Scaling the Hypergeometric Function

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Recall that the hypergeometric function ${}_2F_1$ is defined

$$ {}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} $$

where $(q)_n$ is the Pochhammer symbol

$$ (q)_n = \begin{cases} 1 & n = 0 \\ q(q+1)\cdots(q+n-1) & n \neq 0 \end{cases} $$

From this definition, we see that if $z \sim 0$, then ${}_2F_1(a,b,c;z) \sim 1$.

For our application, we will be using $z = \rho^2 \sim 1$. To get an approximate size for the hypergeometric function near $z = 1$, we use the transformation formula

$$ {}_2F_1(a,b,c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a,b,a+b+1-c;1-z) $$

$$ + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a,c-b,1+c-a;1-z) $$

Since ${}_2F_1(a,b,c;z) \sim 1$ when $z \sim 0$, this formula implies that

$$ {}_2F_1(a,b,c;z) \sim \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} \qquad \text{when } z \sim 1 \tag{1} $$

Now let's move to our application. Recall that we are interested in the function

$$ {}_2F_1(s, s+|n|, 1+|n|; \rho^2) $$

where $s = 1/2 - \nu$ (given that $\lambda = 1/4 - \nu^2$ is the eigenvalue of a Maass form), $n \in \mathbb{Z}$ is bounded by $|n| \leq M$ (M is how far we go out in the finite Fourier series approximation), and ρ is the magnitude of z (which we take to be near 1). If we plug these values into Equation 1, we get

$$ {}_2F_1(s, s+|n|, 1+|n|; \rho^2) \sim \frac{\Gamma(1+|n|)\Gamma(1-2s)}{\Gamma(1+|n|-s)\Gamma(1-s)} + \frac{\Gamma(1+|n|)\Gamma(2s-1)}{\Gamma(s)\Gamma(s+|n|)} (1-\rho^2)^{1-2s} \tag{2} $$

for $\rho \sim 1$. Next, we use Stirling's formula. This is stated in Iwaniec-Kowalski as follows:

$$ \Gamma(s) = \left(\frac{2\pi}{s}\right)^{1/2} \left(\frac{s}{e}\right)^s \left(1 + O\left(\frac{1}{|s|}\right)\right) $$

This is valid in the sector $|\arg s| \leq \pi - \epsilon$ for any $\epsilon > 0$ (the implied constant depends on ϵ). Applying Stirling's formula to Equation 2 gives

$$ {}_2F_1(s, s+|n|, 1+|n|; \rho^2) \sim \frac{(1+|n|)^{\frac{1}{2}+|n|}(1-2s)^{\frac{1}{2}-2s}}{(1+|n|-s)^{\frac{1}{2}+|n|-s}(1-s)^{\frac{1}{2}-s}} + \frac{(1+|n|)^{\frac{1}{2}+|n|}(2s-1)^{2s-\frac{3}{2}}}{s^{s-\frac{1}{2}}(s+|n|)^{s+|n|-\frac{1}{2}}} (1-\rho^2)^{1-2s} \tag{3} $$

Write code to test accuracy of this estimate!!!

To finish, we wish to apply Equation 3 to our "disk version" of Hejhal's algorithm.