UNIVERSIDAD SERGIO ARBOLEDA

MATHEMATICS

DIFFERENTIAL CALCULUS

Limits and Continuity

Alexander Mendoza

January 22, 2024

Contents

Chapter 1

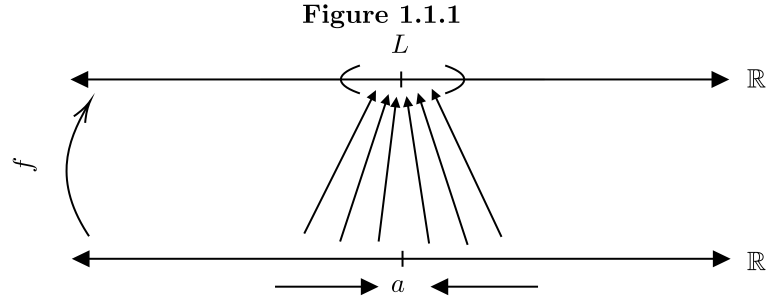
Limits

1.1 An intuitive approach to limits

When we talk about calculus the first thing that comes to our minds are derivatives and integrals; however, the **most fundamental idea** of this discipline, used to formally construct further concepts, is the idea of limits. Let's start by defining limits in an intuitive way, since, in calculus, we cannot entirely separate formal concepts from their intuitive counterparts and its applications, we'll explore this relationship more thoroughly when we start discussing derivatives. Having that clear, we say that the limit of $f(x)$ as x approaches a (gets closer to a) is L if the values of f can get as close as we want to L whenever x is sufficiently close, but different to a.

A limit involves two things, a function $f(x)$ and a point a in which we'll analyze the function. This concept of "analyzing" the function refers to a point-specific review of the behavior of the function for values close to a. Knowing this, we cannot talk about limits in a general way, the definition of limits only applies to a particular point.

Figure 1.1.1

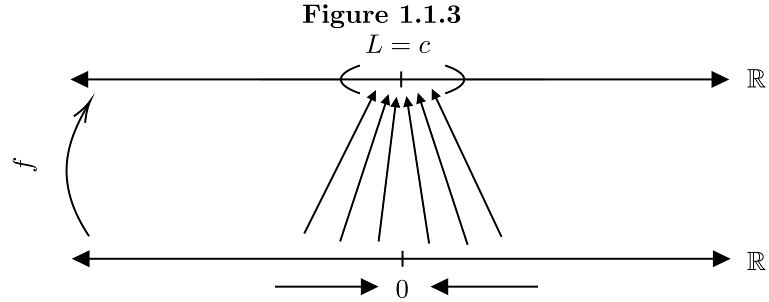


As we can see in Figure 1.1.1, as we get closer to a, $f(x)$ gets closer to L. In limits we gotta get as close as we want to L, this is possible due to the continuity of $\mathbb{R}$.

Now it's worth to elaborate further in the previously discussed idea that a limit is a punctual analysis.

Example 1.1.2 Observe that given $f(x) = c$ where c is constant, then the limit of f as x approaches 0 is $L = c$.

Figure 1.1.3



As shown in Figure 1.1.3, if c is constant, then for any x the value through f is always going to be 0. So the value always gets closer to c as we approach 0.

The limit of a function is often written as

$$\lim_{x \to a} f(x) = L$$

and is read "The limit of $f(x)$, as x approaches a, is equal to L."

Let's review some concrete examples to consolidate more this idea. Consider the function $f(x) = 3x$ and let $a = 5$. Intuitively we see that $f(x)$ must get closer to 15 when x approaches 5.

x	f(x)
4.95	14.85
4.96	14.88
4.97	14.91
4.98	14.94
5	-
5.01	15.03
5.02	15.06
5.03	15.09
5.04	15.12

Notice how we do not evaluate the function at $x = 5$. This is because we are not concerned about what happens at that point; we only want to know the values that are close to that point. Let us now pose a question: if $f(x)$ approaches 15, at a distance less than $\frac{1}{10} = 0.1$, how close to 5 do the values of x need to be?

The answer is $\frac{1}{30} \approx 0.03$. Let's consider why this is the correct answer. Our function is $f(x) = 3x$, meaning that whatever x we input, we'll receive three times that value as an output. With this understanding, we need to find a range, with 5 as its midpoint, such that when we take a value of x within that range, the value of $f(x)$ falls within $\frac{1}{10}$ of 15.

If we first attempt with a range of $\frac{1}{10}$, we notice that

$$3\left(5 - \frac{1}{10}\right) = 14.7 < 14.9 = 15 - \frac{1}{10}.$$

Thus, we chose a range where we found a value for which $f(x)$ does not fall within $\frac{1}{10}$ of 15. Let's now try with $\frac{1}{20}$, a smaller range, and we notice that

$$3\left(5 - \frac{1}{20}\right) = 14.85 < 14.9 = 15 - \frac{1}{10}.$$

Again, we made a poor decision. Now we are going to try with the range $\frac{1}{30}$:

$$3\left(5 - \frac{1}{30}\right) = 14.9 = 15 - \frac{1}{10}.$$

Similarly, for the upper bound:

$$3\left(5 + \frac{1}{30}\right) = 15.1 = 15 + \frac{1}{10}.$$

With this, we have found a range where any number x chosen within that range from 5 ensures that $f(x)$ falls within $\frac{1}{10}$ of 15. The choice of $\frac{1}{30}$ should be evident, given the predictable behavior of $f(x)$. We can also conclude the answer using the following logic. We want $f(x) = 3x$ to be greater than $15 - \frac{1}{10}$ and less than $15 + \frac{1}{10}$:

$$15 - \frac{1}{10} < 3x < 15 + \frac{1}{10}.$$

We now resolve the inequality:

$$\begin{aligned} -\frac{1}{10} &< 3x - 15 &&< \frac{1}{10} \\ \frac{1}{3}\left(-\frac{1}{10}\right) &< \frac{1}{3}(3x - 15) &&< \frac{1}{3}\left(\frac{1}{10}\right) \\ -\frac{1}{30} &< x - 5 &&< \frac{1}{30}. \end{aligned}$$

This is equivalent to $|x - 5| < \dfrac{1}{30}$. Now, if would want to ask the same question but with the range $\frac{1}{100}$ around 15, we know that $\frac{1}{300}$ would be the answer. In general, for any positive number ϵ, the range $\frac{\epsilon}{3}$ can make that all values of $f(x)$, where x is within $\frac{\epsilon}{3}$ from 5, to fall within ϵ apart from 15. In other words, for every positive number ϵ, we can make $|3x - 15| < \epsilon$ simply by requiring that $|x - 5| < \dfrac{\epsilon}{3}$.

Let's try now with a more interesting function, $f(x) = x^2$

With this ideas in mind, the process of verifying a limit now becomes in a game, where given any positive number ϵ, finding another number namely δ, such that we can make $|f(x) - L| < \epsilon$ by requiring $|x - a| < \delta$, would confirm that L is the limit of $f(x)$ as x approaches a. With this, we should have a good intuitive base to talk about the formalization of the definition of a limit.

1.2 The formal definition of a limit

We will now provide a more analytic definition of the concept of a limit, for this it is worth to first define the notion of a neighborhood.

Definition 1.2.1

***Neighborhood*.** A neighborhood of a point a is any open interval containing that point a as its midpoint.

Definition 1.2.2

The function $f(x)$ has a limit L, when x approaches a, if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all x, if $0 < |x-a| < \delta$, then $|f(x)-L| < \epsilon$. We then notate the limit of a function as $\lim_{x\to a} f(x) = L$.

Theorem 1.2.3

Given $f(x)$ a function; if the limit of f as x approaches a, is m and l, then $m = l$. Thus the limit of a function in a given point is unique (inside $\mathbb{R}$).

1.3 Properties of limits

Theorem 1.3.1

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

1. $\lim_{x\to a}(f + g)(x) = L + M$.
2. $\lim_{x\to a}(f \cdot g)(x) = L \cdot M$.
3. $\lim_{x\to a}(\frac{1}{g})(x) = \frac{1}{M}$, when $M \neq 0$.

Chapter 2

Continuity

2.1 Continuous functions

Although in many cases, given an arbitrary function f, $\lim_{x\to a} f(x) = f(a)$, it is not always the case. For example, let f be a function such that

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ x & \text{if } x = x \end{cases}$$

Then is clear that $\lim_{x\to 0} f(x) = 0$ and $f(0) = 1$. The functions where the limit at a point a are equal to the function evaluated at that same point a are very particular and are called continuous functions. The intuitive idea of continuous functions is a function that when you look at the graph it contains no spaces, jumps or wild changes. As we know, defining things from an intuitive point of view is problematic, this is why the formal definition of continuous functions goes as follow:

Definition 2.1.1

A function f is continuous at a if the limit of $f(x)$ as x approaches a exists, f is defined at a and

$$\lim_{x\to a} f(x) = f(a)$$

Let's review some common functions and see if they are continuous.

Example 2.1.2 $f(x) = c$ for some constant $c \in \mathbb{R}$. This function is indeed continuous. $\lim_{x\to a} f(x) = c = f(a)$, this by limit properties of a constant function.

Example 2.1.3 $f(x) = ax + b$ for some $a, b \in \mathbb{R}$ This function is also continuous. $\lim_{x\to a} f(x) = a = f(a)$, this by limit properties of identity function.

Example 2.1.4 $f(x) = a_n x_n^n + a_{n-1} x_{n-1}^{n-1} + \cdots + a_1 x_1 + a_0$ (nth grade polynomial). Any polynomial function is continuous.

Example 2.1.5 $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$ is also a continuous function.

Example 2.1.6 $f(x) = \sin x$ and $g(x) = \cos x$. Are also continuous functions.

Along with this definition, also arises the following theorem.

Theorem 2.1.7

Let f and g be continuous functions at $x = a$, then we have that:

1. $f + g$ is also continuous at a.
2. $f \cdot g$ is also continuous at a.
3. $\frac{1}{g}$ is also continuous at a if $g(a) \neq 0$.

Proof. We'll prove property 1 and the rest follow a similar idea. Let f and g be continuous functions at $x = a$, then

$$\lim_{x \to a} f(x) = f(a) \text{ and } \lim_{x \to a} g(x) = g(a)$$

with this and theorem limits, we have that

$$\lim_{x \to a} (f + g)(x) = f(a) + g(a) = (f + g)(a)$$

There is another property that results very important and dictates how continuity behaves in function compositions.

Theorem 2.1.8

If f is continuous at $g(a)$ and g is continuous at a, then $f \circ g$ is continuous at a.