Alex Jia
157000846

CS 206
Section 05

Homework 04

1. What is the conditional probability that exactly 2 heads appear in 4 independent tosses of a fair coin, given that the first toss lands tails?

Knowing that the first toss of the four independent tosses lands T (tails), then the sample space of possible tosses remaining are:
{ (T,T,T,T), (T,T,T,H), (T,T,H,T), (T,T,H,H), (T,H,T,T), (T,H,T,H), (T,H,H,T), (T,H,H,H) }.
Out of the 8 possibilities, only 3 result in exactly two heads appearing. Therefore, the answer is 3/8.

2. Suppose we have a damaged die in which the 6 shows like a 3. Assume we throw the die twice, the throws being independent, and record the values A and B.

For this problem, I am interpreting the damaged, six sided die to essentially have two '3's (i.e it's sample space will be {1,2,3,3,4,5}).

   a. What is the expected value of A + B?

   To find the E[A+B], we must note that A+B yields a range of values from 2 (or 1+1) to 10(or 5+5). The expected value of A+B will be the summation of the probabilities of all the aforementioned outcomes. E[A] = (1*P(1)+2*P(2)+3*P(3)+4*P(4)+5*P(5)) = 1/6 + 2/6 + 6/6 + 4/6 + 5/6 = 18/6 = 3. Since the only difference between A and B is the fact that they account for separate rolls of the same die, we can also say that E[B] = E[A], which is 3. Thus, E[A+B] = 3 + 3 = 6

   b. What is the expected value of the product AB?

   Utilising the calculations I have produced in part a, I have established that E[A] = E[B] = 3. Thus, E[AB], or E[A*B] = 3*3 = 9

   c. What is the expected value of B given A = 2?

   Since the throws for A and B are independent of each other, E[B|A=2] should not be any different from that of E[B] by itself, as A's result does not affect B's in any way (given the constraints of the current problem). Thus, our answer is still 3.

3. Suppose you bought an intelligent AI machine to solve the homework problems. However, in all machines shipped from the manufacturer, 99% of them are "good" and 1% of them are "bad". The good machines have probability 1/30 of failing after solving a problem. The bad machines are probability ½ of failing after solving a problem. Let T denote the number of problems solved until the machine fails, what is E[T]?

E[T] = 0.99*29/30 + 0.01*1/2 =
0.962 * 100 = 96.2
This means out of 100 problems, the expected number of problems solved before failing is 96.2

4. 100 homeworks are on the table, with two questions to be graded on each homework. Aditya is in charge of grading question one and Jingjing is in charge of grading question two. First, Adytia grades some homeworks at random; each homework has probability 0.3 of being graded. Next, Jingjing randomly grades half the homeworks. That is, he grades 50 homeworks. Assume Aditya and Jinjing make their choices independently.
    a. Let N be the number of homeworks that Aditya graded. What is E[N]?
    b. Let M be the number of homeworks that Aditya graded and Jingjing did not grade. What is E[M]?

5. A 52-card deck is thoroughly shuffled and you are dealt a hand of 13 cards.
    a. If you have one ace, what is the probability that you have a second ace?

Assuming that we are dealing with a traditional 52 card deck (thus having 4 aces), we can rethink the question to ask us: if we are given a 13 card hand, what is the probability of there being at least 2 aces, given that one of the 13 cards is already known to be an ace?

This can be thought of as solving for the possible hands with at least 2 aces / possible hands with at least 1 ace (since 1 ace is already given).

This is:

((4C2)*(48C11) + (4C3)*(48C10) + (4C4)*(48C9)) / ((4C1)*(48C12) + (4C2)*(48C11) + (4C3)*(48C10)+(4C4)*(48C9))

= [6 *(22595200368) + 4*(6540715896) + 1*(1677106640)] / [4*69668534468 + 6*(22595200368) + 4*6540715896 + 1*1677106640]

= 0.36963719133

    b. If you have the ace of spades, what is the probability that you have a second ace?

This is extremely similar to the previous part, except that we know exactly which ace we have in our hand (as opposed to a random one before). We can think of the problem as asking: Give a 13 card hand, what is the probability of getting at least two aces, knowing that you already have one of the aces in your possession (A. of Spades)?

This is:

[(1C1) * (3C1) * (48C11) + (1C1) * (3C2) * (48C10) +(1C1) * (3C3) * (48C9)] / [(1C1) * (52C12)]

= [3 * 22595200368 + 3 * 654071896 + 1 * 1677106640] / 206379406870

= 0.34608551558

6. A random variable X can have the value -1, 1, 2, or ½. Suppose that P(X = 2) = P(X=½)=⅓. Find values a = P(X=1) and b = P(X=-1) such that E(X) = 1.

Knowing the formula for finding the expected value, and the fact that E[X] = 1:

E[X] = -1 * P(X=-1) + 1*P(X=1) + 2 * 1/3 + 1/2 * 1/3 = 1.

1 = 2/3 + 1/6 + -1a + 1b

1 = 5/6 + -1a + 1b.

1/6 + a= b

This is essentially telling us that we can have multiple values for a and b, so long as they follow the relationship of being within 1/6 of each other's values. Thus, a= b - 1/6, and b = a + 1/6 , with reasoning provided by the steps provided above.

7. Urn A contains 1 red and 3 blue balls, urn B contains 2 red and 5 blue balls. One randomly chosen ball is transferred from urn B to urn A. Next, a random ball is chosen from urn A. What is the probability that the ball is red?

Before the transfer: P(red picked from A) = ¼. After the transfer: P(red picked from A) = P(Red taken from B) * 2/5 + P(Blue taken from B) * 1/5. Since we know the ball transfer process is entirely random:

P(Red taken from B) = 2/7, and P(Blue taken from B) = 5/7.

Thus, the entire answer is:

2/7 * 2/5 + 5/7 * 1/5.

=4/35 + 5/35

=9/35.

8. A deck of 52 cards is in random order, and the cards are turned up, one at a time.
   a. What is the probability that the fifth card turned up is an ace?

   1/13. This can be easily deduced from the concept that there's only really 4 cards that can fulfill this condition (one of the four aces, out of fifty-two cards). Thus: 52*51*50*49*4*48! / 52! = 4 / 52 = 1/13

   b. What is the probability that the fifth card turned up is the first ace?

   This one is slightly trickier; we must compute the probability of having the first four cards NOT be an ace, and the fifth card BEING an ace. This can be computed as simply:

   (48 * 47* 46 * 45 * 4) / (52 * 51 * 50 * 49 * 48)

   = 0.0598947271

   c. Given that the fifth card turned up is the first ace, what is the probability that the next card is the ace of spades?

   To rethink of this problem in simpler terms (granted that I am sleep deprived from HackRU), we can consider the problem to be like that of conditional probability: The 'first' card (technically the fifth in this case) is an Ace (either a Spade or not), and we're asked to calculate the probability of drawing an Ace of Spades in the next card:

   P(A. of Spades) * P(A. of Spades next) + ~P(A. of Spades) * P(A. of Spades next)

   = 1/4 * 0 + 3/4 * 1/(52-5)

   = 3/4 * 1/47

   = 0.0159574468