Prove that the domain of natural parameters for the exponential family $p(x|\eta) = \frac{1}{Z(\eta)} h(x) e^{u(x)^\intercal \eta}$ is convex.
Specifically, $\mathcal{D} = \{\eta : Z(\eta) < \infty\}$

We will show directly that the convex combination of $\eta_1, \eta_2 \in \mathcal{D}$ is also in $\mathcal{D}$.

Our approach will be to split up the domain of X into two sets:

$$\begin{aligned} A &= \{x \in X : u(x)^\intercal(\eta_1 - \eta_2) \geq 0\} \\ B &= \{x \in X : u(x)^\intercal(\eta_1 - \eta_2) < 0\} \end{aligned} \tag{1}$$

Then we will use something in the spirit of Markov's inequality to upper bound $Z\left(\lambda\eta_1 + (1-\lambda)\eta_2\right)$.

Rewriting the convex combination in more of a directional form

$$\lambda\eta_1 + (1-\lambda)\eta_2 = \eta_2 + \lambda(\eta_1 - \lambda_2) \tag{2}$$

We can use this to rewrite the normalizer with a $p(x|\eta_2)$ term:

$$\begin{aligned} Z\left(\eta_2 + \lambda(\eta_1 - \eta_2)\right) &= \int h(x) e^{u(x)^\intercal \eta_2} e^{\lambda u(x)^\intercal(\eta_1 - \eta_2)} dx \\ &= \int \frac{Z(\eta_2)}{Z(\eta_2)} h(x) e^{u(x)^\intercal \eta_2} e^{\lambda u(x)^\intercal(\eta_1 - \eta_2)} dx \\ &= Z(\eta_2) \int p(x|\eta_2) e^{\lambda u(x)^\intercal(\eta_1 - \eta_2)} dx \\ &= Z(\eta_2) \int_A p(x|\eta_2) e^{\lambda u(x)^\intercal(\eta_1 - \eta_2)} dx + Z(\eta_2) \int_B p(x|\eta_2) e^{\lambda u(x)^\intercal(\eta_1 - \eta_2)} dx \end{aligned} \tag{3}$$

Since $1 < e^{\lambda u(x)^\intercal(\eta_1 - \eta_2)}$ for $u(x)^\intercal(\eta_1 - \eta_2) < 0$, we can upper bound the integral over B, since it only contains points x for which the dot product is negative (non-positive), as

$$\int_B p(x|\eta_2) e^{\lambda(u(x)^\intercal(\eta_1 - \eta_2)} \leq \int_B p(x|\eta_2) 1 dx \leq 1 \tag{4}$$

We can similarly rewrite the left hand term as

$$Z(\eta_1) \int_A p(x|\eta_1) e^{(1-\lambda) u(x)^\intercal(\eta_1 - \eta_2)} dx \tag{5}$$

and we can similarly upper bound that term.

Hence, we have

$$Z\left(\lambda\eta_1 + (1-\lambda)\eta_2\right) \leq Z(\eta_1) + Z(\eta_2) < \infty \tag{6}$$

Therefore $\mathcal{D}$ is convex.