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CS 206
Section 05

Homework 03

1. Three fair dice colored red, blue and green are rolled. What is the probability that at least two of them roll the same number?

Given three different dice,, the probability of at least two of them rolling the same number is equivalent to two die rolling the same number + all three die rolling the same number. This is equivalent to 6 (3 die) + 90 (2 die) combinations. So out of the 6^3 combinations, we get a probability of 96 / 216, or 4/9.

2. Three fair dice colored red, blue and green are rolled. What is the probability that at exactly two of them roll the same number?

Given that there are 3 distinguishable die, we can say that out of the total 216 possible combinations, about 90 exist that follow this constraint, so our answer is 90/216, or 5/12.

3. Three fair dice colored red, blue and green are rolled. What is the probability that at exactly two of them roll the same number given that the sum of the three numbers is 10?

Given three dice, the probability of exactly two of them rolling the same number is 90/216. However, by adding the additional constraint that the sum of all three numbers be equal to 10, this number reduces drastically, to 9/216, or 1/24.

4. Using the axioms of probability, prove that for any experiment and associated sample space Ω, P(φ) = 0. In other words prove that the probability of the event corresponding to the empty set is 0.

Let's say X represents "any experiment". We already know that the sample space P(Ω) is equal to 1. In this scenario, $P(\Omega) = P(X_1) + P(X_2) + \dots + P(X_n)$, where n goes on until inifinity. By properties of what we already know about the empty set, X U φ = X, and X ∩ φ = φ, which basically means that the two are disjoint, and we can now represent $P(X_k)$ (with k being any integer >=1) as P(φ). Thus, we can conclude with the simplification that

$$\begin{aligned} 1 &= P(X_1) + [P(X_2) + \dots + P(X_n)], \\ &= P(\Omega) + [P(X_2) + \dots + P(X_n)], \\ &= P(\Omega) + [P(\varphi) + \dots + P(\varphi)], \\ &= 1 + P(\varphi) \end{aligned}$$

And finally, P(φ) = 0.

5. 3 rabbits are playing outside their individual holes. An eagle comes and they all randomly go into a hole with one rabbit per hole. What is the probability that no rabbit went to its own hole? Assume that all configurations are equally likely.

This is basically an extension of the inclusion/exclusion principle. P(Rabbit A not entering it's own hole) = ⅓. P(Rabbit A&B not entering their own hole) = ⅓*½. Thus, P(Rabbits A, B, and C not entering their own hole) =

P(A) + P(B) + P(C) - P(AUB) - P(BUC) - P(AUC) + P(AUBUC) =

⅓ + ⅓ + ⅓ - (⅓ * ½) - (⅓ * ½) - (⅓ * ½) + (⅓ * ½ * 1) =

1 - (⅙) - (⅙) - (⅙) - (⅙) + (⅙) = 1/6.

6. You are at a casino table and the rule of the game is the following: At each step a fair coin is tossed. If it comes up H, you get 1 dollar and if it comes up tails you owe 1 dollar. What is the probability that after 100 steps you don't owe anything.

Since flipping a coin in this scenario can only yield you two results (we're not considering the 'landed-on-its-side' possibility), we can apply the Binomial Theorem. We're aiming for a result where we get half H and half T (which means either 50 H or 50 T out of 100 flips, for a total net loss of 0 (i.e not owing anything). This can be thought of as 100C50 * 1/(2^100).

7. 8 identical chocolates are randomly divided among 3 kids. In other words, each possible way to divide is equally likely.
    a. What is the probability that kid 1 gets at least 3 chocolates.

This is equivalent to 1 minus the sum of the probability of kid 1 getting 0, 1, and 2 chocolates. For P(kid1 = 0), we literally calculate the odds of splitting 8 chocolates amongst the remaining 2 kids. This is = (8 + 2 - 1) C (2 - 1), or 9.
For P(kid1 = 1), we calculate the same thing, but this time with 7 chocolates. This is (7+2-1) C (2-1), or 8. Finally, we calculate P(kid1 = 2). This will be (6 + 2 - 1) C (2 - 1), or 7. Now, if we sum these values and subtract from 1, we'll get a number less than 0 (which is not possible). This is because we didn't divide from the total number of possibilities, which is splitting 8 chocolates amongst the three kids. This total is (8 + 3 - 1) C (3 - 1), or 10C2, or 45. Thus, our actual answer is 1- ((9 + 8 + 7) / 45), or 21/45.

    b. What is the probability that kid 1 get at least 3 chocolates given that kid 2 received 3 chocolates.

This is basically further restricting our answer from part a. Our scenario is this:

Kid1 + Kid2 + Kid3 = 8

1. **0 + 3 + Kid3 = 8**
2. **1 + 3 + Kid3 = 8**
3. **2 + 3 + Kid3 = 8**

We just need to subtract the sum of these probabilities from 1.

1. **(5+1-1) C (1-1) = 5C0 = 1**
2. **(4 + 1 - 1) C (1-1) = 4C0 = 1**

3. (3 + 1 - 1) C (1 - 1) = 3C0 = 1

But, as we recall from part a, we didn't account for the total number of ways to split 8 chocolates amongst three kids; This is (5 + 2 - 1) C (2 - 1), or 6C1, or 6.. Our actual answer is this: 1 - ((1+1+1)/6), or 1/2.

8. 12 distinguishable pigeons are randomly distributed among 11 boxes with each outcome being equally likely. What is the probability that no box has more than 2 pigeons.

If the distribution of pigeons to boxes was thought of like this:
2 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 12
Which is equivalent to:
(12C2)(10C1)(9C1)(8C1)(7C1)(6C1)(5C1)(4C1)(3C1)(2C1)(1C1)
2 + 2 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 0 = 12
Which is equivalent to:
(12C2)(10C2)(8C1)(7C1)(6C1)(5C1)(4C1)(3C1)(2C1)(1C1)(0C1)
2 + 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1 + 0 + 0 = 12
Which is equivalent to:
(12C2)(10C2)(8C2)(6C1)(5C1)(4C1)(3C1)(2C1)(1C1)(0C2)
2 + 2 + 2 + 2 + 1 + 1 + 1 + 1 + 0 + 0 + 0 = 12
Which is equivalent to:
(12C2)(10C2)(8C2)(6C2)(4C1)(3C1)(2C1)(1C1)(0C3)
2 + 2 + 2 + 2 + 2 + 1 + 1 + 0 + 0 + 0 + 0 = 12
Which is equivalent to:
(12C2)(10C2)(8C2)(6C2)(4C2)(3C1)(2C1)(1C1)(0C4)
2 + 2 + 2 + 2 + 2 + 2 + 0 + 0 + 0 + 0 + 0 = 12
Which is equivalent to:
(12C2)(10C2)(8C2)(6C2)(4C2)(2C2)(0C5)

9. In the hope of having a dry outdoor wedding, John and Mary decide to get married in the desert, where the number of rainy days per year is 10. Unfortunately, the weather forecaster is predicting rain for tomorrow, the day of John and Mary's wedding. Suppose that the weather forecaster is not perfectly accurate: If it rains the next day, 90% of the time the forecaster predicts rain. If it is dry the next day, 10% of the time the forecaster still (incorrectly) predicts rain. Given this information, what is the probability that it will rain during John and Mary's wedding? [Hint: Use Bayes Theorem.]

Assuming that a year is approximately 365 days, we can utilise Bayes' Theorem to find the solution here. Let A = the event where it will rain tomorrow, and B = the event where rain is predicted for tomorrow. Now, the question is asking us P(A | B), which can be interpreted as [P(B|A)P(A)] / [P(B|A)P(A) + P(B|A)P(~A)]. Since we know that P(A) =

10/365, and P(B|A)=0.9, our final solution is (0.9)(10/365) / [(0.9)(10/365) + (0.9)(355/365)], or (.9)(10/365) / (.9), which is 10/365.

2.1 Probability

Three probability theorists named A, B and C walk into a bar. However, they only have enough money to buy one glass of beer. They come up with the following protocol. First, A flips a fair coin and takes a sip if it comes up Heads. Then B flips a fair coin and takes a sip if it comes up Heads. Finally C takes a sip if the coin tosses of A and B had the same outcome. This process repeats again until the glass is empty. Assume that each sip reduce the beer in the glass by 1 unit and the glass has N units of beer to begin with. Assume that N is a multiple of 6.

a. Prove that the process will eventually end, i.e., all the beer will be consumed in at most N rounds. Here, a round refers to A flipping a coin to decide if he/she gets a sip, then B flipping a coin to decide if he/she gets a sip and then C deciding if he/she gets a sip.

So during a given round, the beer can only really decrease by either 1, or 3 units. This is given via four different scenarios:
A = Heads, B = Heads: A, B, C drink, beer decreases by 3.
A = Heads, B = Tails: A drinks, beer decreases by 1.
A = Tails, B = Heads: B drinks, beer decreases by 1.
A = Tails, B = Tails: C drinks, beer decreases by 1.

The probabilities of each of these scenarios are as follows:
P(everyone drinks) = ½ * ½ = ¼ .
P(only C drinks) = ½ * ½ = ¼.
P(only A drinks) = ½ * ½ = ¼.
P(only B drinks) = ½ * ½ = ¼ .
Therefore, at any given round, we have an average rate of 1.5 sips. Knowing that there are N units of beer to begin with, and N being a multiple of 6, there is an average of 4 rounds here (4 * 1.5 = 6, as we know that the beer has a whole number of sips available).

b. What is the probability that the process ends in less than N/3 rounds?
c. What is the probability that the process ends in exactly N/3 rounds?
d. What is the probability that the process ends in exactly 2N/3 rounds?

2.2 Inclusion/Exclusion

Given an experiment with sample space Ω, and a valid probability distribution P, use the axioms of probability to prove that

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, for any two events $A, B \subseteq \Omega$.

Given that A and B are both subsets of the sample space Ω, we can say that A U B = A U (B ∩ ~A). From here, we can say P(A U B) = P(A) + P(B ∩ ~A). Now let's expand B to be (B ∩ ~A) U (A ∩ B).

From here, we can say that $P(B) = P(B \cap \sim A) + P(A \cap B)$. Substituting $P(B \cap \sim A) = P(B) - P(A \cap B)$ into our expansion for $P(A \cup B)$ gives us:

$$P(A \cup B) = P(A) + (P(B) - P(A \cap B))$$

2. $P(A \cup B \cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$, for any three events A, B, C $\subseteq \Omega$.

This looks to be a more expansive version of the previous question. Let us expand $P(A \cup B \cup C)$ to be $P((A \cup B) \cup C)$, which can be further expanded into $P(A \cup B) + P(C) - P(A \cup B \cap C)$. This can be further expanded to:

$$P(A) + P(B) - P(A \cap B) + P(C) - P((A \cap C) \cap (B \cap C))$$
$$= P(A) + P(B) + P(C) - P(A \cap B) - (P(A \cap C) + P(B \cap C) - P(A \cap B \cap C))$$
$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

2.3 Independence

1. For any experiment and 3 associated events, A, B, C, and if A and B are independent, and B and C are independent then A and C are also independent. Prove this claim or give an example of a random phenomenon where the claim is false.

Let A = the event that I roll a die. Let B = the event that I flip a coin, and let C = event that the value landed by the die is even. In this scenario, A and B are independent of each other, B and C are also independent of each other, but A and C are not (specifically C depends on A).

2. For any experiment and 3 associated events A, B, C, if A and B are independent, B and C are independent, and A and C are independent then A, B, C are also independent. Prove this claim or give an example of a random phenomenon where the claim is false.

Given the above statements, and by definition of independence, $P(A\cap B)=P(A)P(B)$, $P(B\cap C)=P(B)P(C)$, and $P(A\cap C)=P(A)P(C)$. We need to use this information to prove that $P(A\cap B\cap C)=P(A)P(B)P(C)$. In order to do this, we can add the three probabilities together: $P(A\cap B)+P(B\cap C)+P(A\cap C)$. However, we need to subtract $P(A)+P(B)+P(C)$ (because we overcounted those). Thus:

$$P(A\cap B)+P(B\cap C)+P(A\cap C) - P(A) - P(B) - P(C) =$$
$$P(A)P(B) + P(B)P(C) + P(A)P(C) - P(A) - P(B) - P(C) =$$
$$[P(A)][P(B) + P(B)P(C) + P(C) -1 - P(B) - P(C)] =$$
$$[P(A)][P(B)][1 + P(C) + P(C) -1 -1 - P(C)] =$$
$$[P(A)][P(B)][P(C)][1 + 1 + 1 -1 -1 -1] =$$
$$[P(A)][P(B)][P(C)][0] =$$
$$[P(A)][P(B)][P(C)]$$

2.4 Testing Vaccines

A pharmaceutical company has developed a potential vaccine against the H1N1 flu virus. Before any testing of the vaccine, the developers assume that with probability 0.5 their vaccine will be effective and with probability 0.5 it will be ineffective. The developers do an initial laboratory test on the vaccine. The initial lab test is only partially indicative of the effectiveness of the vaccine, with an accuracy of 0.6. Specifically, if the vaccine is effective, then this laboratory test will return “success” with probability 0.6, whereas if the vaccine is ineffective, then this laboratory test will return “failure” with probability 0.6.

a. What is the probability that the laboratory test returns “success”?

Since we know that the the efficacy of the vaccine is effective 50% of the time, we must sum the probability of the test returning success for both an effective and ineffective vaccine. If we let A = event that the vaccine test returns success, and B = event that the vaccine is effective, then we essentially are solving for P(A|B) + P(A|~B), or the P(vaccine test returns success, regardless of the vaccine's effectiveness).
This can be easily gleaned from the information, which tells us that P(A|B) = 0.6, and P(~A|~B) = 0.6. From the second piece of information, we can infer then that P(A|~B) = 0.4. Thus, our solution is (0.5)(0.6 + 0.4), or 50%, aka 0.5.

b. What is the probability that the vaccine is effective, given that the laboratory test returned “success”?

Letting A = event where the vaccine test returns “success”, and B = event where the vaccine itself is effective, this question is asking us P(B|A). Using Bayes' Theorem, we can expand P(B|A) to P(A|B)P(B) / [P(A|B)P(B) + P(A|B)P(~B)]. Luckily, we know that P(B), or the probability of the effectiveness of the vaccine itself, is 0.5. We also know that P(A|B) is 0.6, which helps us to simplify the expression to:
(0.6)(0.5) / [(0.6)(0.5) + (0.6)(0.5)] =
0.3 / (0.3 + 0.3) = 0.3 / 0.6 =
½, or 0.5.