A Geodesic/Horocycle Flow Approach to Hejhal's Algorithm

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Flows in the Disk Model

We wish to describe the geodesic and horocycle flows in the disk model. We do so by recalling the salient facts in the upper half plane model, all the while noting how the Cayley transform moves the behavior into the disk.

Matrix Groups Acting on Hyperbolic Space

The orientation-preserving isometries on the upper half plane $\mathbb{H}$ are exactly those Möbius transformations defined by $\mathrm{PSL}(2, \mathbb{R})$. Recall that we can map from $\mathbb{H}$ to the unit disk $\mathbb{D}$ via the Cayley transform

$$C(z) = \frac{z - i}{z + i}$$

We can of course map back by the inverse

$$C^{-1}(z) = i\frac{1 + z}{1 - z}$$

So for $g \in \mathrm{PSL}(2, \mathbb{R})$ acting on $\mathbb{H}$, the equivalent action on $\mathbb{D}$ is CgC^{-1}. One can show that

$$C\mathrm{PSL}(2, \mathbb{R})C^{-1} = \mathrm{PSU}(1, 1) := \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in M(2, \mathbb{C}) : |\alpha|^2 - |\beta|^2 = 1 \right\} / \{\pm I\}$$

where we are thinking of C as the matrix $\left(\begin{smallmatrix} 1 & -i \\ 1 & i \end{smallmatrix}\right) \in \mathrm{PSL}(2, \mathbb{C})$, i.e., the usual identification of Möbius transformations with matrices.

Next, recall that we extend the action of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathbb{H}$ to one on $T^1\mathbb{H}$ via

$$Dg(z, v) = \left(\frac{az + b}{cz + d}, \frac{1}{(cz + d)^2} v \right)$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. One can check that

1. D is indeed an action
2. D preserves the Riemannian metric on $\mathbb{H} \times \mathbb{C}$
3. $\mathrm{PSL}(2, \mathbb{R})$ acts simply transitively on $T^1\mathbb{H}$

Thus, we can identify $\mathrm{PSL}(2, \mathbb{R})$ with $T^1\mathbb{H}$ via this action by choosing a reference point in $T^1\mathbb{H}$. The typical choice is (i, i), so we identify $(z, v) \in T^1\mathbb{H}$ with the unique $g \in \mathrm{PSL}(2, \mathbb{R})$ such that $Dg(i, i) = (z, v)$.

To think about the corresponding extended action on $T^1\mathbb{D}$, we first need to extend the map $C : \mathbb{H} \to \mathbb{D}$ to a map from $T^1\mathbb{H}$ to $T^1\mathbb{D}$. The most natural such extension is

$$DC(z, v) = (C(z), C'(z)v) = \left(\frac{z - i}{z + i}, \frac{2i}{(z + i)^2} v \right)$$

(see Appendix A for why this extension is natural). One easily verifies that if $(z, v) \in T^1\mathbb{H}$ then $DC(z, v) \in T^1\mathbb{D}$. (Recall that the Riemannian metric on the tangent space at the point $p \in \mathbb{D}$ is given by $\langle v, w\rangle_p = 4v \cdot w/(1 - |p|^2)^2$).

Now that we can map between $T^1\mathbb{H}$ and $T^1\mathbb{D}$, we can extend the action of $\mathrm{PSU}(1, 1)$ on $\mathbb{D}$ to one on $T^1\mathbb{D}$. By passing through $T^1\mathbb{H}$ on the way, we see that the action should be $(DC)(Dg)(DC)^{-1}$ where $\gamma = CgC^{-1}$ is an element of $\mathrm{PSU}(1, 1)$. It is straightforward to check that this is equivalent to the action

$$D\gamma(z, v) = (\gamma(z), \gamma'(z)v) = \left(\frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}, \frac{1}{(\bar{\beta} z + \bar{\alpha})^2} v\right)$$

where

$$\gamma = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{PSU}(1, 1)$$

In other words, the matrix action is the same as the one we are used to in the upper half plane model; it is just a different matrix group for the disk.

Finally, just as we identified $\mathrm{PSL}(2, \mathbb{R})$ with $T^1\mathbb{H}$, we can identify $\mathrm{PSU}(1, 1)$ with $T^1\mathbb{D}$. We will choose the reference point $(0, 1/2) \in T^1\mathbb{D}$ (since this is the image of the old reference point (i, i) under DC). That is, $\mathrm{PSU}(1, 1)$ is identified with $T^1\mathbb{D}$ via

$$\gamma \longleftrightarrow D\gamma\left(0, \frac{1}{2}\right)$$

Geodesic Flow

Recall that the geodesics in $\mathbb{H}$ are vertical lines and semicircles perpendicular to the real line (in other words, the top half of circles with centers on the real axis). One can show that a point in $\mathbb{H}$ paired with a direction uniquely defines a geodesic. In other words, each $(z, v) \in T^1\mathbb{H}$ gives a geodesic. Of course, different points in the tangent bundle can lead to the same geodesic; we typically think of this as taking different *starting points* on the same geodesic.

The *geodesic flow* beginning at a point $(z, v) \in T^1\mathbb{H}$ is defined by flowing along the geodesic defined by (z, v) in the direction of v at a constant speed (speed is defined with respect to the Riemannian metric). Thus, for each starting point $g \sim (z, v) \in T^1\mathbb{H}$ (where we write g for the matrix in $\mathrm{PSL}(2, \mathbb{R})$ corresponding to (z, v)), geodesic flow g_t is a function of time. Note that $g_0 = (z, v)$ and g_t for $t < 0$ refers to flowing in the opposite direction (i.e., $-v$).

We can describe geodesic flow very easily in terms of matrices. First, one computes that the geodesic flow starting at (i, i) is given by $(e^t i, e^t i)$. In terms of $\mathrm{PSL}(2, \mathbb{R})$ - where (i, i) is identified with I - the flow is given by the matrix

$$a_t = \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix}$$

(This is easily checked by verifying that $a_t \sim (e^t i, e^t i)$ in the identification). Next, we can observe that the geodesic flow starting at any point (z, v) is equivalent to mapping the flow starting at (i, i) by the matrix $g \in \mathrm{PSL}(2, \mathbb{R})$ which has $g \sim (z, v)$. This works because Dg is an isometry. Therefore, geodesic flow for time t starting at $g \sim (z, v)$ is succinctly described as

$$g_t = g a_t$$

Next, we would like to describe the geodesic flow in the disk model. Recall that the geodesics in the disk are diameters plus half circles normal to the boundary. We can use the Cayley transform as before to map between the models. The element corresponding to a_t is

$$c_t := C a_t C^{-1} = \begin{pmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}$$

The action of this on $T^1\mathbb{D}$ is thus

$$Dc_t(z,v) = \left(\frac{\left(\cosh \frac{t}{2}\right) z + \sinh \frac{t}{2}}{\left(\sinh \frac{t}{2}\right) z + \cosh \frac{t}{2}}, \frac{1}{\left(\left(\sinh \frac{t}{2}\right) z + \cosh \frac{t}{2}\right)^2} v \right)$$

A quick check that this action makes sense is to note that $Dc_t(0, 1/2)$ does indeed give the interval $(-1, 1)$ on the real line.

Horocycle Flow

Recall that the horocycles in the upper half plane are horizontal lines and circles tangent to the boundary. Given any point $(z, v) \in T^1\mathbb{H}$, the corresponding horocycle is the set of points in the unit tangent bundle for which the distance between it and (z, v) under the geodesic flow goes to 0 as time goes to infinity. So to be more accurate, a horocycle is all the points on the circle paired with unit tangent vectors *normal* to the circle and pointing inward (for a horizontal line, the tangent vectors point up).

The (stable) *horocycle flow* starting at a point in the unit tangent bundle is defined by sliding along a horocycle at constant unit speed. For example, consider the reference point $(i, i) \in T^1\mathbb{H}$. The horocycle for this point is the set

$$\{(x + i, i) : x \in \mathbb{R}\}$$

Sliding along this set at constant (hyperbolic) speed for time s is given by

$$(i, i) \mapsto (s + i, i)$$

In terms of the identification with $\mathrm{PSL}(2, \mathbb{R})$, this is the same as $(s + i, i) = u_s(i, i)$ where

$$u_s = \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}$$

Just as with geodesic flow, we can extend the horocycle flow to any point by mapping from the reference point (i, i). Specifically, let $g \in \mathrm{PSL}_2(\mathbb{R})$ correspond to the point $(z, v) \in T^1\mathbb{H}$. Then the horocycle flow starting at this point is given by

$$h_g(s) = gu_s$$

Note that $u_s^{-1} = u_{-s}$ gives the flow in the other direction.

As usual, we would like to move this flow to the disk model. Recall that the horocycles in the disk are circles tangent to the boundary. We can use the Cayley transform as before to map between the models. The element corresponding to u_s is

$$h_s := Cu_sC^{-1} = \begin{pmatrix} 1 + i\frac{s}{2} & -i\frac{s}{2} \\ i\frac{s}{2} & 1 - i\frac{s}{2} \end{pmatrix}$$

The action of this on $T^1\mathbb{D}$ is thus

$$Dh_s(z,v) = \left(\frac{\left(1 + i\frac{s}{2}\right) z - i\frac{s}{2}}{i\frac{s}{2}z + 1 - i\frac{s}{2}}, \frac{1}{\left(i\frac{s}{2}z + 1 - i\frac{s}{2}\right)^2} v \right)$$

One way to check if this action makes sense would be to show that $Dh_s(0, 1/2)$ gives the circle with center $(1/2, 0)$ and radius $1/2$.

Appendix A: Extension of C to the Tangent Bundles

One way to find an appropriate extension is to consider the unique geodesic going through any point $(z, v) \in T^1\mathbb{H}$. Recall that the geodesic passing through that point is the set $\{ga_t(i) : t \in \mathbb{R}\}$ where g is the unique element of $\mathrm{PSL}(2, \mathbb{R})$ with $g(i, i) = (z, v)$. So the corresponding geodesic in $\mathbb{D}$ is given by $C(g a_t i)$. Taking the derivative with respect to t, we see that the tangent vectors are given by

$$C'(g e^t i) g'(e^t i) e^t i$$

We are interested in the point in $T^1\mathbb{D}$ corresponding to (z, v) so we simply take $t = 0$ and see that the tangent vector should be

$$C'(z) v$$