Riemann Hypotheses

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WMS Talks

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In this talk:

The Riemann zeta function: Euler's work

A brief history of ζ:

- In 1735 Euler solves the **Basel problem** by finding that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

- He also discovered more general formulae for $\sum_{n=1}^{\infty} n^{-2k}$ in terms of the Bernoulli numbers B_{2k} for all natural k.
- In fact, a nice form for

$$\sum_{n=1}^{\infty} n^{-2k-1},$$

is still unknown today.

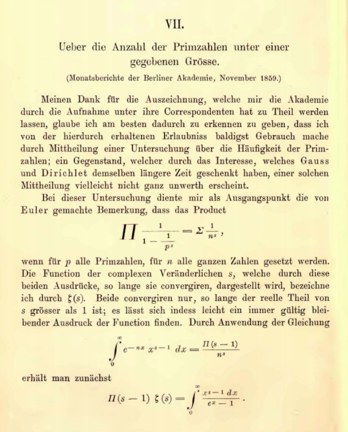
The Riemann zeta function: Along comes Riemann

A brief history of ζ:

- In 1859 Bernhard Riemann, a well known analyst, publishes a paper on his work counting the primes using methods from analysis.
- In the paper he considers

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

- Here the notation ζ for this function is used for the first time.
- Along the way he (essentially) makes four hypotheses.



VII.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859.)

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\Pi \frac{1}{1-\frac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch $\zeta(s)$. Beide convergiren nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^\infty e^{-nx} x^{s-1}\,dx = \frac{\Pi(s-1)}{n^s}$$

erhält man zunächst

$$\Pi(s-1)\,\zeta(s) = \int_0^\infty \frac{x^{s-1}\,dx}{e^x-1}.$$

The Riemann zeta function: What Riemann did

In his work Euler had (more or less) found that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^s}.$$

In his paper Riemann uses this to take the function $\zeta\colon \{\sigma + it \in \mathbb{C} \mid \sigma > 1\} \to \mathbb{C}$ and extend it to all of $\mathbb{C}$. He defines an **analytic continuation** from $\mathbb{C} \to \mathbb{C}$ which matches the series definition given above when the series converges (when $\mathrm{Re}(s) > 1$). Riemann also discovers a **functional equation** for the zeta function by showing that

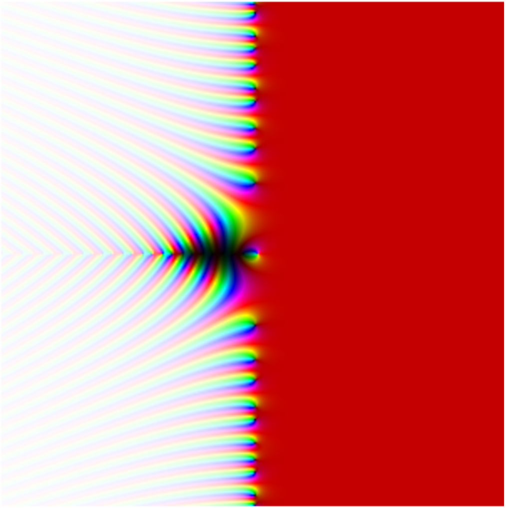
$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

The Riemann zeta function: What does it look like?

$f(s) = s$:

The Riemann zeta function: What does it look like?

$\zeta(s)$:



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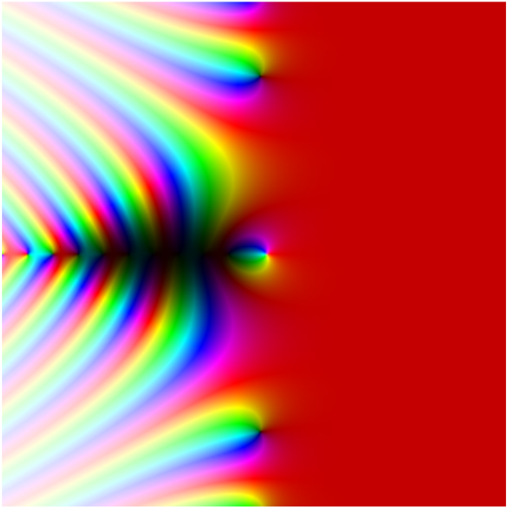

The Riemann zeta function: What does it look like?

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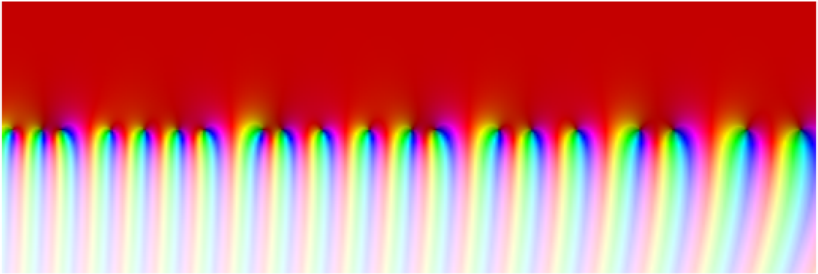


The Riemann zeta function: The hypothesis

The zeta function has "trivial" zeroes at the negative even integers, but also "non-trivial" zeroes lying on the line $\mathrm{Re}(s) = \frac{1}{2}$.

The Riemann Hypothesis (RH)

All non-trivial zeroes of the Riemann zeta function lie on the **critical line**

$$\left\{ s \in \mathbb{C} \colon \ \mathrm{Re}(s) = \frac{1}{2} \right\}.$$

Why do we care?

- It is a natural function to consider, and knowing the zeroes of a complex function are key to understanding it.
- The location of the zeroes of $\zeta(s)$ relates in a strong way to the distribution of the primes.

The Riemann zeta function: Why number theorists care

The prime counting function $\pi(x)$

We define

$$\pi(x) = |\{p \in \mathbb{N} \colon p \text{ prime},\ p \leq x\}|.$$

Gauss noticed that $\pi(x)$ is approximated well by

$$\mathrm{Li}(x) = \int_0^x \frac{\mathrm{d}t}{\log t}$$

which was later confirmed by some of Chebyshev's work.
The Riemann hypothesis is actually equivalent to the statement that there exists $c_2 > c_1 > 0$ such that

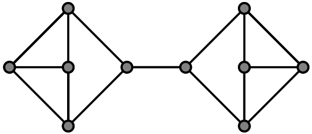
$$\mathrm{Li}(x) + c_1\sqrt{x}\log(x) \leq \pi(x) \leq \mathrm{Li}(x) + c_2\sqrt{x}\log(x)$$

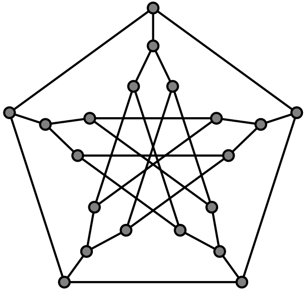
eventually. These are the best possible bounds!

Expander graphs

Suppose we want to design a communications network (for computers, people, phones, etc.) by linking together n entities with wires, such that each object has only k wires from it (this is called k-**regular**).
We shall call our entities **nodes** and wires **edges**, as in graph theory. The set of nodes is V and edges E.

For today we will think of our goal as the following:
If we split our network into two non-empty parts (a partition) there should be lots of edges between the two halves.





The spectrum

Given such a graph G we can fix a numbering of the vertices and form:

The adjacency matrix

Is an $n \times n$ matrix $A_G = (a_{i,j})$ of 1's and 0's given by

$$a_{i,j} = 1 \iff \text{there is an edge between vertex } i \text{ and vertex } j.$$

What is the set of eigenvalues (the **spectrum**) of this matrix?
So we define $\lambda(G)$ to be the next largest eigenvalue after k, i.e.

$$\lambda(G) = \max_{|\lambda_i| < k} |\lambda_i|.$$

The gap between this value and k is very important for optimisation problems of this nature. The larger the better!

Ramanujan graphs

It turns out (Alon, Boppara) that for fixed $\epsilon > 0$ only finitely many graphs do not satisfy

$$\lambda(G) \geq 2\sqrt{k-1} - \epsilon.$$

This set of graphs must be pretty special, so we name them

Ramanujan graphs

A graph G as above which satisfies

$$\lambda(G) \leq 2\sqrt{k-1}.$$

These are highly optimal from our point of view!
Only in 2013 were Ramanujan graphs shown to exist for $k \neq p^n + 1$ (Marcus, Speilman, Srivastava).

The Ihara zeta function

Take a graph G as described above (k-regular for some k, finite and connected). We define:

A prime in G

A prime in G is a **path** in G that is

- **closed**,
- **backtrackless**,
- **tailless** and
- defined up to rotation.

The Ihara zeta function of G

$$\zeta_G(u) = \prod_{P \text{ a prime of } G} \left(1 - u^{\text{length}(p)}\right)^{-1}.$$

The Ihara determinant formula

Though we have defined $\zeta_G(u)$ in a similar way to $\zeta(s)$ the resulting function is a lot simpler!
In fact we always have the following expression for ζ_G:

The Ihara determinant formula

$$\zeta_G(u)^{-1} = (1-u^2)^{|E|-|V|}\det((1-(k-1)u^2)I - Au).$$

The Riemann hypothesis for graphs

The Ihara zeta function has slightly different behaviour than the classical zeta, there are poles instead of zeroes! So we might conjecture

RH for graphs

In the strip $0 < \mathrm{Re}(u) < 1$ the only poles of $\zeta_G(u)$ are on the line $\mathrm{Re}(u) = \frac{1}{2}$.

When is this true?

Theorem

A graph G satisfies the RH for graphs $\iff$ it is Ramanujan.

The sketchiest rough idea of a proof you will ever see

Proof: We are looking for zeroes of $\zeta_G(u)^{-1}$ which is given by the Ihara determinant formula as

$$(1-u^2)^{|E|-|V|}\det(I(1-(k-1)u^2)-Au)$$

which is given by

$$(1-u^2)^{|E|-|V|}\prod_{\lambda}(I(1-(k-1)u^2)-\lambda u).$$

Then check some cases using the fact that $\lambda(G)\leq 2\sqrt{k-1}$ to see when this is zero.

What else has a zeta function?

- Dynamical systems, the Ruelle zeta is actually a generalisation of the Ihara zeta function we saw earlier.
- Function fields (that the analogue of RH is true was proved by André Weil in the 40's).
- Curves over finite fields.
- Fractal strings.
- Schemes (over finite type over $\mathbb{Z}$).

The Dedekind zeta function

Richard Dedekind (1831–1916) wanted to use analysis to study more general fields than just $\mathbb{Q}$, specifically he was interested in number fields.

Number fields

A **number field** is a field that is also a finite dimensional $\mathbb{Q}$-vector space.

e.g. $\{a + bi \mid a, b \in \mathbb{Q}\}$, $\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$,
$\{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{2}\sqrt{3} \mid a, b, c, d \in \mathbb{Q}\}$.
In a general number field the idea of a prime element doesn't work out so well, however if we consider nice subgroups of the field (ideals of the ring of integers) as elements then everything works out nicely. For example we have unique factorisation of ideals into prime ideals.
So we should deal with ideals instead of elements!

The Dedekind zeta function

So Dedekind defined for a number field K

The Dedekind zeta function

$$\zeta_K(s) = \sum_{I \subset \mathcal{O}_K} \frac{1}{N(I)^s}$$

where $N(I)$ is the **norm** of I $(= |\mathcal{O}_K/I|)$.

$\mathbb{Q}$ is a number field too, and it turns out that the ideals of $\mathbb{Q}$ correspond one to one with the set of natural numbers. We then have $N((n)) = |\mathbb{Z}/(n)| = |\mathbb{Z}/n\mathbb{Z}| = n$. Therefore

$$\zeta(s) = \zeta_{\mathbb{Q}}(s).$$

A proof of these hypotheses for all number fields (known as the **extended** Riemann hypothesis) would give approximations for the number of prime ideals of bounded norm, exactly the same as for the original hypothesis.

Closing remarks

- Zeta functions can be used to pack up lots of useful information into one big package (a complex function).
- The properties of this package can tell us about the objects we started with.
- We can also see links between different objects via their zeta functions.
- Due to the abundant computational evidence (over ten trillion non-trivial zeroes found so far, all on the critical line) a huge number of papers have been written that assume the Riemann hypothesis is true. So a proof of the (generalised) hypothesis would imply hundreds of other results true also.

Sources used

I used some of the following when preparing this talk, and so they are possibly good places to look to learn more about the topic:

1. "What is... an expander?" – Peter Sarnak
2. "Problems of the Millennium: The Riemann Hypothesis" – Peter Sarnak
3. "Problems of the Millennium: The Riemann Hypothesis" (Official Millennium prize problem description) – Enrico Bombieri
4. "Zeta Functions of Graphs: A Stroll through the Garden" – Audrey Terras
5. Wikipedia – Enough said
6. http://graphtheoryinlatex.blogspot.com/ – Pretty pictures
7. "Fractal Geometry, Complex Dimensions and Zeta Functions" – Lapidus and van Frankenhuijsen (not used in talk but still cool)