UNIVERSIDAD SERGIO ARBOLEDA

Mathematics

Integral Calculus and Series

Integrability

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Contents

Chapter 1

Integrability

To define the integral we will first need some definitions.

Inferior and Superior Sums

We will use both the superior and inferior sums to define the concept of an integral.

Definition 1.0.1

Partition of an integral. Let a, b be real numbers such that $a < b$ and let P be the following set

$$P = \{x_0, x_1, \dots, x_n\}$$

with $a = x_0 < x_1 < \cdots < x_n = b$, then we say that P is a partition of $[a, b]$

Figure 1, represents graphically the concept of partition.

Definition 1.0.2

Let f be a function bounded on $[a, b]$ and let $P = \{x_0, x_1, \dots x_n\}$ be a partition of $[a, b]$. then
The minimum of a function in $[a, b]$ is defined as

$$m_i := inf\{f(x)|x_{i-1} \leq x \leq x_i\}$$

And the maximum of a function in $[a, b]$ is defined as

$$M_i := sup\{f(x)|x_{i-1} \leq x \leq x_i\}$$

With this we can now define the upper and lower sums.

Definition 1.0.3

Suppose f is bounded on $[a, b]$, let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a, b]$ and let

$$m_i = inf\{f(x)|x_{i-1} \leq x \leq x_i\}$$
$$M_i = sup\{f(x)|x_{i-1} \leq x \leq x_i\}$$

Then we define the lower sum of f for P, denoted by $L(f, P)$, as

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$$

and the upper sum of f for P, denoted by $U(f, P)$, is defined as

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$$

Let's now see which relation the lower and upper sum have. Our intuition may tell us that given f bounded over $[a, b]$ and P a partition of $[a, b]$, $L(f, P) \leq U(f, P)$, in fact this is true for any pair of distinct partitions. But, let's review first a lemma to demonstrate this fact.

Lemma 1.0.4

Let f be a bounded function and P, Q be partitions of $[a, b]$. If $Q \subseteq P$, then

$$L(f, Q) \leq L(f, P)$$
$$U(f, P) \leq U(f, Q)$$

Idea for Proof

The idea for this proof is to first demonstrate that the lemma is true for a partition that contains only one more element, and then construct a succession of partitions, where each new partition will have one more element than the partition before it, and proof the general case.

Proof. Let f be a bounded function and P, Q be partitions of $[a, b]$ where $Q \subseteq P$ and P contains only one more element than Q, namely u. We can represent P and Q as

$$P = \{x_0, x_1, \ldots, x_{k-1}, x_k, \ldots, x_n\}$$
$$Q = \{x_0, x_1, \ldots, x_{k-1}, u, x_k, \ldots, x_n\}$$

Then by definition we have

$$
\begin{aligned}
L(f,P) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{k-1} m_i(x_i - x_{i-1}) + m_k(x_k - x_{k-1}) + \sum_{i=k+1}^{n} m_i(x_i - x_{i-1})
\end{aligned}
$$

Similarly

$$
\begin{aligned}
L(f,Q) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{k-1} m_i(x_i - x_{i-1}) + m'(u - x_{k-1}) + m''(x_k - u) + \sum_{i=k+1}^{n} m_i(x_i - x_{i-1})
\end{aligned}
$$

With this we can state a theorem that will allow us to then define the integral of a function. Before providing the theorem, a useful observation is that, given any partition P

$$L(f,P) \leq U(f,P)$$

We know this because

$$L(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$$

$$U(f,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$$

and by definition, for any i:

$$
\begin{gathered}
m_i \leq M_i \\
m_i(t_i - t_{i-1}) \leq M_i(t_i - t_{i-1})
\end{gathered}
$$

Theorem 1.0.5

Let P_1, P_2 be partitions of an interval $[a,b]$ and let f be bounded over $[a,b]$, then it's true that

$$L(f,P_1) \leq U(f,P_2)$$

Proof. Let $P = P_1 \cup P_2$, then by definition of union, $P_1 \subseteq P$ and $P_2 \subseteq P$, and by the previous lemma:

$$
\begin{gathered}
L(f,P_1) \leq L(f,P) \\
U(f,P) \leq U(f,P_2)
\end{gathered}
$$

By the afore mentioned observation, we have

$$L(f, P) \leq U(f, P)$$

And all together

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2)$$

Finally, by transitivity, we therefore have

$$L(f, P_1) \leq U(f, P_2)$$

Concluding our proof.

With this, we can now define the integral of a function.

Definition 1.0.6

Let f be a function bounded on $[a, b]$, f is said to be integrable on $[a, b]$ if

$$\begin{aligned} &sup\{L(f, P) | P \text{ is a partition of } [a, b]\} \\ = &inf\{U(f, P) | P \text{ is a partition of } [a, b]\} \end{aligned}$$

That common number is what we call the integral of f over $[a, b]$ and is denoted by

$$\int_b^a f$$

And is read "the integral of f over $[a, b]$"