Ruin-based risk measures in discrete-time risk models

Hélène Cossette, Etienne Marceau, Julien Trufin, Pierre Zuyderhoff

École d'actuariat, Université Laval, Québec, Canada
Département de Mathématiques, Université Libre de Bruxelles (ULB), Bruxelles, Belgique

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Abstract

For an insurance company, effective risk management relies on an appropriate measurement of the risk associated to an insurance portfolio. The objective of the present paper is to study properties of ruin-based risk measures defined within discrete-time risk models under a diffrenet perspective at the frontier of the theory of risk measures and ruin theory. Ruin theory is a convenient framework to assess the riskiness of an insurance business. We present and examine desirable properties of ruin-based risk measures. Applications within the classical discrete-time risk model and extensions allowing temporal dependence are investigated. Impact of the temporal dependence on ruin-based risk measurs within those different risk models is also studied. We discuss capital allocation based on Euler's principle for homogeneous and subadditive ruin-based risk measures.

Keywords: Ruin-based risk measures; Discrete-time risk models; Properties; Stochastic orders; Capital allocation.

1 Introduction and motivation

For an insurance company, effective risk management relies on an appropriate measurement of the risk associated to the insurance portfolio. In actuarial science, such assessment is traditionally made with tools developed in the context of ruin theory. One of the main objective of ruin theory is to evaluate the risk of an insurance portfolio within long term dynamic risk models. These risk models describe the evolution over several time periods of the surplus associated to an insurance portfolio, either on a discrete or a continuous time basis. In a ruin context, the risk of a portfolio is quantified in terms of ruin-based risk quantities, based on the event that the portfolio becomes insolvent. For instance, one important objective of ruin theory is the study of ruin probabilities over finite and infinite-time horizons. Standard references on ruin theory include [Gerber, 1979], [Rolski et al., 1999], [Asmussen and Albrecher, 2010] and [Dickson, 2016].

Risk assessment of an insurance portfolio can also be made through risk measures. Risk measures are notably designed to determine minimum capital reserves required to be maintained by financial institutions, such as insurance companies, in order to insure their financial stability. Initially, risk measures were defined over a fixed (and short) period of time. An important literature has been devoted to an axiomatic theory of risk measures. [Artzner et al., 1999] introduced the class of coherent risk measures. Subsequently, [Föllmer and Schied, 2002] and [Frittelli and Gianin, 2002] extended the class of coherent risk measures, defining convex risk measures. In the setting of this axiomatic theory, the considered risk measures are commonly said to be static, since they are defined in terms of random variables (rvs) corresponding to financial losses. Most importantly, no temporal dimension is involved in the risk measurement. For these reasons, authors have considered risk measures defined in terms of risk processes, designated by the rather general expression "dynamic risk measures". Referring to four approaches mentioned in [Cherny, 2009] to define a "dynamic risk measure", ruin-based risk measures considered in this paper are defined as a map from random processes to the set of non-negative real numbers. This approach was considered in different ways. In an application of the study on coherent risk measures for unbounded stochastic processes, [Cheridito et al., 2005] briefly discuss (in Section 5.1) a VaR-type risk measure based on the infinite-time ruin probability itself. [Frittelli and Scandolo, 2006] propose a generalization of the concepts of convex and coherent risk measures to a multiperiod setting, with a careful examination of the axiom of translation invariance. Within the continuous-time compound Poisson risk model and in the vein of [Cheridito et al., 2005], [Trufin et al., 2011] investigate as a function of the claim severity random variable (rv) the smallest amount of capital allowing to ensure that the ultimate ruin probability is less than some acceptance level. [Cossette and Marceau, 2013] examine the capital assessment for an insurance portfolio within the classical discrete-time risk model and within two of its extensions: the classical discrete-time risk model with dependent lines of business and the classical discrete-time risk model with random income. They use finite-time ruin probabilities to define Value-at-Risk (VaR) and Tail-Value-at-Risk (TVaR) dynamic risk measures over a finite-time horizon. They examine the riskiness of the portfolio via the dynamic TVaR.

Following [Trufin et al., 2011], [Gatto and Baumgartner, 2014] focus on the computation of the so-called Value at Ruin and the Tail Value at Ruin, defined within the context of the continuous-time classical compound Poisson risk model perturbed by a Wiener process with infinite-time horizon, using the saddlepoint approximation. Note that [Jiang, 2015] presents a survey of recently proposed risk measures defined from continuous-time risk processes such as the ones in [Trufin et al., 2011] and in [Mitric and Trufin, 2016]. He also proposes four new measures based on continuous-time risk processes. Notably, he analyses properties of a risk measure defined as the reciprocal of the adjustment coefficient within the context of the classical compound Poisson risk model.

In this paper, we aim to study properties of ruin-based risk measures. More specifically, the ruin-based risk measures considered here are mappings from a discrete-time risk process for an insurance portfolio to a positive real number. Ruin-based risk measures have the nice advantage to be defined in a setting at the frontier of the theory of risk measures and ruin theory. A well-known example of ruin-based risk measures is the smallest initial capital that needs to be allocated to a portfolio such that the ruin probability does not exceed a pre-determined value, say 5%, 0.5%, or 0.1%. This procedure has been at the core of ruin theory initiated by the Scandinavian actuaries at the beginning of the twentieth century, in particular [Lundberg, 1903] and [Cramér, 1930]. Risk assessment of an insurance portfolio based on ruin-based risk measures formalizes this rationale. It provides a rigorous approach that lies between the theory of risk measures and ruin theory. It also provides a springboard to the development of new results, and insights from ruin theory. Another interesting application of a positive homogeneous and subadditive ruin-based risk measure is capital allocation using Euler's principle. Capital allocation is useful in the analysis and the computation of the contributions of the components (e.g., lines of business), represented by risk processes, to the riskiness of an insurance portfolio.

The paper is structured as follows. In Section 2, we present definitions and notations used throughout the present paper. In Section 3, we present and discuss desirable properties for a ruin-based risk measure. In Section 4, we consider examples of ruin-based risk measures and examine their properties. In Section 5, we examine the application of ruin-based risk measures within the classical discrete-time risk model, including special cases such as the compound binomial risk model. In Section 6, we study the application of ruin-based risk measures within discrete-time risk models with temporal dependence. In Section 7, we deal with capital allocation based on Euler's principle for homogeneous and subadditive ruin-based risk measures.

2 Definitions and notations

We consider a discrete-time risk model for an insurance portfolio defined on the time index $\mathbb{N} = \{0, 1, 2, ...\}$. The discrete-time risk model is defined by the discrete-time risk process $\underline{X} = \{X_k, k \in \mathbb{N}_+\}$, with $\mathbb{N}_+ = \{1, 2, ...\}$ and where X_k is the net losses rv of the portfolio occurring in period $k \in \mathbb{N}_+$ and has the same distribution as the rv X, i.e., $X_k \sim X$, $k \in \mathbb{N}_+$. The cumulative distribution function (cdf) of the rv X is denoted by F_X and its quantile function is defined by $F_X^{-1}(u) = \inf\{x \in \mathbb{R}, F_X(x) \geq u\}$, $u \in (0, 1)$. When $\underline{X}$ forms a sequence of independent and identically distributed (iid) rvs, the corresponding risk model is the classical (De Finetti) discrete-time risk model (see, e.g. [De Finetti, 1957], [Bühlmann, 2007], and [Dickson, 2016]). A review of discrete-time risk models can be found in [Li et al., 2009].

The objective for an actuary or a quantitative risk manager is to assess the riskiness of a portfolio with a ruin-based risk measure ζ.

Definition 1 *A ruin-based risk measure ζ is a functional mapping from a risk process $\underline{X}$ to $\mathbb{R}_+$.*

From a quantitative risk management perspective, it can be relevant to determine the capital for an insurance portfolio at the initial time by assessing its stochastic behavior, not only for a fixed period of time, but also over a finite or an infinite number of subsequent periods. The common rationale behind Definition 1 of ζ is the evaluation of the capital derived from the ruin-based risk measure ζ, assuming that the structure of the portfolio remains unchanged over a finite number of periods. Note also that we assume that the risk free interest rate is equal to zero, because we do not aim to assess the riskiness associated to stochastic interest rates. To motivate the definitions of these ruin-based risk measures, we first briefly recall the specific ruin theory terminology. Secondly, in Section 3, we discuss the desirable properties for a ruin-based risk measure ζ. Examples of ruin-based risk measures ζ are presented and treated in details in Section 4.

In ruin theory, $\underline{U} = \{U_k, k \in \mathbb{N}\}$ is defined as the (cumulative) surplus (or capital) process of the portfolio, U_k corresponding to the surplus level at period $k \in \mathbb{N}$. For $k = 0$, $U_0 = u \in \mathbb{R}_+$ is the initial amount of capital allocated to the portfolio. Then, in period $k \in \mathbb{N}_+$, the surplus level at time k is $U_k = U_{k-1} - X_k = u - \sum_{j=1}^{k} X_j$. The time of ruin τ is defined as $\inf\{k \in \mathbb{N}_+, U_k < 0\}$, if $\underline{U}$ goes below 0 at least once, or ∞, if $\underline{U}$ never goes below 0. The finite-time ruin probability over n periods is then given by

$$\psi(u, n) = \Pr(\tau \leq n | U_0 = u), \tag{1}$$

for $u \in \mathbb{R}_+$. The infinite-time ruin probability is $\psi(u) = \lim_{n\to\infty} \psi(u, n)$, for $u \in \mathbb{R}_+$. Note that $\psi(u, n) = 1$ and $\psi(u) = 1$, for $u < 0$. Also, $\psi(u, n) \leq \psi(u)$, for $n \in \mathbb{N}_+$. To prevent ruin with certainty over an infinite-time horizon, the expectation of the net losses rv X has to be strictly negative, i.e., $E[X] < 0$, which is called the solvency condition.

The following alternative definition of $\psi(u,n)$ (and $\psi(u)$) is more useful to define ruin-based risk measures. Let $\underline{Y} = \{Y_k, k \in \mathbb{N}\}$ be a random walk with a negative drift and, eventually, with dependent increments, where $Y_0 = 0$ and $Y_k = \sum_{j=1}^{k} X_j$ corresponds to the cumulative sum of the net losses over the first k periods ($k \in \mathbb{N}_+$). The supremum process associated to the random walk $\underline{Y}$ is defined by $\underline{Z} = \{Z_k, k \in \mathbb{N}\}$, where $Z_0 = 0$ and $Z_k = \max\limits_{j=0,1,2,\ldots,k} \{Y_j\}$, with cdf F_{Z_k}, survival function $\overline{F}_{Z_k}$ (with $\overline{F}_{Z_k}(x) = 1 - F_{Z_k}(x)$, $x \in \mathbb{R}_+$), and quantile function $F_{Z_k}^{-1}$. An alternative definition to (1) of the finite-time ruin probability over n periods is then given by

$$\psi(u,n) = \Pr(Z_n > u) = \overline{F}_{Z_n}(u) \tag{2}$$

and can hence be studied through the behavior of the supremum Z_n of the random walk $\underline{Y}$. Similarly, let $Z = \lim\limits_{k\to\infty} Z_k = \max\limits_{j\in\mathbb{N}} \{Y_j\}$, with cdf F_Z, survival function $\overline{F}_Z$ (with $\overline{F}_Z(x) = 1 - F_Z(x)$, $x \in \mathbb{R}_+$), and quantile function F_Z^{-1}. Consequently, the infinite-time ruin probability is also defined by $\psi(u) = \overline{F}_Z(u)$, for $u \geq 0$. Note that, given the definition of Z_n (and Z), its distribution has a probability mass at 0, equal to $F_{Z_n}(0) = 1 - \psi(0,n)$ (and $F_Z(0) = 1 - \psi(0)$).

3 Desirable properties for a ruin-based risk measure

By Definition 1, a ruin-based risk measure ζ is an appraisal of the riskiness of an insurance portfolio via the associated risk process $\underline{X}$ to $\mathbb{R}_+$. Desirable properties for a ruin-based risk measure ζ need to be stated in terms of risk processes and not in terms of rvs (net losses).

Property 1 ***Homogeneity***. *Let $\underline{X}$ be a risk process. A ruin-based risk measure ζ is (positively) homogeneous if $\zeta(a\underline{X}) = a\zeta(\underline{X})$, for all $a > 0$.*

Property 2 ***Subadditivity***. *Let $\underline{X}$ and $\underline{X}'$ be two risk processes. A ruin-based risk measure ζ is subadditive if $\zeta(\underline{X} + \underline{X}') \leq \zeta(\underline{X}) + \zeta(\underline{X}')$.*

Property 3 ***Convexity***. *Let $\underline{X}$ and $\underline{X}'$ be two risk processes. A ruin-based risk measure ζ is convex if $\zeta(\alpha\underline{X} + (1-\alpha)\underline{X}') \leq \alpha\zeta(\underline{X}) + (1-\alpha)\zeta(\underline{X}')$, for all $\alpha \in (0,1)$.*

Property 4 ***Law invariance***. *Let $\underline{X}$ and $\underline{X}'$ be two risk processes with the same distribution, i.e., $\underline{X} \sim \underline{X}'$. A ruin-based risk measure ζ is law invariant if $\zeta(\underline{X}) = \zeta(\underline{X}')$.*

By Definition 1, a ruin-based risk measure also needs to be consistent according to stochastic orders for stochastic processes. We briefly recall basic definitions of stochastic orders that will be useful in the following (see, e.g., [Müller and Stoyan, 2002] and [Shaked and Shanthikumar, 2007] for further details).

Definition 2 ***Usual stochastic orders.***

1. ***Usual univariate stochastic order.*** *Given two univariate rvs V and V', V precedes V' in the usual stochastic order, denoted $V \preceq_{st} V'$, if*

$$\overline{F}_V(x) \leq \overline{F}_{V'}(x), \quad \textit{for all } x \in \mathbb{R}. \tag{3}$$

Note that (3) is verified if, and only if, $\mathbb{E}[\phi(V)] \leq \mathbb{E}[\phi(V')]$ *for any non-decreasing function* ϕ *on* $\mathbb{R}$ *such that the expectations exist. Clearly, if* $V \preceq_{st} V'$*, then* $VaR_\kappa(V) \leq VaR_\kappa(V')$ *for* $\kappa \in (0,1)$.

2. ***Usual multivariate stochastic order.*** *Given two vectors of* n *rvs* $(V_1, ..., V_n)$ *and* $(V'_1, ..., V'_n)$*,* $(V_1, ..., V_n)$ *precedes* $(V'_1, ..., V'_n)$ *in the usual multivariate stochastic order, denoted* $(V_1, ..., V_n) \preceq_{st} (V'_1, ..., V'_n)$*, if*

$$\mathbb{E}[\phi(V_1, ..., V_n)] \leq \mathbb{E}[\phi(V'_1, ..., V'_n)], \tag{4}$$

for any non-decreasing function ϕ *on* $\mathbb{R}^n$ *such that the expectations exist.*

3. ***Usual stochastic order for stochastic processes.*** *Given two stochastic processes* $\underline{V} = \{V_k, k \in \mathbb{N}_+\}$ *and* $\underline{V}' = \{V'_k, k \in \mathbb{N}_+\}$*,* $\underline{V}$ *precedes* $\underline{V}'$ *in the usual stochastic order for stochastic processes, denoted* $\underline{V} \preceq_{st} \underline{V}'$*, if*

$$(V_1, ..., V_n) \preceq_{st} (V'_1, ..., V'_n), \tag{5}$$

for any $n \in \{2, 3, ...\}$.

Property 5 ***Consistency under usual stochastic order for stochastic processes****. Let* $\underline{X}$ *and* $\underline{X}'$ *be two risk processes such that* $\underline{X} \preceq_{st} \underline{X}'$*. A ruin-based risk measure* ζ *is consistent under the usual stochastic order for stochastic processes if* $\zeta(\underline{X}) \leq \zeta(\underline{X}')$.

Remark 1 *By Theorem 6.B.30 of [Shaked and Shanthikumar, 2007], Property 5 is equivalent to the* ***monotonicity*** *property: for two risk processes* $\underline{X}$ *and* $\underline{X}'$*, a ruin-based risk measure* ζ *is monotonic if* $\Pr(X_k \leq X'_k, k \in \mathbb{N}_+) = 1$.

In the following proposition, we summarize several useful results.

Proposition 1 *Let* $\underline{X}$ *and* $\underline{X}'$ *be two risk processes.*

1. *Assume that* $X_k \preceq_{st} X'_k$*, for* $k \in \mathbb{N}_+$ *and that* $(X_1, ..., X_k)$ *and* $(X'_1, ..., X'_k)$ *have the same copula for* $k \in \{2, 3, ...\}$*. Then,* $\underline{X} \preceq_{st} \underline{X}'$.
2. *Assume that* $\underline{X} \preceq_{st} \underline{X}'$*. Then,* $Z_k \preceq_{st} Z'_k$*, for* $k \in \mathbb{N}_+$*, and* $Z \preceq_{st} Z'$.

Proof.

1. From Theorem 4.1 of [Müller and Scarsini, 2001], we have $(X_1, ..., X_k) \preceq_{st} (X'_1, ..., X'_k)$ for $k \in \{2, 3, ...\}$, which, given (5), implies $\underline{X} \preceq_{st} \underline{X}'$.
2. For $k \in \mathbb{N}_+$, the function $\phi : \mathbb{R}^k \to \mathbb{R}_+$, defined by

$$\phi(x_1, ..., x_k) = \max(0, x_1, x_1 + x_2, ..., x_1 + x_2 + ... + x_k),$$

is an increasing function on $\mathbb{R}^k$ for $k \in \mathbb{N}_+$. Then, $Z_k \preceq_{st} Z'_k$, for $k \in \mathbb{N}_+$ and $Z \preceq_{st} Z'$ follows from Theorem 6.B.16 (a)-(d) of [Shaked and Shanthikumar, 2007].

■

We recall definitions of the increasing convex order for rvs, vectors of rvs and discrete-time risk processes.

Definition 3 ***Increasing convex order.***

1. ***Univariate increasing convex order.*** *Given two univariate rvs* V *and* V'*,* V *precedes* V' *in the univariate increasing convex order, denoted* $V \preceq_{icx} V'$*, if* $\mathbb{E}[\phi(V)] \leq \mathbb{E}[\phi(V')]$ *for any non-decreasing convex function* ϕ *on* $\mathbb{R}$ *such that the expectations exist.*

2. ***Multivariate increasing convex order.*** *Given two vectors of n rvs $(V_1, ..., V_n)$ and $(V_1', ..., V_n')$, $(V_1, ..., V_n)$ precedes $(V_1', ..., V_n')$ in the multivariate increasing convex order, denoted $(V_1, ..., V_n) \preceq_{icx} (V_1', ..., V_n')$, if $\mathbb{E}[\phi(V_1, ..., V_n)] \leq \mathbb{E}[\phi(V_1', ..., V_n')]$, for any non-decreasing convex function ϕ on $\mathbb{R}^n$ such that the expectations exist.*
3. ***Increasing convex order for stochastic processes.*** *Given two stochastic processes $\underline{V} = \{V_k, k \in \mathbb{N}\}$ and $\underline{V}' = \{V_k', k \in \mathbb{N}\}$, $\underline{V}$ precedes $\underline{V}'$ in the increasing convex order for stochastic processes, denoted $\underline{V} \preceq_{icx} \underline{V}'$, if*

$$(V_1, ..., V_n) \preceq_{icx} \left(V_1', ..., V_n'\right), \tag{6}$$

for any $n \in \mathbb{N}_+$.

Property 6 ***Consistency under increasing convex order for stochastic processes.*** *Let $\underline{X}$ and $\underline{X}'$ be two risk processes such that $\underline{X} \preceq_{icx} \underline{X}'$. A ruin-based risk measure ζ is consistent under the increasing convex order for stochastic processes if $\zeta(\underline{X}) \leq \zeta(\underline{X}')$.*

Proposition 2 *Let $\underline{X}$ and $\underline{X}'$ be two risk processes.*

1. *Assume that $X_k \preceq_{icx} X_k^{'}$, for $k \in \mathbb{N}_+$ and that the components of $\underline{X}$ (and $\underline{X}'$) form a sequence of independent rvs. Then, $\underline{X} \preceq_{icx} \underline{X}'$.*
2. *Assume that $\underline{X} \preceq_{icx} \underline{X}'$. Then, $Z_k \preceq_{icx} Z_k'$, for $k \in \mathbb{N}_+$, and $Z \preceq_{icx} Z'$.*

Proof.

1. By Theorem 7.A.4 of [Shaked and Shanthikumar, 2007], we have $(X_1, ..., X_k) \preceq_{icx} (X_1', ..., X_k')$ for $k \in \{2, 3, ...\}$ and $\underline{X} \preceq_{icx} \underline{X}'$ follows from (6).
2. The function $\phi : \mathbb{R}^k \to \mathbb{R}_+$, defined by

$$\phi(x_1, ..., x_k) = \max(0, x_1, x_1 + x_2, ..., x_1 + x_2 + ... + x_k)$$

is an increasing continuous convex function for $k \in \mathbb{N}_+$. Then, by Theorem 7.A.5(a) of [Shaked and Shanthikumar, 2007], $Z_k \preceq_{icx} Z_k'$, for $k \in \mathbb{N}_+$. Applying the monotone convergence theorem, $E[Z_k]$ (resp. $E[Z_k']$) tends to $E[Z]$ (resp. $E[Z']$) and $Z \preceq_{icx} Z'$ follows from Theorem 7.A.5(c) of [Shaked and Shanthikumar, 2007].

■

A ruin-based risk measure ζ needs to appropriately quantify the temporal dependence between the losses occurring over a discrete-time horizon. In other words, a ruin-based risk measure ζ needs to be consistent under a proper dependence stochastic order, the so-called supermodular order. The latter is defined in terms of supermodular functions. A function $\phi : \mathbb{R}^m \to \mathbb{R}$ is said to be supermodular if

$$\begin{aligned} & \phi(x_1, ..., x_i + \varepsilon, ..., x_j + \delta, ..., x_m) - \phi(x_1, ..., x_i + \varepsilon, ..., x_j, ..., x_m) \\ \geq \quad & \phi(x_1, ..., x_i, ..., x_j + \delta, ..., x_m) - \phi(x_1, ..., x_i, ..., x_j, ..., x_m) \end{aligned}$$

holds for all $\underline{x} = (x_1, ..., x_m) \in \mathbb{R}^m$, $1 \leq i \leq j \leq m$ and all ε, $\delta > 0$. See [Marshall et al., 1979] for examples of supermodular functions.

Definition 4 ***Supermodular order.*** *Let $\underline{V} = (V_1, ..., V_m)$ and $\underline{V}' = \left(V_1', ..., V_m^{'}\right)$ be two random vectors where, for each i, V_i and $V_i^{'}$ have the same marginal distribution (i.e., $V_i \sim V_i^{'}$ for $i = 1, 2, ..., m$). Then, $\underline{V}$ is smaller than $\underline{V}'$ under the supermodular order, denoted $\underline{V} \preceq_{sm} \underline{V}'$, if $E[\phi(\underline{V})] \leq E[\phi(\underline{V}')]$ for all supermodular functions ϕ, given that the expectations exist.*

The supermodular order is used to compare random vectors $\underline{V}$ and $\underline{V}'$ with different levels of dependence. See e.g. [Shaked and Shanthikumar, 2007], [Müller and Stoyan, 2002] or [Denuit et al., 2006] for details on supermodular ordering.

Property 7 ***Consistency under supermodular order***. *Let $\underline{X}$ and $\underline{X}'$ be two risk processes such that $(X_1, ..., X_k) \preceq_{sm} (X'_1, ..., X'_k)$, for $k \in \mathbb{N}_+$. A ruin-based risk measure ζ is consistent under the supermodular order if $\zeta(\underline{X}) \leq \zeta(\underline{X}')$.*

The following proposition will be useful in the next sections.

Proposition 3 *Let $\underline{X}$ and $\underline{X}'$ be two risk processes such that $(X_1, ..., X_k) \preceq_{sm} (X'_1, ..., X'_k)$, for $k \in \{2, 3, ...\}$. Then, $Z_k \preceq_{icx} Z'_k$, for $k \in \{2, 3, ...\}$, and $Z \preceq_{icx} Z'$.*

Proof. Let $\phi(x_1, ..., x_n) = \max(0, x_1, x_1 + x_2, ..., x_1 + x_2 + ... + x_k)$, for $k \in \{2, 3, ...\}$. Since ϕ is a supermodular function, it follows from Theorem 9.A.16 of [Shaked and Shanthikumar, 2007] that $Z_k \preceq_{icx} Z'_k$, for $k \in \{2, 3, ...\}$. Letting $k \to \infty$, we also have $Z \preceq_{icx} Z'$. ■

4 Examples of ruin-based risk measures

4.1 Preliminaries

Before introducing specific ruin-based risk measures, we recall two of the most popular risk measures (over a fixed period of time). Let the rv X denote the net losses of an insurance portfolio over a fixed period of time. The Value-at-Risk of the rv X is defined by $VaR_\kappa(X) = F_X^{-1}(\kappa)$, for $\kappa \in (0, 1)$. Assuming $E[X] < \infty$, the Tail Value-at-Risk of the rv X is defined by

$$TVaR_\kappa(X) = \tfrac{1}{1-\kappa} \int_\kappa^1 VaR_u(X)\,\mathrm{d}u, \quad \text{for } \kappa \in (0, 1). \tag{7}$$

Two convenient representations of $TVaR_\kappa(X)$ can be derived from (7). First, with $(u)_+ = \max(u; 0)$ and using the probability integral transform, (7) can be written as

$$TVaR_\kappa(X) = VaR_\kappa(X) + \tfrac{1}{1-\kappa} E\left[(X - VaR_\kappa(X))_+\right], \quad \text{for } \kappa \in (0, 1). \tag{8}$$

Then, defining $E\left[X \times 1_{\{X>b\}}\right]$ as the truncated expectation of X, where 1_A is the indicator function such that $1_A(X) = 1$, if $X \in A$, and $1_A(X) = 0$, if $X \notin A$, (8) becomes

$$TVaR_\kappa(X) = \frac{1}{1-\kappa}\left(E\left[X \times 1_{\{X>VaR_\kappa(X)\}}\right] + VaR_\kappa(X)\left(F_X(VaR_\kappa(X)) - \kappa\right)\right), \quad \text{for } \kappa \in (0, 1). \tag{9}$$

When the rv X is continuous, we have $F_X(VaR_\kappa(X)) - \kappa = 0$ such that (9) becomes

$$TVaR_\kappa(X) = \tfrac{1}{1-\kappa} E\left[X \times 1_{\{X>VaR_\kappa(X)\}}\right], \quad \text{for } \kappa \in (0, 1). \tag{10}$$

See e.g. [McNeil et al., 2015] for details on the VaR and TVaR, and their properties.

In this section, we consider four ruin-based risk measures: the ruin-based VaR, the ruin-based TVaR, the Lundberg-Aumann-Serrano index of riskiness, and the risk measure derived from the expected negative part (ENP).

4.2 Ruin-based VaR

Definition 5 *The finite-time ruin-based VaR, denoted $\zeta_{\kappa,n}^{VaR}$, is defined by $\zeta_{\kappa,n}^{VaR}\left(\underline{X}\right) = VaR_{\kappa}\left(Z_n\right)$, for $\kappa \in (0,1)$. The infinite-time ruin-based VaR is given by $\zeta_{\kappa}^{VaR}(\underline{X}) = \lim_{n\to\infty} \zeta_{\kappa,n}^{VaR}(\underline{X})$, for $\kappa \in (0,1)$.*

Proposition 4 *The finite-time (infinite-time) ruin-based VaR satisfies Properties 1, 4, and 5 given in Section 3.*

Proof. Property 1 is obvious since, for $a > 0$, we have

$$\zeta_{\kappa,n}^{VaR}\left(a\underline{X}\right) = VaR_{\kappa}\left(aZ_n\right) = aVaR_{\kappa}\left(Z_n\right) = a\zeta_{\kappa,n}^{VaR}\left(\underline{X}\right)$$

by the positive homogeneity of the VaR. The proof of Property 4 is direct. By Proposition 1.2, $\zeta_{\kappa,n}^{VaR}$ is clearly consistent with the usual stochastic order for stochastic processes. ■

However, the finite-time (infinite-time) ruin-based VaR is not subadditive, as discussed in Section 7.1, which implies that it also fails to satisfy the convexity property. The finite-time ruin-based VaR is not consistent under the increasing convex order and the supermodular order. Currently, we can neither prove or find a counterexample showing that the infinite-time ruin-based VaR is consistent under the increasing convex order and that it is consistent under the supermodular order.

The finite-time (infinite-time) ruin-based VaR can be interpreted as the smallest amount of capital u needed such that finite-time (infinite-time) ruin probability $\psi(u,n)$ $(\psi(u))$ is at most equal to $1-\kappa$. Historically, one of the key tasks of ruin theory was devoted to the derivation and the computation of ruin probabilities within different risk models, with the purpose of computing the initial amount of capital u. In the actuarial literature, focus has been devoted to the study of properties of ruin probabilities (see, e.g. [Asmussen and Albrecher, 2010]) and the surplus analysis (see, e.g. [Gerber and Shiu, 1998] and [Willmot and Woo, 2017]) rather than looking at the resulting capital obtained by inverting the cdf of Z_n (or Z). Here, we believe it is also very interesting to look at the properties of the resulting capital, which we call the finite-time (infinite-time) ruin-based VaR. It helps us to have a fresher look at the riskiness of a risk process describing the stochastic evolution of an insurance portfolio. It also allows to define (introduce) more relevant ruin-based risk measures than the finite-time (infinite-time) ruin-based VaR in order to provide an appropriate appraisal of the riskiness of an insurance portfolio. An example of such ruin-based risk measures is the ruin-based TVaR.

4.3 Ruin-based TVaR

Definition 6 *The finite-time ruin-based TVaR, denoted $\zeta_{\kappa,n}^{TVaR}$, is defined by*

$$\zeta_{\kappa,n}^{TVaR}\left(\underline{X}\right) = TVaR_{\kappa}\left(Z_n\right) = \frac{1}{1-\kappa}\int_{\kappa}^{1} VaR_u\left(Z_n\right)\mathrm{d}u = \frac{1}{1-\kappa}\int_{\kappa}^{1}\zeta_{\kappa,n}^{VaR}\left(\underline{X}\right)\mathrm{d}u, \tag{11}$$

for $\kappa \in (0,1)$. The infinite-time ruin-based TVaR is given by

$$\zeta_{\kappa}^{TVaR}(\underline{X}) = \lim_{n\to\infty}\zeta_{\kappa,n}^{TVaR}(\underline{X}) = TVaR_{\kappa}\left(Z\right) = \frac{1}{1-\kappa}\int_{\kappa}^{1} VaR_u\left(Z\right)\mathrm{d}u = \frac{1}{1-\kappa}\int_{\kappa}^{1}\zeta_{\kappa}^{VaR}\left(\underline{X}\right)\mathrm{d}u, \tag{12}$$

for $\kappa \in (0,1)$.

As mentioned in [Gatto and Baumgartner, 2014] and using (8), (11) becomes

$$\zeta_{\kappa,n}^{TVaR}\left(\underline{X}\right) = VaR_{\kappa}\left(Z_n\right) + \frac{1}{1-\kappa} E\left[\left(Z_n - VaR_{\kappa}\left(Z_n\right)\right)_+\right] = \zeta_{\kappa,n}^{VaR}\left(\underline{X}\right) + \frac{1}{1-\kappa} E\left[\left(Z_n - \zeta_{\kappa,n}^{VaR}\left(\underline{X}\right)\right)_+\right], \tag{13}$$

for $\kappa \in (0,1)$. Now, for a positive rv V with $E[V] < \infty$, we have

$$E\left[(V-d)_+\right] = \int_d^{\infty} \overline{F}_V(x)\,\mathrm{d}x, \tag{14}$$

for $d \geq 0$. Combining (13) and (14), we find

$$\zeta_{\kappa,n}^{TVaR}\left(\underline{X}\right) = \zeta_{\kappa,n}^{VaR}\left(\underline{X}\right) + \frac{1}{1-\kappa}\int_{\zeta_{\kappa,n}^{VaR}\left(\underline{X}\right)}^{\infty} \overline{F}_{Z_n}(x)\,\mathrm{d}x = \zeta_{\kappa,n}^{VaR}\left(\underline{X}\right) + \frac{1}{1-\kappa}\int_{\zeta_{\kappa,n}^{VaR}\left(\underline{X}\right)}^{\infty} \psi(x,n)\,\mathrm{d}x, \tag{15}$$

for $\kappa \in (0,1)$.

Similarly, letting $n \to \infty$ in (13) and (15), we have

$$\zeta_{\kappa}^{TVaR}\left(\underline{X}\right) = \zeta_{\kappa}^{VaR}\left(\underline{X}\right) + \frac{1}{1-\kappa} E\left[\left(Z - \zeta_{\kappa}^{VaR}\left(\underline{X}\right)\right)_+\right] = \zeta_{\kappa}^{VaR}\left(\underline{X}\right) + \frac{1}{1-\kappa}\int_{\zeta_{\kappa}^{VaR}\left(\underline{X}\right)}^{\infty} \psi(x)\,\mathrm{d}x, \tag{16}$$

for $\kappa \in (0,1)$.

The representations of $\zeta_{\kappa,n}^{TVaR}\left(\underline{X}\right)$ and $\zeta_{\kappa}^{TVaR}\left(\underline{X}\right)$ given in (15) and (16) are very useful, since they can be applied, in conjunction with expressions for ruin probabilities $\psi(x,n)$ and $\psi(x)$ to derive expressions for $\zeta_{\kappa,n}^{TVaR}\left(\underline{X}\right)$ and $\zeta_{\kappa}^{TVaR}\left(\underline{X}\right)$ and compute their values.

The finite-time ruin-based risk measure $\zeta_{\kappa,n}^{TVaR}\left(\underline{X}\right)$ (infinite-time ruin-based risk measure $\zeta_{\kappa}^{TVaR}\left(\underline{X}\right)$) can be interpreted as the "mean" of the values taken over the first n periods by the supremum Z_n (Z) which exceeds the finite-time ruin-based risk measure $\zeta_{\kappa,n}^{VaR}\left(\underline{X}\right)$ (infinite-time ruin-based risk measure $\zeta_{\kappa}^{VaR}\left(\underline{X}\right)$). Also, the finite-time ruin-based risk measure $\zeta_{\kappa,n}^{TVaR}\left(\underline{X}\right)$ (infinite-time ruin-based risk measure $\zeta_{\kappa}^{TVaR}\left(\underline{X}\right)$) has the advantage of being more sensitive to the stochastic behavior of Z_n (Z) in the right tail of its distribution. Moreover, the finite-time ruin-based risk measure $\zeta_{\kappa,n}^{TVaR}\left(\underline{X}\right)$ (infinite-time ruin-based risk measure $\zeta_{\kappa}^{TVaR}\left(\underline{X}\right)$) satisfies the seven desirable properties presented in Section 3.

Proposition 5 *The finite-time (infinite-time) ruin-based TVaR satisfies Properties 1 to 7.*

Proof. Property 1 directly follows from the positive homogeneity of the TVaR. Now, for Property 3, since the function $\phi : \mathbb{R}^n \to \mathbb{R}_+$, defined by

$$\phi(x_1,...,x_n) = \max(0, x_1, x_1+x_2, ..., x_1+x_2+...+x_n)$$

is an increasing continuous convex function for $n \in \mathbb{N}_+$, we directly have

$$\phi(\alpha X_1 + (1-\alpha)X_1', \ldots, \alpha X_n + (1-\alpha)X_n') \leq \alpha\phi(X_1,\ldots,X_n) + (1-\alpha)\phi(X_1',\ldots,X_n').$$

Hence, we have

$$\begin{aligned}
\zeta_{\kappa,n}^{TVaR}(\alpha\underline{X}+(1-\alpha)\underline{X}') &= TVaR_{\kappa}(\phi(\alpha X_1+(1-\alpha)X_1',\ldots,\alpha X_n+(1-\alpha)X_n'))\\
&\leq TVaR_{\kappa}(\alpha\phi(X_1,\ldots,X_n)+(1-\alpha)\phi(X_1',\ldots,X_n'))\\
&= TVaR_{\kappa}(\alpha Z_n+(1-\alpha)Z_n')\\
&\leq \alpha TVaR_{\kappa}(Z_n)+(1-\alpha)TVaR_{\kappa}(Z_n')\\
&= \alpha\zeta_{\kappa,n}^{TVaR}(\underline{X})+(1-\alpha)\zeta_{\kappa,n}^{TVaR}(\underline{X}'),
\end{aligned}$$

since the TVaR is a convex risk measure (see Section 2.4.3.5 in [Denuit et al., 2006]). Property 2 is a direct consequence of Properties 1 and 3 with $\alpha = 1/2$. Property 4 is obvious. Proposition 1 implies that $\zeta_{\kappa,n}^{TVaR}$ is consistent with the usual stochastic order for stochastic processes (Property 5). Finally, Properties 6 and 7 are direct consequences of Propositions 2 and 3, respectively, since the TVaR is a risk measure that is consistent with the univariate increasing convex order (see for example Proposition 3.4.8 in [Denuit et al., 2006]). ■

4.4 Lundberg-Aumann-Serrano index of riskiness

The Lundberg-Aumann-Serrano index of riskiness is defined in terms of the adjustment coefficient. It can also be seen as an extension to the Aumann-Serrano index of riskiness. We follow the large deviation approach (see [Müller and Pflug, 2001] for details) to define just below the adjustment coefficient.

Definition 7 ***Adjustment coefficient.*** *Let us define the convex function*

$$c_n(r) = \frac{1}{n}\ln\left(E\left[e^{rY_n}\right]\right). \tag{17}$$

We make the two following assumptions:

1. ***A1:*** $c(r) = \lim_{n\to\infty} c_n(r)$ *exists for* $r \in (0, r_0)$*, where* r_0 *is a strictly positive constant;*
2. ***A2:*** *There is a unique solution* $r \in (0, r_0)$ *such that* $c(r) = \lim_{n\to\infty} c_n(r) = 0$*.*

The adjustment coefficient $r_{AC}(\underline{X})$ *is the solution to*

$$c(r) = \lim_{n\to\infty} c_n(r) = 0. \tag{18}$$

In Theorem 2.1 of [Müller and Pflug, 2001] (see also [Nyrhinen, 1998]), the following asymptotic Lundberg-type result for the ruin probability is obtained:

$$\lim_{u\to\infty} -\frac{\ln(\psi(u))}{u} = r_{AC}(\underline{X}). \tag{19}$$

In ruin theory, the adjustment coefficient, also called Lundberg exponent, is considered as a measure of dangerousness of an insurance portfolio. According to [Gerber, 1979] (p.118), it "plays an important role in ruin theory". Also, [Rolski et al., 2009] stated on p. 18: "the adjustment coefficient is some sort of measure of risk". It is crucial for the computation of ruin-related quantities (see, e.g., [Gerber and Shiu, 1998], [Cheung et al., 2010], [Willmot and Woo, 2017], etc.). The adjustment coefficient is also useful in reinsurance (see, e.g., [Hesselager, 1990], [Hald and Schmidli, 2004], and

[Schmidli, 2017]). The expression of $c_n(r)$ defined in (17) incorporates the information about the temporal dependence structure for $\underline{X}$.

Recently, [Aumann and Serrano, 2008] proposed a new economic index of riskiness. The link between the adjustment coefficient and the Aumann-Serrano economic index of riskiness was established by [Homm and Pigorsch, 2012] and [Meilijson et al., 2009]. In this section, motivation is to examine the adjustment coefficient under a new perspective and study its properties.

Definition 8 ***Lundberg-Aumann-Serrano index of riskiness.*** *Consider a risk process $\underline{X}$ such that the adjustment coefficient $r_{AC}(\underline{X})$ exists. The Lundberg-Aumann-Serrano index of riskiness is defined by*

$$\zeta^{LAS}(\underline{X}) = \frac{1}{r_{AC}(\underline{X})}. \tag{20}$$

The initial definition of the Aumann-Serrano Index (given by the representation given by [Homm and Pigorsch, 2012]) implicitely assumes in the context of the classical discrete-time risk model with a sequence $\underline{X}$ of iid rvs. [Homm and Pigorsch, 2012]) also provide an operational justification to the Aumann-Serrano Index. According to Definition 8, the Lundberg-Aumann-Serrano index of riskiness is an extension of the original Aumann-Serrano Index, in the sense that it is defined in a more general setting, allowing notably for temporal dependence.

In his Master's thesis, [Jiang, 2015] proposed and studied a similar risk measure in the context of continuous-time risk models. The proposed risk measure is defined as a decreasing function of the adjustment coefficient, including the reciprocal of the adjustment coefficient. The risk measure is seen as a mapping from the underlying risk process to $\mathbb{R}_+$. No links are made with the Aumann-Serrano economic index of riskiness. [Dhaene et al., 2003] derive from the classical risk model a risk measure related to the adjustment coefficient. However, it does not have the same properties as $\zeta^{LAS}(\underline{X})$.

Proposition 6 *The Lundberg-Aumann-Serrano index of riskiness $\zeta^{LAS}(\underline{X})$ satisfies Properties 1 to 7.*

Proof. For $a > 0$, let $\frac{1}{\zeta^{LAS}(a\underline{X})}$ be the solution to

$$c(r) = \lim_{n\to\infty} \frac{1}{n} \ln\left(E\left[e^{r(aY_n)}\right]\right) = 0. \tag{21}$$

Since $ra = r_{AC}(\underline{X}) = \frac{1}{\zeta^{LAS}(\underline{X})}$ is also the solution to (21), it follows that $\zeta^{LAS}(a\underline{X}) = a\zeta^{LAS}(\underline{X})$, which proves Property 1.

We have

$$\begin{aligned}
\frac{1}{n}\ln\left(E\left[\mathrm{e}^{\frac{Y_n+Y_n'}{\zeta^{LAS}(\underline{X})+\zeta^{LAS}(\underline{X}')}}\right]\right) &= \frac{1}{n}\ln\left(E\left[\mathrm{e}^{\alpha\frac{Y_n}{\zeta^{LAS}(\underline{X})}+(1-\alpha)\frac{Y_n'}{\zeta^{LAS}(\underline{X}')}}\right]\right) \\
&\leq \alpha\left(\frac{1}{n}\ln\left(E\left[\mathrm{e}^{\frac{Y_n}{\zeta^{LAS}(\underline{X})}}\right]\right)\right) + (1-\alpha)\left(\frac{1}{n}\ln\left(E\left[\mathrm{e}^{\frac{Y_n'}{\zeta^{LAS}(\underline{X}')}}\right]\right)\right),
\end{aligned}$$

for $n \in \mathbb{N}_+$ and with $\alpha = \frac{\zeta^{LAS}(\underline{X})}{\zeta^{LAS}(\underline{X})+\zeta^{LAS}(\underline{X}')}$. Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(E\left[\mathrm{e}^{\frac{Y_n+Y_n'}{\zeta^{LAS}(\underline{X})+\zeta^{LAS}(\underline{X}')}}\right]\right) \le \alpha\left(\lim_{n\to\infty}\frac{1}{n}\ln\left(E\left[\mathrm{e}^{\frac{Y_n}{\zeta^{LAS}(\underline{X})}}\right]\right)\right)+(1-\alpha)\left(\lim_{n\to\infty}\frac{1}{n}\ln\left(E\left[\mathrm{e}^{\frac{Y_n'}{\zeta^{LAS}(\underline{X}')}}\right]\right)\right) = 1.$$

It means that

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(E\left[\mathrm{e}^{\frac{Y_n+Y_n'}{\zeta^{LAS}(\underline{X})+\zeta^{LAS}(\underline{X}')}}\right]\right) \le 1 = \lim_{n\to\infty}\frac{1}{n}\ln\left(E\left[\mathrm{e}^{\frac{Y_n+Y_n'}{\zeta^{LAS}(\underline{X}+\underline{X}')}}\right]\right). \tag{22}$$

From (22), it follows that

$$\frac{1}{\zeta^{LAS}\left(\underline{X}\right)+\zeta^{LAS}\left(\underline{X}'\right)} \le \frac{1}{\zeta^{LAS}\left(\underline{X}+\underline{X}'\right)},$$

which is equivalent to $\zeta^{LAS}\left(\underline{X}+\underline{X}'\right) \le \zeta^{LAS}\left(\underline{X}\right)+\zeta^{LAS}\left(\underline{X}'\right)$, proving Property 2. Since $\zeta^{LAS}(\underline{X})$ is both homogeneous and subadditive, it implies that it also satisfies Property 3. Clearly, $\zeta^{LAS}(\underline{X})$ is law invariant, consistent under the usual stochastic order, and consistent under the increasing convex order. By Corollary 3.6 of [Müller and Pflug, 2001] and given (20), $\zeta^{LAS}(\underline{X})$ is consistent under the supermodular order. ■

Based on the asymptotic result in (19), $\psi(u)$ can be approximated by

$$\psi(u) \simeq e^{-r_{AC}(\underline{X})u}, \tag{23}$$

for large values of u. Define $\zeta_\kappa^{Lund}\left(\underline{X}\right) = -\zeta^{LAS}\left(\underline{X}\right)\times\ln\left(1-\kappa\right)$, for $\kappa \in (0,1)$. Then, from (23) and for values of κ close to 1 (e.g., $\kappa in(0.95,1)$), we obtain the following approximation of the ruin-based VaR :

$$\zeta_\kappa^{VaR}\left(X\right) \simeq \zeta_\kappa^{Lund}\left(\underline{X}\right) = -\zeta^{LAS}\left(\underline{X}\right)\times\ln\left(1-\kappa\right). \tag{24}$$

The traditional version of the approximation in (24) (with adjustment coefficient $r_{AC}\left(\underline{X}\right)$ at the denominator) is well-known in ruin theory. However, the version in (24), with Proposition 6, provides us with a new perspective on the approximation. One can use the Euler principle to find the contributions to $\zeta_\kappa^{VaR}\left(X\right)$.

4.5 Risk measure derived from the expected negative part (ENP)

Inspired from [Loisel and Trufin, 2014], we introduce here the last ruin-based risk measure. It is derived from the expected negative part (ENP) of the surplus process over n periods, defined in terms of the random walk $\underline{Y}$ as

$$E\left[I_n(u)\right] = \sum_{k=1}^{n} E\left[|U_k|1_{\{U_k<0\}}\right] = \sum_{k=1}^{n} E\left[(Y_k-u)_+\right]. \tag{25}$$

Note that the expected negative part is a discrete version of the expected area in red defined in [Loisel, 2005] for continuous risk processes. In the following, when needed to make explicit the dependence on $\underline{X}$, we will denote $I_n(u)$ as $I_n^{(\underline{X})}(u)$.

Definition 9 ***Risk measure derived from the ENP (RM-ENP).*** *For a fixed n, the risk measure derived from the ENP, denoted $\zeta_{A,n}^{ENP}$, is defined by $\zeta_{A,n}^{ENP}(\underline{X}) = \inf\{u \geq 0 | E[I_n(u)] \leq A\}$ for $A > 0$.*

In words, $\zeta_{A,n}^{ENP}(\underline{X})$ is the smallest amount of capital needed such that the expected negative part of the surplus process over n periods is at most equal to some specified level A.

The expected negative part provides an interesting risk indicator for quantifying the liquidity risk of an insurance portfolio over a given time horizon. When the surplus of a portfolio gets below some level, the insurer needs to finance some kind of debt and get supported by fresh money from the parent company or from another entity. Of course, this help cannot last for too long or concern a too high amount. Hence, if the negative part is too large, the insurer is likely to fail to pay its liabilities in the short term since it will not be able to get help to refinance.

Proposition 7 *The risk measure derived from the ENP (RM-ENP) satisfies Properties 3, 4, 5, 6 and 7.*

Proof. For Property 3, we note that

$$E\left[\left(\alpha Y_k + (1-\alpha)Y_k' - u - u'\right)_+\right] \leq E\left[(\alpha Y_k - u)_+\right] + E\left[\left((1-\alpha)Y_k' - u'\right)_+\right]$$

for all $k \in \mathbb{N}_+$ and $u, u' \in \mathbb{R}_+$. Hence, by (25), we have

$$E\left[I_n^{(\alpha\underline{X}+(1-\alpha)\underline{X}')}(u+u')\right] \leq E\left[I_n^{(\alpha\underline{X})}(u)\right] + E\left[I_n^{((1-\alpha)\underline{X}')}(u')\right].$$

Now, by taking $u = \zeta_{\beta A,n}^{ENP}(\alpha\underline{X})$ and $u' = \zeta_{(1-\beta)A,n}^{ENP}((1-\alpha)\underline{X}')$, with $\beta \in (0,1)$, we find

$$E\left[I_n^{(\alpha\underline{X}+(1-\alpha)\underline{X}')}\left(\zeta_{\beta A,n}^{ENP}(\alpha\underline{X}) + \zeta_{(1-\beta)A,n}^{ENP}\left((1-\alpha)\underline{X}'\right)\right)\right] \leq A.$$

Consequently, we have

$$\begin{aligned} \zeta_{A,n}^{ENP}\left(\alpha\underline{X} + (1-\alpha)\underline{X}'\right) &\leq \zeta_{\beta A,n}^{ENP}(\alpha\underline{X}) + \zeta_{(1-\beta)A,n}^{ENP}\left((1-\alpha)\underline{X}'\right) \\ &= \alpha\zeta_{\frac{\beta}{\alpha}A,n}^{ENP}(\underline{X}) + (1-\alpha)\zeta_{\frac{1-\beta}{1-\alpha}A,n}^{ENP}\left(\underline{X}'\right) \end{aligned} \tag{26}$$

since $\zeta_{A,n}^{ENP}(c\underline{X}) = c\,\zeta_{\frac{A}{c},n}^{ENP}(\underline{X})$ for all $c \in \mathbb{R}_+$ as shown in Proposition 8. Property 3 thus follows by taking $\beta = \alpha$. Property 4 is again obvious. Then, for Property 6, we have that $\underline{X} \preceq_{icx} \underline{X'}$ which implies $Y_k \preceq_{icx} Y_k'$ for all $k \in \mathbb{N}_+$ by Theorem 7.A.5 in [Shaked and Shanthikumar, 2007] with $g(X_1, \ldots, X_k) = X_1 + \ldots + X_k$. Hence, by definition of the increasing convex order, we have $E\left[(Y_k - u)_+\right] \leq E\left[(Y_k' - u)_+\right]$ for all $k \in \mathbb{N}_+$ and $u \in \mathbb{R}_+$. So, from (25), we get $E\left[I_n^{(\underline{X})}(u)\right] \leq E\left[I_n^{(\underline{X}')}(u)\right]$. Property 5 directly follows from Property 6. Finally, for Property 7, it suffices to refer to Theorem 9.A.16 in [Shaked and Shanthikumar, 2007] which tells us that $\underline{X} \preceq_{sm} \underline{X'}$ implies $Y_k \preceq_{icx} Y_k'$ for all $k \in \mathbb{N}_+$.

■

Note that (26) is actually more general than the convexity property obtained when $\beta = \alpha$.

Proposition 8 *The risk measure derived from the ENP (RM-ENP) also fulfills modified versions of Properties 1 and 2.*

1. ***Modified homogeneity.*** $\zeta_{A,n}^{ENP}\left(c\underline{X}\right)=c\,\zeta_{\frac{A}{c},n}^{ENP}\left(\underline{X}\right)$ *for all* $c\in\mathbb{R}_{+}$.
2. ***Modified subadditivity.*** $\zeta_{A,n}^{ENP}\left(\underline{X}+\underline{X}'\right)\leq\zeta_{\beta A,n}^{ENP}\left(\underline{X}\right)+\zeta_{(1-\beta)A,n}^{ENP}\left(\underline{X}'\right)$ *for all* $\beta\in(0,1)$.

Proof. For the homogeneity property, we need to show that $E\left[I_{n}^{(c\underline{X})}(u)\right]=c\,E\left[I_{n}^{(\underline{X})}(\frac{u}{c})\right]$, which, by (25), immediately follows since $E\left[(c\,Y_{k}-u)_{+}\right]=c\,E\left[\left(Y_{k}-\frac{u}{c}\right)_{+}\right]$ for all $k\in\mathbb{N}_{+}$ and $u\in\mathbb{R}_{+}$. The subbaditivity property is a direct consequence of the homogeneity property and (26) with $\alpha=\frac{1}{2}$.
■

This last proposition calls for the two following comments. Firstly, the notion of positive homogeneity can be interpreted as the independence with respect to the monetary unit used. When considering two different monetary units u_1 and u_2 say, with exchange rate c (i.e. with $u_1=c\,u_2$), the risk limit A set in the first unit becomes logically A/c in the second one, as the risk process $c\underline{X}$ becomes $\underline{X}$. Secondly, the subadditivity property reflects the idea that the risk can be reduced by diversification. Therefore, for a fair comparison, the risk limit considered for the aggregate portfolio (i.e. with risk process $\underline{X}+\underline{X}'$) must be equal to the sum of the risk limits of the two components (with risk processes $\underline{X}$ and $\underline{X}'$).

5 Classical discrete-time risk model

5.1 Lundberg-Auman-Serrano index of riskiness

As mentioned in Section 2, the classical (De Finetti) discrete-time risk model is defined by a risk process, denoted by $\underline{X}=\{X_k,k\in\mathbb{N}_+\}$, which forms a sequence of iid rvs with $X_k\sim X$, for $k\in\mathbb{N}_+$. As mentioned at, e.g., [Müller and Pflug, 2001], (18) becomes $c\left(r\right)=\ln\left(\mathcal{M}_X\left(r\right)\right)$. Let the net loss rv at period k be defined by $X_k=W_k-\pi$, for $k\in N$, where $\underline{W}=\{W_k,k\in\mathbb{N}_+\}$ is a sequence of iid rvs with $W_k\sim W$, for $k\in\mathbb{N}_+$, and $\pi=(1+\eta)\,E\left[W\right]$ is the premium income per period with $\eta>0$. Then, $c\left(r\right)=\ln\left(\mathcal{M}_W\left(r\right)\right)-\pi$. For comparison purposes in Section 6, we consider the following special case: if W follows a compound Poisson distribution where $\mathcal{M}_W\left(r\right)=\exp\left(\lambda\left(\frac{\beta}{\beta-r}-1\right)\right)$, then it is well known that $r_{AC}\left(\underline{X}\right)=\beta\frac{\eta}{1+\eta}$ (see, e.g., [Dickson, 2016]), which implies that the Lundberg-Auman-Serrano index of riskiness is given by

$$\zeta_{\kappa}^{LAS}\left(\underline{X}\right)=\frac{1}{\beta}\frac{1+\eta}{\eta}. \tag{27}$$

5.2 Numerical computation and recursive relations

For sake of completeness, we briefly discuss the numerical computation of the finite-time ruin-based VaR and the finite-time ruin-based TVaR. To simplify the presentation, we assume that loss rv $X_k=W_k-\pi$, where the aggregate claim rv W_k is defined on $\mathbb{N}$ and the premium income $\pi\in\mathbb{N}_+$, for $k\in\mathbb{N}_+$. Then, for $n\in\mathbb{N}_+$, the cdf of Z_n is given by $F_{Z_n}\left(k\right)=1-\psi\left(k,n\right)=\phi\left(k,n\right)$, for $k\in\mathbb{N}$. The probability mass function (pmf) of W is denoted by $f_W\left(k\right)=\Pr\left(W=k\right),\,k\in\mathbb{N}$. Finite-time non-ruin probabilities are recursively computed with $\varphi\left(k,n\right)=\sum_{j=0}^{k+\pi}f_W\left(j\right)\varphi\left(k+\pi-j,n-1\right)$, for $k\in\mathbb{N}_+$, with $\varphi\left(kh,0\right)=1$, for $k\in\mathbb{N}$ (see, e.g., [Dickson and Waters, 1991] for details). The pmf of Z_n is given by $f_{Z_n}(kh) = \varphi(kh,n) - \varphi((k-1)h,n)$, $k \in \mathbb{N}^+$, with $f_{Z_n}(0) = \varphi(0,n)$. Let $\zeta_{\kappa,n}^{VaR}(\underline{X}) = VaR_\kappa(Z_n) = k_0$, for $k_0 \in \mathbb{N}$. Then, the expression (16) for $TVaR_\kappa(Z_n)$ becomes

$$\zeta_{\kappa,n}^{TVaR}(\underline{X}) = \zeta_{\kappa,n}^{VaR}(\underline{X}) + \frac{1}{1-\kappa} E\left[\max\left(Z_n - \zeta_{\kappa,n}^{VaR}(\underline{X});0\right)\right] = k_0 + \frac{1}{1-\kappa} E\left[\max\left(Z_n - k_0;0\right)\right], \tag{28}$$

where $E[\max(Z_n - k_0;0)] = \sum_{k=k_0+1}^{\infty}(k-k_0) f_{Z_n}(kh)$. Details and illustrations are provided in [Cossette and Marceau, 2013].

5.3 Compound binomial risk model

A special case of the classical discrete-time risk model is the well-known compound binomial risk model, introduced by [Gerber, 1988]. Here, we adopt the definition of ruin used by [Shiu, 1989] and by [Willmot, 1993]. For this specific classical discrete-time risk model, the net losses rv X is defined by $X = W - 1$ with $W = \begin{cases} B & , \quad I = 1 \\ 0 & , \quad I = 0 \end{cases}$, where I follows a Bernoulli distribution with parameter $q \in (0,1)$ and B is a discrete rv defined on $\mathbb{N}_+$, with $E[B] < \infty$. The rvs I and B are independent. The parameters of the distributions of I and B are fixed such that the solvency condition

$$E[X] = E[W] - 1 = E[I]E[B] - 1 < 0 \tag{29}$$

is satisfied, which implies $qE[B] < 1$. Define the finite-time and infinite-time non-ruin probabilities $\phi(u,n) = 1 - \psi(u,n) = F_{Z_n}(u)$ and $\phi(u) = 1 - \psi(u) = F_Z(u)$, for an initial capital $u \in \mathbb{N}$. The finite-time and the infinite-time non-ruin probabilities can be computed recursively. In [Shiu, 1989] and [Willmot, 1993], it is shown that

$$\phi(u) = \frac{\phi(u-1) - q\sum_{j=1}^{u}\phi(u-j)f(j)}{1-q}, \tag{30}$$

for $u \in \mathbb{N}_+$ with initial value

$$\phi(0) = \frac{1 - qE[B]}{1-q}. \tag{31}$$

Similarly, we have

$$\phi(u,n) \quad = \quad (1-q)\,\phi(u+1,n-1) + q\sum_{j=1}^{u+1}\phi(u+1-j,n-1)\,f(j), \tag{32}$$

for $u \in \mathbb{N}$ and $n \in \mathbb{N}_+$ where $\phi(u,0) = 1$, for $u \in \mathbb{N}$. Using (30) (or (32)), one can easily compute the values of $\zeta_k^{VaR}(\underline{X})$ and $\zeta_k^{TVaR}(\underline{X})$ (or, $\zeta_{k,n}^{VaR}(\underline{X})$ and $\zeta_{k,n}^{TVaR}(\underline{X})$).

Example 1 *Assume that the rv B follows a zero-truncated geometric distribution with $\Pr(B=k) = (1-\upsilon)\upsilon^{k-1}$, for $k \in \mathbb{N}_+$ and $E[B] = \frac{1}{1-\upsilon}$, with $\upsilon \in (0,1)$. Due to (29), the parameters q and υ are fixed such that $\frac{q}{1-\upsilon} < 1$. [Willmot, 1993] has shown that*

$$\psi(u) = \overline{F}_Z(u) = 1 - F_Z(u) = \psi(0)\left(\frac{\upsilon}{1-q}\right)^u, \tag{33}$$

for $u \in \mathbb{N}$ and with $\psi(0) = \frac{1-\upsilon}{\upsilon}\frac{1-q}{q}$. From (33), we can easily find that the expression of $\zeta_\kappa^{VaR}(\underline{X})$

is given by

$$\zeta_{\kappa}^{VaR}\left(\underline{X}\right)=F_{Z}^{-1}\left(\kappa\right)=\begin{cases}0 & ,\quad 0<\kappa\leq 1-\psi\left(0\right)\\ \left\lceil\frac{\ln\left(\frac{(1-\kappa)}{\psi(0)}\right)}{\ln\left(\frac{\upsilon}{1-q}\right)}\right\rceil & ,\quad 1-\psi\left(0\right)<\kappa<1\end{cases},\tag{34}$$

for $\kappa\in(0,1)$ *and where* $\lceil x\rceil$ *is the smallest integer greater than* $x\in\mathbb{R}_{+}$. *Also, we have*

$$\pi_{Z}\left(u\right)=E\left[\max\left(Z-u;0\right)\right]=\sum_{k=u}^{\infty}\psi\left(k\right)=\psi\left(0\right)\frac{1-q}{1-q-\upsilon}\left(\frac{\upsilon}{1-q}\right)^{u},\tag{35}$$

for $u\in\mathbb{N}$. *Replacing (34) and (35) in (16), we obtain the following expression for* $\zeta_{\kappa}^{TVaR}\left(\underline{X}\right)$:

$$\zeta_{\kappa}^{TVaR}\left(\underline{X}\right)=\begin{cases}\psi\left(0\right)\frac{1-q}{1-q-\upsilon} & ,\quad 0<\kappa\leq 1-\psi\left(0\right)\\ \left\lceil\frac{\ln\left(\frac{(1-\kappa)}{\psi(0)}\right)}{\ln\left(\frac{\upsilon}{1-q}\right)}\right\rceil+\frac{\psi(0)}{1-\kappa}\frac{1-q}{1-q-\upsilon}\left(\frac{\upsilon}{1-q}\right)^{\left\lceil\frac{\ln\left(\frac{(1-\kappa)}{\psi(0)}\right)}{\ln\left(\frac{\upsilon}{1-q}\right)}\right\rceil} & ,\quad 1-\psi\left(0\right)<\kappa<1\end{cases},$$

for $\kappa\in(0,1)$.

6 Discrete-time risk models with temporal dependence

6.1 Motivations

Generally, in the actuarial literature, authors have examined the impact of dependence on the ruin probabilities or on the adjustment coefficient. However, the probability of ruin on a finite-time horizon is not "consistent" under the supermodular order. Ruin-based risk measures, which are consistent under the supermodular order, are suitable tools to examine and characterize the impact of the dependence on the amount of additional capital to be allocated to the portfolio of the insurance company.

Within discrete-time risk models with temporal dependence, the risk process $\underline{X}=\{X_k,k\in\mathbb{N}_+\}$ is constructed as follows. We define a sequence of identically distributed but not necessarily independent rvs $\underline{W}=\{W_k,k\in\mathbb{N}_+\}$ where W_k represents the aggregate claim amount in period k, $k\in\mathbb{N}_+$. Let $\underline{N}=\{N_k,k\in\mathbb{N}_+\}$ be a discrete-time claim number process where N_k corresponds to the number of claims in period k, $k\in\mathbb{N}_+$. The aggregate claim amount rv W_k is defined as

$$W_k=\sum_{j=1}^{N_k}B_{k,j},\tag{36}$$

assuming that $\sum_{j=1}^{0}a_j=0$. The claim amounts in period k, denoted $B_{k,1}$, $B_{k,2}$, ..., form a sequence of iid strictly positive rvs with cdf F_B and independent of N_k. Given (36), it implies that W_k follows a compound distribution. The rv W_k (N_k) is distributed as W (N) with cdf F_W (F_N). Assuming $E[N]<\infty$ and $E[B]<\infty$, the expectation of the rv W is $E[W]=E[N]E[B]$. The premium income per period is designated by π and satisfies the usual solvency condition $\pi>E[W]$. The strictly positive relative risk margin is $\eta=\frac{\pi}{E[W]}-1$. Then, the $k-$th component of the risk process

$\underline{X} = \{X_k, k \in \mathbb{N}_+\}$ is defined by

$$X_k = W_k - \pi, \tag{37}$$

for $k \in \mathbb{N}_+$. See, e.g., [Cossette et al., 2010] for details on these classes of discrete-time risk models.

In this section, we examine the impact of temporal dependence on ruin-based risk measures consistent under supermodular order, within risk models based on compound distributions assuming for $\underline{N}$ a Poisson MA(1) process, Poisson AR(1) process, Markov Bernoulli process, and a Markov switching regime process.

Proposition 9 *Let $\underline{X}$ and $\underline{X}'$ be two risk processes such that*

$$(N_1, ..., N_k) \preceq_{sm} \left(N_1', ..., N_k'\right), \tag{38}$$

for $k \in \{2, 3, ...\}$. Consider a ruin-based risk measure consistent under the supermodular order. Then, $\zeta(\underline{X}) \leq \zeta(\underline{X}')$. In particular, $\zeta_{\kappa,n}^{TVaR}(\underline{X}) \leq \zeta_{\kappa,n}^{TVaR}(\underline{X}')$, for $n \in \mathbb{N}_+$ and $\kappa \in (0,1)$.

Proof. By Proposition 2 in [Denuit et al., 2002], (38) implies that

$$(W_1, ..., W_k) \preceq_{sm} \left(W_1', ..., W_k'\right), \tag{39}$$

for $k \in \{2, 3, ...\}$. From (39), we also have $(X_1, ..., X_k) \preceq_{sm} (X_1', ..., X_k')$, for $k \in \{2, 3, ...\}$, and the result follows from Proposition 3. ■

6.2 Risk model based on Poisson MA(1)

6.2.1 Definitions and result

We introduce the operator "$\circ$" used to define the dynamics of the Poisson MA(1) and Poisson AR(1) processes. Let M be a non-negative integer-valued rv and $\alpha \in [0,1]$. The $\circ$-operation of α on M is referred to as the binomial thinning of M and is defined as $\alpha \circ M = \sum_{i=1}^{M} D_i$, where $\{D_i, i \in \mathbb{N}_+\}$ is a sequence of iid Bernoulli distributed rvs with mean α and independent of M. The dynamic of a Poisson MA(1) process $\underline{N} = \{N_k, k \in \mathbb{N}_+\}$ is defined as

$$N_k = \alpha \circ \varepsilon_{k-1} + \varepsilon_k, \quad k \in \mathbb{N}_+, \tag{40}$$

where $\underline{\varepsilon} = \{\varepsilon_k, k \in \mathbb{N}_+\}$ is a sequence of iid rvs following a Poisson distribution with mean $\frac{\lambda}{1+\alpha}$ and $\alpha \in [0,1]$. The rv $\alpha \circ \varepsilon_{k-1}$ in (40) is

$$\alpha \circ \varepsilon_{k-1} = \sum_{j=1}^{\varepsilon_{k-1}} \delta_{k-1,j}, \quad k \in \mathbb{N}_+, \tag{41}$$

where $\{\delta_{k-1,j}\}$ is a sequence of iid Bernoulli distributed rvs with mean α. The sequences $\{\delta_{k,j},\ j \in \mathbb{N}_+\}$ are assumed independent for different periods k, $k \in \mathbb{N}_+$. Given these distribution assumptions, the rv $\alpha \circ \varepsilon_{k-1}$ is Poisson with mean $\frac{\lambda\alpha}{1+\alpha}$. The dynamic in (40) generates a stationary discrete-time process where the marginal distribution of N_k is Poisson with mean λ for $k \in \mathbb{N}$. The autocorrelation function of $\underline{N}$ is $\rho_N(h) = \begin{cases} \frac{\alpha}{1+\alpha}, & h = 1 \\ 0, & h > 1 \end{cases}$, which implies that

$\rho_N(1) \in [0, 0.5]$. If $\alpha = 0$, $\underline{N}$ becomes a sequence iid rvs and hence we are in the presence of the classical discrete-time risk model presented in Section 5.1 and based on the risk process denoted by $\underline{X}^{\perp}$ in the present section.

The number of claims N_k in period k is therefore mainly due to the new arrivals between $k-1$ and k, and a proportion of the new arrivals between $k-2$ and $k-1$ defined by the thinning procedure. The expression for the joint mass probability function of (N_k, N_{k-1}) is given by

$$\Pr(N_k = n_k, N_{k-1} = n_{k-1}) = e^{-\left(2-\frac{\alpha}{1+\alpha}\right)\lambda} \sum_{j=0}^{\min(n_k;n_{k-1})} \frac{\left(\frac{\alpha}{1+\alpha}\right)^j \left(1-\frac{\alpha}{1+\alpha}\right)^{n_k+n_{k-1}-2j} \lambda^{n_k+n_{k-1}-j}}{j!\,(n_k-j)!\,(n_{k-1}-j)!}, \tag{42}$$

for $n_k, n_{k-1} \in \mathbb{N}$ and for $k \in \mathbb{N}$. See e.g. [McKenzie, 1988] and [McKenzie, 2003] for other properties of the Poisson MA(1) discrete-time process. The risk model based on the Poisson MA(1) is examined in, e.g., [Cossette et al., 2010] and applied in the context of reinsurance by [Zhang et al., 2015]. In the proof of the following proposition, we need the concordance order, also called correlation order by [Denuit et al., 2006].

Definition 10 ***Concordance order.*** *Let $\underline{X} = (X_1, X_2)$ and $\underline{X}' = (X_1', X_2')$ be two pairs of rvs, with the same marginals. Then, $\underline{X}$ is less concordant then $\underline{X}'$, written $\underline{X} \preceq_{co} \underline{X}'$, if $F_{\underline{X}}(\underline{x}) \leq F_{\underline{X}'}(\underline{x})$, for $\underline{x} \in \mathbb{R}^2$ (see Definition 3.8.1 in [Müller and Stoyan, 2002]).*

Note that, for $m = 2$, the supermodular order coincides with the concordance order.

Proposition 10 *Let $\underline{X}$ and $\underline{X}'$ be two risk processes, where $\lambda = \lambda'$, $B \sim B'$, and $0 \leq \alpha < \alpha' < 1$. Then,*

$$(N_1, ..., N_k) \preceq_{sm} (N_1', ..., N_k'), \tag{43}$$

and

$$(X_1, ..., X_k) \preceq_{sm} (X_1', ..., X_k'), \tag{44}$$

for $k \in \{2, 3, ...\}$. Consider a ruin-based risk measure ζ consistent under the supermodular order. Then, by Proposition 9, $\zeta(\underline{X}) \leq \zeta(\underline{X}')$.

Proof. Firstly, for $0 \leq \alpha < \alpha' < 1$, we need to prove that $(N_1, N_2) \preceq_{co} (N_1', N_2')$. Let $G(x;\gamma)$, $x \geq 0$, and $G^{-1}(u;\gamma)$, $u \in (0,1)$, be the cdf and the quantile function, respectively, of the Poisson distribution with parameter $\gamma > 0$. Define also the independent rvs U_1, U_2, U_1', U_2', V, W, which follow a standard uniform distribution. Then, we represent the pairs of rvs (N_1, N_2) and (N_1', N_2') as follows:

$$\begin{aligned} N_1 &= G^{-1}\left(U_1; \frac{\lambda}{1+\alpha}\right) + G^{-1}\left(W; \frac{\alpha}{1+\alpha}\lambda\right) = \phi_1(U_1, W) \\ N_2 &= G^{-1}\left(U_2; \frac{\lambda}{1+\alpha}\right) + G^{-1}\left(W; \frac{\alpha}{1+\alpha}\lambda\right) = \phi_2(U_2, W) \end{aligned}$$

and

$$N_1' = G^{-1}\left(U_1'; \frac{\lambda}{1+\alpha'}\right) + G^{-1}\left(V; \left(\frac{\alpha'}{1+\alpha} - \frac{\alpha}{1+\alpha}\right)\lambda\right) + G^{-1}\left(W; \frac{\alpha}{1+\alpha}\lambda\right) = \phi_1'(U_1, V, W)$$

$$N_2' = G^{-1}\left(U_2'; \frac{\lambda}{1+\alpha'}\right) + G^{-1}\left(V; \left(\frac{\alpha'}{1+\alpha'} - \frac{\alpha}{1+\alpha}\right)\lambda\right) + G^{-1}\left(W; \frac{\alpha}{1+\alpha}\lambda\right) = \phi_2'(U_1, V, W),$$

with $0 \leq \alpha < \alpha' < 1$. We observe that $\phi_1'(u,v,w)$ and $\phi_2'(u,v,w)$ are increasing in $v \in (0,1)$. Also, $\phi_i(U_i, w)$ and $\phi_i'(U_i, V, w)$ have the same distribution for $i = 1, 2$, and for $w \in (0,1)$. Since (N_1, N_2) and $(\phi_1(U_1, W), \phi_2(U_2, W))$ have the same distribution and (N_1', N_2') and $(\phi_1'(U_1, V, W), \phi_2'(U_2', V, W))$ have the same distribution, it follows from Theorem 3.1 of [Bäuerle, 1997], that $(N_1, N_2) \preceq_{co} (N_1', N_2')$. Note that $\underline{N}$ and $\underline{N}'$ are two homogeneous Markov chains. Since N_1 (and N_2) is stochastically increasing in N_2 (in N_1), then, by Theorem 3.2 of [Hu and Pan, 2000], we obtain the result in (49) for $k \in \{2, 3, ...\}$. Finally, (44) follows from Proposition 2(iv) of [Denuit et al., 2002]. ■

6.2.2 Lundberg-Aumann-Serrano index of riskiness

In Proposition 1 of [Cossette et al., 2010], the expression for $c(r)$ defined in (18) is given by

$$c(r) = \frac{\lambda(1-\alpha)}{1+\alpha}\mathcal{M}_B(r) + \frac{\lambda\alpha}{(1+\alpha)}\mathcal{M}_B^2(r) - \frac{\lambda}{1+\alpha} - r\pi. \tag{45}$$

In [Cossette et al., 2010], it is also mentioned that "the impact of the dependence parameter α on the Lundberg coefficient could have been studied using the supermodular order. However, after investigation, the proof of this inequality based on supermodular ordering remains an open problem." Proposition 11 provides a solution to that problem and it allows us to conclude that $\alpha \leq \alpha'$ implies that $\zeta^{LAS}(\underline{X}) \leq \zeta^{LAS}(\underline{X}')$, assuming that both indices exist.

Example 2 *We consider the discrete-time risk model based on the Poisson MA(1). The risk process $\underline{X} = \{X_k, k \in \mathbb{N}_+\}$ where X_k is defined in (37). The claim number process is a Poisson MA(1) process with parameter λ. The claim amount rv B follows an exponential distribution with mean $\frac{1}{\beta}$ and mgf $\mathcal{M}_B(r) = \frac{\beta}{\beta - r}$. Using Proposition 2 of [Cossette et al., 2010] and (20), we find that the Lundberg-Aumann-Serrano index of riskiness is given by*

$$\zeta^{LAS}(\underline{X}) = \frac{1}{\beta}\frac{2(1+\eta)}{\left(2(1+\eta) - \frac{1}{1+\alpha} - \sqrt{4\frac{\alpha(1+\eta)}{1+\alpha} + \frac{1}{(1+\alpha)^2}}\right)}. \tag{46}$$

We denote by $\underline{X}^{\perp} = \{X_k^{\perp}, k \in \mathbb{N}_+\}$ the risk process when $\alpha = 0$. Note that $\underline{X}^{\perp}$ is the risk process associated to the classical discrete-time risk model where $\zeta^{LAS}(\underline{X}^{\perp}) = \frac{1}{\beta}\frac{1+\eta}{\eta}$ as provided in (27). Then, $\zeta^{LAS}(\underline{X})$ can be represented in terms of $\zeta^{LAS}(\underline{X}^{\perp})$ as follows:

$$\zeta^{LAS}(\underline{X}) = \zeta^{LAS}(\underline{X}^{\perp})\frac{2\eta(1+\alpha)}{1 + 2\eta(1+\alpha) - \sqrt{4\alpha(1+\eta)(1+\alpha)+1}},$$

where the coeffcient $\frac{2\eta(1+\alpha)}{\left(1+2\eta(1+\alpha)-\sqrt{4\alpha(1+\eta)(1+\alpha)+1}\right)} \geq 1$ (see [Cossette et al., 2010]) aggregates the information about the degree of temporal dependence α. Note that $\frac{2\eta(1+\alpha)}{\left(1+2\eta(1+\alpha)-\sqrt{4\alpha(1+\eta)(1+\alpha)+1}\right)} = 1$

when $\alpha = 0$. The additional value of riskiness due to the temporal dependence is

$$\zeta^{LAS}\left(\underline{X}\right) - \zeta^{LAS}\left(\underline{X}^{\perp}\right) = \zeta^{LAS}\left(\underline{X}^{\perp}\right) \frac{1 - \sqrt{4\alpha\left(1+\eta\right)\left(1+\alpha\right)+1}}{1 + 2\eta\left(1+\alpha\right) - \sqrt{4\alpha\left(1+\eta\right)\left(1+\alpha\right)+1}} \geq 0.$$

By Proposition 10, it is shown that $\zeta^{LAS}\left(\underline{X}\right) - \zeta^{LAS}\left(\underline{X}^{\perp}\right)$ increases with α.

6.2.3 Numerical computation and recursive relations

The numerical computation of the ruin-based risk measures $\zeta^{VaR}_{\kappa,n}$ and $\zeta^{TVaR}_{\kappa,n}$ in the context of the discrete-time risk model based on Poisson MA(1) and in the context of the discrete-time risk model based on Poisson AR(1) is very similar. For that reason, we treat it in details within the second risk model in Section 6.3.3.

6.3 Risk model based on Poisson AR(1)

6.3.1 Definitions and result

In this risk model, $\underline{N} = \{N_k, k \in \mathbb{N}_+\}$ is a Poisson AR(1) process where the rv N_1 has a Poisson distribution with mean λ and the autoregressive dynamic is given by

$$N_k = \varepsilon_k + \alpha \circ N_{k-1}, \tag{47}$$

for $k \in \{2, 3, ...\}$. We assume that $\underline{\varepsilon} = \{\varepsilon_k, k \in \mathbb{N}_+\}$ is a sequence of iid rvs following a Poisson distribution with mean $(1-\alpha)\lambda$ where $\alpha \in [0, 1]$. The dynamic given in (47) yields a stationary sequence of Poisson rvs with mean λ. The autocorrelation function for $\underline{N}$ is equal to $\rho_N(h) = \alpha^h$, for $h \geq 1$ (see [McKenzie, 1988]) with $\rho_N(1) \in [0, 1)$. Letting $\alpha = 0$, $\underline{N}$ becomes a sequence of iid rvs which leads to the risk process, denoted here by $\underline{X}^{\perp}$, of the classical discrete-time risk model, presented in Section 5.1. The joint pmf of two successive components of the Poisson AR(1) process $\underline{N}$, (N_k, N_{k-1}), is given by

$$\Pr\left(N_k = n_k, N_{k-1} = n_{k-1}\right) = e^{-(2-\alpha)\lambda} \sum_{j=0}^{\min(n_k;n_{k-1})} \frac{\alpha^j \left(1-\alpha\right)^{n_k+n_{k-1}-2j} \lambda^{n_k+n_{k-1}-j}}{j!\left(n_k-j\right)!\left(n_{k-1}-j\right)!}, \tag{48}$$

for $n_{k-1}, n_k \in \mathbb{N}$ and $k \in \mathbb{N}_+$ (see, e.g. [McKenzie, 1988]).

Proposition 11 *Let $\underline{X}$ and $\underline{X}'$ be two risk processes, where $\lambda = \lambda'$, $B \sim B'$, and $0 \leq \alpha < \alpha' < 1$. Then,*

$$\left(N_1, ..., N_k\right) \preceq_{sm} \left(N_1', ..., N_k'\right), \tag{49}$$

and

$$\left(X_1, ..., X_k\right) \preceq_{sm} \left(X_1', ..., X_k'\right), \tag{50}$$

for $k \in \{2, 3, ...\}$. Consider a ruin-based risk measure ζ consistent under the supermodular order. Then, by Proposition 9, $\zeta\left(\underline{X}\right) \leq \zeta\left(\underline{X}'\right)$. In particular, $\zeta^{TVaR}_{\kappa,n}\left(\underline{X}\right) \leq \zeta^{TVaR}_{\kappa,n}\left(\underline{X}'\right)$, for $n \in \mathbb{N}_+$ and $\kappa \in (0, 1)$.

Proof. We omit the proof since it is very similar to the proof of Proposition 10. ■

6.3.2 Lundberg-Aumann-Serrano index of riskiness

The expression for $c(r)$ is provided in Proposition 5 of [Cossette et al., 2010]. Assuming that $\alpha\mathcal{M}_B(r) < 1$, the expression for $c(r)$ defined in (18) is given by

$$c(r) = \frac{(1-\alpha)^2 \lambda\mathcal{M}_B(r)}{1-(\alpha\mathcal{M}_B(r))} - (1-\alpha)\lambda - r\pi = \frac{(1-\alpha)^2 \lambda\mathcal{M}_B(r)}{1-(\alpha\mathcal{M}_B(r))} - (1-\alpha)\lambda - r\pi. \tag{51}$$

Applying Proposition 11, $\alpha \le \alpha'$ implies that $\zeta^{LAS}(\underline{X}) \le \zeta^{LAS}(\underline{X}')$, assuming that both indices exist.

Example 3 *The discrete-time risk model is defined with the risk process $\underline{X} = \{X_k, k \in \mathbb{N}_+\}$ where X_k are given in (37)) and (36), respectively. The claim number process is a Poisson AR(1) process with parameter λ and the claim amount rv B is exponentially distributed with mean $\frac{1}{\beta}$ and mgf $\mathcal{M}_B(r) = \frac{\beta}{\beta-r}$. Combining Proposition 4 of [Cossette et al., 2010] and (20), the Lundberg-Aumann-Serrano index of riskiness is given by the following nice and simple expression:*

$$\zeta^{LAS}(\underline{X}) = \frac{1}{\beta}\frac{1+\eta}{\eta}\frac{1}{1-\alpha} = \zeta^{LAS}\left(\underline{X}^{\perp}\right)\frac{1}{1-\alpha}, \tag{52}$$

where $\zeta^{LAS}\left(\underline{X}^{\perp}\right)$, given in (27), is the Lundberg-Aumann-Serrano index of riskiness corresponding to the risk process associated to the classical discrete-time risk model, as presented in Section 5.1. The contribution of riskiness due to the temporal dependence is

$$\zeta^{LAS}(\underline{X}) - \zeta^{LAS}\left(\underline{X}^{\perp}\right) = \zeta^{LAS}\left(\underline{X}^{\perp}\right)\frac{\alpha}{1-\alpha} \ge 0. \tag{53}$$

According to Proposition 10, $\zeta^{LAS}(\underline{X}) - \zeta^{LAS}\left(\underline{X}^{\perp}\right)$ becomes larger as α increases, which is obvious by (53).

6.3.3 Numerical computation and recursive relations

In this section, we examine the numerical computation of the ruin-based risk measures $\zeta^{VaR}_{\kappa,n}$ and $\zeta^{TVaR}_{\kappa,n}$. We make the following additional assumptions: the premium income $\pi \in \mathbb{N}^{\perp}$, the initial surplus $u \in \mathbb{N}$ and the rv B is defined on $\mathbb{N}_+$. All the formulas of the present section can be easily adapted to the case where the arithmetical support is $\{0, 1h, 2h, ...\}$ with $h > 0$. For the computation of $\zeta^{VaR}_{\kappa,n}$ and $\zeta^{TVaR}_{\kappa,n}$, we need the conditional probabilities

$$\varrho_{k_1,k_2} = \Pr(N_2 = k_2 | N_1 = k_1) = \frac{\Pr(N_2 = k_2, N_1 = k_1)}{\Pr(N_1 = k)}, \tag{54}$$

for $k_1, k_2 \in \mathbb{N}$ and where the values of the joint probabilities $\Pr(N_2 = k_2, N_1 = k_1)$ in (54) are computed with (48). The finite-time ruin probabilies are obtained with

$$\psi(u,n) = \overline{F}_{Z_n}(u) = \sum_{k_1=0}^{\infty} p_{k_1}\psi(u,n|N_1 = k_1),$$

where $\psi(u, n|N_1 = k_1) = \Pr(\tau \leq n|N_1 = k_1)$ are the conditional finite-time ruin probabilities, which are computed recursively. For $n = 1$, we have

$$\psi(u, 1|N_1 = k_1) = \sum_{l=u+c+1}^{\infty} f(l, k_1),$$

where

$$f(j; k_1) = \begin{cases} \Pr(B_1 + \ldots + B_{k_1} = j), & , \text{ if } k_1 \in \mathbb{N}_+ \\ 1 & , \text{ if } k_1 = 0 \text{ and } j = 0 \\ 0 & , \text{ if } k_1 = 0 \text{ and } j \neq 0 \end{cases}.$$

For $n \in \{2, 3, 4, \ldots\}$, we have

$$\psi(u, n|N_1 = k_1) = \sum_{l=u+\pi+1}^{\infty} f(l, k_1) + \sum_{k_2=0}^{\infty} \varrho_{k_1,k_2} \sum_{j=0}^{u+\pi} f(j, k_1)\psi(u + \pi - j, n - 1|N_1 = k_2),$$

for $u, k_1 \in \mathbb{N}$. Note that, within the discrete-time risk model defined with Poisson MA(1), the finite-time ruin probabilities $\psi(u, n)$ can be computed using the same procedure, with (42) in (54) rather than (48).

Example 4 *In the discrete-time risk model of this section, let the rv B follow a zero-truncated geometric distribution with $\Pr(B = k) = (1 - \upsilon)\upsilon^{k-1}$, for $k \in \mathbb{N}_+$ and $E[B] = \frac{1}{1-\upsilon}$, with $\upsilon = \frac{1}{3}$. Also, the Poisson parameter is $\lambda = 0.4$ and the premium is $\pi = 1$. In Table 1, we provide the values of $\zeta^{VaR}_{0.95,20}(\underline{X})$ and $\zeta^{TVaR}_{0.95,20}(\underline{X})$ for a dependence parameter $\alpha \in \{0, 0.2, 0.5, 0.8\}$. We also indicate the values of the expectation and the variance of Z_{20}.*

α	$E[Z_{20}]$	$Var(Z_{20})$	$\zeta^{VaR}_{0.95,20}(\underline{X})$	$\zeta^{TVaR}_{0.95,20}(\underline{X})$
0	4.2668	33.2646	12	17.7237
0.2	4.6316	40.9529	13	19.6544
0.5	5.4469	63.2907	16	24.3475
0.8	7.0413	135.7083	22	35.3784

Table 1: Results of Example 4.

The results obtained for $E[Z_{20}]$ and $TVaR_{0.95}(Z_{20})$ increase with the dependence parameter α. By Proposition 11 and because the ruin-based TVaR is consistent under the supermodular order, it follows that when the dependence relation between the losses becomes positively stronger, the risk process is riskier and the required amount of capital to be allocated to the portfolio must increase.

6.4 Risk model defined with a Markov Bernoulli Process

6.4.1 Definitions and result

We assume that the claim number process $\underline{N}$ is a Markov Bernoulli process, i.e., $\underline{N}$ is a Markov chain with state space $\{0, 1\}$ and with transition probability matrix

$$\underline{P} = \begin{pmatrix} 1 - (1 - \alpha) q & (1 - \alpha) q \\ (1 - \alpha)(1 - q) & \alpha + (1 - \alpha) q \end{pmatrix} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}, \quad (55)$$

where α can be seen as the dependence parameter, introducing a positive dependence relation between the claim number rvs. In this risk model, at most one claim can occur over a period. The initial probabilities associated to $\underline{P}$ are $\Pr(N_0 = 1) = q = 1 - \Pr(N_0 = 0)$, where $0 \leq \alpha < 1$ and $0 < q < 1$. When α tends to 1, a period with a (no) claim will be likely followed by a period with a (no) claim. The covariance between N_k and N_{k+h} is given by $Cov(N_k, N_{k+h}) = q(1-q)\alpha^h$, for $k \in \mathbb{N}_+$ and $h \in \mathbb{N}$. When $\alpha = 0$, the claim number process $\underline{N}$ becomes a sequence of iid rvs and the corresponding risk process, denoted by $\underline{X}^{\perp}$, is the risk process of the classical discrete-time risk model, presented in Section 5.1. See, e.g., [Cossette et al., 2010] for details on this risk model with temporal dependence.

Proposition 12 *Let $\underline{X}$ and $\underline{X}'$ be two risk processes, where $q = q'$, $B \sim B'$, and $0 \leq \alpha < \alpha' < 1$. Then,*

$$(N_1, ..., N_k) \preceq_{sm} (N'_1, ..., N'_k), \tag{56}$$

for $k \in \{2, 3, ...\}$. Consider a ruin-based risk measure ζ consistent under the supermodular order. Then, by Proposition 9, $\zeta(\underline{X}) \leq \zeta(\underline{X}')$.

Proof. Let

$$\underline{P}' = \begin{pmatrix} 1 - (1-\alpha')\,q & (1-\alpha')\,q \\ (1-\alpha')(1-q) & \alpha' + (1-\alpha')\,q \end{pmatrix}$$

and $\underline{I}$ be a 2×2 identity matrix. Fix $c = \frac{1-\alpha'}{1-\alpha}$. Since $0 \leq \alpha < \alpha' < 1$, it implies that $c \in (0,1)$. Because $\underline{P}'$ is of the form $\underline{P}' = (1-c) \times \underline{I} + c \times \underline{P}$ and since N_1 (and N_2) is stochastically increasing in N_2 (in N_1), then (56) follows from Corollary 3.1 of [Hu and Pan, 2000]. ■

6.4.2 Lundberg-Auman-Serrano index of riskiness

Proposition 8 of [Cossette et al., 2010] provides the following expression of $c(r)$ in (17):

$$c(r) = \ln\left\{(p_{00} + p_{11}\mathcal{M}_B(r)) + \sqrt{(p_{00} + p_{11}\mathcal{M}_B(r))^2 - 4\mathcal{M}_B(r)(p_{00}p_{11} - p_{10}p_{01})}\right\} - \ln 2 - \pi r, \tag{57}$$

where $(p_{00}p_{11} - p_{10}p_{01}) = \alpha$. When the claim amount rv B follows an exponential distribution, there is no closed-form expression of Lundberg-Auman-Serrano index of riskiness. Numerical optimization has to be used to find the solution of (57) and, afterwards, the value $\zeta^{LAS}(\underline{X})$. By Proposition 12, $\zeta^{LAS}(\underline{X}) \leq \zeta^{LAS}(\underline{X}')$ for $0 \leq \alpha < \alpha' < 1$.

6.4.3 Compound Markov binomial risk model

Introduced by [Cossette et al., 2003], the compound Markov binomial risk model is a special case of the risk model defined with a Markov Bernoulli process and a generalization of the compound binomial risk model discussed in Section 5.3. As shown in [Cossette et al., 2003], finite-time and infinite-time ruin probabilities $\psi(u,n)$ and $\psi(u)$, for $u \in \mathbb{N}$, can be computed recursively. Details on the compound Markov binomial risk model can be found in, e.g. [Cossette et al., 2003], [Cossette et al., 2004], [Yuen and Guo, 2006], and [Arvidsson and Francke, 2007].

Let $\underline{X}^{\perp}$ be the risk process associated to the corresponding compound binomial risk model discussed in Section 5.3. Since the ruin-based risk measure $\zeta_{\kappa,n}^{TVaR}$ (or ζ_{κ}^{TVaR}) is consistent under the supermodular order and by Proposition 12, we can now identify the extra amount of capital that needs to be allocated to the portfolio due to the positive temporal dependence between the components of $\underline{X}$ which is given by $\zeta_{\kappa,n}^{TVaR}(\underline{X}) - \zeta_{\kappa,n}^{TVaR}(\underline{X}^{\perp})$ (or $\zeta_{\kappa}^{TVaR}(\underline{X}) - \zeta_{\kappa}^{TVaR}(\underline{X}^{\perp})$).

Example 5 *We consider the compound Markov binomial risk model. Let the rv B follow a zero-truncated geometric distribution with $\Pr(B=k) = (1-\upsilon)\upsilon^{k-1}$, for $k \in \mathbb{N}_+$ and $E[B] = \frac{1}{1-\upsilon}$, with $\upsilon \in (0,1)$. The parameters q and υ are fixed such that $\frac{q}{1-\upsilon} < 1$. In [Cossette et al., 2004], it is proven that*

$$\psi(u) = \overline{F}_Z(u) = 1 - F_Z(u) = \psi(0)\left(\frac{\upsilon}{p_{00} - \alpha(1-\upsilon)}\right)^u, \tag{58}$$

for $u \in \mathbb{N}$ and with $\psi(0) = q\frac{(E[B]-1)+\alpha\upsilon+(1-\alpha)(E[B]-1)}{p_{00}-\alpha(1-\upsilon)}$. The following expression of $\zeta_{\kappa}^{VaR}(\underline{X})$ is derived from (58):

$$\zeta_{\kappa}^{VaR}(\underline{X}) = F_Z^{-1}(\kappa) = \begin{cases} 0 & , \quad 0 < \kappa \leq 1-\psi(0) \\ \left\lceil \frac{\ln\left(\frac{1-\kappa}{\psi(0)}\right)}{\ln\left(\frac{\upsilon}{p_{00}-\alpha(1-\upsilon)}\right)} \right\rceil & , \quad 1-\psi(0) < \kappa < 1 \end{cases}, \tag{59}$$

for $\kappa \in (0,1)$. From (58), we find that

$$\pi_Z(u) = E[\max(Z-u;0)] = \sum_{k=u}^{\infty} \psi(k) = \psi(0)\frac{p_{00}-\alpha(1-\upsilon)}{p_{00}-\alpha(1-\upsilon)-\upsilon}\left(\frac{\upsilon}{p_{00}-\alpha(1-\upsilon)}\right)^u, \tag{60}$$

for $u \in \mathbb{N}$. Combining (59) and (60) into (16), we obtain

$$\zeta_{\kappa}^{TVaR}(\underline{X}) = \begin{cases} \psi(0)\frac{1-q}{1-q-\upsilon} & , \quad 0 < \kappa \leq 1-\psi(0) \\ \left\lceil \frac{\ln\left(\frac{1-\kappa}{\psi(0)}\right)}{\ln\left(\frac{\upsilon}{p_{00}-\alpha(1-\upsilon)}\right)} \right\rceil + \frac{\psi(0)}{1-\kappa}\frac{p_{00}-\alpha(1-\upsilon)}{p_{00}-\alpha(1-\upsilon)-\upsilon}\left(\frac{\upsilon}{p_{00}-\alpha(1-\upsilon)}\right)^{\left\lceil \frac{\ln\left(\frac{1-\kappa}{\psi(0)}\right)}{\ln\left(\frac{\upsilon}{p_{00}-\alpha(1-\upsilon)}\right)} \right\rceil} & , \quad 1-\psi(0) < \kappa < 1 \end{cases},$$

for $\kappa \in (0,1)$.

6.5 Risk model defined in a Markovian environment

6.5.1 Definitions and result

We assume that the claim number process $\underline{N} = \{N_k, k \in \mathbb{N}_+\}$ is influenced by an underlying Marvovian environment represented by the time homogeneous Markov chain $\underline{\Theta}$ defined over the 2-state space $\{\theta_1, \theta_2\}$ with transition probabilities $p_{ij} = \Pr(\Theta_{k+1} = \theta_j | \Theta_k = \theta_i)$, for $k \in \mathbb{N}_+$. We assume that the conditional pmf of $(N_k | \Theta_k = \theta_j)$ $(k \in \mathbb{N}_+)$ is $f_{N|\Theta=\theta_j}$ and the conditional cdf is $F_{N|\Theta=\theta_j}$, $j \in \{1,2\}$. Assume that the conditional distribution of $(N_k|\Theta_k = \theta_j)$ is Poisson with mean λ_j $(j = 1,2)$ with $\lambda_1 \leq \lambda_2$. The transition probability matrix $\underline{P}$ of $\underline{\Theta}$ is denoted by

$$\underline{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 1-(1-\nu)\chi & (1-\nu)\chi \\ (1-\nu)(1-\chi) & \nu+(1-\nu)\chi \end{pmatrix},$$

with $0 < \chi < 1$. To avoid negative transition probabilities, we assume $0 \leq \nu < 1$. The stationary probabilities associated to $\underline{P}$ are $\Pr(\Theta_k = \theta_1) = \frac{p_{21}}{p_{12}+p_{21}} = 1-\chi$ and $\Pr(\Theta_k = \theta_2) = \frac{p_{12}}{p_{12}+p_{21}} = \chi$. When $\nu = 0$, $\underline{\Theta}$ forms a sequence of iid rvs, where the corresponding risk process, $\underline{X}^{\perp}$, is the risk process of the classical discrete-time risk model, presented in Section 5.1.

Definition 11 *Let* $\underline{V}(\psi)$ *be a finite or infinite sequence of rvs* $\{V_k(\psi), k \in \mathbb{N}_+\}$ *where the vector of parameters* ψ *belongs to a subset of* $\mathbb{R}$. *Then,* $\underline{X}$ *is stochastically increasing in* ψ *if* $V_k(\psi_1) \preceq_{st} V_k(\psi_2)$ *whenever* $\psi_1 \leq \psi_2$, *for* $k \in \mathbb{N}_+$.

Definition 12 *Let* $\underline{\Psi} = \{\Psi_k,\ k \in \mathbb{N}_+\}$ *be a sequence of rvs. Then,* $\underline{\Psi}$ *is sequentially stochastically increasing if, for all* n, $\{\Psi_{n+1} | \Psi_n = y_n, ..., \Psi_1 = y_1\}$ *is stochastically increasing in* $(y_1, ..., y_n)$.

Proposition 13 *Let* $\underline{X}$ *and* $\underline{X}'$ *be two risk processes, where* $\lambda_1 = \lambda_1'$, $\lambda_2 = \lambda_2'$, $\chi = \chi_1'$, $B \sim B'$, *and* $0 \leq \nu < \nu' < 1$. *Then,*

$$(N_1, ..., N_k) \preceq_{sm} \left(N_1', ..., N_k'\right), \tag{61}$$

for $k \in \{2, 3, ...\}$. *Consider a ruin-based risk measure* ζ *consistent under the supermodular order. Then, by Proposition 9,* $\zeta(\underline{X}) \leq \zeta(\underline{X}')$.

Proof. Let

$$\underline{P}' = \begin{pmatrix} 1-(1-\nu')\chi & (1-\nu')\chi \\ (1-\nu')(1-\chi) & \nu' + (1-\nu')\chi \end{pmatrix}.$$

Let $c = \frac{1-\nu'}{1-\nu}$, such that $c \in (0,1)$ since $0 \leq \nu < \nu' < 1$. Because $\underline{P}'$ is of the form $\underline{P}' = (1-c) \times \underline{I} + c \times \underline{P}$ and since Θ_1 (and Θ_2) is stochastically increasing in Θ_2 (in Θ_1), then, by Corollary 3.1 of [Hu and Pan, 2000], we have $(\Theta_1, ..., \Theta_k) \preceq_{sm} (\Theta_1', ..., \Theta_k')$, for $k \in \{2, 3, ...\}$. Clearly, since $\lambda_1 \leq \lambda_2$, $\underline{N}(\underline{\Theta})$ and $\underline{N}'(\underline{\Theta}')$ are stochastically increasing. Then, (61) follows from Theorem 3.1 of [Lillo and Semeraro, 2004]. ■

Note that the claim amounts are assumed not to be affected by the Markovian process. The dependence parameter ν indicates the strength of the dependence relation between the claim number rvs. As the parameter ν increases, the risk process for the portfolio becomes riskier. Then, if a risk manager uses a ruin-based risk measure consistent under the supermodular order, such as $\zeta_{\kappa,n}^{TVaR}$, he must allocate a larger amount of capital to the portfolio in order to hedge the riskiness of the insurance portfolio.

6.5.2 Lundberg-Auman-Serrano index of riskiness

The expression for $c(r)$, provided in Proposition 10, is given by

$$c(r) = \ln\left\{ \frac{\left(p_{11}\mathcal{M}_W^{(1)}(r) + p_{22}\mathcal{M}_W^{(2)}(r)\right)}{+\sqrt{\left(p_{11}\mathcal{M}_W^{(1)}(r) + p_{22}\mathcal{M}_W^{(2)}(r)\right)^2 - 4\mathcal{M}_W^{(1)}(r)\,\mathcal{M}_W^{(2)}(r)\,(p_{11}p_{22} - p_{12}p_{21})}} \right\} - \ln 2 - \pi r,$$

where $(p_{11}p_{22} - p_{12}p_{21}) = \nu$ and $\mathcal{M}_W^{(j)}(r) = E\left[e^{rW} | \Theta = \theta_j\right] = e^{\lambda_j(\mathcal{M}_B(r)-1)}$, $j = 1, 2$. When the distribution of B is exponential, the expression of Lundberg-Auman-Serrano index of riskiness is not closed-form. Numerical optimization must be used to compute the solution of (57) and, afterwards, the value $\zeta^{LAS}(\underline{X})$. From Proposition 13, $\zeta^{LAS}(\underline{X}) \leq \zeta^{LAS}(\underline{X}')$ for $0 \leq \nu < \nu' < 1$. .

7 Classical discrete-time risk model with m lines of business and capital allocation based on Euler principle

7.1 Definitions and result

To simplify the presentation, we consider an insurance portfolio composed of m different lines of business within the context of the classical discrete-time risk model. The risk process for the insurance portfolio is denoted $\underline{X} = \{X_k, k \in \mathbb{N}_+\}$, with $X_k = \sum_{i=1}^{m} C_{i,k}$, where the rv $C_{i,k}$ is the net losses in period k for the risk i ($i = 1, 2, ..., m$, $k \in \mathbb{N}_+$). We assume that $\{(C_{1,k}, ..., C_{m,k}), \ k \in \mathbb{N}_+\}$ forms a sequence of identically distributed random vectors. Also, we have $(C_{1,k}, ..., C_{m,k}) \sim \underline{C} = (C_1, ..., C_m)$ and $X_k \sim X = \sum_{i=1}^{m} C_i$. An homogeneous and subadditive ruin-based risk measure ζ is chosen to make a risk assessment of the portfolio. By subadditivity property, we know that $\zeta(\underline{X}) = \zeta(\sum_{i=1}^{m} \underline{C}_i) \leq \sum_{i=1}^{m} \zeta(\underline{C}_i)$, where $\underline{C}_i = \{C_{k,i}, k \in \mathbb{N}_+\}$, for $i = 1, 2, ..., m$.

Proposition 14 *Consider the classical discrete-time risk model with m lines of business. Assume that*

$$(C_{1,k}, ..., C_{m,k}) \preceq_{sm} \left(C'_{1,k}, ..., C'_{m,k}\right), \tag{62}$$

for $k \in \mathbb{N}_+$. Then, for a ruin-based risk measure ζ consistent under the increasing convex order, we have $\zeta(\underline{X}) \leq \zeta(\underline{X}')$.

Proof. From (9.A.19) of [Shaked and Shanthikumar, 2007], (62) implies that $X_k \preceq_{cx} X'_k$, and, therefore,

$$X_k \preceq_{icx} X'_k, \tag{63}$$

for $k \in \mathbb{N}_+$. Combining (63) and Proposition 2.1, we have $\underline{X} \preceq_{icx} \underline{X}'$. Since ζ consistent under the increasing convex order, it follows that $\zeta(\underline{X}) \leq \zeta(\underline{X}')$. ■

Example 13.2.4 in [Cossette and Marceau, 2013] provides an illustration of the result of Proposition 14 with $\zeta_{\kappa,n}^{VaR}$ and $\zeta_{\kappa,n}^{TVaR}$. It also contains a counterexample showing that the ruin-based VaR is not subadditive.

7.2 Capital allocation based on Euler principle

We examine the computation of the contribution of each line of business to the overall risk of the portfolio. This is an important topic in actuarial science and quantitative risk management and this problem has received much attention over the last decade. In order to compute the contribution of each component, we apply Euler's allocation principle (see e.g. Tasche (1999), McNeil et al. (2005), and Rosen et al. (2011) for details on capital allocation rules and Euler's allocation principle). Let us define

$$\underline{X}\left(\underline{\beta}\right) = \left\{X_k\left(\underline{\beta}\right), k \in \mathbb{N}_+\right\} = \left\{\sum_{i=1}^{m} \beta_i C_{i,k}, k \in \mathbb{N}_+\right\},$$

where $\underline{\beta} = (\beta_1, ..., \beta_m)$. For a given homogeneous ruin-based risk measure ζ, the contribution of the component i to $\zeta(\underline{X})$ is given by

$$\zeta(\underline{X}; \underline{C}_i) = \beta_i \left.\frac{\partial}{\partial \beta_i} \zeta(X(\underline{\lambda}))\right|_{\underline{\lambda}=\underline{1}}, \tag{64}$$

for $i = 1, 2, ..., m$ and where $\underline{1} = (1, ..., 1)$.

Example 6 *We consider the classical discrete-time risk model with $m = 2$ lines of business, where $X = C_1 + C_2$ with $C_1 = W_1 - \pi_1$ and $C_2 = W_2 - \pi_2$. Also, $\underline{W} = (W_1, W_2)$ follows a bivariate compound Poisson distribution with parameters $\lambda_1 > 0, \lambda_2 > 0, 0 \leq \gamma_0 \leq \min(\lambda_1, \lambda_2)$. Denote $\gamma_1 = \lambda_1 - \gamma_0$ and $\gamma_2 = \lambda_2 - \gamma_0$. For line i, a claim amount is exponentially distributed with $\mathcal{M}_{B_i}(r) = (1 + \mu_i r)^{-1}$, $i \in \{1, 2\}$. The joint mgf of $\underline{W}$ is*

$$\mathcal{M}_{\underline{X}}(r_1, r_2) = \mathrm{e}^{\gamma_1 \left(\mathcal{M}_{B_1}(r_1) - 1\right)} \mathrm{e}^{\gamma_2 \left(\mathcal{M}_{B_2}(r_2) - 1\right)} \mathrm{e}^{\gamma_0 \left(\mathcal{M}_{B_1}(r_1) \mathcal{M}_{B_2}(r_2) - 1\right)} \tag{65}$$

and $\mathcal{M}_{\underline{C}}(r_1, r_2) = \mathcal{M}_{\underline{X}}(r, r)\, \mathrm{e}^{-\pi_1 r_1 - \pi_2 r_2}$. Also, the mgf of X is $\mathcal{M}_X(r) = \mathcal{M}_{\underline{W}}(r, r)\, \mathrm{e}^{-\pi r}$ where the premium income is $\pi_1 > \mu_1 \lambda_1$, $\pi_2 > \mu_2 \lambda_2$, and $\pi = \pi_1 + \pi_2$. Using (17) and (18) with (65), the closed-form expression for the Lundberg-Aumann-Serrano index of riskiness is

$$\zeta^{LAS}(\underline{X}) = \frac{2\pi \mu_1 \mu_2}{\chi - \sqrt{\chi^2 - \pi \mu_1 \mu_2 (\pi - \lambda_1 \mu_1 - \lambda_2 \mu_2)}}, \tag{66}$$

with

$$\chi = \pi(\mu_1 + \mu_2) - (\lambda_1 + \lambda_2 - \gamma_0).$$

From Proposition 3 and Example 4 of [Denuit et al., 2002], if $0 \leq \gamma_0 \leq \gamma_0' \leq \min(\lambda_1, \lambda_2)$, then $(C_{1,k}, C_{2,k}) \preceq_{sm} \left(C'_{1,k}, C'_{2,k}\right)$ and, by Proposition 14, $\zeta^{LAS}(\underline{X}) - \zeta^{LAS}(\underline{X}') \geq 0$. Applying (64) to (66), we find the following closed-form expression of Euler's contribution of line 1 to $\zeta^{LAS}(\underline{X})$:

$$\zeta^{LAS}(\underline{X}; \underline{C}_1) = \frac{1}{\chi - \sqrt{\chi^2 - \pi \mu_1 \mu_2 (\pi - \lambda_1 \mu_1 - \lambda_2 \mu_2)}} \times \vartheta, \tag{67}$$

where

$$\vartheta = 4\pi_1 \mu_1 \mu_2 + 2 \times \mu_1 \pi_2 \mu_2 - \zeta^{LAS}(\underline{X})(2\pi_1 \mu_1 + \pi_2 \mu_1 + \pi_1 \mu_2) - \frac{\zeta^{LAS}(\underline{X})}{2} \xi$$

and

$$\xi = \begin{array}{c} 2(\pi_1 \mu_1 + \pi_2 \mu_1 + \pi_1 \mu_2 + \pi_2 \mu_2 - (\lambda_1 + \lambda_2 - \gamma_0))(2\pi_1 \mu_1 + \pi_2 \mu_1 + \pi_1 \mu_2) \\ -3\pi_1 \mu_1 \mu_2 \times (\pi_1 - \lambda_1 \mu_1) - \mu_1 \pi_2 \mu_2 \times (\pi_2 - \lambda_2 \mu_2) \\ -2\pi_1 \mu_1 \mu_2 \times (\pi_2 - \lambda_2 \mu_2) - 2\mu_1 \pi_2 \mu_2 \times (\pi_1 - \lambda_1 \mu_1). \end{array}$$

The expression of the closed-form expression for $\zeta^{LAS}(\underline{X}; \underline{C}_2)$ is similar to the one of $\zeta^{LAS}(\underline{X}; \underline{C}_1)$ in (67).

Generally, there is no closed-form expression for $\zeta(\underline{X}; \underline{C}_i)$. For a given $\varepsilon > 0$ (e.g. $\varepsilon = 10^{-3}$ or 10^{-4}), $\zeta(\underline{X}; \underline{C}_i)$ can be approximated by $\widetilde{\zeta}^{\varepsilon}(\underline{X}; \underline{C}_i)$, i.e.,

$$\zeta(\underline{X}; \underline{C}_i) \simeq \widetilde{\zeta}^{\varepsilon}(\underline{X}; \underline{C}_i) = \frac{\zeta\left(\left\{\sum_{j=1, j \neq i}^{m} C_{j,k} + (1 + \varepsilon) C_{i,k}, k \in \mathbb{N}_+\right\}\right) - \zeta\left(\left\{\sum_{j=1}^{m} C_{j,k}, k \in \mathbb{N}_+\right\}\right)}{\varepsilon}. \tag{68}$$

In the following example, we use approximation (68) with the finite-time ruin-based TVaR.

Example 7 *We assume the classical discrete-time risk model with $m = 2$ lines of business, where $m = 2$, $C_i = W_i - \pi_i$, $W_i \sim Gamma(\alpha_i, \beta_i)$, $\pi_i = (1 + \eta_i) E[W_i]$ and $\eta_i \geq 0$, $i \in \{1, 2\}$. We first consider the case where W_1 and W_2 are independent. The values of $\widetilde{\zeta}_{\kappa,n}^{TVaR;\varepsilon}(\underline{X})$ and $\widetilde{\zeta}_{\kappa,n}^{TVaR;\varepsilon}(\underline{X}; \underline{C}_i)$ (with (68)) for $n = 5$ periods, $\varepsilon = 10^{-4}$, and $\kappa \in \{0, 0.5, 0.9, 0.99, 0.999\}$ are provided in Table 2. Secondly, we suppose that the joint cdf of (W_1, W_2) is given by $F_{W_1, W_2}(x_1, x_2) =$*

$C_\theta\left(F_{W_1}(x_1), F_{W_2}(x_2)\right)$, *where* $C_\theta(u_1,u_2) = \left(u_1^{-\theta}+u_2^{-\theta}-1\right)^{-1/\theta}$ *is a Clayton copula. The values of* $\widetilde{\zeta}_{\kappa,n}^{TVaR;\varepsilon}\left(\underline{X};\underline{C}_i\right)$ *for* $n=5$ *periods,* $\varepsilon = 10^{-4}$*, and* $\kappa \in \{0, 0.5, 0.9, 0.99, 0.999\}$ *are provided in Table 3 for a Kendall tau* $\tau = 0.25$ *(*$\theta = 2/3$*) and in Table 4 for a Kendall tau* $\tau = 0.75$ *(*$\theta = 6$*). Values of the three tables are computed with* 10^7 *Monte-Carlo simulations. From [Wei and Hu, 2002], if* $0 \leq \theta \leq \theta$*, then* $(C_{1,k}, C_{2,k}) \preceq_{sm} \left(C'_{1,k}, C'_{2,k}\right)$*, for* $k \in \mathbb{N}_+$*, and, by Proposition 14,* $\zeta_{\kappa,n}^{TVaR}\left(\underline{X}\right) - \zeta_{\kappa,n}^{TVaR}\left(\underline{X}'\right) \geq 0$*, for any* $n \in \mathbb{N}_+$ *(as observed in the three tables).*

i	1		2		Total
(α_i,β_i)	$(1/2, 1/200)$		$(2, 1/50)$		–
$(E[W_i], \eta_i, c_i)$	$(100, 10\%, 110)$		$(100, 10\%, 110)$		–
$\widetilde{\zeta}_{0,5}^{TVaR;0.0001}\left(\underline{X};\underline{C}_i\right)$	129.1255	83.55%	25.4226	16.45%	154.5481
$\widetilde{\zeta}_{0.50,5}^{TVaR;0.0001}\left(\underline{X};\underline{C}_i\right)$	257.9955	84.29%	48.0685	15.70%	306.0640
$\widetilde{\zeta}_{0.9;5}^{TVaR;0.0001}\left(\underline{X};\underline{C}_i\right)$	647.0763	90.17%	70.4942	9.82%	717.5706
$\widetilde{\zeta}_{0.99;5}^{TVaR;0.0001}\left(\underline{X};\underline{C}_i\right)$	1183.6032	93.30%	84.8681	6.69%	1268.4714
$\widetilde{\zeta}_{0.999;5}^{TVaR;0.0001}\left(\underline{X};\underline{C}_i\right)$	1702.8767	94.90%	91.4154	5.09%	1794.2922

Table 2: Results for Example 7, when C_1 and C_2 are independent.

i	1		2		Total
(α_i,β_i)	$(1/2, 1/200)$		$(2, 1/50)$		–
$(E[W_i], \eta_i, \pi_i)$	$(100, 10\%, 110)$		$(100, 10\%, 110)$		–
$\widetilde{\zeta}_{0,5}^{TVaR;0.0001}\left(\underline{X};\underline{C}_i\right)$	133.5568	78.09%	37.4792	21.91%	171.0360
$\widetilde{\zeta}_{0.50,5}^{TVaR;0.0001}\left(\underline{X};\underline{C}_i\right)$	266.1971	78.94%	71.0357	21.06%	337.2330
$\widetilde{\zeta}_{0.9;5}^{TVaR;0.0001}\left(\underline{X};\underline{C}_i\right)$	653.3705	85.10%	114.3534	14.90%	767.7240
$\widetilde{\zeta}_{0.99;5}^{TVaR;0.0001}\left(\underline{X};\underline{C}_i\right)$	1185.3153	88.99%	146.7213	11.01%	1332.0366
$\widetilde{\zeta}_{0.999;5}^{TVaR;0.0001}\left(\underline{X};\underline{C}_i\right)$	1700.7817	91.19%	164.3056	8.81%	1865.0873

Table 3: Results for Example 7, when the joint cdf of (C_1, C_2) is defined with the Clayton copula $(\theta = \frac{2}{3})$.

8 Conclusion

Ruin theory is a convenient framework to assess the riskiness of an insurance business. It accounts for the insured risk during the whole lifetime of the business (infinite-time horizon) or until any given time-horizons (finite-time horizon). This is why actuaries often look at risks in the ruin context when building risk models. Risk measures derived from ruin theory provide robust risk indicators.

In this paper, we have considered four ruin-based risk measures within discrete-time risk models, each of them providing a different angle of the risk covered by the insurance company. We have

i	1		2		Total
(α_i, β_i)	$(1/2, 1/200)$		$(2, 1/50)$		–
$(E[W_i], \eta_i, \pi_i)$	$(100, 10\%, 110)$		$(100, 10\%, 110)$		–
$\tilde{\zeta}_{0,5}^{TVaR;0.0001}(\underline{X}; \underline{C}_i)$	144.5699	70.51%	60.4765	29.49%	205.0465
$\tilde{\zeta}_{0.50,5}^{TVaR;0.0001}(\underline{X}; \underline{C}_i)$	286.5455	71.03%	116.8454	28.97%	403.3909
$\tilde{\zeta}_{0.9;5}^{TVaR;0.0001}(\underline{X}; \underline{C}_i)$	674.8532	75.77%	215.8191	24.23%	890.6723
$\tilde{\zeta}_{0.99;5}^{TVaR;0.0001}(\underline{X}; \underline{C}_i)$	1200.1897	79.80%	303.7203	20.20%	1503.9099
$\tilde{\zeta}_{0.999;5}^{TVaR;0.0001}(\underline{X}; \underline{C}_i)$	1712.8778	82.67%	359.0990	17.33%	2071.9768

Table 4: Results for Example 7, when the joint cdf of (C_1, C_2) is defined with the Clayton copula ($\theta = 6$).

discussed several properties fulfilled by these risk measures and when possible, we have provided explicit forms within the classical discrete-time risk model and the discrete-time risk models with temporal dependence on the one hand, and examined the impact of the temporal dependence throughout different risk models on the other hand. Finally, we have discussed capital allocation issues based on the Euler principle and provided some numerical illustrations with different dependence structures between the subrisks considered.

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