FFSS Problems

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May 17, 2019. There are more problems here than time to work.
Definitely do Problem 1. The rest are for fun.

1 Element Abundances

$$[\text{X/Y}] = \log\epsilon(X) - \log\epsilon(Y) - [\log\epsilon_\odot(X) - \log\epsilon_\odot(Y)] \quad (1)$$

$$[\text{X/Y}] = \log\frac{n_X}{n_Y} - \log\frac{n_{X,\odot}}{n_{Y,\odot}} \quad (2)$$

$$m_X = \mu_X n_X \quad (3)$$

n_X is the number density of element X, μ_X is the mean molecular weight of X.
$\log\epsilon(X)$ is proportional to log number density. It is also often written $A(X)$.

Element	$\log\epsilon_\odot(X)$	μ_X
H	12.00	1.0
C	8.43	12.0
N	7.83	14.0
O	8.69	16.0
Mg	7.60	24.3
Si	7.51	28.1
Ca	6.34	40.1
Ti	4.95	47.9
Fe	7.50	55.8
Ni	6.22	58.7
Sr	2.87	87.6
Ba	2.18	137.3
Eu	0.52	152.0

Table 1: Solar abundances

Photospheric abundances from Asplund+2009 `https://arxiv.org/abs/0909.0948`.

1.1 Write an equation relating the spectroscopic notation [X/H] to ratios of *number* densities and solar abundances

1.2 Write an equation relating [X/H] to ratios of *mass* densities

1.3 If a core-collapse supernova ejects 0.1 $M_\odot$ of Fe, and it sweeps up $10^5 M\odot$ of metal-free hydrogen before forming stars, what is [Fe/H] of stars that form out of that gas?

1.3.1 How much does your answer change if you use specific isotopic weights instead of the mean? How does this compare to a typical abundance error of 0.1-0.2 dex?

2 Cosmology

$$H^2(z) = \frac{\dot{a}}{a} = \frac{8\pi G}{3c^2}\rho(a) \tag{4}$$

$$\rho(a)/\rho_c = \Omega_{M,0}a^{-3} + \Omega_{r,0}a^{-4} + \Omega_{\Lambda,0} + \Omega_{k,0}a^{-2} \tag{5}$$

$$H_0 = 67.8\text{km/s/Mpc} \tag{6}$$

$$\rho_c = \frac{3H_0^2}{8\pi G} \tag{7}$$

Planck 2015: $\Omega_{M,0} = 0.31$, $\Omega_{r,0} = 10^{-4.04}$, $\Omega_{\Lambda,0} = 0.69$, $\Omega_{k,0} = 0.0$
Write a computer program to...

2.1 Calculate the age of the universe ($z = 0$, $a = 1$)

2.2 Calculate the age of the universe at several scale factors a

2.3 Calculate the temperature of the universe at several scale factors a

2.4 Plot the age of the universe vs scale factor

2.5 Plot the age of the universe vs redshift

3 Chemical Evolution 1: analytic metallicity distribution function

The first chemical evolution models were done completely analytically. If we assume instantaneous metal mixing and instantaneous recycling of ejected material, we can write down some basic differential equations describing the flow of mass and metals in a given system.

$$\frac{dM_t}{dt} = f \tag{8}$$

$$\frac{dM_s}{dt} = (1-R)\Psi \tag{9}$$

$$\frac{dM_g}{dt} = -(1-R)\Psi + f \tag{10}$$

$$\frac{d(Z\,M_g)}{dt} = -Z\Psi(1-R) + y\Psi(1-R) + Z_f f \tag{11}$$

where the variables are:

- $M_t(t)$: total mass, which is the sum of stars and gas
- $M_g(t)$: gas mass at a given time
- $M_s(t)$: stellar mass
- $Z(t)$: metal mass fraction in gas (so $M_Z = ZM_g$)
- f: infall rate of gas, Z_f: metallicity of infalling gas
- R: the return fraction, i.e. what fraction of stellar mass formed returns to the ISM
- y: the yield fraction, i.e. what fraction of stellar mass formed that is returned to the gas as metals (due to stellar nucleosynthesis)
- Ψ: star formation rate.

For simplicity we have neglected the fact that some mass gets trapped in remnants (white dwarfs, black holes, neutron stars).

The equations can be further simplified if we assume that $f = 0$, i.e. that no gas falls into our system. This is called the "closed box" model. Here we will derive the *metallicity distribution function* (i.e. the distribution of stellar metallicities) for the closed box model.

3.1 Draw a picture that schematically describes the differential equations as flows of mass and metals between the gas and stars.

3.2 Expand the derivative for Equation 11 and obtain an expression for $M_g \frac{dZ}{dt}$

3.3 Show that $\frac{dZ}{dM_g} = \frac{-y}{M_g}$

Note that the SFR has cancelled out.

3.4 Integrate your expression for $\frac{dZ}{dM_g}$.

assuming that the initial gas mass is $M_{g,i}$ that starts at zero metallicity $Z = 0$, and the final gas mass $M_{g,f} = 0$ and gas metallicity Z_f.

3.5 Calculate $\frac{d(M_s/M_t)}{dZ}$

This is the metallicity distribution function. You may also want to calculate $\frac{d(M_g/M_t)}{dt}$.

3.6 Calculate $\frac{dM_s}{d\log Z}$

Observations of metallicity are always in log, so this is the distribution that can be compared to observations.

4 Chemical Evolution 2: numerical chemical evolution model.

Write a simple numerical chemical evolution model that tracks just H, Mg, and Fe from core-collapse supernovae and Type Ia supernovae. This problem is based on one given by Evan Kirby (Caltech). We will assume a "leaky box" model, where we start with a ball of gas, then no gas falls in but supernova feedback removes gas. This is a toy model that does not accurately represent real galaxies, but is still extremely useful for building intuition.

4.1 Basic Galactic Chemical Evolution (GCE) Model Parameters

Use a Salpeter initial mass function:

$$\phi(m) = \frac{dN}{dm} = 0.17 \left(\frac{m}{M_\odot} \right)^{-2.35} \tag{12}$$

Let the gas mass be $M_g(t)$, with $M_g(0) = 10^9 M_\odot$ of pure hydrogen. Let the stellar mass be $M_s(t)$ with $M_s(0) = 0$.

Let the star formation rate $\Psi(t) = \frac{dM_s}{dt}$ be given by this law:

$$\frac{dM_s}{dt} = (5 \times 10^5 M_\odot \mathrm{Gyr}^{-1}) \left(\frac{M_g}{10^6 M_\odot} \right)^{-0.8} \tag{13}$$

where the coefficients have been chosen to be pretty close to observed data (Kennicutt-Schmidt star formation law), although in general they can vary.

Element	Type II	Type IIa
H	$8.64\ M_\odot$	0
Mg	$1.92 \times 10^{-1} M_\odot$	$8.57 \times 10^{-3} M_\odot$
Fe	$8.10 \times 10^{-2} M_\odot$	$7.49 \times 10^{-1} M_\odot$

Table 2: Supernova yields for H, Mg, Fe

We also need to assume supernova rates. Assume Type Ia supernovae follow this distribution measured in Maoz et al. (2010, ApJ, 722, 1879):

$$\Psi_{Ia} = \begin{cases} 0, & t_{\rm delay} < 0.1{\rm Gyr} \\ (10^{-3}\ {\rm SN\ Gyr^{-1}} M_\odot^{-1}) \left(\frac{t_{\rm delay}}{\rm Gyr}\right)^{-1.1}, & t_{\rm delay} \geq 0.1{\rm Gyr} \end{cases} \tag{14}$$

Let us implement feedback only using supernovae. For simplicity, let's assume that each supernova removes a constant amount of gas from the system, $10^4 M_\odot$ of gas for every supernova that forms. Let's also assume this number is the same for both Type Ia and Type II SNe.

We need yields for the different elements. Let's just assume they are constant for every supernova.

4.2 Make a computational model

The GCE equations are differential equations that you can set up to numerically integrate efficiently with things like Runge Kutta algorithms, but for concreteness and simplicity let's use Euler's method. (If you don't know what that means, don't worry.)

Initialization: Start by creating a timestep array from 0 to 10 Gyr, in timesteps of Δt =1 Myr. Create additional arrays that keeps track of the mass in gas of every element of interest at every time step, and the total mass formed in stars at every time step. Create two other arrays that keep track of the number of Type II and Type Ia supernovae at every timestep.

Model loop: Then, at each time step t_i:

1. Calculate what mass of stars should form in this timestep using the Kennicut-Schmidt relation.

$$\begin{aligned} \Delta M_s(t_i) &= \Delta t \Psi(t) \\ &= 1\ {\rm Myr} \times (5 \times 10^5 M_\odot {\rm Gyr}^{-1}) \left(\frac{M_g(t_i)}{10^6 M_\odot}\right)^{-0.8} \end{aligned}$$

2. Calculate how many Type II supernovae should form at that time step using the initial mass function:

$$N_{II}(t_i) = \Delta M_s(t_i) \int_{10M_\odot}^{100M_\odot} \phi(m) dm$$

   Record this number and add the resulting supernova yields to gas mass at the current timestep.

3. Calculate how many Type Ia supernovae explode at this timestep. This integral is:

$$\begin{aligned} N_{Ia}(t_i) &= \Delta t \int_0^{t_i} \frac{dM_s}{dt}(t - t_{\rm delay}) \Psi_{Ia}(t_{\rm delay}) dt_{\rm delay} \\ &= \Delta t \int_{0.1{\rm Gyr}}^{t_i} 10^{-3} t_{\rm delay}^{-1.1} \frac{dM_s}{dt}(t_i - t_{\rm delay}) dt_{\rm delay} \end{aligned}$$

   In other words, a loop through all previous timesteps to see how many Type Ia at each of those timesteps contribute to the current timestep. Record the number of Ias and add their yields of H, Mg, and Fe to the gas mass at the current timestep.

4. Remove $10^4 M_\odot$ of gas (split proportionally amongst the current H, Mg, and Fe) for each Type II and Type Ia SN.

Repeat until the model runs out of gas.

4.3 Plot

Plot [Mg/Fe] vs [Fe/H]. Do you see a "knee" in [Mg/Fe]? If so, what [Fe/H] does it occur at, and what does that mean? If not, why not?

Plot [Fe/H] vs time. What do you see?

Plot [Mg/H] vs time. What do you see?

Plot the metallicity distribution function. You can do this by assuming that all stars formed at a given timestep have the same metallicity. Another way to think of this is to add the gas metallicity at every timestep together, weighted by the stellar mass formed at every timestep.

You can play with the different parameters (yields, SN feedback, star formation efficiency equation, adding gas at each timestep, etc). You can also add other elements of interest or sources of elements to the problem.

5 Playing with stellar abundance data

JINA investigators have collected a bunch of stellar abundances into one big table. You can look at the data from this interactive website interface: `https://jinabase.pythonanywhere.com`

Buyer beware: this is a literature compilation, with little quality control on individual stellar abundances. (For instance, many of the outliers are analysis errors that no one has gone back to fix.)

These elements are roughly associated with specific types of nucleosynthesis. Plot some abundance trends based on these and see if they make sense with your intuition.

Mg: Mostly hydrostatic CCSN

Ca: Mostly explosive CCSN

Fe: Both Type Ia+CCSN

Sr: a whole bunch of different things

Eu: mostly r-process

Ba: mostly s-process

Some suggestions: plot only MW Halo stars, and remove upper limits.