ABSCHLUSSBERICHT

1 Allgemeine Angaben

DFG-Geschäftszeichen: KA 6129/1-1:1

Projektnummer: 525389262

Titel des Projekts:

- Englischer Titel: *Numerical methods for physical problems with variable growth conditions*
- Deutscher Titel: *Numerische Methoden für physikalische Probleme mit variablen Wachstumsbedingungen*

Name des/r Antragstellenden: Dr. Alex Kaltenbach

Dienstanschrift/en: Institut für Mathematik, Technische Universität Berlin, Straße des 17. Junis 135, 10623 Berlin.

Name(n) der aufnehmenden Person und Institution: Prof. Dr. Luigi C. Berselli, Universität Pisa, Italien.

Berichtszeitraum (gesamte Förderdauer): 01.07.2023 bis 31.03.2024 (9 Monate)

2 Summary

2.1 English summary

This research project aims at theoretical and experimental investigations on numerical methods for the physical models for *smart fluids*, *e.g.*, electro-rheological fluids (*cf.* [1, 2]), micro-polar electro-rheological fluids (*cf.* [3, 4]), magneto-rheological fluids (*cf.* [5]), chemically reacting fluids (*cf.* [6, 7]), and thermo-rheological fluids (*cf.* [8, 9]). Smart fluids have the potential for an application in numerous areas, *e.g.*, in electronic, automobile, heavy machinery, military, and biomedical industry (*cf.* [10], for an overview). All models mentioned above have a common characteristic: all models consist of coupled systems of elliptic or parabolic partial differential equations containing the generalized Navier–Stokes equations with a time- and space-dependent power-law index, the so-called *unsteady* $p(t,x)$*-Navier–Stokes equations*. The latter seek in a bounded domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, and over a time interval $I := (0,T)$, $0 < T < \infty$, abbreviating $Q_T := I \times \Omega$ and $\Gamma_T := I \times \partial\Omega$, for a *velocity vector field* $\mathbf{v}\colon \overline{Q_T} \to \mathbb{R}^d$ and a *kinematic pressure* $q\colon Q_T \to \mathbb{R}$ such that

$$\begin{aligned} \partial_t \mathbf{v} - \operatorname{div}_x \mathbf{S}(\cdot,\cdot,\mathbf{D}_x\mathbf{v}) + \operatorname{div}_x(\mathbf{v}\otimes\mathbf{v}) + \nabla_x q &= \mathbf{f} &&\text{in } Q_T\,,\\ \operatorname{div}_x \mathbf{v} &= 0 &&\text{in } Q_T\,,\\ \mathbf{v} &= \mathbf{0} &&\text{on } \Gamma_T\,,\\ \mathbf{v}(0) &= \mathbf{v}_0 &&\text{in } \Omega\,, \end{aligned} \tag{1}$$

where $\mathbf{f}\colon Q_T \to \mathbb{R}^d$ describes external forces and $\mathbf{v}_0\colon \Omega \to \mathbb{R}^d$ is the initial velocity vector field at time $t = 0$. In the system (1), the *extra stress tensor* $\mathbf{S}\colon Q_T \times \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ depends on the *strain rate tensor* $\mathbf{D}_x\mathbf{v} := \frac{1}{2}(\nabla_x\mathbf{v} + \nabla_x\mathbf{v}^\top)\colon Q_T \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and assumes for a.e. $(t,x)^\top \in Q_T$ and all $\mathbf{A} \in \mathbb{R}^{d\times d}$, *e.g.*, the form

$$\mathbf{S}(t,x,\mathbf{A}) := (\delta + |\mathbf{A}^{\mathrm{sym}}|)^{p(t,x)-2}\mathbf{A}^{\mathrm{sym}}\,,$$

where $\delta \geq 0$ and the *power-law index* $p\colon Q_T \to (1,+\infty)$ is at least (Lebesgue) measurable with

$$1 < p^- := \operatorname{ess\,inf}_{(t,x)^\top\in Q_T} p(t,x) \leq p^+ := \operatorname{ess\,sup}_{(t,x)^\top\in Q_T} p(t,x) < \infty\,.$$

The time- and space-dependency of the power-law index in (1) results in a number of mathematical hurdles; *e.g.*, the invalidity of a Poincaré inequality, a Korn inequality, and a negative norm inequality (*cf.* [11, 12]). Due to these hurdles, for two decades merely partial existence results could be established (*cf.* [13–16]). Consequently, only a few numerical investigations have been carried out (*cf.* [17–23]). In the PhD thesis [11], methods to overcome the hurdles mentioned above were developed and a thorough existence analysis for the unsteady $p(t,x)$-Navier–Stokes equations (1) was established. The methods from the PhD thesis [11], in the present research project, provide a firm foundation for a thorough numerical analysis: More precisely, the present research project comprises the following two mutually connected work areas:

Work Area A: Finite element (FE) approximation of the steady $p(x)$-Navier–Stokes equations.

Work Area B: Finite element (FE) approximation of the unsteady $p(t,x)$-Navier–Stokes equations.

2.2 German summary

Dieses Forschungsprojekt zielt auf theoretische und experimentelle Untersuchungen zu numerischen Methoden für die physikalischen Modelle von *intelligenten Flüssigkeiten* ab, z. B. elektro-rheologische Flüssigkeiten (*cf.* [1, 2]), mikro-polar elektro-rheologische Flüssigkeiten (*cf.* [3, 4]), magneto-rheologische Flüssigkeiten (*cf.* [5]), chemisch reagierende Flüssigkeiten (*cf.* [6, 7]) und thermo-rheologische Flüssigkeiten (*cf.* [8, 9]). Intelligente Flüssigkeiten haben das Potenzial für die Anwendung in zahlreichen Bereichen, z. B. in der Elektronik-, Automobil-, Schwermaschinen-, Militär- und biomedizinischen Industrie (*cf.* [10], für einen Überblick). Alle oben genannten Modelle haben ein gemeinsames Merkmal: alle Modelle bestehen aus gekoppelten Systemen elliptischer oder parabolischer partieller Differentialgleichungen, welche die verallgemeinerten Navier–Stokes-Gleichungen mit einem zeit- und ortsabhängigen Fließ-Index enthalten, die sogenannten *instationären $p(t,x)$-Navier–Stokes-Gleichungen*. Letztere suchen in einem beschränkten Gebiet $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, und in einem endlichen Zeitintervall $I \coloneqq (0,T)$, $0 < T < \infty$, nach einem *Geschwindigkeitsvektorfeld* $\mathbf{v}\colon \overline{Q_T} \to \mathbb{R}^d$ und einem *kinematischen Druck* $q\colon Q_T \to \mathbb{R}$, sodass

$$
\begin{aligned}
\partial_t \mathbf{v} - \operatorname{div}_x \mathbf{S}(\cdot,\cdot,\mathbf{D}_x\mathbf{v}) + \operatorname{div}_x(\mathbf{v}\otimes\mathbf{v}) + \nabla_x q &= \mathbf{f} && \text{in } Q_T\,,\\
\operatorname{div}_x \mathbf{v} &= 0 && \text{in } Q_T\,,\\
\mathbf{v} &= \mathbf{0} && \text{auf } \Gamma_T\,,\\
\mathbf{v}(0) &= \mathbf{v}_0 && \text{in } \Omega\,,
\end{aligned}
\tag{2}
$$

wobei $\mathbf{f}\colon Q_T \to \mathbb{R}^d$ äußere Kräfte beschreibt und $\mathbf{v}_0\colon \Omega \to \mathbb{R}^d$ die Anfangsgeschwindigkjeit zum Zeitpunkt $t = 0$ ist. In dem System (2) hängt der *Extra-Stresstensor* $\mathbf{S}\colon Q_T \times \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ vom *Scheerratentensor* $\mathbf{D}_x\mathbf{v} \coloneqq \frac{1}{2}(\nabla_x\mathbf{v} + \nabla_x\mathbf{v}^\top)\colon Q_T \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ ab und hat für f.a. $(t,x)^\top \in Q_T$ und alle $\mathbf{A} \in \mathbb{R}^{d\times d}$, z. B. die Form

$$
\mathbf{S}(t,x,\mathbf{A}) \coloneqq (\delta + |\mathbf{A}^{\mathrm{sym}}|)^{p(t,x)-2}\mathbf{A}^{\mathrm{sym}}\,,
$$

wobei $\delta \geq 0$ und der *Fließ-Index* $p\colon Q_T \to (1,+\infty)$ mindestens (Lebesgue-)messbar ist mit

$$
1 < p^- \coloneqq \operatorname{ess\,inf}_{(t,x)^\top \in Q_T} p(t,x) \leq p^+ \coloneqq \operatorname{ess\,sup}_{(t,x)^\top \in Q_T} p(t,x) < \infty\,.
$$

Die Zeit- und Ortsabhängigkeit des Fließ-Indexes in (2) führt zu einer Reihe von mathematischen Hürden; z. B. die Ungültigkeit einer Poincaré-Ungleichung, einer Korn-Ungleichung und einer Negativen-Norm-Ungleichung (*cf.* [11, 12]). Wegen dieser Hürden konnten für zwei Jahrzehnte nur Teilexistenzresultate bewiesen werden (*cf.* [13–16]). Folglich wurden nur wenige numerische Untersuchungen durchgeführt (*cf.* [17–23]). In der Dissertation [11] wurden Methoden zur Überwindung der oben genannten Hürden entwickelt und eine vollständige Existenzanalyse für die instationären $p(t,x)$-Navier–Stokes-Gleichungen (2) durchgeführt. Die Methoden der Dissertation [11] bilden in diesem Forschungsprojekt die Grundlage für numerische Analysen. Genauer umfasst dieses Forschungsprojekt die folgenden zwei miteinander verbundenen Arbeitsbereiche:

Arbeitsbereich A: Finite-Elemente-(FE)-Approximation der stationären $p(x)$-Navier–Stokes-Gleichungen.

Arbeitsbereich B: Finite-Elemente-(FE)-Approximation der instationären $p(t,x)$-Navier–Stokes-Gleichungen.

3 Scientific work and results report

3.1 Initial objectives and results of the project

Work Area A: Finite element (FE) approximation of the steady $p(x)$-Navier–Stokes equations.

This work area aims at theoretical and experimental investigations on a finite element (FE) approximation of the steady $p(x)$-Navier–Stokes equations and is divided into three work packages:

Work Package A1: Weak convergence of a finite element (FE) approximation.

Objective A1: The objective of this work package is to establish a general framework that allows to conclude the well-posedness (*i.e.*, existence of discrete solutions), stability (*i.e.*, *a priori* estimates), and weak convergence of a FE approximation of a steady problem, containing the steady $p(x)$-Navier–Stokes equations as a particular case, for a possibly discontinuous power-law index $p \in L^\infty(\Omega)$ with $p^- > 1$ (or $p^- > \frac{3d}{d+2}$ if the convective term is present) and general approximations $(p_h)_{h>0} \subseteq L^\infty(\Omega)$ satisfying $p_h \to p$ $(h \to 0)$.

Result A1: This work package was substituted by Work Package B3, which was added to Work Area B in the course of this research project. However, the findings of Work Package B2 can straightforwardly be reduced to a finite element approximation of the steady $p(x)$-Navier–Stokes equations, so that Work Package A1 indirectly is already partially completed and will be completed in the future.

Work Package A2: Error estimates for a finite element (FE) approximation.

Objective A2.1: An objective of this work package is to derive *a priori* error estimates for a finite element approximation of the steady $p(x)$-Navier–Stokes equations, using Témam's modification (*cf.* [24]) to discretize the convective term and a one-point quadrature rule to discretize the power-law index, imposing slightly more resitrictive assumptions on the regularity of the kinematics pressure than $q \in W^{1,p'(\cdot)}(\Omega)$, *e.g.*, required in [21].

Result A2.1: In the case of an only Hölder continuous power-law index $p \in C^{0,\alpha}(\overline{\Omega})$, $\alpha \in (0,1]$, one cannot hope for the *"full"* regularity $\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) \in W^{1,2}(\Omega;\mathbb{R}^{d\times d})$ and $q \in W^{1,p'(\cdot)}(\Omega)$, but instead it is reasonable to expect (*cf.* [20, Rem. 4.5]) the *"partial"* regularity $\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) \in N^{\beta,2}(\Omega;\mathbb{R}^{d\times d})$, $\beta \in (0,1]$, where $N^{\beta,2}(\Omega;\mathbb{R}^{d\times d})$ denotes the Nikolskiĭ space (*cf.* [25]). Concerning the kinematic pressure, it is proposed to consider, *e.g.*, the regularity $q \in H^{\gamma,p'(\cdot)}(\Omega)$, $\gamma \in (0,1]$, where $H^{\gamma,p'(\cdot)}(\Omega)$ denotes the fractional variable Hajłasz–Sobolev space (*cf.* [26, 27]). If $p \in C^{0,\alpha}(\overline{\Omega})$, $\alpha \in (0,1]$, with $p^- \geq \frac{3d}{d+2}$, $\delta > 0$, $\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) \in N^{\beta,2}(\Omega;\mathbb{R}^{d\times d})$, $\beta \in (0,1]$, and $q \in H^{\gamma,p'(\cdot)}(\Omega)$, $\gamma \in (\frac{\alpha}{\min\{2,(p^+)'\}}, 1]$, it was established (*cf.* [28, Cor. 6.2(6.3)]) that there exists a constant $s > 1$, which can chosen to be close to 1 if $h > 0$ is close to 0, and a constant $c_0 > 0$ such that if $\|\mathbf{D}\mathbf{v}\|_{r(\cdot),\Omega} \leq c_0$, where $r := \min\{2,p\} \in C^{0,\alpha}(\overline{\Omega})$, then

$$\begin{aligned}\|\mathbf{F}_h(\cdot,\mathbf{D}\mathbf{v}_h) - \mathbf{F}_h(\cdot,\mathbf{D}\mathbf{v})\|_{2,\Omega}^2 + \|q_h - q\|_{p'(\cdot),\Omega}^{(r^-)'} \lesssim\; & h^{2\alpha}\,\|1 + |\mathbf{D}\mathbf{v}|^{p(\cdot)s}\|_{1,\Omega} \\ & + h^{2\beta}\,[\mathbf{F}(\cdot,\mathbf{D}\mathbf{v})]_{N^{\beta,2}(\Omega)}^2 \\ & + h^{\min\{2,(p^+)'\}\gamma}\big(\|(\varphi_{|\mathbf{D}\mathbf{v}|})^*(|\nabla^\gamma q|)\|_{1,\Omega} + \||\nabla^\gamma q|\|_{p'(\cdot),\Omega}^{(r^-)'}\big)\,.\end{aligned} \tag{3}$$

The proof of the *a priori* error estimate (3) resorts to several fractional interpolation error estimates, including special fractional interpolation error estimates tailored to the convective term. More precisely, the latter was newly established in [28, Sec. 5]. Apart from that, different from initially expected, it was possible to derive the *a priori* error estimate (3) without imposing any further regularity of the pressure than the natural regularity one can usually expect as above.

Objective A2.2: An objective of this work package is to derive *a priori* error estimates for a finite element approximation of the steady $p(x)$-Navier–Stokes equations (similar to (3)) in which the derived error decay rates do not depend critically on the maximal and the minimal value of the power-law index, *i.e.*, on $p^-, p^+ \in (1,\infty)$.

Result A2.2: If $p \in C^{0,\alpha}(\overline{\Omega})$, $\alpha \in (0,1]$, with $p^- \geq 2$, $\delta > 0$, $\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) \in N^{\beta,2}(\Omega;\mathbb{R}^{d\times d})$, $\beta \in (0,1]$, and $q \in H^{\gamma,p'(\cdot)}(\Omega)$, $\gamma \in (\frac{\alpha}{\min\{2,(p^+)'\}},1]$, and $(\delta+|\mathbf{D}\mathbf{v}|)^{\frac{p(\cdot)-2}{2}}|\nabla^\gamma q| \in L^2(\Omega)$, it was established (*cf.* [28, Cor. 6.2(6.4)]) that there exists a constant $s>1$, which can chosen to be close to 1 if $h>0$ is close to 0, and a constant $c_0>0$ such that if $\|\mathbf{D}\mathbf{v}\|_{r(\cdot),\Omega} \leq c_0$, where $r \coloneqq \min\{2,p\} \in C^{0,\alpha}(\overline{\Omega})$, then

$$\begin{aligned} \|\mathbf{F}_h(\cdot,\mathbf{D}\mathbf{v}_h) - \mathbf{F}_h(\cdot,\mathbf{D}\mathbf{v})\|_{2,\Omega}^2 + \|q_h - q\|_{p'(\cdot),\Omega}^{(r^-)'} \lesssim h^{2\alpha}\, \|1+|\mathbf{D}\mathbf{v}|^{p(\cdot)s}\|_{1,\Omega} \\ + h^{2\beta}\,[\mathbf{F}(\cdot,\mathbf{D}\mathbf{v})]_{N^{\beta,2}(\Omega)}^2 \\ + h^{2\gamma}\left(\|(\delta+|\mathbf{D}\mathbf{v}|)^{\frac{2-p(\cdot)}{2}}|\nabla^\gamma q|\|_{2,\Omega}^2 + \||\nabla^\gamma q|\|_{p'(\cdot),\Omega}^2\right). \end{aligned} \tag{4}$$

In the case $p^- \geq 2$, $\mathbf{f} \in L^2(\Omega;\mathbb{R}^d)$, and $\alpha = \beta = \gamma = 1$, it has been established (*cf.* [28, Lem. 3.5(ii)]) that from $\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) \in W^{1,2}(\Omega;\mathbb{R}^{d\times d})$, it follows that $(\delta+|\mathbf{D}\mathbf{v}|)^{\frac{p(\cdot)-2}{2}}|\nabla q| \in L^2(\Omega)$, so that, in this case, the additional regularity assumption on the kinematic pressure is reasonable. A similar statement for the case $\alpha \in (0,1)$ could not yet be established (due to technical tools yet missing, the derivation of which would have gone beyond the scope of this research project), but remains as a conjecture.

Work Package A3: Experimental investigation of a finite element (FE) approximation.

Objective A3.1: An objective of this work package is to investigate the influence of the α-Hölder regularity of the power-law index $p \in C^{0,\alpha}(\overline{\Omega})$, $\alpha \in (0,1]$, on the error decay rates *via* numerical experiments.

Result A3.1: According to [20, Rem. 4.5], if $p \in C^{0,\alpha}(\overline{\Omega})$, $\alpha \in (0,1]$, one can only hope for the *"partial"* regularity $\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) \in N^{\alpha,2}(\Omega;\mathbb{R}^{d\times d})$. In the case $p^- \geq 2$ and $\alpha = 1$, it has been established (*cf.* [28, Lem. 3.5(i)]) that from $\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) \in W^{1,2}(\Omega;\mathbb{R}^{d\times d})$, it follows that $q \in W^{1,p'(\cdot)}(\Omega)$. A similar statement for the case $\alpha \in (0,1)$ could not yet been established (due to technical tools yet missing, the derivation of which would have gone beyond the scope of this research project), but remains as a conjecture. Thus, in the *a priori* error estimates (3), (4), especially the case $\alpha = \beta = \gamma$ is relevant and, in this case, the expected error decay rate depending on $\alpha \in (0,1]$ could be measured in numerical experiments (*cf.* Result A3.2 and [28, Subsec. 7.2]). This dependence, however, may be traced back to the reduced regularity of the velocity vector field and the kinematic pressure as, in the case $\alpha \in (0,1)$ and $\beta = \gamma = 1$, the expected error decay rate depending on $\alpha \in (0,1]$ could not be measured yet (*cf.* Figure 1).

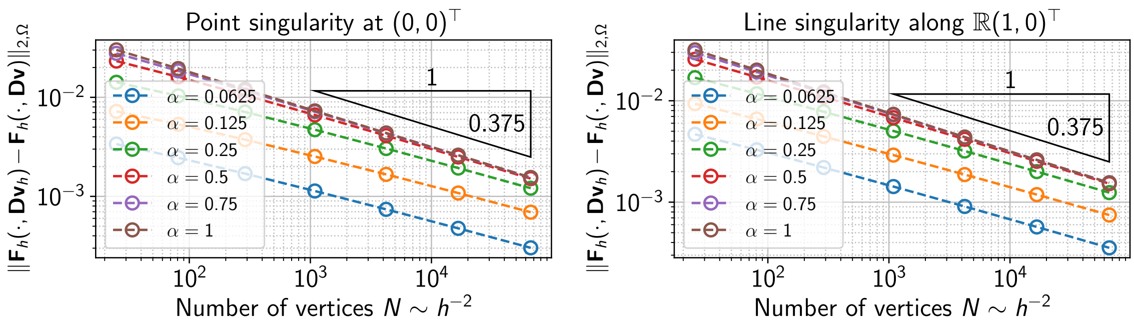

Figure 1: Error decay for the same manufactured solution as in [28, Subsec. 7.2, (Case 1)], in the case $\alpha \in \{0.0625, 0.125, 0.25, 0.5, 0.75, 1.0\}$ and $\beta = \gamma = 1$: *left*: the power-lax index has a point singularity at $(0.0)^\top$; *right*: the power-lax index has a line singularity along $\mathbb{R}(1.0)^\top$.

Objective A3.2: An objective of this work package is to confirm the quasi-optimality of the *a priori* error estimates (3), (4) derived in Work Package A3.1 *via* numerical experiments.

Result A3.2: The quasi-optimality of the *a priori* error estimates for the velocity vector field (3), (4) derived in Work Package A3.1 was confirmed *via* numerical experiments (*cf.* [28, Subsec. 7.2]). However, the quasi-optimality of the *a priori* error estimates (3), (4) for the kinematic pressure derived in Work Package A3.1 could not yet be confirmed (*cf.* [28, Subsec. 7.2]).

Objective: A3.3: An objective of this work package is to investigate the influence of the choice of quadrature points employed to discretize the power-law index in a finite element approximation of the steady $p(x)$-Navier–Stokes equations on the error decay rates *via* numerical experiments.

Result A3.3: For the same manufactured solution as in [28, Subsec. 7.2, (Case 1)], numerical experiment were carried out in which the power-law index is approximated either using nodal interpolation into globally continuous element-wise polynomial functions or using a (local) L^2-projection operator onto element-wise polynomial functions, each of polynomial degrees $p_{\text{deg}} \in \{1, \ldots, 6\}$. In each case, no improvement compared to [28, Subsec. 7.2, (Case 1)] could be reported, neither in terms of the smallness of error nor in terms of the asymptotic behaviour of the error decay (*cf.* Figure 2).

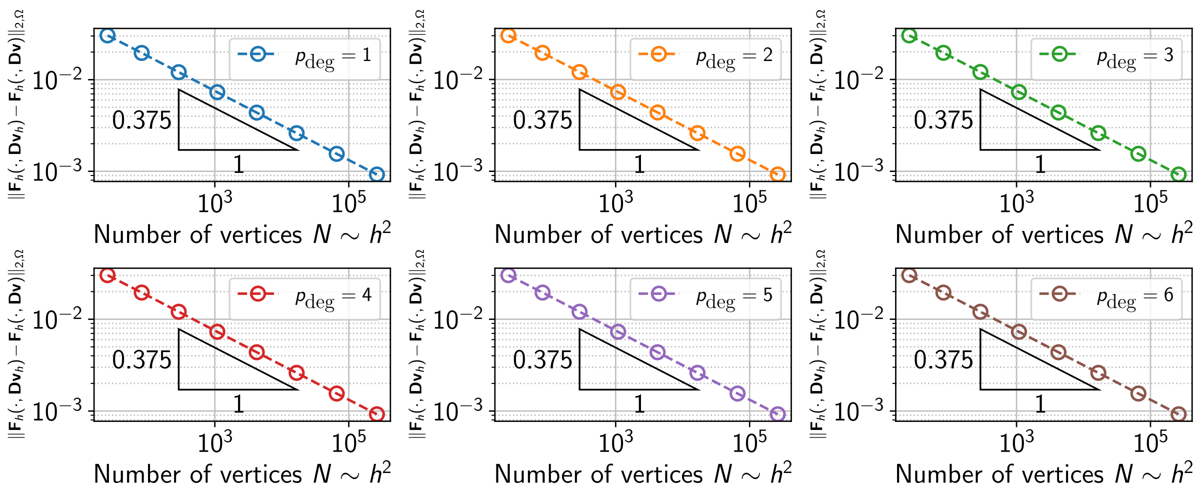

Figure 2: Error decay for the same manufactured solution as in [28, Subsec. 7.2, (Case 1)] when the power-law index is approximated using nodal interpolation into globally continuous element-wise polynomial functions of polynomial degrees $p_{\text{deg}} \in \{1, \ldots, 6\}$. The same results were reported when the power-law index is approximated using a (local) L^2-projection operator onto element-wise polynomial functions of polynomial degrees $p_{\text{deg}} \in \{1, \ldots, 6\}$.

Work Area B: FE approximation of the unsteady $p(t,x)$-Navier–Stokes equations.

This work area aims at theoretical and experimental investigations on a fully-discrete Rothe–Galerkin approximation of the unsteady $p(t,x)$-Navier–Stokes equations (1):

Work Package B1: Weak convergence of a fully-discrete Rothe–Galerkin approximation.

Objective B1: The objective of this work package is to establish a general framework that allows to conclude the well-posedness (*i.e.*, existence of iterates), stability (*i.e.*, *a priori* estimates), and weak convergence of a fully-discrete Rothe–Galerkin approximation of an unsteady problem, containing the unsteady $p(t,x)$-Navier–Stokes equations (1) as a particular case, for a $\log$-Hölder continuous power-law index $p\colon Q_T \to (1,+\infty)$ with $p^- \geq 2$ (or $p^- \geq \frac{3d+2}{d+2}$ if the convective term is present) and general approximations $(p_h^\tau)_{\tau,h>0} \subseteq L^\infty(Q_T)$ satisfying $p_h^\tau \to p$ $(\tau, h \to 0)$.

Result B1: On the basis of the concept of the non-conforming Bochner condition (M) (*cf.* [29, Sec. 6]), a general framework (*cf.* [29, Sec. 7]) was established that allows to conclude the well-posedness, stability, and weak convergence of a fully-discrete Rothe–Galerkin approximation for a $\log$-Hölder continuous power-law index $p\colon Q_T \to (1,+\infty)$ with $p^- \geq 2$ (or $p^- \geq \frac{3d+2}{d+2}$ if the convective term is present) and general approximations $(p_h^\tau)_{\tau,h>0} \subseteq L^\infty(Q_T)$ satisfying $p_h^\tau \to p$ $(\tau, h \to 0)$. This framework applies to a large class of unsteady problems, including, *e.g.*, the unsteady $p(t,x)$-Stokes equations (*cf.* [29, Subsec. 8.4]) and the unsteady $p(t,x)$-Navier–Stokes equations (*cf.* [29, Subsec. 8.5]).

Work Package B2: Experimental investigations on a fully-discrete Rothe–Galerkin approximation.

Objective B2: The objective of this work package is to confirm experimentally the weak convergence of a fully-discrete Rothe–Galerkin approximation of the unsteady $p(t,x)$-Navier–Stokes equations (1), in the case of low regularity data.

Result B2: Since the weak convergence of a fully-discrete Rothe–Galerkin approximation is difficult to measure in numerical experiments, manufactured solutions with fractional regularity properties were constructed, the fractional regularity parameters were gradually reduced, and error decay rates that reduce with decreasing (but are stable for fixed) value of the fractional regularity parameter were measured (*cf.* [29, Subsec. 9.3]). This gave an indication that weak convergence is –at least– likely.

Work Package B3: Error estimates for a fully-discrete Rothe–Galerkin approximation.

Objective B3: The objective of this work package is to derive *a priori* error estimates (similar to (3), (4)) for a fully-discrete Rothe–Galerkin approximation of the unsteady $p(t,x)$-Navier–Stokes equations (1), using Témam's modification (*cf.* [24]) to discretize the convective term and a one-point quadrature rule to discretize the power-law index, imposing only fractional regularity assumptions on the velocity vector field and the kinematic pressure (similar to the Objectives A2.1 and A2.2 or the contributions [23, 28]).

Result B3: A priori error estimates (similar to (3), (4)) for a fully-discrete Rothe–Galerkin approximation of the unsteady $p(t,x)$-Stokes equations, using a one-point quadrature rule to discretize the power-law index, imposing only fractional regularity assumptions on the velocity vector field and the pressure have been derived: more precisely, if $p\in C^{0,\alpha_{\mathrm{t}},\alpha_{\mathrm{x}}}(\overline{Q_T})$, $\alpha_{\mathrm{t}},\alpha_{\mathrm{x}}\in(0,1]$, *i.e.*, for every $(t,x)^\top,(s,y)^\top\in Q_T$, we have that

$$|p(t,x)-p(s,y)|\lesssim |t-s|^{\alpha_{\mathrm{t}}}+|x-y|^{\alpha_{\mathrm{x}}}\,,$$

with $p^->1$,$\mathbf{v}\in L^{p(\cdot,\cdot)}(Q_T;\mathbb{R}^d)\cap L^\infty(I;N^{\delta_{\mathrm{x}},2}(\Omega;\mathbb{R}^d))$, $\delta_{\mathrm{x}}\in(0,1]$, with $\mathbf{v}(t)\in W^{1,p(t,\cdot)}_0(\Omega;\mathbb{R}^d)$ for a.e. $t\in I$, $\mathbf{F}(\cdot,\cdot,\mathbf{D}_x\mathbf{v})\in N^{\beta_{\mathrm{t}},2}(I;(L^2(\Omega))^{d\times d})\cap L^2(I;(N^{\beta_{\mathrm{x}},2}(\Omega))^{d\times d})$, $\beta_{\mathrm{t}}\in(\frac{1}{2},1]$, $\beta_{\mathrm{x}}\in(0,1]$, and $q\in L^{p'(\cdot,\cdot)}(Q_T)$ with $q(t)\in H^{\gamma_{\mathrm{x}},p'(t,\cdot)}(\Omega)$ for a.e. $t\in I$ and $|\nabla_x^{\gamma_{\mathrm{x}}}q|\in L^{p'(\cdot,\cdot)}(Q_T)$, $\gamma_{\mathrm{x}}\in(\frac{\alpha_{\mathrm{x}}}{\min\{2,(p^+)'\}},1]$, it was established (*cf.* [30]) that there exists a constant $s>1$, which can chosen to be close to 1 if $\tau+h>0$ is close to 0, such that

$$\begin{aligned}\max_{k=0,\dots,K}\|\mathbf{v}_h^k-\mathbf{v}(t_k)\|_{2,\Omega}^2+\sum_{k=0}^{K}\tau\,\|\mathbf{F}_h(t_k,\cdot,\mathbf{D}_x\mathbf{v}_h^k)-\mathbf{F}_h(t_k,\cdot,\mathbf{D}_x\mathbf{v}(t_k))\|_{2,\Omega}^2&\\ \lesssim \tfrac{h^{2\delta_{\mathrm{x}}}}{\tau}[\mathbf{v}]^2_{L^\infty(I;N^{\delta_{\mathrm{x}},2}(\Omega))}&\\ +(\tau^{2\alpha_{\mathrm{t}}}+h^{2\alpha_{\mathrm{x}}})\sup\nolimits_{t\in I}\|1+|\mathbf{D}_x\mathbf{v}|^{p(t,\cdot)s}\|_{1,\Omega}&\\ +\tau^{2\beta_{\mathrm{t}}}\,[\mathbf{F}(\cdot,\cdot,\mathbf{D}_x\mathbf{v})]^2_{N^{\beta_{\mathrm{t}},2}(I;L^2(\Omega))}&\\ +h^{2\beta_{\mathrm{x}}}\,[\mathbf{F}(\cdot,\cdot,\mathbf{D}_x\mathbf{v})]^2_{L^2(I;N^{\beta_{\mathrm{x}},2}(\Omega))}&\\ +\|(\varphi_{|\mathbf{D}_x\mathbf{v}|})^*(h_x^{\gamma_{\mathrm{x}}}|\nabla^{\gamma_{\mathrm{x}}}q|)\|_{1,Q_T}\,.&\end{aligned}\tag{5}$$

In the *a priori* error estimate (5), by $(\mathbf{v}_h^k)_{k=0,\dots,K}$, $K\in\mathbb{N}$, we denote the iterates of a Rothe–Galerkin approximation of the unsteady $p(t,x)$-Stokes equations employing a backward Euler step in time and discretely inf-sup stable finite elements in space with respect to discretizations $\{\mathcal{Q}_h^\tau:=\mathcal{I}_\tau\times\mathcal{T}_h\}_{\tau,h>0}$ of the time-space cylinder Q_T. Here, $\{\mathcal{I}_\tau:=\{(t_{k-1},t_k)\}_{k=1,\dots,K}\}_{\tau>0}$ with $t_k=k\tau$ for all $k=0,\dots,K$, *i.e.*, $\tau=\frac{T}{K}$, and $\{\mathcal{T}_h\}_{h>0}$ a family of shape-regular and conforming triangulations of $\overline{\Omega}$. Moreover, the non-linear mapping $\mathbf{F}_h\colon Q_T\times\mathbb{R}^{d\times d}\to\mathbb{R}^{d\times d}_{\mathrm{sym}}$, for every $t\in(t_{k-1},t_k)$, $k=1,\dots,K$, $x\in T$, $T\in\mathcal{T}_h$, and $\mathbf{A}\in\mathbb{R}^{d\times d}$, is defined by

$$\mathbf{F}_h(t,x,\mathbf{A}):=\mathbf{F}(t_k,\xi_T,\mathbf{A})\,,$$

where the point $\xi_T\in T$, for every $T\in\mathcal{T}_h$, is an arbitrary quadrature point and the non-linear mapping $\mathbf{F}\colon Q_T\times\mathbb{R}^{d\times d}\to\mathbb{R}^{d\times d}_{\mathrm{sym}}$, for every $(t,x)^\top\in Q_T$ and $\mathbf{A}\in\mathbb{R}^{d\times d}$, is defined by

$$\mathbf{F}(t,x,\mathbf{A}):=(\delta+|\mathbf{A}^{\mathrm{sym}}|)^{\frac{p(t,x)-2}{2}}\mathbf{A}^{\mathrm{sym}}\,.$$

3.2 Description of the handling of research data generated in the project and any data infrastructures used

The resulting data from the numerical experiments were published in preliminary publications (*cf.* [28, 29]). To create the experiments the open source software package `fenics` has been used. The program code used will be made publicly available after acceptance of the articles in scientific journals.

3.3 Bibliography

[1] K. R. Rajagopal and M. Růžička. “On the Modeling of Electrorheological Materials”. In: *Mech. Research Comm.* 23 (1996), pp. 401–407.

[2] M. Růžička. *Electrorheological Fluids: Modeling and Mathematical Theory*. Vol. 1748. Lecture Notes in Math. Springer, Berlin, 2000.

[3] W. Eckart and M. Růžička. “Modeling micropolar electrorheological fluids”. English. In: *Int. J. Appl. Mech. Eng.* 11.4 (2006), pp. 813–844.

[4] A. C. Eringen. *Microcontinuum field theories. I,II.* New York: Springer-Verlag, 1999.

[5] I. A. Brigadnov and A. Dorfmann. “Mathematical modeling of magnetorheological fluids”. English. In: *Contin. Mech. Thermodyn.* 17.1 (2005), pp. 29–42.

[6] W. M. Lai, S. C. Kuei, and V. C. Mow. “Rheological Equations for Synovial Fluids”. In: *Journal of Biomechanical Engineering* 100.4 (1978), pp. 169–186.

[7] J. Hron, J. Málek, P. Pustějovská, and K. R. Rajagopal. “On the Modeling of the Synovial Fluid”. In: *Advances in Tribology* 2010 (2010).

[8] V. V. Zhikov. “Meyer-type estimates for solving the nonlinear Stokes system”. In: *Differ. Uravn.* 33.1 (1997), 107–114.

[9] S. Antontsev and J. Rodrigues. “On stationary thermo-rheological viscous flows”. In: *ANNALI DELL’UNIVERSITA’ DI FERRARA* 52 (2006), pp. 19–36.

[10] A. Behera. *Text Book: Advanced Materials (ISBN: 978-3-030-80359-9).* 2021.

[11] **A. Kaltenbach**. “Theory of Pseudo-Monotone Operators for Unsteady Problems in Variable Exponent Spaces”. PhD thesis. Institute of Applied Mathematics, University of Freiburg, 2021, p. 235.

[12] **A. Kaltenbach**. *Pseudo-Monotone Operator Theory for Unsteady Problems with Variable Exponents*. Vol. 2329. Lecture Notes in Mathematics. Springer, Cham, 2023, pp. xiii+358.

[13] M. Růžička. *Electrorheological fluids: modeling and mathematical theory*. Vol. 1748. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2000, pp. xvi+176.

[14] V. V. Zhikov. “On the solvability of the generalized Navier-Stokes equations”. In: *Dokl. Akad. Nauk* 423.5 (2008), 597–602.

[15] S. E. Pastukhova. “Compensated compactness principle and solvability of generalized Navier-Stokes equations”. In: vol. 173. 6. 2011, pp. 769–802.

[16] D. Breit and F. Gmeineder. “Electro-rheological fluids under random influences: martingale and strong solutions”. In: *Stochastics and Partial Differential Equations: Analysis and Computations* 7 (2019).

[17] L. Diening. “Theoretical and numerical results for electrorheological fluids”. In: *Freidok Univ. Freiburg im Breisgau* (2002).

[18] E. Carelli, J. Haehnle, and A. Prohl. “Convergence analysis for incompressible generalized Newtonian fluid flows with nonstandard anisotropic growth conditions”. In: *SIAM J. Numer. Anal.* 48.1 (2010), pp. 164–190.

[19] L. M. Del Pezzo, A. L. Lombardi, and S. Martínez. “Interior penalty discontinuous Galerkin FEM for the $p(x)$-Laplacian”. In: *SIAM J. Numer. Anal.* 50.5 (2012), pp. 2497–2521.

[20] D. Breit, L. Diening, and S. Schwarzacher. “Finite element approximation of the $p(\cdot)$-Laplacian”. In: *SIAM J. Numer. Anal.* 53.1 (2015), pp. 551–572.

[21] L. C. Berselli, D. Breit, and L. Diening. “Convergence analysis for a finite element approximation of a steady model for electrorheological fluids”. In: *Numer. Math.* 132.4 (2016), pp. 657–689.

[22] S. Ko, P. Pustějovská, and E. Süli. “Finite element approximation of an incompressible chemically reacting non-Newtonian fluid”. In: *ESAIM Math. Model. Numer. Anal.* 52.2 (2018), pp. 509–541.

[23] D. Breit and P. Mensah. “Space-time approximation of parabolic systems with variable growth”. In: *IMA Journal of Numerical Analysis* 40.4 (2019), pp. 2505–2552.

[24] R. Temam. *Navier-Stokes equations. Theory and numerical analysis*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977, pp. x+500.

[25] S. M. Nikolskii. *Approximation of functions of several variables and imbedding theorems*. Springer-Verlag, New York-Heidelberg, 1975, pp. viii+418.

[26] D. Yang. “New characterizations of Hajłasz-Sobolev spaces on metric spaces”. In: *Sci. China Ser. A* 46.5 (2003), pp. 675–689.

[27] X. Zhang and Q. Sun. *Characterizations of variable fractional Hajłasz–Sobolev spaces*. 2022.

[28] L. C. Berselli and **A. Kaltenbach**. “Error analysis for a finite element approximation of the steady $p(\cdot)$-Navier–Stokes equations”. 2023.

[29] L. C. Berselli and **A. Kaltenbach**. “Convergence analysis for a fully-discrete finite element approximation of the unsteady $p(\cdot,\cdot)$-Navier-Stokes equations”. 2024.

[30] L. C. Berselli, **A. Kaltenbach**, and S. Ko. “Error analysis for a fully-discrete finite element approximation of the unsteady $p(\cdot,\cdot)$-Stokes equations”. 2024.

[31] S. Ko and E. Süli. “Finite element approximation of steady flows of generalized Newtonian fluids with concentration-dependent power-law index”. In: *Math. Comp.* 88.317 (2019), pp. 1061–1090.
[32] J. Jeßberger and A. Kaltenbach. “Finite element discretization of the steady, generalized Navier-Stokes equations with inhomogeneous Dirichlet boundary conditions”. 2023.
[33] J. Jeßberger and A. Kaltenbach. “Finite element discretization of the steady, generalized Navier–Stokes equations for small shear stress exponents”. 2024.
[34] J. Wichmann. “Temporal regularity of symmetric stochastic p-Stokes systems”. English. In: *J. Math. Fluid Mech.* 26.2 (2024), p. 28.
[35] K.-N. Le and J. Wichmann. *A class of space-time discretizations for the stochastic p-Stokes system*. 2023.
[36] J. Wichmann. “Temporal regularity of symmetric stochastic p-Stokes systems”. PhD thesis. Universität Bielefeld, 2022.
[37] O. Hennigh, S. Narasimhan, M. A. Nabian, A. Subramaniam, K. Tangsali, Z. Fang, M. Rietmann, W. Byeon, and S. Choudhry. “NVIDIA SimNet™: An AI-accelerated multi-physics simulation framework”. In: *International Conference on Computational Science*. Springer, 2021, pp. 447–461.
[38] M. Ohlberger and S. Rave. “Reduced Basis Methods: Success, Limitations and Future Challenges”. In: *Proceedings of the Conference Algoritmy* (2016), pp. 1–12.
[39] C. Ma, L. Wu, et al. “The Barron Space and the Flow-Induced Function Spaces for Neural Network Models”. In: *Constructive Approximation* 55.1 (2022), pp. 369–406.
[40] D. Yarotsky. “Error bounds for approximations with deep ReLU networks”. In: *Neural Networks* 94 (2017), pp. 103–114.
[41] I. Gühring and M. Raslan. “Approximation Rates for Neural Networks with Encodable Weights in Smoothness Spaces”. In: *Neural Networks* 134 (2021), pp. 107–130.
[42] I. Gühring, G. Kutyniok, and P. Petersen. “Error bounds for approximations with deep ReLU neural networks in W s, p norms”. In: *Analysis and Applications* 18.05 (2020), pp. 803–859.
[43] W. E and B. Yu. “The Deep Ritz Method: A Deep Learning-Based Numerical Algorithm for Solving Variational Problems”. English (US). In: *Communications in Mathematics and Statistics* 6.1 (2018), pp. 1–12.
[44] M. Raissi, P. Perdikaris, and G. E. Karniadakis. “Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations”. In: *Journal of Computational physics* 378 (2019), pp. 686–707.

4 Published project results

4.1 Peer-reviewed publications

None.

4.2 Further publications and published results

The following two pre-prints contain project results and are currently under review or in preparation:

- L. C. Berselli and A. Kaltenbach. *Error analysis for a finite element approximation of the steady $p(\cdot)$-Navier–Stokes equations*, submitted, 2023.
  URL: https://arxiv.org/abs/2311.00534
- L. C. Berselli and A. Kaltenbach. *Convergence analysis of a fully-discrete finite element approximation of the unsteady $p(\cdot,\cdot)$-Navier–Stokes equations*, submitted, 2024.
  URL: https://arxiv.org/abs/2402.16606
- L. C. Berselli, A. Kaltenbach, and S. Ko. *Error analysis for a fully-discrete finite element approximation of the unsteady $p(\cdot,\cdot)$-Stokes equations*, in preparation, 2024.

4.3 Patents (applied for and granted)

None.

5 Further information on the project, qualifications and outlook

5.1 Description of the course of the project, including any problems with the organization or implementation

Due to an almost two-week illness-related downtime as a consequence of a mosquito bite in the first month, I decided to adjust the originally scheduled course of the project slightly: more precisely, I skipped Work Package A1 and proceeded directly with Work Package A2, which could then be completed within the originally scheduled time frame (*cf.* Table 1, line 1 & 2). There were further delays in Work Package A3: more precisely, in the realization of the Objectives A3.1 and A3.2, it turned out to be much more delicate than originally expected to construct a manufactured solution that is critical enough to confirm the theoretical findings experimentally. This resulted in a delay of about one month (*cf.* Table 1, line 3 & 4). Moreover, I interchanged the order of the Objectives A3.1 and A3.2 (*cf.* Table 1, line 4), since the manufactured solutions from Objective A3.2 were required for Objective A3.1. For the same reasons, I decided to start with Work Package B2 already in the second month of Work Package B1 (*cf.* Table 1, line 4). While Work Package B1 could be completed within the originally scheduled time frame, there was a further delay of about one month in Work Package B2 (*cf.* Table 1, line 6). The delays of one month each in Work Package A3 and Work Package B2, can also be traced back to the fact that, in addition to the originally planned numerical experiments, I decided to carry out numerical experiments for an electro-rheological fluid flow (*cf.* [28, Sec. 7.3] and [29, Sec. 9.4]).

Since Prof. Dr. Seungchan Ko (Inha University, Republic of Korea) approached me during this research project to jointly derive *a priori* error estimates for a fully-discrete Rothe–Galerkin approximation of the unsteady $p(t,x)$-Navier–Stokes equations (1), I decided to add this objective to Work Area B as Work Package B3 (*cf.* Table 1, line 7). More precisely, in Work Package B3, I aim to derive first *a priori* error estimates for a fully-discrete Rothe–Galerkin approximation of the unsteady $p(t,x)$-Stokes equations and, subsequently, for a fully-discrete Rothe–Galerkin approximation of the unsteady $p(t,x)$-Navier–Stokes equations (1). This collaboration began in the last two months of the research project (*cf.* Table 1, line 7) and the first part, *i.e.*, the derivation of *a priori* error estimates for a fully-discrete Rothe–Galerkin approximation of the unsteady $p(t,x)$-Stokes equations will be published in the next few months. In favor of this collaboration, I decided to further postpone Work Package A1 and possibly to assign it later, *e.g.*, as a master's thesis project. However, the findings of Work Package B2 can straightforwardly be reduced to a finite element approximation of the steady $p(x)$-Navier–Stokes equations, so that Work Package A1 indirectly is already partially completed.

<table>
<tr><th>WA \ month</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th></tr>
<tr><td rowspan="4">A</td><td colspan="3">WP A2</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="2">A2.1</td><td>A2.2</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td colspan="4">WP A3</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td colspan="2">A3.2</td><td>A3.1</td><td>A3.3</td><td></td><td></td><td></td><td></td></tr>
<tr><td rowspan="3">B</td><td></td><td></td><td></td><td></td><td colspan="4">WP B1</td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td colspan="3">WP B2</td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td colspan="2">WP B3</td></tr>
</table>

Table 1: Actual course of the research project (WP $\widehat{=}$ Working Package, WA $\widehat{=}$ Work Area).

5.2 Development of their own academic career as well as the type and extent of support provided by the host researcher

The nine-month exemption from any teaching duties gave me sufficient flexibility in terms of time and space, which enabled me to take part in several international conferences. This helped me to further expand my academic network and to present my research.

My host at the University of Pisa (Prof. Dr. Luigi C. Berselli) has always supported me in making new contacts with local scientists and in further educating myself. For example, the University of Pisa funded my participation in a workshop on the virtual element method (link to the workshop: https://www.cism.it/en/activities/courses/C2317/) as well as one trip back to Germany to maintain scientific contacts.

Moreover, during my stay at the University of Pisa I have been invited to two job interviews for professorships in Germany. I believe that these first experience abroad was helpful for receiving these invitations. Apart from that, this research project was helpful to further sharpen my research profile and to find new future research directions. During my preparation for these interviews and in identifying new research directions, my host at the University of Pisa (Prof. Dr. Luigi C. Berselli) always helped me with advice and support.

5.3 Presentation of the thematic development, the use of the scientific environment at the host institution and the expansion of the own network

My host at the University of Pisa (Prof. Dr. Luigi C. Berselli) opened up various thematic development opportunities and scientific contacts for me:

First, he is currently helping me to set up connections to physicists and engineers (especially in Italy) that may help us to perform more realistic and application-oriented numerical experiments for the various physical models for smart fluids. During this research project, we already tried to perform more application-oriented numerical experiments for an electro-rheological fluid flow (*cf.* [28, Sec. 7.3] and [29, Sec. 9.4], respectively), but had difficulties to a find reasonable and realistic set-up. In this connection, Prof. Dr. Vladimir S. Gueorguiev (University of Pisa, Italy) offered support to set up connections to local engineers.

Second, he convinced me to investigate higher regularity properties for solutions to steady and unsteady problems with variable growth conditions on computational (*e.g.*, polygonal/polyhedral and convex) domains. For linear problems, these regularity results are typically well-known. For non-linear problems, there are only a few results. Therefore, to justify the fractional regularity assumptions in the *a priori* error estimates (3), (4), in the future, we aim to derive such regularity results.

Third, he set up a connection to Prof. Dr. Endre Süli (University of Oxford, United Kingdom) and, thus, indirectly to his former student Prof. Dr. Seungchan Ko (Inha University, Republic of Korea), who are both experts in the analysis and numerical analysis of chemically reacting fluids (*cf.* [22, 31]). This led to a joint project with Prof. Dr. Seungchan Ko given *via* the new Work Package B3. Moreover, he set up a connection to Prof. Dr. Luca Heltai (University of Pisa, Italy) who supported me and gave advice while carrying out the numerical experiments in Work Package B2.

5.4 Possible follow-up studies or presentation of possible application perspectives especially with regard to knowledge transfer

The results of this research project open up several future research directions, of which I would like to briefly present the most important ones (in which I am also already active):

Direction 1: Improved error analysis for a FE approximation of the steady $p(x)$-Navier–Stokes equ.
The lower bound $p^- \geq \frac{3d}{d+2}$ in the *a priori* error estimates (3), (4) is not sharp and may be improved to the lower bound $p^- \geq \frac{2d}{d+1}$, when working with the standard Temam modification (*cf.* [32], for the case of a constant power-law index), or even to the optimal lower bound $p^- \geq \frac{2d}{d+2}$, when working with exactly divergence-free finite elements or with a different approximation of the convective term incorporating a divergence correction operator (*cf.* [33], for the case of a constant power-law index).

Direction 2: Regularity theory on computational (*e.g.*, polygonal/polyhedral and convex) domains.
The (fractional) regularity properties of the velocity vector field and the kinematic pressure assumed in the *a priori* error estimates (3), (4) still need to be established, especially, for computational domains, *i.e.*, polygonal (if $d = 2$) or polyhedral (if $d = 3$) (and possibly convex) domains. Establishing such (fractional) regularity properties for the velocity vector field and the kinematic pressure is a future research direction I will follow in collaboration with Prof. Dr. Luigi C. Berselli (University of Pisa, Italy).

Direction 3: Analytical and numerical investigations on the fully-coupled systems for smart fluids.
After this research project has enabled me to investigate the unsteady $p(t,x)$-Navier–Stokes equations (1) from a numerical point of view, the natural next step is the numerical investigation of the fully-coupled systems for smart fluids, *e.g.*, electro-rheological fluids (*cf.* [1, 2]), micro-polar electro-rheological fluids (*cf.* [3, 4]), magneto-rheological fluids (*cf.* [5]), chemically reacting fluids (*cf.* [6, 7]), and thermo-rheological fluids (*cf.* [8, 9]). Here, additional difficulties arise from the coupling, especially in the power-index, which further complicates the passage to the limit in numerical approximations.

Direction 4: Application-oriented numerical experiments for fully-coupled systems for smart fluids.
In [28, Sec. 7.3] and [29, Sec. 9.4], I tried to carry out numerical experiments for an electro-rheological fluid flow that are of less academic nature. However, as no real-world reference data was available, the results of these numerical experiments are only of limited informative value. The next step should, therefore, be to carry out numerical experiments in the numerous application areas of smart fluids (*cf.* [10], for an overview), under conditions that are as realistic as possible. Here, my host at the University of Pisa (Prof. Dr. Luigi C. Berselli) is already supporting me in establishing contacts with engineers and physicists in order to create suitable test cases.

Direction 5: Consideration of stochastic influences into the physical models for smart fluids.
Another research direction is to consider stochastic influences into the physical models for smart fluids,

for example, generalizing the findings of the contributions [34, 35] or the PhD thesis [36], in which the unsteady p-Stokes equations with stochastic influences in the external forcing is considered, to the unsteady $p(t,x)$-Stokes equations. With the support of Dr. Jörn Wichmann (Monash University, Australia), I am currently familiarizing myself with the topic. Our first joint project consists in the derivation of pressure error estimates for a fully-discrete Rothe–Galerkin approximation of the unsteady p-Stokes equations with stochastic influences in the external forcing, where the power-law index $p\in(1,\infty)$ is initially a constant.

Direction 6: Machine learning-based approximation of smart fluids with parametric dependencies. Another research direction is the consideration of machine learning-based approximations for the physical models for smart fluids, especially in the presence of high-dimensional parametric dependencies, which are ubiquitous in engineering applications and occur, *e.g.*, in varying material properties, varying forcing terms or in the form of varying geometries (*cf.* [37, 38]): a prototypical example constitute the *parametric (steady)* $p(\boldsymbol{p},x)$*-Stokes equations*. The latter seek, given a (high-dimensional) *parameter space* $\mathscr{P}\subseteq\mathbb{R}^{d_{\mathscr{P}}}$, $d_{\mathscr{P}}\in\mathbb{N}$, *parametric domains* $\Omega(\boldsymbol{p})\subseteq\mathbb{R}^{d_\Omega}$, $d\in\mathbb{N}$, $\boldsymbol{p}\in\mathscr{P}$, a *parametric power-law index* $p\in L^\infty(Q_{\mathscr{P}})$, $p^->1$, where $Q_{\mathscr{P}}\coloneqq\bigcup_{\boldsymbol{p}\in\mathscr{P}}\{\boldsymbol{p}\}\times\Omega(\boldsymbol{p})$ denotes the *parametric cylinder*, and a *parametric right-hand side* $\boldsymbol{f}\in(L^{p'(\cdot,\cdot)}(Q_{\mathscr{P}}))^{d_\Omega}$, for a.e. parameter $\boldsymbol{p}\in\mathscr{P}$, for a velocity vector field $\boldsymbol{v}(\boldsymbol{p})\colon\overline{\Omega(\boldsymbol{p})}\to\mathbb{R}^{d_\Omega}$ and a kinematic pressure $\boldsymbol{q}(\boldsymbol{p})\colon\Omega(\boldsymbol{p})\to\mathbb{R}$ such that

$$\begin{aligned}-\mathrm{div}_x\,\mathbf{S}(\boldsymbol{p},\cdot,\mathbf{D}_x\boldsymbol{v}(\boldsymbol{p}))+\nabla_x\,\boldsymbol{q}(\boldsymbol{p})&=\boldsymbol{f}(\boldsymbol{p}) &&\text{in }\Omega(\boldsymbol{p})\,,\\ \mathrm{div}_x\,\boldsymbol{v}(\boldsymbol{p})&=0 &&\text{in }\Omega(\boldsymbol{p})\,,\\ \boldsymbol{v}(\boldsymbol{p})&=\mathbf{0} &&\text{on }\partial\Omega(\boldsymbol{p})\,.\end{aligned}\tag{6}$$

In the systems (6), the *parametric extra stress tensor* $\mathbf{S}\colon Q_{\mathscr{P}}\times\mathbb{R}^{d_\Omega\times d_\Omega}\to\mathbb{R}^{d_\Omega\times d_\Omega}_{\mathrm{sym}}$ assumes for a.e. $(\boldsymbol{p},x)^\top\in Q_{\mathscr{P}}$ and all $\mathbf{A}\in\mathbb{R}^{d_\Omega\times d_\Omega}$, *e.g.*, the form

$$\mathbf{S}(\boldsymbol{p},x,\mathbf{A})\coloneqq(\delta+|\mathbf{A}^{\mathrm{sym}}|)^{p(\boldsymbol{p},x)-2}\mathbf{A}^{\mathrm{sym}}\,,$$

where $\delta\geq0$.

The difficulty with formulation (6), however, is that it is high-dimensional and, thus, hardly can be treated with standard grid-based methods (*e.g.*, finite elements) as they face the *curse of dimensionality*, *i.e.*, the computational costs increase exponentially with the dimension of the problem (*i.e.*, $d_{\mathscr{P}}+d_\Omega$). Then, deploying artificial neural networks, which have shown great potential in the approximation of high-dimensional functions and are known to possess extraordinary approximation capabilities, with the possibility to achieve dimension-independent approximation rates for certain function classes (*cf.* [39–42]), is an option. Here, the most prominent methods are the *Deep Ritz Method* (cf. [43]) and *Physics-Informed Neural Networks (PINNs)* (cf. [44]). In collaboration with Prof. Dr. Seungchan Ko (Inha University, Republic of Korea), I aim to establish first the well-posedness of the parametric (steady) $p(\boldsymbol{p},x)$-Stokes equations (6) in an appropriate weak sense (*i.e.*, the existence of minimizers of the respective energy formulations) and, subsequently, to carry out a thorough (weak) convergence and error analysis for different neural network approximations based on the Deep Ritz Method and/or Physics-Informed Neural Networks.