The Ky Fan Norms and Beyond: Dual Norms and Combinations for Matrix Optimization

Introduction

LLM training requires more scalable and efficient optimizers. The recent Muon [2] sometimes doubles AdamW's efficiency due to Muon's unique design :

- **Matrix-Aware Design:** Muon is constructed specifically for weight matrices rather than for generic vectors.
- **Theory-Based Update Rule:** Muon's update is not a mere heuristic. It is derived as the solution to a linear minimization problem constrained by the operator (spectral) norm, giving it a solid mathematical foundation.

Muon's success has inspired variants like Dion [1] algorithm and frameworks of Scion and Gluon [4]. We build on this work by asking a fundamental question:

> Why should the choice of a matrix norm in the Muon's update derivation be restricted to the spectral norm? How might other matrix norms influence the algorithm's performance and computational cost?

To investigate this, we first analyze the link between Muon-like algorithms and their defining norms. We then propose **F-Fanions**, a novel family of optimizers derived from exotic, mixed norms. Finally, we present an empirical comparison between Muon and F-Fanions.

How Norms Shape the Update

Many optimizers are defined by a norm-constrained Linear Minimization Oracle (LMO) [3], which computes the update $\Delta\mathbf{X}^t$ based on the gradient (possibly stochastic / with momentum) $\mathbf{G}^t$:

$$\Delta X^t = \mathbf{X}^{t+1} - \mathbf{X}^t \in \eta \arg\min_{\|\mathbf{X}\|\leq 1} \langle \mathbf{G}^t, \mathbf{X} \rangle .$$

The choice of norm dictates the algorithm. For matrix methods, the update often uses the Singular Value Decomposition (SVD) of the gradient, $\mathbf{G}^t = \mathbf{U}\Sigma\mathbf{V}^\top$.

Case	Method	Norm	Update Formula
Vec.	Normalized SGD	ℓ_2	$-\eta g^t/\|g^t\|_2$
	SignSGD	ℓ_∞	$-\eta\,\mathbf{sign}(g^t)$
	Gauss-Southwell CD	ℓ_1	$-\eta \sum_{i\in\text{argmax}\|g_i^t\|}\mathbf{sign}(g_i^t)e_i$
Mat.	Normalized SGD	$\|\cdot\|_\text{F}$	$-\eta\mathbf{G}^t/\|\mathbf{G}^t\|_\text{F}$
	Muon	$\|\cdot\|_\text{op}$	$-\eta\mathbf{U}\mathbf{V}^\top$
	Neon	$\|\cdot\|_{nuc}$	$-\eta u_1 v_1^\top$
	Dion (without EF)	$\|\cdot\|^\dagger_{KF-k}$	$-\eta\mathbf{U}_k\mathbf{V}_k^\top$

Here $\|\cdot\|_{KF-k}$ denotes Ky Fan k norm, $\|\cdot\|^\dagger$ is the dual norm.

We introduce a generalized norm that unifies and interpolates between these methods:

$$\|\mathbf{X}\|_\text{gen} = (\alpha\|\mathbf{X}\|_{\text{KF}-k} + (1-\alpha)\|\mathbf{X}\|_\text{F})^\dagger .$$

This defines the **F-Fanions**, a new family of optimizers parameterized by $\alpha \in [0,1]$ and k, with the update rule:

$$\Delta\mathbf{X}^t = -\eta\alpha\mathbf{U}_k\mathbf{V}_k^\top - \eta(1-\alpha)\frac{\mathbf{G}^t}{\|\mathbf{G}^t\|_\text{F}} .$$

This framework recovers existing algorithms as special cases:

- **Normalized SGD:** $\alpha = 0$.
- **Dion** (without error feedback): $\alpha = 1, k < \min\{m,n\}$.
- **Muon:** $\alpha = 1, k = \min\{m,n\}$.

We also identify and analyze **F-Neon**, a computationally efficient member of this family where $k = 1$.

Efficiently Computing the Update

Computing the F-Fanion update is efficient at the extremes of the rank parameter k. We propose using the GPU-friendly **Thick-Restarted Lanczos (TRLan)** method in two distinct ways:

- **For small k (e.g., F-Neon):** TRLan is used directly to find the few leading singular vectors.
- **For large k (e.g., Muon):** TRLan is used to compute trailing singular values, they are substracted from a full-rank term, obtained through Newton-Shulz iterations.

The table below demonstrates the speed of the direct approach for finding k leading singular vectors, from a 5000×5000 matrix.

Method	k=1	k=10	k=100
TRLan (our choice)	0.18s	0.47s	1.96s
Randomized SVD (RSVD)	1.15s	19.4s	170.0s
Power Iterations	7.70s	—	—

TRLan's dramatic speed advantage in this direct computation makes low-rank F-Fanions, like F-Neon, highly efficient.

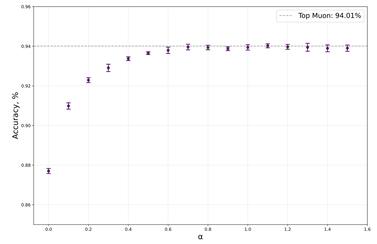
Modded NanoGPT Speedrun

We benchmarked our method against Muon and NSGD on the `modded nanogpt 2024` speedrun, which is pretraining of a real LLM. Below are the results after 1,750 steps. The objective was to achieve a loss lower than 3.28.

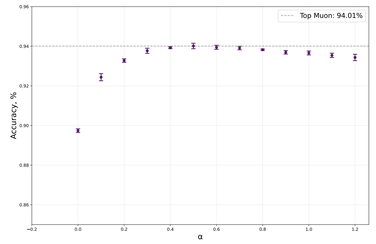
Method	LR	Momentum	Final Loss
Muon (Baseline)	0.05	0.95	3.279
F-Muon ($\alpha = 0.5$)	0.07	0.95	3.281
NSGD	0.07	0.96	3.4651
F-Muon ($\alpha = 0.5$)	0.07	0.96	3.2824

CIFAR-10 Airbench Experiments

We evaluate F-Muon against Muon by training a Convolutional Neural Network (CNN) on the CIFAR-10 Airbench. All results are averaged over multiple runs and reported after 8 epochs with batch size = 2,000.



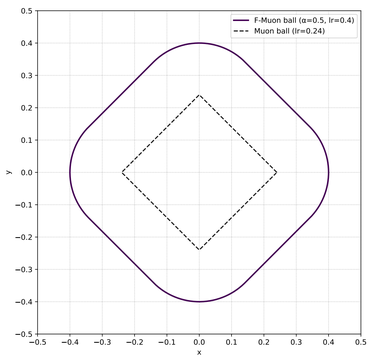
With parameters from tuned version of Muon



With parameters tuned for F-Muon, $\alpha = 0.5$

We observe that a properly tuned F-Muon performs on par with Muon and achieves the same 94.01% accuracy.

Moreover, the tuned F-Muon has a significantly larger trust region than Muon, and the learning rate is so high that Muon yields lower test accuracy when trained with it (see the figure on the right).



Conclusion & Outlook

- We introduced the **F-Fanions** that unify LMO-based optimizers like Muon and Normalized SGD via mixed norms.
- The choice of a norm is a flexible design parameter rather than a fixed rule. Our results show that other norms may be as efficient as the spectral norm.
- Our work poses a crucial question of how to theoretically guide norm selection. Future work could also explore adapting the norm (via α and k) dynamically during training.

Key References

[1] Ahn, K. & Xu, B. *Dion: A Communication-Efficient Optimizer for Large Models.* arXiv:2504.05295, 2025.

[2] Bernstein, J. *Deriving Muon.* 2025. URL: jeremybernste.in/writing/deriving-muon

[3] Bernstein, J. & Newhouse, L. *Old Optimizer, New Norm: An Anthology.* arXiv:2409.20325, 2024.

[4] Riabinin, A. et al. *Gluon: Making Muon & Scion Great Again!* arXiv:2505.13416, 2025.

Alexey Kravatskiy[1] kravtskii.aiu@phystech.edu

Ivan Kozyrev[1] kozyrev.in@phystech.edu

Nikolai Kozlov[1] kozlov.na@phystech.edu

Alexander Vinogradov[1] vinogradov.am@phystech.edu

Daniil Merkulov[1,2,3,4] daniil.merkulov@phystech.edu

Ivan Oseledets[5,2] i.oseledets@skoltech.ru

[1]MIPT [2]Skoltech [3]HSE [4]AI4Science [5]AIRI