# A Disk Model Version of Hejhal's Algorithm

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**Note:** These are pretty old notes. While I do believe they are correct, we use the less common definition which takes $\Delta = -y^2(\partial_{xx} + \partial_{yy})$. I had done this because it makes the operator $\Delta$ positive definite.

Instead, most authors take $\Delta = y^2(\partial_{xx} + \partial_{yy})$ but define the $\lambda$ values by the equation $\Delta u + \lambda u = 0$. In either case, the values of interest $\lambda$ are strictly positive.

I should update these notes at some point to match the convention.

## 1 Fourier Expansion in the Disk Model

Recall that we can map our work in the upper half plane to the unit disk via

$$z \mapsto \frac{z-i}{z+i}$$

(The inverse map is $\tau \mapsto i(1+\tau)/(1-\tau)$, where $\tau$ is taken in the unit disk).

The new idea here is to apply Hejhal's algorithm to the fundamental domain in $\mathbb{D}$. To do this, we begin by writing a Maass form $f$ in polar form $f(\rho, \theta)$. Then $f$ is $2\pi$-periodic in $\theta$, and so it has a Fourier expansion

$$f(\rho, \theta) = \sum_n a_n(\rho) \exp(in\theta)$$

Next, since $f$ is a Maass form, it is an eigenfunction of the hyperbolic Laplace operator, say $\Delta f = s(1-s)f$. After mapping to $\mathbb{D}$ and converting to polar coordinates, this operator is

$$\Delta = -\left( \frac{(1-\rho^2)^2}{4} \frac{\partial^2}{\partial \rho^2} + \frac{(1-\rho^2)^2}{4\rho} \frac{\partial}{\partial \rho} + \frac{(1-\rho^2)^2}{4\rho^2} \frac{\partial^2}{\partial \theta^2} \right) \tag{1}$$

If we would like to treat our function with $\theta$ taking values in the range $[0, \pi]$, then the operator changes according to the map $\theta \mapsto \theta/2$. The new operator is then

$$\Delta = -\left( \frac{(1-\rho^2)^2}{4} \frac{\partial^2}{\partial \rho^2} + \frac{(1-\rho^2)^2}{4\rho} \frac{\partial}{\partial \rho} + \frac{(1-\rho^2)^2}{16\rho^2} \frac{\partial^2}{\partial \theta^2} \right)$$

This version is often used when one identifies the upper half plane with $PSL(2, \mathbb{R})$, for then the $\theta$ in the $KAK$ decomposition corresponds to half of the polar $\theta$.

Applying (1) to the Fourier expansion, then using uniqueness of Fourier coefficients, we have that the $a_n(\rho)$'s satisfy the differential equation

$$\frac{(1-\rho^2)^2}{4} a_n''(\rho) + \frac{(1-\rho^2)^2}{4\rho} a_n'(\rho) - \frac{n^2(1-\rho^2)^2}{4\rho^2} a_n(\rho) = s(s-1) a_n(\rho)$$

This equation has solution

$$a_n(\rho) = a_n (1-\rho^2)^s \rho^{|n|} {}_2F_1(s, s+|n|, 1+|n|; \rho^2) \tag{2}$$

where ${}_2F_1$ is a hypergeometric function.

To get a linear system, we first approximate $f$ by a finite Fourier expansion with $|n| \leq M$ (we are no longer assuming $f$ is a cusp form). We choose $M$ large enough that

$$f(\rho, \theta) = \sum_{|n| \leq M} a_n(\rho) \exp(in\theta) + E(\rho)$$

where $E(\rho)$ is "small" (like, smaller than $10^{-D}$ if we are looking for $D$ digits of precision). The reason we can make this error small is that the $a_n(\rho)$'s are decaying exponentially as $|n| \to \infty$.

Next, we fix a distance $\rho = P$ from the origin and take $N \geq 2M + 1$ equally spaced $\theta$'s to produce sample points. For example, we could take the points $(P, \theta_j)$ where

$$\theta_j = \frac{2\pi}{N}\left(j + \frac{1}{2}\right) \qquad \text{for} \qquad 0 \leq j < N$$

We can then extract an expression for the Fourier coefficients by taking an appropriate linear combination of the function evaluated at the sample points. This expression is

$$a_n(P) = \frac{1}{N} \sum_{j=0}^{N-1} f(P, \theta_j) e^{-in\theta_j} + E(P)$$

Finally, we can set up a $(2M+1) \times (2M+1)$ linear system by using the $\Gamma$-automorphicity of $f$. Let $(P_j^*, \theta_j^*)$ be the point in the chosen fundamental domain which is equivalent to $(P, \theta_j)$ mod $\Gamma$. Some straightforward computation gives the following system

$$a_n c_n(P) = \sum_{|\ell| \leq M} a_\ell V_{n\ell} + 2E(P)$$

where

$$c_n(\rho) = (1 - \rho^2)^s \rho^{|n|} \, {}_2F_1(s, s + |n|, 1 + |n|; \rho^2) \qquad \text{and} \qquad V_{n\ell} = \frac{1}{N} \sum_{j=0}^{N-1} c_\ell(P_j^*) e^{i(\ell\theta_j^* - n\theta_j)}$$

**Note**: recall that the code is varying the parameter $\nu$ in $\lambda = (1/2 + \nu)(1/2 - \nu)$; so we can set $s = 1/2 + \nu$ or $s = 1/2 - \nu$ so that $\lambda = s(1 - s)$.

Since the $a_n$'s should give the Fourier coefficients for a Maass form, which is an eigenfunction of the Laplacian, any scalar multiple of these coefficients will be a solution as well. We assume the scaling $a_2 = 1$ then solve the resulting system.

**Remark**: it is most natural to scale $a_2$ for the following reason.
**FINISH THIS IDEA!**

# Verification of Formula (1)

In the upper half plane model of hyperbolic space, the Laplace-Beltrami operator is

$$\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

where $z = x + iy \in \mathbb{H}$.

We can map from the upper half plane to the unit disk via the conformal map

$$z \mapsto \tau = \frac{z - i}{z + i}$$

Now if we write $\tau = u + iv \in \mathbb{D}$, we can rewrite $(u, v)$ in polar coordinates $(\rho, \theta)$ so that

$$u = \rho \cos\theta \qquad v = \rho \sin\theta$$

(these are the normal polar coordinates in $\mathbb{R}^2$).

We can go from $(\rho, \theta)$ coordinates to $(x, y)$ coordinates via the formulas

$$x = \frac{-2\rho\sin\theta}{1+\rho^2 - 2\rho\cos\theta} \qquad y = \frac{1-\rho^2}{1+\rho^2-2\rho\cos\theta}$$

For the other direction, we have

$$\rho = \sqrt{\frac{x^2+(y-1)^2}{x^2+(y+1)^2}} \qquad \tan\theta = \frac{-2x}{x^2+y^2-1}$$

(Note: the sign of $x^2 + y^2 - 1$ matches the sign of $u$ in $(u, v)$ coordinates in the disk). I wrote some code to verify these formulas.

Next, we wish to rewrite the Laplace-Beltrami operator in our new $(\rho, \theta)$ coordinates. To achieve this, let us be very clear about what we are trying to do. If $f : \mathbb{H} \to \mathbb{C}$ is a function in $(x, y)$ coordinates and $g : \mathbb{D} \to \mathbb{C}$ is a function in $(r, \theta)$ coordinates, then we think of $f$ and $g$ as the same function if

$$f(x, y) = g(\rho(x, y), \theta(x, y)) \ \ \forall\ x + iy \in \mathbb{H}$$

Now, the functions we are interested in are those which are differentiable in the $\mathbb{R}^2$ sense. So by applying the chain rule for functions from $\mathbb{R}^2$ to $\mathbb{R}^2$, we conclude that

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2\rho}{\partial x^2}\frac{\partial g}{\partial\rho} + \frac{\partial^2\theta}{\partial x^2}\frac{\partial g}{\partial\theta} + \left(\frac{\partial\rho}{\partial x}\right)^2\frac{\partial^2 g}{\partial\rho^2} + 2\frac{\partial\rho}{\partial x}\frac{\partial\theta}{\partial x}\frac{\partial^2 g}{\partial\rho\partial\theta} + \left(\frac{\partial\theta}{\partial x}\right)^2\frac{\partial^2 g}{\partial\theta^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2\rho}{\partial y^2}\frac{\partial g}{\partial\rho} + \frac{\partial^2\theta}{\partial y^2}\frac{\partial g}{\partial\theta} + \left(\frac{\partial\rho}{\partial y}\right)^2\frac{\partial^2 g}{\partial\rho^2} + 2\frac{\partial\rho}{\partial y}\frac{\partial\theta}{\partial y}\frac{\partial^2 g}{\partial\rho\partial\theta} + \left(\frac{\partial\theta}{\partial y}\right)^2\frac{\partial^2 g}{\partial\theta^2}$$

In the computation for the Laplacian, we will end up adding these two expressions together. I used sympy (python's symbolic programming package) to do the computations. One finds that

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial\rho}{\partial x}\frac{\partial\theta}{\partial x} + \frac{\partial\rho}{\partial y}\frac{\partial\theta}{\partial y} = 0$$

So, the $\partial g/\partial\theta$ and $\partial^2 g/\partial\rho\partial\theta$ terms vanish. For the other three terms, we get

$$\frac{\partial^2\rho}{\partial x^2} + \frac{\partial^2\rho}{\partial y^2} = \mathbf{???}$$

$$\left(\frac{\partial\rho}{\partial x}\right)^2 + \left(\frac{\partial\rho}{\partial y^2}\right)^2 = \frac{4}{(x^2+(y+1)^2)^2}$$

$$\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y^2}\right)^2 = \frac{4}{(x^2+(y-1)^2)(x^2+(y+1)^2)}$$

Multiplying by $y^2$, then substituting in the expressions for $x$ and $y$ in terms of $\rho$ and $\theta$, we get

$$y^2\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = \frac{(1-\rho^2)^2}{4}\frac{\partial^2 g}{\partial\rho^2} + (\mathbf{???})\frac{\partial g}{\partial\rho} + \frac{(1-\rho^2)^2}{4\rho^2}\frac{\partial^2 g}{\partial\theta^2}$$

It turns out sympy has a little trouble with one of the terms, but following the pattern and plugging in a few test points suggests that the missing piece should be $(1-\rho^2)^2/4\rho$.

## Verification of Formula (2)

In the literature, it is claimed that the differential equation

$$\frac{(1-\rho^2)^2}{4}a_n''(\rho) + \frac{(1-\rho^2)^2}{4\rho}a_n'(\rho) - \frac{n^2(1-\rho^2)^2}{16\rho^2}a_n(\rho) = s(1-s)a_n(\rho) \tag{3}$$

has solution

$$a_n(\rho) = (1-\rho^2)^s \rho^{|n|}\, {}_2F_1(s, s+|n|, 1+|n|; \rho^2) \tag{4}$$

We will first verify this, as it is a simple change of variables to verify our solution (2).

### An Aside on the Hypergeometric Function

Here, we list a few properties of the hypergeometric function which we shall use. The hypergeometric function ${}_2F_1$ is defined for $|z| < 1$ by

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

where $(q)_n$ is the Pochhammer symbol

$$(q)_n = \begin{cases} 1 & n = 0 \\ q(q+1)\cdots(q+n-1) & n \neq 0 \end{cases}$$

It can also be defined as a solution to the differential equation

$$z(1-z)\frac{d^2w}{dz^2} + [c - (a+b+1)z]\frac{dw}{dz} - abw = 0 \tag{5}$$

Let us verify that our power series solves this differential equation. First, notice that $(q)_{n+1} = q(q+1)_n$ (this is a simple exercise). Hence,

$$\frac{d}{dz}\,{}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}}\frac{z^n}{n!} = \frac{ab}{c}\sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n}\frac{z^n}{n!} = \frac{ab}{c}\,{}_2F_1(a+1, b+1, c+1; z)$$

and

$$\frac{d^2}{dz^2}\,{}_2F_1(a, b, c; z) = \frac{a(a+1)b(b+1)}{c(c+1)}\,{}_2F_1(a+2, b+2, c+2; z)$$

Plugging this into the differential equation, one finds that there are five series to deal with. Computing coefficients of $z^n$ for every $n$ is a tedious exercise, but I went for it. All of them came out to be zero, verifying that the hypergeometric function is a solution to (5).

### Verifying Equation (4)

Next, we return to the differential equation (3). We take $q_k$ to be the $k^{th}$ coefficient in the power series expansion of ${}_2F_1$; that is,

$$q_k = \frac{(s)_k(s+|n|)_k}{(1+|n|)_k k!}$$

Then our conjectured solution is

$$a_n(\rho) = (1-\rho^2)^s \sum_{k=0}^{\infty} q_k \rho^{2k+|n|}$$

Now we can take $n \geq 0$, since this solution (as well as the differential equation) is invariant under the map $n \mapsto -n$.

We begin with the case $n \geq 2$, since for $n = 0$ and $n = 1$, the first one or two terms are annihilated by taking derivatives. In the case $n \geq 2$, we have

$$a_n'(\rho) = -2s(1-\rho^2)^{s-1}\sum_{k=0}^{\infty} q_k \rho^{2k+n+1} + (1-\rho^2)^s \sum_{k=0}^{\infty}(2k+n)q_k\rho^{2k+n-1}$$

$$\begin{aligned} a_n''(\rho) = 4s(s-1)(1-\rho^2)^{s-2}\sum_{k=0}^{\infty} q_k\rho^{2k+n+2} - 2s(1-\rho^2)^{s-1}\sum_{k=0}^{\infty}(4k+2n+1)q_k\rho^{2k+n} \\ + (1-\rho^2)^s\sum_{k=0}^{\infty}(2k+n)(2k+n-1)q_k\rho^{2k+n-2} \end{aligned}$$

# Old Ideas

First, note that we are working with the function $h(\rho) = {}_2F_1(a,b,c;\rho^2)$; that is, we have a square in the last component instead of just a $z$. Applying the chain rule, one can verify that the differential equation satisfied by the original hypergeometric function turns into the following for $h(\rho)$:

$$\frac{1-\rho^2}{4}h''(\rho) + \left(\frac{2c-2(a+b)\rho^2-\rho^2-1}{4\rho}\right)h'(\rho) - abh(\rho) = 0 \tag{6}$$

This will come up in our computation.

Next, to solve the differential equation, we make the guess

$$a_n(\rho) = g(\rho)h(\rho)$$

where $h(\rho) = {}_2F_1(a,b,c;\rho^2)$ as before, and $g(\rho)$ is to be determined. Plugging this into (7) gives

$$\begin{aligned} \left[\frac{(1-\rho^2)^2}{4}g(\rho)\right]h''(\rho) + \left[\frac{(1-\rho^2)^2}{4}\left(2g'(\rho)+\frac{1}{\rho}g(\rho)\right)\right]h'(\rho) \\ + \left[\frac{(1-\rho^2)^2}{4}\left(g''(\rho)+\frac{1}{\rho}g'(\rho)-\frac{n^2}{\rho^2}g(\rho)\right)\right]h(\rho) = s(1-s)g(\rho)h(\rho) \end{aligned}$$

It seems to me that this differential equation could be solved by choosing a $g$ which makes the coefficient of $h(\rho)$ on the left hand side equal to 0. Then the hope is that we can choose $a,b,c$ in the definition of $h(\rho)$ so that applying equation (6) gives the solution we are looking for. **This step is just a guess; if this method fails to provide a solution, may need to come back and change this.**

Making the coefficient of $h(\rho)$ equal to zero in the equation above is equivalent to the differential equation

$$\rho^2 g''(\rho) + \rho g'(\rho) - n^2 g(\rho) = 0$$

This is an Euler differential equation with solutions $g_1(\rho) = \rho^n$ and $g_2(\rho) = \rho^{-n}$. Thinking ahead to the Fourier expansion in the disk model, it will be far preferable to have $\rho^{|n|}$ as opposed to $\rho^{-|n|}$, so we will start with this guess. **At this step, we have made the assumption $n \neq 0$; we will have to return to the DE later to consider the case $n = 0$.**

## Solving the Differential Equation for the Fourier Coefficients

We are trying to solve the differential equation

$$\frac{(1-\rho^2)^2}{4}a_n''(\rho) + \frac{(1-\rho^2)^2}{4\rho}a_n'(\rho) - \frac{n^2(1-\rho^2)^2}{4\rho^2}a_n(\rho) = s(1-s)a_n(\rho) \tag{7}$$

for the function $a_n(\rho)$.

Since the coefficients of $a_n(\rho)$ and its derivatives in Equation (7) are all polynomials in $\rho$, we attempt to find a power series solution. So we guess

$$a_n(\rho) = \sum_{m=0}^{\infty} c_m \rho^m$$

Plugging this in and putting everything into one series, we get

$$A(c_0, c_1, c_2, c_3, \rho, \rho^2, \rho^3) + \sum_{m=4}^{\infty} B_m(c_m, c_{m-2}, c_{m-4})\rho^m = 0$$

where $A$ and $B_m$ are the following polynomials:

$$\begin{aligned} A = (&-4s(1-s)c_1 + 2c_1 n^2 - 2c_1 + 6c_2 - c_3 n^2 + 3c_3)\rho^3 \\ &+ (-4s(1-s)c_0 + 2c_0 n^2 - c_2 n^2 + 4c_2)\rho^2 + (c_1 - c_1 n^2)\rho - c_0 n^2 \end{aligned}$$

$$B_m = (m^2 - n^2)c_m - (2(m-2)^2 + 4s(1-s) - 2n^2)c_{m-2} + ((m-4)^2 - n^2)c_{m-4}$$

Now, to solve the differential equation, we need every power of $\rho$ to have a coefficient of 0. In other words, the coefficients in $A$ as a polynomial of $\rho$ need to be 0, and $B_m$ must be equivalently 0 for all $m$. We proceed by making some observations about the $c_m$'s which will simplify the computation.

**Observation 1:** The first coefficient $c_k$ which can be nonzero is for $k = |n|$. To see this, consider the four cases $n^2 = 0, n^2 = 1, n^2 = 4$, and $n^2 > 4$ separately. In the first three cases, the result follows from studying the polynomial $A$. In the fourth case, one finds that $c_0 = c_1 = c_2 = c_3 = 0$ by studying $A$. Then since $c_m$ depends linearly on $c_{m-2}$ and $c_{m-4}$ for $m \geq 4$, we must have that $c_m = 0$ until the coefficient of $c_m$ is 0. This happens when $m = |n|$.