Differentially private modification of SignSGD

Alexey Kravatskiy

Moscow Institute of Physics and Technology

Course: My first scientific paper
(Strijov's practice) & Innovative Practicum /Group 205

Expert: A. N. Beznosikov

Consultant: S. A. Chezhegov

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Distributed, Private, and Noise-resistant

Goal

A communication-efficient and private algorithm for distributed optimization converging under heavy-tailed noise (noise with unbounded variance).

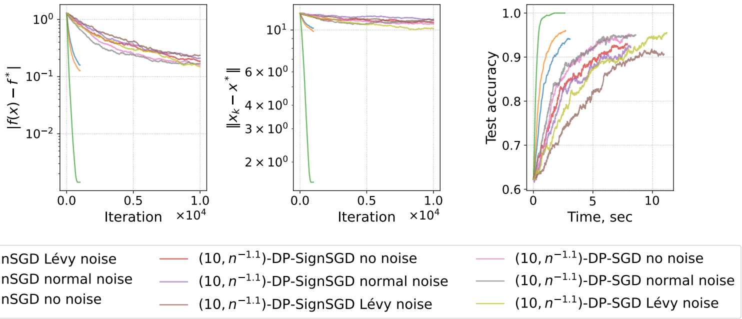
Problem

The only proposed private sign-based algorithm DP-SignSGD either is not private or does not converge.

Solution

Rényi differential privacy and Bernoulli subsampling afford lower noise to ensure convergence.

Differential privacy of DP-SignSGD



DP-SignSGD with our DP-SIGN compressor is (ε, δ)-private and converges under heavy-tailed noise. It behaves very much like DP-SGD with the same privacy mechanism.

Literature

- Jin, Richeng et al. (Feb. 25, 2020). "Stochastic-Sign SGD for Federated Learning with Theoretical Guarantees". In: *Part of this work is published in IEEE Transactions on Neural Networks and Learning Systems, 2024* 36.2, pp. 3834–3846. ISSN: 2162-2388. DOI: 10.1109/tnnls.2023.3345367. arXiv: 2002.10940 [cs.LG].
- Mironov, Ilya (Aug. 2017). "Rényi Differential Privacy". In: *2017 IEEE 30th Computer Security Foundations Symposium (CSF)*. IEEE, pp. 263–275. DOI: 10.1109/csf.2017.11. URL: http://dx.doi.org/10.1109/CSF.2017.11.
- Mironov, Ilya, Kunal Talwar, and Li Zhang (2019). *Rényi Differential Privacy of the Sampled Gaussian Mechanism*. arXiv: 1908.10530 [cs.LG]. URL: https://arxiv.org/abs/1908.10530.

Differential Privacy

Definition

Given a set of local datasets $\mathcal{D}$ provided with a notion of neighboring local datasets $\mathcal{N}_{\mathcal{D}} \subset \mathcal{D} \times \mathcal{D}$ that differ in only one data point. For a query function $f : \mathcal{D} \to \mathcal{X}$, a mechanism $\mathcal{M} : \mathcal{X} \to \mathcal{O}$ to release the answer of the query is defined to be (ϵ, δ)-locally differentially private if for any measurable subset $\mathcal{S} \subseteq \mathcal{O}$ and two neighboring local datasets $(D_1, D_2) \in \mathcal{N}_{\mathcal{D}}$,

$$P(\mathcal{M}(f(D_1)) \in \mathcal{S}) \leq e^{\epsilon} P(\mathcal{M}(f(D_2)) \in \mathcal{S}) + \delta.$$

A key quantity in characterizing local differential privacy for many mechanisms is the sensitivity of the query f in a given norm l_r, which is defined as

$$\Delta_r = \max_{(D_1, D_2) \in \mathcal{N}_{\mathcal{D}}} ||f(D_1) - f(D_2)||_r.$$

Incorrect version of DP-SIGN (Jin et al. 2020)

Definition

For any given gradient $\boldsymbol{g}_m^{(t)}$, the compressor $\mathrm{dp\text{-}sign}$ outputs $\mathrm{dp\text{-}sign}(\boldsymbol{g}_m^{(t)}, \epsilon, \delta)$. The i-th entry of $\mathrm{dp\text{-}sign}(\boldsymbol{g}_m^{(t)}, \epsilon, \delta)$ is given by

$$\mathrm{dp\text{-}sign}(\boldsymbol{g}_m^{(t)}, \epsilon, \delta)_i = \begin{cases} 1, & \text{with probability } \Phi\left(\frac{(\boldsymbol{g}_m^{(t)})_i}{\sigma}\right) \\ -1, & \text{with probability } 1 - \Phi\left(\frac{(\boldsymbol{g}_m^{(t)})_i}{\sigma}\right) \end{cases}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution; $\sigma = \frac{\Delta_2}{\epsilon}\sqrt{2\ln\left(\frac{1.25}{\delta}\right)}$, where ϵ and δ are the differential privacy parameters and Δ_2 is the sensitivity measure.

Theorem 6 from (Jin et al. 2020)

Theorem

Let $u_1, u_2, \cdots, u_M$ be M known and fixed real numbers. Further define random variables $\hat{u}_i = dp\text{-}sign(u_i, \epsilon, \delta), \forall 1 \le i \le M$. Then there always exist a constant σ_0 such that when $\sigma \ge \sigma_0$, $P(sign(\frac{1}{M}\sum_{m=1}^{M}\hat{u}_i) \ne sign(\frac{1}{M}\sum_{m=1}^{M}u_i)) < \left(1 - x^2\right)^{\frac{M}{2}}$, where $x = \frac{|\sum_{m=1}^{M}u_m|}{2\sigma M}$.

Fault From the proof, it follows that the constant σ_0 depends on the values of $\{u_i\}$, which precludes from constructing DP-SIGN compressor.

Fault The authors have not guaranteed the overall (ε, δ)-privacy.

Fault The authors have not proved the convergence of their DP-SignSGD.

Rényi divergence (Mironov 2017)

Definition (Rényi divergence)

Let P and Q be two distributions on $\mathcal{X}$ defined over the same probability space, and let p and q be their respective densities. The Rényi divergence of a finite order $\alpha \neq 1$ between P and Q is defined as

$$\mathrm{D}_\alpha(P \parallel Q) \triangleq \frac{1}{\alpha - 1} \ln \int_{\mathcal{X}} q(x) \left(\frac{p(x)}{q(x)} \right)^{\alpha} \mathrm{d}x.$$

Rényi divergence at orders $\alpha = 1, \infty$ are defined by continuity.

Rényi differential privacy (Mironov 2017)

Definition (Rényi differential privacy (RDP))

We say that a randomized mechanism $\mathcal{M}\colon \mathcal{S} \to \mathcal{R}$ satisfies (α, ε)-Rényi differential privacy (RDP) if for any two *adjacent* inputs $S, S' \in \mathcal{S}$ it holds that

$$\mathrm{D}_\alpha\big(\mathcal{M}(S) \,\|\, \mathcal{M}(S')\big) \leq \varepsilon.$$

Bernoulli sampling + Gaussian Mechanism (Mironov, Talwar, and Zhang 2019)

Definition (Sampled Gaussian Mechanism (SGM))

Let f be a function mapping subsets of $\mathcal{S}$ to $\mathbb{R}^d$. We define the Sampled Gaussian mechanism (SGM) parameterized with the sampling rate $0 < q \leq 1$ and the noise $\sigma > 0$ as

$$\mathrm{SG}_{q,\sigma}(S) \overset{\Delta}{=} f(\{x \colon x \in S \text{ is sampled with probability } q\}) + \mathcal{N}(0, \sigma^2 \mathbb{I}^d),$$

where each element of S is sampled independently at random with probability q without replacement, and $\mathcal{N}(0, \sigma^2 \mathbb{I}^d)$ is spherical d-dimensional Gaussian noise with per-coordinate variance σ^2.

Criterion of a private algorithm

Following the procedure from Mironov, Talwar, and Zhang 2019, we get:

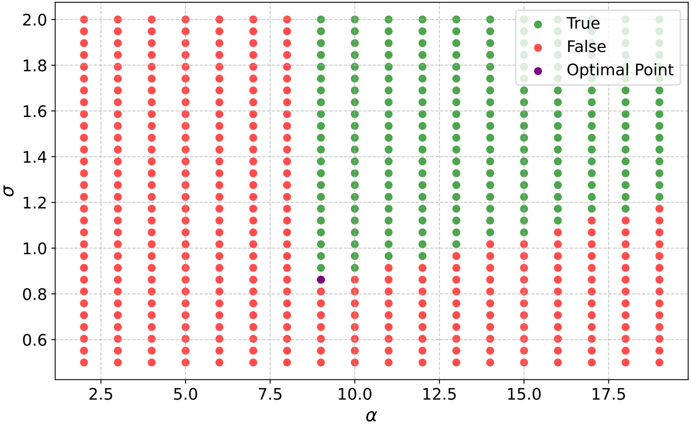
$$\varepsilon_R = \frac{1}{\alpha - 1} \log \left(\sum_{k=0}^{\alpha} \binom{\alpha}{k} (1-q)^{\alpha-k} q^k \exp\left(\frac{k^2 - k}{2\sigma^2} \right) \right)$$

While according to the advanced composition theorem and conversion from Rényi privacy to (ε, δ)-privacy, ε_R must satisfy:

$$\varepsilon_R \leq \varepsilon / T - \frac{\log 1/\delta}{T(\alpha - 1)}$$

.

Grid Search to find minimal σ



(α, σ) that guarantee (ϵ, δ)-dp of T subsamplings for $\epsilon = 1$, $\delta = 1/n^{1.1}$, $T = 1000$, $q = 1/n$, where $n = 649$ (10% of the Mushroom dataset).

Our version of DP-SIGN compressor

Input: coordinate w, loss function l, user database D, (ε, δ)-privacy requirement, number of iterations T, sampling rate q.
Prepare subsample S: add each element $(x, y) \in D$ with probability q.
Compute the gradient $\boldsymbol{g}$ of the subsample for $\frac{1}{|S|}\sum_{(x,y)\in S} l(w;(x,y))$. If S is empty, let $\boldsymbol{g} = 0$.
Grid search $\sigma(q, T, \varepsilon, \delta)$ to satisfy 2 expressions for ε_R stated earlier.
$sign_{noised} = sign(\boldsymbol{g}) + \mathcal{N}(0, (2\sqrt{d}\sigma)^2\mathbb{I}^d)$
Output: $sign(sign_{noised})$

UCI Mushroom Dataset

There are 6,449 training samples equally distributed over 10 workers. Test consists of 1,625 samples. Each sample has $d = 112$ features. $q = 1/n$.
We solve the binary classification problem with $\lambda = 10^{-3}$ regularization.

Optimization problem:

$$\min_{x \in \mathbb{R}^n} \left(\frac{1}{m} \sum_{i=1}^{m} \ln(1 + \exp(-b_i \langle a_i, x \rangle)) + \frac{\lambda}{2} \|x\|^2 \right)$$

Classification rule: $b(a) := \mathrm{sign}(\langle a, x \rangle)$

For each algorithm (SignSGD, DP-SGD, and DP-SignSGD), we set the learning rate $\eta = \frac{1}{\sqrt{100d}}$. The goal is $(10, 1/n^{1.1})$ privacy.

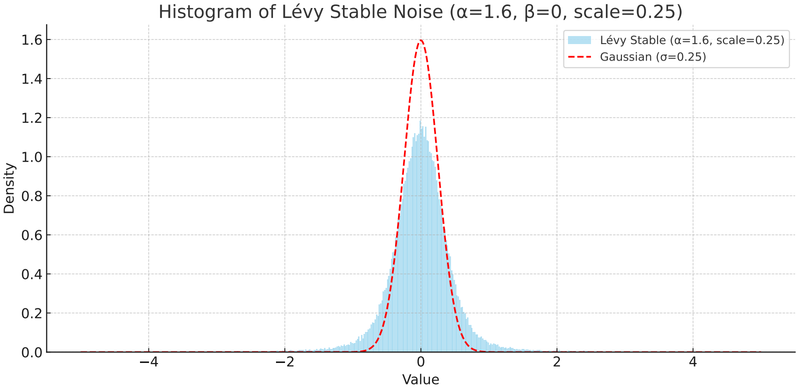
Heavy-tailed noise in gradient estimates

The unbiased estimate $\nabla f(x, \xi)$ has bounded κ-th moment $\kappa \in (1, 2]$ for each coordinate, i.e., $\forall x \in \mathbb{R}^d$:

- $\mathbb{E}_\xi[\nabla f(x, \xi)] = \nabla f(x)$,
- $\mathbb{E}_\xi[|\nabla f(x, \xi)_i - \nabla f(x)_i|^\kappa] \leq \sigma_i^\kappa, i \in \overline{1, d}$,

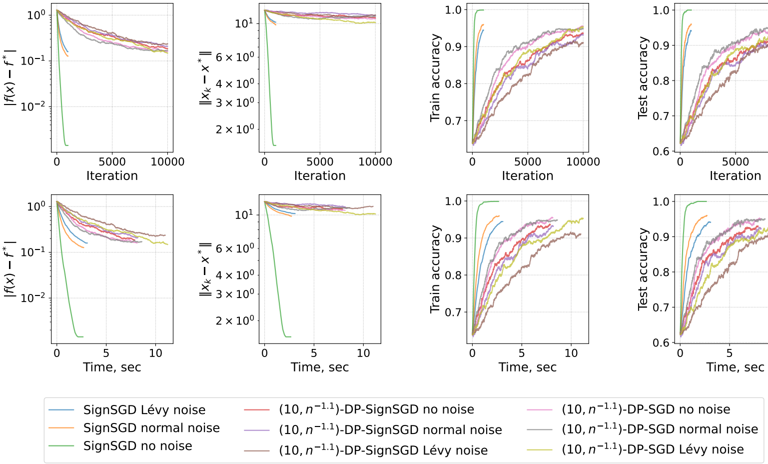
where $\vec{\sigma} = [\sigma_1, \ldots, \sigma_d]$ are non-negative constants. If $\kappa = 2$, then the noise is called a bounded variance.

Synthetic heavy-tailed noise



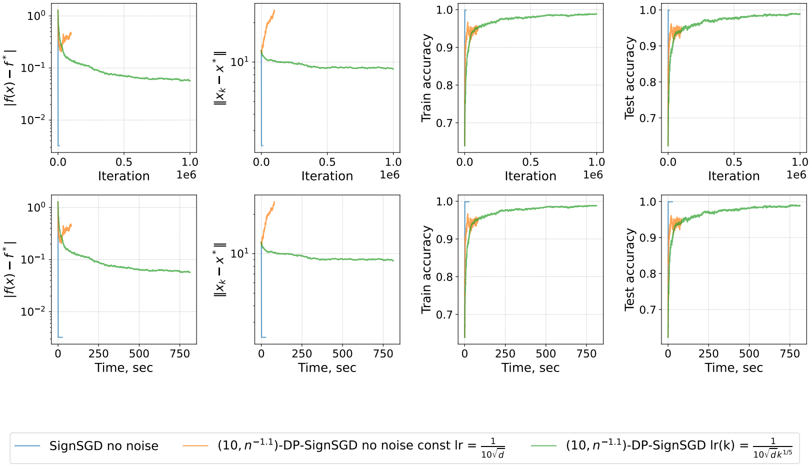
We add to the gradients coordinatewise noise with Lévy stable distribution with $\sigma_l = 1/4$, $\alpha_l = 1.6$, which corresponds to $\kappa = 1.5$, and $\beta_l = 0$ (this distribution is defined by its characteristic function $\varphi(t) = \exp\left(-0.25^{1.6}|t|^{1.6}\right)$).
We compare settings with no noise, $\mathcal{N}(0, 0.25^2\mathbb{I}^d)$ noise and this noise.

Computational experiment



DP-SignSGD converges very slowly, but it depends on a dataset. Lower q leads to much lower σ and better convergence.

Constant vs Dynamic learning rate



DP-SignSGD converges better, when the learning rate is dynamic. $T^{-1/5}$ instead of $T^{-1/2}$ factor is required, because σ depends on T.

Torch experiments

Effective simulation of federated learning via torch.multiprocessing (up to 10x speedup).

Kith and kin of DP-SignSGD

- Add Top-k compressor
- σs depend on the importance of the coordinate
- Several local steps & send $sign(\Delta W)$

Summer plans

- Prove convergence of DP-SignSGD
- Test CNN on CIFAR-10 (code is almost ready)
- Test other modifications of the algoirthm
- Try different compressor operators to tighten privacy accounting

DP-SignSGD with Bernoulli sampling $q = 1/n$

- $(\varepsilon, 1/n^{1.1})$ differentially-private
- empirically converges on logistic regression problem even with heavy-tailed noise
- trains MLP and CNN on MNIST to 70% accuracy
- empirically converges with the same type of convergence like DP-SGD with Bernoulli subsampling
- benefits from slowly decreasing learning rate