

# Differentially private modification of SignSGD

Alexey Kravatskiy

Moscow Institute of Physics and Technology

*Course:* My first scientific paper  
(Strijov's practice) & Innovative Practicum /Group 205

*Expert:* A. N. Beznosikov

*Consultant:* S. A. Chezhegov

2025

# Distributed, Private, and Noise-resistant

## Goal

A communication-efficient and private algorithm for distributed optimization converging under heavy-tailed noise (noise with unbounded variance).

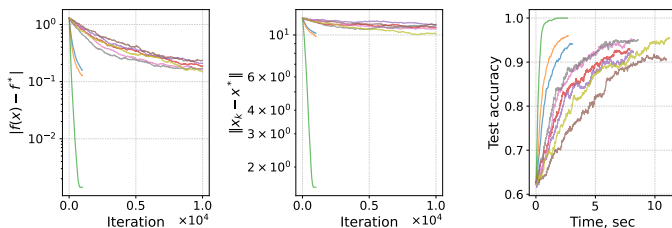
## Problem

The only proposed private sign-based algorithm DP-SignSGD either is not private or does not converge.

## Solution




A new privacy accountant for DP-SignSGD that affords lower noise to ensure convergence.

# Differential privacy of DP-SignSGD



DP-SignSGD with our DP-SIGN compressor is  $(\epsilon, \delta)$ -private and converges under heavy-tailed noise. It behaves very much like DP-SGD with the same privacy mechanism.

# Literature

-  Jin, Richeng et al. (Feb. 25, 2020). “Stochastic-Sign SGD for Federated Learning with Theoretical Guarantees”. In: *Part of this work is published in IEEE Transactions on Neural Networks and Learning Systems, 2024* 36.2, pp. 3834–3846. ISSN: 2162-2388. DOI: 10.1109/tnnls.2023.3345367. arXiv: 2002.10940 [cs.LG].
-  Mironov, Ilya (Aug. 2017). “Rényi Differential Privacy”. In: *2017 IEEE 30th Computer Security Foundations Symposium (CSF)*. IEEE, pp. 263–275. DOI: 10.1109/csf.2017.11. URL: <http://dx.doi.org/10.1109/CSF.2017.11>.
-  Mironov, Ilya, Kunal Talwar, and Li Zhang (2019). *Rényi Differential Privacy of the Sampled Gaussian Mechanism*. arXiv: 1908.10530 [cs.LG]. URL: <https://arxiv.org/abs/1908.10530>.

# Differential Privacy

## Definition

Given a set of local datasets  $\mathcal{D}$  provided with a notion of neighboring local datasets  $\mathcal{N}_{\mathcal{D}} \subset \mathcal{D} \times \mathcal{D}$  that differ in only one data point. For a query function  $f : \mathcal{D} \rightarrow \mathcal{X}$ , a mechanism  $\mathcal{M} : \mathcal{X} \rightarrow \mathcal{O}$  to release the answer of the query is defined to be  $(\epsilon, \delta)$ -locally differentially private if for any measurable subset  $\mathcal{S} \subseteq \mathcal{O}$  and two neighboring local datasets  $(D_1, D_2) \in \mathcal{N}_{\mathcal{D}}$ ,

$$P(\mathcal{M}(f(D_1)) \in \mathcal{S}) \leq e^{\epsilon} P(\mathcal{M}(f(D_2)) \in \mathcal{S}) + \delta.$$

A key quantity in characterizing local differential privacy for many mechanisms is the sensitivity of the query  $f$  in a given norm  $l_r$ , which is defined as

$$\Delta_r = \max_{(D_1, D_2) \in \mathcal{N}_{\mathcal{D}}} \|f(D_1) - f(D_2)\|_r.$$

## Incorrect version of DP-SIGN (Jin et al. 2020)

### Definition

For any given gradient  $\mathbf{g}_m^{(t)}$ , the compressor dp-sign outputs  $\text{dp-sign}(\mathbf{g}_m^{(t)}, \epsilon, \delta)$ . The  $i$ -th entry of  $\text{dp-sign}(\mathbf{g}_m^{(t)}, \epsilon, \delta)$  is given by

$$\text{dp-sign}(\mathbf{g}_m^{(t)}, \epsilon, \delta)_i = \begin{cases} 1, & \text{with probability } \Phi\left(\frac{(\mathbf{g}_m^{(t)})_i}{\sigma}\right) \\ -1, & \text{with probability } 1 - \Phi\left(\frac{(\mathbf{g}_m^{(t)})_i}{\sigma}\right) \end{cases}$$

where  $\Phi(\cdot)$  is the cumulative distribution function of the standard normal distribution;  $\sigma = \frac{\Delta_2}{\epsilon} \sqrt{2 \ln \left( \frac{1.25}{\delta} \right)}$ , where  $\epsilon$  and  $\delta$  are the differential privacy parameters and  $\Delta_2$  is the sensitivity measure.

## Theorem 6 from (Jin et al. 2020)

### Theorem

Let  $u_1, u_2, \dots, u_M$  be  $M$  known and fixed real numbers. Further define random variables  $\hat{u}_i = \text{dp-sign}(u_i, \epsilon, \delta), \forall 1 \leq i \leq M$ . Then there always exist a constant  $\sigma_0$  such that when  $\sigma \geq \sigma_0$ ,

$$P(\text{sign}(\frac{1}{M} \sum_{m=1}^M \hat{u}_i) \neq \text{sign}(\frac{1}{M} \sum_{m=1}^M u_i)) < (1 - x^2)^{\frac{M}{2}}, \text{ where}$$
$$x = \frac{|\sum_{m=1}^M u_m|}{2\sigma M}.$$

**Fault** From the proof, it follows that the constant  $\sigma_0$  depends on the values of  $\{u_i\}$ , which precludes from constructing DP-SIGN compressor.

**Fault** The authors have not guaranteed the overall  $(\epsilon, \delta)$ -privacy.

**Fault** The authors have not proved the convergence of their DP-SignSGD.

# Rényi divergence (Mironov 2017)

## Definition (Rényi divergence)

Let  $P$  and  $Q$  be two distributions on  $\mathcal{X}$  defined over the same probability space, and let  $p$  and  $q$  be their respective densities. The Rényi divergence of a finite order  $\alpha \neq 1$  between  $P$  and  $Q$  is defined as

$$D_{\alpha}(P \parallel Q) \triangleq \frac{1}{\alpha - 1} \ln \int_{\mathcal{X}} q(x) \left( \frac{p(x)}{q(x)} \right)^{\alpha} dx.$$

Rényi divergence at orders  $\alpha = 1, \infty$  are defined by continuity.



# Rényi differential privacy (Mironov 2017)

## Definition (Rényi differential privacy (RDP))

We say that a randomized mechanism  $\mathcal{M}: \mathcal{S} \rightarrow \mathcal{R}$  satisfies  $(\alpha, \varepsilon)$ -Rényi differential privacy (RDP) if for any two *adjacent* inputs  $S, S' \in \mathcal{S}$  it holds that

$$D_{\alpha}(\mathcal{M}(S) \parallel \mathcal{M}(S')) \leq \varepsilon.$$

# Bernoulli sampling + Gaussian Mechanism (Mironov, Talwar, and Zhang 2019)

## Definition (Sampled Gaussian Mechanism (SGM))

Let  $f$  be a function mapping subsets of  $\mathcal{S}$  to  $\mathbb{R}^d$ . We define the Sampled Gaussian mechanism (SGM) parameterized with the sampling rate  $0 < q \leq 1$  and the noise  $\sigma > 0$  as

$$\text{SG}_{q,\sigma}(S) \triangleq f(\{x: x \in S \text{ is sampled with probability } q\}) + \mathcal{N}(0, \sigma^2 \mathbb{I}^d),$$

where each element of  $S$  is sampled independently at random with probability  $q$  without replacement, and  $\mathcal{N}(0, \sigma^2 \mathbb{I}^d)$  is spherical  $d$ -dimensional Gaussian noise with per-coordinate variance  $\sigma^2$ .

## Criterion of a private algorithm

Following the procedure from Mironov, Talwar, and Zhang 2019, we get:

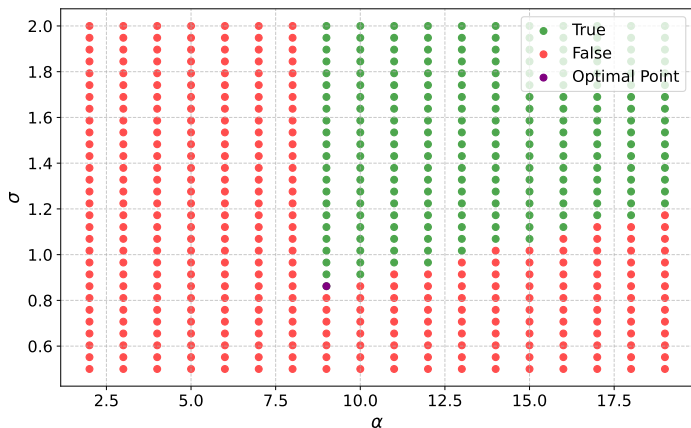
$$\varepsilon_R = \frac{1}{\alpha - 1} \log \left( \sum_{k=0}^{\alpha} \binom{\alpha}{k} (1 - q)^{\alpha - k} q^k \exp \left( \frac{k^2 - k}{2\sigma^2} \right) \right)$$

While according to the advanced composition theorem and conversion from Rényi privacy to  $(\varepsilon, \delta)$ -privacy,  $\varepsilon_R$  must satisfy:

$$\varepsilon_R \leq \varepsilon / T - \frac{\log 1/\delta}{T(\alpha - 1)}$$

.

## Grid Search to find minimal $\sigma$



$(\alpha, \sigma)$  that guarantee  $(\epsilon, \delta)$ -dp of  $T$  subsamplings for  $\epsilon = 1$ ,  $\delta = 1/n^{1.1}$ ,  $T = 1000$ ,  $q = 1/n$ , where  $n = 649$  (10% of the Mushroom dataset).

## Our version of DP-SIGN compressor

**Input:** coordinate  $w$ , loss function  $l$ , user database  $D$ ,  $(\varepsilon, \delta)$ -privacy requirement, number of iterations  $T$ , sampling rate  $q$ , clipping level  $C$ .

Prepare subsample  $S$ : add each element  $(x, y) \in D$  with probability  $q$ .

Compute the gradient  $\mathbf{g}$  of the subsample:

$\frac{1}{|S|} \sum_{(x,y) \in S} l(w; (x, y))$ . If  $S$  is empty, let  $\mathbf{g} = 0$ .

If  $\|\mathbf{g}\|_2 > C$ :  $\mathbf{g} = C \frac{\mathbf{g}}{\|\mathbf{g}\|_2}$ .

Grid search  $\sigma(q, T, \varepsilon, \delta)$  that satisfies 2 expressions for  $\varepsilon_R$  stated earlier.

$\mathbf{g}_{priv} = \mathbf{g} + \mathcal{N}(0, (C\sigma)^2 \mathbb{I}^d)$

**Output:**  $\text{sign}(\mathbf{g}_{priv})$

# UCI Mushroom Dataset

There are 6,449 training samples equally distributed over 10 workers. Test consists of 1,625 samples. Each sample has  $d = 112$  features.  $q = 1/n$ .

We solve the binary classification problem with  $\lambda = 10^{-3}$  regularization.

Optimization problem:

$$\min_{x \in \mathbb{R}^n} \left( \frac{1}{m} \sum_{i=1}^m \ln(1 + \exp(-b_i \langle a_i, x \rangle)) + \frac{\lambda}{2} \|x\|^2 \right)$$

Classification rule:  $b(a) := \text{sign}(\langle a, x \rangle)$

For each algorithm (SignSGD, DP-SGD, and DP-SignSGD), we set the learning rate  $\eta = \frac{1}{\sqrt{100d}}$ . The goal is  $(10, 1/n^{1.1})$  privacy.

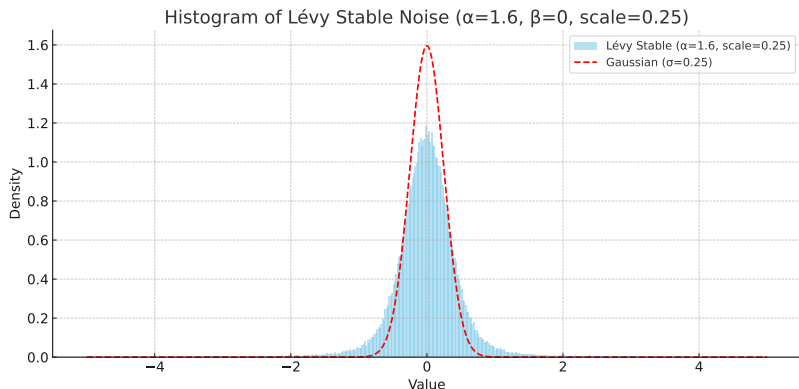
# Heavy-tailed noise in gradient estimates

The unbiased estimate  $\nabla f(x, \xi)$  has bounded  $\kappa$ -th moment  $\kappa \in (1, 2]$  for each coordinate, i.e.,  $\forall x \in \mathbb{R}^d$ :

- ▶  $\mathbb{E}_{\xi}[\nabla f(x, \xi)] = \nabla f(x),$
- ▶  $\mathbb{E}_{\xi}[|\nabla f(x, \xi)_i - \nabla f(x)_i|^{\kappa}] \leq \sigma_i^{\kappa}, i \in \overline{1, d},$

where  $\vec{\sigma} = [\sigma_1, \dots, \sigma_d]$  are non-negative constants. If  $\kappa = 2$ , then the noise is called a bounded variance.

# Synthetic heavy-tailed noise

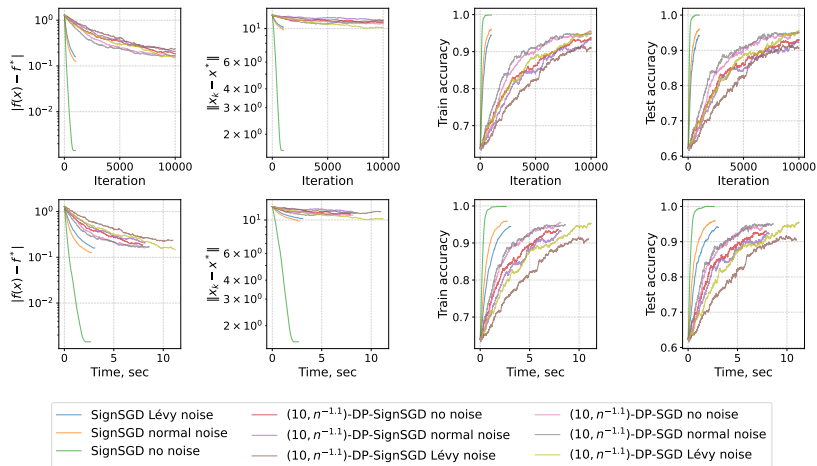


We add to the gradients coordinatewise noise with Lévy stable distribution with  $\sigma_l = 1/4$ ,  $\alpha_l = 1.6$ , which corresponds to  $\kappa = 1.5$ , and  $\beta_l = 0$  (this distribution is defined by its characteristic function  $\varphi(t) = \exp(-0.25^{1.6}|t|^{1.6})$ ).

We compare settings with no noise,  $\mathcal{N}(0, 0.25^2 \mathbb{I}^d)$  noise and this noise.

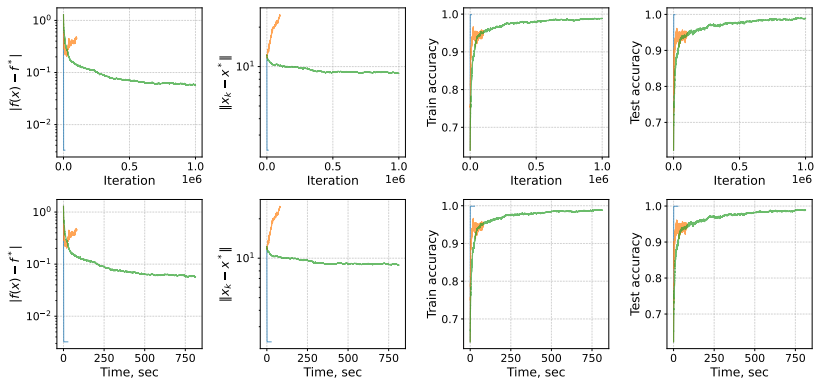


# Computational experiment



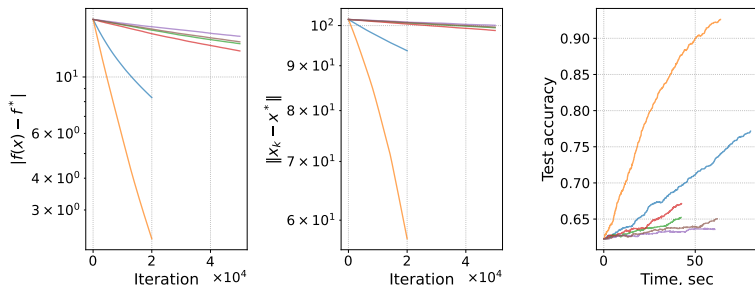
DP-SignSGD converges very slowly, but it depends on a dataset.  
Lower  $q$  leads to much lower  $\sigma$  and better convergence.

# Constant vs Dynamic learning rate



DP-SignSGD converges better, when the learning rate is dynamic.  $T^{-1/5}$  instead of  $T^{-1/2}$  factor is required, because  $\sigma$  depends on  $T$ .

# Our DP-SGD and DP-SignSGD are not feasible yet



The current approach to construct DP-SGD might be not optimal: currently, our DP-SGD fails to converge on MLP problem.

## DP-SignSGD with Bernoulli sampling $q = 1/n$

- ▶  $(10, 1/n^{1.1})$  differentially-private
- ▶ empirically converges on logistic regression problem even with heavy-tailed noise
- ▶ empirically converges with the same type of convergence like DP-SGD with Bernoulli subsampling
- ▶ benefits from slowly decreasing learning rate