4. *for finite morphisms of complete non-singular curves $/k$ $u\colon C' \to C$ we get $r_{C'}(u^*f, u^*g) = u^* r_C(f,g)$.*

.which is well defined as (g) has degree 0.giving

$$r_C\colon K_2(\overline{k}(C)) \to \mathrm{Hom}(H^0(C, \Omega^1_{C/\overline{k}}), \overline{k}).$$

Comparison of the p-adic and C theories

We now move to a very special situation, where the above regulators can be shown to be related to L-values.

$C = E/\kappa$ will be an elliptic curve with CM by $\mathcal{O}_\kappa$. $\Psi = \Psi_{E/\kappa}$ the corresponding Grossencharacter with conductor $\mathfrak{f}$ and assume

$$w_{\mathfrak{f}} = \#\{\zeta \in \mu(K) : \zeta \equiv 1 \pmod{\mathfrak{f}}\} = 1.$$

let ω be a κ-rational invariant differential, $\mathscr{L}$ the period lattice of (E, ω).

.

The theorem

Theorem (Rohrlich, others?)

$$r_\infty(f,g) = \underbrace{12(\mathrm{Nm}\,\mathfrak{a} - \Psi^{-1}(\mathfrak{a})) \sum_{\textit{orbits } \langle Q \rangle} \mathrm{ord}_Q\, g \cdot \Omega(Q) \prod_{\mathfrak{l} \mid \mathfrak{g}_Q} (1 - \Psi(\mathfrak{l}))}_{c_{f,g}} L_\infty(\Psi, 0)$$

$\mathfrak{g}_Q$ ideal of annihilators of Q.

Theorem (Coleman-de Shalit)
We have the formula

$$r_{p,E}(f,g)(\omega) = c_{f,g}\Omega_p(1 - (\Psi(\mathfrak{p})p)^{-1})^{-1} L_p(\Psi).$$

Where do these terms come from? (in the p-adic case)

The rest of the talk: proof overview, see where the terms come in.

.the $c_{f,g}$ is the to the $c_{f,g}$ in the first theorem, I haven't just abused notation, this was one of the most surprising aspects of this theorem to me, personally my main goal was to understand this

Proof

We use a specific class of f's (for $(\mathfrak{a}, \mathfrak{f}\mathfrak{p}) = 1$), the functions

$$f(P) = \Theta_{\mathfrak{a}}(P) = \Delta(\mathcal{L})\Delta(\mathfrak{a}^{-1}\,\mathcal{L})^{-1} \prod_{R \in E[\mathfrak{a}]}{}' \frac{\Delta(\mathcal{L})}{(x(P) - x(R))^6} \in \kappa(E)^\times$$

whose values are **elliptic units**, the divisor of $\Theta_{\mathfrak{a}}$ is

$$12\left((\operatorname{Nm}\mathfrak{a} - 1)\cdot(0) - \sum_{R \in E[\mathfrak{a}]}{}'(R)\right)$$

and we have the **distribution relation**

$$f(\pi P) = \prod_{v \in E[\mathfrak{p}]} f(P + v)$$

These functions generate the set of all functions with divisors supported on torsion.

We also take $g \in \kappa(E)^\times$ with divisor supported on torsion and $Q \in |\operatorname{div} g| \implies \mathfrak{f} | \mathfrak{g}_Q$, $(\mathfrak{g}_Q, \mathfrak{a}\mathfrak{p}) = 1$.

.see de Shalit.

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Coleman-de Shalit's p-adic regulator

Take E with the $\mathfrak{a}$-torsion points removed,

$$X(\mathfrak{a}) = E \setminus \bigcup_{P \in E[\mathfrak{a}]} B(P, 1) \subseteq U_r(\mathfrak{a}) = E \setminus \bigcup_{P \in E[\mathfrak{a}]} B(P, r).$$

Take D to be a derivation that is dual to ω (so $DF_\omega = 1$). Then

$$F_{f,\omega}$$

is the unique (up to constant) $F \in A_2(U_r(\mathfrak{a}))$ for which

$$DF = \log f$$

Then we have

$$D(F(\pi P)) = \pi \cdot (DF)(\pi P)$$

and the distribution relation gives

$$D(F(\pi P)) = \pi \sum_{v \in E[\pi]} (DF)(P + v)$$

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By definition $\pi = \psi(\mathfrak{p})$ is a lift of frobenius (which is algebraic!). As $F \in A_2(U_r(\mathfrak{a}))$, for some (possibly different) r close to 1 we have

$$F(\pi P) - \pi \sum_{v \in E[\pi]} F(P + v) \in A_2(U_r(\mathfrak{a}))$$

the above implies this is locally constant, hence constant! So we change F to get that

$$F(\pi P) - \pi \sum_{v \in E[\pi]} F(P + v) = 0.$$

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Coleman-de Shalit's p-adic regulator

$$F(\pi P) - \pi \sum_{v \in E[\pi]} F(P+v) = 0.$$

Now define

$$F^{\#}(P) = F(P) - p^{-1} \sum_{v \in E[\pi]} F(P+v)$$

so that as $Q \in |\operatorname{div} g|$ is Galois conjugate to πQ over κ:

$$r_p(f,g) = -\sum_Q \operatorname{ord}_Q g F(Q) = -\sum_Q \operatorname{ord}_Q g F(\pi Q)$$

giving

$$\left(1 - \frac{1}{\pi p}\right) r_p(f,g) = -\sum_Q \operatorname{ord}_Q g F^{\#}(Q).$$

We also have

$$\log(f)^{\#}(P) = \log f(P) - p^{-1} \sum_{v \in E[\pi]} \log f(P+v).$$

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Coleman-de Shalit's p-adic regulator

If Q is a torsion point in $X(\mathfrak{a})$ relatively prime to $\mathfrak{p}$ order, then de Shalit has associated a

$$\eta_Q \colon \widehat{\mathbf{G}}_m \xrightarrow{\sim} \widehat{E}$$

so $Q + \eta_Q(S)$ parameterises the residue disk of Q and a $W = W(\overline{\mathbf{F}}_p)$ valued measure μ_Q on $\mathbf{Z}_p^\times$ s.t.

$$\log(f)^{\#}(Q + \eta_Q(S)) = \int_{\mathbf{Z}_p^\times} (1+S)^x \,\mathrm{d}\mu_Q(x) \in W[[S]]$$

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Then work of de Shalit shows that

$$F^{\#}(Q + \eta_Q(S)) = \underbrace{\eta_Q'(0)}_{\Omega_p(Q)} \int_{\mathbf{Z}_p^\times} (1+S)^x x^{-1}\, \mathrm{d}\mu_Q(x) + c$$

for some constant c, and that $F^{\#}(P)$ is rigid analytic on $X(\mathfrak{a})$.

So we get

$$\left(1 - \frac{1}{\pi p}\right) r_p(f,g) = -\sum_Q \operatorname{ord}_Q g\, \Omega_p(Q) \int_{\mathbf{Z}_p^\times} x^{-1}\, \mathrm{d}\mu_Q(x).$$

We need to move to the correct group and remove the dependence on Q, by identifying $G = \mathrm{Gal}(\kappa(\mathfrak{g}\mathfrak{p}^\infty)/\kappa(\mathfrak{g})) \cong \mathbf{Z}_p^\times$ so that

$$= -\sum_{\langle Q \rangle} \operatorname{ord}_Q g\, \Omega_p(Q) \sum_{\tau \in \mathscr{G}/G} \int_G \Psi^{-1}(\sigma)\, \mathrm{d}\mu_{\tau(Q)}(\sigma)$$

$$= -\sum_{\langle Q \rangle} \operatorname{ord}_Q g\, \Omega_p(Q) \int_{\mathscr{G}(\mathfrak{g}_Q)} \Psi^{-1}(\sigma)\, \mathrm{d}\mu_e(\sigma)$$

eta is parameterising the residue disk around Qe is norm compatible sequence of elliptic units (question for later: an euler system?!)

Coleman-de Shalit's p-adic regulator

Theorem (Coates-Wiles)

$$\mu_e = 12(\sigma_{\mathfrak{a}} - \mathrm{Nm}\,\mathfrak{a})\mu(\mathfrak{g}_Q)$$

where $\mu(\mathfrak{g}_Q)$ is the measure which defines the p-adic L-function of conductor $\mathfrak{g}_Q$,so removing those factors we reach

$$\overbrace{(1-(\pi p)^{-1})}^{\text{Distribution relations}} \ r_p(f,g) = \underbrace{12(\mathrm{Nm}\,\mathfrak{a} - \Psi^{-1}(\mathfrak{a}))}_{\text{Coates-Wiles}} \sum_{\text{orbits } \langle Q \rangle} \mathrm{ord}_Q\, g \Omega_p(Q) \underbrace{\prod_{\mathfrak{l} \mid \mathfrak{g}_Q} (1-\Psi(\mathfrak{l}))\, L_p(\Psi)}_{L_{p,\mathfrak{g}_Q} \rightsquigarrow L_p},$$

Fin.

this isn't the exact formula we saw earlier, need to factor out a Ω_p to get something algebraic