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CS 206
Section 05

Homework 01

1. Given S = {1,2, … , n}. How many unordered pairs {A, B} are there where A, B are nonempty subsets of S with A intersect B = NULL.

2^n - 1 pairs.

2. Consider the word "OHMYGODIMONFIRE". How many distinguishable ways are there to rearrange the letters?

15! / (3!*1!*2!*1!*1!*1!*2!*1!*1!*1!)
= 15! / (3!*2!*2!)
= 15! / 24
= **54,486,432,000 ways**.

3. How many sequences of 1s and -1s of length 10 are there that sum up to 2?

tests a combo of '11-11-11-11-11', sees 6 ones, 4 negative ones
= *rearrange question to ask for total # ways to rearrange six '1's and four '-1's
= *sees similarity with solving #2*
= 10! / (6!* 4!)
= (10 * 9 * 8 * 7) / (4 * 3 * 2 * 1)
= (10 * 3 * 1) / (1 * 1)
= **30 sequences**

4. How many 4-letter words can be obtained using any of the 26 letters of the alphabet, if repetition of letters is allowed?

26^4
= **456976 words**

5. How many 4-letter words can be obtained using any of the 26 letters of the alphabet, if repetition is not allowed?

26*25*24*23
= **358800 words**

6. How many 4-letter words contain at least one repeated letter?

Total combos (#4, essentially) - combos w/ no repeat (#5)
= 26^4 - 26*25*24*23
= **98176 words**

7. How many 4-letter words contain the letter X?

Total # 4-letter words - Total # 4-letter words w/out *any* 'X's
= $26^4 - 25^4$
= **66351 words**

8. How many 4-letter words consist of only the letters X and/or Y? (The words XXXX and YYYY are included in this count.)

Treating this like "how many 4-digit combos are available with binary digits"
= let X be 0, Y be 1

= max 4-digit binary combo is 1111, or the decimal equiv of 15

= 15 + 1 (to include ‘0000’)

= **16 words**

2.1 Bad Induction Proofs

a. Claim: ln 2 is a rational number. (Note that ln 2 = 1 - ½ + ⅓ - ¼ + … + $((-1)^{n+1})$ / n) = ∑(n=1, ∞) $((-1)^{n+1})$ / n. This is done by proving that 1 - ½ + ⅓ - ¼ + … + $((-1)^{n+1})$ / n is rational.

Proof by induction on n.

Base case: n = 1: 1 is obviously rational.

Inductive hypothesis: Suppose that 1 - ½ + ⅓ - ¼ + … + $(-1)^{k+1}$) / k is rational.

Inductive step: We need to show that 1 - ½ + ⅓ - ¼ + … + $(-1)^{k+2}$) / k+1 is a rational number. Observe that 1 - ½ + ⅓ - ¼ + … + $(-1)^{k+2}$) / k+1 = (1 - ½ + ⅓ - ¼ +...+$(-1)^{k+1}$ / k) + $(-1)^{k+2}$) / k+1. By the induction hypothesis, 1 - ½ + ⅓ - ¼ + … + $(-1)^{k+2}$) / k+1 is a rational number. Furthermore, $(-1)^{k+2}$) / k+1 is a rational number, as it can be expressed in a fraction. Thus, summing two rational numbers will result in another rational number, and by induction we have proven that ln 2 is rational.

Answer: The error exists in the *Inductive step*. The proof claims that $(-1)^{k+2}$) / k+1 is a rational number based off the reasoning that “it can be expressed as a fraction”. This is in no way based off the inductive hypothesis or anything else that has been mentioned explicitly earlier in the proof. Additionally, the final line does not take indicate how the sum of a rational number (based off the inductive hypothesis’ claim) with $(-1)^{k+2}$) / k+1 indicates a rational number, and how that links back to the original claim that “ln 2 is a rational number”.

b. Claim: log(base 15) n = log(base 251)1 for all natural numbers n.

Proof (by strong induction)

The inductive hypothesis is “log(base 15)n = log(base 251)n”

Base case: log(base 15)1 = 0 = log(base 251)1

Induction Hypothesis: log(base 15)k = log(base 251)k for all natural numbers k <= n

Inductive step: We wish to show that the claim is true for n + 1. Write n + 1 as a product of two natural numbers p and q so that we have:

log(base 15)(n+1) = log(base 15)(pq) = log(base 15)p+log(base 15)q = log(base 251)p + log(base 251)q = log(base 251)(pq) = log(base 251)(n+1) which is true by inductive hypothesis.

Answer: So according to the lecture slides, *strong induction* requires one to prove that given a premise P(n); P(base case), P(base case+1), P(base case+2), …, P(base case+n) implies P(base case+n+1) for all non-negative integers n. The aforementioned proof, however, does not do this (in the *Inductive step*). This alone renders the proof wrong (at least, in terms of strong induction).

c. Claim: 1 / 1*2 + 1 / 2*3 + … + 1/ (n-1) * n = 3/2 = 1/n

Proof by induction on n.

Base case: For n=1, 3/2 - 1/n = 1 / 1*2.
Induction Hypothesis: Assume the theorem is true for n.
Inductive step: 1 / 1*2 + 1 / 2*3 + … + 1 / (n-1) * n + 1 / n * (n+1) = 3/2 - 1/n + 1 / (n * (n+1)
= 3/2 - 1/n + (1/n - 1/n+1)
= 3/2 - 1/n+1

Answer: The base case is wrong. (the claim indicates a summation of various fractions up to (1 / (n - 1) * n) equals 3/2 - 1/n. Since n = 1 in this scenario, the left hand side of the claim would sum the various fractions up to 1 / (1-1) * 1, which is 1 / 0*1, which is a division by zero (the right hand side of the claim checks out, though). Therefore, either the claim was not explicit enough, or the base case is flawed (I suspect the former, but the latter works with the information given). Additionally, the *Inductive Hypothesis* does not

2.2 Piecewise Monotonic Sequences

Let a1, a2, …, an be a permutation of 1, 2, … n. We say that this sequence is piecewise monotonic if each element ai is either larger than all elements before it, or smaller than all elements before it in the permutation. How many piecewise monotonic permutations are there of 1, 2, …, n.

2.3 Balanced Strings

Given a binary string S, let ∆ (S) = number of 1's in S - number of 0's in S. S is balanced if -2 <= ∆ (S) <= 2. How many balanced binary strings of length n are there. Provide a formal proof of your answer.

2.4 Induction

Show that every number in the sequence 1007, 10017, 100117, 1001117, … is divisible by 53.

Claim:

2.5 Counting

200 students participated in a math contest. They had six problems to solve. Each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these students.