Project Overview

Alex Karlovitz

This document contains a brief overview of my thesis project. The task is to extend Hejhal's algorithm for computing the Fourier coefficients and Laplace eigenvalues of Maass forms to a more general situation.

Goals:

- To compute the Hausdorff dimension of the Appolonian circle packing to high precision
  - The Appolonian circle packing can be recognized as the limit set of a group action on hyperbolic 3-space
  - There is a relationship (due to Patterson-Sullivan) between the Hausdorff dimension of a limit set and the base eigenvalue of the Laplace operator
- To extend Hejhal's algorithm to work in hyperbolic 3-space

Strategy:

✓ Review previous work done by Kontorovich and Strömbergsson which extends Hejhal's algorithm to infinite volume fundamental domains (in hyperbolic 2-space)

- Modify K-S's algorithm so that it does not use the cuspidal expansion (since there is no appropriate analog of this in hyperbolic 3-space)
- Extend the new algorithm to work in 3 dimensions
- Apply the new algorithm to the Appolonian case

Algorithm Setup

We have been using Hecke triangle groups as an example for testing. The Hecke triangle group Γ_r is the one-parameter subgroup of $\mathrm{SL}(2,\mathbb{R})$ defined by

$$\Gamma_r = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -r \\ \frac{1}{r} & 0 \end{pmatrix} \right\rangle$$

We are interested in computing the base eigenvalue λ for the action of the differential operator Δ acting on $L^2(\Gamma_r \backslash \mathbb{H})$.

Note: we will often use s or ν when referring to the base eigenvalue; these are defined by

$$\lambda = s(1-s) \qquad \nu = \frac{1}{2} - s$$

To get our hands on some data, we use Fourier expansions of the base eigenfunction. We have three methods of getting an expansion. First, since automorphic functions are invariant under $z \mapsto z+1$, we have an expansion in the x-variable:

$$f(x,y) = \sum_{n \in \mathbb{Z}} a_n W_n(y) e(nx)$$

We could also conjugate our group so that it contains a non-identity diagonal element. Since this causes automorphic functions to be invariant under the map $z \mapsto \kappa z$ for some $\kappa > 1$, we get a logarithmic Fourier expansion in the ρ-variable where (ρ, θ) denotes polar coordinates on $\mathbb{H}$:

$$g(\rho, \theta) = \sum_{n \in \mathbb{Z}} b_n W_n^{(f)}(\theta) e\left(n \frac{\log \rho}{\log \kappa}\right)$$

Finally, we could use a Cayley transform to take $\mathbb{H}$ to $\mathbb{D}$. Functions on the disk are automatically invariant under $\theta \mapsto \theta + 2\pi$, so the base eigenfunction - written in polar coordinates on the disk - will have a Fourier expansion in the θ-variable:

$$h(\rho, \theta) = \sum_{n \in \mathbb{Z}} c_n W_n^{(d)}(\rho) e^{in\theta}$$

More details on the Whittaker functions $W_n, W_n^{(f)}$, and $W_n^{(d)}$ are given below.

Note: although this is not made explicit in the notation, all three Whittaker functions are dependent on the eigenvalue λ, which is of course an unknown.

The Algorithm

The basic idea of Hejhal's algorithm is that *given the eigenvalue* λ, we could set up a linear system to solve for the Fourier coefficients a_n, b_n, or c_n (or a combination thereof).

Two problems immediately come to mind:

1. We don't know the eigenvalue λ.
2. We cannot input an infinite Fourier expansion into a computer.

To rectify the second problem, we choose a cutoff M and only use terms in the expansions with $|n| \leq M$ (when we use multiple expansions, we may use different cutoffs for each expansion). The algorithm proceeds by guessing values of λ, then attempting to step towards the true eigenvalue.

The Linear System

For a fixed test point z in $\mathbb{H}$ (or $\mathbb{D}$), we have two methods of obtaining a linear equation for the Fourier coefficients:

1. Take another point z^* in the orbit of this test point, then set a Fourier expansion to be equal when evaluated at z and at z^*.
   - The base eigenfunction is equal at these two points by automorphy.
   - We will often take test points outside of a fixed fundamental domain, then choose their pullback as the other point in the orbit.
2. Set two different Fourier expansions equal to each other, both evaluated at z.

If we choose enough distinct test points, we will have more equations than variables. Then we can use least squares (or a similar method) to solve for the Fourier coefficients of interest.

Iteration of the Eigenvalue

Let us fix two sets of test points $\vec{z}_1$ and $\vec{z}_2$. Using a guess $\lambda = s(1-s)$ for the eigenvalue, we can set up a linear system as described above using the first set of test points, and solve for the Fourier coefficients. Put these in a vector $\vec{v}_1(s)$.

Note: the coefficients $\vec{v}_1(s)$ will not be the exact coefficients; this is because the value of s is just a guess, and because the expansions are computed only to a finite cutoff.

We can then do the same process using the second set of test points to get a nother vector of coefficients $\vec{v}_2(s)$. Finally, we define the function

$$E(s) = ||\vec{v}_1(s) - \vec{v}_2(s)||_2$$

Note that $E(s) \approx 0$ when $\lambda = s(1-s)$ is the true eigenvalue. So we can employ any method (like the secant method or a grid search) to try to find the zero of this function $E(s)$.

The Whittaker Functions

The base eigenfunction f is an eigenfunction of the hyperbolic Laplace operator Δ. By applying the equation

$$\Delta f + \lambda f = 0$$

to the Fourier expansion of f, we can solve a differential equation for the Fourier coefficients. In all three of our expansions, the Fourier coefficient ends up being a constant times a Whittaker function which depends on λ and the other variable.

Cuspidal Expansion

In $\mathbb{H}$, the hyperbolic Laplacian is

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

and the Whittaker function solving the differential equation is

$$W_n(y) = \begin{cases} y^{1-s} & n = 0 \\ \sqrt{y} K_\nu(2\pi|n|y) & n \neq 0 \end{cases}$$

where K_ν is the K-Bessel function.

Flare Expansion

In (ρ, θ) coordinates on $\mathbb{H}$, the hyperbolic Laplacian is

$$\Delta = \sin^2\theta \left(r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} \right)$$

and the Whittaker function solving the differential equation is

$$W_n^{(f)}(\theta) = \sqrt{\sin\theta} P_{\mu_n}^{-\nu}(\cos\theta)$$

where

$$\mu_n = -\frac{1}{2} + \frac{2\pi i n}{\log \kappa}$$

and $P_\mu^{-\nu}$ is the associated Legendre function of the first kind.

Disk Expansion

In (ρ, θ) coordinates on $\mathbb{D}$, the hyperbolic Laplacian is

$$\Delta = \frac{(1-\rho^2)^2}{4} \frac{\partial^2}{\partial \rho^2} + \frac{(1-\rho^2)^2}{4\rho} \frac{\partial}{\partial \rho} + \frac{(1-\rho^2)^2}{4\rho^2} \frac{\partial^2}{\partial \theta^2}$$

and the Whittaker function solving the differential equation is

$$W_n^{(d)}(\theta) = (1-\rho^2)^s \rho^{|n|} \, {}_2F_1(s, s+|n|, 1+|n|; \rho^2)$$

where ${}_2F_1$ is a hypergeometric function.

Computing the Special Functions

We compute the K-Bessel function using PARI's built-in function "besselk."

We compute the hypergeometric function ${}_2F_1(a,b,c;x)$ using its power series definition as given in Gradshteyn-Ryzhik, 9.100. We sum up to and including the z^N term. N is chosen using a heuristic method: stop when p (the next term being added) has decreased in value at least 10 times in a row and

$$|p| < (\text{current PARI precision}) * |x|$$

We compute the Legendre-P function using the formula Gradshteyn-Ryzhik 8.704, which relates it to the hypergeometric function.

Decay Rates of the Fourier Coefficients

Next, we list results on how quickly the Fourier coefficients decay in the three expansions. This allows us to choose the cutoff M with an accurate idea of the error in taking a finite expansion.

For proofs of the following bounds, see fourierAsymptotics.pdf.

Cuspidal Expansion

It can be shown that $K_\nu(Y)$ decays exponentially as $Y \to \infty$. On the other hand, we show that

$$|a_n| \ll_f n^{\delta-1/2}$$

(I think there is a result that in fact, $|a_n| = O(1)$, where the implied constant depends on f; this appears to be true experimentally). So, the entire Fourier coefficient

$$a_n \sqrt{y} K_\nu(2\pi|n|y)$$

decays exponentially with n. In particular, once $n \approx 1/y$, the coefficients will begin to decay very quickly.

Flare Expansion

It can be shown that

$$b_n \ll |n|^s e^{-\pi|n|\alpha/\log\kappa}$$

where α is chosen so that the flare

$$\{re^{i\theta} : 1 < r < \kappa, 0 < \theta < \alpha\}$$

is entirely contained in a single fundamental domain for $\Gamma\backslash\mathbb{H}$. On the other hand, one finds that

$$P_{\mu_n}^{-\nu}(\cos\theta) \sim (\sin\theta)^{-1/2}|n|^{-\nu-1/2}e^{2\pi|n|\theta/\log\kappa}$$

Thus the entire Fourier coefficient

$$b_n\sqrt{\sin\theta}P_{\mu_n}^{-\nu}(\cos\theta)$$

decays exponentially as $n \to \infty$ so long as $\theta < \frac{\alpha}{2}$.

To Do: think about this! I don't think I'm forcing $\theta < \alpha/2$ in the code!

Disk Expansion

In the course of the proof in fourierAsymptotics.pdf, we find that

$$|c_n W_n^{(d)}(\rho)| \leq c_0 W_0^{(d)}(\rho)$$

for all ρ and n. This is a nice, quick test to see if the code is acting as we think it should.

We also show that

$$W_n^{(d)}(\rho) \sim \rho^{|n|}$$

Plugging this into the previous equation, we see that

$$c_n \ll \rho^{-|n|}$$

as $n \to \infty$. Now, this was true for any $\rho \in [0, 1)$. The idea now is to take $\rho = 1 - \delta$, with $\delta > 0$ small. Then for any $\rho < 1 - \delta$, the entire Fourier coefficient has exponential decay:

$$c_n W_n(\rho) \ll \left(\frac{\rho}{1-\delta}\right)^{|n|}$$

Thus, inside any fixed radius $\rho \leq P$, there is a constant M so that taking the finite Fourier expansion $|n| \leq M$ gives a good approximation to the Maass form.

Question: can I take $\rho \to 1$ in the bound on c_n? I think the answer is no... Think about this!

Facts about the Base Eigenfunction

Here, we list a few facts about the base eigenfunction which allow us to reduce the number of computations. For proofs of these facts, see baseEigenfunc.pdf.

Facts:

- the base eigenfunction is real-valued and non-negative
- in the upper half plane model, the base eigenfunction $f(x + iy)$ is even in x
- in all three expansions, the constant part of the Fourier coefficients are real-valued

Using the above facts, it is a simple argument to show that we can "fold" the Fourier expansions to only have nonzero coefficients for nonnegative n. Moreover, if we choose the Cayley transform from $\mathbb{H} \to \mathbb{D}$ to force automorphic functions to be invariant under $w \mapsto -w$, we can take the odd coefficients in the disk expansion to be 0.

Here are the new expansions.

Cuspidal expansion.

$$f(x + iy) = \sum_{n=0}^{\infty} a_n W_n(y) \cos(2\pi n x)$$

Flare expansion.

$$g(r, \theta) = \sum_{n=0}^{\infty} b_n W_n^{(f)}(\theta) \cos\left(2\pi n \frac{\log r}{\log \kappa}\right)$$

Disk expansion.

$$h(\rho, \theta) = \sum_{n=0}^{\infty} a_{2n} W_{2n}^{(d)}(\rho) \cos(2n\theta)$$

Choosing Test Points

From playing around with Strömbergsson's code (which uses the cuspidal and flare expansions together), it has become clear that the choice of test points is very important to the success of the algorithm.

Some heuristics to keep in mind when choosing test points:

- For a given test point, we only want to use a specific Fourier expansion if it converges quickly at that point.
- We need to be careful that there are no redundancies between test points. Technically, if we have many more points that the number of coefficients we are solving for, the least squares won't care about the redundancies; but of course that would increase the number of computations unnecessarily.
  - Obvious example: two test points in the same orbit.
  - Example: evenness plus invariance under $x \mapsto x + 1$ in the cuspidal expansion causes invariance under reflection across the line $x = \frac{1}{2}$.
  - Similar examples occur in the flare and disk models.

Pictures of Fundamental Domains

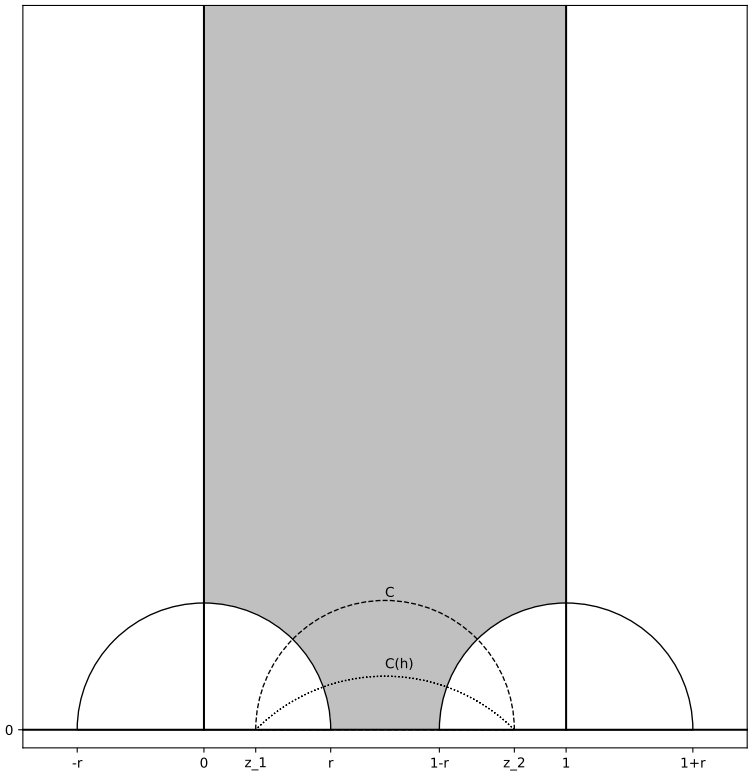

Figure 1: $\mathcal{F}_r$ for $r = 7/20$ with C and $C(h)$ for some $h > 0$

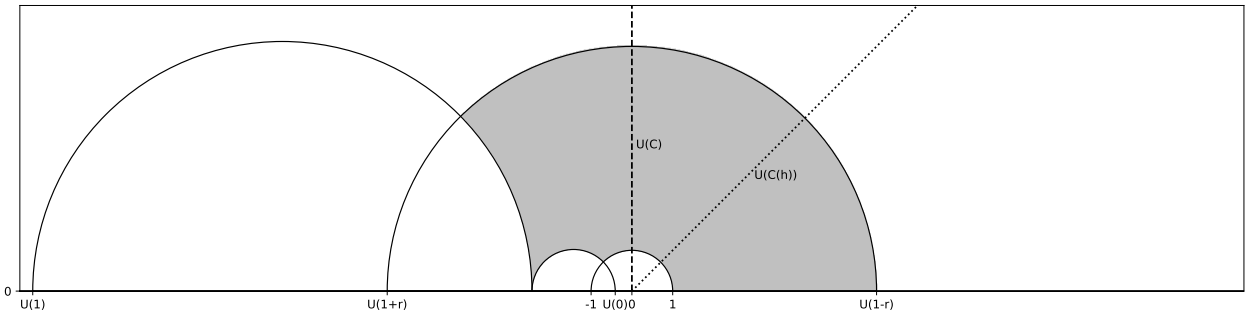

Figure 2: $U(\mathcal{F}_r)$ with $\mathcal{F}_r$ as in Figure 1

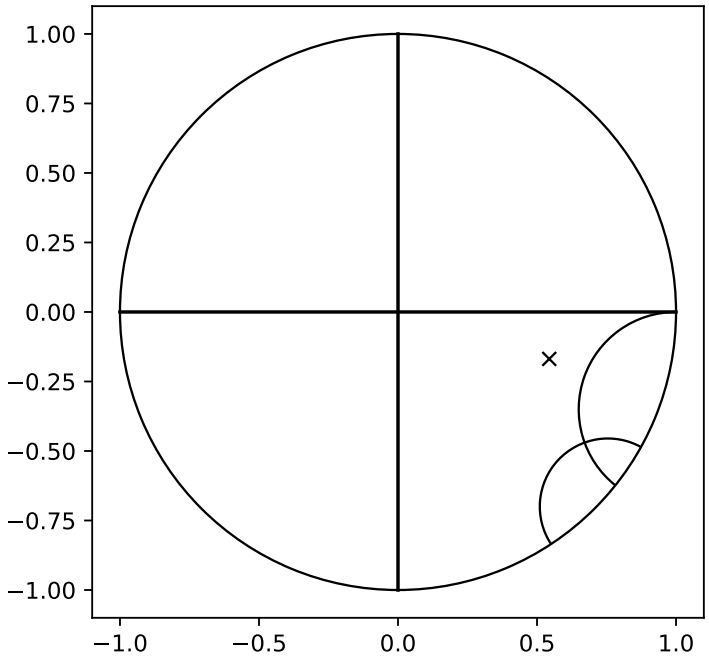

Figure 3: $\mathcal{F}_r$ in the disk

Choices of Test Points

- In the one version of the code which is working, we are using the cuspidal and flare expansions together; we use a set of test points obtained from each expansion
  - We take half a horocycle at the height where C intersects the fundamental domain in Figure 1
  - We take points on a ray out from the origin at a fixed angle and with radii between 1 and $\sqrt{\kappa}$ in Figure 2
- The idea of admissible test points: call a point *admissible* with respect to an expansion if it is in an area where the expansion is expected to converge quickly
  - A point is admissible with respect to the cuspidal expansion if its height is above some fixed lower bound y_0
  - A point is admissible with respect to the flare expansion if its angle is less that some fixed upper bound θ_0

  - A point is admissible with respect to the disk expansion if its radius is less than some fixed upper bound ρ_0

- Ideas for the new algorithm which uses the disk and flare models together
  - Keep the flare test points as they are, and replace the cuspidal test points with something which makes sense for the disk model
  - Perhaps some points in the third and fourth quadrants at a fixed radius ρ, while allowing the angle θ to vary

Questions for Andreas

- I noticed that you chose way more test points already in the chosen fundamental domain than outside of it. So most of the equations in the linear system are obtained by comparing the cuspidal expansion to the flare expansion (as opposed to comparing a single expansion to itself at two equivalent points). Do you have a heuristic reason for this decision?
- I noticed that when you chose your two sets of test points, you chose them to be very similar (i.e., very close together). I could see an argument for choosing the two sets much farther apart: that would make the linear systems look very different, so the only chance they should have of giving the same solution is when we input the true eigenvalue. I tried this, and it did not work as well. Do you have a heuristic reason for choosing these closer together?
- Generally, do you have any thoughts for tests that could be done to determine why the algorithm is failing in the disk case?

Ideas Sparked by the Meeting

- Andreas told us a heuristic idea for how he thinks about using different expansions
  - Break the fundamental domain into finitely pieces such that, in each piece, we have chosen a "best" expansion for test points in that area
  - Example: in a neighborhood of a flare, we could use a hyperbolic expansion, while in a neighborhood of a cusp, we could use a parabolic expansion
- Recall Hejhal's idea for cutting down on computations: summing over a horocycle can pick out the n^{th} Fourier coefficient (up to very small errors; see Booker-Strömbergsson-Venkatesh for a good discussion)
  - Can doing the same thing in the flare and disk models save computation time?
  - I think I already thought about this a while ago; problem is that we cannot compare expansions in this way
  - Could still be useful to use this method for *some* of the equations, while getting the rest of the equations by comparing expansions
- Review previous work by K-S. Why are they only able to find 40ish decimal places?
  - Check if computation of hypergeometric function is inaccurate to more decimal places (Strömbergsson made a note about this in his code)
    - Maybe compare to Maple/Mathematica?
  - If not that, what else could it be? Think about how to determine this.
- Try using all three expansions together in the Hecke group examples.
  - This could be a useful test of the disk model
  - Perhaps this could give faster convergence!

- We could also start trying our algorithm out on a variety of examples
  - McMullen 1998 discusses various examples in Section 3; I should take time to read through these examples
  - One example which jumps out is a Schottky group with multiple flares and no cusps!
- Moving up to hyperbolic 3-space
  - Keep reading about 3-space with Brooke, since the eventual goal is to go up a dimension
  - Could be interesting to move to 3-space soon and try examples which avoid the problems inherent in the Appolonian case (Kontorovich suggested an infinite volume fundamental domain with a bounding square)
- Is there some way of capturing L^2 decay at the cusp without explicitly using the parabolic expansion?
  - Perhaps some kind of sum across a horocycle?
  - This seems to me to defeat the purpose of avoiding the cusp in computations, but I should think more about it