Motivation for p-adic modular forms

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STAGE

Overview

These slides are available online (in handout form) at

`https://alexjbest.github.io/talks/motivation-for-p-adic-modular-forms/slides_h.pdf`

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Goal: Introduce, post hoc, motivation for Katz's definition of p-adic modular forms, especially to motivate Serre's ∂ operator.

Recall

A *modular form* of weight k is a function

$$f\colon \{(E \xrightarrow{\pi} R \text{ an ell. curve}, \omega \in \Gamma(E, \Omega^1_{E/R}) \text{ nowhere vanishing})\} \to R$$

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s.t.

1. $\forall \lambda \in R^\times, f(E, \lambda\omega) = \lambda^{-k} f(E, \omega)$
2. $f(E, \omega)$ is isomorphism invariant.
3. f is functorial w.r.t. R.

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2. $f(E, \omega)$ is isomorphism invariant.
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we then have

$$f(E, \omega) \cdot \omega^{\otimes k} \in \Gamma(R, \underbrace{\pi_*(\Omega^1_{E/R})}_{\underline{\omega}_{E/R}}{}^{\otimes k})$$

De Rham cohomology

The sheaf of values $\underline{\omega}_{E/R}$ is a subsheaf of the de Rham cohomology of E/R:

$$0 \to \underline{\omega}_{E/R} \to \overbrace{H^1_{\mathrm{dR}}(E/R)}^{:=\mathbb{H}^1(E,\Omega^\bullet_{E/R})} \to \underbrace{H^1(E,\mathcal{O}_E)}_{=\underline{\omega}_{E/R}^{\otimes -1}} \to 0$$

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assuming $1/6 \in R$ we can canonically split this sequence:

Fixing $(E,\omega)/R$ we have a unique pair of meromorphic functions with poles only at ∞, of orders 2 and 3 resp., denoted by X, Y so that

$$\omega = \frac{\mathrm{d}X}{Y} \text{ and } E\colon Y^2 = 4X^3 - g_2X - g_3,\ g_i \in R$$

Then we have an inclusion of 2-term complexes

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this induces an isomorphism on $\mathbb{H}^1$. Moreover for $i > 0$,

$$H^i(E, \mathcal{O}_E(\infty)) = 0$$

$$H^i(E, \Omega^1_{E/R}(2\infty)) = 0$$

giving

$$\begin{aligned} H^1_{\mathrm{dR}}(E/R) &\cong \mathbb{H}^1(E, \mathcal{O}_E(\infty) \to \Omega^1_{E/R}(2\infty)) \\ &= \operatorname{coker}(H^0(E, \mathcal{O}_E(\infty)) \to H^0(E, \Omega^1_{E/R}(2\infty))) \\ &= \operatorname{coker}(R \xrightarrow{0} H^0(E, \Omega^1_{E/R}(2\infty))) \\ &= H^0(E, \Omega^1_{E/R}(2\infty)) \\ &\ni \underbrace{\frac{\mathrm{d}X(E,\omega)}{Y(E,\omega)}}_{=\omega}, \underbrace{X(E,\omega)\cdot\omega}_{=\eta} \end{aligned}$$

How does $R^\times$ act?

By uniqueness

$$\begin{aligned} X(E, \lambda\omega) &= \lambda^{-2} \cdot X(E, \omega) \\ Y(E, \lambda\omega) &= \lambda^{-3} \cdot Y(E, \omega) \\ g_2(E, \lambda\omega) &= \lambda^{-4} g_2(E, \omega) \\ g_3(E, \lambda\omega) &= \lambda^{-6} g_3(E, \omega) \end{aligned}$$

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hence

$$\lambda\omega = \frac{\mathrm{d}X(E,\lambda\omega)}{Y(E,\lambda\omega)} \quad \text{and} \quad \lambda^{-1}\eta = \frac{X(E,\lambda\omega)\mathrm{d}X(E,\lambda\omega)}{Y(E,\lambda\omega)}$$

$$H^1_{\mathrm{dR}}(E/R) \simeq \underline{\omega}_{E/R} \oplus \underline{\omega}_{E/R}^{-1}$$

$$\mathrm{Symm}^k\left(H^1_{\mathrm{dR}}(E/R)\right) \simeq \left(\underline{\omega}_{E/R}\right)^{\otimes k} \oplus \left(\underline{\omega}_{E/R}\right)^{\otimes k-2} \oplus \cdots \oplus \left(\underline{\omega}_{E/R}\right)^{\otimes -k}$$

Connections

Let $f : S \to T$ be a smooth T-scheme, $\mathcal{E}$ a quasi-coherent sheaf of $\mathcal{O}_S$ -modules. A *connection* on $\mathcal{E}$ is a homomorphism

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{S/T}$$

of abelian sheaves satisfying the "Leibniz rule"

$$\nabla(ge) = g\nabla(e) + e \otimes \mathrm{d}g$$

where g and e are sections of $\mathcal{O}_S$ and $\mathcal{E}$, respectively, over an open subset of S and $\mathrm{d} : \mathcal{O}_S \to \Omega^1_{S/T}$ the exterior derivative.

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Given an element of the tangent bundle $t \in (\Omega^1_{S/T})^*$ we can define

$$\nabla_t \colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{S/T} \to \mathcal{E}$$

by “contraction”.

The Gauss-Manin connection – complex case

Let R be the ring of holomorphic functions of τ, and E be the relative elliptic curve

$$\boldsymbol{C}/\boldsymbol{Z} + \boldsymbol{Z}\tau$$

which can be expressed as $y^2 = 4x^3 - \frac{E_4}{12}x + \frac{E_6}{216}$, $E_i \in R$.

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To give the Gauss-Manin connection in this context we need only define $\nabla_\tau = \nabla_{\mathrm{d}/\mathrm{d}\tau}$, we do so via the dual connection on $H_1^{\mathrm{dR}}(E/R)$ as

$$\int_{\gamma_i} \nabla_\tau(\xi) = \frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\gamma_i} \xi \quad \text{for} \quad \xi \in H^1_{\mathrm{dR}}(E/R), \text{ and } i = 1, 2$$

Computation of the connection matrix

Let

$$\omega = \frac{\mathrm{d}x}{y}, \; \eta = \frac{x\,\mathrm{d}x}{y}$$

Poincaré duality gives elements $\gamma_i \in H^1_{\mathrm{dR}}(E/R)$ also, satisfying

$$\langle \gamma_2, \gamma_1 \rangle = 1 = -\langle \gamma_1, \gamma_2 \rangle$$

$$\langle \gamma_1, \gamma_1 \rangle = 0 = \langle \gamma_2, \gamma_2 \rangle$$

we then define

$$\omega_i = \int_{\gamma_i} \omega = \langle \omega, \gamma_i \rangle \,, \; \eta_i = \int_{\gamma_i} \eta = \langle \eta, \gamma_i \rangle \in R$$

so that we have

$$\begin{pmatrix} \omega_1 & -\omega_2 \\ \eta_1 & -\eta_2 \end{pmatrix} \begin{pmatrix} \gamma_2 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

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to invert this we note that

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$$

so

$$2\pi i \begin{pmatrix} \gamma_2 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} -\eta_2 & \omega_2 \\ -\eta_1 & \omega_1 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

to which we want to apply ∇_τ

$$\int_{\gamma_i} \nabla_\tau(\gamma_j) = \frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\gamma_i} \gamma_j = 0$$

$$
\begin{aligned}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \nabla_\tau \left(\begin{pmatrix} -\eta_2 & \omega_2 \\ -\eta_1 & \omega_1 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix} \right) \\
&= \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}\tau}\eta_2 & \frac{\mathrm{d}}{\mathrm{d}\tau}\omega_2 \\ -\frac{\mathrm{d}}{\mathrm{d}\tau}\eta_1 & \frac{\mathrm{d}}{\mathrm{d}\tau}\omega_1 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix} + \begin{pmatrix} -\eta_2 & \omega_2 \\ -\eta_1 & \omega_1 \end{pmatrix} \begin{pmatrix} \nabla_\tau(\omega) \\ \nabla_\tau(\eta) \end{pmatrix}
\end{aligned}
$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \nabla_\tau \left(\begin{pmatrix} -\eta_2 & \omega_2 \\ -\eta_1 & \omega_1 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix} \right)$$

$$= \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}\tau}\eta_2 & \frac{\mathrm{d}}{\mathrm{d}\tau}\omega_2 \\ -\frac{\mathrm{d}}{\mathrm{d}\tau}\eta_1 & \frac{\mathrm{d}}{\mathrm{d}\tau}\omega_1 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix} + \begin{pmatrix} -\eta_2 & \omega_2 \\ -\eta_1 & \omega_1 \end{pmatrix} \begin{pmatrix} \nabla_\tau(\omega) \\ \nabla_\tau(\eta) \end{pmatrix}$$

so we get

$$\begin{pmatrix} \nabla_\tau(\omega) \\ \nabla_\tau(\eta) \end{pmatrix} = \frac{-1}{2\pi i} \begin{pmatrix} \omega_1 & -\omega_2 \\ \eta_1 & -\eta_2 \end{pmatrix} \begin{pmatrix} -\eta_2' & \omega_2' \\ -\eta_1' & \omega_1' \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

$$= \frac{-1}{2\pi i} \begin{pmatrix} \eta_1'\omega_2 - \eta_2'\omega_1 & \omega_1\omega_2' - \omega_2\omega_1' \\ \eta_2\eta_1' - \eta_1\eta_2' & \eta_1\omega_2' - \eta_2\omega_1' \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

In fact $\omega_1 = \tau$ and $\omega_2 = 1$ so that $\omega_1' = 1, \omega_2' = 0$ and $\eta_1 - \tau\eta_2 = 2\pi i$, giving $\eta_1' - \tau\eta_2' = \eta_2$ and we simplify

$$\begin{aligned}
\begin{pmatrix} \nabla_\tau(\omega) \\ \nabla_\tau(\eta) \end{pmatrix} &= \frac{-1}{2\pi i} \begin{pmatrix} \eta_1'\omega_2 - \eta_2'\omega_1 & \omega_1\omega_2' - \omega_2\omega_1' \\ \eta_2\eta_1' - \eta_1\eta_2' & \eta_1\omega_2' - \eta_2\omega_1' \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix} \\
&= \frac{-1}{2\pi i} \begin{pmatrix} \eta_1' - \eta_2'\tau & -1 \\ \eta_2\eta_1' - \eta_1\eta_2' & -\eta_2 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix} \\
&= \frac{-1}{2\pi i} \begin{pmatrix} \eta_2 & -1 \\ (\eta_2)^2 - 2\pi i\eta_2' & -\eta_2 \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}
\end{aligned}$$

purely in terms of η_2.

Determining η_2

Let $q = e^{2\pi i\tau}$, and

$$P(q) = E_2(q) = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n, \text{ where } \sigma_1(n) = \sum_{d\geq 1, d|n} d$$

then

Lemma

$$\eta_2 = -\sum_m \sum_n{}' \frac{1}{(m\tau + n)^2} = \frac{-\pi^2}{3}P$$

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Proof.

$$\eta = X \, \mathrm{d}X / Y = \wp(z) \, \mathrm{d}z = -\mathrm{d}\zeta$$

$\square$

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where

$$\zeta = \frac{1}{z} + \sum_m \sum_n{}' \left(\frac{1}{z - m\tau - n} + \frac{1}{m\tau + n} + \frac{z}{(m\tau + n)^2} \right)$$

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$$\eta_2 = \int_{\gamma_2} \eta = \int_0^1 (-\mathrm{d}\zeta(z)) = \int_{\mathrm{z}}^{\mathrm{z}+1} (-\mathrm{d}\zeta(\mathrm{z})) = \zeta(\mathrm{z}) - \zeta(\mathrm{z}+1)$$

$$= \frac{1}{2} - \frac{1}{z+1} + \sum_m \sum_n{}' \left\{ \frac{1}{z - m\tau - n} - \frac{1}{z - m\tau - n + 1} - \frac{1}{(m\tau + n)^2}\right.$$

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$$\eta_2 = \frac{1}{z} - \frac{1}{z+1} + \sum_{m\neq 0} \sum_n \frac{-1}{(m\tau+n)^2} + \sum_{n \neq 0} \left\{ \frac{-1}{n^2} + \frac{1}{z-n} - \frac{1}{z+1-n} \right\}$$

$\square$

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Proof.

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$$\begin{aligned}\eta_2 &= \frac{1}{z} - \frac{1}{z+1} + \sum_{m \neq 0} \sum_n \frac{-1}{(m\tau + n)^2} + \sum_{n \neq 0} \left\{ \frac{-1}{n^2} + \frac{1}{z - n} - \frac{1}{z + 1 - n} \right\} \\ &= -\sum_m \sum_n{}' \frac{1}{(m\tau + n)^2}\end{aligned}$$

□

Aside on η_1

Similarly

$$\eta_1 = \zeta(z) - \zeta(z+\tau) = -\sum_n \sum_m{}' \frac{\tau}{(m\tau+n)^2}$$

giving

$$\eta_2(-1/\tau) = \tau_{\eta_1}(\tau)$$

so

$$\frac{\eta_2(-1/\tau)}{\tau} - \tau\eta_2(\tau) = 2\pi i$$

$$\implies \eta_2(-1/\tau) = \tau^2\eta_2(\tau) + 2\pi \mathrm{i}\tau$$

$$P(-1/\tau) = \tau^2 P(\tau) - \frac{6i\tau}{\pi}$$

In conclusion we have

$$\begin{pmatrix} \nabla_\tau(\omega) \\ \nabla_\tau(\eta) \end{pmatrix} = \frac{1}{2\pi i} \begin{pmatrix} \frac{\pi^2 P}{3} & 1 \\ \frac{\pi^4}{9}P^2 - \frac{12}{2\pi i}P' & -\frac{\pi^2}{3}P \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

We can consider this in terms of $\omega_{\mathrm{can}} = 2\pi i\omega$, $\eta_{\mathrm{can}} = \frac{1}{2\pi i}\eta$ and $\theta = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau} = q\frac{\mathrm{d}}{\mathrm{d}q}$ For the Tate curve over $\boldsymbol{C}((q))$, we have $\omega_{can} = \mathrm{d}t/t$.

$$\nabla(\theta)\begin{pmatrix} \omega \\ \eta \end{pmatrix} = \begin{pmatrix} \frac{-P}{12} & \frac{-1}{4\pi^2} \\ \frac{\pi^2}{36}\left(P^2 - 12\theta P\right) & \frac{P}{12} \end{pmatrix} \begin{pmatrix} \omega \\ \eta \end{pmatrix}$$

so

$$\nabla(\theta)\begin{pmatrix} \omega_{\mathrm{can}} \\ \eta_{\mathrm{can}} \end{pmatrix} = \begin{pmatrix} \frac{-P}{12} & 1 \\ \frac{P^2-12\theta P}{144} & \frac{P}{12} \end{pmatrix} \begin{pmatrix} \omega_{\mathrm{can}} \\ \eta_{\mathrm{can}} \end{pmatrix}$$

Kodaira-Spencer

As before let $f\colon S \to T$ be a smooth T-scheme, E/S an elliptic curve.

We can take a nowhere vanishing invariant differential $\omega \in \Omega^1_{E/S}$ and a derivation $D \in Der(S/T)$ and form the cup product

$$\langle \omega, \nabla(D)\omega \rangle \in \mathcal{O}_S$$

as we have ω in both sides this defines a pairing between $Der(S/T)$ and $\underline{\omega}^2$ for

$$\underline{\omega} = \pi_* \Omega^1_{E/S}$$

this gives a map

$$\underline{\omega}^{\otimes 2} \to \Omega^1_{\mathrm{S/T}}$$

Lemma
On the Tate curve over $\boldsymbol{Z}((q))$

$$\omega_{can}^{\otimes 2} \mapsto \mathrm{d}\boldsymbol{q}/\boldsymbol{q}$$

Proof.
We must check that

$$\langle \omega_{can}, \nabla(\theta)\omega_{can} \rangle_{\mathrm{dR}} = 1$$

but we already found

$$\nabla(\theta)\left(\omega_{\mathrm{can}}\right) = \frac{-P}{12}\omega_{\mathrm{can}} + \eta_{\mathrm{can}}$$

$\square$

The Gauss-Manin connection

To determine if a q-expansion $f(q) \in \boldsymbol{C}[[q]]$ is a modular form of weight k we must check if

$$f(q)(\omega_{can})^{\otimes k}$$

extends to all of $\underline{\omega}^{\otimes k}$. Viewing this inside of H^1_{dR} we ask instead that there exist $a, b \in \boldsymbol{N}$ with $a - b = k$ such that

$$f(q)(\omega_{can})^{\otimes a}(\eta_{can})^{\otimes b}$$

extends to

$$\mathrm{Symm}^{a+b}\left(H^1_{\mathrm{dR}}(E/S)\right)$$

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The Gauss-Manin connection is

$$\nabla : \mathrm{H}^1_{\mathrm{dR}}(\mathrm{E/S}) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(\mathrm{E/S}) \otimes \Omega^1_{\mathrm{S}/T}$$

We can tensor to get

$$\nabla \colon \mathsf{Symm}^{\mathrm{k}}\left(\mathrm{H}^1\right) \longrightarrow \mathsf{Symm}^{\mathrm{k}}\left(\mathrm{H}^1\right) \otimes \Omega^1_{S/T}$$

if

$$\Omega^1_{S/T} \simeq \underline{\omega}^{\otimes 2}$$

we can view this as

$$\sum_{j=0}^{k} \underline{\omega}^{\otimes k-2j} \longrightarrow \sum_{j=0}^{k} \underline{\omega}^{\otimes k-2j} \otimes \omega^{\otimes 2} = \sum_{j=0}^{k} \underline{\omega}^{\otimes k+2-2j}$$

The image of $f(q)$

Under the Gauss-Manin connection

$$
\begin{aligned}
f \mapsto & \theta(f) \cdot (\omega_{can})^{\otimes 2} \cdot (\omega_{can})^{\otimes a} \cdot (\eta_{can})^{\otimes b} \\
& + f \cdot a \cdot (\omega_{can})^{\otimes a-1} \left(\frac{-P}{12} \omega_{can} + \eta_{can} \right) \otimes (\omega_{can})^{\otimes 2} \otimes (\eta_{can})^{b} \\
& + f \cdot (\omega_{can})^{\otimes a} \cdot b \cdot (\eta_{can})^{\otimes b-1} \left(\frac{P^2 - 12\theta P}{144} \omega_{can} + \frac{P}{12} \eta_{can} \right) \cdot (\omega_{can})^{\otimes}
\end{aligned}
$$

$$
\begin{aligned}
= & \left(\theta(f) - (a-b) f \frac{P}{12} \right) (\omega_{can})^{\otimes a+2} (\eta_{can})^{\otimes b} \\
& + (af) (\omega_{can})^{\otimes a+1} (\eta_{can})^{\otimes b+1} \\
& + \left(bf \frac{P^2 - 12\theta P}{144} \right) (\omega_{can})^{\otimes a+3} (\eta_{can})^{\otimes b-1}
\end{aligned}
$$

So if f is modular of weight $k = a - b$ then

$$\begin{cases} \theta(f) - (a-b)f\frac{P}{12}, & \text{is modular of weight } a+2-b \\ af, & \text{is modular of weight } a-b \\ bf\frac{P^2-12\theta P}{144}, & \text{is modular of weight } a+3-b+1 \end{cases}$$

the operator

$$\partial(F) = 12\theta(f) - kPf$$

due to Serre therefore raises the weight by 2.

Corollary

$$P^2 - 12\theta P$$

is modular of weight 4 and hence

$$P^2 - 12\theta P = E_4 = Q$$

Proof.
Start with $f = 1$ modular of weight $1 - 1 = 0$ to see $P^2 - 12\theta P$ has weight 4. $\square$

Corollary (Deligne)

$$P = \frac{\theta\Delta}{\Delta}$$

Proof.

$$\theta(\Delta) - \Delta P = 0$$

as it lies in weight 14 and level 1.

$\square$