5 Colouring

How many colours do we need to colour the countries of a map in such a way that adjacent countries are coloured differently? How many days have to be scheduled for committee meetings of a parliament if every committee intends to meet for one day and some members of parliament serve on several committees? How can we find a school timetable of minimum total length, based on the information of how often each teacher has to teach each class?

A *vertex colouring* of a graph $G = (V, E)$ is a map $c: V \to S$ such that $c(v) \neq c(w)$ whenever v and w are adjacent. The elements of the set S are called the available *colours*. All that interests us about S is its size: typically, we shall be asking for the smallest integer k such that G has a *k-colouring*, a vertex colouring $c: V \to \{1, \ldots, k\}$. This k is the *(vertex-) chromatic number* of G; it is denoted by $\chi(G)$. A graph G with $\chi(G) = k$ is called *k-chromatic*; if $\chi(G) \leqslant k$, we call G *k-colourable*.

vertex colouring

chromatic number $\chi(G)$

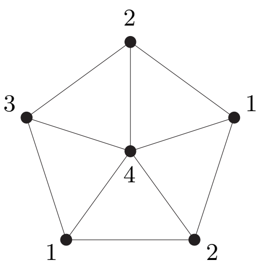

Fig. 5.0.1. A vertex colouring $V \to \{1, \ldots, 4\}$

Note that a k-colouring is nothing but a vertex partition into k independent sets, now called *colour classes*; the non-trivial 2-colourable graphs, for example, are precisely the bipartite graphs. Historically, the colouring terminology comes from the map colouring problem stated

colour classes

above, which leads to the problem of determining the maximum chromatic number of planar graphs. The committee scheduling problem, too, can be phrased as a vertex colouring problem – how?

edge colouring

An *edge colouring* of $G = (V, E)$ is a map $c\colon E \to S$ with $c(e) \neq c(f)$ for any adjacent edges e, f. The smallest integer k for which a *k-edge-colouring* exists, i.e. an edge colouring $c\colon E \to \{1, \dots, k\}$, is the *edge-chromatic number*, or *chromatic index*, of G; it is denoted by $\chi'(G)$. The third of our introductory questions can be modelled as an edge colouring problem in a bipartite multigraph (how?).

chromatic index $\chi'(G)$

Clearly, every edge colouring of G is a vertex colouring of its line graph $L(G)$, and vice versa; in particular, $\chi'(G) = \chi(L(G))$. The problem of finding good edge colourings may thus be viewed as a restriction of the more general vertex colouring problem to this special class of graphs. As we shall see, this relationship between the two types of colouring problem is reflected by a marked difference in our knowledge about their solutions: while there are only very rough estimates for χ, its sister χ' always takes one of two values, either Δ or $\Delta + 1$.

5.1 Colouring maps and planar graphs

If any result in graph theory has a claim to be known to the world outside, it is the following *four colour theorem* (which implies that every map can be coloured with at most four colours):

Theorem 5.1.1. (Four Colour Theorem)
Every planar graph is 4-colourable.

Some remarks about the proof of the four colour theorem and its history can be found in the notes at the end of this chapter. Here, we prove the following weakening:

Proposition 5.1.2. (Five Colour Theorem)
Every planar graph is 5-colourable.

(4.1.1)
(4.2.10)

First proof. Let G be a plane graph with $n \geqslant 6$ vertices and m edges. We assume inductively that every plane graph with fewer than n vertices can be 5-coloured. By Corollary 4.2.10,

n, m

$$d(G) = 2m/n \leqslant 2\,(3n-6)/n < 6\,;$$

v H c

let $v \in G$ be a vertex of degree at most 5. By the induction hypothesis, the graph $H := G - v$ has a vertex colouring $c\colon V(H) \to \{1, \dots, 5\}$. If c uses at most 4 colours for the neighbours of v, we can extend it to a 5-colouring of G. Let us assume, therefore, that v has exactly 5 neighbours, and that these have distinct colours.

D $s_1, \ldots, s_5$ $v_1, \ldots, v_5$

Let D be an open disc around v, so small that it meets only those five straight edge segments of G that contain v. Let us enumerate these segments according to their cyclic position in D as $s_1, \ldots, s_5$, and let vv_i be the edge containing s_i ($i = 1, \ldots, 5$; Fig. 5.1.1). Without loss of generality we may assume that $c(v_i) = i$ for each i.

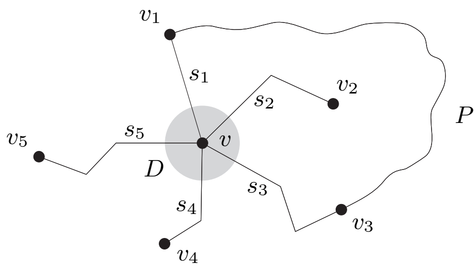

Fig. 5.1.1. The proof of the five colour theorem

P C

Let us show first that every v_1–v_3 path $P \subseteq H - \{v_2, v_4\}$ separates v_2 from v_4 in H. Clearly, this is the case if and only if the cycle $C := vv_1Pv_3v$ separates v_2 from v_4 in G. We prove this by showing that v_2 and v_4 lie in different faces of C.

Let us pick an inner point x_2 of s_2 in D and an inner point x_4 of s_4 in D. Then in $D \smallsetminus (s_1 \cup s_3) \subseteq \mathbb{R}^2 \smallsetminus C$ every point can be linked by a polygonal arc to x_2 or to x_4. This implies that x_2 and x_4 (and hence also v_2 and v_4) lie in different faces of C: otherwise D would meet only one of the two faces of C, which would contradict the fact that v lies on the frontier of both these faces (Theorem 4.1.1).

$H_{i,j}$

Given $i, j \in \{1, \ldots, 5\}$, let $H_{i,j}$ be the subgraph of H induced by the vertices coloured i or j. We may assume that the component C_1 of $H_{1,3}$ containing v_1 also contains v_3. Indeed, if we interchange the colours 1 and 3 at all the vertices of C_1, we obtain another 5-colouring of H; if $v_3 \notin C_1$, then v_1 and v_3 are both coloured 3 in this new colouring, and we may assign colour 1 to v. Thus, $H_{1,3}$ contains a v_1–v_3 path P. As shown above, P separates v_2 from v_4 in H. Since $P \cap H_{2,4} = \emptyset$, this means that v_2 and v_4 lie in different components of $H_{2,4}$. In the component containing v_2, we now interchange the colours 2 and 4, thus recolouring v_2 with colour 4. Now v no longer has a neighbour coloured 2, and we may give it this colour. □

(4.2.10)
(4.2.11)
(4.4.6)
v

Second proof. As in the first proof, we assume inductively that every planar graph with fewer than $|G|$ vertices can be 5-coloured, and find a vertex v in G of degree 5. Since $K^5 \not\subseteq G$ by Corollary 4.2.11, this vertex v has non-adjacent neighbours u, w. Let $P = \{u, v, w\}$.

Since minors of planar graphs are again planar, e.g. by Kuratowski's theorem, G/P has a 5-colouring by the induction hypothesis. This induces a 5-colouring of $G - v$ in which u and w receive the same colour, that of v_P from our colouring of G/P. The neighbours of v in this colouring of $G - v$ thus use at most four colours, and we can colour v with the fifth colour. $\square$

As a backdrop to the two famous theorems above, let us cite another well-known result:

Theorem 5.1.3. (Grötzsch 1959)
Every planar graph not containing a triangle is 3-colourable.

5.2 Colouring vertices

How do we determine the chromatic number of a given graph? How can we *find* a vertex-colouring with as few colours as possible? How does the chromatic number relate to other graph invariants, such as average degree, connectivity or girth?

Straight from the definition of the chromatic number we may derive the following upper bound:

Proposition 5.2.1. *Every graph G with m edges satisfies*

$$\chi(G) \leqslant \tfrac{1}{2} + \sqrt{2m + \tfrac{1}{4}}\,.$$

Proof. Let c be a vertex colouring of G with $k = \chi(G)$ colours. Then G has at least one edge between any two colour classes: if not, we could have used the same colour for both classes. Thus, $m \geqslant \frac{1}{2}k(k-1)$. Solving this inequality for k, we obtain the assertion claimed. $\square$

greedy algorithm

One obvious way to colour a graph G with not too many colours is the following *greedy algorithm*: starting from a fixed vertex enumeration $v_1, \ldots, v_n$ of G, we consider the vertices in turn and colour each v_i with the first available colour – e.g., with the smallest positive integer not already used to colour any neighbour of v_i among $v_1, \ldots, v_{i-1}$. In this way, we never use more than $\Delta(G) + 1$ colours, even for unfavourable choices of the enumeration $v_1, \ldots, v_n$. If G is complete or an odd cycle, then this is even best possible.

In general, though, this upper bound of $\Delta + 1$ is rather generous, even for greedy colourings. Indeed, when we come to colour the vertex v_i in the above algorithm, we only need a supply of $d_{G[v_1,\ldots,v_i]}(v_i) + 1$ rather than $d_G(v_i) + 1$ colours to proceed; recall that, at this stage, the algorithm ignores any neighbours v_j of v_i with $j > i$. Hence in most graphs, there will be scope for an improvement of the $\Delta+1$ bound by choosing a particularly suitable vertex ordering to start with: one that picks vertices of large degree early (when most neighbours are ignored) and vertices of small degree last. Locally, the number $d_{G[v_1,\dots,v_i]}(v_i)+1$ of colours required will be smallest if v_i has minimum degree in $G[v_1,\dots,v_i]$. But this is easily achieved: we just choose v_n first, with $d(v_n)=\delta(G)$, then choose as v_{n-1} a vertex of minimum degree in $G-v_n$, and so on.

The least number k such that G has a vertex enumeration in which each vertex is preceded by fewer than k of its neighbours is called the *colouring number* $\mathrm{col}(G)$ of G. The enumeration we just discussed shows that $\mathrm{col}(G) \leqslant \max_{H\subseteq G}\delta(H)+1$. But for $H\subseteq G$ clearly also $\mathrm{col}(G)\geqslant \mathrm{col}(H)$ and $\mathrm{col}(H)\geqslant \delta(H)+1$, since the 'back-degree' of the last vertex in any enumeration of H is just its ordinary degree in H, which is at least $\delta(H)$. So we have proved the following: *colouring number* $\mathrm{col}(G)$

Proposition 5.2.2. *Every graph G satisfies*

$$\chi(G)\leqslant \mathrm{col}(G)=\max\{\,\delta(H)\mid H\subseteq G\,\}+1\,.$$

□

The colouring number of a graph is closely related to its arboricity; see Exercise 14 and the remark following Theorem 2.4.3.

Proposition 5.2.2 shows that every k-chromatic graph has a subgraph of minimum degree at least $k-1$. In fact, it has a k-chromatic such subgraph:

[7.3] [9.2.1] [9.2.3] [11.2.3]

Lemma 5.2.3. *Every k-chromatic graph has a k-chromatic subgraph of minimum degree at least $k-1$.*

Proof. Given G with $\chi(G)=k$, let $H\subseteq G$ be minimal with $\chi(H)=k$. If H had a vertex v of degree $d_H(v)\leqslant k-2$, we could extend a $(k-1)$-colouring of $H-v$ to one of H, contradicting the choice of H. □

As we have seen, every graph G satisfies $\chi(G)\leqslant\Delta(G)+1$, with equality for complete graphs and odd cycles. In all other cases, this general bound can be improved a little:

Theorem 5.2.4. (Brooks 1941)
Let G be a connected graph. If G is neither complete nor an odd cycle, then

$$\chi(G)\leqslant\Delta(G)\,.$$

Proof. We apply induction on $|G|$. If $\Delta(G) \leqslant 2$, then G is a path or a cycle, and the assertion is trivial. We therefore assume that $\Delta := \Delta(G) \geqslant 3$, and that the assertion holds for graphs of smaller order. Suppose that $\chi(G) > \Delta$.

Δ

Let $v \in G$ be a vertex and $H := G - v$. Then $\chi(H) \leqslant \Delta$: by induction, every component H' of H satisfies $\chi(H') \leqslant \Delta(H') \leqslant \Delta$ unless H' is complete or an odd cycle, in which case $\chi(H') = \Delta(H') + 1 \leqslant \Delta$ as every vertex of H' has maximum degree in H' and one such vertex is also adjacent to v in G.

v, H

Since H can be Δ-coloured but G cannot, we have the following:

Every Δ-colouring of H uses all the colours $1, \dots, \Delta$ on the neighbours of v; in particular, $d(v) = \Delta$. (1)

Given any Δ-colouring of H, let us denote the neighbour of v coloured i by v_i, $i = 1, \dots, \Delta$. For all $i \neq j$, let $H_{i,j}$ denote the subgraph of H spanned by all the vertices coloured i or j.

$v_1, \dots, v_\Delta$
$H_{i,j}$

For all $i \neq j$, the vertices v_i and v_j lie in a common component $C_{i,j}$ of $H_{i,j}$. (2)

$C_{i,j}$

Otherwise we could interchange the colours i and j in one of those components; then v_i and v_j would be coloured the same, contrary to (1).

$C_{i,j}$ is always a v_i–v_j path. (3)

Indeed, let P be a v_i–v_j path in $C_{i,j}$. As $d_H(v_i) \leqslant \Delta - 1$, the neighbours of v_i have pairwise different colours: otherwise we could recolour v_i, contrary to (1). Hence the neighbour of v_i on P is its only neighbour in $C_{i,j}$, and similarly for v_j. Thus if $C_{i,j} \neq P$, then P has an inner vertex with three identically coloured neighbours in H; let u be the first such vertex on P (Fig. 5.2.1). Since at most $\Delta - 2$ colours are used on the neighbours of u, we may recolour u. But this makes $P\mathring{u}$ into a component of $H_{i,j}$, contradicting (2).

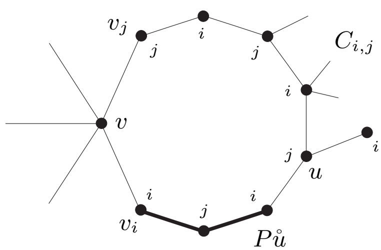

Fig. 5.2.1. The proof of (3) in Brooks's theorem

For distinct i, j, k, the paths $C_{i,j}$ and $C_{i,k}$ meet only in v_i. (4)

For if $v_i \neq u \in C_{i,j} \cap C_{i,k}$, then u has two neighbours coloured j and two coloured k, so we may recolour u. In the new colouring, v_i and v_j lie in different components of $H_{i,j}$, contrary to (2).

The proof of the theorem now follows easily. If the neighbours of v are pairwise adjacent, then each has Δ neighbours in $N(v) \cup \{v\}$ already, so $G = G[N(v) \cup \{v\}] = K^{\Delta+1}$. As G is complete, there is nothing to show. We may thus assume that $v_1 v_2 \notin G$, where $v_1, \ldots, v_\Delta$ derive their names from some fixed Δ-colouring c of H. Let $u \neq v_2$ be the neighbour of v_1 on the path $C_{1,2}$; then $c(u) = 2$. Interchanging the colours 1 and 3 in $C_{1,3}$, we obtain a new colouring c' of H; let v'_i, $H'_{i,j}$, $C'_{i,j}$ etc. be defined with respect to c' in the obvious way. As a neighbour of $v_1 = v'_3$, our vertex u now lies in $C'_{2,3}$, since $c'(u) = c(u) = 2$. By (4) for c, however, the path $\mathring{v}_1 C_{1,2}$ retained its original colouring, so $u \in \mathring{v}_1 C_{1,2} \subseteq C'_{1,2}$. Hence $u \in C'_{2,3} \cap C'_{1,2}$, contradicting (4) for c'. $\square$

$v_1, \ldots, v_\Delta$

c

u

c'

We have so far seen some necessary conditions for high chromaticity, in the form of upper bounds on χ. If $\chi(G) \geqslant k$, for example, then also $\Delta \geqslant k$ (unless G is complete or an odd cycle), and G has a subgraph of minimum degree at least $k-1$. These conditions are far from sufficient, though: if $G = K_{n,n}$, say, they hold for all $k \leqslant n$ but $\chi(G) = 2$.

It would be nice also to have some sufficient conditions for $\chi \geqslant k$. If they are easy to check, they might provide useful certificates for why we are unable to colour a given graph with few colours. If they could even be shown to be necessary too, they would 'explain' why certain graphs are highly chromatic – just as the marriage condition in Hall's theorem 'explains' why certain matchings in bipartite graphs fail: its violation clearly prevents a graph from having the desired matching, and it is violated every time such a matching fails to exist.

For example, we might try to determine the class $\mathcal{X}_k$ of $\subseteq$-minimal graphs that cannot be coloured with fewer than k colours. As is easy to check (cf. Lemma 12.6.1.), a given graph G satisfies $\chi(G) \geqslant k$ if and (12.6.1) only if it has a subgraph in $\mathcal{X}_k$, just as in Kuratowski's planarity theorem with minors or topological minors. So containing any graph from $\mathcal{X}_k$ is a certificate for $\chi \geqslant k$, and these certificates together 'explain' this phenomenon in the sense discussed.

But will these certificates be easy to find in an arbitrary k-chromatic graph, or at least easy to check? That is, will it be easy to verify that a given graph $X \in \mathcal{X}_k$ is indeed in $\mathcal{X}_k$, or even just that $\chi(X) \geqslant k$? We shall return to this question in a moment.

One obvious sufficient condition for $\chi(G) \geqslant k$ is that $K^k \subseteq G$. But this condition is not necessary: as Theorem 5.2.5 will show, k-chromatic graphs need not even contain a triangle. Hence while K^k certainly lies in $\mathcal{X}_k$, it is not its only element. Conversely, Lemma 5.2.3 implies that all the graphs in $\mathcal{X}_k$ have minimum degree at least $k-1$; but not all graphs of minimum degree $k-1$ are in $\mathcal{X}_k$, since they need not satisfy $\chi \geqslant k$.

The following theorem of Erdős implies that $\mathcal{X}_k$ cannot be finite. In fact, it implies that for no k is there a finite set $\mathcal{X}$ of graphs X with $\chi(X) \geqslant 3$ such that every k-chromatic graph has a subgraph in $\mathcal{X}$:

[9.2.3] **Theorem 5.2.5.** (Erdős 1959)
For every integer k there exists a graph G with girth $g(G) > k$ and chromatic number $\chi(G) > k$.

Theorem 5.2.5 was first proved non-constructively using random graphs, and we shall give this proof in Chapter 11.2. Constructing graphs of large chromatic number and girth directly is not easy; cf. Exercise 22 for the simplest case.

The message of Erdős's theorem is that, contrary perhaps to what we had hoped, large chromatic number can occur as a purely global phenomenon: locally, around each vertex, a graph of large girth looks just like a tree, and in particular is 2-colourable there. But what exactly can cause high chromaticity as a global phenomenon remains a mystery.

k-constructible

Nevertheless, there exists a simple – though not always short – procedure to construct all the graphs of chromatic number at least k. For each $k \in \mathbb{N}$, let us define the class of *k-constructible* graphs recursively as follows:

(i) K^k is k-constructible.

(ii) If G is k-constructible and two vertices x, y of G are non-adjacent, then also $(G+xy)/xy$ is k-constructible.

(iii) If G_1, G_2 are k-constructible and there are vertices x, y_1, y_2 such that $G_1 \cap G_2 = \{x\}$ and $xy_1 \in E(G_1)$ and $xy_2 \in E(G_2)$, then also $(G_1 \cup G_2) - xy_1 - xy_2 + y_1y_2$ is k-constructible (Fig. 5.2.2).

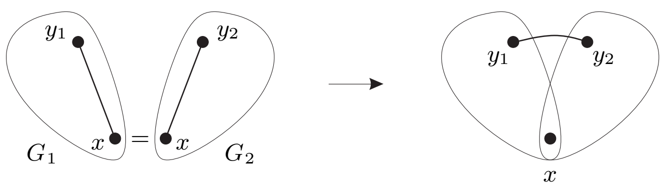

Fig. 5.2.2. The *Hajós construction* (iii)

One easily checks inductively that all k-constructible graphs – and hence their supergraphs – are at least k-chromatic. For example, any colouring of the graph $(G+xy)/xy$ in (ii) induces a colouring of G, and hence by inductive assumption uses at least k colours. Similarly, in any colouring of the graph constructed in (iii) the vertices y_1 and y_2 do not both have the same colour as x, so this colouring induces a colouring of either G_1 or G_2 and hence uses at least k colours.

It is remarkable, though, that the converse holds too:

Theorem 5.2.6. (Hajós 1961)
Let G be a graph and $k \in \mathbb{N}$. Then $\chi(G) \geqslant k$ if and only if G has a k-constructible subgraph.

Proof. Let G be a graph with $\chi(G) \geqslant k$; we show that G has a k-constructible subgraph. Suppose not; then $k \geqslant 3$. Adding some edges if necessary, let us make G edge-maximal with the property that none of its subgraphs is k-constructible. Now G is not a complete r-partite graph for any r: for then $\chi(G) \geqslant k$ would imply $r \geqslant k$, and G would contain the k-constructible graph K^k.

Since G is not a complete multipartite graph, non-adjacency is not an equivalence relation on $V(G)$. So there are vertices y_1, x, y_2 such that $y_1x, xy_2 \notin E(G)$ but $y_1y_2 \in E(G)$. Since G is edge-maximal without a k-constructible subgraph, each edge xy_i lies in some k-constructible subgraph H_i of $G + xy_i$ $(i = 1, 2)$. x, y_1, y_2 H_1, H_2

Let H_2' be an isomorphic copy of H_2 that contains x and $H_2 - H_1$ but is otherwise disjoint from G, together with an isomorphism $v \mapsto v'$ from H_2 to H_2' that fixes $H_2 \cap H_2'$ pointwise. Then $H_1 \cap H_2' = \{x\}$, so H_2' v' *etc.*

$$H := (H_1 \cup H_2') - xy_1 - xy_2' + y_1y_2'$$

is k-constructible by (iii). One vertex at a time, let us identify in H each vertex $v' \in H_2' - G$ with its partner v; since vv' is never an edge of H, each of these identifications amounts to a construction step of type (ii). Eventually, we obtain the graph

$$(H_1 \cup H_2) - xy_1 - xy_2 + y_1y_2 \subseteq G\,;$$

this is the desired k-constructible subgraph of G. □

Does Hajós's theorem solve our Kuratowski-type problem for highly chromatic graphs, which was to find a class of graphs of chromatic number at least k with the property that every such graph has a subgraph in this class? Formally it does, albeit with an infinite characterizing class: the class of k-constructible graphs, which contains $\mathcal{X}_k$. Unlike Kuratowski's characterization of planar graphs, however, this does not – at least not obviously – make Hajós's theorem a good characterization of the graphs of chromatic number $< k$: as one can show, proving that a given k-constructible graph is indeed k-constructible is just as hard as proving that a graph of chromatic number $\geqslant k$ does indeed need at least k colours. See the notes for details.

5.3 Colouring edges

Clearly, every graph G satisfies $\chi'(G) \geqslant \Delta(G)$. For bipartite graphs, we have equality here:

[5.4.5] **Proposition 5.3.1.** (König 1916)
Every bipartite graph G satisfies $\chi'(G) = \Delta(G)$.

(1.6.1) *Proof.* We apply induction on $\|G\|$. For $\|G\| = 0$ the assertion holds. Now assume that $\|G\| \geqslant 1$, and that the assertion holds for graphs with

Δ, xy fewer edges. Let $\Delta := \Delta(G)$, pick an edge $xy \in G$, and choose a Δ-edge-colouring of $G - xy$ by the induction hypothesis. Let us refer to

α-*edge* the edges coloured α as α-*edges*, etc.

In $G - xy$, each of x and y is incident with at most $\Delta - 1$ edges.

α, β Hence there are $\alpha, \beta \in \{1, \dots, \Delta\}$ such that x is not incident with an α-edge and y is not incident with a β-edge. If $\alpha = \beta$, we can colour the edge xy with this colour and are done; so we may assume that $\alpha \neq \beta$, and that x is incident with a β-edge.

Let us extend this edge to a maximal walk W from x whose edges are coloured β and α alternately. Since no such walk contains a vertex twice (why not?), W exists and is a path. Moreover, W does not contain y: if it did, it would end in y on an α-edge (by the choice of β) and thus have even length, so $W + xy$ would be an odd cycle in G (cf. Proposition 1.6.1). We now recolour all the edges on W, swapping α with β. By the choice of α and the maximality of W, adjacent edges of $G - xy$ are still coloured differently. We have thus found a Δ-edge-colouring of $G - xy$ in which neither x nor y is incident with a β-edge. Colouring xy with β, we extend this colouring to a Δ-edge-colouring of G. $\square$

Theorem 5.3.2. (Vizing 1964)
Every graph G satisfies

$$\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1 .$$

V, E *Proof.* We prove the second inequality by induction on $\|G\|$. For $\|G\| = 0$

Δ it is trivial. For the induction step let $G = (V, E)$ with $\Delta := \Delta(G) > 0$ be given, and assume that the assertion holds for graphs with fewer edges.

colouring Instead of '$(\Delta + 1)$-edge-colouring' let us just say 'colouring'.

For every edge $e \in G$ there exists a colouring of $G - e$, by the induction hypothesis. In such a colouring, the edges at a given vertex v use at most $d(v) \leqslant \Delta$ colours, so some colour $\beta \in \{1, \dots, \Delta + 1\}$ is

missing *missing at v*. For any other colour α, there is a unique maximal walk (possibly trivial) starting at v, whose edges are coloured alternately α

α/β-*path* and β. This walk is a path; we call it the α/β-*path from v*.

Suppose that G has no colouring. Then the following holds:

Given $xy \in E$, and any colouring of $G - xy$ in which the colour α is missing at x and the colour β is missing at y, the α/β-path from y ends in x. $(*)$

Otherwise we could interchange the colours α and β along this path and colour xy with α, obtaining a colouring of G (contradiction).

Let $xy_0 \in G$ be an edge. By induction, $G_0 := G - xy_0$ has a colouring c_0. Let α be a colour missing at x in this colouring. Further, let $y_0, \dots, y_k$ be a maximal sequence of distinct neighbours of x in G such that $c_0(xy_{i+1})$ is missing in c_0 at y_i for every $i < k$. For each of the graphs $G_i := G - xy_i$ we define a colouring c_i, setting α G_i

$$c_i(e) := \begin{cases} c_0(xy_{j+1}) & \text{for } e = xy_j \text{ with } j \in \{0, \dots, i-1\} \\ c_0(e) & \text{otherwise;} \end{cases}$$ c_i

note that in each of these colourings the same colours are missing at x as in c_0.

Now let β be a colour missing at y_k in c_0. By $(*)$, the α/β-path P from y_k in G_k (with respect to c_k) ends in x, with an edge yx coloured β since α is missing at x. Since y cannot serve as y_{k+1}, by the maximality of the sequence $y_0, \dots, y_k$, we thus have $y = y_i$ for some $0 \leqslant i < k$ (Fig. 5.3.1). By definition of c_k, therefore, $\beta = c_k(xy_i) = c_0(xy_{i+1})$. By the choice of y_{i+1} this means that β was missing at y_i in c_0, and hence also in c_i. Now the α/β-path P' from y_i in G_i with respect to c_i starts with $y_i P y_k$, since the edges of $P\mathring{x}$ are coloured the same in c_i as in c_k. But in c_0, and hence in c_i, there is no edge at y_k coloured β. Therefore P' ends in y_k, contradicting $(*)$. β □

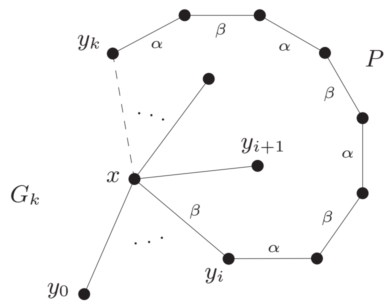

Fig. 5.3.1. The α/β-path P in $G_k = G - xy_k$

Vizing's theorem divides the finite graphs into two classes according to their chromatic index; graphs satisfying $\chi' = \Delta$ are called (imaginatively) *class 1*, those with $\chi' = \Delta + 1$ are *class 2*. There is no good characterization theorem that enables us to tell these classes apart, because no easily checkable 'certificate' is known for a graph to be class 2.

Regular graphs of large even order and large degree are class 1:

Theorem 5.3.3. (Csaba, Kühn, Lo, Osthus, Treglown 2016)
There exists an $n_0 \in \mathbb{N}$ such that, for all even $n \geqslant n_0$ and $d \geqslant n/2$, every d-regular graph G of order n satisfies $\chi'(G) = \Delta(G)$.

5.4 List colouring

In this section, we take a look at a relatively recent generalization of the concepts of colouring studied so far. This generalization may seem a little far-fetched at first glance, but it turns out to supply a fundamental link between the classical (vertex and edge) chromatic numbers of a graph and its other invariants.

Suppose we are given a graph $G = (V, E)$, and for each vertex of G a list of colours permitted at that particular vertex: when can we colour G (in the usual sense) so that each vertex receives a colour from its list? More formally, let $(S_v)_{v \in V}$ be a family of sets. We call a vertex colouring c of G with $c(v) \in S_v$ for all $v \in V$ a colouring *from the lists* S_v. The graph G is called *k-list-colourable*, or *k-choosable*, if, for every family $(S_v)_{v \in V}$ with $|S_v| = k$ for all v, there is a vertex colouring of G from the lists S_v. The least integer k for which G is k-choosable is the *list-chromatic number*, or *choice number* $\mathrm{ch}(G)$ of G.

k-choosable

choice number $\mathrm{ch}(G)$

List-colourings of edges are defined analogously. The least integer k such that G has an edge colouring from any family of lists of size k is the *list-chromatic index* $\mathrm{ch}'(G)$ of G; formally, we just set $\mathrm{ch}'(G) := \mathrm{ch}(L(G))$, where $L(G)$ is the line graph of G.

$\mathrm{ch}'(G)$

In principle, showing that a given graph is k-choosable is more difficult than proving it to be k-colourable: the latter is just the special case of the former where all lists are equal to $\{1, \dots, k\}$. Thus,

$$\mathrm{ch}(G) \geqslant \chi(G) \quad \text{and} \quad \mathrm{ch}'(G) \geqslant \chi'(G)$$

for all graphs G.

In spite of these inequalities, many of the known upper bounds for the chromatic number have turned out to be valid for the choice number, too. Examples for this phenomenon include Brooks's theorem and Proposition 5.2.2; in particular, graphs of large choice number still have subgraphs of large minimum degree. On the other hand, it is easy to construct graphs for which the two invariants are wide apart (Exercise 30). Taken together, these two facts indicate a little how far those general upper bounds on the chromatic number may be from the truth.

The following theorem shows that, in terms of its relationship to other graph invariants, the choice number differs fundamentally from the chromatic number. As mentioned before, there are 2-chromatic graphs of arbitrarily large minimum degree, e.g. the graphs $K_{n,n}$. The choice number, however, will be forced up by large values of invariants like δ, ε or κ:

Theorem 5.4.1. (Alon 1993)
There exists a function $f \colon \mathbb{N} \to \mathbb{N}$ such that, given any integer k, all graphs G with average degree $d(G) \geqslant f(k)$ satisfy $\mathrm{ch}(G) \geqslant k$.

The proof of Theorem 5.4.1 uses probabilistic methods as introduced in Chapter 11.

Although statements of the form $\mathrm{ch}(G) \leqslant k$ are formally stronger than the corresponding statement of $\chi(G) \leqslant k$, they can be easier to prove. A pretty example is the list version of the five colour theorem: every planar graph is 5-choosable. The proof of this does not use the five colour theorem (or even Euler's formula, on which the proof of the five colour theorem is based). We thus reobtain the five colour theorem as a corollary, with a very different proof.

Theorem 5.4.2. (Thomassen 1994)
Every planar graph is 5-choosable.

Proof. We shall prove the following assertion for all plane graphs G with at least 3 vertices: (4.2.8)

> *Suppose that every inner face of G is bounded by a triangle and its outer face by a cycle $C = v_1 \dots v_k v_1$. Suppose further that v_1 has already been coloured with the colour 1, and v_2 has been coloured 2. Suppose finally that with every other vertex of C a list of at least 3 colours is associated, and with every vertex of $G - C$ a list of at least 5 colours. Then the colouring of v_1 and v_2 can be extended to a colouring of G from the given lists.* (∗)

Let us check first that (∗) implies the assertion of the theorem. Let any plane graph be given, together with a list of 5 colours for each vertex. Add edges to this graph until it is a maximal plane graph G. By Proposition 4.2.8, G is a plane triangulation; let $v_1 v_2 v_3 v_1$ be the boundary of its outer face. We now colour v_1 and v_2 (differently) from their lists, and extend this colouring by (∗) to a colouring of G from the lists given.

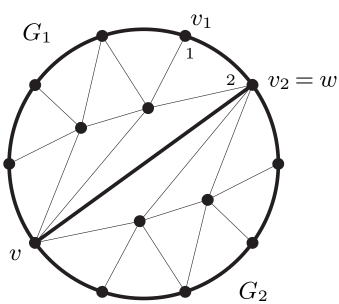

Fig. 5.4.1. The induction step with a chord vw; here the case of $w = v_2$

Let us now prove (∗), by induction on $|G|$. If $|G| = 3$, then $G = C$ and the assertion is trivial. Now let $|G| \geqslant 4$, and assume (∗) for smaller graphs. If C has a chord vw, then vw lies on two unique cycles $C_1, C_2 \subseteq C + vw$ with $v_1v_2 \in C_1$ and $v_1v_2 \notin C_2$. For $i = 1, 2$, let G_i denote the subgraph of G induced by the vertices lying on C_i or in its inner face (Fig. 5.4.1). Applying the induction hypothesis first to G_1 and then – with the colours now assigned to v and w – to G_2 yields the desired colouring of G.

vw

$u_1, \ldots, u_m$

If C has no chord, let $v_1, u_1, \ldots, u_m, v_{k-1}$ be the neighbours of v_k in their natural cyclic order around v_k;[1] by definition of C, all those neighbours u_i lie in the inner face of C (Fig. 5.4.2). As the inner faces of C are bounded by triangles, $P := v_1u_1 \ldots u_mv_{k-1}$ is a path in G, and $C' := P \cup (C - v_k)$ a cycle.

C'

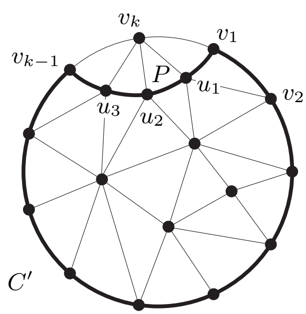

Fig. 5.4.2. The induction step without a chord

We now choose two different colours $j, \ell \neq 1$ from the list of v_k and delete these colours from the lists of all the vertices u_i. Then every list of a vertex on C' still has at least 3 colours, so by induction we may colour C' and its interior, i.e. the graph $G - v_k$. At least one of the two colours j, ℓ is not used for v_{k-1}, and we may assign that colour to v_k. □

As is often the case with induction proofs, the key to the proof above lies in its delicately balanced strengthening of the assertion proved. Compared with ordinary colouring, the task of finding a suitable strengthening is helped greatly by the possibility to give different vertices lists of different lengths, and thus to tailor the colouring problem more fittingly to the structure of the graph. This suggests that maybe in other unsolved colouring problems too it might be of advantage to aim straight for their list version, i.e. to prove an assertion of the form $\mathrm{ch}(G) \leqslant k$ instead of the formally weaker $\chi(G) \leqslant k$. Unfortunately, this approach fails for the four colour theorem: planar graphs are *not* in general 4-choosable.

[1] as in the first proof of the five colour theorem

As mentioned before, the chromatic number of a graph and its choice number may differ a lot. Surprisingly, however, no such examples are known for edge colourings. Indeed it has been conjectured that none exist:

List Colouring Conjecture. *Every graph G satisfies* $\mathrm{ch}'(G) = \chi'(G)$.

We shall prove the list colouring conjecture for bipartite graphs. As a tool we shall use orientations of graphs, defined in Chapter 1.10. If D is a directed graph and $v \in V(D)$, we denote by $N^+(v)$ the set, and by $d^+(v)$ the number, of vertices w such that D contains an edge directed from v to w. $N^+(v)$ $d^+(v)$

To see how orientations come into play in the context of colouring, recall the greedy algorithm from Section 5.2. This colours the vertices of a graph G in turn, following a previously fixed ordering $(v_1, \ldots, v_n)$, with the smallest available colour. This ordering defines an orientation of G if we orient every edge v_iv_j 'backwards', that is, from v_j to v_i if $i < j$. Then to determine a colour for v_j the algorithm only looks at previously coloured neighbours of v_j, those to which v_j sends a directed edge. In particular, if $d^+(v) < k$ for all vertices v, the algorithm will use at most k colours.

If we rewrite the proof of this fact (rather awkwardly) as a formal induction on k, the essential property of the set U of vertices coloured 1 is that every vertex in $G - U$ sends an edge to U: this ensures that $d^+_{G-U}(v) < d^+_G(v)$ for all $v \in G - U$, so we can colour $G - U$ with the remaining $k - 1$ colours by the induction hypothesis.

The following lemma generalizes these observations to list colouring, and to orientations D of G that do not necessarily come from a vertex enumeration but may contain some directed cycles. Let us call an independent set $U \subseteq V(D)$ a *kernel* of D if, for every vertex $v \in D - U$, there is an edge in D directed from v to a vertex in U. Note that kernels of non-empty directed graphs are themselves non-empty. *kernel*

Lemma 5.4.3. *Let H be a graph and $(S_v)_{v \in V(H)}$ a family of lists. If H has an orientation D with $d^+(v) < |S_v|$ for every v, and such that every induced subgraph of D has a kernel, then H can be coloured from the lists S_v.*

Proof. We apply induction on $|H|$. For $|H| = 0$ we take the empty colouring. For the induction step, let $|H| > 0$. Let α be a colour occurring in one of the lists S_v, and let D be an orientation of H as stated. The vertices v with $\alpha \in S_v$ span a non-empty subgraph D' in D; by assumption, D' has a kernel $U \neq \emptyset$. α D' U

Let us colour the vertices in U with α, and remove α from the lists of all the other vertices of D'. Since each of those vertices sends an edge to U, the modified lists S'_v for $v \in D - U$ again satisfy the condition

$d^+(v) < |S'_v|$ in $D - U$. Since $D - U$ is an orientation of $H - U$, we can thus colour $H - U$ from those lists by the induction hypothesis. As none of these lists contains α, this extends our colouring $U \to \{\alpha\}$ to the desired list colouring of H. $\square$

In our proof of the list colouring conjecture for bipartite graphs we shall apply Lemma 5.4.3 only to colourings from lists of uniform length k. However, note that keeping list lengths variable is essential for the proof of the lemma itself: its simple induction could not be performed with uniform list lengths.

Theorem 5.4.4. (Galvin 1995)
Every bipartite graph G satisfies $\mathrm{ch}'(G) = \chi'(G)$.

(2.1.4) *Proof.* Let $G =: (X \cup Y, E)$, where $\{X, Y\}$ is a vertex bipartition of G.
X, Y, E We say that two edges of G *meet in* X if they share an end in X, and correspondingly for Y.
k, c Let $\chi'(G) =: k$, and let c be a k-edge-colouring of G.

Clearly, $\mathrm{ch}'(G) \geqslant k$; we prove that $\mathrm{ch}'(G) \leqslant k$. Our plan is to use
H Lemma 5.4.3 to show that the line graph H of G is k-choosable. To apply the lemma, it suffices to find an orientation D of H with $d^+(e) < k$ for every vertex e of H, and such that every induced subgraph of D has a
D kernel. To define D, consider adjacent $e, e' \in E$, say with $c(e) < c(e')$. If e and e' meet in X, we orient the edge $ee' \in H$ from e' towards e; if e and e' meet in Y, we orient it from e to e' (Fig 5.4.3).

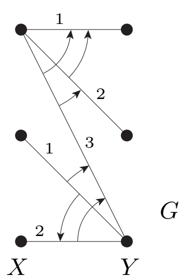

Fig. 5.4.3. Orienting the line graph of G

Let us compute $d^+(e)$ for given $e \in E = V(D)$. If $c(e) = i$, say, then every $e' \in N^+(e)$ meeting e in X has its colour in $\{1, \dots, i-1\}$, and every $e' \in N^+(e)$ meeting e in Y has its colour in $\{i+1, \dots, k\}$. As any two neighbours e' of e meeting e either both in X or both in Y are themselves adjacent and hence coloured differently, this implies $d^+(e) < k$ as desired.

D' It remains to show that every induced subgraph D' of D has a kernel. This, however, is immediate by the stable marriage theorem (2.1.4) for G, if we interpret the directions in D as expressing preference. Indeed, given a vertex $v \in X \cup Y$ and edges $e, e' \in V(D')$ at v, write $e <_v e'$ if the edge

ee' of H is directed from e to e' in D. Then any stable matching in the graph $(X \cup Y, V(D'))$ for this set of preferences is a kernel in D'. □

By Proposition 5.3.1, we now know the exact list-chromatic index of bipartite graphs: (5.3.1)

Corollary 5.4.5. *Every bipartite graph G satisfies* $\mathrm{ch}'(G) = \Delta(G)$. □

5.5 Perfect graphs

As discussed in Section 5.2, a high chromatic number may occur as a purely global phenomenon: even when a graph has large girth, and thus locally looks like a tree, its chromatic number may be arbitrarily high. Since such 'global dependence' is obviously difficult to deal with, one may become interested in graphs where this phenomenon does not occur, i.e. whose chromatic number is high only when there is a local reason for it.

Before we make this precise, let us define two new invariants for a graph G. The greatest integer r such that $K^r \subseteq G$ is the *clique number* $\omega(G)$ of G. The greatest size of a set of independent vertices $\omega(G)$
in G is the *independence number* $\alpha(G)$ of G. Clearly, $\alpha(G) = \omega(\overline{G})$ and $\alpha(G)$
$\omega(G) = \alpha(\overline{G})$.

A graph is called *perfect* if every induced subgraph $H \subseteq G$ has *perfect*
chromatic number $\chi(H) = \omega(H)$, i.e. if the trivial lower bound of $\omega(H)$ colours always suffices to colour the vertices of H. Thus, while proving an assertion of the form $\chi(G) > k$ may in general be difficult, even in principle, for a given graph G, it can always be done for a perfect graph simply by exhibiting some K^{k+1} subgraph as a 'certificate' for non-colourability with k colours.

At first glance, the structure of the class of perfect graphs appears somewhat contrived: although it is closed under induced subgraphs (if only by explicit definition), it is not closed under taking general subgraphs or supergraphs, let alone minors (examples?). However, perfection is an important notion in graph theory: the fact that several fundamental classes of graphs are perfect (as if by fluke) may serve as a superficial indication of this.[2]

What graphs, then, are perfect? Bipartite graphs are, for instance. Less trivially, the complements of bipartite graphs are perfect, too – a fact equivalent to König's duality theorem 2.1.1 (Exercise 41). The

[2] The class of perfect graphs has duality properties with deep connections to optimization and complexity theory, which are far from understood. Theorem 5.5.6 shows the tip of an iceberg here; for more, the reader is referred to Lovász's survey cited in the notes.

so-called *comparability graphs* are perfect, and so are the *interval graphs* (see the exercises); both these turn up in numerous applications.

In order to study at least one such example in some detail, we prove here that the chordal graphs are perfect: a graph is *chordal* (or *triangulated*) if each of its cycles of length at least 4 has a chord, i.e. if it contains no induced cycles other than triangles.

chordal

To show that chordal graphs are perfect, we shall first characterize their structure. If G is a graph with induced subgraphs G_1, G_2 and S, such that $G = G_1 \cup G_2$ and $S = G_1 \cap G_2$, we say that G arises from G_1 and G_2 by *pasting* these graphs together along S.

pasting

[12.3.6]

Proposition 5.5.1. *A graph is chordal if and only if it can be constructed recursively by pasting along complete subgraphs, starting from complete graphs.*

Proof. If G is obtained from two chordal graphs G_1, G_2 by pasting them together along a complete subgraph, then G is clearly again chordal: any induced cycle in G lies in either G_1 or G_2, and is hence a triangle by assumption. Since complete graphs are chordal, this proves that all graphs constructible as stated are chordal.

Conversely, let G be a chordal graph. We show by induction on $|G|$ that G can be constructed as described. This is trivial if G is complete. We therefore assume that G is not complete, in particular that $|G| > 1$, and that all smaller chordal graphs are constructible as stated. Let $a, b \in G$ be two non-adjacent vertices, and let $X \subseteq V(G) \setminus \{a, b\}$ be a minimal a–b separator. Let C denote the component of $G - X$ containing a, and put $G_1 := G[V(C) \cup X]$ and $G_2 := G - C$. Then G arises from G_1 and G_2 by pasting these graphs together along $S := G[X]$.

a, b, X
C
G_1, G_2
S

Since G_1 and G_2 are both chordal (being induced subgraphs of G) and hence constructible by induction, it suffices to show that S is complete. Suppose, then, that $s, t \in S$ are non-adjacent. By the minimality of $X = V(S)$ as an a–b separator, both s and t have a neighbour in C. Hence, there is an X-path from s to t in G_1; we let P_1 be a shortest such path. Analogously, G_2 contains a shortest X-path P_2 from s to t. But then $P_1 \cup P_2$ is a chordless cycle of length $\geqslant 4$ (Fig. 5.5.1), contradicting our assumption that G is chordal. □

s, t

Proposition 5.5.2. *Every chordal graph is perfect.*

Proof. Since complete graphs are perfect, it suffices by Proposition 5.5.1 to show that any graph G obtained from perfect graphs G_1, G_2 by pasting them together along a complete subgraph S is again perfect. So let $H \subseteq G$ be an induced subgraph; we show that $\chi(H) \leqslant \omega(H)$.

Let $H_i := H \cap G_i$ for $i = 1, 2$, and let $T := H \cap S$. Then T is again complete, and H arises from H_1 and H_2 by pasting along T. As

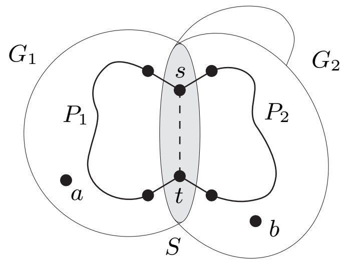

Fig. 5.5.1. If G_1 and G_2 are chordal, then so is G

an induced subgraph of G_i, each H_i can be coloured with $\omega(H_i)$ colours. Since T is complete and hence coloured injectively, two such colourings, one of H_1 and one of H_2, may be combined into a colouring of H with $\max\{\omega(H_1), \omega(H_2)\} \leqslant \omega(H)$ colours – if necessary by permuting the colours in one of the H_i. $\square$

By definition, every induced subgraph of a perfect graph is again perfect. The property of perfection can therefore be characterized by forbidden induced subgraphs: there exists a set $\mathcal{H}$ of imperfect graphs such that any graph is perfect if and only if it has no induced subgraph isomorphic to an element of $\mathcal{H}$. (For example, we may choose as $\mathcal{H}$ the set of all imperfect graphs with vertices in $\mathbb{N}$.)

Naturally, one would like to keep $\mathcal{H}$ as small as possible. It is one of the deepest results in graph theory that $\mathcal{H}$ need only contain two types of graph: the odd cycles of length $\geqslant 5$ and their complements. (Neither of these are perfect; cf. Theorem 5.5.4 below.) This fact, the famous *strong perfect graph conjecture* of Berge (1963), was proved only 40 years later:

strong perfect graph theorem

Theorem 5.5.3. (Chudnovsky, Robertson, Seymour & Thomas 2006) *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an odd cycle of length at least 5 as an induced subgraph.*

hole antihole

In the context of perfect graphs, induced cycles of length at least 4 in G are usually referred to as *holes* in G, while holes in $\overline{G}$ are *antiholes* of G. In this jargon, the strong perfect graph theorem says that a graph is perfect if and only if it has neither odd holes nor odd antiholes.

The proof of the strong perfect graph theorem is long and technical, and it would not be too illuminating to attempt to sketch it. Its overall design is not unlike our proof that chordal graphs are perfect: it decomposes the graph under consideration into simpler pieces, which are shown to be perfect first, in ways that preserve perfection when the pieces are put back together – much as the decomposition along complete subgraphs did in the case of chordal graphs.

To shed more light on the notion of perfection, we instead give two direct proofs of its most important consequence: the *perfect graph theorem*, formerly Berge's *weak perfect graph conjecture*:

perfect graph theorem

Theorem 5.5.4. (Lovász 1972)
A graph is perfect if and only if its complement is perfect.

We give two proofs for Theorem 5.5.4. The first is based on ideas from Lovász's original proof. The second is a magical linear algebra proof, due to Gasparian (1996), of a slightly stronger theorem of Lovász's.

replicating a vertex

The first proof of Theorem 5.5.4 is based on a crucial lemma. Let G be a graph and $x \in G$ a vertex, and let G' be obtained from G by adding a vertex x' and joining it to x and all the neighbours of x. We say that G' is obtained from G by *replicating* the vertex x to the pair $\{x, x'\}$ (Fig. 5.5.2).

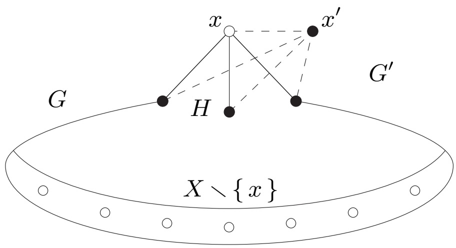

Fig. 5.5.2. Replicating the vertex x in the proof of Lemma 5.5.5

Lemma 5.5.5. (Replication Lemma)
Any graph obtained from a perfect graph by replicating a vertex is again perfect.

x, x'

Proof. We use induction on the order of the perfect graph considered. Replicating the vertex of K^1 yields K^2, which is perfect. For the induction step, let G be a non-trivial perfect graph, and let G' be obtained from G by replicating a vertex $x \in G$ to a pair $\{x, x'\}$. For our proof that G' is perfect it suffices to show $\chi(G') \leqslant \omega(G')$. Indeed, every *proper* induced subgraph H' of G' is either isomorphic to an induced subgraph H of G and hence perfect by assumption, or it is obtained from a proper induced subgraph of G by replicating x and hence perfect by the induction hypothesis. Either way, it can be coloured with $\omega(H')$ colours.

ω

Let $\omega(G) =: \omega$; then $\omega(G') \in \{\omega, \omega+1\}$. If $\omega(G') = \omega+1$, then

$$\chi(G') \leqslant \chi(G) + 1 = \omega + 1 = \omega(G')$$

X

and we are done. So let us assume that $\omega(G') = \omega$. Then x lies in no $K^\omega \subseteq G$: together with x', this would yield a $K^{\omega+1}$ in G'. Let us colour G with ω colours. Since every $K^\omega \subseteq G$ meets the colour class X of x but not x itself, the graph $H := G - (X \setminus \{x\})$ has clique number $\omega(H) < \omega$ (Fig. 5.5.2). Since G is perfect, we may thus colour H with $\omega - 1$ colours. Now X is independent, so the set $(X \setminus \{x\}) \cup \{x'\} = V(G' - H)$ is also independent. We can therefore extend our $(\omega - 1)$-colouring of H to an ω-colouring of G', showing that $\chi(G') \leqslant \omega = \omega(G')$ as desired. □ H

Interestingly, while replicating a vertex preserves perfection, it does not preserve the property of $\chi = \omega$ if this holds only for the graph itself: we need that all induced subgraphs have this property too (Exercise 47).

Proof of Theorem 5.5.4. Given a perfect graph G, let us show that $\overline{G}$ is perfect too. We prove the following statement first:

Every induced subgraph H of G has a complete subgraph K that meets all its independent vertex sets of size $\alpha(H)$. (∗)

For our proof of (∗), let $\mathcal{A}$ be the set of all independent vertex sets of size $\alpha(H)$ in H. Let $\mathcal{A}' = \{ A' \mid A \in \mathcal{A} \}$ be a set of disjoint copies of the sets in $\mathcal{A}$, one new set A' for each $A \in \mathcal{A}$, and let $U := \bigcup \mathcal{A}'$ be their disjoint union. For every vertex v of H write U_v for the set of its clones in U, one in each A' with $v \in A$.[3] These sets U_v may be empty, but every vertex of H' lies in some U_v. $\mathcal{A}$ $\mathcal{A}'$ U_v

Define a graph H' on U by making U_v complete for every $v \in H$, and adding all $U_u - U_v$ edges whenever uv is an edge of H. The sets $A' \in \mathcal{A}'$ are independent in H', because the $A \in \mathcal{A}$ are independent in H. And H' contains no larger independent sets than these A'. Indeed, any independent set in H' meets each U_v in at most one vertex; the vertices v whose U_v it meets are independent in H, so there are at most $\alpha(H) = |A'|$ such v; and H' has no vertices outside the sets U_v. Since the A' are disjoint and all have the same size, the fact that H' contains no larger independent sets than these implies that it cannot be coloured with fewer than $|\mathcal{A}'|$ colours. Thus, $\chi(H') = |\mathcal{A}'|$. H'

Since H' arises from an induced subgraph of H by vertex replication, Lemma 5.5.5 implies that it is perfect. It thus has a complete subgraph K' of order $\chi(H')$. As $\chi(H') = |\mathcal{A}'|$ and K' has at most one vertex in each A', it meets every $A' \in \mathcal{A}'$.

Let K be the subgraph of H induced by the vertices v for which K' meets U_v. This is a complete subgraph of H that meets every $A \in \mathcal{A}$, because K' meets every $A' \in \mathcal{A}'$. This completes the proof of (∗).

Let us now show that $\overline{G}$ is perfect. Expressed in terms of G, we have to show that we can cover the vertices of any induced subgraph H of G by at most $\alpha(H)$ complete subgraphs K of H. These can be found inductively by $\alpha(H)$ applications of (∗), since $\alpha(H - K) < \alpha(H)$ if $K \subseteq H$ satisfies (∗). □

[3] This can be expressed formally by letting $A' := \{ (v, A) \mid v \in A \}$ for every $A \in \mathcal{A}$, and $U_v := \{ (v, A) \mid v \in A \in \mathcal{A} \} = \{ (v, A) \in A' \mid A \in \mathcal{A} \}$ for every $v \in V(H)$.

The proof of the perfect graph theorem ist best understood if we start from the last paragraph: once we have shown (∗), the proof that $\overline{G}$ is perfect is easy.

The existence of a complete graph K as in (∗) follows at once from the perfection of H if the sets in $\mathcal{A}$ provide an optimal colouring of H: if they cover its vertices and $|\mathcal{A}| \leqslant \chi(H)$. The first of these requirements, however, is immaterial. We can just replace H with its subgraph induced by those of its vertices that are covered by $\mathcal{A}$: that subgraph, too, will be perfect, and any K found in it will satisfy (∗) also for the original H.

So assuming that $\mathcal{A}$ covers H, how can we prove that a colouring it induces is optimal? This would be immediate if the $A \in \mathcal{A}$ were disjoint: since they all have the same size, any colouring with fewer colours would need a larger colour class, which does not exist since the sets in $\mathcal{A}$ have size $\alpha(H)$. The trick, now, is simply to *make* the sets in $\mathcal{A}$ disjoint, as is done first thing in the proof of (∗). The obvious way to copy the structure of H to the enlarged vertex set without creating any larger independent sets, then, is to endow it with the edges as in the definition of H' in the proof. This, in turn, motivates the notion of vertex replication.

All we need to complete the proof now is that the expanded graph H' is again perfect. This is Lemma 5.5.5.

Since the following characterization of perfection is symmetrical in G and $\overline{G}$, it clearly implies Theorem 5.5.4. As our proof of Theorem 5.5.6 will again be from first principles, we thus obtain a second and independent proof of the perfect graph theorem.

Theorem 5.5.6. (Lovász 1972)
A graph G is perfect if and only if

$$|H| \leqslant \alpha(H) \cdot \omega(H) \qquad (*)$$

for all induced subgraphs $H \subseteq G$.

v_i, n *Proof.* Let us write $V(G) =: \{v_1, \dots, v_n\}$, and put $\alpha := \alpha(G)$ and
α, ω $\omega := \omega(G)$. The necessity of (∗) is immediate: if G is perfect, then every induced subgraph H of G can be partitioned into at most $\omega(H)$ colour classes each containing at most $\alpha(H)$ vertices, and (∗) follows.

To prove sufficiency, we apply induction on $n = |G|$. Assume that every induced subgraph H of G satisfies (∗), and suppose that G is not perfect. By the induction hypothesis, every proper induced subgraph of G is perfect. Hence, every non-empty independent set $U \subseteq V(G)$ satisfies

$$\chi(G-U) = \omega(G-U) = \omega\,. \qquad (1)$$

Indeed, while the first equality is immediate from the perfection of $G-U$, the second is easy: '$\leqslant$' is obvious, while $\chi(G-U) < \omega$ would imply $\chi(G) \leqslant \omega$, so G would be perfect contrary to our assumption.

Let us apply (1) to a singleton $U = \{u\}$ and consider an ω-colouring of $G - u$. Let K be the vertex set of any K^ω in G. Clearly,

$$\textit{if } u \notin K \textit{ then } K \textit{ meets every colour class of } G - u; \tag{2}$$

$$\textit{if } u \in K \textit{ then } K \textit{ meets all but exactly one colour class of } G - u. \tag{3}$$

Let $A_0 = \{u_1, \dots, u_\alpha\}$ be an independent set in G of size α. A_0
Let $A_1, \dots, A_\omega$ be the colour classes of an ω-colouring of $G - u_1$, let $A_{\omega+1}, \dots, A_{2\omega}$ be the colour classes of an ω-colouring of $G - u_2$, and so on; altogether, this gives us $\alpha\omega + 1$ independent sets $A_0, A_1, \dots, A_{\alpha\omega}$ A_i
in G. For each $i = 0, \dots, \alpha\omega$, there exists by (1) a $K^\omega \subseteq G - A_i$; we denote its vertex set by K_i. K_i

Note that if K is the vertex set of any K^ω in G, then

$$K \cap A_i = \emptyset \textit{ for exactly one } i \in \{0, \dots, \alpha\omega\}. \tag{4}$$

Indeed, if $K \cap A_0 = \emptyset$ then $K \cap A_i \neq \emptyset$ for all $i \neq 0$, by definition of A_i and (2). Similarly if $K \cap A_0 \neq \emptyset$, then $|K \cap A_0| = 1$, so $K \cap A_i = \emptyset$ for exactly one $i \neq 0$: apply (3) to the unique vertex $u \in K \cap A_0$, and (2) to all the other vertices $u \in A_0$.

Let $A = (a_{ij})$ be the real $(\alpha\omega + 1) \times n$ matrix whose rows are the A
incidence vectors of the sets A_i with $V(G)$: where $a_{ij} = 1$ if $v_j \in A_i$, and $a_{ij} = 0$ otherwise. Similarly, let B denote the real $n \times (\alpha\omega + 1)$ matrix B
whose columns are the incidence vectors of the sets K_j with $V(G)$. Now while $|A_i \cap K_i| = 0$ for all i by the choice of K_i, we have $A_i \cap K_j \neq \emptyset$ and hence $|A_i \cap K_j| = 1$ whenever $i \neq j$, by (4). Thus,

$$J := AB$$

is the $(\alpha\omega + 1) \times (\alpha\omega + 1)$ matrix with zero entries in the main diagonal and all other entries 1.

It is easy to see that J is non-singular, and hence has rank $\alpha\omega + 1$. But A, which has n columns, cannot have lower rank than J. Hence $n \geqslant \alpha\omega + 1$, which contradicts $(*)$ for $H := G$. □

5.6 χ-bounded graph properties

Our motivation for considering perfect graphs in the last section was that high chromaticity is easier to understand for these graphs than in general: if a perfect graph has chromatic number at least r, it has a local reason for this in the form of a K^r subgraph. It turned out that perfection has structural implications far beyond this. We may ask, therefore, whether graph properties defined by a slightly weaker condition might still imply some interesting structure for the graphs that have them.

Consider any graph property $\mathcal{G}$ which, like perfection, is closed under taking induced subgraphs: a class that contains all induced subgraphs of its elements. We call $\mathcal{G}$ *χ-bounded* if there exists a function $f: \mathbb{N} \to \mathbb{N}$, called a *$\chi$-bounding* function, such that $\chi(G) \leqslant f(r)$ for all graphs $G \in \mathcal{G}$ of clique number $\omega(G) \leqslant r$, for all $r \in \mathbb{N}$. In particular, then, $\chi(G) \leqslant f(\omega(G))$; but note that for individual functions f this is a weaker condition. Perfection, for example, is χ-bounded with χ-bounding function $f(r) = r$.

χ-bounded
χ-bounding

Qualitatively, graphs in a χ-bounded class $\mathcal{G}$ have complete subgraphs as large as desired as soon as their chromatic number is big enough: that is the idea behind the notion of χ-boundedness. Quantitatively, if f is a χ-bounding function for $\mathcal{G}$, then any $G \in \mathcal{G}$ with $\chi(G) > f(r)$ contains a K^{r+1}, for every $r \in \mathbb{N}$.

Not all graph classes that are closed under taking induced subgraphs are χ-bounded: the triangle-free graphs, for example, are not, by Erdős's Theorem 5.2.5. So which classes are?

Every graph property $\mathcal{G}$ that is closed under induced subgraphs is characterized by the class $\mathcal{H}$ of graphs that are not in $\mathcal{G}$ and are minimal with this property under the induced-subgraph relation: an arbitrary graph G lies in $\mathcal{G}$ if and only if it has no induced subgraph in $\mathcal{H}$. One way to search for χ-bounded graph properties, therefore, is to examine those characterized in this way by some particularly simple classes $\mathcal{H}$.

The strong perfect graph theorem implies that $\mathcal{G}$ is χ-bounded when $\mathcal{H}$ consists of all odd holes and antiholes. But since χ-boundedness is weaker than perfection, we can make $\mathcal{H}$ smaller, thereby making $\mathcal{G}$ larger:

Theorem 5.6.1. (Scott & Seymour 2016)
The graphs with no odd hole are χ-bounded.

Theorem 5.6.1 is one of the earliest results of its kind proved in the wake of the strong perfect graph theorem. It has since been strengthened in various ways, see the notes. The notion of χ-boundedness itself is much older. It was introduced by Gyárfás in 1975, who later proved the following pretty result:

Proposition 5.6.2. *If P is any fixed path, then the graphs not containing P as an induced subgraph are χ-bounded.*

We shall derive Proposition 5.6.2 from a lemma that weakens the assumption of not containing the given path to better support induction:

Lemma 5.6.3. *There is a function $g\colon \mathbb{N}^2 \to \mathbb{N}$ such that $\chi(G) \leqslant g(c,k)$ for every connected graph G satisfying*

(i) *$\chi(G[N(v)]) \leqslant c$ for every vertex $v \in G$;*

(ii) *G has a vertex v at which no induced path of length k starts.*

Proof. We prove by induction on k that the function g defined recursively by $g(c,1) := 1$ and $g(c,k) := g(c,k-1)+c+1$, for all $c \in \mathbb{N}$ and $k > 1$, is as desired. The induction starts with $k = 1$, in which case $G = K^1$ by (ii).

For the induction step pick $v \in G$ as in (ii). Since G is connected, every component C of $G - N(v) - v$ has a neighbour u in $N(v)$. By induction, $\chi(G[V(C) \cup \{u\}]) \leqslant g(c,k-1)$, so also $\chi(C) \leqslant g(c,k-1)$. Combining such colourings of all these C with a $(c+1)$-colouring of v and its neighbours, which exists by (i), yields a $g(c,k)$-colouring of G. □

Proof of Proposition 5.6.2. Fix any $k \in \mathbb{N}$. Let $f\colon \mathbb{N} \to \mathbb{N}$ be defined recursively by $f(1) := 1$ and $f(r) := g(f(r-1),k)$ for $r > 1$, where g is the function from Lemma 5.6.3. We prove by induction on r that every graph G that contains no path of length k induced and is such that $\omega(G) \leqslant r$ satisfies $\chi(G) \leqslant f(r)$.

The induction starts at $r = 1$, in which case G consists of isolated vertices. For the induction step let $G_v := G[N(v)]$ for vertices v of G. As $\omega(G) \leqslant r$ we have $\omega(G_v) \leqslant r-1$, and hence $\chi(G_v) \leqslant f(r-1)$, for every v. By Lemma 5.6.3 with $c = f(r-1)$ this yields $\chi(G) \leqslant f(r)$. □

Proposition 5.6.2 is a special case of a more sweeping conjecture. As we noted earlier for triangles, excluding a graph H will not define a χ-bounded graph property if H contains a cycle: by Theorem 5.2.5 there are graphs of arbitrarily large chromatic number and girth greater than $g(H)$, which thus do not contain H but have clique number 2.

The following conjecture claims that this is the only obstruction to χ-boundedness: that excluding any graph that does *not* contain a cycle will define a χ-bounded class.

Conjecture. (Gyárfás 1975; Sumner 1981)
If F is any fixed forest, then the graphs not containing F as an induced subgraph are χ-bounded.

Exercises

1.$^-$ Show that the four colour theorem does indeed solve the map colouring problem stated in the first sentence of the chapter. Conversely, does the 4-colourability of every map imply the four colour theorem?

2.$^-$ Show that, for the map colouring problem above, it suffices to consider maps such that no point lies on the boundary of more than three countries. How does this affect the proof of the four colour theorem?

3. Try to turn the proof of the five colour theorem into one of the four colour theorem, as follows. Defining v and H as before, assume inductively that H has a 4-colouring; then proceed as before. Where does the proof fail?

4. Calculate the chromatic number of a graph in terms of the chromatic numbers of its blocks.

5.$^-$ Let G be a graph, and let $k \in \mathbb{N}$.

   (i) Show that G has chromatic number at most k if and only if there exists a homomorphism from G to K^k.

   (ii) Show that G is bipartite if and only if there exists a homomorphism from G to K^2 or to an even cycle.

   (iii) Are there homomorphisms from C^{17} to C^7, from C^7 to C^{17}, from C^{16} to C^7, and from C^{17} to C^6?

6. Show that graphs of large girth and without a given minor are 'nearly bipartite' in the following sense. Let H be a fixed graph and C a fixed odd cycle. Use Theorem 7.2.6 to show that if G is a graph of sufficiently large girth (depending only on H and C) that does not contain H as a minor, then there is a homomorphism from G to C.

7. For every $n > 1$, find a bipartite graph on $2n$ vertices, ordered in such a way that the greedy algorithm uses n rather than 2 colours.

8. Consider the following approach to vertex colouring. First, find a maximal independent set of vertices and colour these with colour 1; then find a maximal independent set of vertices in the remaining graph and colour those 2, and so on. Compare this algorithm with the greedy algorithm: which is better?

9. Show that the bound of Proposition 5.2.2 is always at least as sharp as that of Proposition 5.2.1.

A k-chromatic graph G is called *critically k-chromatic*, or just *critical*, if $\chi(G-v) < k$ for every $v \in V(G)$.

10. Determine the critical 3-chromatic graphs.

11.$^+$ Show that every critical k-chromatic graph is $(k-1)$-edge-connected.

12. Formalize and prove the following statement: assuming large average degree drives the colouring number up but not the chromatic number.

13. Write $\mathrm{col}'(G)$ for the least number of colours used by the greedy algorithm for a suitable vertex ordering of a graph G. Does every G satisfy $\mathrm{col}'(G) = \mathrm{col}(G)$ or $\mathrm{col}'(G) = \chi(G)$?

14. Find a function f such that every graph of arboricity at least $f(k)$ has colouring number at least k, and a function g such that every graph of colouring number at least $g(k)$ has arboricity at least k, for all $k \in \mathbb{N}$.

15. Given $k \in \mathbb{N}$, find a constant $c_k > 0$ such that every large enough graph G with $\alpha(G) \leqslant k$ contains a cycle of length at least $c_k\,|G|$.

16.$^-$ Find a graph G for which Brooks's theorem yields a significantly weaker bound on $\chi(G)$ than Proposition 5.2.2.

17.$^+$ Show that, in order to prove Brooks's theorem for a graph $G = (V, E)$, we may assume that $\kappa(G) \geqslant 2$ and $\delta(G) \geqslant 3$. Then prove the theorem under these assumptions, showing first the following two lemmas.

(i) Let $v_1, \ldots, v_n$ be an enumeration of V. If every v_i $(i < n)$ has a neighbour v_j with $j > i$, and if $v_1v_n, v_2v_n \in E$ but $v_1v_2 \notin E$, then the greedy algorithm uses at most $\Delta(G)$ colours.

(ii) If G is not complete, it has a vertex v_n with non-adjacent neighbours v_1, v_2 that do not separate G.

18.$^+$ Show that the following statements are equivalent for a graph G:

(i) $\chi(G) \leqslant k$;

(ii) G has an orientation without directed paths of length k;

(iii) G has an acyclic such orientation (one without directed cycles).

19. Given a graph G and $k \in \mathbb{N}$, let $P_G(k)$ denote the number of vertex colourings $V(G) \to \{1, \ldots, k\}$. Show that P_G is a polynomial in k of degree $n := |G|$, in which the coefficient of k^n is 1 and the coefficient of k^{n-1} is $-\|G\|$. (P_G is called the *chromatic polynomial* of G.)

(Hint. Apply induction on $\|G\|$.)

20.$^+$ Determine the class of all graphs G for which $P_G(k) = k\,(k-1)^{n-1}$. (As in the previous exercise, let $n := |G|$, and let P_G denote the chromatic polynomial of G.)

21. Show that for every $k \in \mathbb{N}$ there is a unique $\subseteq$-minimal 'Kuratowski class' $\mathcal{X}_k$ of k-chromatic graphs such that every k-chromatic graph has a subgraph in $\mathcal{X}_k$, but that for $k \geqslant 3$ this class $\mathcal{X}_k$ is never finite.

22. For every $k \in \mathbb{N}$, construct a triangle-free k-chromatic graph. Can you even make them C^4-free?

23. Consider an infinite matrix whose rows and columns are both indexed by $\mathbb{N} \setminus \{0\}$. For all $i, j \in \mathbb{N} \setminus \{0\}$ join the element v_{ij} in row i and column j to every element of column $i + j$. Prove that the resulting graph contains no triangle. What is its chromatic number?

24. In the definition of 'k-constructible', replace axioms (ii) and (iii) by

    (ii)$'$ *Every supergraph of a k-constructible graph is k-constructible.*

    (iii)$'$ *If x, y_1, y_2 are distinct vertices of a graph G and $y_1y_2 \in E(G)$, and if both $G + xy_1$ and $G + xy_2$ are k-constructible, then G is k-constructible.*

    Show that a graph is k-constructible with respect to this new definition if and only if its chromatic number is at least k.

25.$^-$ An $n \times n$-matrix with entries from $\{1, \ldots, n\}$ is called a *Latin square* if every element of $\{1, \ldots, n\}$ appears exactly once in each column and exactly once in each row. Recast the problem of constructing Latin squares as a colouring problem.

26. Without using Proposition 5.3.1, show that $\chi'(G) = k$ for every k-regular bipartite graph G.

27. Prove Proposition 5.3.1 from the statement of the previous exercise.

28.$^+$ A forest is called *linear* if all its components are paths. Show that every cubic graph has an edge-decomposition into two linear forests.

29.$^-$ Without using Theorem 5.4.2, show that every plane graph is 6-list-colourable.

30. For every integer k, find a 2-chromatic graph whose choice number is at least k.

31.$^-$ Find a general upper bound for $\mathrm{ch}'(G)$ in terms of $\chi'(G)$.

32. Compare the choice number of a graph with its colouring number: which is greater? Can you prove the analogue of Theorem 5.4.1 for the colouring number (directly, without using any major theorem)?

33.$^+$ Prove that the choice number of K_2^r is r.

34. The *total chromatic number* $\chi''(G)$ of a graph $G = (V, E)$ is the least number of colours needed to colour the vertices and edges of G simultaneously so that any adjacent or incident elements of $V \cup E$ are coloured differently. The *total colouring conjecture* says that $\chi''(G) \leqslant \Delta(G) + 2$. Bound the total chromatic number from above in terms of the list-chromatic index, and use this bound to deduce a weakening of the total colouring conjecture from the list colouring conjecture.

35.$^-$ Does every oriented graph have a kernel? If not, does every graph have an orientation in which every induced subgraph has a kernel? If not, does every graph have an orientation that has a kernel?

36.$^+$ Prove that every directed graph without odd directed cycles has a kernel.

37. Show that every bipartite planar graph is 3-list-colourable.

    (Hint. Apply the previous exercise and Lemma 5.4.3.)

38.$^-$ Show that perfection is closed neither under edge deletion nor under edge contraction.

39.$^-$ Deduce Theorem 5.5.6 from the strong perfect graph theorem.

40. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two sets of imperfect graphs, each minimal with the property that a graph is perfect if and only if it has no induced subgraph in $\mathcal{H}_i$ $(i = 1, 2)$. Do $\mathcal{H}_1$ and $\mathcal{H}_2$ contain the same graphs, up to isomorphism?

41. Use König's Theorem 2.1.1 to show that the complement of any bipartite graph is perfect.

42. Using the results of this chapter, find a one-line proof of the following theorem of König, the dual of Theorem 2.1.1: in any bipartite graph without isolated vertices, the minimum number of edges meeting all vertices equals the maximum number of independent vertices.

43. A graph is called a *comparability graph* if there exists a partial ordering of its vertex set such that two vertices are adjacent if and only if they are comparable. Show that every comparability graph is perfect.

44. A graph G is called an *interval graph* if there exists a set $\{ I_v \mid v \in V(G) \}$ of real intervals such that $I_u \cap I_v \neq \emptyset$ if and only if $uv \in E(G)$.

    (i) Show that every interval graph is chordal.

    (ii) Show that the complement of any interval graph is a comparability graph.

    (Conversely, a chordal graph is an interval graph if its complement is a comparability graph; this is a theorem of Gilmore and Hoffman (1964).)

45. Show that $\chi(H) \in \{\omega(H), \omega(H)+1\}$ for every line graph H.

46.$^+$ Characterize the graphs whose line graphs are perfect.

47. Let G' be obtained from a graph G by replicating a vertex. If G satisfies $\chi = \omega$, does G' too?

48. Show that a graph G is perfect if and only if every non-empty induced subgraph H of G contains an independent set $A \subseteq V(H)$ such that $\omega(H - A) < \omega(H)$.

49.$^+$ Consider the graphs G for which every induced subgraph H has the property that every maximum-size complete subgraph of H meets every maximum-size independent vertex set in H.

    (i) Show that these graphs G are perfect.

    (ii) Show that these graphs G are precisely the graphs not containing an induced copy of P^3.

50.$^+$ Show that in every perfect graph G one can find a set $\mathcal{A}$ of independent vertex sets and a set $\mathcal{K}$ of vertex sets of complete subgraphs such that $\bigcup \mathcal{A} = V(G) = \bigcup \mathcal{K}$ and every set in $\mathcal{A}$ meets every set in $\mathcal{K}$.
    (Hint. Lemma 5.5.5.)

51.$^+$ Let G be a perfect graph. As in the proof of Theorem 5.5.4, replace every vertex v of G with a perfect graph G_v (not necessarily complete), adding all G_u–G_v edges whenever uv is an edge of G. Show that the resulting graph G' is again perfect.

52. Show that $\chi(G) \geqslant \max\{\omega(G), \lceil |G|/\alpha(G\rceil)\}$ for all graphs G.
    (i)$^-$ Find an example of G where this inequality is strict.
    (ii)$^+$ Prove that all graphs G obtained from cycles by repeated vertex replication satisfy it with equality.

53.$^-$ Discuss why χ-boundedness is defined only for graph properties that are closed under induced subgraphs. Support your arguments by the construction of a class that satisfies the definition of χ-boundedness but is *not* closed under induced subgraphs, so that the graphs in this class are very similar to the triangle-free graphs.

54.$^-$ Reduce the Gyárfás-Sumner conjecture to trees: prove that if not containing any fixed tree as an induced subgraph is a χ-bounded property then so is that of not containing any fixed forest.

Notes

The authoritative reference work on all questions of graph colouring is T.R. Jensen & B. Toft, *Graph Coloring Problems*, Wiley 1995. Starting with a brief survey of the most important results and areas of research in the field, this monograph gives a detailed account of over 200 open colouring problems, complete with extensive background surveys and references. Most of the remarks below are discussed comprehensively in this book, and all the references for this chapter can be found there. A book specifically on edge colouring is L.M. Favrholdt, D. Scheide, M. Stiebitz & B. Toft, *Graph Edge Coloring: Vizing's Theorem and Goldberg's Conjecture*, Wiley 2012.

The *four colour problem*, whether every map can be coloured with four colours so that adjacent countries are shown in different colours, was raised by a certain Francis Guthrie in 1852. He put the question to his brother Frederick, who was then a mathematics undergraduate in Cambridge. The problem was first brought to the attention of a wider public when Cayley presented it to the London Mathematical Society in 1878. A year later, Kempe published an incorrect proof, which was in 1890 modified by Heawood into our first proof of the five colour theorem. In 1880, Tait announced 'further proofs' of the four colour conjecture, which never materialized; see the notes for Chapter 10. Our second proof of the five colour theorem dates back at least to 1972, when Woodall used it in his first graph theory course.

The first widely accepted proof of the four colour theorem was published by Appel and Haken in 1977. The proof builds on ideas that can be traced back as far as Kempe's paper, and were developed largely by Birkhoff and Heesch. Very roughly, the proof sets out first to show that every plane triangulation must contain at least one of 1482 certain 'unavoidable configurations'. In a second step, a computer is used to show that each of those configurations is 'reducible', i.e., that any plane triangulation containing such a configuration can be 4-coloured by piecing together 4-colourings of smaller plane triangulations. Taken together, these two steps amount to an inductive proof that all plane triangulations, and hence all planar graphs, can be 4-coloured.

Appel & Haken's proof has not been immune to criticism, not only because of their use of a computer. The authors responded with a 741 page long algorithmic version of their proof, which added more configurations to the 'unavoidable' list: K. Appel & W. Haken, *Every Planar Map is Four Colorable*, American Mathematical Society 1989. A much shorter proof, which is based on the same ideas (and, in particular, uses a computer in the same way) but can be more readily verified both in its verbal and its computer part, has been given by N. Robertson, D. Sanders, P.D. Seymour & R. Thomas, The four-colour theorem, *J. Comb. Theory, Ser. B* **70** (1997), 2–44.

A relatively short proof of Grötzsch's theorem was found by C. Thomassen, A short list color proof of Grötzsch's theorem, *J. Comb. Theory, Ser. B* **88** (2003), 189–192. Although not touched upon in this chapter, colouring problems for graphs embedded in surfaces other than the plane form a substantial and interesting part of colouring theory; see B. Mohar & C. Thomassen, *Graphs on Surfaces*, Johns Hopkins University Press 2001.

The k-chromatic subgraph H with $\delta(H) \geqslant k-1$ in Lemma 5.2.3 cannot in general be chosen with $\delta(H) = k-1$. See Jensen & Toft, Chapter 5. In conjunction with Theorem 1.4.3, Lemma 5.2.3 implies that graphs of large chromatic number have highly connected subgraphs. Some of these also have large chromatic number themselves; this was proved by Alon, Kleitman, Saks, Seymour and Thomassen, Subgraphs of large connectivity and chromatic number in graphs of large chromatic number, *J. Graph Theory* **11** (1987), 367–371.

The proof of Brooks's theorem indicated in Exercise 17, where the greedy algorithm is applied to a carefully chosen vertex ordering, is due to Lovász (1973). Lovász (1968) was also the first to *construct* graphs of arbitrarily large girth and chromatic number, graphs whose existence Erdős had proved by probabilistic methods ten years earlier in Graph theory and probability, *Can. J. Math.* **11** (1959), 34–38. Another constructive proof can be found in J. Nešetřil & V. Rödl, Sparse Ramsey graphs, *Combinatorica* **4** (1984), 71–78.

A. Urquhart, The graph constructions of Hajós and Ore, *J. Graph Theory* **26** (1997), 211–215, showed that not only do the graphs of chromatic number at least k each *contain* a k-constructible graph (as by Hajós's theorem); they are in fact all themselves k-constructible. Note that, in the course of constructing a given graph, the order of the graphs constructed on the way can go both up and down, depending on which rule is applied at each step. This means that there is no obvious upper bound on the number of steps needed to construct a given graph, and indeed no such bound is known. In particular, Hajós's theorem does not provide bounded-length 'certificates' for the property of having chromatic number at least k. Unlike Kuratowski's theorem, it is therefore not a 'good characterization' in the sense of complexity theory. (See Chapter 12.7, the notes for Chapter 10, and the end of the notes for Chapter 12 for more details.)

Algebraic tools for showing that the chromatic number of a graph is large have been developed by Kleitman & Lovász (1982), by Alon & Tarsi (see Alon's paper cited below), and by Babson & Kozlov (2007).

Theorem 5.3.3 was proved by B. Csaba, D. Kühn, A. Lo, D. Osthus and A. Treglown, Proof of the 1-factorization and Hamilton decomposition conjectures, *Memoirs of the AMS* **244** (2016), arXiv:1401.4164.

List colourings were first introduced in 1976 by Vizing. Among other things, Vizing proved the list-colouring equivalent of Brooks's theorem. Voigt (1993) constructed a plane graph of order 238 that is not 4-choosable; thus, Thomassen's list version of the five colour theorem is best possible. A stimulating survey on the list-chromatic number and how it relates to the more classical graph invariants (including a proof of Theorem 5.4.1) is given by N. Alon, Restricted colorings of graphs, in (K. Walker, ed.) *Surveys in Combinatorics*, LMS Lecture Notes **187**, Cambridge University Press 1993. Both the list colouring conjecture and Galvin's proof of the bipartite case are originally stated for multigraphs. Kahn (1994) proved that the conjecture is asymptotically correct, as follows: given any $\epsilon > 0$, every graph G with large enough maximum degree satisfies $\mathrm{ch}'(G) \leqslant (1+\epsilon)\Delta(G)$.

The total colouring conjecture (Exercise 34) was proposed around 1965 by Vizing and by Behzad; see Jensen & Toft for details.

A comprehensive treatment of perfect graphs is given in A. Schrijver, *Combinatorial optimization*, Springer 2003. See also *Perfect Graphs* by J. Ramírez-Alfonsín & B. Reed (eds.), Wiley 2001.

Our proof of the perfect graph theorem, Theorem 5.5.4, is taken from the article of M. Preissmann and A. Sebő in that volume. Our proof of Theorem 5.5.6, which implies the perfect graph theorem, is due to G.S. Gasparian, Minimal imperfect graphs: a simple approach, *Combinatorica* **16** (1996), 209–212.

Theorem 5.5.3 is proved in M. Chudnovsky, N. Robertson, P.D. Seymour and R. Thomas, The strong perfect graph theorem, *Ann. Math.* **164** (2006), 51–229, arXiv:math/0212070. This proof is elucidated by N. Trotignon in his 2013 survey on the arXiv:1301.5149. Chudnovsky, Cornuejols, Liu, Seymour and Vušković, Recognizing Berge graphs, *Combinatorica* **25** (2005), 143–186, constructed an $O(n^9)$ algorithm testing for odd holes and antiholes, and thus by the strong perfect graph theorem also for perfection.

Theorem 5.6.1 is due to A. Scott and P.D. Seymour, Induced subgraphs of graphs with large chromatic number. I. Odd holes, *J. Comb. Theory, Ser. B* **121** (2016), 68–84, arXiv:1410.4118. It has since been shown that it suffices to exclude holes of certain lengths, such as large odd holes, or more generally holes of any specified length modulo k, for any fixed k. See, A. Scott and P.D. Seymour, Induced subgraphs of graphs with large chromatic number. X. Holes of specific residue, *Combinatorica* **39** (2019), 1105–1132, arXiv:1705.04609. Our proof of Proposition 5.6.2 is from A. Scott and P.D. Seymour, A survey on χ-boundedness, *J. Graph Theory* **95** (2020), 473–504, arXiv:1812.07500. References for the Gyárfás-Sumner conjecture and the origin of Proposition 5.6.2 can also be found there.

The property of not containing *any* subdivision of some fixed tree as an induced subgraph was shown to be χ-bounded by A.D. Scott, Induced trees in graphs of large chromatic number, *J. Graph Theory* **24** (1997), 297–311. Not containing any subdivision of an arbitrary fixed graph H induced, however, is not a χ-bounded property; see J. Kozik et al, Triangle-free intersection graphs of line segments with large chromatic number, *J. Comb. Theory, Ser. B* **105** (2014), 6–10, arXiv:1209.1595.

The structure of graphs forced by forbidding some fixed induced subgraph or subgraphs, as in the strong perfect graph theorem and Section 5.6, has been studied more generally. One of the central problems is the *Erdős-Hajnal conjecture* that the graphs without some fixed induced subgraph have much larger sets of vertices that are either independent or induce a complete subgraph than arbitrary graphs do. See Chapter 9.1 for a precise statement.