Homework 2 - Berkeley STAT 157

Handout 1/29/2019, due 2/5/2019 by 4pm in Git by committing to your repository.

In [1]:
```
from mxnet import nd, autograd, gluon
%matplotlib inline
from matplotlib import pyplot as plt
```

1. Multinomial Sampling

Implement a sampler from a discrete distribution from scratch, mimicking the function `mxnet.ndarray.random.multinomial`. Its arguments should be a vector of probabilities p. You can assume that the probabilities are normalized, i.e. tha they sum up to 1. Make the call signature as follows:

```
samples = sampler(probs, shape)

probs   : An ndarray vector of size n of nonnegative numbers summing up to 1
shape   : A list of dimensions for the output
samples : Samples from probs with shape matching shape
```

Hints:

1. Use `mxnet.ndarray.random.uniform` to get a sample from $U[0, 1]$.
2. You can simplify things for `probs` by computing the cumulative sum over `probs`.

```
In [2]: def sampler(probs, shape):
            ## Add your codes here
            n = len(probs)
            s = nd.zeros(shape).size
            temp = nd.zeros(s)
            ps = nd.random.uniform(0, 1, shape=s)
            for i in range(s):
                p = ps[i]
                v = 0
                j = 0
                for prob in probs:
                    v += prob
                    if p <= v:
                        temp[i] = j
                        break
                    j += 1
            return temp.reshape(shape)

        # a simple test
        sampler(nd.array([0.2, 0.3, 0.5]), (2,3))
```

```
Out[2]: [[2. 2. 2.]
         [2. 2. 2.]]
        <NDArray 2x3 @cpu(0)>
```

2. Central Limit Theorem

Let's explore the Central Limit Theorem when applied to text processing.

- Download https://www.gutenberg.org/ebooks/84 (https://www.gutenberg.org/files/84/84-0.txt) from Project Gutenberg
- Remove punctuation, uppercase / lowercase, and split the text up into individual tokens (words).
- For the words `a`, `and`, `the`, `i`, `is` compute their respective counts as the book progresses, i.e.

$$n_{\text{the}}[i] = \sum_{j=1}^{i} \{w_j = \text{the}\}$$

- Plot the proportions $n_{\text{word}}[i]/i$ over the document in one plot.
- Find an envelope of the shape $O(1/\sqrt{i})$ for each of these five words. (Hint, check the last page of the sampling notebook (http://courses.d2l.ai/berkeley-stat-157/slides/1_24/sampling.pdf))
- Why can we **not** apply the Central Limit Theorem directly?
- How would we have to change the text for it to apply?
- Why does it still work quite well?

In [10]:

```
filename = gluon.utils.download('https://www.gutenberg.org/files/84/8
4-0.txt')
with open(filename) as f:
    book = f.read()
print(book[0:100])

## Add your codes here
```

```
Project Gutenberg's Frankenstein, by Mary Wollstonecraft (Godwin) She
lley

This eBook is for the u
```

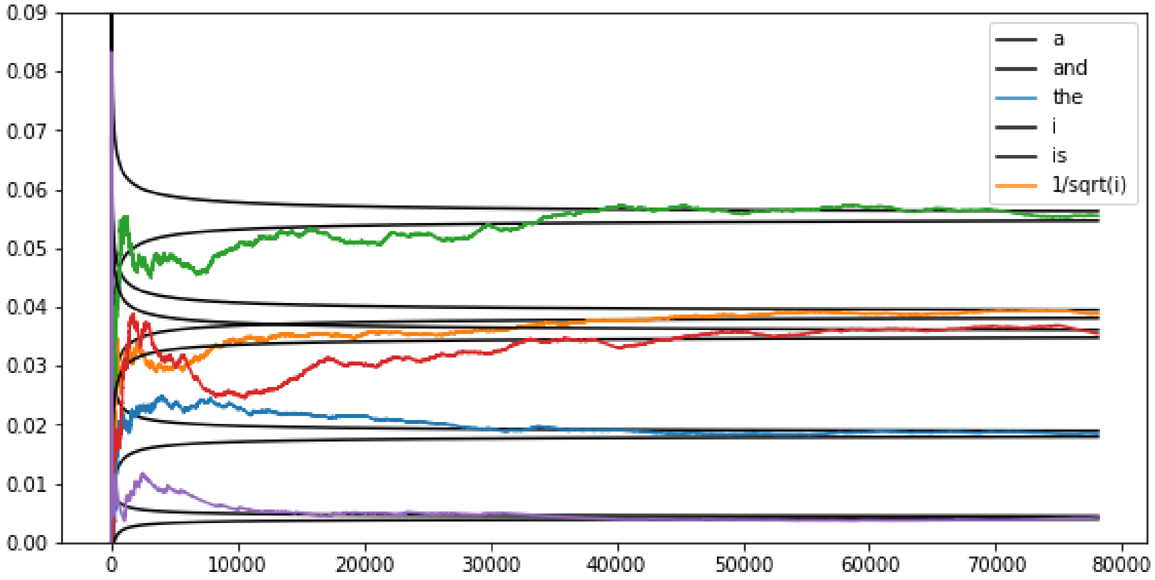
In [11]:

```
import re
import string
import numpy as np
tokens = '\n'.join(book.split(' ')).split('\n')
tokens = list(map(\
             lambda x: ''.join(c for c in x if c not in string.punctu
ation).lower()\
               , tokens))
tokens = list(filter(lambda x: x is not '', tokens))
tokens[:6]
tokens = np.array(tokens)
```

In [23]:

```
plt.figure(figsize=(10,5))
words = ['a', 'and', 'the', 'i', 'is']
for word in words:
    x = 1*(tokens == word)
    mean = np.mean(x)
    variance = mean * (1 - mean)
    y = np.arange(1, len(x)+1)
    plt.plot(y,(variance**0.5) * np.power(y,-0.5) + mean,'black')
    plt.plot(y,-(variance**0.5) * np.power(y,-0.5) + mean,'black')
    z = np.cumsum(x)/y
    plt.plot(y,z)
plt.ylim(0, .09)

plt.legend(words + ['1/sqrt(i)'])
plt.show()
```



- Why can we **not** apply the Central Limit Theorem directly?

The subsequent proportions are not independant of each other. For example, the next value has either the same number or the same number + 1 sucesses as last value.

- How would we have to change the text for it to apply?

Have each x value be an independant sample from the corpus of words where we measure the proportion of a certain word. So we could randomize the text before we calculate each p

- Why does it still work quite well?

We converge to the actual probability since the first half of the corpus proportion is a really good estimate for the second half of the corpus.

3. Denominator-layout notation

We used the numerator-layout notation for matrix calculus in class, now let's examine the denominator-layout notation.

Given $x, y \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, we have

$$\frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix}, \qquad \frac{\partial \mathbf{y}}{\partial x} = \left[\frac{\partial y_1}{\partial x}, \frac{\partial y_2}{\partial x}, \ldots, \frac{\partial y_m}{\partial x} \right]$$

and

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{y}}{\partial x_1} \\ \frac{\partial \mathbf{y}}{\partial x_2} \\ \vdots \\ \frac{\partial \mathbf{y}}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1}, \frac{\partial y_2}{\partial x_1}, \ldots, \frac{\partial y_m}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2}, \frac{\partial y_2}{\partial x_2}, \ldots, \frac{\partial y_m}{\partial x_2} \\ \vdots \\ \frac{\partial y_1}{\partial x_n}, \frac{\partial y_2}{\partial x_n}, \ldots, \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

Questions:

1. Assume $\mathbf{y} = f(\mathbf{u})$ and $\mathbf{u} = g(\mathbf{x})$, write down the chain rule for $\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$

$\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$ = $\frac{\partial \mathbf{y}}{\partial \mathbf{u}} * \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$

1. Given $\mathbf{X} \in \mathbb{R}^{m \times n},\ \mathbf{w} \in \mathbb{R}^n,\ \mathbf{y} \in \mathbb{R}^m$, assume $z = \|\mathbf{X}\mathbf{w} - \mathbf{y}\|^2$, compute $\frac{\partial z}{\partial \mathbf{w}}$.

$\frac{\partial z}{\partial \mathbf{w}} = 2 * \mathbf{X}^T(\mathbf{X}\mathbf{w} - \mathbf{y})$

4. Numerical Precision

Given scalars x and y, implement the following `log_exp` function such that it returns

$$-\log\left(\frac{e^x}{e^x + e^y}\right)$$

.

In [80]:
```
def log_exp(x, y):
    return -1*nd.log(nd.exp(x)/(nd.exp(x)+nd.exp(y)))
```

Test your codes with normal inputs:

```
In [118]: x, y = nd.array([2]), nd.array([3])
          z = log_exp(x, y)
          z
```

```
Out[118]: [1.3132617]
          <NDArray 1 @cpu(0)>
```

Now implement a function to compute $\partial z/\partial x$ and $\partial z/\partial y$ with `autograd`

```
In [119]: def grad(forward_func, x, y):
              x.attach_grad()
              y.attach_grad()
              with autograd.record():
                  z = forward_func(x, y)
              z.backward()
              print('x.grad =', x.grad)
              print('y.grad =', y.grad)
```

Test your codes, it should print the results nicely.

```
In [122]: grad(log_exp, x, y)
```

```
x.grad =
[-0.7310586]
<NDArray 1 @cpu(0)>
y.grad =
[0.7310586]
<NDArray 1 @cpu(0)>
```

But now let's try some "hard" inputs

```
In [123]: x, y = nd.array([50]), nd.array([100])
          grad(log_exp, x, y)
```

```
x.grad =
[nan]
<NDArray 1 @cpu(0)>
y.grad =
[nan]
<NDArray 1 @cpu(0)>
```

Does your code return correct results? If not, try to understand the reason. (Hint, evaluate `exp(100)`). Now develop a new function `stable_log_exp` that is identical to `log_exp` in math, but returns a more numerical stable result.

In [124]:

```
def stable_log_exp(x, y):
    b = max(x, y)
    return -x + b + nd.log(nd.exp(x - b) + nd.exp(y - b))

grad(stable_log_exp, x, y)
```

```
x.grad =
[-1.]
<NDArray 1 @cpu(0)>
y.grad =
[1.]
<NDArray 1 @cpu(0)>
```

In [126]:

```
x, y = nd.array([2]), nd.array([3])
z = stable_log_exp(x, y)
z
```

Out[126]:

```
[1.3132616]
<NDArray 1 @cpu(0)>
```

In [127]:

```
grad(stable_log_exp, x, y)
```

```
x.grad =
[-0.7310586]
<NDArray 1 @cpu(0)>
y.grad =
[0.7310586]
<NDArray 1 @cpu(0)>
```

In [128]:

```
x, y = nd.array([50]), nd.array([100])
grad(stable_log_exp, x, y)
```

```
x.grad =
[-1.]
<NDArray 1 @cpu(0)>
y.grad =
[1.]
<NDArray 1 @cpu(0)>
```