CS-E4740 - Federated Learning

L2 - FL Design Principle

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Calendar **Glossary** **Book** **GitHub**

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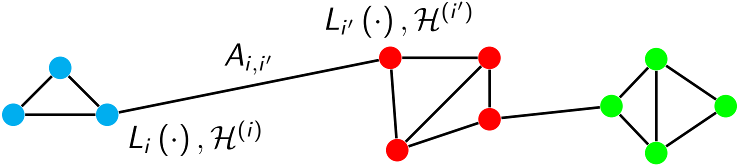
Some FL Network



- devices represented by nodes $i = 1, \ldots, n$
- some i, i' connected by an edge with weight $A_{i,i'} > 0$
- device i learns hypothesis $h^{(i)} \in \mathcal{H}^{(i)}$
- usefulness of $h^{(i)}$ measured by local loss $L_i(\cdot)$

FL via Regularization

consider an federated learning network (FL network) where

- each node trains a linear model $h^{(\mathbf{w}^{(i)})}(\mathbf{x}) := \mathbf{x}^T \mathbf{w}^{(i)}$
- each node carries m_i labelled data points
- each data point is characterized by d features

$\Longrightarrow$ node-wise training fails if $m_i \ll d$ (overfitting)

Idea:

$$\boxed{\text{use the neighbors } \mathcal{N}^{(i)} := \{i' : \{i, i'\} \in \mathcal{E}\} \text{ for regularization!}}$$

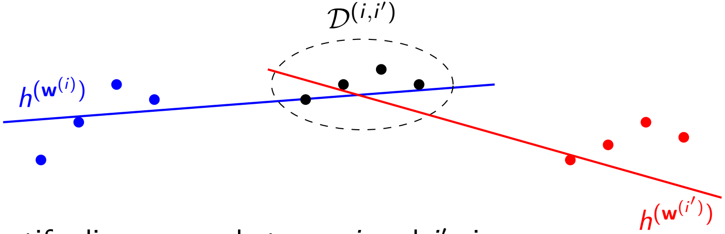
FL via Regularization (ctd.)

regularization can be done either via

- **data augmentation** using information from neighbors,
- **pruning local models** by requiring agreement across edges,
- **adding a penalty term** to the local loss function

Building a Penalty Across Edges

- consider two connected nodes i, i' with datasets $\mathcal{D}^{(i)}, \mathcal{D}^{(i')}$
- assume a non-empty overlap $\mathcal{D}^{(i,i')} = \mathcal{D}^{(i)} \cap \mathcal{D}^{(i')}$



quantify discrepancy between i and i' via

$$\sum_{\mathbf{x} \in \mathcal{D}^{(i,i')}} \left(h^{(\mathbf{w}^{(i)})}(\mathbf{x}) - h^{(\mathbf{w}^{(i')})}(\mathbf{x}) \right)^2 = \sum_{\mathbf{x} \in \mathcal{D}^{(i,i')}} \left(\mathbf{x}^T \mathbf{w}^{(i)} - \mathbf{x}^T \mathbf{w}^{(i')} \right)^2$$

$$= \left(\mathbf{w}^{(i)} - \mathbf{w}^{(i')} \right)^T \left[\sum_{\mathbf{x} \in \mathcal{D}^{(i,i')}} \mathbf{x}\mathbf{x}^T \right] \left(\mathbf{w}^{(i)} - \mathbf{w}^{(i')} \right).$$

Discrepancy Measure for Parametric models

use "norm-like" function $\phi(\mathbf{w}^{(i)} - \mathbf{w}^{(i')})$ of difference between model parameter at two nodes i, i'

- $\phi(\mathbf{u}) = \|\mathbf{u}\|_2^2$, or
- $\phi(\mathbf{u}) = \mathbf{u}^T\mathbf{Q}\mathbf{u}$ with positive semi-definite (psd) $\mathbf{Q}$, or
- $\phi(\mathbf{u}) = \|\mathbf{u}\|_1$, or
- $\phi(\mathbf{u}) = \|\mathbf{u}\|$, or
- $\ldots$

Discrepancy Measure for Federated GMM

- node i carries Gaussian mixture model (GMM) with model parameters $\mathbf{w}^{(i)}$
- measure discrepancy via Kullback–Leibler divergence (KL divergence)
$$\frac{1}{2}\Big(D^{(\mathrm{KL})}\big(\mathbf{w}^{(i)}, \mathbf{w}^{(i')}\big) + D^{(\mathrm{KL})}\big(\mathbf{w}^{(i')}, \mathbf{w}^{(i)}\big)\Big)$$
- useful for federated soft clustering
- more about this in Lecture 5 - Federated Clustering

Discrepancy Measure for Federated k-means

- node i carries local cluster centroids $\mathbf{w}^{(i,1)}, \ldots, \mathbf{w}^{(i,k)}$
- define discrepancy via

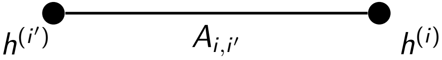
$$\sum_{c \in [k]} \min_{c' \in [k]} \left\| \mathbf{w}^{(i,c)} - \mathbf{w}^{(i',c')} \right\|_2^2 + \sum_{c \in [k]} \min_{c' \in [k]} \left\| \mathbf{w}^{(i',c)} - \mathbf{w}^{(i,c')} \right\|_2^2$$

- first term vanishes when $\{\mathbf{w}^{(i,c)}\}_{c=1}^{k} \subseteq \{\mathbf{w}^{(i',c)}\}_{c=1}^{k}$
- second term vanishes when $\{\mathbf{w}^{(i,c)}\}_{c=1}^{k} \supseteq \{\mathbf{w}^{(i',c)}\}_{c=1}^{k}$
- discrepancy vanishes when $\{\mathbf{w}^{(i',c)}\}_{c=1}^{k} = \{\mathbf{w}^{(i,c)}\}_{c=1}^{k}$

more about this in Lecture 5 - Federated Clustering

Generalized Total Variation (GTV)

consider some discrepancy measure $d^{(h^{(i)},h^{(i')})}$ for hypotheses $h^{(i)}, h^{(i')}$ at two connected nodes



we then define the generalized total variation (GTV),

$$\sum_{\{i,i'\}\in\mathcal{E}} A_{i,i'} d^{(h^{(i)},h^{(i')})}$$

by summing the scaled discrepancy measures over all edges

GTVMin

generalized total variation minimization (GTVMin) balances local losses with GTV:

$$\min_{\substack{h^{(1)} \in \mathcal{H}^{(1)} \\ \dots \\ h^{(n)} \in \mathcal{H}^{(n)}}} \sum_{i=1}^{n} L_i \left(h^{(i)}\right) + \alpha \sum_{\{i,i'\} \in \mathcal{E}} A_{i,i'} d^{(h^{(i)},h^{(i')})}$$

allows for VERY heterogeneous FL networks, e.g., $\mathcal{H}^{(1)} =$ lin.model, $\mathcal{H}^{(2)} =$ LLM, $\mathcal{H}^{(3)} =$ decision tree

GTVMin - Extreme Cases

GTVMin balances local losses with GTV:

$$\min_{\substack{h^{(1)} \in \mathcal{H}^{(1)} \\ \dots \\ h^{(n)} \in \mathcal{H}^{(n)}}} \sum_{i=1}^{n} L_i\left(h^{(i)}\right) + \alpha \sum_{\{i,i'\} \in \mathcal{E}} A_{i,i'} d^{(h^{(i)}, h^{(i')})}$$

different regimes depending on regularization strength α

- $\alpha = 0$: GTVMin splits into independent empirical risk minimization (ERM) at each i
- $\alpha \rightarrow \infty$: GTVMin becomes single global ERM with each node i holding a copy of the global learnt hypothesis
- for intermediate α, GTVMin becomes cluster-wise ERM

GTVMin for Parametric models

$$\min_{\substack{\mathbf{w}^{(1)} \in \mathbb{R}^d \\ \dots \\ \mathbf{w}^{(n)} \in \mathbb{R}^d}} \sum_{i=1}^{n} L_i\left(\mathbf{w}^{(i)}\right) + \alpha \sum_{\{i,i'\} \in \mathcal{E}} A_{i,i'} \phi\big(\mathbf{w}^{(i)} - \mathbf{w}^{(i')}\big)$$

note that this special case of GTVMin requires local models to be parametrized by the same Euclidean space $\mathbb{R}^d$

GTVMin for Federated Linear regression

consider FL network where each node trains a linear model using local dataset $\left(\mathbf{x}^{(1)}, y^{(1)}\right), \ldots, \left(\mathbf{x}^{(m_i)}, y^{(m_i)}\right)$

federated linear regression via GTVMin

$$\min_{\substack{\mathbf{w}^{(1)} \in \mathbb{R}^d \\ \ldots \\ \mathbf{w}^{(n)} \in \mathbb{R}^d}} \sum_{i=1}^{n} \frac{1}{m_i} \left\| \mathbf{y}^{(i)} - \mathbf{X}^{(i)} \mathbf{w}^{(i)} \right\|_2^2 + \alpha \sum_{\{i,i'\} \in \mathcal{E}} A_{i,i'} \left\| \mathbf{w}^{(i)} - \mathbf{w}^{(i')} \right\|_2^2$$

here, we used the local feature matrix $\mathbf{X}^{(i)} := \left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\right)^T$ and the local label vector $\mathbf{y}^{(i)} := \left(y^{(1)}, \ldots, y^{(n)}\right)^T$

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Interpretations

we next discuss some interpretations of GTVMin

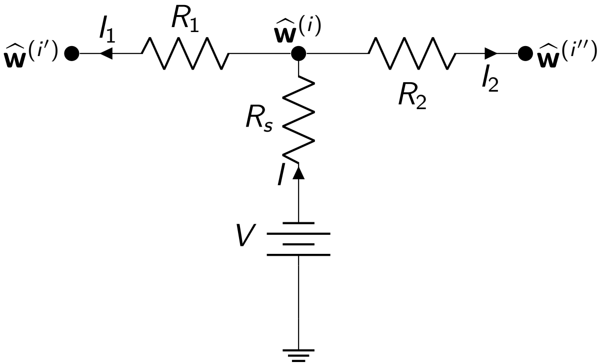
$$\min_{\mathbf{w}^{(1)},\ldots,\mathbf{w}^{(n)} \in \mathbb{R}^d} \sum_{i=1}^{n} L_i\left(\mathbf{w}^{(i)}\right) + \alpha \sum_{\{i,i'\} \in \mathcal{E}} A_{i,i'} \left\| \mathbf{w}^{(i)} - \mathbf{w}^{(i')} \right\|_2^2$$

with smooth and convex loss functions $L_i\left(\mathbf{w}^{(i)}\right)$

we assume that there exists a solution $\widehat{\mathbf{w}}^{(1)}, \ldots, \widehat{\mathbf{w}}^{(n)}$ (do we really need this assumption?)

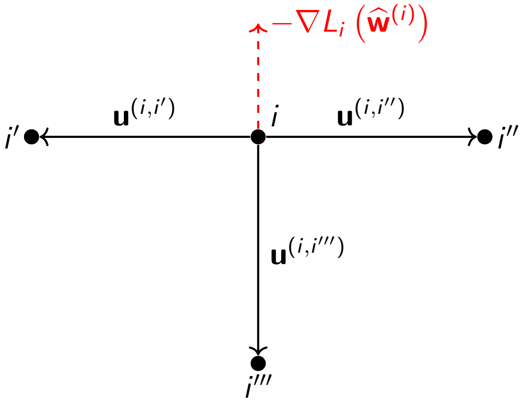
Electronic Circuit

Consider a node i with neighbours $\mathcal{N}^{(i)} = \{i', i''\}$.



$$\underbrace{-\nabla L_i\left(\widehat{\mathbf{w}}^{(i)}\right)}_{I} = \underbrace{A_{i,i'}\left(\widehat{\mathbf{w}}^{(i)} - \widehat{\mathbf{w}}^{(i')}\right)}_{I_1} + \underbrace{A_{i,i''}\left(\widehat{\mathbf{w}}^{(i)} - \widehat{\mathbf{w}}^{(i'')}\right)}_{I_2}$$

Vector-Valued Flows

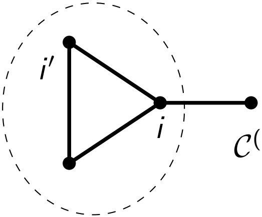


Vector-valued flow $\mathbf{u}^{(i,i')} := \nabla \phi\big(\mathbf{u}\big)\Big|_{\mathbf{u}=\widehat{\mathbf{w}}^{(i)}-\widehat{\mathbf{w}}^{(i')}}$.

AJ, "On the Duality Between Network Flows and Network Lasso," in IEEE Signal Processing Letters, 2020.

Locally Weighted Learning

GTVMin delivers model parameters $\widehat{\mathbf{w}}^{(i)}$ that form clusters



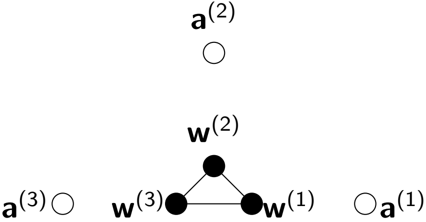
$$\mathcal{C}^{(i)} := \left\{ i' \in \mathcal{V} : \widehat{\mathbf{w}}^{(i)} \approx \widehat{\mathbf{w}}^{(i')} \right\}.$$

for node i, GTVMin is the same as locally weighted learning

$$\min_{\mathbf{w}^{(i)} \in \mathbb{R}^d} \sum_{i'=1}^{n} L_{i'}\left(\mathbf{w}^{(i)}\right) \rho_{i'} \text{ with } \rho_{i'} = \begin{cases} 1 & \text{if } i' \in \mathcal{C}^{(i)} \\ 0 & \text{otherwise.} \end{cases}$$

C. G. Atkeson, S. A. Schaal and Andrew W, Moore, Locally Weighted Learning, AI Review,Volume 11, Pages 11-73 (Kluwer Publishers) 1997.

Generalized Convex Clustering



$$\min_{\mathbf{w}^{(1)},\ldots,\mathbf{w}^{(n)}} \sum_{i=1}^{n} \left\|\mathbf{w}^{(i)} - \mathbf{a}^{(i)}\right\|_2^2 + \alpha \sum_{i,i' \in \mathcal{V}} \left\|\mathbf{w}^{(i)} - \mathbf{w}^{(i')}\right\|_2.$$

D. Sun, et.al, Convex Clustering: Model, Theoretical Guarantee and Efficient Algorithm, JMLR, 2021.

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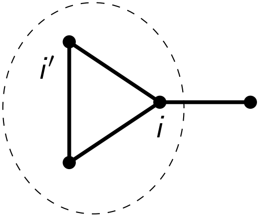
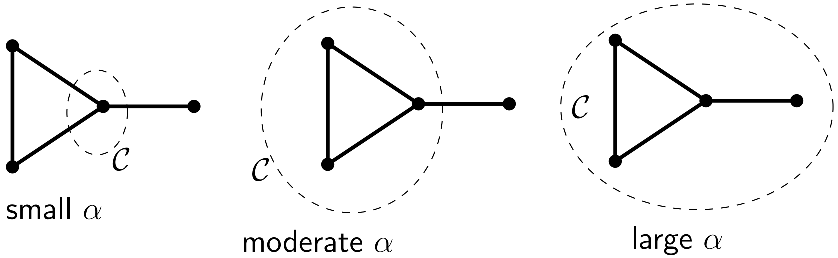
Statistical Aspects

- GTVMin solution yields model parameters $\widehat{\mathbf{w}}^{(i)}$, $i = 1, \ldots, n$
- how useful are these?
- loss value $L_i\left(\widehat{\mathbf{w}}^{(i)}\right)$ can be misleading (why?)
- better to use aggregate loss $\sum_{i \in \mathcal{C}^{(i)}} L_i\left(\widehat{\mathbf{w}}^{(i)}\right)$, with cluster



$$\mathcal{C}^{(i)} := \left\{ i' : \widehat{\mathbf{w}}^{(i)} \approx \widehat{\mathbf{w}}^{(i')} \right\}.$$

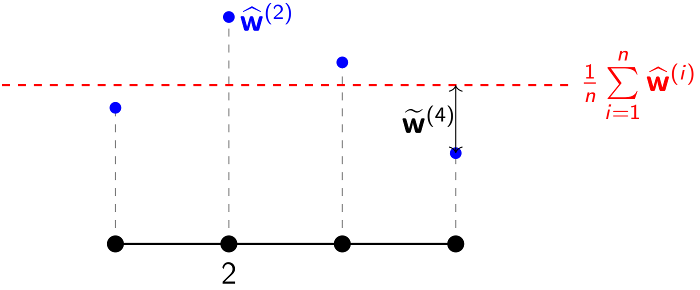
Clustering of GTVMin



Y. SarcheshmehPour, Y. Tian, L. Zhang and A. Jung, "Clustered Federated Learning via Generalized Total Variation Minimization," in IEEE Transactions on Signal Processing, 2023.

Analysis of GTVMin - Assumptions

- consider a connected FL network with $\lambda_2 > 0$
- assume loss functions satisfy $\min_{\mathbf{v} \in \mathbb{R}^d} \sum_{i=1}^{n} L_i(\mathbf{v}) \leq \varepsilon$
- use GTVMin to learn model parameters $\widehat{\mathbf{w}}^{(i)}$
- define variation $\widetilde{\mathbf{w}}^{(i)} := \widehat{\mathbf{w}}^{(i)} - \frac{1}{n} \sum_{i=1}^{n} \widehat{\mathbf{w}}^{(i)}$



Analysis of GTVMin - Upper Bound

the variation $\widetilde{\mathbf{w}}^{(i)}$ is upper bounded as

$$\sum_{i=1}^{n} \left\| \widetilde{\mathbf{w}}^{(i)} \right\|_2^2 \leq \frac{\varepsilon}{\alpha \lambda_2}$$

this bound involves the

- connectivity of FL network (via λ_2)
- the properties of local loss functions (via ε)
- the GTVMin parameter α

$\boxed{\text{Large } \alpha\lambda_2 \text{ enforces similar model parameters } \widehat{\mathbf{w}}^{(i)}.}$

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Computational Aspects

$$\min_{\substack{h^{(1)} \in \mathcal{H}^{(1)} \\ \dots \\ h^{(n)} \in \mathcal{H}^{(n)}}} \sum_{i=1}^{n} L_i \left(h^{(i)} \right) + \alpha \sum_{\{i,i'\} \in \mathcal{E}} A_{i,i'} d^{(h^{(i)}, h^{(i')})}$$

- how can we solve it efficiently over an FL network?
- how much compute/communication is needed at least?
- what is the effect of edges $\mathcal{E}$, loss functions $L_i(\cdot)$, and discrepancy measure $d^{(\cdot,\cdot)}$?

Fixed-Point Characterization

consider hypotheses $\widehat{h}^{(i)}$, for $i = 1, \ldots, n$, that solve

$$\min_{h^{(1)},\ldots,h^{(n)}} \sum_{i=1}^{n} L_i\left(h^{(i)}\right) + \alpha \sum_{\{i,i'\}\in\mathcal{E}} A_{i,i'} d^{(h^{(i)},h^{(i')})}$$

trivially, for each node i, solution $\widehat{h}^{(i)}$ must satisfy

$$\widehat{h}^{(i)} \in \underbrace{\operatorname*{arg\,min}_{h\in\mathcal{H}^{(i)}} L_i\left(h\right) + \alpha \sum_{i'\in\mathcal{N}^{(i)}} A_{i,i'} d^{(h,\widehat{h}^{(i')})}}_{\mathcal{F}^{(i)}\left(\widehat{h}^{(1)},\ldots,\widehat{h}^{(n)}\right)}$$

(when is this necessary condition also sufficient?)

Fixed-Point Characterization (ctd.)

necessary condition for GTVMin solution:

$$\widehat{h}^{(i)} \in \underbrace{\underset{h \in \mathcal{H}^{(i)}}{\arg\min}\, L_i\left(h\right) + \alpha \sum_{i' \in \mathcal{N}^{(i)}} A_{i,i'} d^{(h,\widehat{h}^{(i')})}}_{\mathcal{F}^{(i)}\left(\widehat{h}^{(1)},\ldots,\widehat{h}^{(n)}\right)}$$

- when is this necessary condition also sufficient?
- nothing but regularization of $L_i\left(h\right)$ using penalty term

$$\alpha \sum_{i' \in \mathcal{N}^{(i)}} A_{i,i'} d^{(h,\widehat{h}^{(i')})}$$

Fixed-Point Characterization for Parametric Models

consider GTVMin with a smooth and convex $L_i(\cdot)$,

$$\min_{\mathbf{w}^{(1)},\ldots,\mathbf{w}^{(n)} \in \mathbb{R}^d} \sum_{i=1}^{n} L_i\left(\mathbf{w}^{(i)}\right) + \alpha \sum_{\{i,i'\} \in \mathcal{E}} A_{i,i'} \left\| \mathbf{w}^{(i)} - \mathbf{w}^{(i')} \right\|_2^2 \quad (1)$$

$$\boxed{\underbrace{\widehat{\mathbf{w}}}_{\left(\widehat{\mathbf{w}}^{(1)},\ldots,\widehat{\mathbf{w}}^{(n)}\right)} \quad \text{solves (1)} \Leftrightarrow \widehat{\mathbf{w}} = \mathcal{F}^{(\eta)} \widehat{\mathbf{w}}}$$

$\mathcal{F}^{(\eta)}$ maps $\mathbf{u} = \left(\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(n)}\right)^T$ to $\mathbf{v} = \left(\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}\right)^T$,

$$\mathbf{v}^{(i)} = \mathbf{u}^{(i)} - \eta \left[\nabla L_i\left(\mathbf{u}^{(i)}\right) + 2\alpha \sum_{i' \in \mathcal{N}^{(i)}} A_{i,i'} \left(\mathbf{u}^{(i)} - \mathbf{u}^{(i')}\right) \right].$$

different learning rate $\eta > 0$ yields different $\mathcal{F}^{(\eta)}$

Fixed-Point Iterations

Q: how to compute a fixed point $\widehat{\mathbf{w}}$ of $\mathcal{F}$?

A: start with initial guess $\widehat{\mathbf{w}}^{(0)}$ and iterate

$$\widehat{\mathbf{w}}^{(t)} = \mathcal{F}\widehat{\mathbf{w}}^{(t-1)}, \text{ for } t = 1, 2, \ldots.$$

if $\mathcal{F}$ is **firmly non-expansive operator**, $\lim_{t\to\infty} \widehat{\mathbf{w}}^{(t)} = \widehat{\mathbf{w}}$

if $\mathcal{F}$ is a **contractive operator** with constant $\kappa < 1$,

$$\left\|\widehat{\mathbf{w}}^{(t)} - \widehat{\mathbf{w}}\right\|_2 \leq \kappa^t \left\|\widehat{\mathbf{w}}^{(0)} - \widehat{\mathbf{w}}\right\|_2$$

H. Bauschke, P. Combettes, "Convex Analysis and Monotone Operator Theory in Hilbert Spaces," Springer, 2017.

GD as Fixed-point iteration

gradient descent (GD) for smooth and convex $f(\mathbf{w})$,

$$\mathbf{w}^{(t)} = \mathbf{w}^{(t-1)} - \eta \nabla f\left(\mathbf{w}^{(t-1)}\right)$$

is a fixed-point iteration with $\mathcal{F}^{(\eta)} : \mathbf{w} \mapsto \mathbf{w} - \eta \nabla f(\mathbf{w})$

- convergence can be ensured if η is sufficiently small
- e.g., use varying learning rate $\eta_t = 1/t$

FL Algorithm as Fixed-point iteration

turn GTVMin optimality condition into FL algorithm

$$\widehat{h}^{(i,t+1)} = \mathcal{F}^{(i)}\left(\widehat{h}^{(1,t)}, \ldots, \widehat{h}^{(n,t)}\right) \text{ for } t = 0, 1, \ldots$$

- the node-wise update operator $\mathcal{F}^{(i)}$ depends on
  - local loss function $L_i(\cdot)$
  - neighbors $\mathcal{N}^{(i)}$ and edge weights
  - discrepancy measure $d^{(\cdot,\cdot)}$
- design choices ensure that iteration solves GTVMin
- update is an instance of regularized empirical risk minimization (RERM)

Online FL Algorithms

a more general form of FL algorithms is

$$\widehat{h}^{(i,t+1)} = \mathcal{F}^{(i,t)}\left(\widehat{h}^{(1,t)}, \ldots, \widehat{h}^{(n,t)}\right)$$

with time-varying update operators $\mathcal{F}^{(i,t)}$

- allows for data points arriving continuously
- relevant for online learning or reinforcement learning (RL)
- $\mathcal{F}^{(i,t)}$ depends on data arriving at node i and time t

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The FL Workflow of this Course

1. formulate FL application as GTVMin
2. GTVMin solutions are trained local models
3. find a fixed-point characterization of GTVMin solutions
4. solve GTVMin via fixed-point iteration

Two Research Question

two core questions:

- (statistical) where do fixed-point iterations converge to ?
- (compute) how to efficiently compute fixed-point iterations ?

What's Next?

the next lecture discusses the design and study of fixed-point iterations for solving GTVMin

Further Resources

- **YouTube:** @alexjung111
- **LinkedIn:** Alexander Jung
- **GitHub:** alexjungaalto