COURSERA
INTRODUCTION TO MATHEMATICAL THINKING
TEST FLIGHT ASSIGNMENT

ALEXANDER LADERMAN

Disclaimer. In the process of completing this assignment I have availed myself of various aids, including but not limited to the following:

- Course materials provided by the instructor or available through the course resources.
- Online platforms such as Wolfram Alpha, Wikipedia, and other educational websites.
- Conversations with ChatGPT or similar language models for clarifications and guidance.
- General content available on the Internet relevant to the topics covered in the assignment.

I acknowledge that the ideas, concepts, and information presented in this assignment may be influenced by the aforementioned aids. However, the final work is a result of my own understanding, analysis, and synthesis of the provided material.

1. Question

Say whether the following is true or false and support your answer by a proof:

$$(\exists m \in \mathbb{N})(\exists n \in \mathbb{N})(3m + 5n = 12)$$

Answer

It's false. We set n to it's smallest possible value, which is $n = 1$. We then check what happens as we increase m:

$$n = 1 \begin{cases} \text{for } m = 1, & 3 + 5 = 8 \neq 12 \\ \text{for } m = 2, & 6 + 5 = 11 \neq 12 \\ \text{for } m = 2, & 9 + 5 = 14 > 12 \end{cases}$$

This demonstrates that $\forall m > 2$ the expression is always greater than, and thus not equal to, 12.

Now we check for n with the smallest m

$$m = 1 \begin{cases} \text{for } n = 1, & 3 + 5 = 8 \neq 12 \\ \text{for } n = 2, & 3 + 10 = 13 > 12 \end{cases}$$

This likewise demonstrates that $\forall n > 2$ the expression is always greater than, and thus again not equal to, 12.

We have tested for $m, n = 1$. To exhaust all possibilities we must finally try with $1 < m, n \leq 2$. The only natural number value we can give m, n to satisfy that condition is 2, so:

$$\text{for } m, n = 2, 6 + 10 = 16 > 12$$

This exhausts all possibilities proving the statement is false.

2. Question

Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).

Answer

It's true. Let five consecutive integers be expressed as:

$$n + (n + 1) + (n + 2) + (n + 3) + (n + 4)$$

We factor out n to get:

$$n(1 + (1 + 1) + (1 + 2) + (1 + 3) + (1 + 4)) = n(15) = n \times 3 \times 5$$

This shows that the sum is a multiple of 5, which by the rules of divisibility is divisible by 5.

3. Question

Say whether the following is true or false and support your answer by a proof: For any integer n, the number $n^2 + n + 1$ is odd.

Answer

It's true. Factor out n:

$$n(n+1)+1$$

Either n is even or n is odd:

For n even

Let $(n+1) = m$. Substituting into the above expression produces $n \times m + 1$. The term $n \times m$ is a multiple of the even number n, which as such is divisible by 2 and thus also even. When to an even number we add the last term of 1 **we will get an odd number**. I.e. the statement is true when n even;

For n odd

Adding 1 to n is going to give us an even number $m = (n+1)$ Just as like before the term $n \times m$ is a multiple of an even number and so also even. To which adding 1 from the last term also **produces an odd number**. Thus the statement is true also when n odd;

Hence the statement is true for all integers.

4. Question

Prove that every odd natural number is of one of the forms $4n + 1$ or $4n + 3$, where n is an integer.

Answer

It's true by application of the Division Theorem:

$$m = 4n + r, 0 \leq r < 4$$

In other words all natural numbers will be one of $(4n + 0)$, $(4n + 1)$, $(4n + 2)$ or $(4n + 3)$

The first term in every expression is a multiple of 4, and will as such always be even. Adding the remainder to an even number will give us either an odd number or an even number, depending on whether the remainder added is even or odd. Hence, adding either 1 or 3 as stated in the question will produce an odd number m, which holds for all natural numbers m.

5. Question

Prove that for any integer n, at least one of the integers $n, n+2, n+4$ is divisible by 3.

Answer

It's true. By the Division Theorem:

$$a = bq + r, 0 \leq r < b$$

b = 3 will divide all integers, leaving a remainder $r \mid 0 \leq r < 3$ to produce one of $(3q+0)$, $(3q+1)$ or $(3q+2)$

We consider the case for each one of the three integers:

$$n \begin{cases} \text{for } (3q+0), & n = 3q \mid 3 \\ \text{for } (3q+1), & n = 3q+1 \\ \text{for } (3q+2), & n = 3q+2 \end{cases}$$

n is divisible for $(3q+0)$

$$n+2 \begin{cases} \text{for } (3q+0), & n+2 = 3q \\ \text{for } (3q+1), & n+2 = 3q+1 \rightarrow n = 3q+3 = 3(q+1) \mid 3 \\ \text{for } (3q+2), & n+2 = 3q+2 \end{cases}$$

n is divisible for $(3q+1)$

$$n+4 \begin{cases} \text{for } (3q+0), & n+4 = 3q \\ \text{for } (3q+1), & n+4 = 3q+1 \\ \text{for } (3q+2), & n+4 = 3q+2 \rightarrow n = 3q+6 = 3(q+2) \mid 3 \end{cases}$$

n is divisible for $(3q+2)$

This demonstrates that for any n, at least one of n, $(n+2)$ or $(n+4)$ will be divisible by 3, proving the statement as true.

6. Question

A classic unsolved problem in number theory asks if there are infinitely many pairs of 'twin primes', pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

Answer

It's true. We established in the last problem that at least one of a triple of whole numbers each 2 from the next will be divisible by 3. In which case all such triplets apart from the one that, as stated, includes 3 itself cannot satisfy the requirement for each of the numbers to be prime because at least one of those numbers will be a multiple of the prime number 3.

7. Question

Prove that for any natural number n,

$$2 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 2$$

Answer

It's true. By induction.

$$2 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 2$$

For the base case $n = 1$, the identity checks out $2 = 2^{1+1} - 2 = 2^2 - 2 = 4 - 2 = 2$ For the inductive step we assume that the formula holds for some arbitrary k:

$$2 + 2^2 + 2^3 + ... + 2^k = 2^{k+1} - 2$$

Then using the inductive hypothesis we check if it holds for $k + 1$

$$2 + 2^2 + 2^3 + ... + 2^k + 2^{k+1} = 2^{(k+1)+1} - 2$$

We move 2^{k+1} to the right hand side and simplify

$$2 + 2^2 + 2^3 + ... + 2^k = 2^{(k+1)+1} - 2^{k+1} - 2$$

$$2 + 2^2 + 2^3 + ... + 2^k = 2^{k+2} - 2^{k+1} - 2$$

$$2 + 2^2 + 2^3 + ... + 2^k = 2^{k+2} - 2^{k+1} - 2$$

$$2 + 2^2 + 2^3 + ... + 2^k = 2 * 2^{k+1} - 2^{k+1} - 2$$

$$2 + 2^2 + 2^3 + ... + 2^k = 2^{k+1} - 2$$

The formula holds for $k + 1$ and the induction is complete.

8. Question

Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number $M > 0$, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.

Answer

For a sequence $\{a_n\}$ the limit $\lim_{n\to\infty} a_n = L$ is defined as follows:

For every positive number ϵ, there exists a positive integer N such that for all $n > N$, the inequality $|a_n - L| < \epsilon$ holds.
Assuming the limit does approach ML, we can multiply both sides by M:

$$\{Ma_n\}_{n=1}^{\infty} \implies M|a_n - L| < M\epsilon$$

Given $\epsilon > 0$. By the Archimedean property we can simply choose a value that satisfies the upper bound. Let us rewrite ϵ with $\frac{\epsilon}{M}$

$$|Ma_n - ML| < M * \frac{\epsilon}{M}$$

$$|Ma_n - ML| < \epsilon$$

This shows the limit as ML which proves the statement.

9. Question

Given an infinite collection $A_n, n = 1, 2, ...$ of intervals of the real line, their *intersection* is defined to be $\bigcap_{n=1}^{\infty} A_n = \{x \mid (\forall n)(x \in A_n)\}$. Give an example of a family of intervals $An, n = 1, 2, ...$, such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Prove that your example has the stated property.

Answer

If the intervals are nested then each subsequent interval in the sequence will be included in it's parent and thus by recursion $\forall n, A_{n+1}$ will be a subset of A_n. If the intervals become increasingly small and approach some limit the intersection as $n \to \infty$ of an open interval will be the empty set. For example:

$$\{A_1\} = (0, \frac{1}{1})$$
$$\{A_2\} = (0, \frac{1}{2})$$
$$\{A_3\} = (0, \frac{1}{3})$$
$$\vdots$$
$$\{A_n\} = (0, \frac{1}{n})$$

10. Question

Give an example of a family of intervals $A_n, n = 1, 2, ...,$ such that such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number. Prove that your example has the stated property.

Answer

Again if the intervals are nested then $\forall n, A_{n+1} \subset A_n$. And by completeness, if the intervals are closed and become increasingly small they will approach a point as $n \to \infty$. For example:

$$\{A_n\}[0, \frac{1}{n}], n \to \infty = [0, 0]$$

The intersection is then called a singleton, a set containing the single real number at that point:

$$\bigcap_{n=1}^{\infty} A_n = \{0\}$$