UNIVERSIDAD SERGIO ARBOLEDA

Mathematics

Linear Algebra I

Vector Spaces

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Chapter 1

Vector Spaces

1.1 Vector Spaces

We now are ready to start looking at a direction that is closer to linear algebra by reviewing vector spaces. We define a vector space is a composite object consisting of a field, a set of vectors and two operations with certain special properties. The formal definition of a vector space goes as follows.

Definition

Vector Space. A vector space consists of:

1. A field F of scalars;
2. A set V of objects, called vectors;
3. A binary operation $\oplus : V \times V \to V$ called vector addition, which associates with each pair of vectors $\vec{v}, \vec{u} \in V$ to a vector $\vec{v} + \vec{u} \in V$, called the sum of $\vec{v}$ and $\vec{u}$, in such a way that V together with $\oplus$, $(V, \oplus)$ form an abelian group:
   (a) Addition is associative, $\vec{v} + (\vec{u} + \vec{w}) = (\vec{v} + \vec{u}) + \vec{w}$;
   (b) There is a unique vector 0 in V, called the zero vector, such that $\vec{v} + 0 = \vec{v}$ for all $\vec{v}$ in V.
   (c) For each vector $\vec{v}$ in V there is a unique vector $-a$ in V such that $\vec{v} + (-\vec{v}) = 0$.
   (d) Addition is commutative, $\vec{v} + \vec{u} = \vec{u} + \vec{v}$;
4. A function $\odot : F \times V \to V$ called scalar multiplication, which associates with each scalar α in F and vector $\vec{v} \in V$ to a vector $\alpha\vec{v}$ in V, called the product of α and $\vec{v}$ in such a way that:
   (a) $1\vec{v} = \vec{v}$ for every $\vec{v}$ in V;
   (b) $(\alpha, \beta)\vec{v} = \alpha(\beta\vec{v})$;
   (c) $c(\vec{v} + \vec{u}) = \alpha\vec{v} + c\vec{u}$;
   (d) $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$;

We then say that $(V, \oplus, \odot)$ is a vector space over the field $(F, +, \cdot)$.

Notice how the scalar multiplication behaves like an operation, but is not an operation, this is because it includes elements from the field whichi is external to V. Also some characteristics of the vector addition is that the addition is performed component by component. Take for example the abelian group $\mathbb{R}^2$ with the vector addition $\oplus$, the latter can be defined as $(a, b) \oplus (c, d) = (a + c, b + d)$ for all $a, b, c, d \in \mathbb{R}$.