Computing the Hausdorff Dimension for Symmetric Schottky Groups

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Schottky Groups in the Unit Disk

We can form a Schottky group in the unit disk as follows: let $\mathcal{C}$ be a finite collection of disjoint circles which are orthogonal to the unit circle (actually, we will allow two circles in $\mathcal{C}$ to be tangent, so their intersections with the open unit disk are disjoint). Then the group $\Gamma(\mathcal{C})$ generated by reflections across these circles in the unit disk is an example of a *Schottky group*.

Example: Symmetric Circles

A simple example of a Schottky group in the unit disk is obtained by taking circles of the same size spaced symmetrically about the disk. We could parameterize such sets of circles by the two variables n and θ, where n is the number of circles and θ is the angle along the unit circle cut out by one of the circles. See Figure 1 for an example.

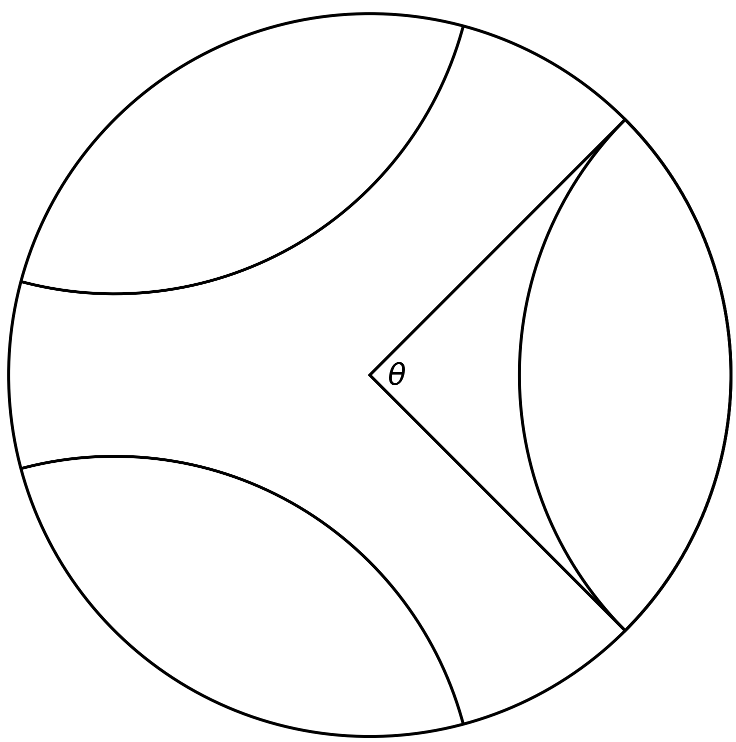

Figure 1: Example with $n = 3$ circles cutting out arcs of angle $\theta = \frac{\pi}{2}$

A fundamental domain for such a Schottky group is the exterior of all the circles (inside the unit disk of course). To see this, label the reflections across the n circles $R_1, \ldots, R_n$. We will also use R_i to denote the actual reflections; the meaning should be clear from context. Let

$$\gamma = R_{i_1} \cdots R_{i_k}$$

be a reduced word of length k in the reflections (by *reduced* we mean that $R_{i_j} \neq R_{i_{j+1}}$ for all j, since reflections are involutions). Note that every element of the Schottky group can be written as such a reduced word. Now suppose z is any element of the fundamental domain. If $k = 1$, then it is clear that $\gamma(z)$ is inside the circle R_{i_1}. By induction, it is then easy to see that for $k \geq 1$, $\gamma(z)$ is always in the circle R_{i_1}. Thus, the only way for $\gamma(z)$ to be in the fundamental domain is t o have $k = 0$; i.e., γ is the identity.

To see that every orbit has a point in the fundamental domain, one must check that given a point in one of the circles R_i, reflection through that circle *decreases* the distance from the point to 0. Thus, since the Schottky group acts discretely on hyperbolic space, we can eventually move any point to the fundamental domain by repeatedly reflecting outside of any circle it falls in.

See Figure 2 for an example.

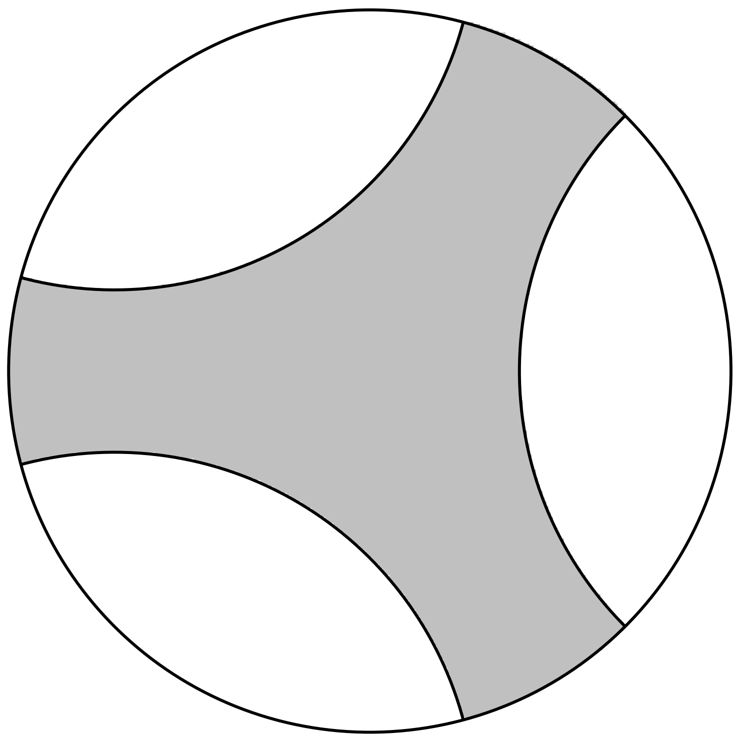

Figure 2: The example from Figure 1 with a fundamental domain filled in.

Note: For the rest of the document, we will focus on the case $n = 3$. The discussion extends easily to $n > 3$.

Rotational Symmetry of Maass Forms

Recall that Maass forms for Γ are eigenfunctions of the Laplace-Beltrami operator acting on $L^2(\Gamma\backslash\mathbb{H})$. We claim that for a given eigenvalue, a Maass form for the symmetric Schottky group of parameters (n, θ) can be taken to be invariant under rotation by $2\pi/n$.

To verify this claim, first recall that the Laplace operator on the disk model (in polar coordinates) is

$$\Delta = -\left(\frac{(1-\rho^2)^2}{4}\frac{\partial^2}{\partial\rho^2} + \frac{(1-\rho^2)^2}{4\rho}\frac{\partial}{\partial\rho} + \frac{(1-\rho^2)^2}{4\rho^2}\frac{\partial^2}{\partial\theta^2}\right)$$

From this formula, one easily sees that rotations are Δ-invariant. Thus, if we compose any eigenfunction of the Laplacian with a rotation, we still have an eigenfunction of the same eigenvalue.

Next, note that a rotation of $2\pi/n$ is smooth on the fundamental domain for the (n, θ) symmetric Schottky group. Thus, given a Maass form f, we can construct a $2\pi/n$ rotation-invariant Maass form of the same eigenvalue by taking the linear combination

$$\frac{1}{n}\sum_{i=0}^{n-1} f\left(e^{2\pi i/n} z\right)$$

So we can assume that any Maass form we are working with is invariant under rotation by $2\pi/n$. Even further, we can use the multiplicity one principle to conclude that the base eigenfunction is itself invariant under this rotation.

Consequences for Hejhal's Algorithm

When running Hejhal's algorithm on the symmetric Schottky groups, we need only consider test points in one $2\pi/n$ sector. We can of course take this sector to entirely contain one flare, while not intersecting any others.

If the pullback z^* of a test point z falls outside of the chosen sector, we can simply replace z^* with the appropriate rotation to make it fall in this sector, since the Maass form will be equivalent at those different points.

In the Appendix, we work through what this means for choosing test points for Hejhal's algorithm. As we show there, the only admissible points come from a very specific area in a flare domain. Unfortunately, these test points end up giving very slow convergence in the flare expansion, thus making them useless for Hejhal's algorithm.

These considerations lead one to consider looking at covers of the Schottky group, perhaps by building the rotational invariance into the group itself. We consider this idea in the next section.

On Covers of the Symmetric Schottky Group

In this section, we assume we are interested in a symmetric Schottky group with three circles. We will construct another group which contains the Schottky group as a finite index subgroup. Because it is a finite degree cover, it will share its bottom eigenvalue with the Schottky group. **DOUBLE CHECK THAT FACT WITH AK!** Moreover, our cover will admit a doubling which comprises only of Möbius transformations, making it more suitable for use in Hejhal's algorithm. Finally, the hope is that the larger group (with a smaller fundamental domain) will allow for a greater variety of admissible test points.

A Larger Reflection Group

We begin by considering a reflection group which contains our symmetric Schottky group as a proper subset. We will describe this group in the disk model. We define three geodesics by stating their endpoints:

- D_1 connects -1 and 1
- D_2 connects $-e^{i\pi/3}$ and $e^{i\pi/3}$
- R connects $e^{-i\theta}$ and $e^{-\theta}$

where θ is the parameter for our symmetric Schottky group. Let Γ be the group generated by the reflections through these geodesics.

$$\Gamma = \langle D_1, D_2, R\rangle$$

(Note that we are abusing notation and letting D_1, D_2, and R refer to the reflection through the corresponding geodesics). See Figure 3 for an example with $\theta = \pi/2$.

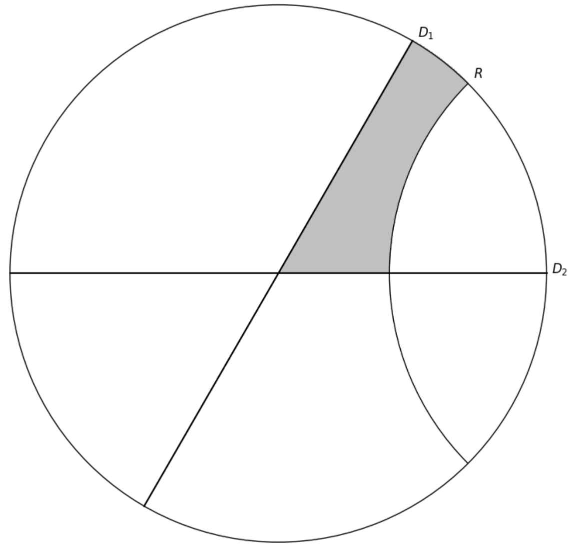

Figure 3: Fundamental Domain for Γ with $\theta = \pi/2$

To see that Γ contains our Schottky group, first observe that the composition $S = D_1 D_2$ is just rotation by $2\pi/3$. Thus, $S(R)$ and $S^2(R)$ give the other two circles in the Schottky group. In particular, $S^{-1}RS$ and SRS^{-1} give the reflections through those two circles. Thus, all three generators of the symmetric Schottky group are contained in Γ.

Doubling Across D_1

The group Γ will have the same bottom eigenvalue as the symmetric Schottky group; however, since Γ is a reflection group, it contains elements which are not Möbius transformations. To rectify this, we will double the group across D_1. Since a composition of two reflections is a Möbius transformation, the doubled group will comprise solely of these. Moreover, doubling a group does not affect its bottom eigenvalue, so we will still be able to study the doubled group when applying Hejhal's algorithm to discover the base eigenvalue for the symmetric Schottky group.

Specifically, we are looking at the group

$$\Gamma_0 = \langle D_1 D_2, D_1 R \rangle$$

This is called *doubling* across D_1 since the fundamental domain for the doubled group is exactly the old fundamental domain plus its reflection across D_1 (see Figure 4). Note that the composition $D_1 D_2$ is just rotation by angle $2\pi/3$. It will be necessary to also determine the action of another generator. To simplify this, we first note that we can replace $D_1 R$ with $(D_1 D_2)^{-1} D1R = D_2 R$ as a generator. This is helpful since the action of D_2 is simply conjugation. Next, we note that the circle containing the geodesic R has center $\sec(\theta/2)$ and radius $\tan(\theta/2)$. Thus, reflection through R can be written as

$$R(z) = \frac{\tan^2(\theta/2)}{\bar{z} - \sec(\theta/2)} + \sec(\theta/2)$$

We can now explicitly write down the action of $D_2 R$.

$$D_2 R(z) = \overline{R(z)} = \frac{\sec(\theta/2) z - 1}{z - \sec(\theta/2)}$$

One notices that multiplying the numerator and denominator by $i \cot(\theta/2)$ converts this to a Möbius transformation where the corresponding matrix is in $\mathrm{PSU}(1,1)$ (this is the matrix group which is conjugate via

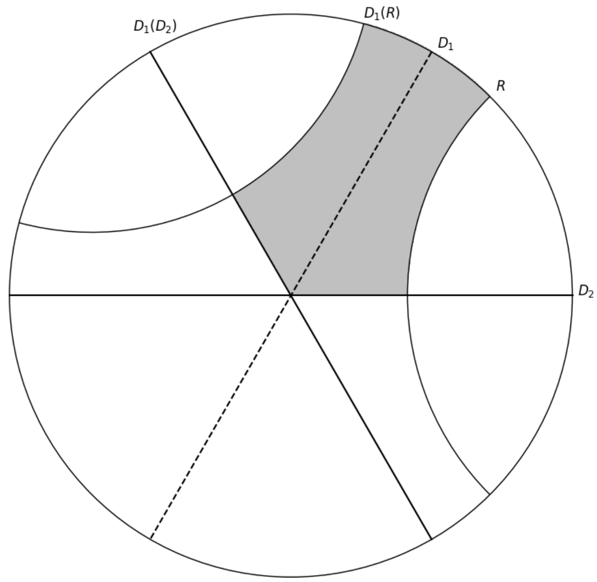

Figure 4: Fundamental Domain for Γ_0 with $\theta = \pi/2$

the Cayley transform to $\mathrm{PSL}(2, \mathbb{R})$). Specifically,

$$D_2R(z) = \frac{i\csc(\theta/2)z - i\cot(\theta/2)}{i\cot(\theta/2) - i\csc(\theta/2)}$$

In this way, we see that $\Gamma_0 = \langle D_1D_2, D_2R \rangle$ is isomorphic to the matrix group

$$\left\langle \begin{pmatrix} e^{i\pi/3} & \\ & e^{-i\pi/3} \end{pmatrix}, \begin{pmatrix} i\csc(\theta/2) & -i\cot(\theta/2) \\ i\cot(\theta/2) & -i\csc(\theta/2) \end{pmatrix} \right\rangle < \mathrm{SU}(1,1)$$

OLD STUFF PAST HERE

Beyond this point, everything is work from before I realized the rotational symmetry made the algorithm break. I am still saving everything in case I need to look back at it later...

Mapping to the Upper Half Plane

Recall that we can map from the upper half plane model to the disk model via the *Cayley transform*

$$C(z) = \frac{z-i}{z+i}$$

We can go the other direction by taking the inverse

$$C^{-1}(w) = i \cdot \frac{1+w}{1-w}$$

Also recall that C and C^{-1} send geodesics to geodesics.

Next, note that we are choosing to center one of our circles of reflection about 1. Since $C^{-1}(1) = \infty$, this has the effect of forcing our fundamental domain to be bounded inside C^{-1} of that circle. See Figure 5 for an example.

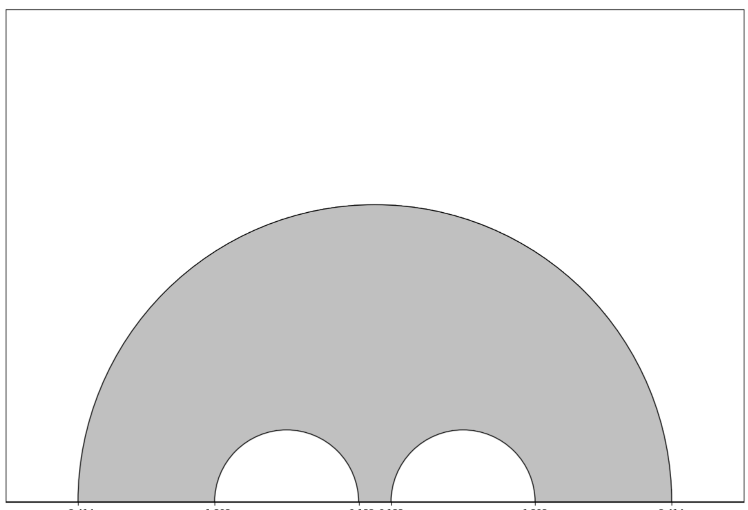

Figure 5: The example from Figure 2 mapped to the upper half plane.

Note: this image seems to single out the middle flare as different from the other two, but this is just an artifact of our choice of Cayley transform. We could just as easily have mapped $e^{\pi i/3}$ or $e^{-\pi i/3}$ to -1 instead.

Finally, let's give a name to each of the circles involved. In the disk, we will call the rightmost circle (which contains the point 1) R_1. We then proceed counterclockwise and name the next two circles R_2 and R_3. The resulting labels in the upper half plane are shown in Figure 6.

Identifying Flare Domains

Let us call the group generated by our three reflections

$$\Gamma = \langle R_1, R_2, R_3 \rangle$$

To get a flare domain, we need to identify a hyperbolic element in Γ whose axis[1] "cuts off" a flare in our fundamental domain. To do this, we first recall that even length words in circle reflections give Möbius

[1]Recall that the *axis* of a hyperbolic matrix is the geodesic whose endpoints are the fixed points of the matrix.

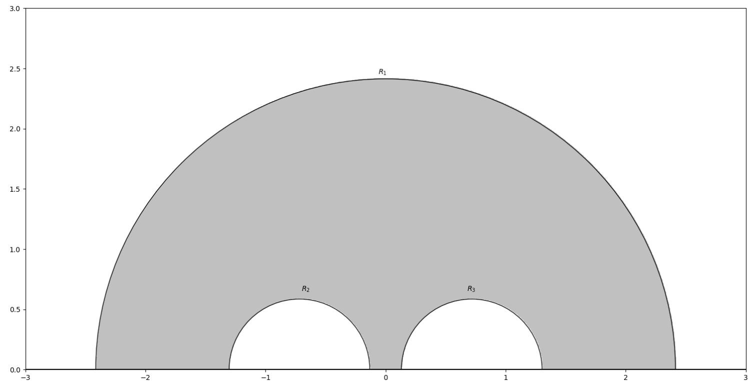

Figure 6: Labeling of the circles from Figure 5.

transformations. Thus, we could look for appropriate hyperbolic matrices by checking small words of even length in our generators. Indeed, one quickly finds that R_1R_2, R_1R_3, and R_2R_3 are hyperbolic matrices with appropriate axes (see Figure 7).

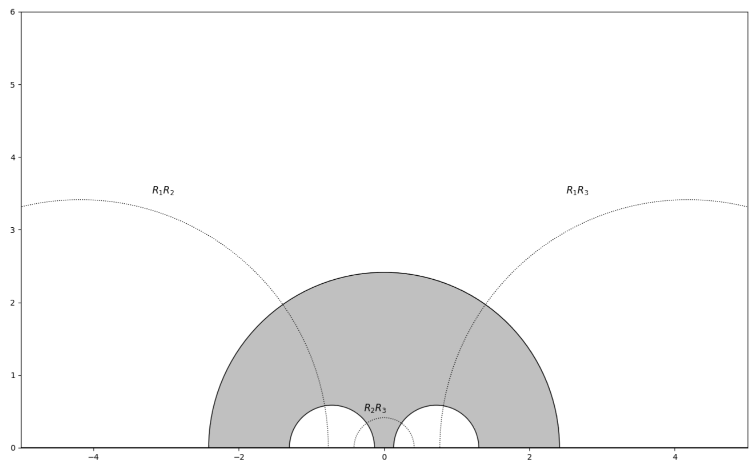

Figure 7: Axes cutting off flares labeled with the associated hyperbolic matrices.

Let's be a bit more formal in our discussion of flares. To identify a flare between geodesics C_1 and C_2, one may look for a geodesic with the following properties:

- one endpoint of the geodesic is on the other side of C_1 from the flare
- the other endpoint is on the other side of C_2 from the flare
- the geodesic meets C_1 and C_2 at right angles

In our situation, it is always possible to find a hyperbolic element in Γ whose axis satisfies these properties. To see this, one looks in inversive coordinates (see Kontorovich's letter to Bill Duke). Letting v_1 and v_2 denote the inversive coordinates for C_1 and C_2, respectively, one looks for the inversive coordinates v such that

$$v_1^t Q v = 0 \qquad v_2^t Q v = 0 \qquad v^t Q v = -1$$

where Q is the quadratic form defining inversive coordinates. The first two equations give the orthogonality, and the third gives a true vector of inversive coordinates. Finally, these coordinates give the endpoints of our geodesic. To find the corresponding Möbius transformation in Γ, one searches over all transformations with this axis for the one taking the intersection point with C_1 to that of C_2.

Mapping to a Flare Domain

Let us look at a generic flare: two geodesics separated by positive length interval on the real line, plus a geodesic cutting through both of these at a right angle. Let the two points on this last geodesic be labeled $z_1 < z_2$, and call the rightmost point of the first geodesic t. See Figure 8. Now consider the Möbius

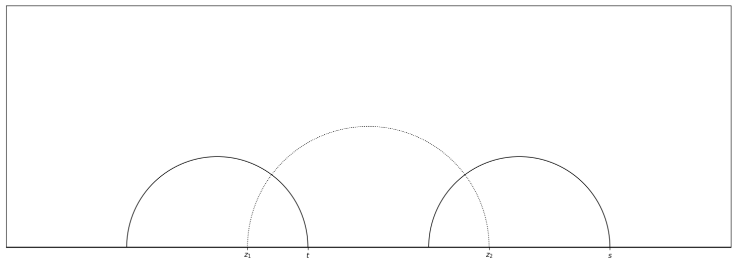

Figure 8: A generic flare.

transformation

$$U(z) = \left(\frac{t - z_2}{t - z_1}\right) \frac{z - z_1}{z - z_2}$$

This function is chosen so that

$$U(z_1) = 0 \qquad U(z_2) = \infty \qquad U(t) = 1$$

Thus, applying such a U to a generic flare gives us a *flare domain*. See Figure 9 for an example.

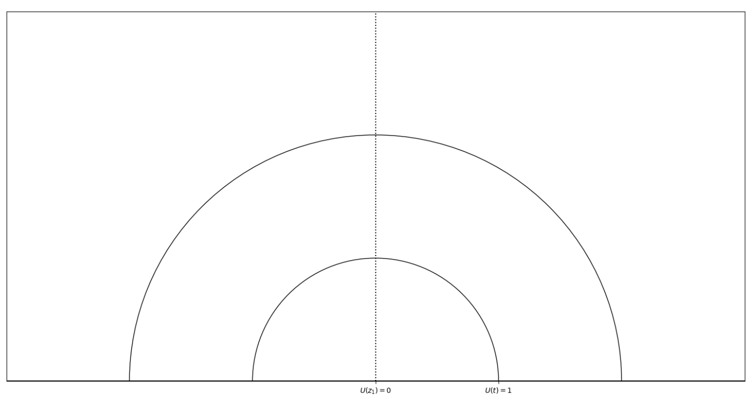

Figure 9: A flare domain.

Let's consider our symmetric Schottky group example. Using the flare cut off by the axis of R_2R_3, we get the flare domain pictured in Figure 10.

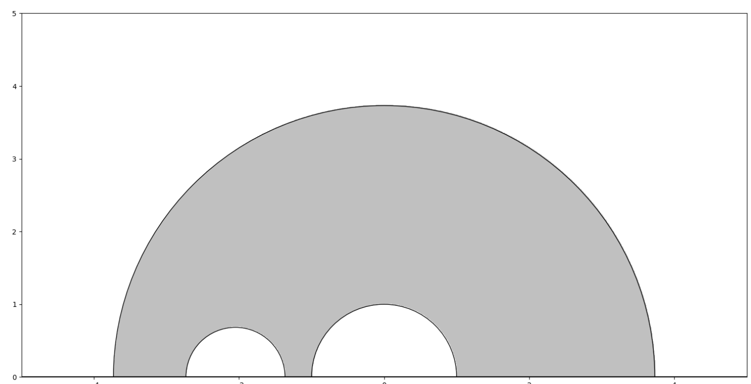

Figure 10: The flare domain obtained from middle flare in Figure 7.

Appendix: Considerations for Choosing Test Points

Let's consider the action of $R_2 R_3$ in the flare domain. Note that the main purpose of mapping to the flare domain is to convert this hyperbolic transformation into a scaling matrix (of course this is still a hyperbolic transformation, just a very specific and simple one). Specifically, the action is converted to one of the form

$$z \mapsto \kappa z$$

for some $\kappa > 1$. Furthermore, recall that it is this action which leads to a logarithmic Fourier expansion of Maass forms. Specifically, if ϕ is a Maass form for such a Schottky group, where $\phi(r, \theta)$ is a function of the polar coordinates (r, θ) in the flare, then

$$\phi(r, \theta) = \sum_{n \in \mathbb{Z}} g_n(\theta) e\left(n \frac{\log r}{\log \kappa}\right)$$

But note that $\phi(r, \theta)$ is then automatically invariant under the map $z \mapsto \kappa z$. So if we take a test point with pullback equal to $\kappa^n z$ for some $n \in \mathbb{Z}, n \neq 0$, we gain *no new information* from equating ϕ at this point and its pullback.

Next, recall that our Maass form is also invariant *in the disk model* under rotation by $2\pi/3$. Let's draw the $2\pi/3$-sector containing our flare after mapping over to the flare domain. This is shown in Figure 11. As

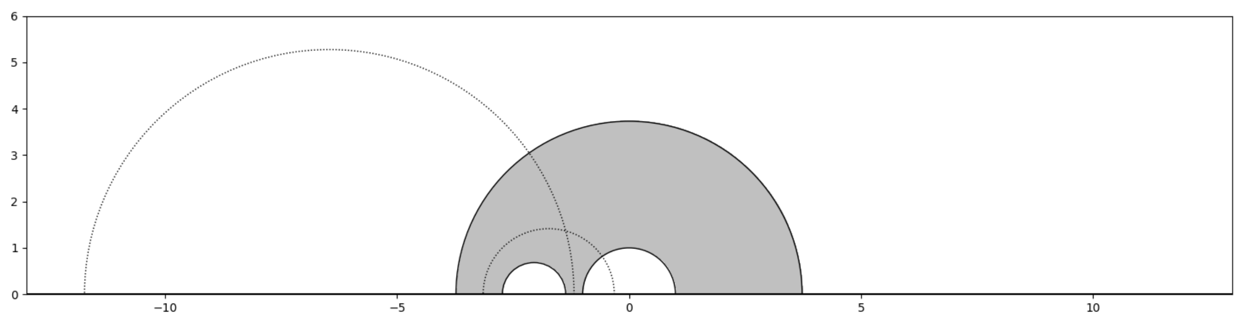

Figure 11: The $2\pi/3$-sector containing our flare.

we have argued, the only viable test points are those which are inside our chosen sector and whose pullback is not just multiplication by a power of κ. Looking at Figure 11, these points will occur outside the leftmost dotted geodesic, and with argument relatively close to π, so that multiplication by a negative power of κ puts them inside the smallest circle.

To do: rewrite that last paragraph. Pretty clunky...