Hejhal's Algorithm in $\mathbb{H}^3$

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In this document, we describe Hejhal's algorithm as it applies to hyperbolic 3-space. It is essentially the same as in 2-space; the differences occur only in the details of discrete groups acting on $\mathbb{H}^3$. We begin with some background regarding hyperbolic 3-space, then we move on to Maass forms in this context. After that, we will be ready to discuss the algorithm.

Preliminaries on $\mathbb{H}^3$

By *hyperbolic 3-space* (or $\mathbb{H}^3$), we mean the unique simply-connected Riemannian 3-manifold with constant sectional curvature -1. There are various models of this space, but for now we will work with the upper half-space model. Let the set U^3 consist of the subset of $\mathbb{R}^3$ with positive third component

$$U^3 = \{(x_1, x_2, y) \in \mathbb{R}^3 : y > 0\}$$

We endow U^3 with the metric

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dy^2}{y^2}$$

It is well known that (U^3, ds^2) is then isometric to $\mathbb{H}^3$. Moreover, we note that $\mathbb{H}^3$ can be identified with the Hamiltonian quaternions whose k-term is 0. That is, we take the subalgebra of

$$\{x_1 + x_2 i + x_3 j + x_4 k : x_1, x_2, x_3, x_4 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}$$

whose elements have $x_3 > 0$ and $x_4 = 0$. Replacing the name "x_3" with "y," we evidently have the same set U^3.

Möbius Transformations in 3-space

Recall that $\mathrm{SL}(2, \mathbb{R})$ acts on the upper half plane $\mathbb{H}^2$ via Möbius transformations. It is well known that these give all orientation-preserving isometries of hyperbolic 2-space. In a similar fashion, $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$. Specifically, a given matrix acts on $\mathbb{H}^3$ via Möbius transformations, where we interpret division as multiplication by the inverse in the quaternions:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(z) = (\alpha z + \beta)(\gamma z + \delta)^{-1}$$

($z^{-1} = \bar{z}/|z|^2$). If interested, one can find proofs of these facts online. (See, e.g., Holodyn 2013 for an outline).

To get a geometric feel for these transformations, it is instructive to consider the *frame bundle* $\mathrm{F}\mathbb{H}^3$ of hyperbolic space. This is the set

$$\mathrm{F}\mathbb{H}^3 = \{(z, v, w) \in U^3 \times \mathbb{R}^3 \times \mathbb{R}^3 : ||(v, w)||_z = 1,\ v \perp w\}$$

where the norm $||(\cdot, \cdot)||_z$ is derived from the hyperbolic metric. **(ASK AK FOR DETAILS HERE)!** One can show that $\mathrm{PSL}(2, \mathbb{C})$ acts simply transitively on $\mathrm{F}\mathbb{H}^3$, where the action is by

$$g(z, v, w) = (g(z), \mathbf{?}, \mathbf{?}) \qquad \text{Maybe } (g(z), g'(z)v, g'(z)w)\text{? But need to think about derivative...}$$

(ASK AK FOR DETAILS HERE)! Thus, we can identify $\mathrm{PSL}(2,\mathbb{C})$ with $\mathrm{F}\mathbb{H}^3$ by associating a matrix g with the point $g(j,j,1)$. Intuitively, one thinks of z as the base point in upper half-space, v as a tangent vector pointing in the direction of a geodesic, and w as a "frame vector" which is perpendicular to the tangent vector. (Recall that geodesics in $\mathbb{H}^3$ are semicircles or vertical lines which are perpendicular to the x_1x_2-plane).

We now describe the geometry of the matrix action by describing how it moves the frame bundle. First, recall the decomposition $\mathrm{SL}(2,\mathbb{C}) = NAK$ where

$$N = \left\{ \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} : u \in \mathbb{C} \right\} \qquad A = \left\{ \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\} \qquad K = M\mathrm{SO}(2)M$$

where $\mathrm{SO}(2)$ is the special orthogonal group and

$$M = \left\{ \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

Matrices from these three groups move the frame bundle in fairly simple ways.

- A matrix in N moves the point $(j,j,1)$ to $(j+u,j,1)$; that is, the base point is shifted by $\mathrm{Re}(u)$ in the x_1 direction and by $\mathrm{Im}(u)$ in the x_2 direction. In other words, N moves points along the horosphere determined by the base point (horospheres in $\mathbb{H}^3$ are planes parallel to the x_1x_2-plane or spheres which are tangent to the x_1x_2-plane); the real part of u determines the translation in the direction of the tangent vector, and the imaginary part determines the translation in the direction of the frame vector.
- Just as in hyperbolic 2-space, a matrix in A moves points along geodesics at unit speed for time t.
- Matrices in K fix the base point $j \in \mathbb{H}^3$ in $(j,j,1)$ but can rotate the frame bundle to any position. Specifically, matrices in M rotate the frame vector while leaving the tangent vector fixed, while matrices in $\mathrm{SO}(2)$ have the opposite effect.

The Action of Hyperbolic Elements

We will be interested in subgroups $\Gamma \subseteq \mathrm{PSL}(2,\mathbb{C})$ which act discontinuously on $\mathbb{H}^3$. Such subgroups always contain *loxodromic* elements (diagonalizable matrices with eigenvalues of norm not equal to ± 1). In the special case where a loxodromic element has *real* eigenvalues, we call the matrix *hyperbolic*. Now suppose Γ contains a hyperbolic element γ, and let $g \in \mathrm{SL}(2,\mathbb{C})$ diagonalize it. Specifically, suppose

$$g\gamma g^{-1} = \begin{pmatrix} \sqrt{\kappa} & \\ & \sqrt{\kappa}^{-1} \end{pmatrix}$$

where $\kappa > 1$. Then there exists a fundamental domain for $g\Gamma g^{-1}/\mathbb{H}^3$ contained in

$$F = \{ z \in \mathbb{H}^3 : 1 \le ||z||_2 \le \kappa \}$$

since $g\gamma g^{-1}$ acts on the base points by scaling by κ. When the group $\Gamma\backslash\mathbb{H}^3$ has infinite covolume, then such a fundamental domain must contain a positive-measure set in the x_1x_2-plane as part of its boundary. We call these fundamental domains *flare domains*, as the definition follows the same lines as those in $\mathbb{H}^2$.

Fourier Expansion in a Flare

In the upper-half plane model of $\mathbb{H}^3$, the Laplacian is

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y^2} \right) + y \frac{\partial}{\partial y}$$

To make use of the invariance under $z \mapsto \kappa z$ in a flare domain, we will utilize spherical coordinates. Specifically, we write (r,θ,φ) for a point in $\mathbb{H}^3$, where r is the (Euclidean) distance from the origin, θ is the angle of the projection onto the x_1x_2-plane measured counterclockwise from the positive x_1-axis, and φ is the smaller angle measured off of the positive y-axis. Note that since we are working with the upper-half plane model, $\varphi \in [0, \pi/2]$ with $\varphi = \pi/2$ indicating a point on the boundary. In these coordinates, one can check that the Laplacian becomes

$$\Delta = -\left(r^2\cos^2\varphi\frac{\partial^2}{\partial r^2} + \cot^2\varphi\frac{\partial^2}{\partial\theta^2} + \cos^2\varphi\frac{\partial^2}{\partial\varphi^2} + r\cos^2\varphi\frac{\partial}{\partial r} + \cot\varphi\frac{\partial}{\partial\varphi}\right)$$

Next, suppose f is a Maass form for $\Gamma\backslash\mathbb{H}^3$, where $\Gamma \subseteq \mathrm{PSL}(2,\mathbb{C})$ contains the diagonal matrix whose action is $z \mapsto \kappa z$. Then f has a logarithmic Fourier expansion in r.

$$f(r,\theta,\varphi) = \sum_{n\in\mathbb{Z}} a_n(\theta,\varphi)e\left(n\frac{\log r}{\log\kappa}\right)$$

Since f is an eigenfunction of the Laplacian, we have $\Delta f = \lambda f$ for some $\lambda \in \mathbb{C}$. This results in the following PDE for $a_n(\theta,\varphi)$

$$\cot^2\varphi\frac{\partial^2}{d\theta^2}a_n(\theta,\varphi) + \cos^2\varphi\frac{\partial^2}{\partial\varphi^2}a_n(\theta,\varphi) + \cot\varphi\frac{\partial}{\partial\varphi}a_n(\theta,\varphi) + \left(\lambda - \frac{4\pi^2n^2}{\log^2\kappa}\cos^2\varphi\right)a_n(\theta,\varphi) = 0$$

Let us fix n for the moment and write $a_n(\theta,\varphi) = f(\theta)g(\varphi)$. This separation of variables results in the two ODEs

$$\begin{cases} \frac{f''(\theta)}{f(\theta)} = C \\ \sin^2\varphi\frac{g''(\varphi)}{g(\varphi)} + \tan\varphi\frac{g'(\varphi)}{g(\varphi)} + \lambda\tan^2\varphi - \frac{4\pi^2n^2}{\log^2\kappa}\sin^2\varphi = C \end{cases}$$

where $C \in \mathbb{R}$ is some constant shared between the two equations. Solving the equation in θ, and using the fact that $a_n(\theta,\varphi)$ is invariant under $\theta \mapsto \theta + 2\pi$, one finds that

$$f(\theta) = e^{im\theta}$$

for some $m \in \mathbb{Z}$. Note that this restricts the constant to $C = -m^2$.

Alternatively, one could simply note that the invariance under $\theta \mapsto \theta + 2\pi$ implies $a_n(\theta,\varphi)$ has a Fourier expansion in θ. This lets us assume

$$f(r,\theta,\varphi) = \sum_{m,n\in\mathbb{Z}} g_{m,n}(\varphi)e^{im\theta}e\left(n\frac{\log r}{\log\kappa}\right)$$

Applying the differential equation $\Delta f + \lambda f = 0$, we can reduce to an ODE in $g_{m,n}(\theta)$ for each pair of $m, n \in \mathbb{Z}$. Fixing m and n for the moment, we write $g = g_{m,n}$. The ODE is then

$$\sin^2\varphi g''(\varphi) + \tan\varphi g'(\varphi) + \left(m^2 + \lambda\tan^2\varphi - \frac{4\pi^2n^2}{\log^2\kappa}\sin^2\varphi\right)g(\varphi) = 0 \tag{1}$$

which is the same as we found using separation of variables above.

In both methods, we are left trying to find a solution to (1). This can be converted to a hypergeometric differential equation as follows. First, make the change of variables

$$h(\cos^2\varphi) = \sin\varphi g(\varphi) \quad x = \cos^2\varphi$$

Then equation (1) reduces to

$$4x^2(x-1)^2h''(x) + \left(x^2 - \frac{4\pi^2n^2}{\log^2\kappa}x(1-x) + m^2x + \lambda(1-x)\right)h(x) = 0$$

Some straightforward calculations show that $x = 0, 1, \infty$ are the three regular singular points of this differential equation. Thus, the ODE has a hypergeometric function as a solution.