SHEARS

1. Equidistribution of H-translates of a closed geodesic

Let $\Gamma < G = \mathrm{SL}(2,\mathbb{R})$ be a lattice, say co-compact for now, and assume after conjugation that Γ contains the hyperbolic element $\mathrm{diag}(e^{\ell/2}, e^{-\ell/2})$. Then the unit tangent bundle $T_1(\Gamma\backslash\mathbb{H})$ has a closed geodesic of length ℓ from the point $(i, \uparrow)$. Assume for simplicity that $L^2(\Gamma\backslash G)$ has no exceptional spectrum. Let $w = k_{\pi/4}$ denote the element which maps $(i,\uparrow)$ to $(i,\rightarrow)$, and $H = w^{-1}Aw$ be the geodesic flow orthogonal to A. (Acting on $(i,\uparrow)$ by H and A, we can move to any point in $\mathbb{H}$ with tangent vector pointing away from the origin, so this is the same as polar coords, $z = re^{i\theta}$, with r determined by A and θ by H. Here's the equidistribution theorem.

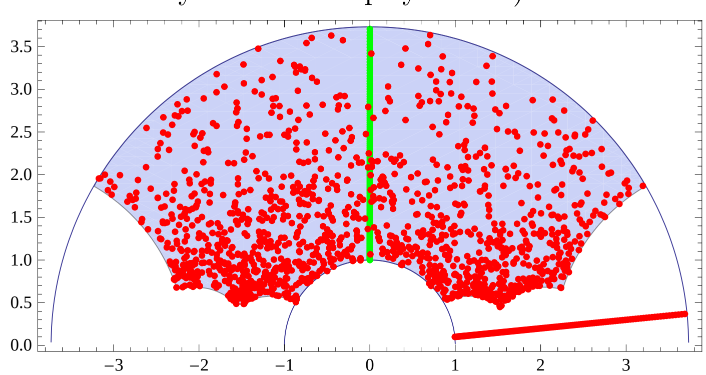
Theorem 1. *For a nice test function f on $\Gamma\backslash G$,*

$$\int_{(A\cap\Gamma)\backslash A} f(ah_t)da = \frac{1}{\mathrm{vol}}\int_{\Gamma\backslash G} f + O_f(e^{(-1/8+\varepsilon)t}),$$

as $t \to \infty$.

The optimal bound here would replace $-1/8$ by $-1/2$, and is what Valentin did in his notes by spectral methods. The point of this note is to do something very soft and easy to get *some* power savings.

Geometrically, this is the following equidistribution: (green is the starting closed geodesic, red is its h-flow, shown in $\mathbb{H}$ and $\Gamma\backslash\mathbb{H}$; I can send the Mathematica code if anyone wants to play with it)



Let $\bar{N}$ be the group $w^{-1}Nw$, where $N = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\}$. We have the decomposition

$$G = AH\bar{N}$$

(this is easy to see algebraically or from geometry in $T^1(\mathbb{H})$). For $\varepsilon > 0$, let $\psi = \psi_\varepsilon$ be a function on $A\backslash G$, defined in the above coordinates by

$$\psi(ah\bar{n}) = \eta_\varepsilon(h)\phi_\varepsilon(\bar{n}),$$

where η and ϕ are smooth normalized bump functions of a ball of radius ε in H and $\bar{N}$, respectively. Normalized means

$$\int_{(\Gamma\cap A)\backslash G} \psi = 1.$$

Average ψ over Γ to Ψ, that is, let

$$\Psi(g) := \sum_{\gamma\in(\Gamma\cap A)\backslash\Gamma} \psi(\gamma g).$$

Consider the matrix coefficient:

$$\langle \pi(h_t).f, \Psi\rangle = \int_{\Gamma\backslash G} f(gh_t)\Psi(g)dg = \int_{(\Gamma\cap A)\backslash G} f(gh_t)\psi(g)dg,$$

where we unfolded. Parametrize $G = AH\bar{N}$:

$$= \int_{(\Gamma\cap A)\backslash A}\int_H\int_{\bar{N}} f(ah\bar{n}h_t)\psi(ah\bar{n})d\bar{n}dhda = \int_{(\Gamma\cap A)\backslash A}\int_H\int_{\bar{N}} f(ah\bar{n}h_t)\eta_\varepsilon(h)\phi_\varepsilon(\bar{n})d\bar{n}dhda.$$

Now for the "wavefront lemma": note that if $h_t = w^{-1}a_tw$ and $\bar{n}_x = w^{-1}n_xw$, then

$$h_t^{-1}\bar{n}_xh_t = w_{-1}a_t^{-1}n_xa_tw = \bar{n}_{xe^{-t}}.$$

So if $\bar{n} \in \operatorname{supp}\phi_\varepsilon$, then so is its h_t conjugate. (If we wanted the limit the other way, $t \to -\infty$, then just choose the lower-triangular N.) Assume f is uniformly Lipschitz, so

$$f(ah\bar{n}h_t) = f(ah_th\bar{n}') = f(ah_t) + O_f(\varepsilon),$$

where $\bar{n}'$ is the conjugate. So we have

$$\langle \pi(h_t).f, \Psi\rangle = \int_{(\Gamma\cap A)\backslash A} f(ah_t)da\cdot\int_H\int_{\bar{N}} \eta_\varepsilon(h)\phi_\varepsilon(\bar{n})d\bar{n}dh + O(\varepsilon) = \int_{(\Gamma\cap A)\backslash A} f(ah_t)da + O(\varepsilon).$$

On the other hand, expand spectrally and apply Howe-Moore:

$$\langle \pi(h_t).f, \Psi\rangle = \frac{1}{\mathrm{vol}}\langle f, \mathbf{1}\rangle\langle \mathbf{1}, \Psi\rangle + O(e^{-t/2}\ \mathcal{S}f\ \mathcal{S}\Psi),$$

where $\mathcal{S}$'s are Sobolev 1-norms. (I'm ignoring an epsilon here; in general the error term is $e^{-(1/2-\theta)t}$, where θ controls the exceptional spectrum, $\theta = 0$ being Ramanujan, which we assumed.) By construction of ψ, we have $\mathcal{S}\Psi \ll \varepsilon^{-3}$ (I did this calculation quickly, so check if you agree).

Putting it together, we have

$$\int_{(\Gamma\cap A)\backslash A} f(ah_t)da = \frac{1}{\mathrm{vol}}\int_{\Gamma\backslash G} f + O(e^{-t/2}\ \mathcal{S}f\ \varepsilon^{-3} + \varepsilon),$$

and optimizing ε gives the claim.