UNIVERSIDAD SERGIO ARBOLEDA

MATHEMATICS

NUMBER THEORY

Foundations

Subtitle example.

Alexander Mendoza

February 7, 2024

Contents

Chapter 1

Foundations

1.1 Definition of the natural numbers and its operations

Let's begin by defining our workbench, the natural numbers. We'll define the natural numbers in the most standard way, that is using Peano's axioms.

Peano's Axioms

There exists the set of natural numbers $\mathbb{N}$

1. $0 \in \mathbb{N}$.
2. If x is a natural number, then $s(x) \in \mathbb{N}$.
3. Does not exists $x \in \mathbb{N}$ such that $s(x) = 0$.
4. Let $x, y \in \mathbb{N}$, if $s(x) = s(y)$, then $x = y$.
5. Let $B \subseteq \mathbb{N}$ such that:
   - $0 \in B$
   - If $n \in B$, then $s(n) = B$

   Note

   B is said to be an inductive set. The original Axioms stated B as any set instead of a subset of $\mathbb{N}$, and then, by a theorem, $\mathbb{N}$ would be the smallest of all inductive sets. To avoid all this work, we state that B is a subset of $\mathbb{N}$ and from this we can conclude that $B = \mathbb{N}$

The last axiom is known as the Principle of Mathematical Induction (PMI), we'll elaborate further in this topic since it's going to be used everywhere from now on.

Addition

Addition is a recursive function $+$ with the following properties:

$$+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

where given $m, n \in \mathbb{N}$:

$$+(m, 0) = m$$
$$+(m, s(n)) = s(+(m, n))$$

The output of the function is called the sum of m and n and is denoted as $m + n$ for convenience, with this notation our definition can be viewed as:

$$m + 0 = m$$
$$m + s(n) = s(m, n)$$

Let's give a couple of examples to verify that this function is indeed the sum we are familiar with.

Example 1.1.1 Let $m = 2$ and $n = 3$, then by definition

$$\begin{aligned} 2 + 3 &= s(2 + 2) \\ &= s(s(2 + 1)) \\ &= s(s(s(2 + 0))) \\ &= s(s(s(2))) \\ &= s(s(3)) \\ &= s(4) \\ &= 5 \end{aligned}$$