Machine Learning Crash Course: 4. Tree-based Methods

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This module

In this module we look at tree-based methods for both regression and classification. Trees divide the feature space into partitions and apply simple local models to the partitititions of the data. We introduce methods such as bagging, random forests, and boosting which combine multiple trees in order to get better results.

A Decision Tree

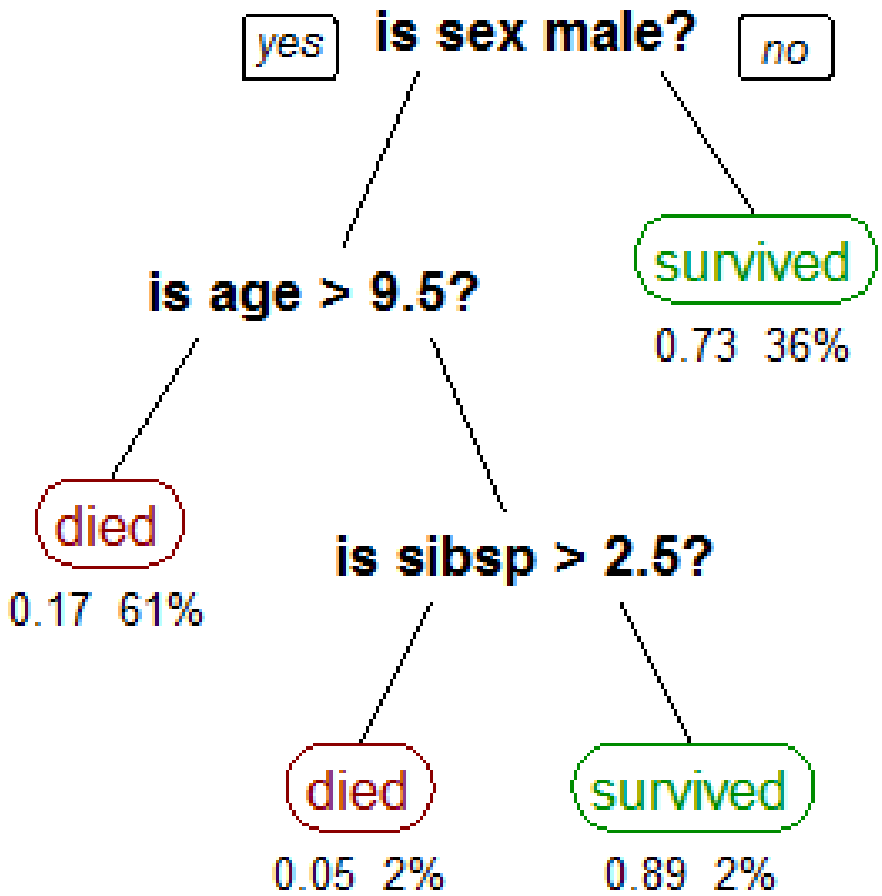

Figure: A Decision Tree for Titanic Survivors (Source: Wikipedia)

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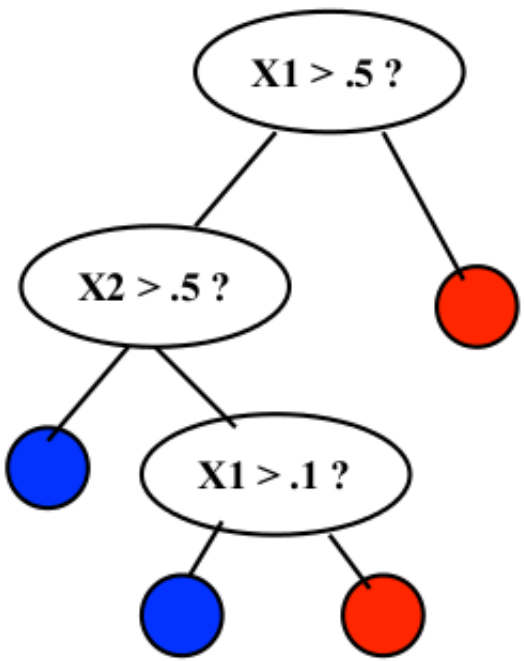
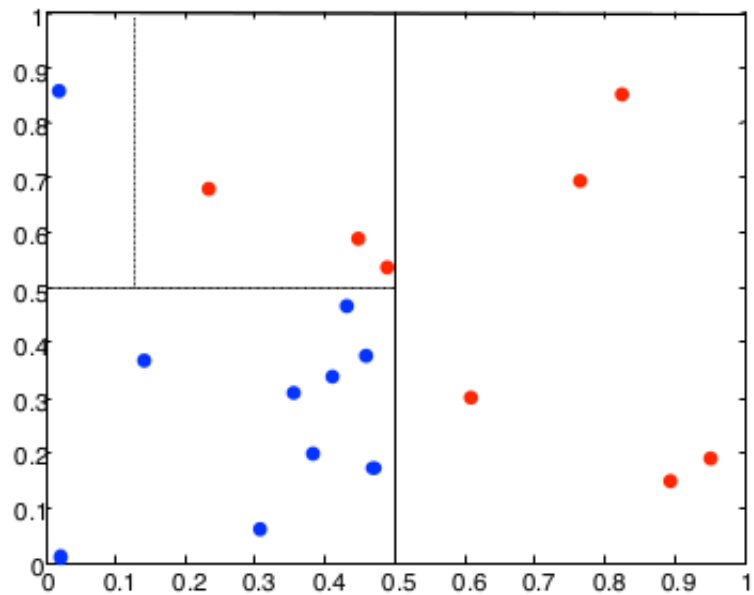
- Trees partition the feature space $X_1, X_2, \ldots$ into J distinct and non-overlapping regions $R_1, \ldots, R_J$:





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- Let's look at regression trees for now.

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- For each predictor X_j and given a cut-off value s define the two half-planes

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- Now we look for the predictor X_j and the value of s that minimizes the sum of the RSS in the two regions R_1 and R_2:

$$\sum_{i:x_i \in R_1} (y_i - \hat{y}_{R_1})^2 + \sum_{i:x_i \in R_2} (y_i - \hat{y}_{R_2})^2.$$

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- This algorithm is called recursive binary splitting.

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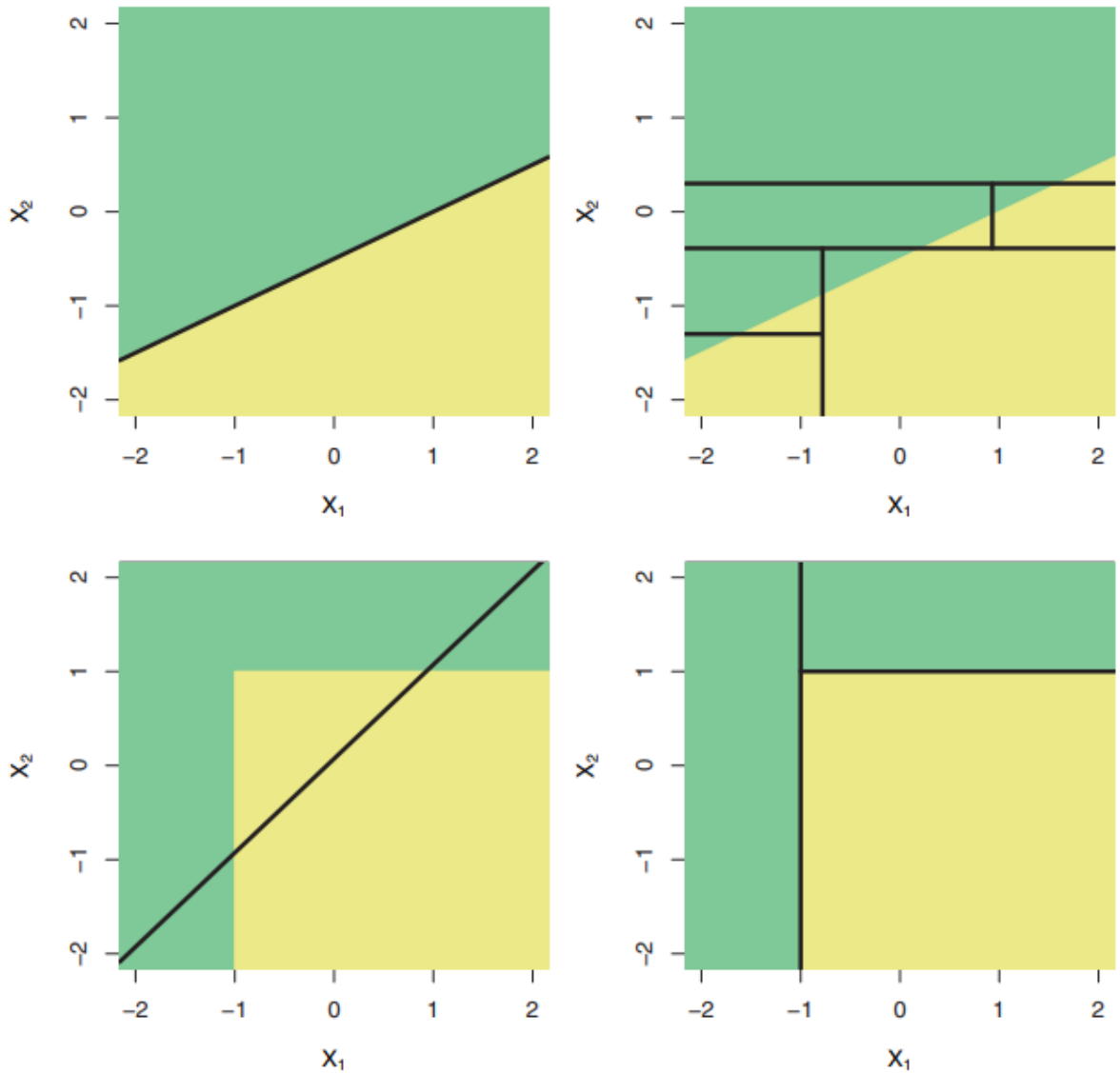
- How to determine α?

Trees vs Linear Methods



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- Bad news: overall quite bad performance, not competitive with most methods
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- Next we explore methods that combine multiple trees to give better performance

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- Method is general but works especially well with trees

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- This means we don't need to use cross-validation to estimate the test error because bootstrapping is already "kind of cross-validation"

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- Different set of predictors at every step, for every tree.
- Usually $p \approx \sqrt{m}$ (but in principle this is a tuning parameter).

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- In terms of decision trees, a weak learner is a shallow tree (i.e. only one or two splits)

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- This is called a slow learner

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- Parameters are chosen by Cross-validation

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- Take more time to compute because trees are grown sequentially