Assignment A3 — FL Algorithms as Fixed-Point Iterations

CS-E4740 Federated Learning

Read Carefully

There is no submission. Your work is assessed via a multiple-choice quiz.

1 Purpose

This assignment studies how a one-dimensional optimization problem

$$\min_{w \in \mathbb{R}} f(w) \tag{1}$$

can be solved via a fixed-point iteration

$$w^{(t+1)} = \mathcal{F} w^{(t)}. \tag{2}$$

In general, there are different choices for the operator $\mathcal{F}$ such that (2) converges to a solution of (1). We focus on two constructions:

- a gradient step for a smooth and strongly convex objective function,
- a proximal operator for a convex (possibly non-smooth) objective function.

2 Part I: Gradient-Based Fixed-Point Iteration

Consider the strongly convex and smooth objective function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(w) = 2w^2 - 8w.$$

Tasks

1. Compute the minimizer $w^\star$ using the zero-gradient condition
$$f'(w^\star) = 0.$$
2. Consider the gradient step
$$T^{(f,\eta)}(w) = w - \eta f'(w),$$
with learning rate $\eta > 0$. Show that the iteration
$$w^{(t+1)} = T^{(f,\eta)}(w^{(t)})$$
always satisfies
$$w^{(t+1)} - w^\star = q(\eta)(w^{(t)} - w^\star).$$
3. Compute numerically:
   - the largest number $\alpha \geq 0$ such that $T^{(f,\eta)}$ is a contractive operator for any $\eta \in (0, \alpha)$.
   - the value $q(1/8)$,
4. Starting from $w^{(0)} = 18$, perform 3 iterations using $T^{(f,\eta)}$ with $\eta = 1/8$ and compute $w^{(3)} - w^\star$.

3 Part II: Proximal Fixed-Point Iteration

For a continuous convex objective function f, we can define the proximal operator

$$\mathbf{prox}_{\eta f}(w) := \arg\min_{u\in\mathbb{R}} \left(f(u) + \frac{1}{2\eta}(u-w)^2 \right) \text{ with some } \eta > 0.$$

3.1 Strongly Convex Case

Consider the strongly convex objective function

$$f(w) = 2w^2.$$

1. Try to find the corresponding proximal operator in closed-form.
2. For $\eta = 1/2$, compute the smallest number κ such that $|w^{(t+1)}| \le \kappa |w^{(t)}|$ for all $t = 0, 1, \ldots$.
3. Starting from $w^{(0)} = 81$, perform 4 iterations

   $$w^{(t+1)} = \mathbf{prox}_{0.5f}(w^{(t)}).$$

   and report

   $$|w^{(4)}|.$$

3.2 Non-Strongly Convex Case

Now consider the convex objective function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(w) = |w|.$$

Note that this function is neither differentiable nor strongly convex.

1. Compute explicitly

   $$T(w) := \mathbf{prox}_{0.5f}(w).$$

   Hint: Recall that the proximal operator is defined as the minimizer of a one-dimensional convex optimization problem. For a fixed w, consider three possible cases where the minimizer lies in $(-\infty, 0)$, $\{0\}$, or $(0, \infty)$. For each possible case, simplify the objective accordingly and determine the candidate minimizer.
2. Compute the maximal ratio

   $$\sup_{u\neq v} \frac{|T(u) - T(v)|}{|u - v|}.$$

4 Summary

This assignment demonstrates that:

- The gradient step for a strongly convex and smooth objective function is a contractive operator
- The proximal operator of a convex (possibly non-differentiable) function is always non-expansive.