

# Heuristic Argument For Why Disk Model Approach Fails

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## 1 Recalling Some Numerics from Hejhal's Algorithm

In the original version of Hejhal's Algorithm, we use the Fourier expansion at the cusp at infinity and get

$$f(x + iy) = \sum_{n=-\infty}^{\infty} a_n \sqrt{y} K_{\nu}(2\pi|n|y) e(nx)$$

where  $f$  has eigenvalue  $\lambda = (1/2 - \nu)(1/2 + \nu)$ .

Now, the  $K$ -Bessel function has well-understood asymptotics. I need to look up the exact asymptotics, but the gist is that it decays exponentially. So, for relatively large values of  $y$ , most of the information in the Fourier expansion is contained in relatively small values of  $|n|$ .

With this in mind, we take points on a low-lying horocycle as our sample points, and sum over these. Then, we play this off of the expansions at the pullbacks to the standard fundamental domain. This has the following effects:

- **Good:** since the sample points have a small height, they will hit a large variety of fundamental domains; this ensures their pullbacks aren't "too close" together, which could cause the system to be numerically singular
- **Bad:** since the sample points are low, the  $K$ -Bessel function will be relatively large, causing us to need a relatively large amount of terms in the Fourier expansion
- **Good:** since the fundamental domain is bounded off of the real axis, the  $y$ -values of the pullbacks are relatively large; this ensures that we need relatively few terms in the Fourier expansion
- **Good:** since the sample points are all at the same height, the number of times we have to compute the  $K$ -Bessel function at the "bad" points is much smaller than at the "good" points

## 2 Comparing the Disk Model Approach to the Original

In the disk model approach, we instead use the model of hyperbolic space in the unit disk. Then, since our function is  $2\pi$ -periodic, we have a Fourier expansion in the  $\theta$  variable. The expansion ends up being

$$f(\rho, \theta) = \sum_{n=-\infty}^{\infty} a_n (1 - \rho^2)^2 \rho^{|n|} {}_2F_1(s, s + |n|, 1 + |n|; \rho^2) \exp(n\theta)$$

In my notes on the Hausdorff dimension of limit sets, I derive the asymptotics of the hypergeometric function for  $\rho \sim 1$ .

In short, the hypergeometric function is very small for large  $\rho$  values, and it is close to 1 for  $\rho$  close to 0. To try to mimic the original method, we choose sample points on a circle of fixed radius  $P$  close to 1 and sum over these. Then we play these off of the expansions at the pullbacks to some fundamental domain. This has the following effects:

- **Good:** since the sample points have a large radius, they will hit a large variety of fundamental domains

- **Good:** the sample points have relatively large radii, so the hypergeometric function will be very small at these points; this means we will need relatively few terms in the Fourier expansion
- **Bad:** if we choose the standard fundamental domain, we will have points of *any* radius from 0 to 1; this means some of the pullbacks will have a relatively small radius, causing the hypergeometric function to be relatively large, which in turn causes us to need a large number of terms in the Fourier expansion
- **Bad:** since the pullbacks can have such a variety of radii, we may have to compute the hypergeometric function at a large number of “bad” points