Math Department

Fundamentals of Mathematics

Chronicals

Set Theory

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1 Sets

The intuitively idea of a set is that a set is a collection of objects, not necessarily mathematical ones. Almost all mathematical contexts are constructed from this idea of a set. We can take the set of all people in Colombia. The objects contained in the set are called the elements of the set. If A is a set and a is an element of A, we write $a \in A$. If a is not in the set A, we write $a \notin A$.

Logically, either $a \in A$ or $a \notin A$ but not both.

The two most common way of presenting a set is:

- To list its elements $A = \{a, b, c\}$.
- To provide a statement that all the elements of the set must satisfy in order to be part of it. $A = \{x \mid x > 0\}$.

1.1 Subsets

A subset has the following definition. $A \subseteq B \Leftrightarrow \forall x(x \in A \rightarrow x \in B)$. That is, a set A is a subset of a set B if and only if for all x that belongs to A, x also belongs to B.

The following is the most common way to prove that a set A is a subset of a set B. First let $x \in A$ ... (argumentation) ... then $a \in B$. Hence $A \subseteq B$.

1.2 Properties of sets

1. $A \subseteq B$. *Proof.* Let $a \in A$, A being the left side of $A \subseteq A$. It then follows that $a \in A$, being this A the right side of $A \subseteq A$. Hence, $A \subseteq A$ by definition of subsets.

2. $\emptyset \subseteq A$. *Proof.* The following is a proof by contradiction. Suppose $\emptyset \not\subseteq A$. Then by definition of subsets it follows that there exists an element $x \in \emptyset$ and $x \notin A$. By definition of empty set this statement cannot be true. Hence, $\emptyset \subseteq A$.

3. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. *Proof.* Let $a \in A$. Because $A \subseteq B$, it follows that $a \in B$. Then because $B \subseteq C$, it follows that $a \in C$. Thus, we see that $a \in A$ implies $a \in C$, $A \subseteq C$.

4. Let A and B be sets, $A = B$ if $A \subseteq B$ and $B \subseteq A$.

5. Let A and B be sets, A is a proper subset of B if $A \subseteq B$ and $A \neq B$.

6. Let A be a set. The power set of A, denoted $\mathcal{P}(A)$ is the set defined by $\mathcal{P}(A) = \{X | X \subseteq A\}$

1.3 Set Operations

We can get new sets out of old ones. This can be achieved by performing certain operations in these sets, the most common operations with sets correspond to the *or* and *and* operators in logic.

1. $A \cup B$. Let A and B be sets. The union of A and B, written as $A \cup B$, is the set with elements x such that $x \in A$ or $x \in B$, that is $\{x | x \in A \vee x \in B\}$

2. $A \cap B$. Let A and B be sets. The intersection of A and B, written as $A \cap B$, is the set with elements x such that $x \in A$ and $x \in B$, that is $\{x | x \in A \wedge x \in B\}$

We can utilize Venn diagrams to visualize sets.

1.3.1 Properties of set operations

1. $A \cap B \subseteq A$ and $A \cap B \subseteq B$. If X is a set such that $X \subseteq A$ and $X \subseteq B$, then $X \subseteq A \cap B$.

2. $A \subseteq A \cup B$ and $B \subseteq A \cup B$. If X is a set such that $X \subseteq A$ and $X \subseteq B$, then $X \subseteq A \cap B$.

3. $A \cup B = B \cup A$ and $A \cap B = B \cap A$. (*Commutative Laws*).