NOTES ON FOURIER EXPANSIONS ETC

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Contents

1. A Fourier expansion in a wedge

In this note we consider a special Fourier expansion of eigenfunctions of the Laplace-Beltrami operator in the hyperbolic upper half space $\mathbb{H}^3 = \{(x_1, x_2, y) \,:\, y > 0\}$. Precisely (stated in a normalized setting) we expand a function $f : \mathbb{H}^3 \to \mathbb{C}$ in the coordinates

$$x = x_1 \in \mathbb{R}, \qquad \rho = \sqrt{x_2^2 + y^2} > 0, \qquad \theta = \arg(x_2 + iy) \in (0, \pi).$$

These are exactly the coordinates which Lax-Phillips consider in [11, proof of Thm 4.8]; however our treatment differs from theirs in that we *assume that f is invariant under the parabolic map $S : (x_1, x_2, y) \mapsto (x_1 + 1, x_2, y)$*. (In the x, ρ, θ-coordinates: $S : (x, \rho, \theta) \mapsto (x + 1, \rho, \theta)$.) We expect that our Fourier expansion will work well (and lead to good control) in any "wedge" region $\{(x, \rho, \theta) \,:\, \theta \le \theta_0\}$ such that f is L^2 in

$$\{(x, \rho, \theta) \,:\, \theta \le \theta_0\}/[S] = \{(x, \rho, \theta) \,:\, x \in \mathbb{R}/\mathbb{Z},\ \rho > 0,\ 0 < \theta \le \theta_0\}.$$

Possibly we may need to here also assume "θ_0 sufficiently small", but I don't think so; actually I expect also $\theta_0 \in (\frac{\pi}{2}, \pi)$ should be ok (i.e. an "obtuse angled wedge").

(By contrast, Lax-Phillips' discussion in [11, proof of Thm 4.8] leads to good control if we assume f is L^2 in the *whole* wedge $\{(x, \rho, \theta) \,:\, \theta \le \theta_0\}$, which of course the S-invariant functions which we consider *cannot* be, unless they vanish identically.)

2. Is the expansion useful for the Apollonian group?

We normalize the Apollonian group to be the discrete subgroup $\Gamma := \langle s_1, s_2, s_3, s_4 \rangle \subset \mathrm{Isom}(\mathbb{H}^3)$, where s_1, s_2, s_3, s_4 are the inversions in the four mutually tangent circles

$$c_1 = \{\mathrm{Re}\ z = 1\};\ c_2 = \{\mathrm{Re}\ z = -1\};\ c_3 = \{|z - i| = 1\};\ c_4 = \{|z + i| = 1\}.$$

(Thus I use the same notation as Rikard in his 20100909 email; in particular I write $z = x_1 + ix_2 \in \mathbb{C}$ for points in $\partial\mathbb{H}^3$.) Explicitly we have, *for the action on $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$:*

$$s_1(z) = 2 - \bar{z}; \qquad s_2 = -2 - \bar{z}; \qquad s_3 = i + \frac{1}{\bar{z} + i}; \qquad s_4 = -i + \frac{1}{\bar{z} - i},$$

and the "standard" choice of fundamental domain for $\Gamma\backslash\mathbb{H}^3$ is

(2.1)
$$F := \Big\{(x_1, x_2, y) \in \mathbb{H}^3 \;:\; -1 < x_1 < 1,\; x_1^2 + (x_2-1)^2 + y^2 > 1,\; x_1^2 + (x_2+1)^2 + y^2 > 1\Big\}.$$

Note that Γ contains the parabolic map $s_1 s_2 : (x_1, x_2, y) \mapsto (x_1 + 4, x_2, y)$; hence the Fourier expansion described in the previous section can be applied to any wedge region

(2.2)
$$W_{h,\theta_0} := \big\{(x_1, x_2, y) \in \mathbb{H}^3 \;:\; x_1 \in \mathbb{R}/4\mathbb{Z},\; 0 < \arg(x_2 - h + iy) \leq \theta_0\big\} \qquad (h \in \mathbb{R},\; \theta_0 \in (0, \pi)),$$

for any h, θ_0 such that our sought-for eigenfunction f can be proved to be L^2 in W_{h,θ_0}. In other words: For any h, θ_0 such that W_{h,θ_0} intersects only finitely many Γ-translates of F. Now we know that the limit set $\Lambda(\Gamma)$ is contained in $\{-1 \leq \operatorname{Im} z \leq 1\}$; hence (thinking about the geometry...) any choice of $h > 1$ should be ok, with $\theta_0 = \frac{\pi}{2}$!!

Of course a similar construction can be done for each of the six pairs of circles c_j, c_k ($1 \leq j < k \leq 4$) and the corresponding parabolic element $s_j s_k \in \Gamma$. The interesting question is: *Can the resulting six wedge regions be made to together cover all of F??* If necessary, we may allow to also add the regions associated to the "parabolic rank one Fourier expansions" considered previously by Rikard...

I find it a bit difficult to visualize these six wedge regions and their unions, and to simplify things I will utilize the fact that "all six cusps of $\Gamma\backslash\mathbb{H}^3$ look exactly the same"; i.e. for any pair of cusps there is some isometry of $\Gamma\backslash\mathbb{H}^3$ onto itself which takes the first cusp to the second. Of course any such isometry must come from the normalizer of Γ in $\operatorname{Isom}(\mathbb{H}^3)$. Note that the Patterson-Sullivan eigenfunction is *invariant under this normalizer!* (Since the eigenvalue is simple.) And also when looking for larger eigenvalues we may utilize the extra symmetries coming from the normalizer.

3. The normalizer of Γ in $\operatorname{Isom}(\mathbb{H}^3)$

I am sure that the material in this section (like that in most other sections of this note) is "well-known" although at present I don't know exactly where to find it... (Cf. Remark 3.2 below for a connection with [14, Ch. 7]. I should also dig deeper in e.g. [4, pdf] and [5, pdf]...)

Given any permutation σ of $\{1, 2, 3, 4\}$ there is a unique map $T_\sigma \in \operatorname{Isom}(\mathbb{H}^3)$ which maps c_j to $c_{\sigma(j)}$ for each j:

$$T_\sigma(c_j) = c_{\sigma(j)}, \qquad \forall j \in \{1, 2, 3, 4\}. \tag{3.1}$$

[Proof: To prove the existence of T_σ it suffices to treat the case when σ is a transposition, since these generate the permutation group $\mathfrak{S}_4$. Thus assume $\sigma = (j, k)$ for some $1 \leq j < k \leq 4$. Now one checks (by a case by case analysis, or by in one way or other noticing that it actually suffices to consider the case $j = 1$, $k = 2$) that there exists a unique 'circle' $c \subset \mathbb{C} \cup \{\infty\}$ which is tangent to c_j and c_k at their point of tangency, and which is orthogonal to c_ℓ for $\ell \in \{1, 2, 3, 4\} \setminus \{j, k\}$. We define $T = T_\sigma$ to be inversion in this circle σ. (This defines the action of T on $\mathbb{C} \cup \{\infty\}$; of course the action of T on $\mathbb{H}^3$ is defined to be the inversion in the hemisphere based on c.) Clearly this map T_σ satisfies (3.1).

Next, to prove the uniqueness of T_σ (for general $\sigma \in \mathfrak{S}_4$) it suffices to prove that if $T \in \operatorname{Isom}(\mathbb{H}^3)$ satisfies $T(c_j) = c_j$ for $j \in \{1, 2, 3, 4\}$, then T is the identity map. But note that for any $1 \leq j < k \leq 4$, the conditions force the point of tangency between c_j and c_k to be a *fix-point* of T. Hence $T(z) = z$ in fact holds for each of the six points $z \in \{\infty, 0, 1+i, 1-i, -1+i, -1-i\}$! If T is orientation preserving then this forces T to be the identity map (indeed already *three* fixpoints would suffice to get this); hence it only remains to consider the case when T is orientation reversing, i.e. $T(z) = \frac{a\overline{z}+b}{c\overline{z}+d}$ for some $a, b, c, d \in \mathbb{C}$ ($ad - bc \neq 0$). Now $T(\infty) = \infty$ forces $c = 0$; thus we may also assume $d = 1$; and now $T(0) = 0$ forces $b = 0$; thus $f(z) = a\overline{z}$ for some $a \in \mathbb{C} \setminus \{0\}$, i.e. f is a reflection in a line through 0, followed by a scaling by a factor $\delta = |a| > 0$. Clearly such a map cannot have all four points $1+i, 1-i, -1+i, -1-i$ as fix-points, and we are done.]

It follows that the map

$$\mathfrak{S}_4 \ni \sigma \mapsto T_\sigma \in \mathrm{Isom}(\mathbb{H}^3)$$

is a group isometry.

Next, since s_j denotes inversion in c_j, it follows that $T_\sigma s_j T_\sigma^{-1}$ is inversion in $c_{\sigma(j)}$, [1] viz:

$$T_\sigma s_j T_\sigma^{-1} = s_{\sigma(j)}, \qquad \forall \sigma \in \mathfrak{S}_4,\ j \in \{1,2,3,4\}. \tag{3.2}$$

This implies that each map T_σ lies in the normalizer of Γ:

$$T_\sigma \in \Gamma_0 := N_{\mathrm{Isom}(\mathbb{H}^3)}(\Gamma) \qquad \forall \sigma \in \mathfrak{S}_4.$$

Now it turns out that all the maps T_σ have the wonderfully convenient property of preserving our fundamental domain F (which we defined in (2.1)):

Lemma 3.1.

$$T_\sigma(F) = F, \qquad \forall \sigma \in \mathfrak{S}_4.$$

Proof. (The following beautiful proof was suggested by Rikard 20101119.) F is the unique open connected subset of $\mathbb{C} \cup \{\infty\}$ which has boundary $= c_1 \cup c_2 \cup c_3 \cup c_4$. But T_σ is a homeomorphism of $\mathbb{C} \cup \{\infty\}$ mapping the set $c_1 \cup c_2 \cup c_3 \cup c_4$ onto itself; hence also $T_\sigma(F)$ is an open connected subset of $\mathbb{C} \cup \{\infty\}$ which has boundary $= c_1 \cup c_2 \cup c_3 \cup c_4$. This implies $T_\sigma(F) = F$.

My original, much longer proof: Since $\mathfrak{S}_4$ is generated by the three transpositions (12), (23) and (34), it suffices to prove the lemma in the three cases $\sigma = (12)$, $\sigma = (23)$, $\sigma = (34)$. Now by inspection we see that $T_{(12)}$ acts on $\mathbb{C} \cup \{\infty\}$ as $z \mapsto -\overline{z}$; hence its action on $\mathbb{H}^3$ is reflection in the plane $x_1 = 0$:

$$T_{(12)}(x_1, x_2, y) = (-x_1, x_2, y).$$

Hence $T_{(12)}(F) = F$. Similarly $T_{(34)}$ acts on $\mathbb{C} \cup \{\infty\}$ as $z \mapsto \overline{z}$; hence

$$T_{(34)}(x_1, x_2, y) = (x_1, -x_2, y),$$

and thus $T_{(34)}(F) = F$. Finally we see that $T_{(23)}$ acts on $\mathbb{C} \cup \{\infty\}$ by inversion in the circle $\{|z - (1+i)| = 2\}$; hence $T_{(23)}$ acts on $\mathbb{H}^3$ by inversion in the Euclidean hemisphere H with center at $(x_1, x_2, y) = (1, 1, 0)$ and radius 2. It follows that $T_{(23)}$ maps $\mathbb{H}^3$-half spaces as follows:

$$\begin{aligned}
&T_{(23)}\big(\{x_1^2 + (x_2 - 1)^2 + y^2 > 1\}\big) = \{x_1 > -1\};\\
&T_{(23)}\big(\{x_1 > -1\}\big) = \{x_1^2 + (x_2 - 1)^2 + y^2 > 1\};\\
&T_{(23)}\big(\{x_1^2 + (x_2 + 1)^2 + y^2 > 1\}\big) = \{x_1^2 + (x_2 + 1)^2 + y^2 > 1\};\\
&T_{(23)}\big(\{x_1 < 1\}\big) = \{x_1 < 1\}.
\end{aligned}$$

By taking the intersection of these four half spaces we conclude $T_{(23)}(F) = F$, and we are done. □

Before going on I here record the explicit form of all T_σ (the action on $\mathbb{C} \cup \{\infty\}$) with σ a transposition. From these it is easy to compute T_σ for any σ.

$$\begin{aligned}
&T_{(12)}(z) = -\overline{z} && = \big[\text{inversion in } c_{(12)} = \{\mathrm{Re}\, z = 0\}\big]\\
&T_{(13)}(z) = \frac{4}{\overline{z} + 1 + i} - 1 + i && = \big[\text{inversion in } c_{(13)} = \{|z - (-1+i))| = 2\}\big]\\
&T_{(14)}(z) = \frac{4}{\overline{z} + 1 - i} - 1 - i && = \big[\text{inversion in } c_{(14)} = \{|z - (-1-i)| = 2\}\big]\\
&T_{(23)}(z) = \frac{4}{\overline{z} - 1 + i} + 1 + i && = \big[\text{inversion in } c_{(23)} = \{|z - (1+i)| = 2\}\big]\\
&T_{(24)}(z) = \frac{4}{\overline{z} - 1 - i} + 1 - i && = \big[\text{inversion in } c_{(24)} = \{|z - (1-i)| = 2\}\big]\\
&T_{(34)}(z) = \overline{z} && = \big[\text{inversion in } c_{(34)} = \{\mathrm{Im}\, z = 0\}\big].
\end{aligned} \tag{3.3}$$

[1] I suppose this is completely obvious, but here is a pedantic proof: $T_\sigma s_j T_\sigma^{-1}$ is an orientation reversing map, and for every $z \in c_{\sigma(j)}$ we have $T_\sigma^{-1}(z) \in c_j$, thus $s_j T_\sigma^{-1}(z) = T_\sigma^{-1}(z)$ and $T_\sigma s_j T_\sigma^{-1}(z) = z$. Hence it remains to prove that any orientation reversing "Möbius transformation" which preserves a certain circle c pointwise, must in fact be inversion in c. By a conjugation it suffices to treat the case $c = \{\mathrm{Im}\ z = 0\}$, and now the claim follows by a direct computation.

Lemma 3.2. *The maps T_σ together generate $N_{Isom(\mathbb{H}^3)}(\Gamma)$ over Γ, and in fact we have*

$$\Gamma_0 = N_{Isom(\mathbb{H}^3)}(\Gamma) = \bigsqcup_{\sigma\in\mathfrak{S}_4} \Gamma T_\sigma \tag{3.4}$$

(disjoint union); thus $[N_{Isom(\mathbb{H}^3)}(\Gamma) : \Gamma] = 24$.

Proof. To prove that the union on the right is disjoint, viz. that $\Gamma T_\sigma \neq \Gamma T_\tau$ for all $\sigma \neq \tau \in \mathfrak{S}_4$, it suffices to prove that $T_\sigma \notin \Gamma$ for all $\sigma \in \mathfrak{S}_4 \setminus \{I\}$. However this follows from Lemma 3.2. Next, since each T_σ normalizes Γ we have $\Gamma T_\sigma = T_\sigma \Gamma$ and $\Gamma T_\sigma \Gamma T_\tau = \Gamma T_\sigma T_\tau = \Gamma T_{\sigma\tau}$ for all $\sigma, \tau \in \mathfrak{S}_4$; hence the union on the right hand side in (3.4) is a group. It now follows that the right hand side of (3.4) is a supergroup of Γ of index 24. It remains to prove that $N_{\mathrm{Isom}(\mathbb{H}^3)}(\Gamma)$ has no element outside $\cup_\sigma \Gamma T_\sigma$. *** I BELIEVE I SEE HOW TO PROVE IT BUT I DON'T TAKE THE TIME NOW TO WRITE IT OUT (if I am wrong then all of the following discussion will work anyway, with "$\Gamma_0 = \cup_\sigma \Gamma T_\sigma \subsetneq N_{\mathrm{Isom}(\mathbb{H}^3)}(\Gamma)$". *** □

Now if $F' \subset F$ is a fundamental domain for the action of the group $\{T_\sigma\}$ on F, then F is partitioned into the 24 subsets $T_\sigma F'$ ($\sigma \in \mathfrak{S}_4$) [2] and F' is a fundamental domain for $\Gamma_0 \backslash \mathbb{H}^3$!

A nice choice of F' is as follows:

$$F' := \Big\{ P \in F \,:\, x_1(P) > 0,\ x_2(P) > 0,\ y(P) > y(T_\sigma(P)),\ \forall \sigma \in \mathfrak{S}_4 \setminus S \Big\}, \tag{3.5}$$

where

$$S := \{I, (12), (34), (12)(34)\} \subset \mathfrak{S}_4.$$

Lemma 3.3. *The set F' in* (3.5) *is a fundamental domain for the action of $\{T_\sigma \,:\, \sigma \in \mathfrak{S}_4\}$ on F.*

Proof. First assume $T_\tau(F') \cap F' \neq \emptyset$ for some fixed $\tau \in \mathfrak{S}_4 \setminus \{I\}$, and let $P = (x_1, x_2, y) \in T_\tau(F') \cap F'$. Then $x_1, x_2 > 0$. Recall that

$$T_{(12)}(x_1, x_2, y) = (-x_1, x_2, y);$$
$$T_{(34)}(x_1, x_2, y) = (x_1, -x_2, y);$$
$$T_{(12)(34)}(x_1, x_2, y) = (-x_1, -x_2, y);$$

hence if $\tau \in S \setminus \{I\}$ then either $x_1(T_\tau^{-1}(P)) = -x_1 < 0$ or $x_2(T_\tau^{-1}(P)) = -x_2 < 0$ (or both), and this contradicts $T_\tau^{-1}(P) \in F'$. Hence $\tau \in \mathfrak{S}_4 \setminus S$ must hold. Hence also $\tau^{-1} \notin S$, and now $P \in F'$ implies $y(P) > y(T_{\tau^{-1}}(P)) = y(T_\tau^{-1}(P))$, while $T_\tau^{-1}(P) \in F'$ implies $y(T_\tau^{-1}(P)) > y(T_\tau(T_\tau^{-1}(P))) = y(P)$, and this is a contradiction. Hence our initial assumption must be false, i.e. we have proved that $T_\tau(F') \cap F' = \emptyset$ for all $\tau \in \mathfrak{S}_4 \setminus \{I\}$. Hence also $T_\sigma(F') \cap T_\tau(F') = \emptyset$ whenever $\sigma \neq \tau \in \mathfrak{S}_4$, i.e. the 24 regions $T_\sigma(F')$ ($\sigma \in \mathfrak{S}_4$) are pairwise disjoint.

It remains to prove that the images of $\overline{F'} \cap F$ cover all of F, i.e. that for any given point $P \in F$ there exists some $\sigma \in \mathfrak{S}_4$ such that $T_\sigma(P) \in \overline{F'}$. Thus let $P \in F$ be given. By replacing P with $T_\tau(P)$ for some appropriate $\tau \in \mathfrak{S}_4$ we may assume that $y(P) \geq y(T_\sigma(P))$ for all $\sigma \in \mathfrak{S}_4$. Next, by replacing (our new) P with $T_\sigma(P)$ for some appropriate $\sigma \in S$ (note that any such replacement keeps $y(P)$ fixed), we may *also* assume $x_1(P) \geq 0$ and $x_2(P) \geq 0$. Hence we are "done"! [To be extremely pedantic – and probably the following can be almost completely reduced away, but we anyway need to make things explicit: To complete our proof it remains to prove that

$$\overline{F'} \supset \Big\{ P \in F \,:\, x_1(P) \geq 0,\ x_2(P) \geq 0,\ y(P) \geq y(T_\sigma(P)),\ \forall \sigma \in \mathfrak{S}_4 \setminus S \Big\}. \tag{3.6}$$

(Here $\overline{F'}$ is the closure of F' inside $\mathbb{H}^3$.) Note that

$$S = \{\sigma \in \mathfrak{S}_4 \,:\, T_\sigma(\infty) = \infty\}. \tag{3.7}$$

(Proof: If $T_\sigma(\infty) = \infty$ then using (3.1) and $\infty \notin c_1$, $\infty \notin c_2$, $\infty \in c_3$, $\infty \in c_4$ it follows that $\sigma(3), \sigma(4) \in \{3, 4\}$; hence $\sigma \in S$. The converse is obvious, using $T_{(12)}(z) = -\overline{z}$ and

[2] as usual, the more precise statement is that $\overline{F} = \cup_\sigma \overline{T_\sigma F'}$ and the sets $T_\sigma F'$ have pairwise disjoint interiors.

$T_{(34)}(z) = \overline{z}$.) Now consider any $\sigma \in \mathfrak{S}_4 \setminus S$; then there is $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C})$ such that either $T_\sigma(z) \equiv \frac{az+b}{cz+d}$ or $T_\sigma(z) \equiv \overline{(\frac{az+b}{cz+d})}$, and here $c \neq 0$, because of (3.7). Hence, writing $P = (x_1, x_2, y) \in \mathbb{H}^3$ and $z = x_1 + ix_2$ we have the following equivalences (cf. [1, p.3(1.10)], and note that if $T_\sigma(z) \equiv \overline{(\frac{az+b}{cz+d})}$ then the action of T_σ on $\mathbb{H}^3$ is given by first acting with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C})$ and then applying $(x_1, x_2, y) \mapsto (x_1, -x_2, y)$.

$$(3.8) \qquad y(P) > y(T_\sigma(P)) \Longleftrightarrow |cz+d|^2 + |c^2|y^2 > 1 \Longleftrightarrow \Big|z + \frac{d}{c}\Big| + y^2 > |c|^{-2},$$

and the last condition says that (x_1, x_2, y) lies outside the Euclidean ball with center $-d/c \in \mathbb{C} \subset \partial\mathbb{H}^3$ and radius $|c|^{-1}$. Similarly $y(P) \geq y(T_\sigma(P)) \Leftrightarrow |z + d/c| + y^2 \geq |c|^{-2}$. Hence we conclude that if $P = (x_1, x_2, y) \in \mathbb{H}^3$ satisfies $y(P) \geq y(T_\sigma(P))$ then for *for every $\varepsilon > 0$, the point $P' = (x_1, x_2, y + \varepsilon)$ satisfies $y(P') > y(T_\sigma(P'))$!* It now follows that if $P = (x_1, x_2, y)$ is an arbitrary point in the set in the right hand side of (3.6), then for every $\varepsilon > 0$ we have that for every sufficiently small $\varepsilon' > 0$, the point $P' = (x_1 + \varepsilon', x_2 + \varepsilon', y)$ lies in F', and this implies that P lies in $\overline{F'}$. Hence we have proved that (3.6) holds.] $\square$

In fact we can determine F' explicitly. I have a simple Maple program which goes through all the 20 elements $T_\sigma \in \mathfrak{S}_4 \setminus S$ and compute the corresponding Euclidean ball (cf. (3.8) above), and it turns out that the only ball complements appearing are

$$R_0 = \big\{x_1^2 + x_2^2 + y^2 > 2\big\}$$

and

$$R_{(\alpha_1,\alpha_2)} = \big\{(x_1 - \alpha_1)^2 + (x_2 - \alpha_2)^2 + y^2 > 4\big\} \quad \text{for } (\alpha_1, \alpha_2) \in A := \{(1,1),\ (1,-1),\ (-1,1),\ (-1,-1)\}.$$

Here note that

$$\{x_1 > 0, x_2 > 0\} \cap \Big(\bigcap_{(\alpha_1,\alpha_2) \in A} R_{(\alpha_1,\alpha_2)} \Big) \cap R_0 = \{x_1 > 0, x_2 > 0\} \cap R_{(1,1)}.$$

(Proof: If (x_1, x_2, y) belongs to the set in the right hand side then $x_1, x_2 > 0$ and hence for any $(\alpha_1, \alpha_2) \in A$ we have $(x_1 - \alpha_1)^2 + (x_2 - \alpha_2)^2 + y^2 \geq (x_1 - 1)^2 + (x_2 - 1)^2 + y^2 > 4$, since $(x_1, x_2, y) \in R_{(1,1)}$. Hence $(x_1, x_2, y) \in R_{(\alpha_1,\alpha_2)}$ for all $(\alpha_1, \alpha_2) \in A$. Furthermore $(x_1 - 1)^2 + (x_2 - 1)^2 + y^2 > 4$ implies $x_1^2 + x_2^2 + y^2 > 2 + 2x_1 + 2x_2 > 2$, and thus $(x_1, x_2, y) \in R_0$. Hence (x_1, x_2, y) also belongs to the set in the left hand side, and we are done.)

Hence we conclude:

$$F' = \Big\{(x_1, x_2, y) \in F \,:\, x_1 > 0,\ x_2 > 0,\ (x_1 - 1)^2 + (x_2 - 1)^2 + y^2 > 4\Big\}.$$

Recalling also the definition of F, (2.1), and noticing that if $x_1 > 0$ then $(x_1 - 1)^2 + (x_2 - 1)^2 + y^2 > 4 \Rightarrow x_1^2 + (x_2 - 1)^2 + y^2 > 3 + 2x_1 > 3 > 1$, we finally conclude:

$$(3.9) \qquad F' = \Big\{(x_1, x_2, y) \in F \,:\, 0 < x_1 < 1,\ x_2 > 0,\ (x_1 - 1)^2 + (x_2 - 1)^2 + y^2 > 4\Big\}.$$

Remark 3.1. Note that F' has four sides; namely parts of the four hyperbolic hyperplanes $\{x_1 = 0\}$, $\{x_1 = 1\}$, $\{x_2 = 0\}$ and $\{(x_1 - 1)^2 + (x_2 - 1)^2 + y^2 = 4\}$. Hence, since F' is a fundamental domain for $\Gamma_0 \backslash \mathbb{H}^3$, *** it surely follows from Poincaré's theorem (in the generality allowing orientation reversing isometries) that Γ_0 is generated by the four inversions in these hyperplanes, i.e. generated by the four maps $T_{(12)}$, s_1, $T_{(34)}$ and $T_{(23)}$; and also it should be easy to read off exactly which relations these generators satisfy. I do not check the relations now, but just note that *indeed,* $\Gamma_0 = \langle T_{(12)}, s_1, T_{(34)}, T_{(23)} \rangle$. [Direct proof: Since (12), (34), (23) generate $\mathfrak{S}_4$, the maps $T_{(12)}, T_{(34)}, T_{(23)}$ generate $\{T_\sigma \,:\, \sigma \in \mathfrak{S}_4\}$. Also $T_{(1j)} s_1 T_{(1j)} = s_j$ for $j = 2, 3, 4$ (cf. (3.2)) hence s_1 together with $\{T_\sigma \,:\, \sigma \in \mathfrak{S}_4\}$ generate all s_2, s_3, s_4. Hence $\langle T_{(12)}, s_1, T_{(34)}, T_{(23)} \rangle = \langle s_1, s_2, s_3, s_4, T_\sigma \,:\, \sigma \in \mathfrak{S}_4 \rangle = \Gamma_0$, as claimed.]

Remark 3.2. In the last remark we proved $\Gamma_0 = \langle T_{(12)}, T_{(23)}, T_{(34)}, s_1\rangle$. Let us now note that *the subgroup* $\langle T_{(12)}, T_{(23)}, T_{(34)}, s_1\rangle$ *is conjugate to the modular group* $\mathrm{PSL}(2,\mathbb{Z})$ *extended with reflection* $z \mapsto -\overline{z}$*!* Geometrically this is clear by looking at the "bottom" of F', viz. the region

$$F'_0 = \{z \in \mathbb{C} \,:\, 0 < \mathrm{Re}\, z < 1,\ |z - (1+i)| > 2\}$$

and how the translates of F'_0 under $\langle T_{(12)}, T_{(23)}, T_{(34)}, s_1\rangle$ tesselate all of $\{\mathrm{Im}\, z > 1\} \subset \mathbb{C}$. If we transform F'_0 by the map $z \mapsto \frac{1}{2}(z-(1+i))$ (and conjugate the group $\langle T_{(12)}, s_1, T_{(23)}\rangle$ correspondingly) we get back exactly the tesselation of the upper half plane $\{\mathrm{Im}\, z > 0\}$ by the images of the triangle $\{z \in \mathbb{C} \,:\, -\frac{1}{2} < \mathrm{Re}\, z < 0,\ |z| > 1\}$ under the modular group $\mathrm{PSL}(2,\mathbb{Z})$ extended with the reflection $z \mapsto -\overline{z}$.

Surely this must correspond to the appearance of the modular group in [14, pp. 206–]! (*** Check this more explicitly!)

4. Conclusions

Note that we may cover F' in (3.9) by one wedge W_{h,θ_0} as in (2.2) with an appropriate $h > 1$ and θ_0 slightly larger than $\frac{\pi}{2} + \arctan(2^{-1/2})$!

[Let us compute the exact θ_0 needed to cover F' for a given $h > 1$! The "lowest" point in $\{x_2 = 0\} \cap \{(x_1-1)^2 + (x_2-1)^2 + y^2 \geq 4\}$ is at $x_1 = 0$; thus $1 + 1 + y^2 \geq 4$, viz. $y \geq \sqrt{2}$. We need our plane to get below this; hence we need $\theta_0 \geq \frac{\pi}{2} + \arctan(h/\sqrt{2})$.]

As another example, note that we may also cover F' in (3.9) by one wedge W_{h,θ_0} as in (2.2) with an appropriate $h > 1$ and $\theta_0 = \frac{\pi}{2}$, together with the region

$$\{(x_0, x_1, y) \in \mathbb{H}^3 \,:\, 0 < x_1 < 1, y > y_0\}/[T_{(34)}] \tag{4.1}$$

with an appropriate (not too small) $y_0 > 0$. Since the latter region is a region attached to the "parabolic rank one Fourier expansion" for the parabolic element $T_{(12)}s_1$ (using also the $T_{(12)}$-invariance), it now seems *very hopeful that this should suffice to determine the Patterson-Sullivan eigenfunction!* In fact, if we are indeed able to work with $\theta_0 = \frac{\pi}{2}$ then W_{h,θ_0} covers *all* of $\{x_2 > h\}$, and hence all of (4.1) outside of $\{0 < x_1 < 1,\ 0 < x_2 \leq h,\ y > y_0\}$; this huge overlap makes it hopeful that we should be able to easily get good control on the rank one parabolic expansion despite its lack of "natural discreteness"!

5. Carry over Lax-Phillips discussion to an $[x \mapsto x+1]$-invariant setting?

The remainder of this note is concerned with working out the expansion which we "announced" above in Section 1. Recall the normalized setting in Section 1. Also recall that Lax-Phillips, [11, proof of Thm 4.8], work by identifying each submanifold $\{\theta = \text{fixed}\} \subset \mathbb{H}^3$ with a copy of the hyperbolic upper half plane $\mathbb{H}^2$, and using decomposition with respect to the Laplace-Beltrami operator in $\mathbb{H}^2$.

In our setting with invariance under $(x_1, x_2, y) \mapsto (x_1+1, x_2, y)$, the task will be to spectrally decompose an "arbitrary" function $v : \mathbb{H}^2 \to \mathbb{C}$ which is invariant under $z \mapsto z+1$. Note that the "non-Euclidean Fourier transform" (explained in Helgason's book, [6, Thm 4.2]) is *not* natural for this task. Instead, it seems natural to seek a spectral expansion

$$v(x+iy) = \int_0^\infty \sum_{n \neq 0} c_n(t) K_{it}(2\pi|n|y) e(nx)\, dt,$$

or similar, where we wish to determine the functions $c_n(t)$.

This is clearly very closely related to the following result in Iwaniec's book: [7, Prop. 1.4]. (I don't seem to have any previous notes about that result; see my notes; nor is there anything in Dennis' notes.)

6. Working through the details of proof of Iwaniec's Prop 1.4 – CORRECTED!

(I have to double check the computation below, but if it is correct it shows that Iwaniec's Prop 1.4 must be corrected by replacing $W_s(rz)$ by $W_s(-rz)$ in *either* (1.29) or (1.30) – but NOT in both...)

In the following we work a bit "formally", i.e. not checking exact conditions of convergence. However it seems clear that everything should indeed work for $f \in C_0^\infty(\mathbb{H}^2)$ as Iwaniec has in his statement.

Recall that ([7, (1.26),(1.27)])

$$W_s(x+iy) := 2\sqrt{|y|}K_{s-\frac{1}{2}}(2\pi|y|)e(x).$$

Hence the right hand side of [7, Prop. 1.4] can be written out explicitly as follows (writing $s = \frac{1}{2} + it$, $z = x + iy$ and $w = u + iv$):

$$\frac{1}{2\pi i}\int_{(1/2)}\int_{\mathbb{R}} W_s(rz)f_s(r)\gamma_s(r)\,dr\,ds$$
$$= \frac{1}{2\pi i}\int_{(1/2)}\int_{\mathbb{R}} W_s(rz)\Big(\int_{\mathbb{H}^2} f(w)W_s(rw)\,d\mu(w)\Big)(2\pi|r|)^{-1}t\sinh(\pi t)\,dr\,ds$$
$$= \frac{2}{\pi i}\int_{(1/2)}\int_{\mathbb{R}} \sqrt{|r|y}K_{s-\frac{1}{2}}(2\pi|r|y)e(rx)\Big(\int_0^\infty\int_{\mathbb{R}} f(u+iv)\sqrt{|r|v}K_{s-\frac{1}{2}}(2\pi|r|v)e(ru)\,du\,\frac{dv}{v^2}\Big)(2\pi|r|)^{-1}t\sinh(\pi t)$$
$$= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{\mathbb{R}} \sqrt{y}K_{it}(2\pi|r|y)e(rx)\Big(\int_0^\infty\int_{\mathbb{R}} f(u+iv)\sqrt{v}K_{it}(2\pi|r|v)e(ru)\,du\,\frac{dv}{v^2}\Big)t\sinh(\pi t)\,dr\,dt$$

Let us write

$$\tilde{f}(r,v) = \int_{\mathbb{R}} f(u+iv)e(ru)\,du.$$

Then by Fourier inversion, assuming f is nice, we have

$$f(u+iv) = \int_{\mathbb{R}} \tilde{f}(r,v)e(-ur)\,dr.$$

Now the above expression can be written:

$$= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{\mathbb{R}} \sqrt{y}K_{it}(2\pi|r|y)e(rx)\Big(\int_0^\infty \tilde{f}(r,v)K_{it}(2\pi|r|v)\,\frac{dv}{v^{3/2}}\Big)\,t\sinh(\pi t)\,dr\,dt$$
$$= \sqrt{y}\int_{\mathbb{R}} e(rx)\int_{-\infty}^{\infty} K_{it}(2\pi|r|y)\Big(\int_0^\infty v^{-\frac{1}{2}}\tilde{f}(r,v)K_{it}(2\pi|r|v)\,\frac{dv}{v}\Big)\,\pi^{-2}t\sinh(\pi t)\,dt\,dr$$
$$= \sqrt{y}\int_{\mathbb{R}} (2\pi|r|)^{\frac{1}{2}}e(rx)\int_{-\infty}^{\infty} K_{it}(2\pi|r|y)\Big(\int_0^\infty v^{-\frac{1}{2}}\tilde{f}\Big(r,\frac{v}{2\pi|r|}\Big)K_{it}(v)\,\frac{dv}{v}\Big)\,\pi^{-2}t\sinh(\pi t)\,dt\,dr$$

Here the inner double integral can be evaluated using Kontorovitch-Lebedev inversion; cf. (7.2), (7.3) below ([7, (1.31) CORRECTED]). We get:

$$= \sqrt{y}\int_{\mathbb{R}} (2\pi|r|)^{\frac{1}{2}}e(rx)\Big((2\pi|r|y)^{-\frac{1}{2}}\tilde{f}(r,y)\Big)\,dr = \int_{\mathbb{R}} \tilde{f}(r,y)e(rx)\,dr = f(-x+iy).$$

Hence we have proved ALMOST the desired formula, and by reviewing the above computations we see that we get "$f(x+iy)$" in the end if we only replace $W_s(rz)$ by $W_s(-rz)$ in *exactly one* of (1.29) or (1.30) in Iwaniec's book! □

7. Discussion about Kontorovitch-Lebedev inversion

Note that a fairly detailed presentation (without proofs but with precise references) of this is given on Encycl of Maths. Also a less detailed presentation is on wikipedia. Below, to start with I follow old handwritten notes, [17, pp. 88–94], slightly polished.

We first require the following lemma.

Lemma 7.1. *For any $t \in \mathbb{R} \setminus \{0\}$ and $A > 1$ we have*

$$\int_0^\infty K_{it}(y) e^{-Ay} \, dy = \frac{\pi}{\sqrt{A^2-1}} \frac{\sin(\operatorname{arccosh}(A)t)}{\sinh(\pi t)}.$$

Proof. I prove this in [17, pp. 88–89]; but it is also in [3, 6.611.3].[3] □

The following is one (weak) version of Kontorovitch-Lebedev inversion.

Lemma 7.2. *If $L : \mathbb{R} \to \mathbb{C}$ is an even function such that $h(t) := L(t)\sinh(\pi t)$ satisfies $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\widehat{h} \in L^1(\mathbb{R})$, then*

$$\int_0^\infty K_{it}(y) \int_{\mathbb{R}} L(w) K_{iw}(y) \pi^{-2} \sinh(\pi w) w \, dw \, \frac{dy}{y} = L(t) \tag{7.1}$$

for almost all $t \in \mathbb{R}$, and the left hand side is a continuous function of t.

(Note that since the left hand side is continuous, the relation (7.1) holds for *all* t if we add the condition that L be continuous.)

Proof. We have (for any $t \in \mathbb{R} \setminus \{0\}$, $y > 0$):

$$K_{it}(y) = \int_0^\infty e^{-y\cosh w} \cos(tw) \, dw = \frac{1}{2} \int_{\mathbb{R}} e^{-y\cosh w} e^{-itw} \, dw = \frac{i}{2t} y \int_{\mathbb{R}} e^{-y\cosh w} \sinh w e^{-itw} \, dw,$$

where we integrated by parts in the last step. Hence

$$tK_{it}(y) = \frac{iy}{2} \int_{\mathbb{R}} e^{-y\cosh w} \sinh w e^{-itw} \, dw$$

(also for $t = 0$), and so by Fourier inversion (noticing that for any fixed $y > 0$ the function $w \mapsto e^{-y\cosh w} \sinh w$ is of Schwarz class),

$$\int_{\mathbb{R}} tK_{it}(y) e^{itw} \, dt = 2\pi \cdot \frac{iy}{2} \cdot e^{-y\cosh w} \sinh w = \pi i y e^{-y\cosh w} \sinh w, \qquad \forall w \in \mathbb{R}.$$

In other words, if we fix define for $y > 0$, $t \in \mathbb{R}$,

$$g_y(t) = tK_{it}(y);$$

then

$$\widehat{g_y}(w) = \int_{\mathbb{R}} g_y(t) e^{-itw} = -\pi i y e^{-y\cosh w} \sinh w, \qquad \forall w \in \mathbb{R}.$$

Now consider an arbitrary function $L : \mathbb{R} \to \mathbb{C}$ as in the statement of the lemma. Without loss of generality we may assume that L is real valued. Set

$$h(t) := L(t)\sinh(\pi t),$$

so that $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\widehat{h} \in L^1(\mathbb{R})$. Now set

$$f(y) = \int_{\mathbb{R}} L(t) K_{it}(y) \pi^{-2} \sinh(\pi t) t \, dt = \frac{1}{\pi^2} \int_{\mathbb{R}} g_y(t) \overline{h(t)} \, dt$$

Here $g_y, h \in L^2(\mathbb{R})$ (indeed, g_y is even of Schwarz class, since its Fourier transform is so), and hence by Parseval's formula ([16, p.187 (10)])

$$f(y) = \frac{1}{2\pi^3} \int_{\mathbb{R}} \widehat{g_y}(w) \overline{\widehat{h}(w)} \, dw = -\frac{iy}{2\pi^2} \int_{\mathbb{R}} e^{-y\cosh w} (\sinh w) \overline{\widehat{h}(w)} \, dw.$$

Here h is odd and real valued; hence $\widehat{h}(w) \in i\mathbb{R}$ for all $w \in \mathbb{R}$, and thus

$$f(y) = \frac{iy}{2\pi^2} \int_{\mathbb{R}} e^{-y\cosh w} (\sinh w) \widehat{h}(w) \, dw.$$

[3] Actually if I am not mistaken now there is a sign mistake in [3, 6.611.3]; however I note that Maple numerical evaluation gives answers consistent with the formula we state here.

It follows that

$$\int_0^\infty K_{it}(y)f(y)\,\frac{dy}{y} = \frac{i}{2\pi^2}\int_0^\infty\int_{\mathbb{R}} K_{it}(y)e^{-y\cosh w}(\sinh w)\widehat{h}(w)\,dw\,dy.$$

Let us here assume $t \in \mathbb{R}\setminus\{0\}$; then $\sup_{y>0}|K_{it}(y)| < \infty$. Also $\int_0^\infty e^{-y\cosh w}(\sinh w)\,dy = \tanh w \leq 1$, and using these facts together with $\widehat{h} \in L^1(\mathbb{R})$ we see that the above double integral is absolutely convergent. Hence by Fubini we get

$$= \frac{i}{2\pi^2}\int_{\mathbb{R}}\Big(\int_0^\infty K_{it}(y)e^{-y\cosh w}\,dy\Big)(\sinh w)\widehat{h}(w)\,dw = \frac{i}{2\pi^2}\int_{\mathbb{R}}\frac{\pi}{|\sinh w|}\frac{\sin(|w|t)}{\sinh(\pi t)}(\sinh w)\widehat{h}(w)\,dw$$

$$= \frac{i}{2\pi\sinh(\pi t)}\int_{\mathbb{R}}\widehat{h}(w)\sin(wt)\,dw = \frac{1}{2\pi\sinh(\pi t)}\int_{\mathbb{R}}\widehat{h}(w)e^{iwt}\,dw,$$

where we used Lemma 7.1, and in the last step we also used that h is odd and hence $\widehat{h}$ is odd. Now the last function is continuous with respect to t ([16, Theorem 9.6]), and we have for almost all $t \in \mathbb{R}$ ([16, Theorem 9.11]):

$$= \frac{h(t)}{\sinh(\pi t)} = L(t).$$

□

Note that the above result says that, under appropriate conditions, the transform which takes $f : \mathbb{R}^+ \to \mathbb{C}$ to [the even function]

$$L(t) := \int_0^\infty f(y)K_{it}(y)\,\frac{dy}{y}. \tag{7.2}$$

is inverse to the transform which takes L to

$$f(y) := \int_{\mathbb{R}} L(w)K_{iw}(y)\pi^{-2}\sinh(\pi w)w\,dw. \tag{7.3}$$

Hence we have (under appropriate conditions) agreement with Iwaniec's statement [7, (1.31) CORRECTED by inserting a factor 2 in the right hand side!] – simply by composing the transform and its inverse the other way.

Furthermore, replacing $f(y)$ with $f(y)/y$ we get: The transform which takes $f : \mathbb{R}^+ \to \mathbb{C}$ to [the even function]

$$L(t) := \int_0^\infty f(y)K_{it}(y)\,dy \tag{7.4}$$

is inverse to the transform which takes L to

$$f(y) := \frac{1}{\pi^2 y}\int_{\mathbb{R}} L(w)K_{iw}(y)\sinh(\pi w)w\,dw = \frac{2}{\pi^2 y}\int_0^\infty L(w)K_{iw}(y)\sinh(\pi w)w\,dw. \tag{7.5}$$

It is in this latter form that the transform is stated on Encycl of Maths and wikipedia.

We will postpone a detailed discussion about exact conditions for the transform to hold. However let us already now note that nice L^2-statement given in Encycl of Maths: The transform (7.4) extends to an isomorphism from $L^2(\mathbb{R}^+, x)$ onto $L^2(\mathbb{R}^+, \frac{2}{\pi^2}w\sinh(\pi w))$; thus in particular we have for any $f \in L^2(\mathbb{R}^+, x)$:

$$\int_0^\infty y|f(y)|^2\,dy = \frac{2}{\pi^2}\int_0^\infty w\sinh(\pi w)|L(w)|^2\,dw.$$

Thus in the original form of the transform, (7.2)–(7.3):

Lemma 7.3. *For any nice function $f : \mathbb{R}^+ \to \mathbb{C}$, if*

$$L(t) := \int_0^\infty f(y)K_{it}(y)\,\frac{dy}{y}$$

then

$$\int_0^\infty |f(y)|^2 \frac{dy}{y} = \frac{2}{\pi^2}\int_0^\infty w\sinh(\pi w)|L(w)|^2\,dw.$$

Proof. *** POSTPONE... *** □

8. Spectral decomposition of functions on the 'cylinder' $[z \mapsto z+1]\backslash\mathbb{H}$

Note this will turn out to be a very direct analogue of Iwaniec's [7, Prop. 1.4]! However: *Surely there must exist some PRECISE reference working out this expansion???*

(Another question: What is the correct "name" for the quotient space $[z \mapsto z+1]\backslash\mathbb{H}$? It seems that "hyperbolic cylinder" is standard terminology for the quotient $[\gamma]\backslash\mathbb{H}$ where γ is a *hyperbolic* (or loxodromic) map! Cf., e.g., [19, pdf], [19, (2.6)], [18, pdf], and also [12, Sec. 8].)

Given a "nice" function $f:\mathbb{H}^2 \to \mathbb{C}$ satisfying $f(z+1)\equiv f(z)$ we wish to find a formula for the functions $c_n:\mathbb{R}^+\to\mathbb{C}$ ($n\in\mathbb{Z}\setminus\{0\}$) such that

$$f(z)=f(x+iy)=\int_{-\infty}^\infty c_0(t)y^{\frac12+it}\,dt+\int_0^\infty\sum_{n\neq0}c_n(t)\sqrt{y}K_{it}(2\pi|n|y)e(nx)\,dt. \tag{8.1}$$

We note that if (8.1) holds then for any $w\in\mathbb{R}$ and $m\in\mathbb{Z}\setminus\{0\}$ we have

$$\begin{aligned}
&\int_{(\mathbb{R}/\mathbb{Z})\times\mathbb{R}^+} f(z)\sqrt{y}K_{iw}(2\pi|m|y)e(-mx)\,d\mu(z)\\
&=\int_0^\infty K_{iw}(2\pi|m|y)\int_{\mathbb{R}/\mathbb{Z}}\Big(\sum_{n\neq0}e(nx)\int_0^\infty c_n(t)K_{it}(2\pi|n|y)\,dt\Big)e(-mx)\,dx\,\frac{dy}{y}\\
&=\int_0^\infty K_{iw}(2\pi|m|y)\int_0^\infty c_m(t)K_{it}(2\pi|m|y)\,dt\,\frac{dy}{y}\\
&=\int_0^\infty K_{iw}(y)\int_0^\infty c_m(t)K_{it}(y)\,dt\,\frac{dy}{y}
\end{aligned}$$

In order to make this more similar to the K-L transform we make the Ansatz

$$c_m(t)=2\pi^{-2}\sinh(\pi t)t\delta_m(t),$$

for some function $\delta_m:\mathbb{R}^+\to\mathbb{C}$. Then the above equals

$$2\int_0^\infty K_{iw}(y)\int_0^\infty \delta_m(t)K_{it}(y)\pi^{-2}\sinh(\pi t)t\,dt\,\frac{dy}{y},$$

and by the K-L inversion formula (cf. (7.2), (7.3)) this is

$$=\delta_m(w)=\frac{\pi^2}{2}\frac{c_m(w)}{w\sinh(\pi w)}.$$

Hence we have proved that in order for (8.1) to hold, we should take

$$c_n(t)=\frac{2}{\pi^2}t\sinh(\pi t)\int_{(\mathbb{R}/\mathbb{Z})\times\mathbb{R}^+}f(z)\sqrt{y}K_{it}(2\pi|n|y)e(-nx)\,d\mu(z),\qquad \forall n\neq0. \tag{8.2}$$

Let us also note that (8.1) implies that if we defined $F(y):=\int_{\mathbb{R}/\mathbb{Z}}f(x+iy)\,dx=\int_{-\infty}^\infty c_0(t)y^{\frac12+it}\,dt$ then

$$c_0(t)=\frac{1}{2\pi}\int_0^\infty F(y)y^{-it-\frac32}\,dy=\frac{1}{2\pi}\int_{(\mathbb{R}/\mathbb{Z})\times\mathbb{R}^+}f(z)y^{\frac12-it}\,d\mu(z). \tag{8.3}$$

[Proof: The second equality is trivial, and the first is (of course) Mellin inversion. Let us write this out completely explicitly since we will probably need it later: We have $F(y)=\int_{-\infty}^\infty c_0(t)y^{\frac12+it}\,dt=i^{-1}\int_{(1/2)}c_0(i^{-1}(s-\frac12))y^s\,ds$; hence by Mellin inversion $c_0(i^{-1}(s-\frac12))=\frac{1}{2\pi i}\int_0^\infty(iF(y))y^{-s-1}\,dy=\frac{1}{2\pi}\int_0^\infty F(y)y^{-s-1}\,dy$. Setting here $s=\frac12+it$ we get the claimed relation.

Let us also write out the same proof using Fourier inversion: Substituting $y = e^u$ in our formula for $F(y)$ we get $F(e^u)e^{-\frac{1}{2}u} = \int_{-\infty}^{\infty} c_0(t)e^{iut}\,dt$;[4] hence by Fourier inversion $c_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(e^u)e^{-\frac{1}{2}u}e^{-iut}\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(y)y^{-it-\frac{3}{2}}\,dy$.]

But we wish to prove that (8.1) holds for "arbitrary" functions $f : [S]\backslash\mathbb{H} \to \mathbb{C}$ satisfying suitable conditions, and for this it seems we should work "the other way". Thus let $f : [S]\backslash\mathbb{H} \to \mathbb{C}$ be given, and *define* $c_n(t)$ by (8.2), (8.3), or in other words (for $n \neq 0$):

$$c_n(t) = \frac{2}{\pi^2} t\sinh(\pi t)\int_0^\infty K_{it}(2\pi|n|v)\int_{\mathbb{R}/\mathbb{Z}} f(u+iv)\,e(-nu)\,du\,\frac{dv}{v^{3/2}} \tag{8.4}$$

[Note that this converges for all f satisfying $f(z) = O(y^{\frac{1}{2}+\varepsilon})$ as $y \to 0$, and $f(z) = O(e^{cy})$ as $y \to \infty$ with a sufficiently small constant $c > 0$.] With this choice of $c_n(t)$, we find that, for any $z = x + iy \in \mathbb{H}$:

$$\int_0^\infty \sum_{n\neq 0} c_n(t)\sqrt{y}K_{it}(2\pi|n|y)e(nx)\,dt$$

$$= \frac{2}{\pi^2}\sum_{n\neq 0}\Bigl(\int_0^\infty t\sinh(\pi t)\Bigl(\int_0^\infty K_{it}(2\pi|n|v)\int_{\mathbb{R}/\mathbb{Z}} f(u+iv)\,e(-nu)\,du\,\frac{dv}{v^{3/2}}\Bigr)K_{it}(2\pi|n|y)\,dt\Bigr)\sqrt{y}e(nx)$$

$$= \frac{2}{\pi^2}\sum_{n\neq 0}\Bigl(\int_0^\infty t\sinh(\pi t)\Bigl(\int_0^\infty K_{it}(2\pi|n|v)v^{-\frac{1}{2}}\int_{\mathbb{R}/\mathbb{Z}} f(u+iv)\,e(-nu)\,du\,\frac{dv}{v}\Bigr)K_{it}(2\pi|n|y)\,dt\Bigr)\sqrt{y}e(nx)$$

$$= \frac{2}{\pi^2}\sum_{n\neq 0}(2\pi|n|)^{\frac{1}{2}}\Bigl(\int_0^\infty t\sinh(\pi t)\Bigl(\int_0^\infty K_{it}(v)v^{-\frac{1}{2}}\int_{\mathbb{R}/\mathbb{Z}} f\Bigl(u+i\frac{v}{2\pi|n|}\Bigr)\,e(-nu)\,du\,\frac{dv}{v}\Bigr)K_{it}(2\pi|n|y)\,dt\Bigr)\sqrt{y}e(nx$$

By the K-L transform (cf. (7.2), (7.3)) this is

$$= \sum_{n\neq 0}(2\pi|n|)^{\frac{1}{2}}\Bigl((2\pi|n|y)^{-\frac{1}{2}}\int_{\mathbb{R}/\mathbb{Z}} f(u+iy)e(-nu)\,du\Bigr)\sqrt{y}e(nx)$$

$$= \sum_{n\neq 0}\Bigl(\int_{\mathbb{R}/\mathbb{Z}} f(u+iy)e(-nu)\,du\Bigr)e(nx)$$

Also, for $c_0(t)$ as in (8.3), we have

$$\int_{-\infty}^{\infty} c_0(t)y^{\frac{1}{2}+it}\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Bigl(\int_0^\infty\int_{\mathbb{R}/\mathbb{Z}} f(u+iv)\,du\,v^{-\frac{3}{2}-it}\,dv\Bigr)y^{\frac{1}{2}+it}\,dt$$

$$= \frac{\sqrt{y}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Bigl(e^{-\frac{1}{2}\alpha}\int_{\mathbb{R}/\mathbb{Z}} f(u+ie^{\alpha})\,du\Bigr)\,e^{-it\alpha}\,d\alpha\,e^{i(\log y)t}\,dt$$

$$= \sqrt{y}e^{-\frac{1}{2}(\log y)}\int_{\mathbb{R}/\mathbb{Z}} f(u+ie^{\log y})\,du = \int_{\mathbb{R}/\mathbb{Z}} f(u+iy)\,du$$

Adding these we obtain:

$$\int_{-\infty}^{\infty} c_0(t)y^{\frac{1}{2}+it}\,dt + \int_0^\infty \sum_{n\neq 0} c_n(t)\sqrt{y}K_{it}(2\pi|n|y)e(nx)\,dt$$

$$= \sum_{n\in\mathbb{Z}}\Bigl(\int_{\mathbb{R}/\mathbb{Z}} f(u+iy)e(-nu)\,du\Bigr)e(nx) = f(x+iy).$$

This proves that for $f : [S]\backslash\mathbb{H} \to \mathbb{C}$ satisfying suitable conditions, the formula (8.1) holds when $c_n(t)$ are defined by (8.2) ($\Leftrightarrow$(8.4)).

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[4]For use in program: If $F(y)$ is real valued then $c_0(-t) = \overline{c_0(t)}$ and hence $F(e^u)e^{-\frac{1}{2}u} = \int_{-\infty}^{\infty} c_0(t)e^{iut}\,dt = \int_{-\infty}^{\infty}((\mathrm{Re}\ c_0(t))\cos(ut) - (\mathrm{Im}\ c_0(t))\sin(ut))\,dt + i\int_{-\infty}^{\infty}((\mathrm{Re}\ c_0(t))\sin(ut) + (\mathrm{Im}\ c_0(t))\cos(ut))\,dt$
$= 2\int_0^\infty(\mathrm{Re}\ c_0(t))\cos(ut)\,dt - 2\int_0^\infty(\mathrm{Im}\ c_0(t))\sin(ut)\,dt.$

Next, in order to mimic Lax-Phillips [11, proof of Thm 4.8], it seems that we need a Parseval's formula for $\int_{[S]\backslash\mathbb{H}} |f(z)|^2\,d\mu(z)$!

Note that Lemma 7.3 applies with $f_{Lemma}(y) = 2\int_0^\infty \delta_m(t)K_{it}(y)\pi^{-2}\sinh(\pi t)t\,dt$ and $L_{Lemma}(t) = \delta_m(t)$; hence

$$\int_0^\infty \left|2\int_0^\infty \delta_m(t)K_{it}(y)\pi^{-2}\sinh(\pi t)t\,dt\right|^2 \frac{dy}{y} = \frac{2}{\pi^2}\int_0^\infty w\sinh(\pi w)|\delta_m(w)|^2\,dw.$$

In other words:

$$\int_0^\infty \left|\int_0^\infty c_m(t)K_{it}(y)\,dt\right|^2 \frac{dy}{y} = \frac{\pi^2}{2}\int_0^\infty \frac{|c_m(w)|^2}{w\sinh(\pi w)}\,dw. \tag{8.5}$$

Now we have:

$$\begin{aligned}
\int_{[S]\backslash\mathbb{H}} |f(z)|^2\,d\mu(z) &= \int_0^\infty \int_{\mathbb{R}/\mathbb{Z}} \left|\int_{-\infty}^\infty c_0(t)y^{\frac12+it}\,dt + \sum_{n\neq 0}\Big(\int_0^\infty c_n(t)\sqrt{y}K_{it}(2\pi|n|y)\,dt\Big)e(nx)\right|^2 dx\,\frac{dy}{y^2}.\\
&= \int_0^\infty \left(\left|\int_{-\infty}^\infty c_0(t)y^{\frac12+it}\,dt\right|^2 + \sum_{n\neq0}\left|\int_0^\infty c_n(t)\sqrt{y}K_{it}(2\pi|n|y)\,dt\right|^2\right)\frac{dy}{y^2}\\
&= \int_{-\infty}^\infty \left|\int_{-\infty}^\infty c_0(t)e^{itu}\,dt\right|^2 du + \sum_{n\neq0}\int_0^\infty \left|\int_0^\infty c_n(t)K_{it}(2\pi|n|y)\,dt\right|^2\frac{dy}{y}\\
&= \int_{-\infty}^\infty \left|\int_{-\infty}^\infty c_0(t)e^{itu}\,dt\right|^2 du + \sum_{n\neq0}\int_0^\infty \left|\int_0^\infty c_n(t)K_{it}(y)\,dt\right|^2\frac{dy}{y},
\end{aligned}$$

and using here (8.5) and "the usual" Parseval's formula (cf. [8, p.155 (Lemma)] and also the definition [8, p.133(1.1)]) we get

$$= 2\pi\int_{-\infty}^\infty |c_0(t)|^2\,dt + \frac{\pi^2}{2}\sum_{n\neq0}\int_0^\infty \frac{|c_n(w)|^2}{w\sinh(\pi w)}\,dw.$$

Hence we have proved:

Lemma 8.1. *For $f(z)$ given by (8.1), we have*

$$\int_{[S]\backslash\mathbb{H}^2} |f(z)|^2\,d\mu(z) = 2\pi\int_{-\infty}^\infty |c_0(t)|^2\,dt + \frac{\pi^2}{2}\sum_{n\neq0}\int_0^\infty \frac{|c_n(t)|^2}{t\sinh(\pi t)}\,dt.$$

Although it certainly remains to sort out several technical details, I am convinced that the above type of arguments will actually show that our transform $f \leftrightarrow \{c_n(t)\}$ extends to an *isometry between* $L^2([S]\backslash\mathbb{H}^2,\mu)$ *and *** CORRECT THE FOLLOWING TO ALSO INCLUDE* $n=0$*; NOTATION??? **** $L^2(\prod_{n\neq0}\mathbb{R}^+, \frac{\pi^2}{2}(t\sinh(\pi t))^{-1})$, where the latter space is the space of all sequences $\{c_n(t)\}_{n\neq0}$ of functions $\mathbb{R}^+\to\mathbb{C}$, with L^2-norm

$$\|\{c_n(t)\}\|^2 := \frac{\pi^2}{2}\sum_{n\neq0}\int_0^\infty \frac{|c_n(t)|^2}{t\sinh(\pi t)}\,dt.$$

***

9. Working out the $x \mapsto x+1$ invariant analogue of Lax-Phillips expansion

We now try to mimic Lax-Phillips [11, proof of Thm 4.8], using the transform $f \leftrightarrow \{c_n(t)\}$ worked out in Section 8. Recall that Lax-Phillips' L equals $L_0 + I$ where

$$L_0 = y^2\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y^2}\Big) - y\frac{\partial}{\partial y}$$

is the Laplace-Beltrami operator (and L' is the self-adjoint extension of L); hence the equation which is studied on [11, p.314], $Lu = -\mu^2 u$, is equivalent with $L_0 u = -(1+\mu^2)u$. Lax-Phillips work with $\mu \geq 0$, and prove that no such eigenfunctions can exist, viz. that $-L_0$ has no discrete eigenvalues ≥ 1. We will now allow general $\mu \in \mathbb{R}_{\geq 0} \cup [-1,0]i$, i.e. we consider an arbitrary eigenvalue $\lambda = 1+\mu^2 \in \mathbb{R}_{\geq 0}$ of $-L_0$. Let us also write $s_\mu = 1+i\mu$, so that $s_\mu \in [1,2] \cup (1+i\mathbb{R}_{\geq 0})$ and

$$\lambda = 1+\mu^2 = s_\mu(2-s_\mu).$$

(This s_μ should not be mixed up with the arclength coordinate $s > 0$ used on [11, p.314].) Now in Lax-Phillips coordinates x, ρ, s we have, if we write $u = (\operatorname{sech} s)v$ (viz., $v = (\cosh s)u$) ([11, (4.22)']):

$$\big(\partial_s^2 + \rho^2(\operatorname{sech} s)^2(\partial_x^2 + \partial_\rho^2)\big)v = -\mu^2 v \tag{9.1}$$

We are studying the eigenfunction in the half-space $\{x_2 > 0, y > 0\} \subset \mathbb{H}^3$, and we are assuming that it is invariant under $x_1 \mapsto x_1 + 1$. In the x, ρ, s-coordinates this corresponds to the region $\{x \in \mathbb{R},\ \rho, s > 0\}$, and $v(x,\rho,s)$ is invariant under $x \mapsto x+1$. Also the fact that $\int_{\mathbb{R}/\mathbb{Z}} \int_0^\infty \int_0^\infty |u(x_1,x_2,y)|^2 \frac{dx_1\,dx_2\,dy}{y^3} < \infty$ corresponds to (cf. [11, (4.23)])

$$\int_0^\infty \int_0^\infty \int_{\mathbb{R}/\mathbb{Z}} |v(x,\rho,s)|^2 \, \frac{dx\,d\rho}{\rho^2}\, ds < \infty. \tag{9.2}$$

Now for each fixed $s > 0$ we consider $(x+i\rho) \mapsto v(x,\rho,s)$ as a function in $[S]\backslash\mathbb{H}^2$, and express it as in (8.1), viz.

$$v(x,\rho,s) = \int_{-\infty}^{\infty} c_0(t,s)\rho^{\frac12+it}\,dt + \int_0^\infty \sum_{n\neq 0} c_n(t,s)\sqrt{\rho}K_{it}(2\pi|n|\rho)e(nx)\,dt. \tag{9.3}$$

We know

$$\rho^2(\partial_x^2+\partial_\rho^2)\Big(\rho^{\frac12+it}\Big) = -(\tfrac14+t^2)\Big(\rho^{\frac12+it}\Big)$$

and

$$\rho^2(\partial_x^2+\partial_\rho^2)\Big(\sqrt{\rho}K_{it}(2\pi|n|\rho)e(nx)\Big) = -(\tfrac14+t^2)\Big(\sqrt{\rho}K_{it}(2\pi|n|\rho)e(nx)\Big),$$

and hence for v as above we have ***FORMALLY***

$$\begin{aligned}\big(\partial_s^2 + \rho^2(\operatorname{sech} s)^2(\partial_x^2+\partial_\rho^2)\big)v &= \int_{-\infty}^{\infty}\Big\{\Big(\partial_s^2 - (\tfrac14+t^2)(\operatorname{sech} s)^2\Big)c_0(t,s)\Big\}\rho^{\frac12+it}\,dt \\ &+ \int_0^\infty \sum_{n\neq 0}\Big\{\Big(\partial_s^2 - (\tfrac14+t^2)(\operatorname{sech} s)^2\Big)c_n(t,s)\Big\}\sqrt{\rho}K_{it}(2\pi|n|\rho)e(nx)\,dt.\end{aligned}$$

Hence for (9.1) to hold we must have (*** almost everywhere, and in some weak sense***)

$$\Big(\partial_s^2 - (\tfrac14+t^2)(\operatorname{sech} s)^2\Big)c_n(t,s) = -\mu^2 c_n(t,s). \tag{9.4}$$

Let us first try to mimic Lax-Phillips' argument in the case $\mu > 0$. They say that the above o.d.e. implies, via a perturbation argument (which I haven't checked but it seems very likely!) that (for any fixed n and a.e. fixed t) $c_n(t,s)$ behaves like $ae^{i\mu s} + be^{-i\mu s}$ as $s \to \infty$. But we also have, by (9.2) and Lemma 8.1,

(9.5)

$$2\pi\int_0^\infty\int_{-\infty}^{\infty}|c_0(t,s)|^2\,dt\,ds + \frac{\pi^2}{2}\sum_{n\neq 0}\int_0^\infty\int_0^\infty \frac{|c_n(t,s)|^2}{t\sinh(\pi t)}\,dt\,ds = \int_0^\infty\int_0^\infty\int_{\mathbb{R}/\mathbb{Z}}|v(x,\rho,s)|^2\,\frac{dx\,d\rho}{\rho^2}\,ds < \infty.$$

Hence for every n and a.e. t we have $\int_0^\infty |c_n(t,s)|^2\,ds < \infty$, and it follows that $c_n(t,s)$ must vanish identically. Hence *Lax-Phillips' argument carries over to our situation in the case $\mu > 0$!* (Of course this is no surprise, for if there would exist $f \in L^2([x_1 \mapsto x_1+1]\backslash\{(x_1,x_2,y) \in \mathbb{H}^3 : x_2 > 0\})$ satisfying $Lf = -\mu^2 f$ then we could apply Lax-Phillips Theorem 4.8 to the restriction of f to the $\mathbb{H}^3$-half space $\{(x_1,x_2,y) \in \mathbb{H}^3 : x_1^2 + (x_2-1)^2 + y^2 < \frac14\}$ and get a contradiction. Similarly we expect that Lax-Phillips' argument also carries over to prove that $v \equiv 0$ in the case $\mu = 0$. Hence from now on we consider the case $\mu \in [-1,0)i$. Now surely (9.4) must again imply ("by a perturbation argument" – but I haven't checked) that $c_n(t,s)$ behaves like $ae^{i\mu s} + be^{-i\mu s}$, and here as above (but "worse"!) we see that $a = 0$ (for all n and a.e. t).

Solve (9.4) explicitly?? Cf. e.g. in Alex' thesis [9, pdf] or Gamburd [2, pp.180–]. (Recall [11, (4.21')]; $s = \int_\theta^{\pi/2} \frac{d\theta'}{\sin\theta'} = \log(\frac{\sin\theta}{1-\cos\theta})$; hence

$$\cosh s = \frac{1}{\sin\theta}; \qquad \sinh s = \cot\theta; \qquad \tanh s = \cos\theta.$$

Hence from [2, p.171] we see that Gamburd's "ϕ" corresponds to our "θ", and Gamburd's "ρ" corresponds to our "s".)

Note that our ODE (9.4) looks rather different, and in fact easier, than [2, (4.17)]... If we mimic Gamburd and substitute $y = \tanh s = \cos\theta$ then $\operatorname{sech}^2 s = \frac{y^2}{1-y^2}$ and then $\partial_s^2 = (1-y^2)(-2y\partial_y + (1-y^2)\partial_y^2)$, and hence we obtain the equivalent ODE:

$$(1-y^2)(-2y\partial_y + (1-y^2)\partial_y^2)c_n(t,y) - (\tfrac14 + t^2)(1-y^2)^{-1}y^2 c_n(t,y) = -\mu^2 c_n(t,y).$$

(We are interested in $y \to 1^-$...)

Let's try to "cheat"; have Maple answer! A quick Maple session:

```
> ode:=diff(c(s),s,s)=A*sech(s)^2*c(s)-B*c(s);
                        2
                       d                   2
             ode := --- c(s) = A sech(s)  c(s) - B c(s)
                         2
                       ds

> dsolve(ode);
memory used=3.8MB, alloc=3.0MB, time=0.29
memory used=7.6MB, alloc=4.9MB, time=0.49
                                          1/2          /        2\1/2
                                (-4 A + 1)       1/2   |sinh(s) |
c(s) = _C1 LegendreP(- 1/2 + -------------, B    I,  |--------|     )
                                      2                |       2|
                                                       \cosh(s) /

                                          1/2          /        2\1/2
                                (-4 A + 1)       1/2   |sinh(s) |
     + _C2 LegendreQ(- 1/2 + -------------, B    I,  |--------|     )
                                      2                |       2|
                                                       \cosh(s) /
```

We look up that these are the "associated Legendre functions of the first (P) and second (Q) kind"; in fact comparing [2, (4.20)] with the ode in `?LegendreP` (in Maple) shows that (most likely) `LegendreP(v,u,x)`$= P_v^u(x)$ and `LegendreQ(v,u,x)`$= Q_v^u(x)$. To correspond with (9.4) we should take $A = \frac14 + t^2$ and $B = \mu^2$; hence $-\frac12 + \frac12\sqrt{1-4A} = -\frac12 + \sqrt{-t^2} = -\frac12 \pm it$; and the general solution which Maple proposes is:

$$c_n(t,s) = C_1 P_{-\frac12 \pm it}^{-\mu i}(\cos\theta) + C_2 Q_{-\frac12 \pm it}^{-\mu i}(\cos\theta).$$

(We could also choose "μi", but the present "$-\mu i$" gives better agreement with Gamburd/Kontorovich.) Note that this is *very* similar to what was found in [2, p.181 line -1]; in a sense the only difference is that the factor $\sqrt{\sin\phi}$ in Gamburd's expression is now replaced by $\sinh s = \sin\theta$! (Coming from our substitution $u = (\operatorname{sech} s)v$.) Recall in particular that *our* μ lies in $[-1,0)i$, since we are considering Laplace-Beltrami eigenvalue $\lambda = 1 + \mu^2 \in [0,1)$; hence $-\mu i \in [-1,0)$.

Now recall that (9.5) implies that $\int_0^\infty |c_n(t,s)|^2\,ds < \infty$ for every n and a.e. t; hence from [2, p.181 lines -7 to -5] we see that $C_2 = 0$ in the above formula; i.e. we have to choose $P^{-\mu i}_{-\frac{1}{2}\pm it}$! On the other hand [2, p.181 lines -7 to -5] says that as $s \to \infty$ ($\Leftrightarrow \theta \to 1^-$):

$$P^{-\mu i}_{-\frac{1}{2}\pm it}(\cos\theta) \asymp (1-\cos\theta)^{\frac{1}{2}\mu i} \asymp (1-\cos^2\theta)^{\frac{1}{2}\mu i} = (\sin\theta)^{\mu i} = (\operatorname{sech} s)^{\mu i};$$

thus we have exponential decay so that $\int_0^\infty |c_n(t,s)|^2\,ds < \infty$ certainly holds!

Recall also that $s_\mu = 1 + i\mu$, $s_\mu \in [1,2] \cup (1 + i\mathbb{R}_{\geq 0})$ and $\lambda = 1 + \mu^2 = s_\mu(2 - s_\mu)$; and for the Patterson-Sullivan eigenvalue for the Apollonian group we have $s_\Gamma \approx 1.305688$ (according to McMullen); hence $\mu \approx -0.305688i$.

To conclude: In our expansion (9.3) we should have

$$c_n(t,s) = \xi_n(t) P^{-\mu i}_{-\frac{1}{2}+it}(\tanh s) \tag{9.6}$$

for some functions $\xi_0 : \mathbb{R} \to \mathbb{C}$ and $\xi_n : \mathbb{R}^+ \to \mathbb{C}$ ($n \in \mathbb{Z}_{\neq 0}$).

Now the expansion (9.3) was for the normalized situation of invariance under $x \mapsto x+1$. For the Patterson-Sullivan eigenfunction invariant under the Apollonian group we only have $x \mapsto x+2$ invariance; but also invariance under $x \mapsto -x$. Hence we may replace (9.3) with:

$$v(x,\rho,s) = \int_{-\infty}^{\infty} c_0(t,s)\rho^{\frac{1}{2}+it}\,dt + 2\int_0^\infty \sum_{n=1}^{\infty} c_n(t,s)\sqrt{\rho}K_{it}(\pi|n|\rho)\cos(\pi n x)\,dt. \tag{9.7}$$

The separation of variables in (9.6) is still ok; hence we should have:

(9.8)

$$v(x,\rho,s) = \int_{-\infty}^{\infty} \xi_0(t) P^{-\mu i}_{-\frac{1}{2}+it}(\tanh s)\rho^{\frac{1}{2}+it}\,dt + 2\int_0^\infty \sum_{n=1}^{\infty} \xi_n(t) P^{-\mu i}_{-\frac{1}{2}+it}(\tanh s)\sqrt{\rho}K_{it}(\pi|n|\rho)\cos(\pi n x)\,dt$$

for some (hopefully provably nice...) functions $\xi_0 : \mathbb{R} \to \mathbb{C}$ and $\xi_n : \mathbb{R}^+ \to \mathbb{C}$ ($n \in \mathbb{Z}_{\neq 0}$). Recall here that $v(x,\rho,s) = (\cosh s)\cdot u(x,\rho,s)$, where $u(x,\rho,s)$ is the Patterson-Sullivan eigenfunction!

10. Storing the non-discrete functions

We will first try Hermite polynomials, although this may in the end turn out to be not optimal for the problem at hand.

We fix some s and we wish to "store" the functions $t \mapsto c_n(t) := c_n(t,s)$. Let us first consider $n = 0$. Recall that $c_0(-t) = \overline{c_0(t)}$ (cf. footnote 4); thus Re $c_0(t)$ is an even function and Im $c_0(t)$ is an odd function. Now recall about the Hermite polynomials. We use the "physicists notation" from there; in particular the *Hermite functions*

$$\psi_k(t) = \frac{1}{\sqrt{2^k k! \sqrt{\pi}}} e^{-\frac{1}{2}t^2} H_k(t),$$

which form an orthonormal basis of $L^2(\mathbb{R}, dt)$. For $n = 0$ we expect $c_0(t) \in L^2(\mathbb{R}, dt)$ (cf. (9.5)), and hence there exist coefficients α_k so that

$$c_0(t) = \sum_{k=0}^{\infty} \alpha_k \psi_k(t).$$

The formula for the coefficients is of course: $\alpha_k = \int_{-\infty}^{\infty} c_0(t)\psi_k(t)\, dt$. In fact, since $\psi_k(t)$ is even for k even and odd for k odd, we have, if k is even:

$$\alpha_k = \int_{-\infty}^{\infty} c_0(t)\psi_k(t)\, dt = \int_0^{\infty} (c_0(t) + \overline{c_0(t)})\psi_k(t)\, dt = 2\int_0^{\infty} (\text{Re } c_0(t))\psi_k(t)\, dt,$$

and for k odd:

$$\alpha_k = \int_{-\infty}^{\infty} c_0(t)\psi_k(t)\, dt = \int_0^{\infty} (c_0(t) - \overline{c_0(t)})\psi_k(t)\, dt = 2i\int_0^{\infty} (\text{Im } c_0(t))\psi_k(t)\, dt.$$

Next, for $n \geq 1$ we expect instead $\int_0^{\infty} \frac{|c_n(t)|^2}{t\sinh(\pi t)}\, dt < \infty$ (cf. (9.5) again). Here we also recall from (8.4) that $c_n(t)$ is an even function (if we wish to consider it also for $t < 0$) and that also $\frac{c_n(t)}{t\sinh(\pi t)}$ is even and smooth on all $\mathbb{R}$. Note that $t \mapsto \sqrt{t\sinh(\pi t)}$ is an *odd* analytic function in a neighbourhood of the real axis! (To be precise, the formula "$\sqrt{t\sinh(\pi t)}$" is only correct for $t \geq 0$; for $t < 0$ we of course have to consider the negative square root in order to get an analytic function.) Of course, $\int_0^{\infty} |\frac{c_n(t)}{\sqrt{t\sinh(\pi t)}}|^2\, dt < \infty$, and the function $t \mapsto \frac{c_n(t)}{\sqrt{t\sinh(\pi t)}}$ is *odd*; hence perhaps we should expand this function as

$$\frac{c_n(t)}{\sqrt{t\sinh(\pi t)}} = \sum_{k=0}^{\infty} \alpha_k \psi_k(t),$$

where we will necessarily have $\alpha_k = 0$ for all even k. And for odd k:

$$\alpha_k = \int_{-\infty}^{\infty} \frac{c_n(t)}{\sqrt{t\sinh(\pi t)}}\psi_k(t)\, dt = 2\int_0^{\infty} \frac{c_n(t)}{\sqrt{t\sinh(\pi t)}}\psi_k(t)\, dt.$$

Note that when $c_n(t)$ "varies between $t \in [0, t_0]$" where $t_0 > 0$ then it seems we will have better convergence if we expand as:

$$\frac{c_n(t)}{\sqrt{t\sinh(\pi t)}} = \sum_{k=0}^{\infty} \alpha_k \psi_k(t_0 t),$$

where now $\alpha_k = 0$ for all even k, and for odd k:

$$\alpha_k = 2\int_0^{\infty} \frac{c_n(t_0^{-1}t)}{\sqrt{t_0^{-1}t\sinh(\pi t_0^{-1}t)}}\psi_k(t)\, dt = 2t_0\int_0^{\infty} \frac{c_n(t)}{\sqrt{t\sinh(\pi t)}}\psi_k(t_0 t)\, dt.$$

OR: Note $\int_0^\infty |\frac{c_n(t)}{t\sinh(\pi t)}|^2 t\sinh(\pi t)\,dt < \infty$; thus perhaps we should expand (the even, and rapidly decaying function)

$$\frac{c_n(t)}{t\sinh(\pi t)} = \sum_{k=0}^{\infty} \alpha_k \psi_k(t)?$$

11. Working notes regarding McMullen's program and further implementation

His paper: [13, pdf].

McMullen in [13, p.692] claims $dim = 1.305688$. (On [13, Table 1] he claims he used max diam $P_i = 0.0005$ (and $|\mathcal{P}| = 1,397,616$) to get his claim.)

`bin/hdim -a -e .01` takes around 5 seconds and returns dimension 1.3058....

`bin/hdim -a -e .005` takes around 30 seconds and returns dimension 1.305727...

Formula for the "Patterson-Sullivan eigenfunction": See [10, (4.2),(4.3)] (my notes; tex). (I am not sure I understand the notation "$x + iy$" therein...) (Note that this is a non-Euclidean Fourier transform *** CONTINUE! *** details on Helgason's GGA (tex)

(In the 2-dim case we used the formula in Patterson, [15, (3.5)].)

My working notes on McMullen's programs: dvi; tex.

Let us discuss how to translate between McMullen's coordinates and the ones we are using. McMullen's circles are $\{|z| = 1\}$ and $\{|z - (1 + r)e^{\frac{2}{3}k\pi i}| = r\}$ for $k = -1, 0, 1$, where $r = \frac{\sqrt{3}}{2-\sqrt{3}} = 6.461\ldots$. Let

$$J(z) = \frac{2\sqrt{3}}{z-1} + \sqrt{3} - 1.$$

Then $J(1) = \infty$, $J(-1) = -1$ and $J(1 + 2r) = 1$; hence since J has real coefficients we have $J(\{|z| = 1\}) = \{\text{Re } z = -1\}$ and $J(\{|z - (1 + r)| = r\}) = \{\text{Re } z = 1\}$. Note also that the two circles $\{|z - (1 + r)e^{\frac{2}{3}k\pi i}| = r\}$, $k = \pm 1$, are tangent to each other at $z = -\frac{1}{2}(1 + r)$, and $J(-\frac{1}{2}(1 + r)) = 0$ (actually this was forced on us...), and furthermore $J(\{\text{Im } z > 0\}) = \{\text{Im } z < 0\}$; hence we must have

$$J(\{|z - (1 + r)e^{-\frac{2}{3}\pi i}| = r\}) = \{|z - i| = 1\}$$

and

$$J(\{|z - (1 + r)e^{\frac{2}{3}\pi i}| = r\}) = \{|z + i| = 1\}.$$

(I double checked this by evaluating J at several points on the circle, in Maple.)

Hence J maps McMullen's fundamental domain to ours! Note that J is translation $z \mapsto z-1$ followed by inversion in the circle $\{|z| = 1\}$ followed by reflection in the real line followed by $z \mapsto 2\sqrt{3}z + \sqrt{3} - 1$. The corresponding map of $\mathbb{H}^3$ is $(x_1, x_2, y) \mapsto (x_1 - 1, x_2, y)$ followed by inversion in the ball $x_1^2 + x_2^2 + y^2 = 1$ (viz., $(x_1, x_2, y) \mapsto \frac{1}{x_1^2+x_2^2+y^2}(x_1, x_2, y)$) followed by reflection $(x_1, x_2, y) \mapsto (x_1, -x_2, y)$ followed by $(x_1, x_2, y) \mapsto (2\sqrt{3}x_1 + \sqrt{3} - 1, 2\sqrt{3}x_2, 2\sqrt{3}y)$. In other words:

$$(x_1, x_2, y) \mapsto \frac{2\sqrt{3}}{(x_1 - 1)^2 + x_2^2 + y^2}(x_1 - 1, -x_2, y) + (\sqrt{3} - 1, 0, 0).$$

The inverse map, which maps OUR fundamental domain onto McMullen's, is:

$$(x_1, x_2, y) \mapsto \frac{2\sqrt{3}}{(x_1 + 1 - \sqrt{3})^2 + x_2^2 + y^2}(x_1 + 1 - \sqrt{3}, -x_2, y) + (1, 0, 0).$$

(Checked in Maple...)

Now we have to note carefully that McMullen's program computes an invariant measure *with respect to the conformal metric* $|dz|$*;* let us call this measure μ! What we need in Patterson-Sullivan's formula is the invariant measure *with respect to the conformal metric* $|dz|/(1+|z|^2)$ (up to proportionality). By [13, p.695 line -4] this measure is: $(1 + |z|^2)^{-\delta} \cdot d\mu(z)$.

Further implementation details: Checking with

```
plainFvalues([0,1,2,3,4,5,6,7,8,9],[0.7],2.5,1.5,20,1,1)
plainFvalues([0,1,2,3,4,5,6,7,8,9],[0.6],2.5,1.5,20,1,1)
plainFvalues([0,1,2,3,4,5,6,7,8,9],[0.5],2.5,1.5,20,1,1)
plainFvalues([0,1,2,3,4,5,6,7,8,9],[0.4],2.5,1.5,20,1,1)
plainFvalues([0,1,2,3,4,5,6,7,8,9],[0.3],2.5,1.5,20,1,1)
```

etc we see that $n = 0, 1, 2, 3, 4, 5, 6, 7$ should suffice to give ~ 5 digits accuracy. Also playing with `th=2.0` etc seem to show that this suffices there also, as expected. We can also check this with:

```
checkplainFvalues(7,[0.2,0.4],0.3,2.5,1.5,20)
checkplainFvalues(7,[0.2,0.4],0.4,2.5,1.5,20)
checkplainFvalues(7,[0.2,0.4],0.5,2.5,1.5,20)
checkplainFvalues(7,[0.2,0.4],0.6,2.5,1.5,20)
```

etc.

We next wish to see which parameters are needed to get a sufficiently good storing of the K-L transforms of each Fourier coefficient. For $n = 2$ it seems we may have to integrate over $t \in [0, 35]$. Now the following tests show that perhaps $u \in [-10, 5]$ with $usplit = 15$ *may* suffice for $t \in [\varepsilon, 30]$, but it certainly does *not* suffice for $t \in [\varepsilon, 35]$.

```
? allocatemem(8000000)
? Fouriercn(2,2.5,1.5,20,-10.0,5.0,20,[0.2,1.0,30.0])
Now calling Feval...
Now Feval done...
%5 = [-0.000009173043245710954338628182885, -0.0003451966600873334097272650441, 24003575311
? Fouriercn(2,2.5,1.5,20,-10.0,5.0,10,[0.2,1.0,30.0])
Now calling Feval...
Now Feval done...
%6 = [-0.000009173043245656741422442821643, -0.0003451966600872721814817934956, -3762995767
? Fouriercn(2,2.5,1.5,20,-10.0,5.0,15,[0.2,1.0,30.0])
Now calling Feval...
Now Feval done...
%7 = [-0.000009173043245588330367633343305, -0.0003451966600869802358710737971, 24032986629
? Fouriercn(2,2.5,1.5,20,-9.0,4.0,15,[0.2,1.0,30.0])
Now calling Feval...
Now Feval done...
%8 = [-0.000009113341911813752730752355958, -0.0003447393833957417922462050236, 24564155879
? Fouriercn(2,2.5,1.5,20,-9.0,4.0,15,[35.0])
Now calling Feval...
Now Feval done...
%9 = [4563012293775336.669603912306]
? Fouriercn(2,2.5,1.5,20,-10.0,5.0,15,[35.0])
Now calling Feval...
Now Feval done...
%10 = [-11175696607474766.48036646839]
? Fouriercn(2,2.5,1.5,20,-10.0,5.0,16,[35.0])
Now calling Feval...
Now Feval done...
%11 = [7617092306308477.404687969482]
?
```

The tests become quite time-consuming, and it seems wise to perhaps store the $F_n(e^u)$-function as a Hermite linear combination or similar!?!

```
Fouriercn(2,2.5,1.5,20,-7.5,3.0,10,[0.2,1.0,30.0])
```

```
Now calling Feval...
Now Feval done...
%56 = [-0.0000088798701974963371248447885 81, -0.00034544044265489027133938644 15, 2226702363
? Fouriercn(2,2.5,1.5,20,-9.0,4.0,20,[0.2,1.0,30.0])
Now calling Feval...
Now Feval done...
%57 = [-0.000009113341911812079364865220836, -0.0003447393833956488732601582774, 2456458156
? Fouriercn(2,2.5,1.5,20,-10.0,5.0,20,[0.2,1.0,30.0])
BUT, MUCH BETTER:
getbackFny(2,[0.2,0.4,0.6,0.8,1.0,1.2,1.4,1.6],2.5,1.5,20,-7.0,2.5,6,30.0,10);
and EVEN better:
getbackFny(2,[0.2,2.0,3.0,4.0],2.5,1.5,20,-7.0,2.5,9,35.0,7);
```

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