*All the code runs using python3 and it uses numpy, scipy and matplotlib as dependencies.*

$\boxed{1}$

$$D(h, m+1) = \frac{4^m}{4^m - 1} D(h, m) - \frac{1}{4^m - 1} D(h-1, m)$$

$$D(h, m) = L + \sum_{k=m}^{\infty} A(k, m) \left(\frac{h}{2^n}\right)^{2k}$$

proving the recursion we will use later

$$A(k, m+1)(4^m - 1) = A(k, m)(4^m - 2^{2k}) \Rightarrow A(k, m+1) = A(k, m) \cdot \frac{4^m - 2^{2k}}{4^m - 1}$$

Inserting the 2nd into the first we get

$$D(h, m+1) = \frac{4^m}{4^m - 1}\left(L + \sum_{k=m}^{\infty} A(k, m)\left(\frac{h}{2^n}\right)^{2k}\right) - \frac{1}{4^m - 1}\left(L + \sum_{k=m}^{\infty} A(k, m)\left(\frac{h}{2^{n-1}}\right)^{2k}\right)$$

rearranging with common multipliers:

$$= \frac{L(4^m - 1)}{4^m - 1} + \sum_{k=m}^{\infty} A(k, m)\left(\frac{h}{2^n}\right)^{2k} \frac{1}{4^m - 1}\left[4^m - 2^{2k}\right]$$

$A(k, m+1)$

using the formula for $A(k, m+1)$:

$$= L + \sum_{k=m}^{\infty} A(k, m+1)\left(\frac{h}{2^n}\right)^{2k}$$

and we have proved the recursion.

## 2

*The code runs at richardson.py and prints the results*

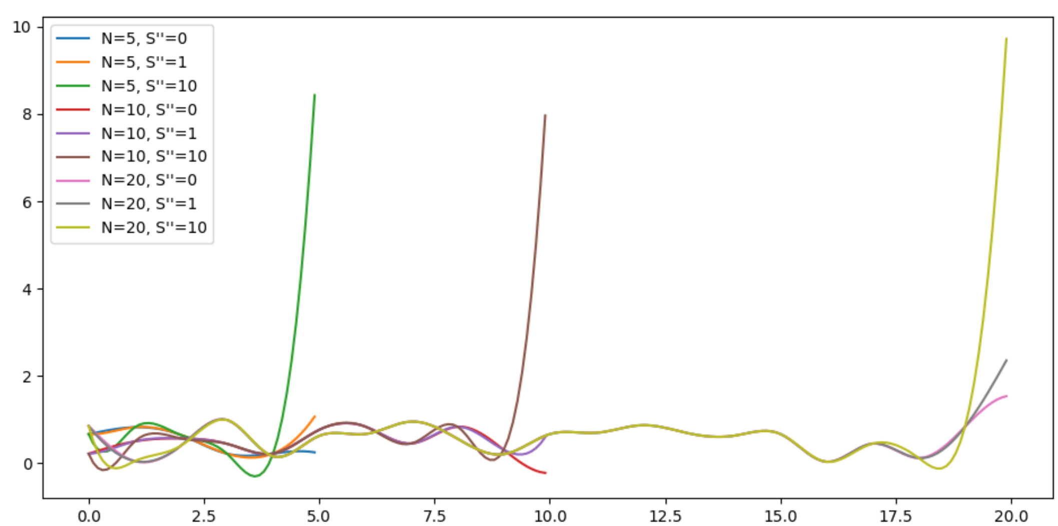
Using RE (getting -0.6865307826651446)  and then comparing it to the derivative calculated using the central differential form we get:

```
Richardson extrapolation N=5, h=0.1 for f(x): -0.6865307826651446
h=0.0001, error=2.4596089969186608e-05
h=1e-06, error=2.459119841091706e-07
h=1e-08, error=1.1543903610800044e-08
h=1e-10, error=4.9518362743583566e-08
h=1e-12, error=8.66469142835058e-05
h=1e-14, error=0.0018074926024524984
```

## 3

*Code located at cubic_spline.py and runs using python(3) and has scipy, numpy, matplotlib as dependencies*

Hope I understood it correctly, I create a random point at each i = 1, 2, … N with N=5,10,20. Getting the following plot:

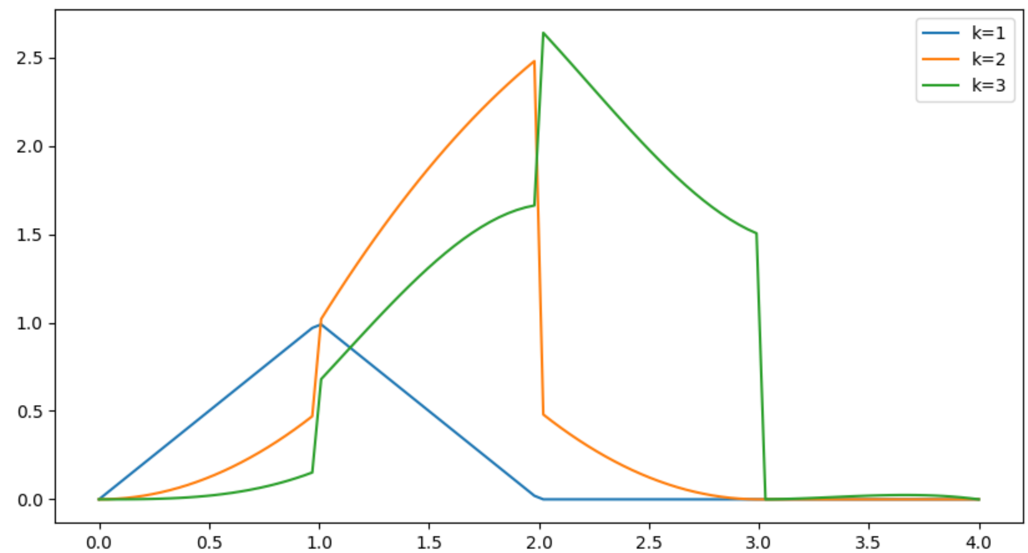



We can see that when S''=10 that ending of the function rises up.

4

I did the calculations manually, hoping I understood everything correctly. At the next pages you will find the parametrized polynomial up to k=3 and then the result when we set i=0.
You can run the code at b_spline.py using python(3)

Here is the plot of all three:



$\boxed{4}$ write down B-spline for $k=1,2,3$

$$N_i^0(t)=\begin{cases}1 & t\in[t_i,t_{i+1})\\ 0 & \text{otherwise}\end{cases}$$

$\underline{k=1}$

$$N_i^1(t)=\left(\frac{t-t_i}{t_{i+1}-t_i}\right)N_i^0(t)+\left(\frac{t_{i+2}-t}{t_{i+2}-t_{i+1}}\right)N_{i+1}^0$$

$$=\begin{cases}\dfrac{(t-t_i)}{t_{i+1}-t_i} & \text{if } t\in[t_i,t_{i+1})\\[2ex] \dfrac{t_{i+2}-t}{t_{i+2}-t_{i+1}} & t\in[t_{i+1},t_{i+2})\\[2ex] 0 & \end{cases}$$

$\underline{k=2}$

$$N_i^2(t)=\left(\frac{t-t_i}{t_{i+2}-t_i}\right)N_i^1(t)+\left(\frac{t_{i+3}-t}{t_{i+3}-t_{i+1}}\right)N_{i+1}^1(t)$$

$$=\begin{cases}\dfrac{(t-t_i)^2}{(t_{i+1}-t_i)(t_{i+2}-t_i)} & t\in[t_i,t_{i+1})\\[2ex] \dfrac{(t-t_i)(t_{i+2}-t)}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}+\dfrac{(t_{i+3}-t)(t-t_{i+1})}{(t_{i+3}-t_{i+1})(t_{i+2}-t_{i+1})} & t\in[t_{i+1},t_{i+2})\\[2ex] \dfrac{(t_{i+3}-t)^2}{(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})} & t\in[t_{i+2},t_{i+3})\\[2ex] 0 & \text{else where}\end{cases}$$

$\underline{K=3}$

$$N_i^3 = \frac{t-t_i}{t_{i+3}-t_i} N_i^2(t) + \frac{t_{i+4}-t}{t_{i+4}-t_{i+1}} N_{i+2}^2 =$$

$$= \frac{(t-t_i)^3}{(t_{i+1}-t_i)(t_{i+2}-t_i)(t_{i+3}-t_i)} \quad , \quad \text{at} \quad t \in [t_i, t_{i+1})$$

$$\frac{t-t_i}{t_{i+3}-t_i} \cdot \left( \frac{t-t_i}{t_{i+2}-t_i} \cdot \frac{t_{i+2}-t}{t_{i+2}-t_{i+1}} + \frac{t_{i+3}-t}{t_{i+3}-t_{i+1}} \frac{t-t_{i+1}}{t_{i+2}-t_{i+1}} \right) \quad \text{at} \quad t \in [t_{i+1}, t_{i+2})$$
$$+ \frac{t_{i+4}-t}{t_{i+4}-t_{i+1}} \frac{(t-t_i)^2}{(t_{i+1}-t_i)(t_{i+2}-t_i)}$$

$$\frac{(t-t_i)}{(t_{i+3}-t_i)} \frac{(t_{i+3}-t)^2}{(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})} + \frac{t_{i+4}-t}{t_{i+4}-t_{i+1}} \left( \frac{t-t_i}{t_{i+2}-t_i} \cdot \frac{t_{i+2}-t}{t_{i+2}-t_{i+1}} + \quad \text{at} \quad t \in [t_{i+2}, t_{i+3})\right.$$
$$\left. + \frac{t_{i+3}-t}{t_{i+3}-t_{i+1}} \cdot \frac{t-t_{i+1}}{t_{i+2}-t_{i+1}} \right)$$

$$\frac{t_{i+4}-t}{t_{i+4}-t_{i+1}} \cdot \frac{(t_{i+3}-t)^2}{(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})} \quad \text{at} \quad t \in [t_{i+3}, t_{i+4})$$

$$0 \quad \text{at elsewhere}$$

$\boxed{4}$ Now for $t_i = i$ and starting from 0 we calculate

k=1

$$N_1^1(t) = \begin{cases} \dfrac{t}{1} & t \in [0,1) \\ \dfrac{2-t}{1} & t \in [1,2) \\ 0 & \text{elsewhere} \end{cases}$$

$$N_2^2 = \begin{cases} \dfrac{t^2}{2} & t \in [0,1) \\ \dfrac{(t)(2-t)}{2} + \dfrac{(3-t)(t-1)}{2 \cdot 1} = \dfrac{-t^2+6t-3}{2} & t \in [1,2) \\ \dfrac{(3-t)^2}{2} & t \in [2,3) \\ 0 & \text{elsewhere} \end{cases}$$

$N_0^3(t) =$

$$
\begin{cases}
\dfrac{t^3}{6} & t \in [0,1) \\[2ex]
\dfrac{(t-0)}{3} \cdot \left( \dfrac{t}{2} \cdot \dfrac{(2-t)}{1} + \underbrace{\dfrac{(3-t)}{2} \cdot \dfrac{(t-1)}{1}}_{\frac{-t^2+4t-3}{2}} \right) + \dfrac{(4-t)}{3} \cdot \dfrac{t^2}{2} & \\[2ex]
= \dfrac{2t^2 - t^3}{6} + \dfrac{3t^2 - 3t - t^3 + t^2}{6} + \dfrac{4t^2 - t^3}{6} = \dfrac{-3t^3 + 10t^2 - 3t}{6} & t \in [1,2) \\[2ex]
\dfrac{t}{3} \cdot \dfrac{(3-t)^2}{2} + \dfrac{(4-t)}{3} \left( \dfrac{2t - t^2 - t^2 + 4t - 3}{2} \right) & \\[2ex]
= \dfrac{9t - 6t^2 + t^3}{6} + \dfrac{(4-t)}{3} \cdot \dfrac{(-2t^2 + 6t - 3)}{2} = \dfrac{3t^3 - 20t^2 + 36t}{6} & t \in [2,3) \\[2ex]
\dfrac{(4-t)}{3} \cdot \dfrac{(3-t)^2}{2 \cdot 1} = \dfrac{(4-t)}{3} \cdot \dfrac{(9-6t+t^2)}{2} = \dfrac{36 - 24t + 4t^2 - 9t + 6t^2 - t^3}{6} & \\[2ex]
= \dfrac{-t^3 + 10t^2 - 33t + 36}{6} & t \in [3,4) \\[2ex]
0 & \text{else}
\end{cases}
$$