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CS 206
Section 05

Homework 02

1. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

Since there are 26 letters in the English language (we're assuming all last names here are spelt with English), and 30 students (we're also assuming that these students all have last names, which are also spelt with the English language), then by the pigeonhole principle there *must* be at least 4 students who share the first letter of their last name with another student.

2. Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

This question is essentially asking that with any five numbers, at least two will share the same remainder when modulo'd with 4. This is actually true, in the sense that any integer modulo'd by 4 will always yield a result of 0, 1, 2, or 3. With four different possibilities and five numbers, we can instantiate the pigeonhole principle to say that there will be at least two numbers that share the same result when modulo'd with 4 (0, 1, 2, or 3).

3. How many solutions are there to the equation x1 +x2 +x3 +x4 = 17 where x1, x2, x3, x4 are non-negative integers?

(17+4-1) C (4 - 1), or 20C3, aka 1140 solutions.

4. In each of the following cases, how many solutions are there to the equation x1 +x2 +x3 + x4 + x5 = 17 where x1, x2, x3, x4, x5 are non-negative integers.
   a. x1 ≥ 1.
      (17 - 5*1 + 5 - 1) C (5 - 1), or 16C4, aka 1820 solutions.
   b. xi ≥ 2 for i = 1, 2, . . . , 5
      ...so this is basically the same thing as the previous question, but with each variable being at least 2:
      (17 - 5*2 + 5-1) C (5 - 1), or 11C4, aka 330 solutions.
   c. 0 ≤ x1 ≤ 10
      ...it's convoluted, but I'm going to just 'combine' 10 combinations:
      (17+4-1)C(4-1) + (16+4-1)C(4-1) + (15+4-1)C(4-1) + (14+4-1)C(4-1) + (13+4-1)C(4-1) + (12+4-1)C(4-1) + (11+4-1)C(4-1) + (10+4-1)C(4-1) + (9+4-1)C(4-1) + (8+4-1)C(4-1) + (7+4-1)C(4-1) = 21C3 + 19C3 + 18C3 + 17C3 + 16C3 + 15C3 + 14C3 + 13C3 + 12C3 + 11C3 +10C3 + 9C3 + 8C3 + 7C3
      Which roughly sums to 6140 solutions.

d. 0 ≤ x1 ≤ 3, 1 ≤ x2 ≤ 4, x3 ≥ 15.
Num Solutions:

0 (x3 = 17) +
1 (x3 = 16, x2 = 1) +
3 (x3 = 15, x2 = 1, x1+x4+x5 = 1) +
1 (x3 = 15, x2 = 2)
= 5 solutions.

e. x1 ≥ x2.

5. How many ways are there to distribute 6 identical balls into 9 different bins?

(6+9-1)C(9-1) = 14C8 = 3,003 ways.

6. How many ways are there to distribute 15 distinguishable objects into five distinguishable boxes so that the boxes have one, two, three, four, and five objects in them, respectively.

Since the constraints happen to align the number of boxes to each box perfectly (every box is filled as requested to sum 15 total objects), we essentially can map out 15 slots for each ball to fill. Since the objects are distinguishable, we can pretty much do this: 15!

7. There are 10 questions on a discrete mathematics final exam. How many ways are there to assign scores to the problems if the sum of the scores is 100 and each question is worth at least 5 points?

This is basically a systems of equations problem, where we have
X1 + X2 + X3 + X4 + … + X10 = 100, where X# >= 5.
This sums to (100-5*10+10-1)C(10-1) = 59C9, or 12,565,671,261 ways.

8. There are 12 signs of the zodiac. How many people are needed to guarantee that at least 6 of these people have the same sign?

At least 12*12 people, or 144. This is because of the possibility that the distribution of zodiac signs amongst a given group of people are exactly equal (i.e every sign has the same number of constituents).

2.1 Counting

How many bit strings of length n, where n ≥ 4, contain exactly two occurrences of 01?

So for exactly two occurrences of ‘01’ to exist in a bit string of size n (minimum 4), we can visualize a sample string as 1...01….010...0, which is essentially any number of ‘1’s, followed by any number of ‘0’s, and then ‘01’, followed yet again by any number of ‘1’s, then any number of ‘0’s, and then ‘01’, and finally any number of ‘1’s and any number of ‘0’s.. If we try to quantify the above generalization into a string of sorts, we get “x1 + x2 +

01 + x3 + x4 + 01 + x5 + x6", where the variables 'x#' correspond to an arbitrary number of '0's or '1's. This sum (x1 + x2 +...+x6), along with the two accounts of '01', equal n-4. But to simplify our answer, we can subtract four from both sides of the equation to just get:

X1 + X2 + X3 + X4 + X5 + X6 = n

Which, thanks to our lecture notes, is basically (n+6-1)C(6-1), or 'n+5 choose 5' (it's implied that n here still follows the >= 4 constraint).

2.2 Bad Counting

In lecture, we showed that the number of ways to divide 20 pieces of gold amongst 5 pirates is 24 C 4 by counting all strings with 20 Gs and 4 /s. The following alternate method was suggested to count such strings: Write down 20 Gs. Each of the 4 /s can go to 21 places, for a total of 21^4 possibilities. The /s are indistinguishable, so their ordering does not matter. Thus, we divide by 4! since we could have placed the 4 /s in any order. So the answer must be

21^4 / 4!

However, this is certainly not correct as it is not equivalent to our answer 24C4, and in fact it is not even an integer! Explain carefully and concisely what is wrong with this approach. In particular, does it overcount or undercount? Why?

2.3 Pigeonhole Principle

Show that in every set of 100 integers, there exist two whose difference is a multiple of 37.

The way the problem is laid out seems to heavily imply the use of the modulo operator; so my interpretation is that given 100 integers, prove that at least two integers have a difference of 37 when modulo'd by 37. If we partition the 100 integers into 37 spaces based off their remainder when modulo'd by 37, (i.e an integer that is a multiple of 37 goes to a space where all its members modulo'd by 37 equal 0, and so on), then there *will* exist at least two numbers that fit into the same category (i.e two numbers that, when subtracted, give a difference value equivalent to a multiple of 37).

A simpler explanation is that I am merely pigeonhol'ing 100 birds into 37 boxes, and the fact that at least 2 pigeons will share a single box is grounds to the above premise.

2.4 Pigeonhole Principle

Your TAs are helping the students to form homework groups, so they have every student fill out a form listing all of the other students who they would be willing to work with. There are 251 students in the class, and every student lists exactly 168 other students who they would be willing to work with. For any two students in the class, if student A puts student B on their list, then student B will also have student A on their list. Show that there must be some group of 4 students who are all willing to work with one another.

If we consider the fact that there are 251 students in the class, and each student has at 168 other students (than themselves) listed as potential collaborators, we get a total of 215*168 = 42168 'links', or student-to-student requests. Since we also know that every 'link' is essentially mutual (i.e linking student A to student B allows us to assume that student B did the same thing to student A), we ought to have 42168 / 2 = 21084 'unique' links. If we divide this number by the 251 students, then we arrive at 21084 / 251 = 84

‘unique’ links per student. Since this is an even number, we can assume that a group of 4 does indeed exist.