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The net Bayes premium with dependence between the risk profiles

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ABSTRACT

In Bayesian analysis it is usual to assume that the risk profiles Θ_1 and Θ_2 associated with the random variables "number of claims" and "amount of a single claim", respectively, are independent. A few studies have addressed a model of this nature assuming some degree of dependence between the two random variables (and most of these studies include copulas). In this paper, we focus on the collective and Bayes net premiums for the aggregate amount of claims under a compound model assuming some degree of dependence between the random variables Θ_1 and Θ_2. The degree of dependence is modelled using the Sarmanov–Lee family of distributions [Sarmanov, O.V., 1966. Generalized normal correlation and two-dimensional Frechet classes. Doklady (Soviet Mathematics) 168, 596–599 and Ting-Lee, M.L., 1996. Properties and applications of the Sarmanov family of bivariate distributions. Communications Statistics: Theory and Methods 25 (6) 1207–1222], which allows us to study the impact of this assumption on the collective and Bayes net premiums. The results obtained show that a low degree of correlation produces Bayes premiums that are highly sensitive.

1. Introduction

It is well known that the collective or compound risk model (hereafter CRM) is described by a frequency distribution for the number of claims N and a sequence of independent and identically distributed random variables representing the size of the single claims X_i. Frequency N and severity X_i are assumed to be independent. Note that this assumed independence is conditional on distribution parameters. There is an extensive body of literature on modelling the risk process, see e.g. McNeil et al. (2005).

Many studies have been published of CRMs in which the primary distribution is a Poisson distribution (for N distribution) with a θ_1 parameter with domain in $(0, \infty)$, taken as a realization of a random variable Θ_1, and in which the secondary distribution (for the claim size) is an exponential distribution with a θ_2 parameter also with domain in $(0, \infty)$, again considered to be a realization of a random variable Θ_2 (hereafter CRM.PE). Θ_1 and Θ_2 are often termed as risk profiles.

A set of interesting results on the sum of random variables with an exponential distribution and a random number of summands is presented in Kozubowski and Panoska (2005). Furthermore, a comprehensive collection of approximate forms for the compound mixed Poisson distribution is found in Nadarajah and Kotz (2006a,b). A robustness analysis for the Bayes premium in the CRM Poisson-exponential can be seen in Hernández-Bastida et al. (2009).

Determination of the premiums is a major and interesting problem in insurance theory. In particular, computation of the Bayes premium is the natural aim when prior information and claim experience are considered (see Klugman (1992, p. 64)). In Bayesian actuarial literature, it is usually assumed that the risk profiles Θ_1 and Θ_2 are independent. However, a small body of literature has been developed in which different degrees of dependence in the model, most of them including copulas, have been developed. The issue of dependence between random variables has been discussed in application areas such as project risk analysis (see e.g. Duffley and van Dorp (1998), Yi and Bier (1998), Clemen and Reilly (1999) and Härdle et al. (2002)). An excellent exposition for actuaries on the concept of copula, including an extensive and valuable bibliography is made in Frees and Valdez (1998).

On the other hand, an attractive way to model the dependence between risks is via the dependence between risk profiles (see Shevchenko and Wüthrich (2006, p. 21)). In this line, the incorporation of the dependence by means of the random variables Θ_1 and Θ_2 in a CRM is studied in the present paper. Specifically, we assume that the structure of the dependence between the parameters is captured in the prior bivariate distribution with specified

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marginal distributions. We focus on the collective and Bayes net premiums for the aggregate amount of claims under a compound model, assuming some degree of dependence between the random variables Θ_1 and Θ_2.

There exists an extensive body of literature on the construction of bivariate distributions (see Drouet and Kotz (2001) or Kotz and van Dorp (2002) and the references therein). We focus on the prior bivariate distribution, with specified marginal distributions from the Sarmanov–Lee family of distributions (Sarmanov, 1966; Ting-Lee, 1996). This family is implemented as follows: assume $f_1(x_1)$ and $f_2(x_2)$ are univariate probability density functions or probability mass functions, with supports defined on $A_1 \subseteq \mathbb{R}$ and $A_2 \subseteq \mathbb{R}$, respectively. Let $\phi_i(t)$, $i = 1, 2$, be bounded nonconstant functions such that $\int_{-\infty}^{\infty} \phi_i(t) f_i(t)\mathrm{d}t = 0$ (the mixing functions). Then, the function defined by

$$h(x_1, x_2) = f_1(x_1) f_2(x_2) \left[1 + \omega\, \phi_1(x_1)\phi_2(x_2)\right] \tag{1}$$

is a genuine bivariate joint density (or a probability mass function) with specified marginals $f_1(x_1)$ and $f_2(x_2)$, provided ω is a real number which satisfies the condition $1 + \omega\, \phi_1(x_1)\phi_2(x_2) \geq 0$ for all x_1 and x_2. The Sarmanov–Lee family is a special case of Cohen's 1984 construction. This family of bivariate densities includes some of the Farlie–Gumbel–Morgenstern (Farlie, 1960) distributions as special cases.

Note that the formulation here is that the uncertainty is described in the marginal distributions with the dependence being captured by means of the mixing functions and the ω parameter.

The remainder of this paper is structured as follows. Section 2 introduces the model to be considered, with special attention to the prior bivariate distribution. Section 3 describes and analyzes the collective and Bayes premiums. The numerical study highlights the influence of the level of dependence between Θ_1 and Θ_2 on the value of the Bayes premium. Furthermore, the following aspect is explored: it is evident that determining the Bayes premium is frequently difficult; it is straightforward to propose the premium as the product of the Bayes premium if only the number of claims is considered, and to propose the Bayes premium if only the severity is considered (see for example Frangos and Vrontos (2001)). This methodology can be seen as an extreme case of independence, with the two random processes in the problem being analyzed separately. Here, we provide a numerical illustration of the difference between the Bayes premium and the premium proposed in the manner indicated above. Section 3 also contains numerical illustrations and our main conclusions are drawn in Section 4. Appendix A describes the constants used in the text and Appendix B contains several proofs to facilitate comprehension of the various results.

2. The model

The CRM is a sequence $N, X_1, X_2, \ldots,$ of random variables, with the following meaning: N is the random variable number of accidents or claims. We assume it has a Poisson distribution with parameter $\theta_1 > 0$, with the moment generating function $M_1(t; \theta_1) = \mathrm{e}^{\theta_1(\mathrm{e}^t - 1)}$. X_i, for $i = 1, 2, \ldots,$ is the random variable claim size, and we assume it follows an exponential distribution of parameter $\theta_2 \geq 0$, with $f(x_i|\theta_2) = \theta_2 \mathrm{e}^{-\theta_2 x_i}$, for positive x_i. Its moment generating function is noted as $M_2(t; \theta_2) = \theta_2/(\theta_2 - t)$.

In the CRM our interest lies in the random variable aggregate amount of claims, S which is given by $S = \sum_{i=1}^{N} X_i$, for $N \geq 1$ and 0 for $N = 0$. Its probability density function is given by $f(s|\theta_1, \theta_2) = \sum_{n=0}^{\infty} p(n|\theta_1) \cdot f^{n\star}(x|\theta_2)$, where $p(n|\theta_1)$ denotes the probability mass function of N and $f^{n\star}$ is the nth convolution of $f(x|\theta_2)$, the probability density function of the claim size. For the case in question, it is well known that the probability density function in the CRM.PE (see Rolski et al. (1999)) is expressed as $f_{crm.PE}(s|\theta_1, \theta_2) = \sqrt{\theta_1\theta_2/s}\, \mathrm{e}^{-(\theta_1+\theta_2 s)} I_1(2\sqrt{\theta_1\theta_2 s})$ for $s > 0$ while $f_{crm.PE}(0|\theta_1, \theta_2) = \mathrm{e}^{-\theta_1}$, where $I_\upsilon(x)$ is the modified Bessel function of the first kind given by $I_\upsilon(x) = \sum_{k=1}^{\infty} \frac{(x/2)^{2k+\upsilon}}{k!\,\Gamma(\upsilon+k+1)}$, $x \in \mathbb{R}$, $\upsilon \in \mathbb{R}$. It should be pointed out that the usual discontinuity in the CRM at $s = 0$, i.e. $f_{crm.PE}(s|\theta_1, \theta_2) \neq \mathrm{e}^{-\theta_1}$ when s tends to zero is presented; which will be considered in advance.

When a Bayesian analysis is carried out it is necessary to specify a prior distribution for the vector of parameters (θ_1, θ_2), frequently assuming for the sake of mathematical convenience that the parameters are independent. Here, however, we shall consider a degree of dependence between the random variables Θ_1 and Θ_2. This can be justified in the following form. Suppose that S is the random variable representing the aggregate amount of claims in an automobile insurance portfolio. The classical collective risk model shown above assumes independence between the random variables N and X, but in practice some degree of dependence is bound to exist between these two random variables. Since it is difficult to implement this assumption in the first stage of the model (the Poisson and exponential likelihoods for N and X, respectively), we translate the non-dependence to the second stage (the prior distributions of the risk profiles). These two risk profiles are assumed to be random variables and, in the model above, they represent the expected number of claims and the reciprocal of the expected claim size. Therefore, the non-independence assumption is presented in the model and can be used in a simple way, as shown below.

Let us assume that the random variables Θ_1 and Θ_2 are dependent, and specify a marginal prior gamma distribution for each of them (which in both cases is the conjugate a priori distribution). Let $\Theta_1 \sim \Gamma(a; b) \equiv \pi_{10}(\theta_1)$ with shape parameter $a > 0$ and scale parameter $b > 0$, and $g_{10}(t) = E_{\pi_{10}}(\mathrm{e}^{-t\theta_1}) = (1 + t/b)^{-a}$ is the Laplace transform of Θ_1. Let $\Theta_2 \sim \Gamma(c; d) \equiv \pi_{01}(\theta_2)$ with shape parameter $c > 0$ and scale parameter $d > 0$, and $g_{01}(t) = E_{\pi_{01}}(\mathrm{e}^{-t\theta_2}) = (1 + t/d)^{-c}$ is the Laplace transform of Θ_2. Henceforth, we shall consider $c > 1$, a necessary and sufficient condition for the existence of the inverse moment for a $\Gamma(c, d)$ (see Piergosch and Casella (1985)). Note that $\pi_0^{(0)}(\theta_1, \theta_2) = \pi_{10}(\theta_1)\pi_{01}(\theta_2)$ represents the case in which we assume independence between Θ_1 and Θ_2.

The following result provides the prior distribution under the Sarmanov–Lee family of distributions.

Lemma 1. *The prior bivariate distribution given by*

$$\pi_0^{(\omega)}(\theta_1, \theta_2) = \pi_0^{(0)}(\theta_1, \theta_2) \left[1 + \omega(\mathrm{e}^{-\theta_1} - k_1)(\mathrm{e}^{-\theta_2} - k_2)\right] \tag{2}$$

is a genuine prior bivariate distribution of (Θ_1, Θ_2) with marginal distributions $\pi_{10}(\theta_1)$ and $\pi_{01}(\theta_2)$, with dependence between Θ_1 and Θ_2 for $\omega \neq 0$. The range of variation of the parameter ω is given by $\omega \in [\omega_1, \omega_2]$, with $\omega_1 = \frac{-1}{\max\{k_1 k_2; (1-k_1)(1-k_2)\}} < 0$ and $\omega_2 = \frac{1}{\max\{(1-k_1)k_2; k_1(1-k_2)\}} > 0$, where

$$k_1 = \left(\frac{b}{b+1}\right)^a, \tag{3}$$

$$k_2 = \left(\frac{d}{d+1}\right)^c. \tag{4}$$

Proof. The result is obtained by using (1) with mixing functions $\phi_1(\theta_1) = \mathrm{e}^{-\theta_1} - k_1$ and $\phi_2(\theta_2) = \mathrm{e}^{-\theta_2} - k_2$. ■

Remark 1. It should be pointed out that the prior bivariate distribution (2) can be expressed as a linear combination of products of gamma distributions.

The following result gives the expression of the correlation coefficient between the random variables Θ_1 and Θ_2 when the prior distribution (2) is assumed.

Table 1
Extreme values of the linear correlation coefficient for several combinations of hyperparameters.

a; b; c; d	Scene	ω_1	ω_2	$\rho(\omega_1)$	$\rho(\omega_2)$	$\rho(\omega_2)-\rho(\omega_1)$
$a=2$; $b=2$; $c=2$; $d=2$	$S3$	−3.24	4.05	−0.142	0.178	0.32
$a=0.5$; $b=1$; $c=2$; $d=2$	$S2$	−3.18	2.54	−0.167	0.133	0.3
$a=0.5$; $b=1$; $c=2$; $d=1$	$S4$	−4.55	1.88	−0.201	0.083	0.28
$a=3$; $b=4$; $c=2$; $d=3$	$S1$	−3.47	3.64	−0.122	0.128	0.25
$a=3$; $b=3$; $c=3$; $d=5$	$S1$	−4.09	2.98	−0.125	0.091	0.22
$a=2$; $b=5$; $c=2$; $d=2$	$S2$	−3.24	2.59	−0.111	0.089	0.2
$a=2$; $b=1$; $c=2$; $d=5$	$S3$	−4.36	1.92	−0.126	0.055	0.18
$a=0.5$; $b=5$; $c=7$; $d=5$	$S2$	−3.92	1.52	−0.051	0.02	0.07
$a=7$; $b=5$; $c=9$; $d=2$	$S4$	−1.42	3.68	−0.004	0.012	0.02
$a=2$; $b=1$; $c=9$; $d=1$	$S4$	−1.33	4.01	−0.001	0.002	0.003

Lemma 2. *For the prior bivariate distribution (2) the correlation coefficient between Θ_1 and Θ_2 is directly proportional to ω with a positive proportionality factor, and is given by*

$$\rho(\omega; a, b, c, d) \equiv \rho(\omega) = \omega \frac{k_1\, k_2 \sqrt{ac}}{(b+1)\,(d+1)} \tag{5}$$

where k_1 and k_2 are as in (3) and (4), respectively.

Proof. From Ting-Lee (1996) we have $\rho = \omega \nu_1 \nu_2/(\sigma_1 \sigma_2)$, where $\nu_1 = \int_{\Theta_1} \theta_1 \phi_1(\theta_1) \pi_{10}(\theta_1) \mathrm{d}\theta_1$ and $\nu_2 = \int_{\Theta_2} \theta_2 \phi_2(\theta_2) \pi_{01}(\theta_2) \mathrm{d}\theta_2$ and σ_1 and σ_2 are the standard deviations of the gamma density function $\pi_{10}(\theta_1)$ and $\pi_{01}(\theta_2)$, respectively. Now, after simple but tedious algebra we obtain the desired result. ■

Analysis of the linear correlation coefficient is especially interesting for our purposes because it will enable us to study the intensity of the dependence between Θ_1 and Θ_2. For this reason, the range of variation of the coefficient, which is clearly a function of the hyperparameters a, b, c and d, is studied below.

It is straightforward to determine the range of variation of the linear correlation coefficient in the specified model. It can be proved that if the prior distributions $\pi_{10}(\theta_1)$ and $\pi_{01}(\theta_2)$ are elicited, four different settings can be obtained. In these, the extreme values ω_1 and ω_2 are as follows.

(S1) If $\frac{k_2}{1-k_2} \geq \frac{1-k_1}{k_1}$ and $\frac{k_2}{1-k_2} \geq \frac{k_1}{1-k_1}$ then $[\omega_1; \omega_2] = [\frac{-1}{k_1 k_2}; \frac{1}{(1-k_1)k_2}]$.
(S2) If $\frac{k_2}{1-k_2} \geq \frac{1-k_1}{k_1}$ and $\frac{k_2}{1-k_2} \leq \frac{k_1}{1-k_1}$ then $[\omega_1; \omega_2] = [\frac{-1}{k_1 k_2}; \frac{1}{k_1(1-k_2)}]$.
(S3) If $\frac{k_2}{1-k_2} \leq \frac{1-k_1}{k_1}$ and $\frac{k_2}{1-k_2} \geq \frac{k_1}{1-k_1}$ then $[\omega_1; \omega_2] = [\frac{-1}{(1-k_1)(1-k_2)}; \frac{k_1}{(1-k_1)}]$.
(S4) If $\frac{k_2}{1-k_2} \leq \frac{1-k_1}{k_1}$ and $\frac{k_2}{1-k_2} \leq \frac{k_1}{1-k_1}$ then $[\omega_1; \omega_2] = [\frac{-1}{(1-k_1)(1-k_2)}. \frac{1}{k_1(1-k_2)}]$

Table 1 shows a summary of a group of 3000 simulations of different gammas, in which it is considered that the hyperparameter c should not be smaller than 1, in order to avoid the non-existence of the inverse moment. Among the simulations, 10 cases were selected: 4 corresponded to the largest difference obtained in the range of correlation, 3 cases represented a moderate level and 3 showed an almost non-correlated situation. As regards the shape of the distributions, all the situations were represented. The prior bivariate distribution considered indicated both positive and negative linear association situations, and in all the situations except the last three, the intensity of the association was moderate.

The derivative of (5) can be used to elicit the parameters of the prior distribution in accordance with our preferences for the behaviour of the correlation coefficient with respect to the values of the parameters, as shown in the following result.

Proposition 1.
- *If $\omega > 0$, then $\rho(\omega)$ is an increasing (decreasing) function of a if $b > (<) 1/(\mathrm{e}^{\frac{1}{2a}} - 1)$.*
- *If $\omega > 0$, then $\rho(\omega)$ is an increasing (decreasing) function of c if $d > (<) 1/(\mathrm{e}^{\frac{1}{2c}} - 1)$.*
- *If $\omega > 0$, then $\rho(\omega)$ is an increasing (decreasing) function of b if $a > (<) b$.*
- *If $\omega > 0$, then $\rho(\omega)$ is an increasing (decreasing) function of d if $c > (<) d$*

and the opposite is obtained when we consider $\omega > 0$.

Proof. The proof is straightforward, taking into account that from (5) we obtain

$$\frac{\partial \rho(\omega)}{\partial a} = \frac{k_1 k_2 c \omega \left[1 + 2a \log\left(\frac{b}{b+1}\right)\right]}{2(b+1)(d+1)\sqrt{ac}},$$

$$\frac{\partial \rho(\omega)}{\partial b} = \frac{k_1 k_2 \omega \sqrt{ac}(a-b)}{b(b+1)^2(d+1)},$$

$$\frac{\partial \rho(\omega)}{\partial c} = \frac{k_1 k_2 a \omega \left[1 + 2c \log\left(\frac{d}{d+1}\right)\right]}{2(b+1)(d+1)\sqrt{ac}},$$

$$\frac{\partial \rho(\omega)}{\partial d} = \frac{k_1 k_2 \omega \sqrt{ac}(c-d)}{d(b+1)(d+1)^2}. \quad ■$$

The following result determines the marginal distribution in the CRM.PE model for the prior distribution $\pi_0^{(\omega)}(\theta_1, \theta_2)$:

Lemma 3. *In the model CRM.PE with dependence, the marginal distribution of $f_{crm.PE}(s|\theta_1, \theta_2)$ for $\pi_0^{(\omega)}(\theta_1, \theta_2)$ is given by*

$$m\left(s \neq 0 | \pi_0^{(\omega)}\right) = m\left(s | \pi_0^{(0)}\right) + \omega \mathcal{P}\left(s; \pi_0^{(0)}\right) \tag{6}$$

and $m(0|\pi_0^{(\omega)}) = k_1$, where

$$\begin{aligned} &\mathcal{P}\left(s; \pi_0^{(0)}\right) \\ &\quad = \left\{ (k_3(s) - k_5(s))\, {}_2F_1\left(a+1, c+1; 2; \frac{s}{(b+2)(d+s+1)}\right) \right. \\ &\qquad \left. - k_4(s) {}_2F_1\left(a+1, c+1; 2; \frac{s}{(b+2)(d+s)}\right) \right\}, \end{aligned} \tag{7}$$

where ${}_2F_1(m, n; c, x)$ is the Gaussian hypergeometric function (see Johnson et al. (2005)).
The constants $k_3(s)$, $k_4(s)$ and $k_5(s)$ are shown in Appendix A.

Proof. See Appendix B. ■

Corollary 1. *If $\omega = 0$ (i.e. independence), then the marginal distribution has the following known expression*

$$\begin{aligned} &m\left(s \neq 0 | \pi_0^{(0)}\right) \\ &\quad = \frac{b^a d^c ac}{(b+1)^{a+1}(d+s)^{c+1}} {}_2F_1\left(a+1, c+1; 2; \frac{s}{(b+1)(d+s)}\right), \end{aligned}$$

and $m\left(0 | \pi_0^{(0)}\right) = k_1$.

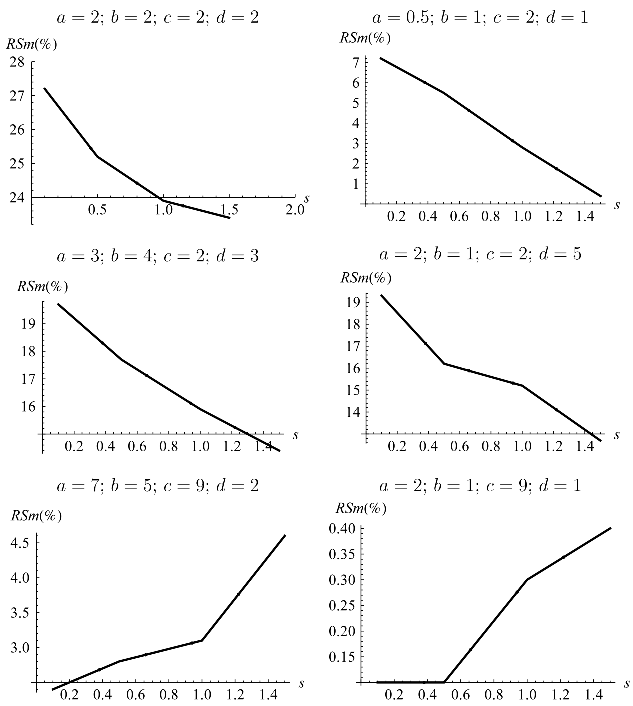

Fig. 1. RS_m factor of marginal distribution.

Proof. It is direct. ■

Observe that the marginal distribution in the CRM.PE with dependence, given by (6), is shown as the sum of two terms. The first of these is the marginal distribution with independence and the second is the product of the ω parameter (referring to the degree of dependence) and another quantity, which is a function of the a priori distribution with independence ($\pi_0^{(0)}$) and the data. This latter term, given in (7), can be interpreted as the contribution to the level of dependence, given by ω, due to the base model (with independence) and the data.

Fig. 1 shows different values of the relative variation of the marginal distribution obtained as $RSm = |m(\omega_1) - m(\omega_2)| / m^{(0)}$ (100%) for different aggregate amounts of claims observed ($s = 0.1, 0.5, 1$ and 1.5), where $m^{(0)}$ is the value of the marginal distribution when independence between the two random variables is assumed. The relative variations are notable. The first four cases correspond to ranges of the linear correlation coefficient of 0.32, 0.28, 0.25 and 0.18, respectively. The last two cases correspond to situations of almost non-correlation. These latter ranges are 0.02 and 0.003, respectively.

3. The collective and Bayes premiums

In this section the collective premium and the Bayes premium are analyzed in the CRM.PE with dependence. Furthermore, in the case of independence, we analyze the difference between the Bayes premium in the CRM.PE and the value of the premium proposed in Frangos and Vrontos (2001), where the premium is obtained as the product of the Bayes premium, calculated separately for the number of claims and for the claim size. Henceforth, the collective premium and the Bayes premium will be denoted by $P'[\cdot]$ and $P^*[\cdot]$, respectively.

It is well known that in the CRM.PE model the net risk premium is given by $E(S) = E(N)E(X) = \theta_1/\theta_2$, i.e. the product of the net risk premiums for the number of claims and the claim size. If $c > 1$, the collective net premiums are given by $P'[\pi_{10}(\theta_1)] = E_{\pi_{10}}(\theta_1) = a/b$ for the number of claims and $P'[\pi_{01}(\theta_2)] = E_{\pi_{01}}(1/\theta_2) = d/(c-1)$ for the claim size. Finally, if independence between the parameters θ_1 and θ_2 is assumed, then the collective net premium for the CRM.PE is given by $P'\left[\pi_0^{(0)}(\theta_1, \theta_2)\right] = E_{\pi_0^{(0)}}(\theta_1/\theta_2) = ad/(b(c-1)) = P'[\pi_{10}(\theta_1)] \cdot P'[\pi_{01}(\theta_2)]$. For a revision of premium calculation principles see Hernández-Bastida et al. (2009) and references therein.

Table 2
Collective premium values with independence between the parameters and in extreme cases of correlation.

$a; b; c; d$	$P'^{(0)}$	$P'(\omega_1)$	$P'(\omega_2)$	RSP' (%)
$a = 2; b = 2; c = 2; d = 2$	2	1.7875	2.266	24
$a = 0.5; b = 1; c = 2; d = 1$	0.5	0.2988	0.5833	56.9
$a = 3; b = 4; c = 2; d = 3$	2.25	2.1	2.4074	13.66
$a = 2; b = 1; c = 2; d = 5$	10	9.2424	0.333	10.91
$a = 7; b = 5; c = 9; d = 2$	0.35	0.3497	0.3508	0.31
$a = 2; b = 1; c = 9; d = 1$	0.25	0.2499	0.2502	0.13

Now, let $X_1, X_2, \ldots, X_k$, be a sample of the amount of claims, such that $\sum_{i=1}^{k} X_i = s$ is the total claim amount generated by k claims produced in a single-period of time. Then it is readily confirmed that the Bayes premiums for the number of claims and the claim size are given by $P^*[\pi_{10}(\theta_1|k)] = E_{\pi_{10}(\theta_1|k)}(\theta_1) = (a+k)/(b+1)$ and $P^*[\pi_{01}(\theta_2|s)] = E_{\pi_{01}(\theta_2|s)}(1/\theta_2) = (d+s)/(c+k-1)$, respectively. Observe that the premium proposed in Frangos and Vrontos (2001) is the product of the two previous premiums, i.e. $P^*_{FV}[\pi_{10}; \pi_{01}; k; s] = P^*[\pi_{10}(\theta_1|k)] \cdot P^*[\pi_{01}(\theta_2|s)]$. Later, we will illustrate the difference between P^*_{FV} and the Bayes premium obtained in the CRM.PE model proposed in this paper.

Now, it is established that the collective premium in the model with dependence is equal to the collective premium in the model with independence plus a term directly proportional to ω, with a positive proportional quantity.

Proposition 2. *In the CRM.PE model with dependence the collective premium is*

$$P'\left[\pi_0^{(\omega)}(\theta_1, \theta_2)\right] = P'\left[\pi_0^{(0)}(\theta_1, \theta_2)\right] + \omega \frac{ak_1k_2}{b(b+1)(c-1)}, \tag{8}$$

where $P'\left[\pi_0^{(0)}(\theta_1, \theta_2)\right] = \frac{ad}{b(c-1)}$.

Proof. See Appendix B. ■

Note that if the linear correlation is negative, i.e., $\omega \in [\omega_1, 0)$ then the collective premium with dependence is strictly lower than the collective premium with independence, with the opposite being obtained if the linear correlation is positive, $\omega \in (0, \omega_2]$. Furthermore, the range of variation of the collective premium is directly proportional to $(\omega_2 - \omega_1)$, where the proportional term, $\frac{ak_1k_2}{b(b+1)(c-1)}$, is always positive if $c > 1$.

Table 2 shows the values of the collective premium, highlighting when independence is considered ($\omega = 0$), noted as $P'^{(0)}$ and the extreme cases of linear association ($P'(\omega_1)$ and $P'(\omega_2)$). The table also shows the robustness of the collective premium, in relation to the dependence, calculated by the expression

$$RSP' = \frac{\left(P'\left[\pi_0^{(\omega_2)}(\theta_1, \theta_2)\right] - P'\left[\pi_0^{(\omega_1)}(\theta_1, \theta_2)\right]\right)}{P'\left[\pi_0^{(0)}(\theta_1, \theta_2)\right]}(100\%).$$

From the first three cases, it follows that even with moderate linear correlation levels (the three cases present ranges $\rho(\omega_2) - \rho(\omega_1)$ equal to 0.32, 0.3 and 0.28 respectively), the variation in the collective premium (8) is considerable, with variations of around 56.9% (the second case considered).

The next result provides the Bayes premium under the proposed new model.

Proposition 3. *In the CRM.PE model with dependence, the best estimation of the risk premium is the Bayes premium, given by*

$$\begin{aligned}P^*\left[\pi_0^{(\omega)}(\theta_1,\theta_2|0)\right] &= P^*\left[\pi_0^{(0)}(\theta_1,\theta_2|0)\right]+\omega k_6\\ &= \frac{ad}{(b+1)(c-1)}+\omega k_6, \qquad (9)\\ P^*\left[\pi_0^{(\omega)}(\theta_1,\theta_2|s\neq 0)\right] &= P^*\left[\pi_0^{(0)}(\theta_1,\theta_2|s\neq 0)\right]\\ &\quad+\frac{\omega}{m\left(s|\pi_0^{(\omega)}\right)}\mathcal{N}\left(s;\pi_0^{(0)}\right), \qquad (10)\end{aligned}$$

where

$$\begin{aligned}\mathcal{N}\left(s;\pi_0^{(0)}\right) &= \left(k_1k_2-\frac{\mathcal{P}\left(s;\pi_0^{(0)}\right)}{m(s|\pi_0^{(0)})}\right)\\ &\times k_7(s)_2F_1\left(a+2,c;2;\frac{s}{(b+1)(d+s)}\right)\\ &+k_8(s)_2F_1\left(a+2,c;2;\frac{s}{(b+2)(d+s+1)}\right)\\ &-k_9(s)_2F_1\left(a+2,c;2;\frac{s}{(b+2)(d+s)}\right)\\ &-k_{10}(s)_2F_1\left(a+2,c;2;\frac{s}{(b+1)(d+s+1)}\right),\end{aligned}$$

with the constants $k_6, k_7(s), k_8(s), k_9(s)$ *and* $k_{10}(s)$ *given in Appendix A.*

Proof. See Appendix B. ■

Observe that k_6 is a positive value and so the Bayes premium (9) is a nondecreasing function for ω, showing that the negative linear association between θ_1 and θ_2 diminishes the value for the Bayes premium in the absence of any claim. On the other hand, the opposite is true if the linear association is positive.

Because they are straightforward, the next results are presented without proofs.

Corollary 2. *If there exists independence* ($\omega=0$) *it follows that the Bayes premium is* $P^*\left[\pi_0^{(0)}(\theta_1,\theta_2|0)\right]=\frac{ad}{(b+1)(c-1)}$, *and* (10) *is given by*

$$\begin{aligned}P^*\left[\pi_0^{(0)}(\theta_1,\theta_2|s\neq 0)\right] &= \frac{k_{7\,2}F_1\left(a+2,c;2;\frac{s}{(b+1)(d+s)}\right)}{m(s/\pi_0^{(0)})}\\ &= \frac{(a+1)(d+s)}{(b+1)c}\,\frac{{}_2F_1\left(a+2,c;2;\frac{s}{(b+1)(d+s)}\right)}{{}_2F_1\left(a+1,c+1;2;\frac{s}{(b+1)(d+s)}\right)}. \qquad (11)\end{aligned}$$

Corollary 3. *If there is no claim* ($s=0$)*, the Bayes premium is strictly lower than the collective premium, i.e.*

$$P^*\left[\pi_0^{(0)}(\theta_1,\theta_2|0)\right] < P'\left[\pi_0^{(0)}(\theta_1,\theta_2)\right].$$

The difference between them, which is equal to $\frac{ad}{b(b+1)(c-1)}$, *is interpreted as the diminution, with respect to the collective premium, obtained if no claim is made.*

Corollary 4. *The range of variation of the Bayes premium when* $\omega\in[\omega_1,\omega_2]$ *can be expressed as*

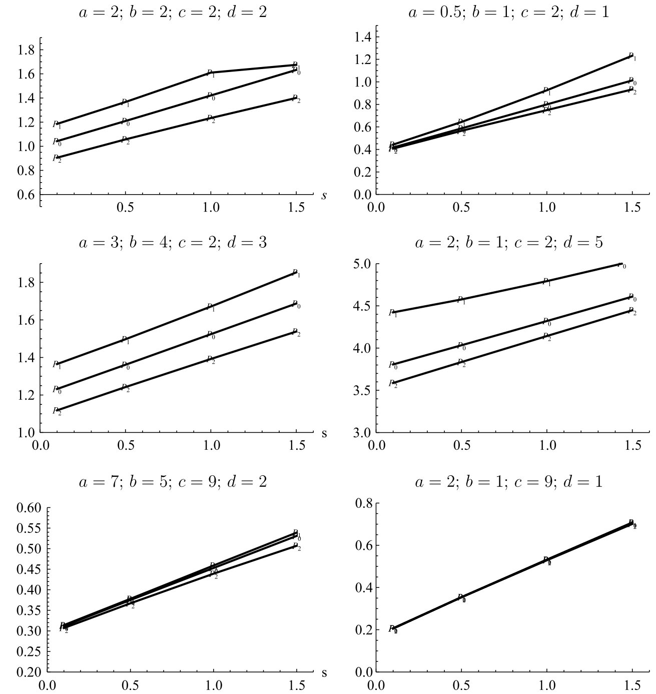

Fig. 2. Bayes premium for different values of s, showing the independence case (P_0) and extreme cases of linear correlation (P_1 for $\rho(\omega_1)$ and P_2 for $\rho(\omega_2)$).

$$\begin{aligned}&P^*\left[\pi_0^{(\omega_2)}(\theta_1,\theta_2|s\neq 0)\right]-P^*\left[\pi_0^{(\omega_1)}(\theta_1,\theta_2|s\neq 0)\right]\\ &=\frac{-(\omega_2-\omega_1)^2\,\mathcal{N}\left(s,\pi_0^{(0)}\right)\mathcal{P}\left(s;\pi_0^{(0)}\right)}{m\left(s/\pi_0^{(\omega_1)}\right)m\left(s/\pi_0^{(\omega_2)}\right)},\end{aligned}$$

while

$$P^*\left[\pi_0^{(\omega_2)}(\theta_1,\theta_2|0)\right]-P^*\left[\pi_0^{(\omega_1)}(\theta_1,\theta_2|0)\right]=(\omega_2-\omega_1)k_6.$$

The Bayes premiums, obtained in the same values as in the previous illustrations, are shown in Fig. 2.

Fig. 3 shows the magnitude $(P_2-P_1)/P_0$ (100%) for the six cases illustrated in Fig. 2, where we have denoted P_1 and P_2 when $\rho(\omega_1)$ and $\rho(\omega_2)$ are considered, respectively.

Fig. 3 corroborates and clarifies the implicit conclusions shown in Fig. 2, where the very marked differences between the different values of the Bayes premiums are apparent. Note that the first four cases correspond to ranges of correlation coefficients of 0.32, 0.28, 0.25 and 0.18; while the last two situations are of almost non-correlation, with ranges of 0.02 and 0.003.

As with the collective premium, it is readily concluded that moderate levels of linear correlation between the risk profiles lead to appreciable variations in the Bayes premium (for the sake of illustration, note the first four cases).

In the following, we derive the expression which enables us to evaluate the difference between the Bayes premium, if there exists independence between Θ_1 and Θ_2, and the premium proposed in Frangos and Vrontos (2001), denoted by P^*_{FV}.

Proposition 4. *Let* $P^*\left[\pi_0^{(0)}(\theta_1,\theta_2|s)\right]$ *be the Bayes premium obtained when independence is assumed and* $P^*_{FV}\left[\pi_{10};\pi_{01};k;s\right]=P^*\left[\pi_{10}(\theta_1|k)\right]P^*\left[\pi_{01}(\theta_2|s)\right]$ *the premium obtained as the product of the two premiums indicated. Then,*

(i) *If no claim is made* ($s=0$) *the two premiums coincide, i.e.*

$$P^*\left[\pi_0^{(0)}(\theta_1,\theta_2|0)\right]=P^*_{FV}\left[\pi_{10};\pi_{01};0;0\right].$$

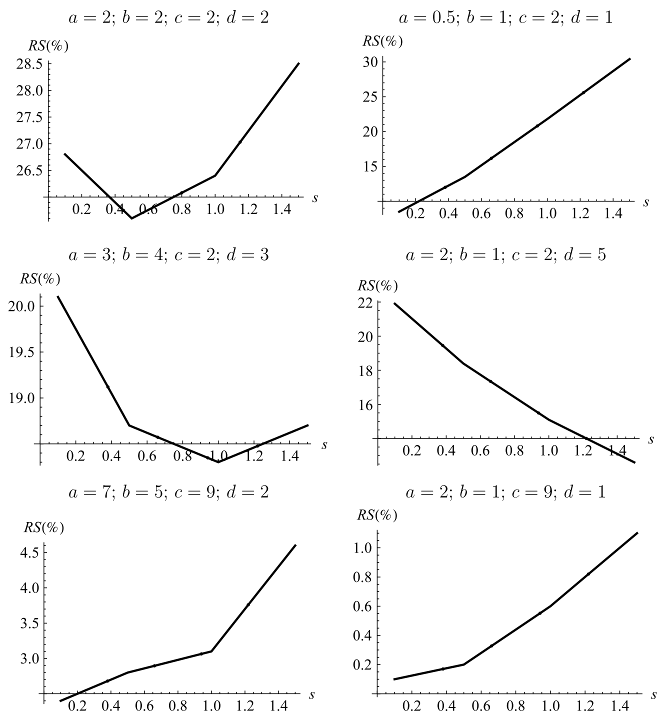

Fig. 3. $(P_2 - P_1)/P_0$ (100%) values.

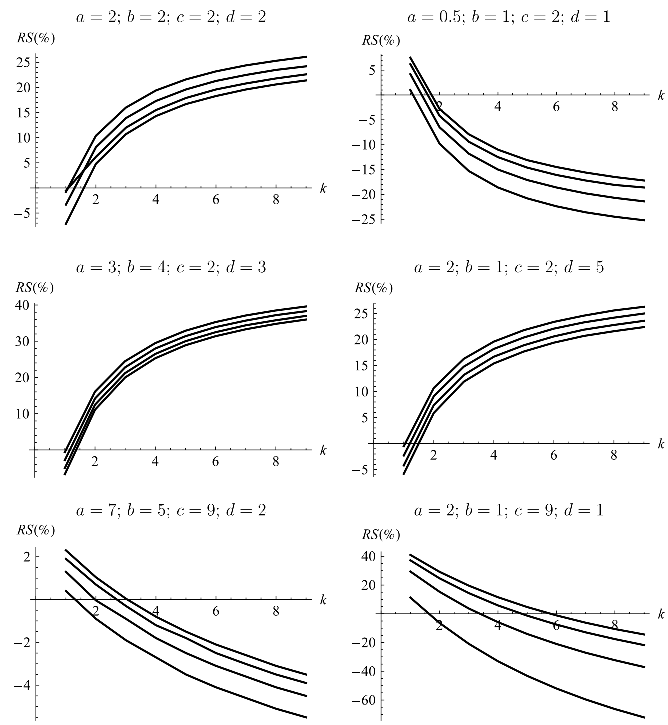

Fig. 4. *RS* values.

(ii) *If* $s \geq 0$, *it follows that the premiums differ and the difference is given by*

$$P^*\left[\pi_0^{(0)}(\theta_1, \theta_2|s \neq 0)\right] - P^*_{FV}\left[\pi_{10}; \pi_{01}; k; s \neq 0\right]$$
$$= \frac{(d+s)}{(b+1)}\left[\frac{{}_2F_1\left(a+2, c; 2; \frac{s}{(b+1)(d+s)}\right)}{{}_2F_1\left(a+1, c+1; 2; \frac{s}{(b+1)(d+s)}\right)} \times \frac{a+1}{c} - \frac{a+k}{c+k-1}\right].$$

Now the difference in Proposition 4 is evaluated. Fig. 4 shows, for the same six cases as in Fig. 3, the values of

$$RS = \frac{P^*\left[\pi_0^{(0)}(\theta_1, \theta_2|s \neq 0)\right] - P^*_{FV}\left[\pi_{10}; \pi_{01}; k; s| \neq 0\right]}{P^*\left[\pi_0^{(0)}(\theta_1, \theta_2|s \neq 0)\right]}(100\%) \tag{12}$$

for four values of s ($s = 0.1; 0.5; 1; 1.5$, corresponding to the curves from left to right in the figure). With the aim of viewing the difference for a fixed $s > 0$ when this aggregate loss comes from different claim numbers, diverse k ($k = 1, 2, \ldots, 9$) values are considered for each $s > 0$.

It can be observed that the differences between (11) and P^*_{FV} for $s \neq 0$, computed by using (12), are positive and negative, and in absolute terms are very considerable. If the sign were constant, we could speak about infra- or over-estimation, but such is not the case. Consequently, it is not appropriate to consider the P^*_{FV} premium as an approximation to the premium that should really be considered, i.e. the $P^*\left[\pi_0^{(0)}(\theta_1, \theta_2|s \neq 0)\right]$ premium.

4. Conclusions

The most important contribution of this paper is that it further allows for dependence between the number of claims and the claim size in the well-known compound Poisson model, assuming an exponential distribution for the claim size random variable and where dependence between these elements has been translated to the profile risks. The degree of dependence is studied by introducing a prior bivariate distribution belonging to the Sarmanov–Lee family of distributions.

This model was used to analyze the resulting effects on the collective and the Bayes net premium. The results obtained lead us to conclude that even at moderate levels of correlation between the risk profiles, the above premiums are highly sensitive.

In addition, we compared the Bayes net premium with independence between the parameters and the Bayes net premium obtained as a product of two independent processes, one for the number of claims and the other for their severity. This latter premium coincides with the one used in Frangos and Vrontos (2001). Both premiums coincide if the aggregate amount of claims is zero, but they differ when this amount is larger than zero. This difference sometimes presents a positive sign and sometimes a negative one, and is very considerable in absolute values. All our results are illustrated numerically.

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Appendix A. Expressions of k_i and $k_i(s)$ appearing in this study

Here, we set out the expressions of k_i, $i = 1, 2$ used in the text and the expressions of $k_i(s)$, $i = 3, \ldots, 10$. All of them are continuous functions in s.

In Lemma 1:

$$k_1 = \left(\frac{b}{b+1}\right)^a,$$

$$k_2 = \left(\frac{d}{d+1}\right)^c.$$

In Lemma 3:

$$k_3(s) = \frac{b^a d^c ac}{(b+2)^{a+1}(d+s+1)^{c+1}},$$
$$k_4(s) = \frac{b^a d^{2c} ac}{(b+2)^{a+1}(d+1)^c(d+s)^{c+1}},$$
$$k_5(s) = \frac{b^{2a} d^c ac}{(b+2)^{2a+1}(d+s+1)^{c+1}}.$$

In Proposition 3:

$$k_6 = \frac{a(b+1)^{a+1}k_2 - a(b+2)^{a+1}k_1k_2}{(b+2)^{a+1}(b+1)(c-1)},$$
$$k_7(s) = \frac{b^a d^c a(a+1)}{(b+1)^{a+2}(d+s)^c},$$
$$k_8(s) = \frac{b^a d^c a(a+1)}{(b+2)^{a+2}(d+s+1)^c},$$
$$k_9(s) = \frac{k_2 b^a d^c a(a+1)}{(b+2)^{a+2}(d+s)^c},$$
$$k_{10}(s) = \frac{k_1 b^a d^c a(a+1)}{(b+1)^{a+2}(d+s+1)^c}.$$

Appendix B. Proof of lemmas and propositions

The following result will be useful in the proofs shown in this Appendix.

Result 1. *For whatever integers r, s, u, v, and the sample observation x, the following equalities are verified*

$$I_1(r,s,u,v) \equiv \int_{\Theta_1}\int_{\Theta_2} \theta_1^r\theta_2^s e^{-u\theta_1}e^{-v\theta_2}\pi_0^{(0)}(\theta_1,\theta_2)d\theta_1 d\theta_2 = \frac{\Gamma(a+r)b^a\Gamma(c+s)d^c}{\Gamma(a)(b+u)^{r+a}\Gamma(c)(d+v)^{c+s}}.$$

$$I_2(r,s,u,v,x) \equiv \int_{\Theta_1}\int_{\Theta_2} \theta_1^r\theta_2^s e^{-u\theta_1}e^{-v\theta_2} f_{crm.PE}(s|\theta_1,\theta_2)\pi_0^{(0)}(\theta_1,\theta_2)d\theta_1 d\theta_2 = \frac{\mathcal{M}(s)}{\Gamma(a)(b+u+1)^{r+a+1}\Gamma(c)(d+v+x)^{c+s+1}},$$

where

$$\mathcal{M}(s) = \Gamma(a+r+1)b^a\Gamma(c+s+1)d^c \times {}_2F_1\left(a+r+1, c+s+1; 2; \frac{x}{(b+1+u)(x+d+v)}\right).$$

Proof. The proof is direct. ■

Proof of Lemma 3. From Lemma 1 and Result 1 it follows that

$$m\left(s\neq 0|\pi_0^{(\omega)}\right) = \int_{\Theta_1}\int_{\Theta_2} f_{crm.PE}(s|\theta_1,\theta_2)\pi_0^{(\omega)}(\theta_1,\theta_2)d\theta_1 d\theta_2$$
$$= m\left(s/\pi_0^{(0)}\right) + \omega\left[k_1k_2 m\left(s/\pi_0^{(0)}\right) + I_2(0,0,1,1,s) - k_2I_2(0,0,1,0,s) - k_1I_2(0,0,0,1,s)\right]$$

and

$$m\left(0|\pi_0^{(\omega)}\right) = \int_{\Theta_1}\int_{\Theta_2} f_{crm.PE}(0|\theta_1,\theta_2)\pi_0^{(\omega)}(\theta_1,\theta_2)d\theta_1 d\theta_2$$
$$= m\left(0|\pi_0^{(0)}\right) + \omega\left[k_1k_2 m\left(0|\pi_0^{(0)}\right) + I_1(0,0,2,1) - k_2I_1(0,0,2,0) - k_1I_1(0,0,1,1)\right].$$

Now, after simple computations, the desired result is obtained. ■

Proof of Proposition 2. Under the net premium principle, the risk premium is $P = \theta_1/\theta_2$. Then, from Lemma 1 and Result 1 we have

$$P'\left[\pi_0^{(\omega)}(\theta_1,\theta_2)\right] = \int_{\Theta_1}\int_{\Theta_2} \frac{\theta_1}{\theta_2}\pi_0^{(\omega)}(\theta_1,\theta_2)d\theta_1 d\theta_2$$
$$= P'\left[\pi_0^{(0)}(\theta_1,\theta_2)\right] + \omega\left[k_1k_2P'\left[\pi_0^{(0)}(\theta_1,\theta_2)\right] + I_1(1,-1,1,1) - k_2I_1(1,-1,1,0) - k_1I_1(1,-1,0,1)\right]$$

and $P'\left[\pi_0^{(0)}(\theta_1,\theta_2)\right] = I_1(1,-1,0,0)$.

Now the result is almost direct. ■

Proof of Proposition 3. From Lemma 1 and Result 1 it follows that

$$P^*\left[\pi_0^{(\omega)}(\theta_1,\theta_2|0)\right] = \int_{\Theta_1}\int_{\Theta_2} \frac{\theta_1}{\theta_2}\pi_0^{(\omega)}(\theta_1,\theta_2)d\theta_1 d\theta_2$$
$$= \frac{I_1(1,-1,1,0)}{k_1} + \frac{1}{k_1}\omega\left[k_1k_2I_1(1,-1,1,0) + I_1(1,-1,2,1) - k_2I_1(1,-1,2,0) - k_1I_1(1,-1,1,1)\right]$$

and

$$P^*\left[\pi_0^{(\omega)}(\theta_1,\theta_2|s\neq 0)\right] = \int_{\Theta_1}\int_{\Theta_2} \frac{\theta_1}{\theta_2}\pi_0^{(\omega)}(\theta_1,\theta_2|s)d\theta_1 d\theta_2 = \frac{I_2(1,-1,0,0,s) + \omega\mathcal{R}(s)}{m(s|\pi_0^{(\omega)})},$$

where

$$\mathcal{R}(s) = I_2(1,-1,0,0,s) + I_2(1,-1,1,1,s) - k_2I_2(1,-1,1,0,s) - k_1I_2(1,-1,0,1,s).$$

Then we have

$$P^*\left[\pi_0^{(\omega)}(\theta_1,\theta_2|s\neq 0)\right]$$
$$= \frac{(k_7(s) + \omega k_1k_2k_7(s))\,{}_2F_1\left(a+2, c; 2; \frac{s}{(b+1)(d+s)}\right)}{m\left(s|\pi_0^{(\omega)}\right)} + \frac{\omega}{m\left(s|\pi_0^{(\omega)}\right)} + \left\{k_8(s)\,{}_2F_1\left(a+2, c; 2; \frac{s}{(b+2)(d+s+1)}\right) - k_9(s)\,{}_2F_1\left(a+2, c; 2; \frac{s}{(b+2)(d+s)}\right) - k_{10}(s)\,{}_2F_1\left(a+2, c; 2; \frac{s}{(b+1)(d+s+1)}\right)\right\}.$$

Now, after simple algebra this latter expression can be rewritten as the sum of two terms where the first one is the Bayes premium with independence between Θ_1 and Θ_2. ■

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