Hejhal's Algorithm in $\mathbb{H}^3$

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In this document, we describe Hejhal's algorithm as it applies to hyperbolic 3-space. It is essentially the same as in 2-space; the differences occur only in the details of discrete groups acting on $\mathbb{H}^3$. We begin with some background regarding hyperbolic 3-space, then we move on to Maass forms in this context. After that, we will be ready to discuss the algorithm.

Preliminaries on $\mathbb{H}^3$

By *hyperbolic 3-space* (or $\mathbb{H}^3$), we mean the unique simply-connected Riemannian 3-manifold with constant sectional curvature -1. There are various models of this space, but for now we will work with the upper half-space model. Let the set U^3 consist of the subset of $\mathbb{R}^3$ with positive third component

$$U^3 = \{(x_1, x_2, y) \in \mathbb{R}^3 : y > 0\}$$

We endow U^3 with the metric

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dy^2}{y^2}$$

It is well known that (U^3, ds^2) is then isometric to $\mathbb{H}^3$. Moreover, we note that $\mathbb{H}^3$ can be identified with the Hamiltonian quaternions whose k-term is 0. That is, we take the subalgebra of

$$\{x_1 + x_2 i + x_3 j + x_4 k : x_1, x_2, x_3, x_4 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}$$

whose elements have $x_3 > 0$ and $x_4 = 0$. Replacing the name "x_3" with "y," we evidently have the same set U^3.

Möbius Transformations in 3-space

Recall that $\mathrm{SL}(2,\mathbb{R})$ acts on the upper half plane $\mathbb{H}^2$ via Möbius transformations. It is well known that these give all orientation-preserving isometries of hyperbolic 2-space. In a similar fashion, $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2,\mathbb{C})$. Specifically, a given matrix acts on $\mathbb{H}^3$ via Möbius transformations, where we interpret division as multiplication by the inverse in the quaternions:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(z) = (\alpha z + \beta)(\gamma z + \delta)^{-1}$$

($z^{-1} = \bar{z}/|z|^2$). If interested, one can find proofs of these facts online. (See, e.g., Holodyn 2013 for an outline).

To get a geometric feel for these transformations, it is instructive to consider the *frame bundle* $\mathrm{F}\mathbb{H}^3$ of hyperbolic space. This is the set

$$\mathrm{F}\mathbb{H}^3 = \{(z, v, w) \in U^3 \times \mathbb{R}^3 \times \mathbb{R}^3 : ||(v, w)||_z = 1,\ v \perp w\}$$

where the norm $||(\cdot,\cdot)||_z$ is derived from the hyperbolic metric. **(ASK AK FOR DETAILS HERE)!** One can show that $\mathrm{PSL}(2,\mathbb{C})$ acts simply transitively on $\mathrm{F}\mathbb{H}^3$, where the action is by

$$g(z, v, w) = (g(z), \mathbf{?}, \mathbf{?}) \qquad \text{Maybe } (g(z), g'(z)v, g'(z)w)\text{? But need to think about derivative...}$$

(ASK AK FOR DETAILS HERE)! Thus, we can identify $\mathrm{PSL}(2,\mathbb{C})$ with $\mathrm{F}\mathbb{H}^3$ by associating a matrix g with the point $g(j,j,1)$. Intuitively, one thinks of z as the base point in upper half-space, v as a tangent vector pointing in the direction of a geodesic, and w as a "frame vector" which is perpendicular to the tangent vector. (Recall that geodesics in $\mathbb{H}^3$ are semicircles or vertical lines which are perpendicular to the x_1x_2-plane).

We now describe the geometry of the matrix action by describing how it moves the frame bundle. First, recall the decomposition $\mathrm{SL}(2,\mathbb{C}) = NAK$ where

$$N = \left\{ \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} : u \in \mathbb{C} \right\} \qquad A = \left\{ \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\} \qquad K = M\mathrm{SO}(2)M$$

where $\mathrm{SO}(2)$ is the special orthogonal group and

$$M = \left\{ \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

Matrices from these three groups move the frame bundle in fairly simple ways.

- A matrix in N moves the point $(j,j,1)$ to $(j+u,j,1)$; that is, the base point is shifted by $\mathrm{Re}(u)$ in the x_1 direction and by $\mathrm{Im}(u)$ in the x_2 direction. In other words, N moves points along the horosphere determined by the base point (horospheres in $\mathbb{H}^3$ are planes parallel to the x_1x_2-plane or spheres which are tangent to the x_1x_2-plane); the real part of u determines the translation in the direction of the tangent vector, and the imaginary part determines the translation in the direction of the frame vector.
- Just as in hyperbolic 2-space, a matrix in A moves points along geodesics at unit speed for time t.
- Matrices in K fix the base point but can rotate the frame bundle to any position. Specifically, matrices in M rotate the frame vector while leaving the tangent vector fixed, while matrices in $\mathrm{SO}(2)$ have the opposite effect.

The Action of Hyperbolic Elements