Networked Federated Learning
for Numerical Weather Prediction

Alexander Jung (Aalto University)

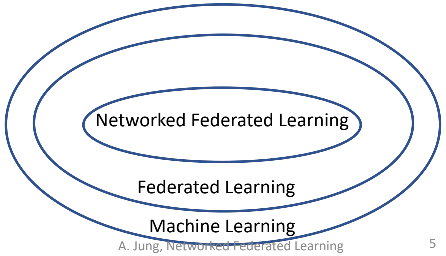
https://www.linkedin.com/in/aljung/
https://www.youtube.com/channel/UC_tW4Z_GfJ2WCnKDtwMuDUA
https://twitter.com/alexjungaalto

- GTVMin as NFL Principle
- The Dual of GTVMin
- Interpretations
- Computational Aspects
- Statistical Aspects

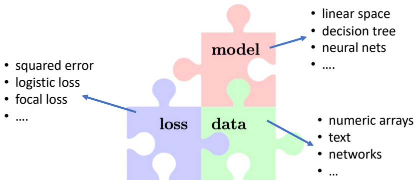
- GTVMin as NFL Principle
- The Dual of GTVMin
- Interpretations
- Computational Aspects
- Statistical Aspects

In a nutshell:

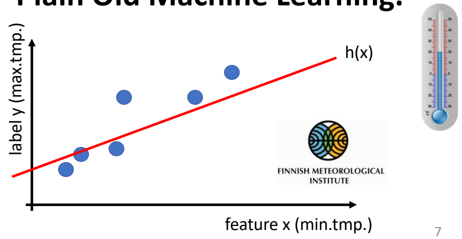
organize data, models and computation for machine learning as networks.



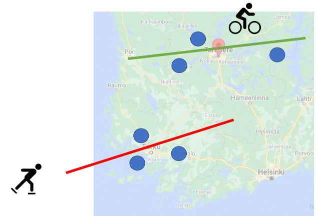
Three Components of ML



Plain Old Machine Learning.



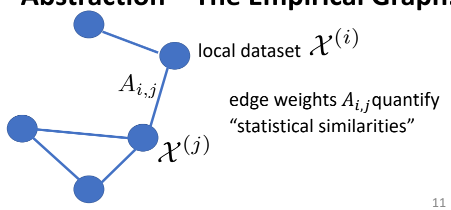
Networked Federated Learning



Networked Data

Weather Stations.

Abstraction – The Empirical Graph.

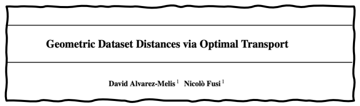


edge weights $A_{i,j}$ quantify "statistical similarities"

How To Measure Statistical Sim.?

```
>>> from scipy.stats import ks_2samp
>>> import numpy as np
>>>
>>> np.random.seed(12345678)
>>> x = np.random.normal(0, 1, 1000)
>>> y = np.random.normal(0, 1, 1000)
>>> z = np.random.normal(1.1, 0.9, 1000)
>>>
>>> ks_2samp(x, y)
Ks_2sampResult(statistic=0.022999999999999909, pvalue=0.95189016804849647)
>>> ks_2samp(x, z)
Ks_2sampResult(statistic=0.41800000000000004, pvalue=3.7081494119242173e-77)
```

https://stackoverflow.com/questions/10884668/two-sample-kolmogorov-smirnov-test-in-python-scipy
https://en.wikipedia.org/wiki/Kolmogorov%E2%80%93Smirnov_test

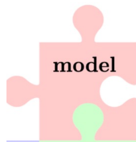

Geometric Dataset Distances via Optimal Transport

David Alvarez-Melis[1] Nicolò Fusi[1]

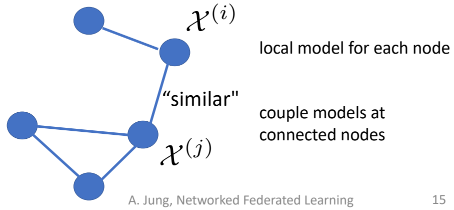
"In this work we propose an alternative notion of distance between datasets that (i) is model-agnostic, (ii) does not involve training,…

https://arxiv.org/pdf/2002.02923.pdf

Networked Models



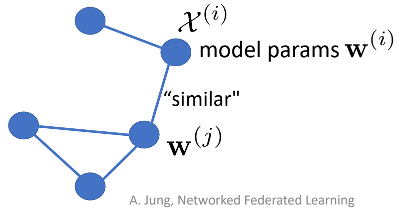
Networked Models.



local model for each node

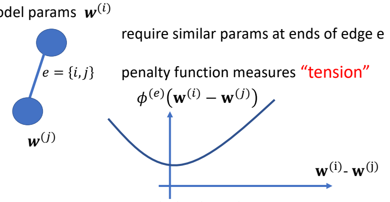
couple models at connected nodes

Networked Parametric Models.



Smoothness/Clustering Assumption.

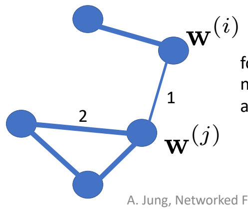
model params $\mathbf{w}^{(i)}$



require similar params at ends of edge e

penalty function measures "tension"

Generalized Total Variation (GTV)



force params of well connected nodes to be similar by requiring a small GTV

$$\sum_{\{i,j\}} A_{i,j}\phi(\mathbf{w}^{(i)} - \mathbf{w}^{(j)})$$

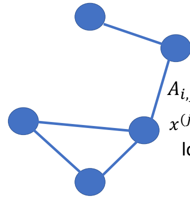
Two Special Cases of GTV.

total variation $\phi(\mathbf{u}) = \|\mathbf{u}\|_2$

graph Laplacian quadratic from is GTV with

$$\phi(\mathbf{u}) = \|\mathbf{u}\|_2^2$$

Smooth Graph Signals.



low pass constraint on

$$\sum_{\{i,j\}} A_{i,j}(\mathbf{w}^{(i)} - \mathbf{w}^{(j)})^2$$

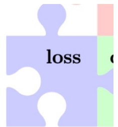
From now on,

GTVMin with penalty being a norm

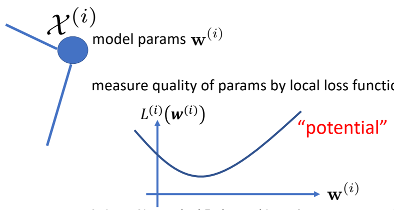
$$\phi(\mathbf{u}) = \|\mathbf{u}\|$$

(unless otherwise stated)

GTV Minimization.



Local Loss Functions.



measure quality of params by local loss function

GTV Minimization.

$$\min_{\mathbf{w}} \sum_{i\in\mathrm{M}} L^{(i)}(\mathbf{w}^{(i)}) + \lambda \sum_{\{i,j\}} A_{i,j}\phi(\mathbf{w}^{(i)} - \mathbf{w}^{(j)})$$

average local loss

training set $\mathcal{M}$

increasing λ

"clusteredness"

Special Case: Network Lasso.

$$\min_{\mathbf{w}} \sum_{i\in\mathrm{M}} L^{(i)}(w^{(i)}) + \lambda \sum_{\{i,j\}} A_{i,j}\|w^{(i)} - w^{(j)}\|$$

Network Lasso: Clustering and Optimization in Large Graphs

by D Hallac · 2015 · Cited by 206 — **Network Lasso**: **Clustering and Optimization in Large Graphs** ... Keywords: Convex **Optimization**, ADMM, **Network Lasso**. Go to: ... 2013 [**Google Scholar**]. 2.

Abstract · INTRODUCTION · CONVEX PROBLEM... · EXPERIMENTS

Special Case: "MOCHA"

$$\min_{w} \sum_{i\in\mathrm{M}} L^{(i)}(w^{(i)}) + \lambda \sum_{\{i,j\}} A_{i,j}\|w^{(i)} - w^{(j)}\|^2$$

https://papers.nips.cc › paper › 7029-federated-m... PDF

Federated Multi-Task Learning - NIPS Proceedings

by V Smith · 2017 · Cited by 501 — 3.2 MOCHA: A Framework for **Federated Multi-Task Learning**. In the **federated** setting, the aim is to train statistical models directly on the edge, and thus we solve (1) while assuming that the data {X1,..., Xm} is distributed across m nodes or devices.

- GTVMin as NFL Principle
- The Dual of GTVMin
- Interpretations
- Computational Aspects
- Statistical Aspects

"Massaging" GTV Minimization.

$$\widehat{\mathbf{w}} \in \arg\min_{\mathbf{w}\in\mathcal{W}} f(\mathbf{w}) + g(\mathbf{D}\mathbf{w})$$

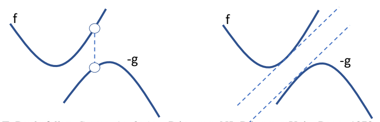
with $f(\mathbf{w}) := \sum_{i\in\mathcal{V}} L_i(\mathbf{w}^{(i)})$, and $g(\mathbf{u}) := \lambda \sum_{e\in\mathcal{E}} A_e\phi(\mathbf{u}^{(e)})$.

with incidence matrix/operator

$\mathbf{D} : \mathcal{W} \rightarrow \mathcal{U} : \mathbf{w} \mapsto \mathbf{u}$ with $\mathbf{u}^{(e)} = \mathbf{w}^{(e_+)} - \mathbf{w}^{(e_-)}$.

Fenchel's Duality

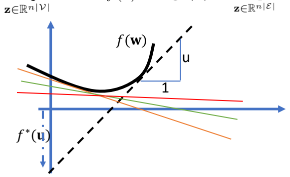
$$\min_{\mathbf{w}\in\mathcal{W}} f(\mathbf{w}) + g(\mathbf{D}\mathbf{w}) = \max_{\mathbf{u}\in\mathcal{U}} -g^*(\mathbf{u}) - f^*(-\mathbf{D}^T\mathbf{u}).$$



R. T. Rockafellar, *Convex Analysis*. Princeton, NJ: Princeton Univ. Press, 1970.
https://en.wikipedia.org/wiki/Fenchel%27s_duality_theorem

Convex Conjugate.

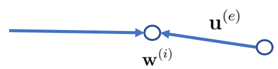
$f^*(\mathbf{w}) := \sup_{\mathbf{z}\in\mathbb{R}^{n|\mathcal{V}|}} \mathbf{w}^T\mathbf{z} - f(\mathbf{z})$ $\quad g^*(\mathbf{u}) := \sup_{\mathbf{z}\in\mathbb{R}^{n|\mathcal{E}|}} \mathbf{u}^T\mathbf{z} - g(\mathbf{z})$



The Dual of GTVMin.

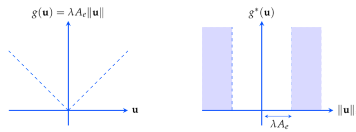
$$\max_{\mathbf{u}\in\mathcal{U}} - \sum_{i\in\mathcal{V}} L_i^*(\mathbf{w}^{(i)}) - \lambda \sum_{e\in\mathcal{E}} A_e\phi^*(\mathbf{u}^{(e)}/(\lambda A_e))$$

$$\text{subject to } -\mathbf{w}^{(i)} = \sum_{e\in\mathcal{E}} \sum_{i=e_\perp} \mathbf{u}^{(e)} - \sum_{i=e_-} \mathbf{u}^{(e)} \text{ for all nodes } i \in \mathcal{V}.$$



dual variables $\mathbf{u}^{(e)}$ for each (oriented) edge $e = (j, i)$

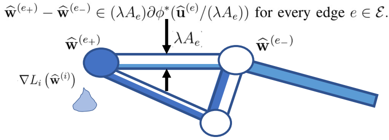
Convex Conjugate of Norm.



Primal and Dual Optimality.

$$\sum_{e\in\mathcal{E}}\sum_{i=e_+}\hat{\mathbf{u}}^{(e)} - \sum_{i=e_-}\hat{\mathbf{u}}^{(e)} = -\nabla L_i\left(\hat{\mathbf{w}}^{(i)}\right) \text{ for all nodes } i\in\mathcal{V}$$

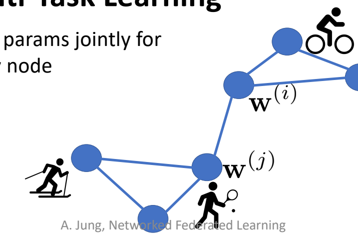
$$\hat{\mathbf{w}}^{(e_+)} - \hat{\mathbf{w}}^{(e_-)} \in (\lambda A_e)\partial\phi^*(\hat{\mathbf{u}}^{(e)}/(\lambda A_e)) \text{ for every edge } e\in\mathcal{E}.$$



- GTVMin as NFL Principle
- The Dual of GTVMin
- Interpretations
- Computational Aspects
- Statistical Aspects

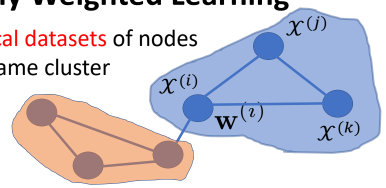
Multi-Task Learning

learn params jointly for every node



Locally Weighted Learning

pool local datasets of nodes in the same cluster



William S. Cleveland, Susan J. Devlin, Eric Grosse, "Regression by local fitting: Methods, properties, and computational algorithms," Journal of Econometrics, Volume 37, Issue 1, 1988.

Generalized Convex Clustering

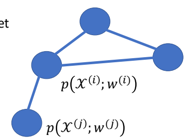
$$\min_{\mathbf{w}} \sum_{i\in M}\left\|w^{(i)} - a^{(i)}\right\|^2 + \lambda\sum_{\{i,j\}} A_{i,j}\left\|w^{(i)} - w^{(j)}\right\|_p$$

D. Sun, K.-C. Toh, Y. Yuan; **Convex Clustering: Model, Theoretical Guarantee and Efficient Algorithm**, JMLR, 22(9):1−32, 2021

(Probabilistic) Graphical Model

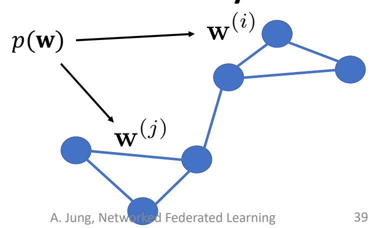
separate prob. space for each local dataset

traditionally, PGMs use a common prob. space for all local datasets



AJ, "**Networked Exponential Families for Big Data Over Networks**," in *IEEE Access*, vol. 8, pp. 202897-202909, 2020, doi: 10.1109/ACCESS.2020.3033817.

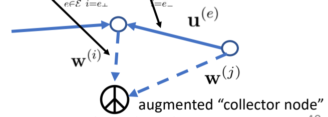
Approx. Hierarch. Bayes' Model



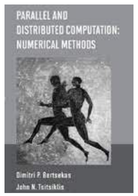
Non-Linear Min-Cost-Flow

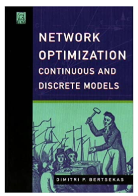
$$\max_{\mathbf{u}\in\mathcal{U}} -\sum_{i\in\mathcal{V}} L_i^*\left(\mathbf{w}^{(i)}\right) - \lambda\sum_{e\in\mathcal{E}} A_e\phi^*\left(\mathbf{u}^{(e)}/(\lambda A_e)\right)$$

subject to $-\mathbf{w}^{(i)} = \sum_{e\in\mathcal{E}}\sum_{i=e_+}\mathbf{u}^{(e)} - \sum_{i=e_-}\mathbf{u}^{(e)}$ for all nodes $i\in\mathcal{V}$.



Non-Linear Min-Cost-Flow





Electrical Network. ("AI is new Electricity!")

Kirchhoff's Current Law

$$\sum_{e\in\mathcal{E}}\sum_{i=e_+}\hat{\mathbf{u}}^{(e)} - \sum_{i=e_-}\hat{\mathbf{u}}^{(e)} = -\nabla L_i\left(\hat{\mathbf{w}}^{(i)}\right) \text{ for all nodes } i\in\mathcal{V}$$

$$\hat{\mathbf{w}}^{(e_+)} - \hat{\mathbf{w}}^{(e_-)} \in (\lambda A_e)\partial\phi^*(\hat{\mathbf{u}}^{(e)}/(\lambda A_e)) \text{ for every edge } e\in\mathcal{E}.$$

Generalized Ohm Law

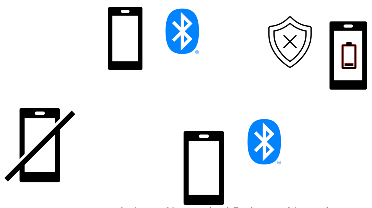
- GTVMin as NFL Principle
- The Dual of GTVMin
- Interpretations
- Computational Aspects
- Statistical Aspects

Computational Aspects.

$$\min_{\mathbf{w}} \sum_{i\in M} L^{(i)}(w^{(i)}) + \lambda\sum_{\{i,j\}} A_{i,j}\phi(w^{(i)} - w^{(j)})$$

- solve in ad-hoc nets of low-cost devices
- robustness against node/link failures
- robustness against "stragglers"

Our Toy NFL Setting



Another NFL Setting…

https://www.google.com/about/datacenters/

https://en.wikipedia.org/wiki/Optical_fiber

Two Main Flavours

- Primal (Gradient) Methods
- Primal-Dual Methods

Primal (Gradient) Methods

Gradient Descent

$$\underbrace{\min_{\mathbf{w}} \sum_{i \in \mathcal{M}} L^{(i)}\left(w^{(i)}\right) + \lambda \sum_{\{i,j\}} A_{i,j}\phi\left(w^{(i)} - w^{(j)}\right)}_{f(w)}$$

optimality condition $\nabla f(w) = 0$

$$w^{(k+1)} = w^{(k)} - \alpha^{(k)} \nabla f\left(w^{(k)}\right)$$

Subgradient Descent (SGD)

$$\underbrace{\min_{\mathbf{w}} \sum_{i \in \mathcal{M}} L^{(i)}\left(w^{(i)}\right) + \lambda \sum_{\{i,j\}} A_{i,j}\phi\left(w^{(i)} - w^{(j)}\right)}_{f(w)}$$

optimality condition $0 \in \partial f(w)$

$$w^{(k+1)} = w^{(k)} - \alpha^{(k)} g^{(k)} \qquad g^{(k)} \in \partial f\left(w^{(k)}\right)$$

Distributed SGD

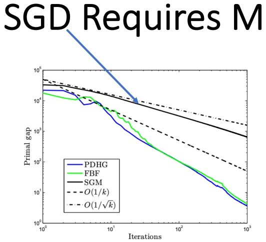
A. Nedić and A. Olshevsky, "Distributed Optimization Over Time-Varying Directed Graphs," in *IEEE Transactions on Automatic Control*, 2015,

A. Nedic (M.S., University of Belgrade, 1991)

A. Nedić and A. Olshevsky, "Stochastic Gradient-Push for Strongly Convex Functions on Time-Varying Directed Graphs," in *IEEE Transactions on Automatic Control*, 2016,

A. Nedic and A. Ozdaglar, "Distributed Subgradient Methods for Multi-Agent Optimization," in *IEEE Transactions on Automatic Control*, Jan. 2009.

SGD Requires Many Iter.



Complexity of SGD

Theorem 3. *Let $L_\ell, R > 0$ and $\gamma \in (0,1]$. There exists a matrix W of eigengap $\gamma(W) = \gamma$, and n functions f_i satisfying (A2), where n is the size of W, such that for all $t < \frac{d-2}{2}\min(\tau/\sqrt{\gamma}, 1)$ and all $i \in \{1, ..., n\}$,*

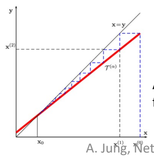
$$\bar{f}(\theta_{i,t}) - \min_{\theta \in B_2(R)} \bar{f}(\theta) \geq \frac{RL_\ell}{108}\sqrt{\frac{1}{(1+\frac{2t\sqrt{\gamma}}{\tau})^2} + \frac{1}{1+t}}. \qquad (19)$$

K. Scaman, F. Bach, S. Bubeck, L. Massoulié, Y Lee, Optimal Algorithms for Non-Smooth Distributed Optimization in Networks, NeurIPS 2018.

SGD as Fixed Point Iteration

$$w^{(k+1)} = \mathcal{T}^{(k)}\left(w^{(k)}\right)$$

with $\mathcal{T}^{(k)}\left(w^{(k)}\right) = w^{(k)} - \alpha^{(k)} \partial f\left(w^{(k)}\right)$



AJ, "A Fixed-Point of View on Gradient Methods for Big Data", Front. Appl. Math. Stat., 2017.

Plenary on Fixed-Point Tools

$$w^{(k+1)} = \mathcal{T}^{(k)}\left(w^{(k)}\right)$$

Jean-Christophe Pesquet

Jean-Christophe Pesquet (I
in 1987, the Ph.D. and HDF
1999, he was a Maître de C
University Paris-Est, and fro
the university. He is curre
Director of the CVN (Inria te
2021. In 2005, J.-C. Pesque
was a member of the SPTM
IEEE SPL (2004-2006). He
journal (2010-2015), and a r
now an associate editor of
methods in data science.

Fixed Point Strategies in Signal and Image Processing

Primal-Dual Methods

Primal-Dual Optimality Conditions.

(assuming convexity of loss functions and GTV penalty)

$$\mathbf{M}^{-1}\begin{pmatrix} \partial f & \mathbf{D}^T \\ -\mathbf{D} & \partial g^* \end{pmatrix}\begin{pmatrix} \widehat{\mathbf{w}} \\ \widehat{\mathbf{u}} \end{pmatrix} \ni \mathbf{0} \text{ with } \mathbf{M} := \begin{pmatrix} \mathbf{T}^{-1} & -\mathbf{D}^T \\ -\mathbf{D} & \mathbf{\Sigma}^{-1} \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} \widehat{\mathbf{w}} \\ \widehat{\mathbf{u}} \end{pmatrix} = \left(\mathbf{I} + \mathbf{M}^{-1}\begin{pmatrix} \partial f & \mathbf{D}^T \\ -\mathbf{D} & \partial g^* \end{pmatrix}\right)^{-1}\begin{pmatrix} \widehat{\mathbf{w}} \\ \widehat{\mathbf{u}} \end{pmatrix}$$

this is again a fixed-point problem !

Proximal Point Algorithm.

primal and dual variables $\widehat{\mathbf{w}}, \widehat{\mathbf{u}}$ optimal if and only if

$$\mathbf{M}^{-1}\begin{pmatrix} \partial f & \mathbf{D}^T \\ -\mathbf{D} & \partial g^* \end{pmatrix}\begin{pmatrix} \widehat{\mathbf{w}} \\ \widehat{\mathbf{u}} \end{pmatrix} \ni \mathbf{0} \text{ with } \mathbf{M} := \begin{pmatrix} \mathbf{T}^{-1} & -\mathbf{D}^T \\ -\mathbf{D} & \mathbf{\Sigma}^{-1} \end{pmatrix}$$

solve iteratively by proximal point algorithm

$$\begin{pmatrix} \widehat{\mathbf{w}}^{(k+1)} \\ \widehat{\mathbf{u}}^{(k+1)} \end{pmatrix} = \left(\mathbf{I} + \mathbf{M}^{-1}\begin{pmatrix} \partial f & \mathbf{D}^T \\ -\mathbf{D} & \partial g^* \end{pmatrix}\right)^{-1}\begin{pmatrix} \widehat{\mathbf{w}}^{(k)} \\ \widehat{\mathbf{u}}^{(k)} \end{pmatrix}$$

A. Chambolle, T. Pock. An introduction to continuous optimization for imaging. Acta Numerica, 2016.

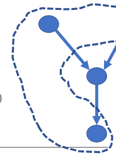
After Some Manipulations.

Algorithm 1 Primal-Dual Method for Networked FL

Input: empirical graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{A})$; training set $\{\mathbf{X}^{(i)}\}_{i \in \mathcal{M}}$; regularization parameter λ; loss $\mathcal{L}$; GTV penalty ϕ

Initialize: $k := 0; \widehat{\mathbf{w}}_0 := \mathbf{0}; \widehat{\mathbf{u}}_0 := \mathbf{0}$; $\sigma_e = 1/2$ and $\tau_i = 1/|\mathcal{N}_i|$

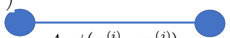
while stopping criterion is not satisfied **do**
  for all nodes $i \in \mathcal{V}$ **do**
    $\widehat{\mathbf{w}}^{(i)}_{k+1} := \widehat{\mathbf{w}}^{(i)}_k - \tau_i \sum_{e \in \mathcal{E}} D_{e,i} \widehat{\mathbf{u}}^{(e)}_k$
  end for
  for nodes in the training set $i \in \mathcal{M}$ **do**
    $\widehat{\mathbf{w}}^{(i)}_{k+1} := \mathcal{PU}^{(i)}\{\widehat{\mathbf{w}}^{(i)}_{k+1}\}$
  end for
  for all edges $e \in \mathcal{E}$ **do**
    $\widehat{\mathbf{u}}^{(e)}_{k+1} := \widehat{\mathbf{u}}^{(e)}_k + \sigma_e\left(2\left(\widehat{\mathbf{w}}^{(e_+)}_{k+1} - \widehat{\mathbf{w}}^{(e_-)}_{k+1}\right) - \left(\widehat{\mathbf{w}}^{(e_+)}_k - \widehat{\mathbf{w}}^{(e_-)}_k\right)\right)$
    $\widehat{\mathbf{u}}^{(e)}_{k+1} := \mathcal{DU}^{(e)}\{\widehat{\mathbf{u}}^{(e)}_{k+1}\}$
  end for
  $k := k+1$
end while



Algorithm 1 is Attractive for NFL…

- decentralized implementation (mess. pass.)
- robust against various imperfections
  - approximate primal/dual updates
  - node/link failures
- privacy friendly; no raw data exchanged

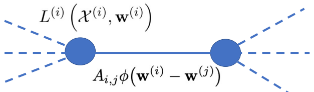
Local Computations in Algorithm 1.



node-wise primal update: $\mathcal{PU}^{(i)}\{\mathbf{v}\} := \underset{\mathbf{z} \in \mathbb{R}^n}{\operatorname{argmin}}\, L^{(i)}(\mathbf{z}) + (1/2\tau_i)\|\mathbf{v} - \mathbf{z}\|^2.$

edge-wise dual update: $\mathcal{DU}^{(e)}\{\mathbf{v}\} := \underset{\mathbf{z} \in \mathbb{R}^n}{\operatorname{argmin}}\, \lambda A_e \phi^*(\mathbf{z}/(\lambda A_e)) + (1/2\sigma_e)\|\mathbf{v} - \mathbf{z}\|^2.$

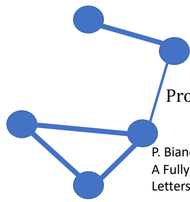
Spreading Local Results.



2: **for** all nodes $i \in \mathcal{V}$ **do**
3: $\widehat{\mathbf{w}}^{(i)}_{k+1} := \widehat{\mathbf{w}}^{(i)}_k - \tau_i \sum_{e \in \mathcal{E}} D_{e,i} \widehat{\mathbf{u}}^{(e)}_k$
4: **end for**

8: **for** all edges $e \in \mathcal{E}$ **do**
9: $\widehat{\mathbf{u}}^{(e)}_{k+1} := \widehat{\mathbf{u}}^{(e)}_k + \sigma_e\left(2\left(\widehat{\mathbf{w}}^{(e_+)}_{k+1} - \widehat{\mathbf{w}}^{(e_-)}_{k+1}\right) - \left(\widehat{\mathbf{w}}^{(e_+)}_k - \widehat{\mathbf{w}}^{(e_-)}_k\right)\right)$

Probabilistic Networked Data

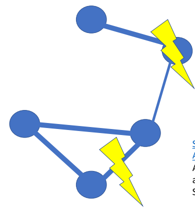


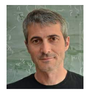
$\mathcal{X}^{(i)}$ iid $\sim p^{(i)}(\mathbf{z})$

$\mathrm{Prob}(A_{i,j} = 1) = p$

P. Bianchi, W. Hachem, A. Salim.
A Fully Stochastic Primal-Dual Algorithm. Optimization Letters, Springer Verlag, 2020,

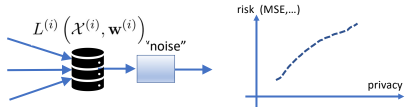
Random Node/Link Failures.





Stochastic Primal-Dual Hybrid Gradient Algorithm with Arbitrary Sampling and Imaging Applications
Antonin Chambolle, Matthias J. Ehrhardt, Peter Richtárik, and Carola-Bibiane Schönlieb
SIAM Journal on Optimization 2018 28:4, 2783-2808

Privacy-Preservation.



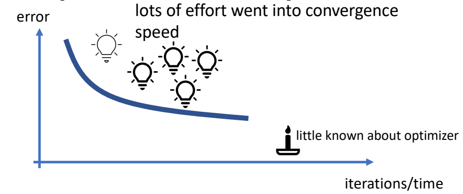
- F. Shang, T. Xu, Y. Liu, H. Liu, L. Shen and M. Gong, "Differentially Private ADMM Algorithms for Machine Learning," in *IEEE Transactions on Information Forensics and Security*, vol. 16, pp. 4733-4745, 2021, doi: 10.1109/TIFS.2021.3113768.
- J. C. Duchi, M. I. Jordan, and M. J. Wainwright, "Local privacy and statistical minimax rates," in Proc. IEEE Annu. Symp. Found. Comput. Sci., pp. 429–438, 2013.

Bottom Line.

PD method solves GTVMin in distributed, robust and privacy-friendly way

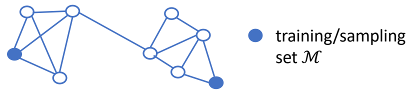
….., however ….

Compute vs. Accuracy



Are GTVMin Solutions Any Good?

$$\min_{\mathbf{w}} \sum_{i\in\mathcal{M}} L^{(i)}(w^{(i)}) + \lambda \sum_{\{i,j\}} A_{i,j}\phi(w^{(i)} - w^{(j)})$$



which combination of signal model (choice of ϕ) and sampling set M ensure solutions of GTVMin are "sensible" ?

- GTVMin as NFL Principle
- The Dual of GTVMin
- Interpretations
- Computational Aspects
- Statistical Aspects

Statistical Aspects of GTVMin

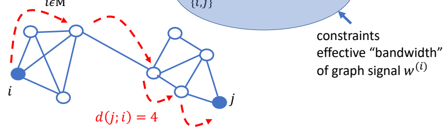
$$min_w \sum_{i\epsilon M} L^{(i)}(w^{(i)}) + \lambda \sum_{\{i,j\}} A_{i,j}\phi(w^{(i)} - w^{(j)})$$

statistical properties of GTVMin solutions?

- sampling theorems (signal processing)
- generalization bounds (ML)

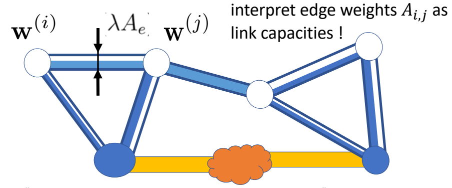
Signal Processing Perspective.

$$\min_{\mathbf{w}} \sum_{i\epsilon M} (w^{(i)} - y^{(i)})^2 + \lambda \sum_{\{i,j\}} A_{i,j}(w^{(i)} - w^{(j)})^2$$



M. Tsitsvero, S. Barbarossa and P. Di Lorenzo, "Signals on Graphs: Uncertainty Principle and Sampling," in *IEEE Transactions on Signal Processing*, 2016,

Our Perspective: Flows.



AJ, "On the Duality Between Network Flows and Network Lasso," in *IEEE Signal Processing Letters*, vol. 27, pp. 940-944, 2020.

Why Flows?

$$\sum_{e\in\mathcal{E}}\sum_{i=e_+} \widehat{\mathbf{u}}^{(e)} - \sum_{i=e_-} \widehat{\mathbf{u}}^{(e)} = -\nabla L_i\left(\widehat{\mathbf{w}}^{(i)}\right) \text{ for all nodes } i \in \mathcal{V}$$

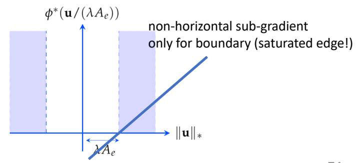
$$\widehat{\mathbf{w}}^{(e_+)} - \widehat{\mathbf{w}}^{(e_-)} \in (\lambda A_e)\partial\phi^*(\widehat{\mathbf{u}}^{(e)}/(\lambda A_e)) \text{ for every edge } e \in \mathcal{E}.$$

the solutions of GTVMin is a flow $\widehat{\mathbf{u}}^{(e)}$,

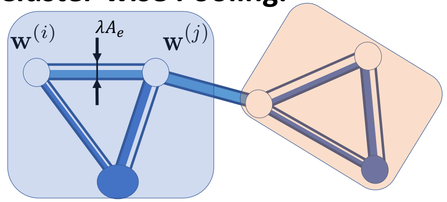
properties of the flow coupled with properties of GTVMin solution!

Primal-Dual Witness

$$\widehat{\mathbf{w}}^{(e_+)} - \widehat{\mathbf{w}}^{(e_-)} \in (\lambda A_e)\partial\phi^*(\widehat{\mathbf{u}}^{(e)}/(\lambda A_e)) \text{ for every edge } e \in \mathcal{E}.$$

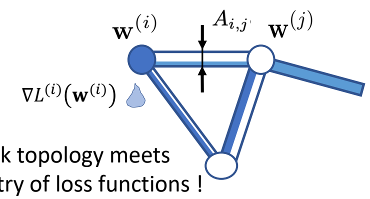


Cluster-wise Pooling.



parameter vectors can only change over saturated links !

Leaky Training Set.

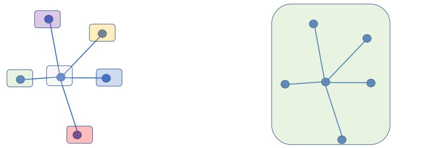


network topology meets geometry of loss functions !

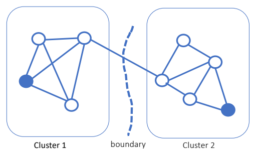
Personalization vs. Globalization

small λ, edges easily saturated

large λ, edges hard to saturate

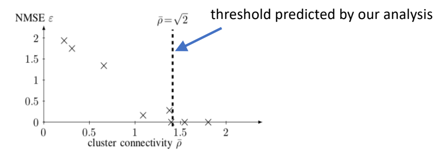


Define Cluster by Boundary Flow.



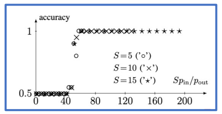
connectivity measured by flow ρ that can be routed over boundary edge

Statistical Error vs. Connectivity.



A. Jung and N. Tran, "Localized Linear Regression in Networked Data," in *IEEE Signal Processing Letters*, vol. 26, no. 7, pp. 1090-1094, July 2019.

Clustering Assumption in SBM.

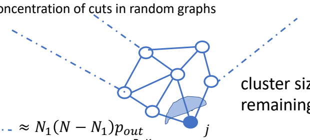


- intra-cluster edge prob p_{in}
- inter-cluster edge prob p_{out}
- S training nodes in each cluster
- critical value for S*pin/pout

A. Jung, "Clustering in Partially Labeled Stochastic Block Models via Total Variation Minimization," *54th Asilomar Conference on Signals, Systems, and Computers*, 2020,

Mathematical Devices.

- flow conservation/Hoffman's circulation theorem
- concentration of cuts in random graphs



cluster size N_1
remaining nodes $N - N_1$

$\approx N_1(N-N_1)p_{out}$

$\approx N_1 p_{in}$

R. Karger,
Random sampling in cut, flow, and network design problems,
Math. Oper. Res., 24 (1999), pp. 383–413.

Wrap Up.

- GTVMin paradigm for NFL
- dual of GTVMin = non-lin. minimum-cost flow
- solve GTVMin. with primal-dual method
- scalable and robust message passing
- GTV min. adaptively pools similar datasets

Thank you for
your attention!