COMPUTER VISION, GRAPHICS, AND IMAGE PROCESSING 28, 58–71 (1984)

Minimal Rectangular Partitions of Digitized Blobs

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Received March 25, 1983; accepted February 15, 1984

An algorithm is presented for partitioning a finite region of the digital plane into a minimum number of rectangular regions. It is demonstrated that the partition problem is equivalent to finding the maximum number of independent vertices in a bipartite graph. The graph's matching properties are used to develop an algorithm that solves the independent vertex problem. The solution of this graph-theoretical problem leads to a solution of the partition problem.

1. INTRODUCTION

Conventional image processing is often more efficient when the image is rectangular. In certain applications the convolving of an image with a point spread function or "mask" can be made particularly efficient by specifying the nonnegative values of the point spread function over a rectangular domain and specifying that function to be zero outside that domain. Consequently, when an image or point spread function is defined over a nonrectangular region it is often advantageous to partition that region into rectangular subregions, preferably a minimal number of subregions. The algorithms presented in this paper may also find applications in computer memory organization [1] and in integrated circuit mask making [2].

Suppose, for example, that one wishes to implement a convolution with a two-valued point spread function $h(\mathbf{x})$ efficiently. If $h(\mathbf{x}) = 1$ over a rectangular region, and zero everywhere else, then one can implement the convolution recursively [3]. Let B denote the region over which $h(\mathbf{x}) = 1$. We refer to B as a digital blob. If B is not rectangular, then one may implement the filtering recursively by partitioning B into a set of rectangular subregions, and storing each subregion as an array in the computer memory [4].

In this paper we derive an algorithm that partitions any finite four-connected subset of an infinite rectangular array (or "mosaic") R into a minimum number of rectangular subregions whose edges are parallel to the coordinate axes of R. That is, we show how to find the smallest order rectangular partition of a blob on R.

We first formulate and define the problem in geometrical terms. Theorem 1 relates the boundary properties of the digitized blob to the order of its minimum partition. The partitioning problem is then transformed to a problem in the application of graph theory. We show how to use the properties of bipartite graphs to find a minimum order rectangular partition. The solution is stated as an algorithm whose validity is established by the theory sketched in the paper. We conclude the paper with an illustrative example.

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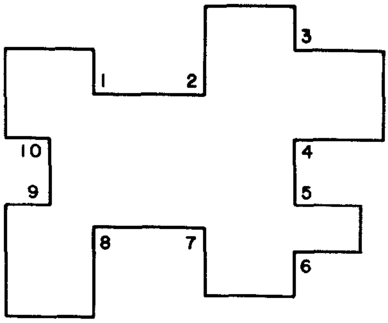

FIG. 1. A digitized blob.

2. DEFINITIONS FROM GEOMETRY

Let R denote a rectangular mosaic, where Λ denotes the conjunction of two propositions.

DEFINITION 1. A blob on R is a four-connected finite region in R. (See Fig. 1.)

DEFINITION 2. Two concave vertices $V_1 = (x_1, y_1)$ and $V_2 = (x_2, y_2)$ on the boundary of a blob on R are *cohorizontal* if $(x_1 = x_2)\Lambda$(The line V_1V_2 is a chord of B.)

DEFINITION 3. Two concave vertices V_1 and V_2 are *covertical* if $(y_1 = y_2)\Lambda$(The line V_1V_2 is a chord of B.)

DEFINITION 4. Two concave vertices V_1 and V_2 are *cogrid* if they are either cohorizontal or covertical.

DEFINITION 5. A *rectangular partition* of a blob on R, B, is a partition $\{P_i\}_{i=1}^{M}$ such that $(\cup_i P_i = B)\Lambda(P_i \cap P_j = \phi i \neq j)\Lambda(P_i$ is a rectangle for all i.)

Note. M is defined as the *order* of the partition $\{P_i\}$. (See Fig. 2.)

We next state two lemmas. Lemma 1 is concerned with the special case of a blob whose boundary contains no cogrid concave vertices. Lemma 2 counts the number of subregions introduced by chords drawn between points on the blob's boundary. Both lemmas are used in the proof of Theorem 1. We assume in the remainder of this paper that the blobs are simply connected [5].

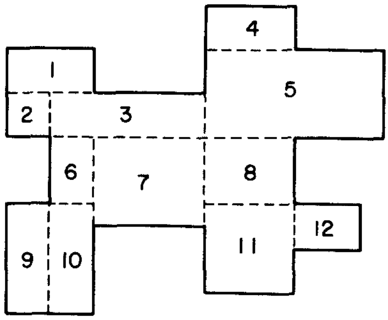

FIG. 2. A nonminimal rectangular partition of the digitized blob of Fig. 1.

3. GEOMETRICAL PROPERTIES OF DIGITAL BLOBS

LEMMA 1. *For a blob on R whose boundary contains N noncogrid concave vertices and no cogrid concave vertices, there exists a minimum order rectangular partition of order $N + 1$.*

Proof

I. Minimality. For each concave vertex, I, we select and extend to the blob's interior one of its edges. The extension will terminate at:

(1) The digital blob's boundary, or

(2) The extension of some other concave vertex, J, into the blob's interior.

Such a procedure will partition the blob's interior into $N + 1$ regions. This is clear since each concave vertex is located in a single subregion and its extension divides the subregion into two new subregions. Then by induction we obtain $N + 1$ subregions.

Each concave vertex must have at least one extension to ensure a rectangular partition. If not, some subregion contains an interior angle of 270° and is clearly not a rectangular region. Since none of these extensions can be cogrid, the number of subregions formed by the procedure is $N + 1$.

II. Rectangularity. If we show the subregions are all rectangles, the proof is complete.

(1) Each of the concave vertices is divided into an angle of 90° and an angle of 180°.

(2) The intersection of a vertex I extension must intersect a vertex J extension at 90°. This is true since all extensions are either horizontal or a vertical. An intersection angle of 180° contradicts the noncogrid assumption. This leaves the only other possibility of 90°.

(3) The intersection of an extension of a vertex I extension must intersect the blob's boundary at 90 or 180°. The case 180° implies we have cogrid concave vertices: a contradiction. (A convex vertex can only be approached from the exterior of the blob.)

This implies all interior subregion angles are 90°. Then the $N + 1$ subregions are all rectangles. □

Let C' denote a set of nonintersecting chords connecting points on the boundary of a blob B. Let $|C'| = L'$ and c_i' denote an element of C', $i = 1, 2, \ldots, L'$.

Let c denote a chord connecting two points on the boundary of B such that c and C' share no boundary points. Let x denote the number of intersections of c with C'. We then state

LEMMA 2. *The set C' and the chord c partition B into $L' + x + 2$ regions.*

Proof

The nonintersecting chords c_i partition B into $L' + 1$ regions (shown easily by induction). Let P_i, $i = 1, 2, \ldots, L' + 1$ denote these subregions. The chord c must divide $x + 1$ of the regions P_i into two subregions. Then the increase in number of subregions introduced by the chord c is just $x + 1$. Hence the total number of regions, s, in the partition defined by C' and c is just

$$s = L' + 1 + x + 1 = L' + x + 2. \square$$

Note. If c and C' share a single boundary point then

$$s = L' + x + 1.$$

If c and C' share two boundary points then

$$s = L' + 1 + x - 1 = L' + x.$$

Using Lemmas 1 and 2 we next prove the main theorem in the geometrical formulation of the partition problem.

4. MINIMUM ORDER RECTANGULAR PARTITION

THEOREM 1. *A blob B on a rectangular mosaic R has a minimum order rectangular partition of order $N - L + 1$ where,*

N = *Total number of concave vertices on the boundary of B.*

L = *Maximum number of nonintersecting chords that can be drawn between cogrid concave vertices.*

Proof

Let $\hat{P}$ denote the order of a minimum order rectangular partition of B.

Part I. $\hat{P} \le N - L + 1$. The L nonintersecting chords partition B into $(L + 1)$ subregions, $\{b_i\}$. Each b_i contains c_i, $i = 1, 2, \ldots, L + 1$ noncogrid concave vertices such that

$$\sum_{i=1}^{L+1} c_i = N - 2L.$$

By Lemma 1, each subregion b_i has a minimum order rectangular partition of order $(c_i + 1)$. Consequently, B has a rectangular covering of order P, where

$$P = \sum_{i=1}^{L+1} (c_i + 1) = \sum_{i=1}^{L+1} c_i + \sum_{i=1}^{L+1} 1.$$

Hence

$$P = N - 2L + L + 1 = N - L + 1.$$

Therefore

$$\hat{P} \le N - L + 1.$$

Part II. Minimality. Every concave vertex on the boundary of B must have at least one extension as described in the proof of Lemma 1.

Let φ denote an arbitrary rectangular partition of B. Let C denote the set of chords joining cogrid vertices in φ.

Let $C' \subseteq C$ be a subset of C such that all chords, $c_i' \in C'$, are nonintersecting. Let L' denote the cardinality of C'.

Let $C'' = \{c'' | c'' \in C - C'$ and c'' intersects $C'\}$. Let x_i $i = 1, 2, \ldots, L''$ be the number of intersections of chord $c_i'' \in C''$ with the set C'. Clearly $x_i \geq 1$.

C' partitions B into $L' + 1$ subregions which are not necessarily rectangular.

Let P denote the number of rectangles in φ.

Let M be the increase in the number of subregions introduced by the set C'' in the partition φ. $M = M_1 + M_2 + M_3$ where

$$M_1 = \sum_{i=1}^{L_1} (x_i + 1) \geq \sum_{i=1}^{L_1} 2 = 2L_1$$

when L_1 chords c_i'' share no boundary points with C'.

$$M_2 = \sum_{i=1}^{L_2} x_i \geq \sum_{i=1}^{L_2} 1 = L_2$$

when L_2 chords c_i'' share one boundary point with C'.

$$M_3 = \sum_{i=1}^{L_3} (x_i - 1) \geq \sum_{i=1}^{L_3} 1 = L_3$$

when L_3 chords c_i'' share two boundary points with C'. (1)

Let S denote the number of regions generated by the construction of the chords c_i' and c_i''. Then $S = M + L' + 1$.

The total number of noncogrid concave vertices, Q, in the S subregions is

$$Q = N - 2L' - 2L_1 - L_2.$$

By Lemma 1 and Part I of this theorem, the number of rectangles, P, in φ is

$$P = S + Q = N - L' + 1 + (M_1 - 2L_1) + (M_2 - L_2) + M_3. \tag{2}$$

By inequality (1) above $M_1 - 2L_1 \geq 0$ with equality for $M_1 = 2L_1 = 0$; $M_2 - L_2 \geq 0$ with equality for $M_2 = L_2 = 0$.

Let $\hat{P} = \min\{P\}$ taken over the variables L', L_1, L_2, L_3, M_1, M_2, and M_3.

With $L' \leq L$ we choose for L' its maximum value to minimize P. Further we let $L_i = 0$ for all i. Then substituting into Eq. (2) above, we obtain

$$\min_{\substack{L', L_1, L_2, L_3 \\ M_1, M_2, M_3}} \{P\} = \hat{P} \geq N - L + 1. \square$$

Theorem 1 shows only the existence of a minimal order rectangular partition. In the remainder of this paper we show how to find an L set of chords which constitutes a maximal set of nonintersecting chords as described in the theorem. Once this set of chords has been found, we use Lemma 1 to obtain the minimum order rectangular partition.

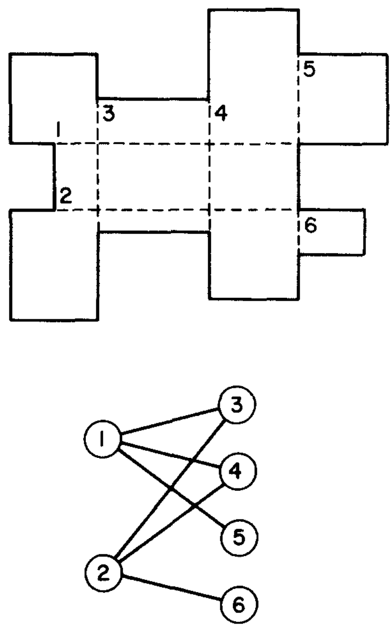

FIG. 3. Graph representation of problem L chord.

We first reduce the problem of finding L nonintersecting chords, problem L Chord, to a graph theory problem (Fig. 3).

Define a graph $G = (V, E)$ such that

(1) Each $v_i \in V$ corresponds to cogrid chord, say i, of B.

(2) Each edge $v_i v_j \in E$ corresponds to the intersection of i and j in B.

We proceed by defining some necessary terms. In addition we state, without proof, two theorems from graph theory which are attributed to Gallai and König.

5. DEFINITIONS FROM GRAPH THEORY

Let $G = (V, E)$ be a graph. See [6, 7].

DEFINITION 6. A set of vertices (edges) which covers all the edges (vertices) of G is called a *vertex cover* (edge cover) for G.

DEFINITION 7. The smallest number of vertices (edges) in any vertex (edge) cover for G is called a *vertex* (*edge*) *covering number* and denoted by $\alpha_0(G)(\alpha_1(G))$.

DEFINITION 8. A set of vertices (edges) in G is called *independent* if no two of its members are adjacent.

DEFINITION 9. The largest number of vertices (edges) in an independent set is called the *vertex* (*edge*) independence number, $\beta_0(\beta_1)$.

DEFINITION 10. A bipartite graph G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2.

THEOREM 2. (Gallai): *For any nontrivial connected graph G, $p = \alpha_0 + \beta_0 = \alpha_1 + \beta_1$ where $p = |V|$.*

DEFINITION 11. A set of β_1 independent edges in G is called a *maximum matching* of G.

THEOREM 3. (König): *If G is bipartite, then the number of edges in a maximum matching equals the vertex covering number, that is* $\beta_1 = \alpha_0$.

6. THE GRAPH REDUCTION OF PROBLEM L CHORD

In graph-theoretic terms, we are able to restate problem L Chord as follows:

Graph Problem. For the graph $G = (V, E)$ defined above, find the largest subset of independent nodes of G, i.e., the largest subset of chords containing no intersections. We note that in the rectangular partition, all chords are either horizontal or vertical. Consequently we have the following lemma.

LEMMA 3. *The graph $G = (V, E)$ is a bipartite graph.*

7. A MATCHING PROPERTY OF BIPARTITE GRAPHS

Suppose we have a maximum matching on a bipartite graph $G = ((U, V), E)$. Let the matching contain k edges (a k matching).

We observe the matching partitions the vertex sets U and V into sets U', U'', V', and V'', respectively such that U' and V' contain only matched vertices and U'' and V'' contain independent vertices (Fig. 4). Let $u_i v_i$ designate the ith pair of vertices in the matching.

LEMMA 4. *There does not exist any path from U'' to V'' that contains an edge $u_i v_i$.*

Proof

Let a path from U'' to V'' be labeled

$$u_i' v_i, u_i v_j, u_k v_l u_m, \ldots, v_q u_r, v_i'$$

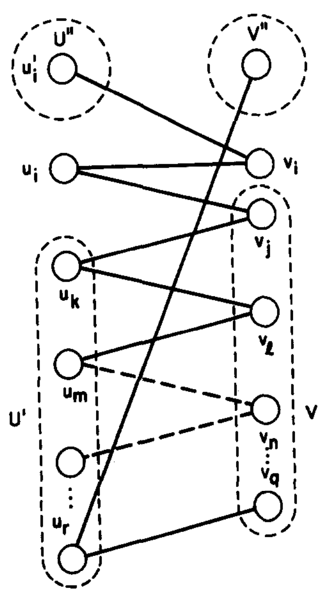

FIG. 4. A matching property of bipartite graphs.

(See Fig. 4.) Construct a new matching, M', which contains

$$u_i'v_i, u_iv_j, u_kv_l, u_mv_n, \ldots, u_rv_i'$$

This matching is of order $k + 1$. This contradicts the assumption that we had a maximum matching of order k. □

We now state Algorithm 1, which describes a procedure for finding the maximum independent set of vertices of a bipartite graph.

8. MAXIMUM INDEPENDENT SET FOR BIPARTITE GRAPHS

Algorithm 1

Find the maximum independent set of vertices for a bipartite graph.

Step 1—Find the maximum matching for the bipartite graph $G = (U, V, E)$.

Step 2—Color each pair of matched vertices (u_i, v_i) red. For each pair of red vertices do the following. (a) *If* there exists an edge from u_i to V'' in G, color u_i green and v_i blue or *if* there exists and edge from v_i to U'' in G, color v_i green and v_i blue. (b) Recursively color each remaining red vertex connected in G to a blue vertex green, and color its matched vertex blue.

Step 3—For all remaining pairs of red colored vertices u_jv_j (color u_j blue and v_j green if v_j is connected to a green vertex). Go to 2b.

Step 4—For all remaining pairs of red vertices color u_j green and v_j blue.

Step 5—Color all vertices $u \in U''$ and all $v \in V''$ blue. □

In Theorem 4 we prove that the algorithm finds a minimum vertex covering set for a bipartite graph. The extension to a maximum independent set of vertices is shown by Corollary 1. In Example 1 we are given a bipartite graph and we demonstrate the utilization of Algorithm 1 through the series of diagrams shown in Fig. 5. The set $\{E\}$ is shown only in Step 1.

THEOREM 4. *The set C containing only green vertices is a minimum vertex covering set of G.*

Proof

I. C is a vertex cover.

(a) All edges from U' to V'' and from V' to U'' are covered by Step 2a.

(b) All edges of G in the k matching are covered by the selection of either u_i or v_i for all i.

(c) The edges u_iv_j when u_j is green are obviously covered.

(d) The edges u_iv_j when u_i is blue are covered by Step 2b (v_j is colored green). Note: By Lemma 4 edges from v_j to U'' cannot exist.

(e) All independent edges are covered by Step 4.

II. C is a *minimum* vertex cover. The number of vertices colored green is just k, one for each vertex pair in the matching. Then, by Theorem 3 the vertex cover is minimum since $k = \alpha_0$. □

COROLLARY 1. *The set of all blue vertices, $\overline{C}$, is a maximum independent set.*

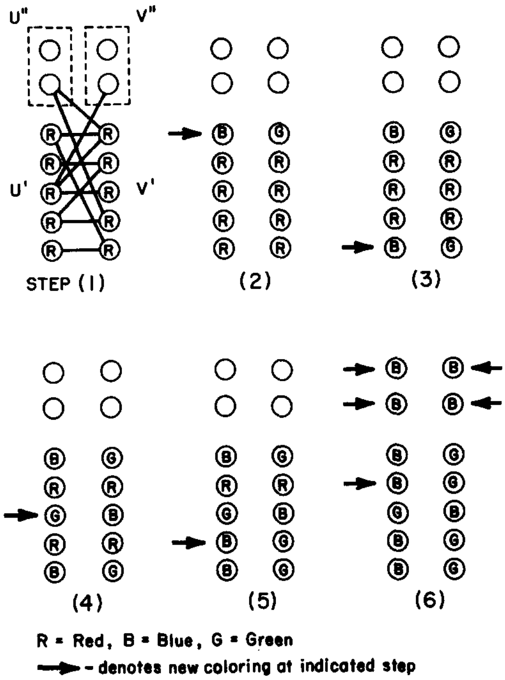

FIG. 5. Example 1.

Proof

$\bar{C}$ is the complement of C. Then

$$|\bar{C}| = p - \alpha_0 = \beta_0.$$

The graph $G' = (\bar{C} = V - C, \bar{E})$ contains no edges. If it does, C is not a vertex cover. □

Example 1. In step (1) of Fig. 5 we find a bipartite graph. The matched pairs of a maximal matching are indicated by the letter R (red). There are five such pairs. Steps (2) through (6) indicate the operation of Algorithm 1 in finding a minimum vertex cover for the bipartite graph. The vertex cover is indicated in step (6) by the vertices labeled G (green). The vertices labeled B (blue) in step (6) are a maximum independent set of vertices.

Example 2. We apply Algorithm 1 to the geometry problem of Fig. 3 to obtain the results shown in Fig. 6. Each blue vertex in set $\bar{C}$ corresponds to a chord in our geometrical formulation. Let the set of blue vertices, $\bar{C}$, have cardinality L, i.e., $|\bar{C}| = L$. If we construct the corresponding chords for our blob, B, we partition B into $L + 1$ not necessarily rectangular subregions (Fig. 7). Each subregion contains only noncogrid concave vertices on its boundary. Through the construction of the extensions (horizontal or vertical) for each noncogrid concave vertex we obtain a rectangular partition of B of order $N - L + 1$. (See Fig. 8.)

We have shown that a minimum order rectangular partition exists for every simply four-connected blob, B, digitized on a rectangular mosaic, R. We find such a

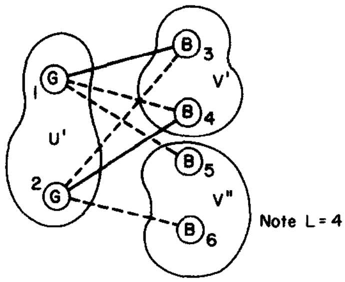

FIG. 6. Maximal independent set of vertices (B).

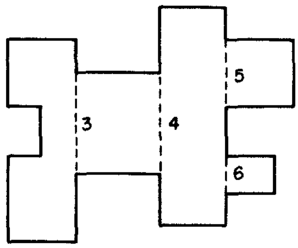

FIG. 7. Construction of chords.

partition by reducing the geometry problem to a graph problem which we solve by an extension of the matching properties of bipartite graphs.

9. NONSIMPLE BLOBS

We wish to extend our results to the case where B is a four-connected region in R. We let D denote the number of holes [5] in B. Let β denote the set of points representing the boundary of B. Let γ_i denote the set of points representing the boundary of a hole d_i in B. We note that according to the definition of β, $\gamma \in \beta$. We term $\sigma = \cup_{i=1}^{D}\gamma_i$ the inner boundary of B and refer to its complement in β as the outer boundary of B. The four-connected property ensures that each boundary γ_i contains at least one segment α_i which is parallel to a segment of $\{\beta - \gamma_i\}$ and that

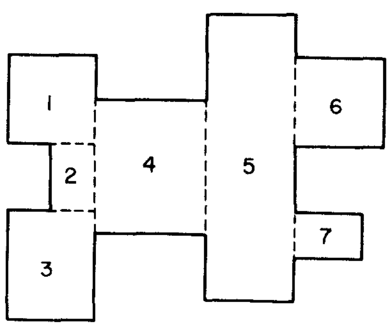

FIG. 8. A minimum order rectangular partition.

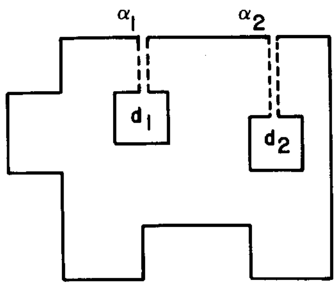

FIG. 9. A nonsimply connected blob.

at least one γ_i has a segment which is parallel to $\{\beta - \sigma\}$. The cardinality of a minimum order rectangular partition of B is given by the following theorem.

THEOREM 5. *The smallest number of rectangles in a rectangular partition of a nonsimply four-connected blob B is given by*

$$N - L + 1 - D$$

where

N is the total number of concave vertices contained in β

and

L is the largest number of nonintersecting chords that can be drawn between cogrid concave vertices in β

and

D is the number of holes in B.

Proof

Let $\hat{P}$ denote the order of a minimum rectangular partition of B.

Part I. $\hat{P} \leq N - L + 1 - D$. Let c_i denote a horizontal or vertical cut drawn orthogonal to and at interior points of α_i and $\{\beta - \alpha_i\}$ as shown in Fig. 9. Assume we have exactly one such cut for each hole $d_.$ If we further assume that at least one cut c_i is drawn between a_i and $\{\beta - \sigma\}$, the resulting blob B' is a simply four-connected blob. The cuts introduce no concave vertices in B' that do not already exist in B. Let φ denote a minimum order rectangular partition of B'. Assume φ has cardinality

$$P' = N - L' + 1.$$

Case 1. $L' = L$. Each cut c_i generates two rectangles in the partition of B'. The removal of each c reduces by one the number of rectangles in the partition. Since there are D cuts,

$$\hat{P} \leq N - L + 1 - D.$$

Case 2. $L' < L$. Assume we have a maximum set of nonintersecting chords for the blob B, of cardinality L. For each i, that c_i cuts a chord from this set, the simple blob B' will have one less pair of cogrid concave vertices; i.e., for F cut chords, $L' = L - F$. The removal of each c_i which cuts a chord reduces the number of rectangles in φ by two. The removal of each remaining cut reduces the number of rectangles in φ by one. We again assume

$$P' = N - (L - F) + 1.$$

But

$$\begin{aligned} P &= P' - 2F - (D - F) \\ &= P' - F - D \\ &= N - L + F - F - D + 1. \end{aligned}$$

Hence,

$$\hat{P} \leq N - L - D + 1.$$

Part II. Minimality. Suppose the partition of B in Part I is not minimal. Then a minimum partition of B will have Q rectangles where $Q < N - L + 1 - D$. This implies $Q + D < N - L + 1$. For Case 1, $Q + D$ is the number of rectangles in φ. This contradicts $P' = N - L + 1$. Hence for Case 1,

$$\hat{P} = N - L + 1 - D.$$

For case 2, $Q + D + F < N - L' + 1$. But for Case 2, $Q + D + F$ is the number of rectangles in φ. This contradicts $P' = N - L' + 1$. □

We see that Algorithm 1 can still be used to determine L and is unaffected by the existence of holes in B. The minimum rectangular partition can be found by exactly the same procedure described in Example 2.

10. THE COMPLEXITY OF ALGORITHM 1

The lowest complexity algorithm for finding the maximum matching of a bipartite graph $G(U, V, E)$ has the complexity of $O(|U + V|^{1/2}|E|)$. Given N concave vertices on the boundary of a blob B, the maximum number of possible cogrid chords is $N/2$. Then in our equivalent graph problem $|V| \leq N/2$. Let $|V| = N/2$ and let W denote the number of horizontal chords. Then $|E|$ can be as large as

$$(N/2 - W)(W) = WN/2 - W^2.$$

Hence, $\max|E|$ can be found by differentiation with respect to W to obtain $W = N/4$. Then $\max |E| = N^2/16$.

This implies Step 1 of Algorithm 1 is $O(N^{2.5})$. The remainder of the algorithm has complexity $O(k)$, where k is the cardinality of the matching found in Step 1. Since $k < N$, the complexity of Algorithm 1 is just $O(N^{2.5})$.

11. CONVEX PARTITIONS OF NONCONVEX POLYGONS

The construction of each chord in the rectangular partition algorithm increases the number of subregions by one and decreases the number of concave vertices by two. Every concave vertex on the boundary of B must have one of its edges extended.

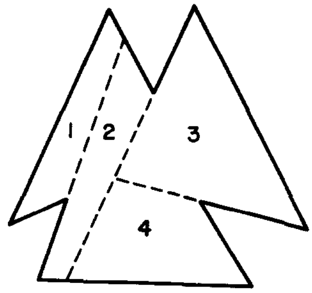

FIG. 10. Convex nonminimal polygon partition of a nonconvex polygon.

The cogrid chordal constructions extend the edges of the concave vertices while minimizing the number of generated subregions.

The problem of decomposing a nonconvex polygon into the union of a minimal number of convex polygons is a generalization of the rectangular partition problem. An algorithm that solves the polygonal decomposition problem is given in [8]. It is interesting to note that this problem was first believed to be NP complete (see [9] for description of NP problems). The algorithm presented in [8] has a complexity of $O(N^6)$ where N is the number of concave vertices on the boundary of the nonconvex polygon.

In Fig. 10 we find a nonconvex blob B. A simple method for partitioning the blob into a nonminimal set of convex polygons is to extend either edge of each concave vertex until it intersects the blob's boundary or the extension of some other concave vertex. This is demonstrated in Fig. 10. It is clear that if N denotes the number of concave vertices on the boundary of B then an upper bound on the cardinality of a convex partition is just $N + 1$.

We examine the possibility of constructing nonintersecting chords between concave vertices in order to divide each concave vertex into two convex angles. Such a construction is a generalization of the cogrid chordal construction used in the solution to the rectangular partition problem. A simple counterexample shows that the procedure suggested by the above construction does not lead to a minimum cardinality solution.

In Fig. 11 we show the three possible chords that can be drawn between the concave vertices. Construction of any of the chords does not eliminate a concave vertex; hence, we are falsely led to the solution shown by Fig. 10. In Fig. 12,

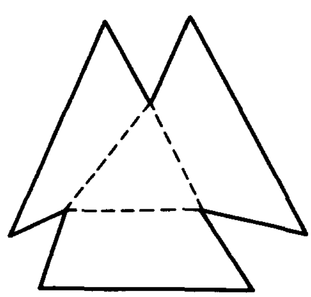

FIG. 11. Three invalid chords.

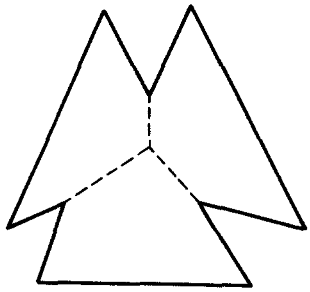

FIG. 12. A minimal partition.

however, we find a minimal solution demonstrating that the solution of Fig. 10 is not minimal. Unlike the more restricted rectangular partition problem, the nonintersecting chordal constructions do not lead to minimal partitions in the polygonal decomposition problem. For further discussions of polygonal decomposition see [8].

12. SUMMARY

We have shown that a minimum order rectangular partition exists for every blob B digitized on a rectangular mosaic R. We find such a partition by reducing the geometry problem to a graph problem which we solve by an extension of the matching properties of bipartite graphs.

The algorithm time complexity is shown to be $O(N^{2.5})$ where N is the number of concave vertices on the boundary of B. The problem can also be solved using a more general convex polygonal partition algorithm with a complexity of $O(N^6)$. The special properties of rectangular partitions yield a substantial improvement in complexity.

ACKNOWLEDGMENTS

We thank Dov Harel for his many helpful suggestions. This research was supported by the National Institute of General Medical Sciences of the U. S. Public Health Service under Grant GM-17632 and National Research Service Award 1F326M08103-01.

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