A priori and *a posteriori* error identities for convex minimization problems based on convex duality relations

Lecture 3

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University of Freiburg,
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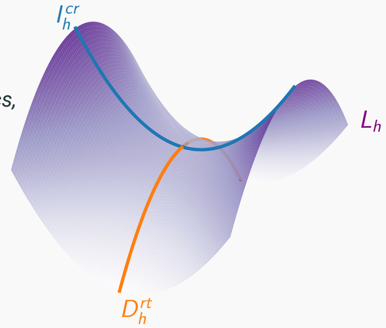


Content of the Lecture 3

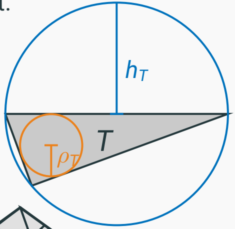
- **Lecture 3: Convex duality theory for discrete integral functionals**
  - Crouzeix–Raviart element and Raviart–Thomas element;
    - Triangulations and discrete spaces;
    - Crouzeix–Raviart element and special features;
    - Raviart–Thomas element and special features;
    - Relations.
  - Fenchel duality theory for discrete integral functionals;
    - Integral representation of discrete dual energy functional;
    - Discrete Fenchel duality relations;
    - Discrete reconstruction formulas;
    - Examples.

Crouzeix–Raviart element and Raviart–Thomas element

Triangulation

- **Triangulation:** Let $\{\mathcal{T}_h\}_{h>0}$ be *shape-regular* triangulations of the *simplicial Lipschitz domain* Ω, *i.e.*, there exists a constant $\omega_0 > 0$ s.t.

$$\sup_{T\in\mathcal{T}_h} \frac{h_T}{\rho_T} \leq \omega_0 \,,$$



where, for every $T \in \mathcal{T}_h$, we denote by

- $\rho_T \coloneqq \sup\{r > 0 \mid \exists x \in T \,:\, B_r^d(x) \subset T\}$;
- $h_T \coloneqq \operatorname{diam}(T)$.

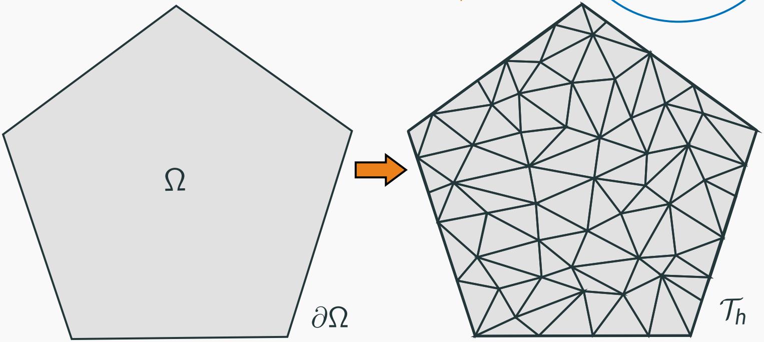


Figure: Triangulation of pentagon $\Omega \subseteq \mathbb{R}^2$.

- **Sets of sides:**

$$\mathcal{S}_h^i \coloneqq \{T \cap T' \mid T, T' \in \mathcal{T}_h : \dim_{\mathscr{H}}(T \cap T') = d-1\}\,,$$

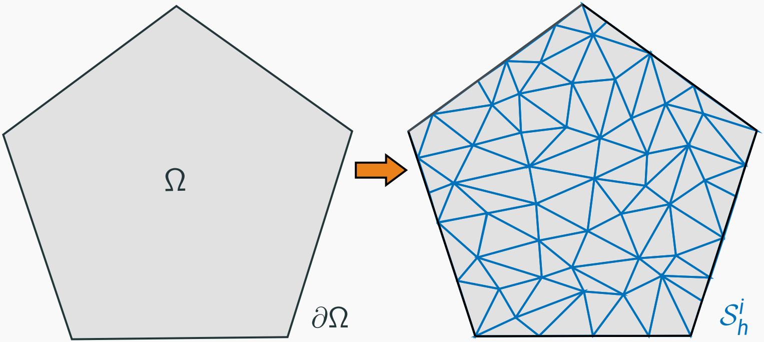


Figure: Triangulation of pentagon $\Omega \subseteq \mathbb{R}^2$.

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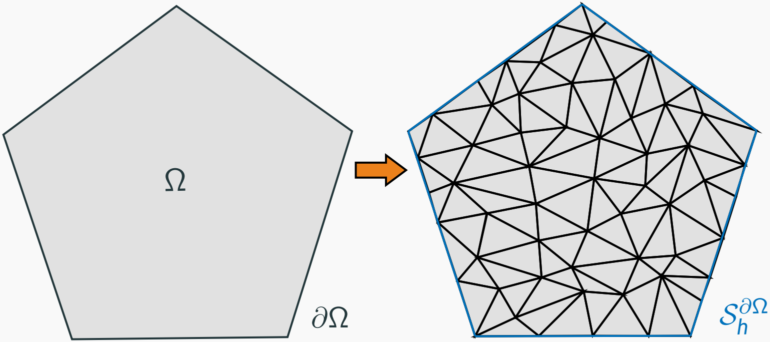
- **Sets of sides:**

$$\mathcal{S}_h^{i} \coloneqq \{T \cap T' \mid T, T' \in \mathcal{T}_h : \dim_{\mathscr{H}}(T \cap T') = d-1\}\,,$$
$$\mathcal{S}_h^{\partial\Omega} \coloneqq \{T \cap \partial\Omega \mid T \in \mathcal{T}_h : \dim_{\mathscr{H}}(T \cap \partial\Omega) = d-1\}\,,$$



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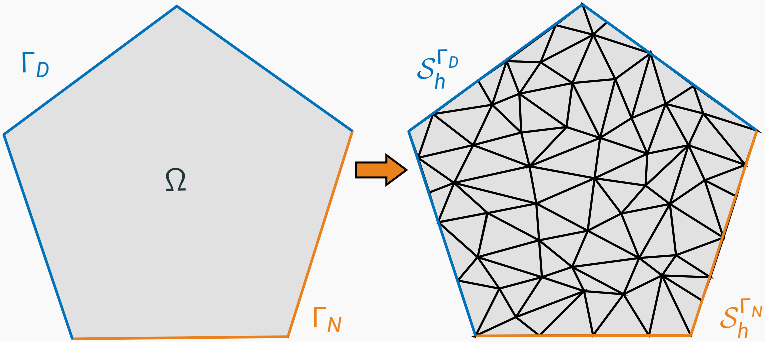
$$\mathcal{S}_h^{\Gamma_D} \coloneqq \{S \in \mathcal{S}_h^{\partial\Omega} \mid \mathrm{int}(S) \subseteq \Gamma_D\}\,,$$
$$\mathcal{S}_h^{\Gamma_N} \coloneqq \{S \in \mathcal{S}_h^{\partial\Omega} \mid \mathrm{int}(S) \subseteq \Gamma_N\}\,,$$



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$$\mathcal{S}_h^{\Gamma_D} \coloneqq \{S \in \mathcal{S}_h^{\partial\Omega} \mid \operatorname{int}(S) \subseteq \Gamma_D\}\,,$$
$$\mathcal{S}_h^{\Gamma_N} \coloneqq \{S \in \mathcal{S}_h^{\partial\Omega} \mid \operatorname{int}(S) \subseteq \Gamma_N\}\,,$$
$$\mathcal{S}_h \coloneqq \mathcal{S}_h^{i} \cup \mathcal{S}_h^{\partial\Omega}\,.$$

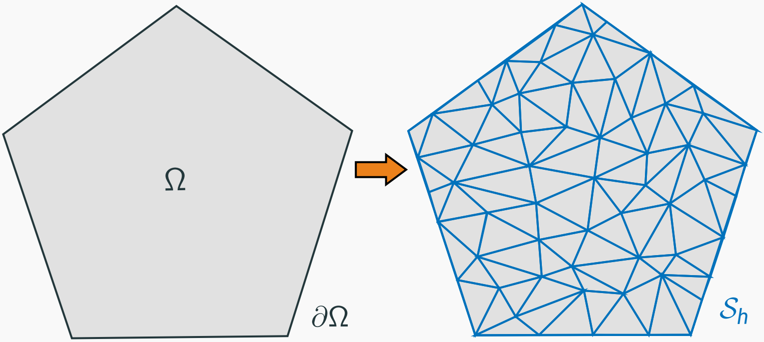

Figure: Triangulation of pentagon $\Omega \subseteq \mathbb{R}^2$.

- **Discrete spaces:** For *polynomial degree* $k \in \mathbb{N} \cup \{0\}$, we define

$$\mathbb{P}^k(\mathcal{T}_h) \coloneqq \left\{ v_h \in L^\infty(\Omega) \mid v_h|_T \in P^k(T) \text{ for all } T \in \mathcal{T}_h \right\},$$

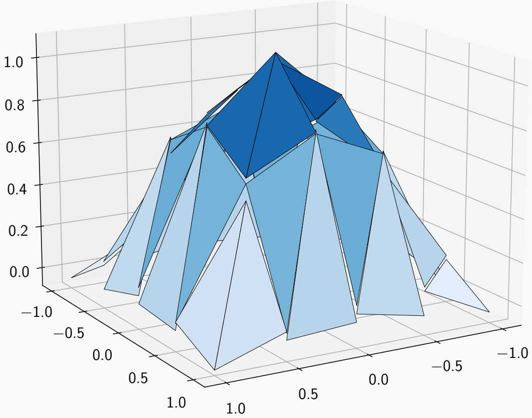

Figure: (local) L^2-projection onto $\mathbb{P}^k(\mathcal{T}_h)$ of $u(x_1, x_2) = (1 - x_1^2)(1 - x_2^2)$.

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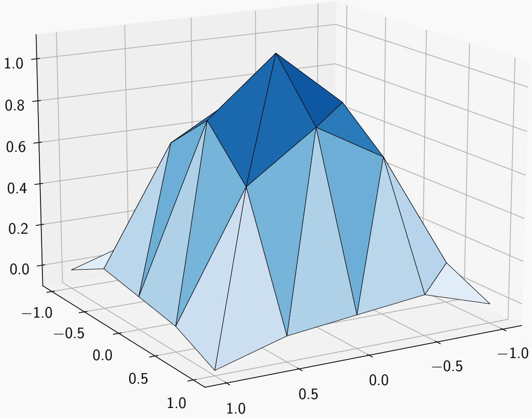
$$\mathcal{S}^k(\mathcal{T}_h) \coloneqq W^{1,1}(\Omega) \cap \mathbb{P}^k(\mathcal{T}_h),$$



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$$\mathcal{S}^k(\mathcal{T}_h) \coloneqq W^{1,1}(\Omega) \cap \mathbb{P}^k(\mathcal{T}_h),$$
$$\mathcal{S}^k_D(\mathcal{T}_h) \coloneqq W^{1,1}_D(\Omega) \cap \mathbb{P}^k(\mathcal{T}_h).$$

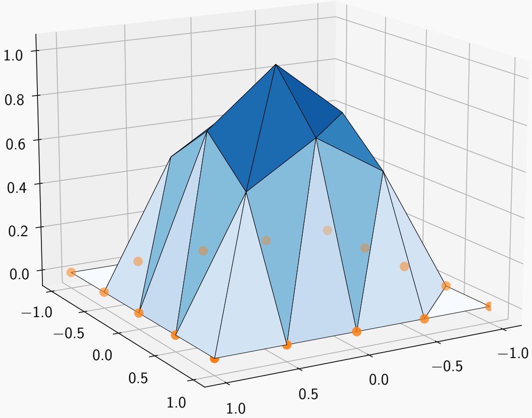
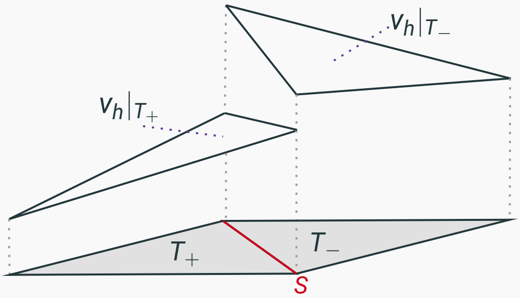


Figure: nodal interpolation into $\mathcal{S}^k_D(\mathcal{T}_h)$ of $u(x_1, x_2) = (1 - x_1^2)(1 - x_2^2)$.

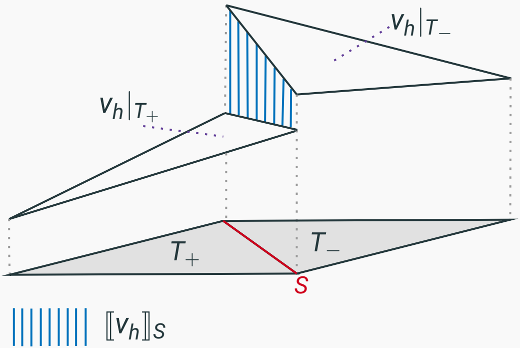
Jumps and averages



Jumps and averages

- **Jumps:** For every $v_h \in \mathbb{P}^k(\mathcal{T}_h)$ and $S \in \mathcal{S}_h$, we define

$$[\![v_h]\!]_S \coloneqq \begin{cases} v_h|_{T_+} - v_h|_{T_-} & \text{if } S \in \mathcal{S}_h^{i}\,,\ T_+, T_- \in \mathcal{T}_h \text{ s.t. } \partial T_+ \cap \partial T_- = S\,,\\ v_h|_T & \text{if } S \in \mathcal{S}_h^{\partial\Omega}\,,\ T \in \mathcal{T}_h \text{ s.t. } S \subseteq \partial T\,. \end{cases}$$



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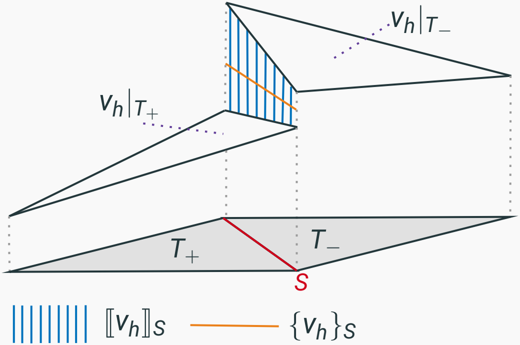
- **Averages:** For every $v_h \in \mathbb{P}^k(\mathcal{T}_h)$ and $S \in \mathcal{S}_h$, we define

$$\{v_h\}_S \coloneqq \begin{cases} \tfrac{1}{2}(v_h|_{T_+} + v_h|_{T_-}) & \text{if } S \in \mathcal{S}_h^{i}\,,\ T_+, T_- \in \mathcal{T}_h \text{ s.t. } \partial T_+ \cap \partial T_- = S\,,\\ v_h|_T & \text{if } S \in \mathcal{S}_h^{\partial\Omega}\,,\ T \in \mathcal{T}_h \text{ s.t. } S \subseteq \partial T\,. \end{cases}$$



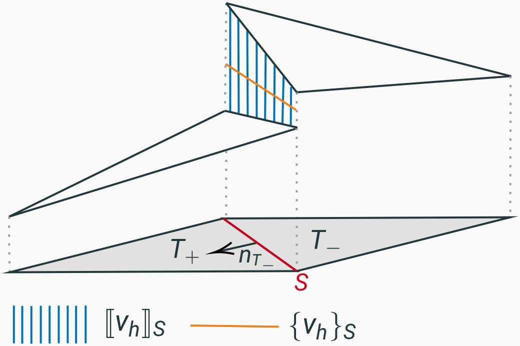
Normal jumps and normal averages

- **Normal jumps:** For every $y_h \in (\mathbb{P}^k(\mathcal{T}_h))^d$ and $S \in \mathcal{S}_h$, we define

$$[\![y_h \cdot n]\!]_S \coloneqq \begin{cases} y_h|_{T_+} \cdot n_{T_+} + y_h|_{T_-} \cdot n_{T_-} & \text{if } S \in \mathcal{S}_h^i\,,\ T_+, T_- \in \mathcal{T}_h \text{ s.t. } \partial T_+ \cap \partial T_- = S\,, \\ y_h|_T \cdot n & \text{if } S \in \mathcal{S}_h^{\partial\Omega}\,,\ T \in \mathcal{T}_h \text{ s.t. } S \subseteq \partial T\,, \end{cases}$$

- **Normal averages:** For every $y_h \in (\mathbb{P}^k(\mathcal{T}_h))^d$ and $S \in \mathcal{S}_h$, we define

$$\{ y_h \cdot n \}_S \coloneqq \begin{cases} \frac{1}{2}(y_h|_{T_+} \cdot n_{T_+} - y_h|_{T_-} \cdot n_{T_-}) & \text{if } S \in \mathcal{S}_h^i\,,\ T_+, T_- \in \mathcal{T}_h \text{ s.t. } \partial T_+ \cap \partial T_- = S\,, \\ y_h|_T \cdot n & \text{if } S \in \mathcal{S}_h^{\partial\Omega}\,,\ T \in \mathcal{T}_h \text{ s.t. } S \subseteq \partial T\,, \end{cases}$$



Crouzeix–Raviart element

- **Crouzeix–Raviart element:** (*cf.* [2, Crouzeix & Raviart, '73])

$$\mathcal{S}^{1,cr}(\mathcal{T}_h) \coloneqq \left\{ v_h \in \mathbb{P}^1(\mathcal{T}_h) \;\middle|\; \fint_S [\![v_h]\!]_S \,\mathrm{d}s = [\![v_h]\!]_S(x_S) = 0 \text{ for all } S \in \mathcal{S}_h^i \right\},$$

$$\mathcal{S}_D^{1,cr}(\mathcal{T}_h) \coloneqq \left\{ v_h \in \mathcal{S}^{1,cr}(\mathcal{T}_h) \;\middle|\; \fint_S v_h \,\mathrm{d}s = v_h(x_S) = 0 \text{ for all } S \in \mathcal{S}_h^{\Gamma_D} \right\}.$$

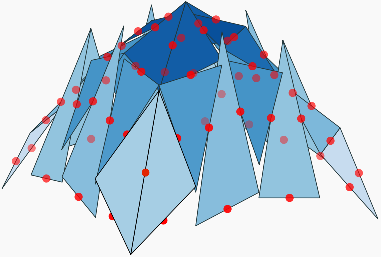


Figure: Crouzeix–Raviart minimizer of Dirichlet energy with $\Omega \coloneqq (-1,1)^2$, $\Gamma_D \coloneqq \partial\Omega$ and $f \coloneqq 1$.

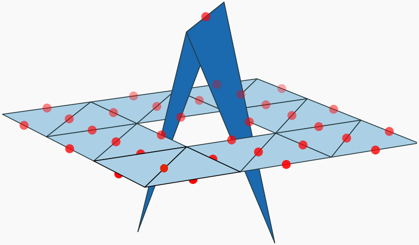


Figure: Crouzeix–Raviart basis function.

- **(Non-)Conformity:** $\mathcal{S}^{1,cr}(\mathcal{T}_h) \not\subseteq W^{1,p}(\Omega)$.

$x_S \coloneqq \frac{1}{d}\sum_{\nu \in \mathcal{N}_h \,:\, \nu \in T} \nu$ for all $S \in \mathcal{S}_h$.

- **Crouzeix–Raviart element:** (*cf.* [2, Crouzeix & Raviart, '73])

$$\mathcal{S}^{1,cr}(\mathcal{T}_h) \coloneqq \left\{ v_h \in \mathbb{P}^1(\mathcal{T}_h) \;\middle|\; \fint_S [\![v_h]\!]_S \,\mathrm{d}s = [\![v_h]\!]_S(x_S) = 0 \text{ for all } S \in \mathcal{S}_h^i \right\},$$

$$\mathcal{S}_D^{1,cr}(\mathcal{T}_h) \coloneqq \left\{ v_h \in \mathcal{S}^{1,cr}(\mathcal{T}_h) \;\middle|\; \fint_S v_h \,\mathrm{d}s = v_h(x_S) = 0 \text{ for all } S \in \mathcal{S}_h^{\Gamma_D} \right\}.$$

- **Basis functions:** $(\varphi_S)_{S\in\mathcal{S}_h} \subseteq \mathcal{S}^{1,cr}(\mathcal{T}_h)$ s.t.

$$\varphi_S(x_{S'}) = \delta_{SS'} \qquad \text{for all } S, S' \in \mathcal{S}_h\,,$$

e.g., for $T \in \mathcal{T}_h$ s.t. $S \subseteq \partial T$,

$$\varphi_S \coloneqq 1 - d\varphi_{\nu_S} \qquad \text{in } T\,,$$

where $\nu_S \in \mathcal{N}_h$ with $\nu_S \in T$ and $\nu_S \notin S$.

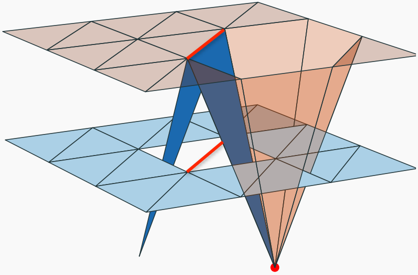


Figure: Crouzeix–Raviart basis function.

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$x_S \coloneqq \frac{1}{d}\sum_{\nu\in\mathcal{N}_h\,:\,\nu\in T} \nu$ for all $S \in \mathcal{S}_h$.

Crouzeix–Raviart element

	Crouzeix–Raviart	Courant
local DOFs	Edges/Facets, *i.e.*, • 3 per element in 2D; • 4 per element in 3D.	Vertices, *i.e.*, • 3 per element in 2D; • 4 per element in 3D.
global DOFs	$\operatorname{card}(\mathcal{S}_h)$ ($\approx 3 \times \operatorname{card}(\mathcal{N}_h)$ in 2D)	$\operatorname{card}(\mathcal{N}_h)$
Sparsity	$\operatorname{supp}(\varphi_S) \subseteq \omega_S$, *i.e.*, on 2 elements	$\operatorname{supp}(\varphi_\nu) \subseteq \omega_\nu$, *i.e.*, on vertex patch
Duality	conforming dual problem	non-conforming dual problem

Figure: Comparison of Crouzeix–Raviart and Courant element in 2D and 3D.

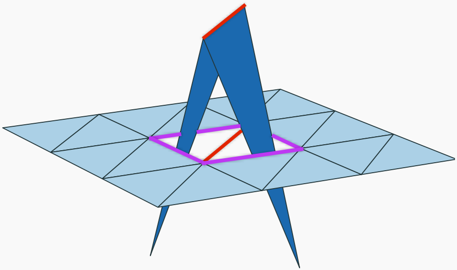


Figure: Crouzeix–Raviart basis function.

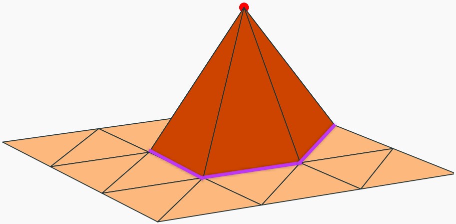


Figure: Courant basis function.

Special features of the Crouzeix–Raviart element

- **Fortin interpolation operator:** $\Pi_h^{cr}\colon W_D^{1,p}(\Omega)\to \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, for every $v\in W_D^{1,p}(\Omega)$, defined by

$$\Pi_h^{cr} v \coloneqq \sum_{S\in\mathcal{S}_h}\langle v\rangle_S\,\varphi_S\,,\quad \text{where}\quad \langle v\rangle_S \coloneqq \fint_S v\,\mathrm{d}s\quad\text{for all } S\in\mathcal{S}_h\,,$$

preserves averages of gradients and moments (on sides), *i.e.*, for every $v\in W_D^{1,p}(\Omega)$, it holds that

$$\begin{aligned}\nabla_h\Pi_h^{cr} v &= \Pi_h\nabla v &&\text{in } (\mathbb{P}^0(\mathcal{T}_h))^d\,,\\ \pi_h\Pi_h^{cr} v &= \pi_h v &&\text{in } \mathbb{P}^0(\mathcal{S}_h)\,,\end{aligned}$$

where

- $\Pi_h\colon L^1(\Omega)\to\mathbb{P}^0(\mathcal{T}_h)$, for every $v\in L^1(\Omega)$, is defined by

$$\Pi_h v \coloneqq \sum_{T\in\mathcal{T}_h}\langle v\rangle_T\,\chi_T\,,\quad \text{where}\quad \langle v\rangle_T \coloneqq \fint_T v\,\mathrm{d}s\quad\text{for all } T\in\mathcal{T}_h\,;$$

- $\pi_h\colon L^1(\cup\mathcal{S}_h)\to\mathbb{P}^0(\mathcal{S}_h)$, for every $v\in L^1(\cup\mathcal{S}_h)$, is defined by

$$\pi_h v \coloneqq \sum_{S\in\mathcal{S}_h}\langle v\rangle_S\,\chi_S\,,\quad \text{where}\quad \langle v\rangle_S \coloneqq \fint_S v\,\mathrm{d}s\quad\text{for all } S\in\mathcal{S}_h\,.$$

Stability with constant 1

Theorem: (stability with constant 1)

If $\phi\colon \mathbb{R}^d \to \mathbb{R}$ is convex, then for every $v \in W^{1,p}_D(\Omega)$, it holds that

$$\int_\Omega \phi(\nabla_h \Pi_h^{cr} v)\,\mathrm{d}x \leq \int_\Omega \phi(\nabla v)\,\mathrm{d}x\,,$$

i.e., with constant 1.

Proof.

- By *Jensen's inequality*, for every $T \in \mathcal{T}_h$, it holds that

$$\phi\Big(\fint_T \nabla v\,\mathrm{d}y\Big) \leq \fint_T \phi(\nabla v)\,\mathrm{d}y\,.$$

- Due to $\nabla_h \Pi_h^{cr} v = \Pi_h \nabla v$, we conclude that

$$\begin{aligned}
\int_\Omega \phi(\nabla_h \Pi_h^{cr} v)\,\mathrm{d}x &= \sum_{T\in\mathcal{T}_h} \int_T \phi\Big(\fint_T \nabla v\,\mathrm{d}y\Big)\,\mathrm{d}x \\
&\leq \sum_{T\in\mathcal{T}_h} \Big(\int_T 1\,\mathrm{d}x\Big) \fint_T \phi(\nabla v)\,\mathrm{d}y \\
&= \int_\Omega \phi(\nabla v)\,\mathrm{d}x\,.
\end{aligned}$$

$\blacksquare$

Theorem: (discrete Poincaré inequality)

If $|\Gamma_D| > 0$, then for every $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$, it holds that

$$\int_\Omega |v_h|^p \,\mathrm{d}x \lesssim_h \int_\Omega |\nabla_h v_h|^p \,\mathrm{d}x\,.$$

♦ **Proof (by sketch).** Show that $\ker(\nabla_h|_{\mathcal{S}^{1,cr}_D(\mathcal{T}_h)}) = \{0\}$.

- Let $v_h \in \ker(\nabla_h|_{\mathcal{S}^{1,cr}_D(\mathcal{T}_h)})$, i.e., for every $T \in \mathcal{T}_h$, it holds that

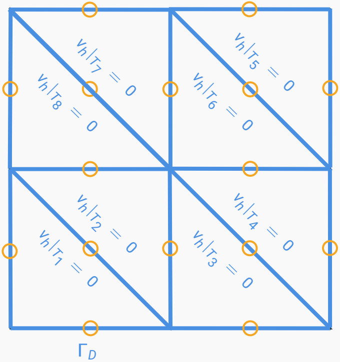
$$\nabla(v_h|_T) = 0 \quad \text{in } T\,.$$

→ For every $T \in \mathcal{T}_h$, there exists $c_T \in \mathbb{R}$ s.t.

$$v_h|_T = c_T \quad \text{in } T\,.$$

→ Due to $v_h(x_S) = 0$ for every $S \in \mathcal{S}^{\Gamma_D}_h \neq \emptyset$, it follows that

$$v_h = 0 \quad \text{a.e. in } \Omega\,. \qquad \blacksquare$$



- **Raviart–Thomas element:** (*cf.* [4, Raviart & Thomas, '77])

$$\mathcal{RT}^0(\mathcal{T}_h) \coloneqq \{y_h \in (\mathbb{P}^1(\mathcal{T}_h))^d \mid y_h|_T \cdot n_T = \mathrm{const} \text{ on } \partial T \text{ for all } T \in \mathcal{T}_h\,,$$
$$[\![y_h \cdot n]\!]_S = 0 \text{ on } S \text{ for all } S \in \mathcal{S}_h^i\}\,,$$
$$\mathcal{RT}^0_N(\mathcal{T}_h) \coloneqq \{y_h \in \mathcal{RT}^0(\mathcal{T}_h) \mid y_h \cdot n = 0 \text{ a.e. on } \Gamma_N\}\,.$$

- **Basis functions:** $(\psi_S)_{S\in\mathcal{S}_h} \subseteq \mathcal{RT}^0(\mathcal{T}_h)$ s.t.

$$\psi_S \cdot n_{S'} = \delta_{SS'} \quad \text{on } S' \quad \text{for all } S, S' \in \mathcal{S}_h\,,$$

e.g.,

$$\psi_S(x) \coloneqq \begin{cases} \pm\frac{|S|}{(d!)|T_\pm|}(\nu_\pm - x) & \text{if } x \in T_\pm\,, \\ 0 & \text{if } x \in \Omega \setminus (T_+ \cup T_-)\,. \end{cases}$$

- **Conformity:** $\mathcal{RT}^0(\mathcal{T}_h) \subseteq W^{p'}(\mathrm{div};\Omega)$.

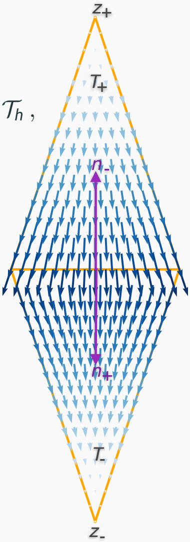


Figure: Raviart–Thomas basis function.

Special features of the Raviart–Thomas element

- **Fortin interpolation operator:** $\Pi_h^{rt}\colon W_N^{p'}(\mathrm{div};\Omega)\cap(W^{1,1}(\Omega))^d\to\mathcal{RT}_N^0(\mathcal{T}_h)$, for every $y\in W_N^{p'}(\mathrm{div};\Omega)\cap(W^{1,1}(\Omega))^d$ defined by

$$\Pi_h^{rt}y\coloneqq\sum_{S\in\mathcal{S}_h}\langle y\cdot n\rangle_S\,\psi_S\,,\qquad\text{where}\qquad\langle y\cdot n\rangle_S\coloneqq\fint_S y\cdot n\,\mathrm{d}s\quad\text{for all }S\in\mathcal{S}_h\,,$$

preserves averages of divergences and moments of normal traces (on sides), *i.e.*, for every $y\in W_N^{p'}(\mathrm{div};\Omega)\cap(W^{1,1}(\Omega))^d$, it holds that

$$\begin{aligned}\mathrm{div}\,\Pi_h^{rt}y&=\Pi_h\mathrm{div}\,y&&\text{in }\mathbb{P}^0(\mathcal{T}_h)\,,\\ \pi_h[\Pi_h^{rt}y\cdot n]&=\pi_h[y\cdot n]&&\text{in }\mathbb{P}^0(\mathcal{S}_h)\,,\end{aligned}$$

where

- $\Pi_h\colon L^1(\Omega)\to\mathbb{P}^0(\mathcal{T}_h)$, for every $v\in L^1(\Omega)$, is defined by

$$\Pi_h v\coloneqq\sum_{T\in\mathcal{T}_h}\langle v\rangle_T\chi_T\,,\qquad\text{where}\qquad\langle v\rangle_T\coloneqq\fint_T v\,\mathrm{d}s\quad\text{for all }T\in\mathcal{T}_h\,;$$

- $\pi_h\colon L^1(\cup\mathcal{S}_h)\to\mathbb{P}^0(\mathcal{S}_h)$, for every $v\in L^1(\cup\mathcal{S}_h)$, is defined by

$$\pi_h v\coloneqq\sum_{S\in\mathcal{S}_h}\langle v\rangle_S\chi_S\,,\qquad\text{where}\qquad\langle v\rangle_S\coloneqq\fint_S v\,\mathrm{d}s\quad\text{for all }S\in\mathcal{S}_h\,.$$

Key ingredient I: discrete surjectivity of divergence operator

Lemma: (key ingredient I: discrete surjectivity of divergence operator)

The following statements apply:

(i) If $\Gamma_N \neq \partial\Omega$, then $\operatorname{div}\colon \mathcal{RT}^0_N(\mathcal{T}_h) \to \mathbb{P}^0(\mathcal{T}_h)$ is surjective;

(ii) If $\Gamma_N = \partial\Omega$, then $\operatorname{div}\colon \mathcal{RT}^0_N(\mathcal{T}_h) \to \mathbb{P}^0_0(\mathcal{T}_h) \coloneqq \mathbb{P}^0(\mathcal{T}_h)/\mathbb{R}$ is surjective.

Proof. If $p<2$, $\Pi_h^{rt}\colon W^{p'}_N(\operatorname{div};\Omega)\to\mathcal{RT}^0_N(\mathcal{T}_h)$ is still well-defined (*cf.* [3, Ern, Guermond, '21]).

ad (i). Since $\operatorname{div}(W^{p'}_N(\operatorname{div};\Omega))=L^{p'}(\Omega)$, for every $f_h\in\mathbb{P}^0(\mathcal{T}_h)$, there is $y\in W^{p'}_N(\operatorname{div};\Omega)$ s.t.

$$\operatorname{div} y = f_h \quad \text{a.e. in } \Omega\,.$$

Then, $y_h \coloneqq \Pi_h^{rt} y \in \mathcal{RT}^0_N(\mathcal{T}_h)$ satisfies

$$\left.\begin{aligned}\operatorname{div} y_h &= \Pi_h \operatorname{div} y\\ &= \Pi_h f_h = f_h\end{aligned}\right\}\quad \text{a.e. in } \Omega\,.$$

ad (ii). Since $\operatorname{div}(W^{p'}_N(\operatorname{div};\Omega))=L^{p'}_0(\Omega)$, for every $f_h\in\mathbb{P}^0_0(\mathcal{T}_h)$, there is $y\in W^{p'}_N(\operatorname{div};\Omega)$ s.t.

$$\operatorname{div} y = f_h \quad \text{a.e. in } \Omega\,.$$

Then, $y_h \coloneqq \Pi_h^{rt} y \in \mathcal{RT}^0_N(\mathcal{T}_h)$ satisfies

$$\left.\begin{aligned}\operatorname{div} y_h &= \Pi_h \operatorname{div} y\\ &= \Pi_h f_h = f_h\end{aligned}\right\}\quad \text{a.e. in } \Omega\,.$$

■

Lemma: (discrete integration-by-parts formula)

For every $v_h \in \mathcal{S}^{1,cr}(\mathcal{T}_h)$ and $y_h \in \mathcal{RT}^0(\mathcal{T}_h)$, it holds that

$$\int_\Omega \nabla_h v_h \cdot y_h \,\mathrm{d}x + \int_\Omega v_h \operatorname{div} y_h \,\mathrm{d}x = \int_{\partial\Omega} v_h\, y_h \cdot n \,\mathrm{d}s\,.$$

Proof.

- Element-wise integration-by-parts yields that

$$\begin{aligned}\int_\Omega \nabla_h v_h \cdot y_h \,\mathrm{d}x + \int_\Omega v_h \operatorname{div} y_h \,\mathrm{d}x &= \sum_{T\in\mathcal{T}_h}\bigg[\int_T \nabla_h v_h \cdot y_h \,\mathrm{d}x + \int_T v_h \operatorname{div} y_h \,\mathrm{d}x\bigg] \\ &= \sum_{T\in\mathcal{T}_h}\int_{\partial T} v_h\, y_h \cdot n \,\mathrm{d}s \\ &= \int_{\partial\Omega} v_h\, y_h \cdot n \,\mathrm{d}s + \sum_{S\in\mathcal{S}_h^i}\int_S [\![v_h y_h \cdot n]\!]_S \,\mathrm{d}s\,.\end{aligned}$$

- The product rule $[\![v_h y_h \cdot n]\!]_S = [\![v_h]\!]_S \{y_h \cdot n\}_S + \{v_h\}_S [\![y_h \cdot n]\!]_S$ for $S \in \mathcal{S}_h^i$ yields that

$$\sum_{S\in\mathcal{S}_h^i}\int_S [\![v_h y_h \cdot n]\!]_S \,\mathrm{d}s = \sum_{S\in\mathcal{S}_h^i}\bigg[\{y_h \cdot n\}_S \underbrace{\int_S [\![v_h]\!]_S \,\mathrm{d}s}_{=0} + \underbrace{[\![y_h \cdot n]\!]_S}_{=0} \int_S \{v_h\}_S \,\mathrm{d}s\bigg] = 0\,. \qquad \blacksquare$$

Lemma: (key ingredient II: discrete orthogonality relation)

$$\ker(\mathrm{div}|_{\mathcal{RT}^0_N(\mathcal{T}_h)}) = (\nabla_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h)))^\perp \qquad \text{(in } (\mathbb{P}^0(\mathcal{T}_h))^d) \,.$$

Proof.

ad '$\subseteq$'. For $y_h \in \ker(\mathrm{div}|_{\mathcal{RT}^0_N(\mathcal{T}_h)})$, for every $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$, it holds that

$$\int_\Omega \nabla_h v_h \cdot y_h \,\mathrm{d}x = -\int_\Omega v_h \underbrace{\mathrm{div}\, y_h}_{=0} \,\mathrm{d}x = 0 \,,$$

i.e., $y_h \in (\nabla_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h)))^\perp$.

ad '$\supseteq$'. For $y_h \in (\nabla_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h)))^\perp$, for every $S \in \mathcal{S}^i_h$, it holds that

$$0 = \int_\Omega \nabla_h \varphi_S \cdot y_h \,\mathrm{d}x = [\![y_h \cdot n]\!]_S \, |S| \,,$$

i.e., $y_h \in \mathcal{RT}^0(\mathcal{T}_h)$ with $\mathrm{div}\, y_h = 0$ a.e. in Ω.

→ For every $S \in \mathcal{S}^{\Gamma_N}_h$, it holds that

$$0 = \int_\Omega \nabla_h \varphi_S \cdot y_h \,\mathrm{d}x = y_h \cdot n|_S \, |S| \,,$$

i.e., $y_h \in \ker(\mathrm{div}|_{\mathcal{RT}^0_N(\mathcal{T}_h)})$. ∎

Fenchel duality theory for discrete integral functionals

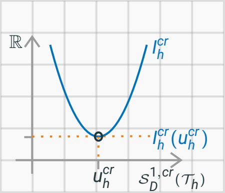
Discrete primal problem

- **Three non-conforming modifications:**
  1. Replace ϕ and ψ by *element-wise approximations* ϕ_h and ψ_h, *i.e.*,
     - $\phi_h\colon \Omega\times\mathbb{R}^d\to\mathbb{R}\cup\{+\infty\}$ is measurable s.t. $\phi_h(x,\cdot)\in\Gamma_0(\mathbb{R}^d)$ for a.e. $x\in\Omega$;
     - $\psi_h\colon \Omega\times\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ is measurable s.t. $\psi_h(x,\cdot)\in\Gamma_0(\mathbb{R}^d)$ for a.e. $x\in\Omega$;
     - $\phi_h(\cdot,t),\psi_h(\cdot,s)\in\mathbb{P}^0(\mathcal{T}_h)$ for all $t\in\mathbb{R}^d$ and $s\in\mathbb{R}$.
  2. *(Local) L^2-projection operator* $\Pi_h\colon L^1(\Omega)\to\mathbb{P}^0(\mathcal{T}_h)$;
  3. *Element-wise gradient operator* $\nabla_h\colon \mathcal{S}^{1,cr}_D(\mathcal{T}_h)\to(\mathbb{P}^0(\mathcal{T}_h))^d$.

- **Discrete primal problem:** Min. $I_h^{cr}\colon\mathcal{S}^{1,cr}_D(\mathcal{T}_h)\to\mathbb{R}\cup\{+\infty\}$, for every $v_h\in\mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ defined by

$$I_h^{cr}(v_h)\coloneqq\int_\Omega\phi_h(\cdot,\nabla_h v_h)\,\mathrm{d}x+\int_\Omega\psi_h(\cdot,\Pi_h v_h)\,\mathrm{d}x\,.$$

- **Assumption:** A minimizer $u_h^{cr}\in\mathcal{S}^{1,cr}_D(\mathcal{T}_h)$, a so-called *discrete primal solution*, exists.



Discrete (Fenchel) primal problem

- **Setup of a discrete (Fenchel) primal problem:**
  - Let $G_h\colon (\mathbb{P}^0(\mathcal{T}_h))^d \to \mathbb{R}\cup\{+\infty\}$, for every $y_h \in (\mathbb{P}^0(\mathcal{T}_h))^d$, be defined by

$$G_h(y_h) \coloneqq \int_\Omega \phi_h(\cdot, y_h)\,\mathrm{d}x\,.$$

→ $G_h \in \Gamma_0((\mathbb{P}^0(\mathcal{T}_h))^d)$;

  - Let $F_h\colon \mathcal{S}_D^{1,cr}(\mathcal{T}_h) \to \mathbb{R}\cup\{+\infty\}$, for every $v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, be defined by

$$F_h(v_h) \coloneqq \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x\,.$$

→ $F_h \in \Gamma_0(\mathcal{S}_D^{1,cr}(\mathcal{T}_h))$;

  - Let $\Lambda_h\colon \mathcal{S}_D^{1,cr}(\mathcal{T}_h) \to (\mathbb{P}^0(\mathcal{T}_h))^d$, for every $v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, be defined by

$$\Lambda_h v_h \coloneqq \nabla_h v_h\,.$$

→ $\Lambda_h \in L(\mathcal{S}_D^{1,cr}(\mathcal{T}_h); (\mathbb{P}^0(\mathcal{T}_h))^d)$.

→ **Discrete (Fenchel) primal problem:** For every $v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, we have that

$$I_h^{cr}(v_h) = G_h(\Lambda_h v_h) + F_h(v_h)\,.$$

- **Discrete (Fenchel) dual problem:** Maximize $D_h^0 : (\mathbb{P}^0(\mathcal{T}_h))^d \to \mathbb{R} \cup \{-\infty\}$, for every $y_h \in (\mathbb{P}^0(\mathcal{T}_h))^d$, defined by

$$D_h^0(y_h) \coloneqq -F_h^*(-\Lambda_h^* y_h) - G_h^*(y_h)\,,$$

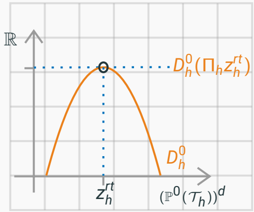
where

  - For every $y_h \in (\mathbb{P}^0(\mathcal{T}_h))^d$, it holds that

$$\begin{aligned} G_h^*(y_h) &= \sup_{\widehat{y}_h \in (\mathbb{P}^0(\mathcal{T}_h))^d} \left\{ \int_\Omega y_h \cdot \widehat{y}_h \,\mathrm{d}x - \int_\Omega \phi_h(\cdot, \widehat{y}_h)\,\mathrm{d}x \right\} \\ &= \sum_{T \in \mathcal{T}_h} \int_T \sup_{t \in \mathbb{R}^d} \{ y_h(x_T) \cdot t - \phi_h(x_T, t) \}\,\mathrm{d}x \\ &= \int_\Omega \phi_h^*(\cdot, y_h)\,\mathrm{d}x\,; \end{aligned}$$

  - For every $y_h \in (\mathbb{P}^0(\mathcal{T}_h))^d$, we have that

$$F_h^*(-\Lambda_h^* y_h) = \sup_{v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)} \left\{ \int_\Omega -y_h \cdot \nabla_h v_h \,\mathrm{d}x - \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x \right\}.$$



Integral representation of dual problem

- **Integral representation of $F_h^* \circ (-\Lambda_h^*)$:** For every $y_h \in \mathcal{R}T_N^0(\mathcal{T}_h)$, we have that

$$F_h^*(-\Lambda_h^* \Pi_h y_h) = \sup_{v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)} \left\{ \int_\Omega -\Pi_h y_h \cdot \nabla_h v_h \,\mathrm{d}x - \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x \right\}$$

$$= \sup_{v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)} \left\{ \int_\Omega \operatorname{div} y_h \, \Pi_h v_h \,\mathrm{d}x - \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x \right\}$$

when? $$\left(= \int_\Omega \psi_h^*(\cdot, \operatorname{div} y_h)\,\mathrm{d}x \right).$$

Fenchel conjugate of discrete integral functionals defined on $\Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h))$

Lemma: (Fenchel conjugate of integral functionals defined on $\Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h))$)

Let one the following two assumptions be satisfied:

- $\Gamma_D \neq \partial\Omega$;
- $\psi_h(x,\cdot) \in C^1(\mathbb{R})$ for a.e. $x \in \Omega$.

Then, for every $\hat{v}_h \in \mathbb{P}^0(\mathcal{T}_h)$, it holds that

$$\int_\Omega \psi_h^*(\cdot,\hat{v}_h)\,\mathrm{d}x = \sup_{v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)} \bigg\{ \int_\Omega \hat{v}_h\, \Pi_h v_h\,\mathrm{d}x - \int_\Omega \psi_h(\cdot,\Pi_h v_h)\,\mathrm{d}x \bigg\}\,.$$

Proof (of the case $\Gamma_D \neq \partial\Omega$).

- Appealing to [1, Bartels & Wang, '21], it holds that

$$\mathbb{P}^0(\mathcal{T}_h) = \Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h))\,.$$

→ For every $\hat{v}_h \in \mathbb{P}^0(\mathcal{T}_h)$, we find that

$$\begin{aligned}\sup_{v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)} \bigg\{ \int_\Omega \hat{v}_h\, \Pi_h v_h\,\mathrm{d}x - \int_\Omega \psi_h(\cdot,\Pi_h v_h)\,\mathrm{d}x \bigg\} &= \sup_{v_h \in \mathbb{P}^0(\mathcal{T}_h)} \bigg\{ \int_\Omega \hat{v}_h\, v_h\,\mathrm{d}x - \int_\Omega \psi_h(\cdot,v_h)\,\mathrm{d}x \bigg\} \\ &= \int_\Omega \psi_h^*(\cdot,\hat{v}_h)\,\mathrm{d}x\,. \qquad \blacksquare\end{aligned}$$

Integral representation of dual problem

- **Integral representation of $F_h^* \circ (-\Lambda_h^*)$:** For every $y_h \in \mathcal{R}T_N^0(\mathcal{T}_h)$, we have that

$$F_h^*(-\Lambda_h^*\Pi_h y_h) = \sup_{v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)} \bigg\{ \int_\Omega -\Pi_h y_h \cdot \nabla_h v_h \,\mathrm{d}x - \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x \bigg\}$$

$$= \sup_{v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)} \bigg\{ \int_\Omega \operatorname{div} y_h \, \Pi_h v_h \,\mathrm{d}x - \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x \bigg\}$$

when? $$\bigg(= \int_\Omega \psi_h^*(\cdot, \operatorname{div} y_h)\,\mathrm{d}x \bigg).$$

- **Assumption:** For every $y_h \in \mathcal{R}T_N^0(\mathcal{T}_h)$, we have that

$$F_h^*(-\Lambda_h^*\Pi_h y_h) = \int_\Omega \psi_h^*(\cdot, \operatorname{div} y_h)\,\mathrm{d}x\,.$$

→ **Integral representation:** For every $y_h \in \mathcal{R}T_N^0(\mathcal{T}_h)$, we have that

$$D_h^0(\Pi_h y_h) = -\int_\Omega \phi_h^*(\cdot, \Pi_h y_h)\,\mathrm{d}x - \int_\Omega \psi_h^*(\cdot, \operatorname{div} y_h)\,\mathrm{d}x\,.$$

- **Assumption:** A maxmizer $\Pi_h z_h^{rt} \in \Pi_h(\mathcal{R}T_N^0(\mathcal{T}_h))$, a so-called *discrete dual solution*, exists.

Integral representation of dual problem

- **Integral representation of $F_h^* \circ (-\Lambda_h^*)$:** For every $y_h \in \mathcal{RT}_N^0(\mathcal{T}_h)$, we have that

$$F_h^*(-\Lambda_h^*\Pi_h y_h) = \sup_{v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)} \left\{ \int_\Omega -\Pi_h y_h \cdot \nabla_h v_h \,\mathrm{d}x - \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x \right\}$$

$$= \sup_{v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)} \left\{ \int_\Omega \operatorname{div} y_h \, \Pi_h v_h \,\mathrm{d}x - \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x \right\}$$

when? $$\left(= \int_\Omega \psi_h^*(\cdot, \operatorname{div} y_h)\,\mathrm{d}x \right).$$

- **Assumption:** For every $y_h \in \mathcal{RT}_N^0(\mathcal{T}_h)$, we have that

$$F_h^*(-\Lambda_h^*\Pi_h y_h) = \int_\Omega \psi_h^*(\cdot, \operatorname{div} y_h)\,\mathrm{d}x\,.$$

→ **Integral representation:** For every $y_h \in \mathcal{RT}_N^0(\mathcal{T}_h)$, we have that

$$D_h^{rt}(y_h) \coloneqq -\int_\Omega \phi_h^*(\cdot, \Pi_h y_h)\,\mathrm{d}x - \int_\Omega \psi_h^*(\cdot, \operatorname{div} y_h)\,\mathrm{d}x\,.$$

- **Assumption:** A maxmizer $z_h^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_h)$, a so-called *discrete dual solution*, exists.

Lemma: (discrete weak duality relation)

There holds a *discrete weak duality relation*, *i.e.*, it holds that

$$\inf_{v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)} I^{cr}_h(v_h) \geq \sup_{y_h \in \mathcal{RT}^0_N(\mathcal{T}_h)} D^{rt}_h(y_h)\,.$$

- **Proof (for integral functionals).** Let $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ and $y_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ be arbitrary.
  - By the *Fenchel–Young inequality*, it holds that

$$\left.\begin{aligned} \Pi_h y_h \cdot \nabla_h v_h &\leq \phi_h^*(\cdot, \Pi_h y_h) + \phi_h(\cdot, \nabla_h v_h) &&\text{a.e. in } \Omega\,,\\ \operatorname{div} y_h\, \Pi_h v_h &\leq \psi_h^*(\cdot, \operatorname{div} y_h) + \psi_h(\cdot, \Pi_h v_h) &&\text{a.e. in } \Omega\,. \end{aligned}\right\} \quad (*)$$

  - Summation of $(*)$ and the *discrete integration-by-parts formula* yield that

$$\begin{aligned} 0 &= \int_\Omega \Pi_h y_h \cdot \nabla_h v_h \,\mathrm{d}x + \int_\Omega \operatorname{div} y_h\, \Pi_h v_h \,\mathrm{d}x \\ &\leq \int_\Omega \phi_h(\cdot, \nabla_h v_h)\,\mathrm{d}x + \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x \\ &\quad + \int_\Omega \phi_h^*(\cdot, \Pi_h y_h)\,\mathrm{d}x + \int_\Omega \psi_h^*(\cdot, \operatorname{div} y_h)\,\mathrm{d}x \\ &= I^{cr}_h(v_h) - D^{rt}_h(y_h)\,. \end{aligned}$$

■

Lemma: (discrete strong duality ⇔ discrete convex optimality relations)

For $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ and $z_h^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_h)$, the following statements are equivalent:

(i) A *discrete strong duality relation* applies, *i.e.*, it holds that

$$I_h^{cr}(u_h^{cr}) = D_h^{rt}(z_h^{rt})\,;$$

(ii) *Discrete convex optimality relations* apply, *i.e.*, it holds that

$$\begin{aligned}
\phi_h^*(\cdot,\Pi_h z_h^{rt}) - \Pi_h z_h^{rt}\cdot\nabla_h u_h^{cr} + \phi_h(\cdot,\nabla_h u_h^{cr}) &= 0 \qquad \text{a.e. in } \Omega\,,\\
\psi_h^*(\cdot,\operatorname{div} z_h^{rt}) - \operatorname{div} z_h^{rt}\,\Pi_h u_h^{cr} + \psi_h(\cdot,\Pi_h u_h^{cr}) &= 0 \qquad \text{a.e. in } \Omega\,.
\end{aligned}$$

- **Proof.** By the *Fenchel–Young inequality* and *discrete integration-by-parts formula*, it holds that

$$\begin{aligned}
\text{(i)} \quad \Leftrightarrow \quad 0 &= I_h^{cr}(u_h^{cr}) - D_h^{rt}(z_h^{rt})\\
\Leftrightarrow \quad 0 &= \int_\Omega \underbrace{\big\{\phi_h^*(\cdot,\Pi_h z_h^{rt}) - \Pi_h z_h^{rt}\cdot\nabla_h u_h^{cr} + \phi_h(\cdot,\nabla_h u_h^{cr})\big\}}_{\geq 0}\,\mathrm{d}x\\
&\quad + \int_\Omega \underbrace{\big\{\psi_h^*(\cdot,\operatorname{div} z_h^{rt}) - \operatorname{div} z_h^{rt}\,\Pi_h u_h^{cr} + \psi_h(\cdot,\Pi_h u_h^{cr})\big\}}_{\geq 0}\,\mathrm{d}x\\
\Leftrightarrow \quad &\text{(ii)}\,.
\end{aligned}$$

∎

Discrete reconstruction formula

Lemma: (discrete reconstruction formula)

Let $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ be a discrete primal solution and let the following be satisfied:

- $\phi_h(x,\cdot) \in C^1(\mathbb{R}^d)$ for a.e. $x \in \Omega$;
- $\psi_h(x,\cdot) \in C^1(\mathbb{R})$ for a.e. $x \in \Omega$;

Then, a dual solution $z_h^{rt} \in \mathcal{R}T_N^0(\mathcal{T}_h)$ is given via

$$z_h^{rt} = D_t\phi_h(\cdot, \nabla_h u_h^{cr}) + \frac{D_t\psi_h(\cdot, \Pi_h u_h^{cr})}{d}(\mathrm{id}_{\mathbb{R}^d} - \Pi_h \mathrm{id}_{\mathbb{R}^d}) \quad \text{a.e. in } \Omega\,.$$

In particular, a discrete strong duality applies, *i.e.*, $I_h^{cr}(u_h^{cr}) = D_h^{rt}(z_h^{rt})$.

Proof.

- There exists $\widehat{z}_h \in \mathcal{R}T_N^0(\mathcal{T}_h)$ s.t.

$$\operatorname{div} \widehat{z}_h = D_t\psi_h(\cdot, \Pi_h u_h^{cr}) \quad \text{a.e. in } \Omega\,.$$

- For every $v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, it holds that

$$\int_\Omega (z_h^{rt} - \widehat{z}_h)\cdot \nabla_h v_h \,\mathrm{d}x = \int_\Omega \underbrace{\Pi_h z_h^{rt}}_{= D_t\phi_h(\cdot, \nabla_h u_h^{cr})} \cdot \nabla_h v_h \,\mathrm{d}x + \int_\Omega \underbrace{\operatorname{div} \widehat{z}_h}_{= D_t\psi_h(\cdot, \Pi_h u_h^{cr})} \Pi_h v_h \,\mathrm{d}x = 0\,,$$

i.e., $z_h^{rt} - \widehat{z}_h \in (\nabla_h(\mathcal{S}_D^{1,cr}(\mathcal{T}_h)))^\perp$.

Discrete reconstruction formula

- Due to the *discrete orthogonality relation*, it follows that

$$z_h^{rt} - \widehat{z}_h \in (\nabla_h(\mathcal{S}_D^{1,cr}(\mathcal{T}_h)))^{\perp} = \ker(\operatorname{div}|_{\mathcal{RT}_N^0(\mathcal{T}_h)}) ,$$

i.e., we have that $z_h^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_h)$ with

$$\left.\begin{aligned} \operatorname{div} z_h^{rt} &= \operatorname{div} \widehat{z}_h \\ &= D_t\psi_h(\cdot, \Pi_h u_h^{cr}) \end{aligned}\right\} \quad \text{a.e. in } \Omega .$$

- In summary, we have that $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ and $z_h^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_h)$ satisfy

$$\begin{cases} \Pi_h z_h^{rt} = D_t\phi_h(\cdot, \nabla_h^{cr} u_h^{cr}) & \text{a.e. in } \Omega , \\ \operatorname{div} z_h^{rt} = D_t\psi_h(\cdot, \Pi_h u_h^{cr}) & \text{a.e. in } \Omega . \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \phi_h^*(\cdot, \Pi_h z_h^{rt}) - \Pi_h z_h^{rt} \cdot \nabla_h u_h^{cr} + \phi_h(\cdot, \nabla_h u_h^{cr}) = 0 & \text{a.e. in } \Omega , \\ \psi_h^*(\cdot, \operatorname{div} z_h^{rt}) - \operatorname{div} z_h^{rt}\, \Pi_h u_h^{cr} + \psi_h(\cdot, \Pi_h u_h^{cr}) = 0 & \text{a.e. in } \Omega . \end{cases}$$

$$\Leftrightarrow \quad I_h^{cr}(u_h^{cr}) = D_h^{rt}(z_h^{rt}) .$$

→ By the *discrete weak duality relation*, we conclude that

$$\begin{aligned} D_h^{rt}(z_h^{rt}) &= I_h^{cr}(u_h^{cr}) \\ &\geq \sup_{y_h \in \mathcal{RT}_N^0(\mathcal{T}_h)} D_h^{rt}(y_h) . \end{aligned}$$

∎

Discrete reconstruction formula

Lemma: (discrete reconstruction formula)

Let $z_h^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_h)$ be a discrete dual solution and let the following be satisfied:

- $\phi_h^*(x,\cdot) \in C^1(\mathbb{R}^d)$ for a.e. $x \in \Omega$;
- $\psi_h^*(x,\cdot) \in C^1(\mathbb{R})$ for a.e. $x \in \Omega$;

Then, a discrete primal solution $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ is given via

$$u_h^{cr} = D_t\psi_h^*(\cdot, \operatorname{div} z_h^{rt}) + D_t\phi_h^*(\cdot, \Pi_h z_h^{rt}) \cdot (\mathrm{id}_{\mathbb{R}^d} - \Pi_h \mathrm{id}_{\mathbb{R}^d}) \quad \text{a.e. in } \Omega\,.$$

In particular, a discrete strong duality applies, *i.e.*, $I_h^{cr}(u_h^{cr}) = D_h^{rt}(z_h^{rt})$.

Proof.

- There exists $\widehat{u}_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ s.t.

$$\nabla_h \widehat{u}_h = D_t\phi_h^*(\cdot, \Pi_h z_h^{rt}) \quad \text{a.e. in } \Omega\,.$$

- For every $y_h \in \mathcal{RT}_N^0(\mathcal{T}_h)$, it holds that

$$\int_\Omega (u_h^{cr} - \widehat{u}_h) \cdot \operatorname{div} y_h \,\mathrm{d}x = \int_\Omega \underbrace{\Pi_h u_h^{cr}}_{= D_t\psi_h^*(\cdot, \operatorname{div} z_h^{rt})} \operatorname{div} y_h \,\mathrm{d}x + \int_\Omega \underbrace{\nabla_h \widehat{u}_h}_{= D_t\phi_h^*(\cdot, \Pi_h z_h^{rt})} \cdot \Pi_h y_h \,\mathrm{d}x = 0\,,$$

i.e., $u_h^{cr} - \widehat{u}_h \in (\operatorname{div}(\mathcal{RT}_N^0(\mathcal{T}_h)))^\perp$.

Discrete reconstruction formula

- Due to the *surjectivity of divergence operator*, it follows that

$$u_h^{cr} - \widehat{u}_h \in (\operatorname{div}(\mathcal{RT}_N^0(\mathcal{T}_h)))^\perp = \begin{cases} \{0\} & \text{if } \Gamma_N \neq \partial\Omega\,, \\ \mathbb{R} & \text{if } \Gamma_N = \partial\Omega\,, \end{cases}$$

i.e., we have that $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ with

$$\left.\begin{aligned} \nabla_h u_h^{cr} &= \nabla_h \widehat{u}_h \\ &= D_t\psi_h^*(\cdot, \Pi_h z_h^{rt}) \end{aligned}\right\} \quad \text{a.e. in } \Omega\,.$$

- In summary, we have that $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ and $z_h^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_h)$ satisfy

$$\begin{cases} \Pi_h u_h^{cr} = D_t\phi_h^*(\cdot, \operatorname{div} z_h^{rt}) & \text{a.e. in } \Omega\,, \\ \nabla_h u_h^{cr} = D_t\psi_h^*(\cdot, \Pi_h z_h^{rt}) & \text{a.e. in } \Omega\,. \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \phi_h^*(\cdot, \Pi_h z_h^{rt}) - \Pi_h z_h^{rt} \cdot \nabla_h u_h^{cr} + \phi_h(\cdot, \nabla_h u_h^{cr}) = 0 & \text{a.e. in } \Omega\,, \\ \psi_h^*(\cdot, \operatorname{div} z_h^{rt}) - \operatorname{div} z_h^{rt}\, \Pi_h u_h^{cr} + \psi_h(\cdot, \Pi_h u_h^{cr}) = 0 & \text{a.e. in } \Omega\,. \end{cases}$$

$$\Leftrightarrow \quad I_h^{cr}(u_h^{cr}) = D_h^{rt}(z_h^{rt})\,.$$

→ By the *discrete weak duality relation*, we conclude that

$$\begin{aligned} I_h^{cr}(u_h^{cr}) &= D_h^{rt}(z_h^{rt}) \\ &\leq \inf_{v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)} I_h^{cr}(v_h)\,. \end{aligned}$$

■

Examples

- **Discrete primal problem:** Minimize $I_h^{cr} : \mathcal{S}_D^{1,cr}(\mathcal{T}_h) \to \mathbb{R}$, for every $v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ defined by

$$I_h^{cr}(v_h) \coloneqq \frac{1}{2}\int_\Omega |\nabla_h v_h|^2 \,\mathrm{d}x - \int_\Omega f_h \,\Pi_h v_h \,\mathrm{d}x\,, \qquad (f_h \in \mathbb{P}^0(\mathcal{T}_h))$$

  i.e., $\phi_h \coloneqq \frac{1}{2}|\cdot|^2 \in C^1(\mathbb{R}^d)$ and $\psi_h(x,\cdot) \coloneqq (t \mapsto -f_h(x)t) \in C^1(\mathbb{R})$ for a.e. $x \in \Omega$.

- **Application:** (Deflection of membrane)

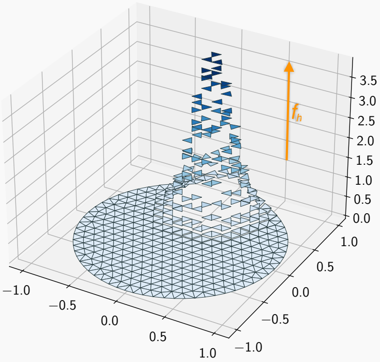


Figure: $f_h \in \mathbb{P}^0(\mathcal{T}_h)$.

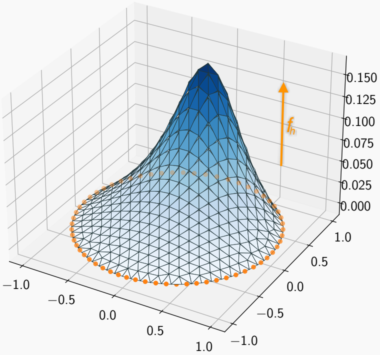


Figure: $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, where $\Gamma_D = \partial\Omega$.

Examples: Poisson problem

- **Discrete dual problem:** Maximize $D_h^{rt}:\mathcal{RT}^0_N(\mathcal{T}_h)\to\mathbb{R}\cup\{-\infty\}$, for every $y_h\in\mathcal{RT}^0_N(\mathcal{T}_h)$ defined by

$$D_h^{rt}(y_h) \coloneqq -\frac{1}{2}\int_\Omega |\Pi_h y_h|^2\,\mathrm{d}x - I^\Omega_{\{-f_h\}}(\operatorname{div} y_h)\,,$$

where $I^\Omega_{\{-f_h\}}:\mathbb{P}^0(\mathcal{T}_h)\to\mathbb{R}\cup\{+\infty\}$, for every $\widehat{v}_h\in\mathbb{P}^0(\mathcal{T}_h)$, is defined by

$$I^\Omega_{\{-f_h\}}(\widehat{v}_h) \coloneqq \begin{cases} 0 & \text{if } \widehat{v}_h = -f_h \text{ a.e. in } \Omega\,,\\ +\infty & \text{else}\,.\end{cases}$$

- **Discr. dual solution, discr. strong duality, discr. convex optimality relations:** There exists a discrete dual solution $z_h^{rt}\in\mathcal{RT}^0_N(\mathcal{T}_h)$ s.t.

$$I_h^{cr}(u_h^{cr}) = D_h^{rt}(z_h^{rt}) \quad\Leftrightarrow\quad \begin{cases} \frac{1}{2}|\Pi_h z_h^{rt}|^2 - \Pi_h z_h^{rt}\cdot\nabla_h u_h^{cr} + \frac{1}{2}|\nabla_h u_h|^2 = 0 & \text{a.e. in } \Omega\,,\\ I^{(\cdot)}_{\{-f_h\}}(\operatorname{div} z_h^{rt}) - \operatorname{div} z_h^{rt}\,\Pi_h u_h^{cr} - f_h\, u_h^{cr} = 0 & \text{a.e. in } \Omega\,.\end{cases}$$

$$\Leftrightarrow\quad \begin{cases} \Pi_h z_h^{rt} = \nabla_h u_h^{cr} & \text{a.e. in } \Omega\,,\\ \operatorname{div} z_h^{rt} = -f_h & \text{a.e. in } \Omega\,.\end{cases}$$

Examples: p-Dirichlet problem

- **Discrete primal problem:** Minimize $I_h^{cr} : \mathcal{S}_D^{1,cr}(\mathcal{T}_h) \to \mathbb{R}$, for every $v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, defined by

$$I_h^{cr}(v_h) \coloneqq \frac{1}{p}\int_\Omega |\nabla_h v_h|^p \,\mathrm{d}x - \int_\Omega f_h\, \Pi_h v_h \,\mathrm{d}x\,, \qquad (f_h \in \mathbb{P}^0(\mathcal{T}_h))$$

i.e., $\phi_h := \frac{1}{p}|\cdot|^p \in C^1(\mathbb{R}^d)$ and $\psi_h(x,\cdot) := (t \mapsto -f_h(x)t) \in C^1(\mathbb{R})$ for a.e. $x \in \Omega$.

- **Application:**

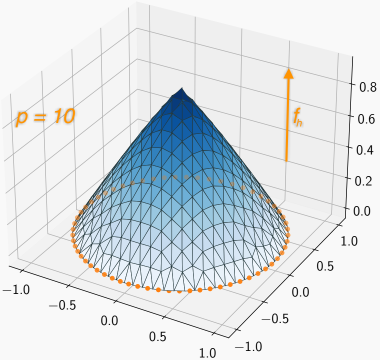


Figure: $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, where $\Gamma_D = \partial B_1^2(0)$ and $f \equiv 1$.

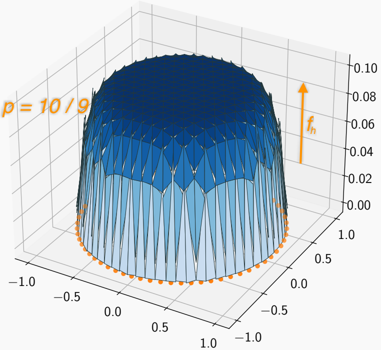


Figure: $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, where $\Gamma_D = \partial B_1^2(0)$ and $f \equiv 1$.

- **Discrete dual problem:** Maximize $D_h^{rt}\colon \mathcal{RT}_N^0(\mathcal{T}_h)\to\mathbb{R}\cup\{-\infty\}$, for every $y_h\in\mathcal{RT}_N^0(\mathcal{T}_h)$, defined by

$$D_h^{rt}(y_h) \coloneqq -\frac{1}{p'}\int_\Omega |\Pi_h y_h|^{p'}\,\mathrm{d}x - I_{\{-f_h\}}^{\Omega}(\operatorname{div} y)\,,$$

where $I_{\{-f_h\}}^{\Omega}\colon \mathbb{P}^0(\mathcal{T}_h)\to\mathbb{R}\cup\{+\infty\}$, for every $\widehat{v}_h\in\mathbb{P}^0(\mathcal{T}_h)$, is defined by

$$I_{\{-f_h\}}^{\Omega}(\widehat{v}) \coloneqq \begin{cases} 0 & \text{if } \widehat{v}_h = -f_h \text{ a.e. in } \Omega\,,\\ +\infty & \text{else}\,.\end{cases}$$

- **Discr. dual solution, discr. strong duality, discr. convex optimality relations:** There exists a discrete dual solution $z_h^{rt}\in\mathcal{RT}_N^0(\mathcal{T}_h)$ s.t.

$$I_h^{cr}(u_h^{cr}) = D_h^{rt}(z_h^{rt}) \quad\Leftrightarrow\quad \begin{cases} \frac{1}{p'}|\Pi_h z_h^{rt}|^{p'} - \Pi_h z_h^{rt}\cdot\nabla_h u_h^{cr} + \frac{1}{p}|\nabla_h u_h^{cr}|^p = 0 & \text{a.e. in } \Omega\,,\\ I_{\{-f_h\}}^{(\cdot)}(\operatorname{div} z_h^{rt}) - \operatorname{div} z_h^{rt}\,\Pi_h u_h^{cr} - f_h\,\Pi_h u_h^{cr} = 0 & \text{a.e. in } \Omega\,.\end{cases}$$

$$\Leftrightarrow\quad \begin{cases} \Pi_h z_h^{rt} = |\nabla_h u_h^{cr}|^{p-2}\nabla_h u_h^{cr} & \text{a.e. in } \Omega\,,\\ \operatorname{div} z_h^{rt} = -f_h & \text{a.e. in } \Omega\,.\end{cases}$$

Examples: Obstacle problem

- **Discrete primal problem**: Minimize $I_h^{cr} : \mathcal{S}_D^{1,cr}(\mathcal{T}_h) \to \mathbb{R} \cup \{+\infty\}$, for every $v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ defined by

$$I_h^{cr}(v_h) \coloneqq \frac{1}{2}\int_\Omega |\nabla_h v_h|^2 \,\mathrm{d}x - \int_\Omega f_h\, \Pi_h v_h \,\mathrm{d}x + I_+^\Omega(\Pi_h v_h)\,, \qquad (f_h \in \mathbb{P}^0(\mathcal{T}_h))$$

where $I_+^\Omega : \mathbb{P}^0(\mathcal{T}_h) \to \mathbb{R} \cup \{+\infty\}$, for every $\widehat{v}_h \in \mathbb{P}^0(\mathcal{T}_h)$, is defined by

$$I_+^\Omega(\widehat{v}_h) \coloneqq \begin{cases} 0 & \text{if } \widehat{v}_h \geq 0 \text{ a.e. in } \Omega\,, \\ +\infty & \text{else}\,. \end{cases}$$

- **Application**: (Deflection of membrane with obstacle)

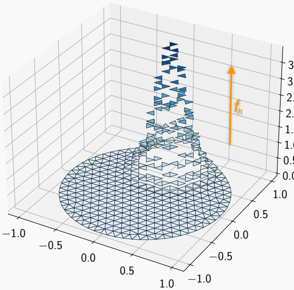


Figure: $f_h \in \mathbb{P}^0(\mathcal{T}_h)$.

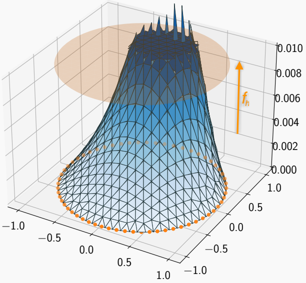


Figure: $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$.

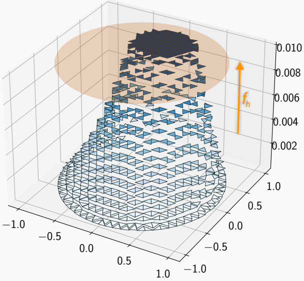


Figure: $\Pi_h u_h^{cr} \in \mathbb{P}^0(\mathcal{T}_h)$.

Examples: Obstacle problem

- **Discrete dual problem:** Maximize $D_h^{rt} : \mathcal{RT}_N^0(\mathcal{T}_h) \to \mathbb{R} \cup \{-\infty\}$, for every $y_h \in \mathcal{RT}_N^0(\mathcal{T}_h)$, defined by

$$D_h^{rt}(y_h) \coloneqq -\frac{1}{2}\int_\Omega |\Pi_h y_h|^2 \,\mathrm{d}x - I_-^\Omega(f_h + \operatorname{div} y_h)\,,$$

where $I_-^\Omega : \mathbb{P}^0(\mathcal{T}_h) \to \mathbb{R} \cup \{+\infty\}$, for every $\widehat{v}_h \in \mathbb{P}^0(\mathcal{T}_h)$, is defined by

$$I_-^\Omega(\widehat{v}_h) \coloneqq \begin{cases} 0 & \text{if } \widehat{v}_h \leq 0 \text{ a.e. in } \Omega\,, \\ +\infty & \text{else}\,. \end{cases}$$

- **Discr. dual solution, discr. strong duality, discr. convex optimality relations:** There exists a discrete dual solution $z_h^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_h)$ s.t.

$$I_h^{cr}(u_h^{cr}) = D_h^{rt}(z_h^{rt}) \quad \Leftrightarrow \quad \begin{cases} \frac{1}{2}|\Pi_h z_h^{rt}|^2 - \Pi_h z_h^{rt} \cdot \nabla_h u_h^{cr} + \frac{1}{2}|\nabla_h u_h^{cr}|^2 = 0 & \text{a.e. in } \Omega\,, \\ \left.\begin{array}{l} I_-^{(\cdot)}(f_h + \operatorname{div} z_h^{rt}) \\ \quad - (f_h + \operatorname{div} z_h^{rt})\,\Pi_h u_h^{cr} + I_-^{(\cdot)}(\Pi_h u_h^{cr}) \end{array}\right\} = 0 & \text{a.e. in } \Omega\,. \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \Pi_h z_h^{rt} = \nabla_h u_h^{cr} & \text{a.e. in } \Omega\,, \\ f_h + \operatorname{div} z_h^{rt} \leq 0 & \text{a.e. in } \Omega\,, \\ \Pi_h u_h^{cr} \geq 0 & \text{a.e. in } \Omega\,, \\ (f_h + \operatorname{div} z_h^{rt})\,\Pi_h u_h^{cr} = 0 & \text{a.e. in } \Omega\,. \end{cases}$$

Thank You for today!

S. Bartels and Z. Wang.
Orthogonality relations of Crouzeix-Raviart and Raviart-Thomas finite element spaces.
Numer. Math., 148(1):127–139, 2021.

M. Crouzeix and P.-A. Raviart.
Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I.
Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge, 7(R-3):33–75, 1973.

A. Ern and J. L. Guermond.
Finite Elements I: Approximation and Interpolation.
Number 1 in Texts in Applied Mathematics. Springer International Publishing, 2021.

P.-A. Raviart and J. M. Thomas.
A mixed finite element method for 2nd order elliptic problems.
In Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975), pages 292–315. Lecture Notes in Math., Vol. 606, 1977.