Problem Set 2

ECON 40201 - Advanced Industrial Organization II

DUE: February 20th, 2025

Please typeset your assignment in LaTeX. For the coding part, feel free to use your preferred programming language. Please comment your code clearly, as the clarity of your code will also be evaluated. Submit both your write-ups and codes through Canvas in a single zip file. You can work in a group of two (please clearly indicate both authors in your submitted PDF) and submit a single file per group.

Exercise 1. Dynamic Discrete Choice - Theory and Implementation

In the Jupyter Notebook file ps2_ex1.ipynb, you will find an implementation of single-agent dynamic discrete choice estimation following Rust (1987)[1]. We mostly follow the original model in the paper, with a few minor changes:

- Set of alternatives: $\mathcal{J} = \{1, 2\}$:
  - Choice 1: Maintain bus engine.
  - Choice 2: Replace bus engine (which regenerates the mileage to 0).
- State variables:
  - Observable state variables, $x_t \in \{0, ..., X-1\}$, are the bus mileages.
- State transition matrix:
  - Choice 1: When the engine is maintained, the transition matrix from state x_t to x_{t+1} is

$$\begin{pmatrix} \varphi_1 & \varphi_2 & 1-\varphi_1-\varphi_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \varphi_1 & \varphi_2 & 1-\varphi_1-\varphi_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \varphi_1 & \varphi_2 & 1-\varphi_1-\varphi_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \varphi_1 & 1-\varphi_1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

  - Choice 2: When the engine is replaced, mileage gets regenerated to 0 (i.e., $x_t = 0$) so that

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

[1] We thank Conroy Lau for providing his Jupyter Notebook code on single-agent dynamic discrete choice estimation.

- Discount factor, β, is assumed to be known.
- The deterministic component of the utility, $\overline{u}_j(x_t;\theta)$, are parameterized as
  - Choice 1: $\overline{u}_1(x_t;\theta) = \theta_1 x_t$.
  - Choice 2: $\overline{u}_2(x_t;\theta) = \theta_2$.

  We can interpret θ_1 as the maintenance cost and θ_2 as the replacement cost.
- Unobservable parts of the flow utility, ϵ_{jt}, are assumed to follow a T1EV distribution with mean 0 for any $j \in \mathcal{J}$ and t.

1. For some different combinations of model parameters (β, θ_1, and θ_2), plot the implied true conditional choice probability of choosing to maintain the engine (i.e., Choice 1) for each state x_t. Compare your plots across different values of β, θ_1, and θ_2. How do plots change as you vary each of those model parameters? Do the patterns you observe make sense? (Hint: Conditional choice probabilities can be calculated using `prob_from_dgp` function in the Jupyter Notebook.)

2. We will now implement a simplified version of Hotz and Miller (1993)'s choice probability inversion theorem to estimate θ_1 and θ_2.[2]. Let $\overline{v}_j(x_t;\theta,\varphi)$ be the deterministic part of the value function for choosing choice j at state x_t so that the Bellman equation representation becomes

$$\overline{v}_j(x_t;\theta,\varphi) = \overline{u}_j(x_t;\theta) + \beta \int \ln\left(\sum_{k\in\mathcal{J}} \exp\{\overline{v}_k(x_{t+1};\theta,\varphi)\}\right) \Pr[dx_{t+1}|x_t, a_t = j, \varphi].$$

Let $\widehat{\Pr}[a_t = j|x_t]$ and $\widehat{v}_j(x_t)$ be the sample analogue of $\Pr[a_t = j|x_t;\theta,\varphi]$ and $\overline{v}_j(x_t;\theta,\varphi)$ respectively. Then, the sample analogue of the conditional choice probability can be written as

$$\widehat{\Pr}[a_t = j|x_t] = \frac{\exp(\widehat{v}_j(x_t))}{\sum_{k\in\mathcal{J}} \exp(\widehat{v}_k(x_t))}.$$

Let J be the reference choice, or outside option, in which we normalize $\overline{v}_J(x_t;\theta,\varphi) = 0$. Then, using the above sample analogue representation, we can write

$$\ln\left(\frac{\widehat{\Pr}(a_t = j|x_t)}{\widehat{\Pr}(a_t = J|x_t)}\right) = \widehat{v}_j(x_t) - \underbrace{\widehat{v}_J(x_t)}_{= 0 \text{ by normalization}}.$$

As a result, the maximum-likelihood estimator would proceed with the objective function

$$\sum_t \sum_{j\in\mathcal{J}} \mathbf{1}(a_t = j|x_t) \log\left\{\frac{\exp(\overline{\widehat{v}}_j(x_t;\theta,\varphi))}{\sum_{k\in\mathcal{J}} \exp(\overline{\widehat{v}}_k(x_t;\theta,\varphi))}\right\}$$

Complete the functions `csvf_from_ccp`, `likelihood_two_step_ccp`, and `two_step_ccp` under "Estimation 2: Conditional choice probability inversion" section of the Jupyter Notebook to implement the estimation method illustrated above. (The suggested flow of implementation in the Jupyter Notebook is there only to help you. Therefore, you are free to implement the estimation method above with your own functions instead of completing the functions in the Jupyter Notebook if you would prefer.)

[2]We follow the example illustrated in the Section 2.3.2 of the textbook.

3. Choose your true model parameters and simulate data using `draw` function in the Jupyter Notebook. Use the choice probability inversion method you implemented in Part 2 to estimate $\theta = (\theta_1, \theta_2)$ and plot the implied conditional choice probabilities. How do they compare to your true values of θ and the corresponding true conditional choice probabilities?

4. Repeat the Part 3 using (i) nested fixed-point algorithm, (ii) nested pseudo-likelihood, and (iii) MPEC (all of them are already coded-up in the Jupyter Notebook). Compare their speed and accuracy for different values of the model parameters and discuss.

Exercise 2. Dynamic Discrete Choice - Empirical Exercise

In the file ps2_ex2.csv, you will find mileage data for the bus managed by one Harold Zurcher. For this exercise, there are no covariates. Each time period, Harold Zurcher chooses to perform maintenance on the bus, or to replace the engine. Let his flow utilities be given by the following function

$$u\left(x_t, d_t\right) + \epsilon_{a,t} = \begin{cases} -\theta_1 x_t - \theta_2 \left(\frac{x_t}{100}\right)^2 + \epsilon_{0,t} & \text{if } d_t = 0 \\ -\theta_3 + \epsilon_{1,t} & \text{if } d_t = 1 \end{cases}$$

Where x_t is the current mileage of the bus, d_t is the choice of Harold Zurcher, and θ is a vector of parameters. Each choice also contains unobserved utility $\epsilon_{a,t}$ that are distributed independent T1EV.

Harold Zurcher maximizes his lifetime discounted utility, discounted by β, where the state x_t evolves according to

$$p\left(x_{t+1} \mid x_t, d_t\right) = \begin{cases} g\left(x_{t+1} - 0\right) & \text{if } d_t = 1 \\ g\left(x_{t+1} - x_t\right) & \text{if } d_t = 0 \end{cases}$$

That is, replacing the engine regenerates the mileage to 0 .

1. How can you recover engine replacement from the mileage data? Store these decisions in a separate variable.

2. Discuss the conditional independence assumption.

3. Discretize the domain of x_t into K chunks. Estimate the Markov Transition probability $p\left(x_{t+1} \mid x_t, d_t\right)$. This should be two $K \times K$ stochastic matrices, depending on action d_t. Make your own choice of K.

4. Define the expected value function $EV(x, d) = \int V_\theta(y, \epsilon) p(d\epsilon) p(dy \mid x, d)$. Show the following fixed point equation holds:

$$EV(x, d) = \int \log\left(\sum_j \exp(u(y, j) + \beta EV(y, j))\right) p(dy \mid x, d)$$

5. Derive the conditional choice probabilities using $EV(x, d)$ and θ.

6. Reduce the state space of EV using the regenerative property.

7. Rewrite the fixed point equation as a matrix equation.

8. Write a function that solves the fixed-point equation using Rust's algorithm (Hint: Jupyter Notebook from the Question 1 might be a good starting point, although you will need to change some model specifications.).

9. Write a function that computes the likelihood of the sample for any parameter θ.

10. Estimate θ using $\beta = 0.999$.

Exercise 3. Static Discrete Games with Simultaneous and Sequential Entry

Consider a static entry game between J firms operating in T markets. Upon entry, the profit of firm j in market t depends on the number of entrants and is given by:

$$\pi_{jt}\left(n_{-jt}\right) = x_{jt}\beta - \phi_j - \delta_j \log\left(1 + n_{-jt}\right) + \epsilon_{jt}$$

where x_{jt} are market-firm specific profit shifters, ϕ_j is a firm specific fixed cost, n_{-jt} is the number of competitors in market t and $\epsilon_{jt} \sim F$ is an idiosyncratic shock. Given n_{-jt} firm j enters if and only if $\pi_{jt} \geq 0$.

1. For a given market t, find the pure strategy Nash Equilibrium (or Equilibria) as a function of the realization of $(\epsilon_{1t}, \ldots, \epsilon_{Jt})$. For simplicity assume that $J = 2$.

2. Suppose you have data on the entry decisions of both firms and some relevant characteristics $\{y_{1t}, y_{2t}, x_{1t}, x_{2t}\}_{t=1}^{T}$, where $y_{jt} = 1$ if firm j entered market t. Write the likelihood of the data as $\prod_{t=1}^{T} l\left(y_{1t}, y_{2t} \mid x_{1t}, x_{2t}, \beta, \delta_1, \delta_2\right)$, where for each $t, (y_{1t}, y_{2t})$ is a Nash Equilibrium. Is the likelihood well-defined? Explain.

3. Now suppose that entry is sequential and assume that firm 1 moves before firm 2. Let $\prod_{t=1}^{T} l\left(y_{1t}, y_{2t} \mid x_{1t}, x_{2t}, \beta, \delta_1, \delta_2\right)$ the likelihood of the data where, for each $t, (y_{1t}, y_{2t})$ is a Subgame Perfect Nash Equilibrium. Is the likelihood well-defined? Explain.

4. Assume we have market level data on T markets. For each market t, we only observe the number of firms n_t and some market level characteristic x_t. Moreover, assume that firms are homogeneous and play a sequential entry game in each market with the same entry profits as before:
$$\pi_t\left(n_t\right) = x_t\beta - \phi - \delta \log\left(n_t\right) + \epsilon_t.$$

   (a). What are the conditions under which n_t is a Nash Equilibrium?

   (b). Assuming that $\epsilon_t \overset{i.i.d}{\sim} N(0, 1)$, use the data in ps2_ex3.csv to estimate the parameters (β, ϕ, δ) using MLE.