Insurance: Mathematics and Economics 87 (2019) 153–168

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Collective risk models with dependence

Hélène Cossette, Etienne Marceau *, Itre Mtalai

École d'Actuariat, Université Laval, Québec, Canada

ARTICLE INFO

Article history:
Received May 2018
Received in revised form April 2019
Accepted 30 April 2019
Available online 14 May 2019

JEL classification:
C600
C630
C690

Keywords:
Random sums
Collective risk models
Dependence
Copulas
Archimedean copulas
Hierarchical Archimedean copulas
Mixed Erlang distributions

ABSTRACT

In actuarial science, collective risk models, in which the aggregate claim amount of a portfolio is defined in terms of random sums, play a crucial role. In these models, it is common to assume that the number of claims and their amounts are independent, even if this might not always be the case. We consider collective risk models with different dependence structures. Due to the importance of such risk models in an actuarial setting, we first investigate a collective risk model with dependence involving the family of multivariate mixed Erlang distributions. Other models based on mixtures involving bivariate and multivariate copulas in a more general setting are then presented. These different structures allow to link the number of claims to each claim amount, and to quantify the aggregate claim loss. Then, we use Archimedean and hierarchical Archimedean copulas in collective risk models, to model the dependence between the claim number random variable and the claim amount random variables involved in the random sum. Such dependence structures allow us to derive a computational methodology for the assessment of the aggregate claim amount. While being very flexible, this methodology is easy to implement, and can easily fit more complicated hierarchical structures.

1. Introduction

Collective risk models are fundamental in actuarial science to model the aggregate claim amount of an insurance company. For a given portfolio of policyholders and over a fixed period of time, the aggregate claim amount random variable (rv) S is defined as a random sum, i.e.,

$$S=\sum_{i=1}^{N} X_i, \tag{1}$$

where $\underline{X}=\{X_i, i\in\mathbb{N}\}$ is a sequence of non-negative rvs, and N is a positive counting rv (with the convention that $\sum_{i=1}^{0} a_i = 0$). In an insurance context, the rv N represents the number of claims and X_i corresponds to the amount of the ith claim ($i\in\mathbb{N}$).

Let $\aleph=\aleph(F_N, F_X)$ be a class of collective risk models defined with the sequence $(N,\underline{X})$. The cumulative distribution function (cdf), the probability mass function (pmf), and the probability generating function (pgf) of the rv N are respectively denoted by F_N, γ_N, and $\mathcal{P}_N$, with support $\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and where

$$\gamma_N(n)=\Pr(N=n), n\in\mathbb{N}_0,$$

and by

$$\mathcal{P}_N(s)=E\left[s^N\right]=\sum_{n=0}^{\infty} s^n\gamma_N(n), s\in[0,1].$$

The rv X_i is distributed as the strictly positive rv X, with cdf F_X, for $i\in\mathbb{N}$. To simplify the presentation, we assume that $F_X(0)=0$.

In the classical collective risk model, the components of $\underline{X}$ are assumed to be independent of the counting rv N, and also independent and identically distributed (iid) rvs (see, e.g., Rolski et al. (1999) and Klugman et al. (2009)). However in practice, these assumptions are not always verified. For example, while analyzing a car insurance data set, Gschlossl and Czado (2007) found that the number and the size of claims are significantly dependent. See also, e.g., Kousky and Cooke (2009) for other related examples, such as the highlighted dependency between flood damage and wind damage, using catastrophic loss data.

While several papers proposed models that only account for the dependence between claim amounts (see e.g., Denuit et al. (2006)), few others considered an extra dependency between claim amounts and claim counts as well. For example, claim counts can be considered as predictors for the claim amounts, see e.g., Gschlossl and Czado (2007), Frees et al. (2011) and Garrido et al. (2016). In another setting, inter-claim times and claim sizes are assumed to be dependent in a compound Poisson process, see e.g., Albrecher et al. (2014), Boudreault et al. (2006), Cossette et al. (2008) , and Landriault et al. (2014).

* Corresponding author.
E-mail address: etienne.marceau@act.ulaval.ca (E. Marceau).

https://doi.org/10.1016/j.insmatheco.2019.04.008
0167-6687/

Due to their interesting properties, copulas have been widely used to model a dependence structure between rvs. In a collective risk context, such a structure has been used by several researchers to model the dependence between claim amounts and their related sizes. For example, Czado et al. (2012) and Krämer et al. (2013) propose to use families of bivariate copulas to model the dependency relationship between the number of claims and the average claim amount. In the same context, we aim in this paper to use copulas to account for the dependence between all the components of $(N, \underline{X})$. In this paper, we propose different approaches to account for a dependence structure between the components of the sequence $(N, \underline{X})$ in a collective risk model setting. We first present a dependent collective risk model in which $\left(N, \underline{X}\right) \in \aleph$ such that, for any $k \in \mathbb{N}$, the random vector $(X_1, \ldots, X_k)$ follows a multivariate mixed Erlang distribution. The dependence between the components of $(N, X_1, \ldots, X_k)$ comes from the dependence construction between the counting rv N and the mixing weights defining the multivariate mixed Erlang distribution. Another approach used to introduce a dependence structure is through mixing, more precisely by introducing a dependence relation between N and a positive discrete mixing rv. Bivariate copulas will be used within this methodology. Lastly, multivariate copulas and hierarchical Archimedean copulas will also be at the basis of other constructions.

The outline of the paper is as follows. In Section 2, we present basic definitions and collective risk models with specific dependence structures. Risk assessment and the impact of the dependence relation between the components of the sequence $(N, \underline{X})$ on the distribution of the aggregate claim amount rv S is discussed in Section 3. In Section 4, we investigate three collective risk models with dependence: the first one is defined with multivariate mixed Erlang; the construction of the second one is based on mixing; and the dependence structure of the third one is defined with a Farlie–Gumbel–Morgenstern (FGM) multivariate copula. Finally, in Section 5, we present and examine a collective risk model where the dependence structure for the components of the sequence $\left(N, \underline{X}\right) \in \aleph$ is defined with either an Archimedean or a hierarchical Archimedean copula. Numerical examples are presented to illustrate different aspects of the collective risk models with dependence discussed in the present paper.

2. Definitions and properties

2.1. Basic definitions and properties

Consider a collective risk model defined with a sequence $\left(N, \underline{X}\right) \in \aleph$. As suggested in the proof of Theorem 4.3.5 of Müller and Stoyan (2002), it is convenient to represent the random sum defined in (1) by

$$S = \phi(N, X_1, X_2, \ldots) = \sum_{i=1}^{\infty} X_i \times 1_{\{N \geq i\}}, \tag{2}$$

where $\phi(n, x_1, x_2, \ldots) = \sum_{i=1}^{\infty} x_i \times 1_{\{n \geq i\}}$ with $1_{\{n \geq i\}} = \begin{cases} 1, & n \geq i \\ 0, & \text{otherwise} \end{cases}$.

The cdf of the rv S is given by

$$F_S(x) = \Pr(N=0) + \sum_{n=1}^{\infty} \Pr(N=n, X_1 + \cdots + X_n \leq x) \tag{3}$$

$$= \Pr(N=0) + \sum_{n=1}^{\infty} \Pr(N=n) \times \Pr(X_1 + \cdots + X_n \leq x | N=n), \; x \geq 0, \tag{4}$$

with the expectation of the rv S given by

$$E[S] = \sum_{n=1}^{\infty} \Pr(N=n) \times n \times E[X|N=n]. \tag{5}$$

For $k \in \mathbb{N}$, let

$$\mathcal{L}_{X_1,\ldots,X_k}(t_1, \ldots, t_k) = E\left[e^{-t_1 X_1} \ldots e^{-t_k X_k}\right]$$

be the joint Laplace–Stieltjes transform (LST) of the vector of rvs $(X_1, \ldots, X_k)$. Conditioning on $\{N=n\}$, the LST of the rv S is given by

$$\mathcal{L}_S(t) = \Pr(N=0) + \sum_{n=1}^{\infty} \Pr(N=n) \, \mathcal{L}_{X_1,\ldots,X_n|N=n}(t, \ldots, t), \; t \geq 0, \tag{6}$$

where

$$\mathcal{L}_{X_1,\ldots,X_n|N=n}(t_1, \ldots, t_n) = E\left[e^{-t_1 X_1} \ldots e^{-t_n X_n} | N=n\right], \; t_1, \ldots, t_n \geq 0,$$

is the conditional LST of $(X_1, \ldots, X_n | N=n)$, for $n \in \mathbb{N}$.

Also, as we shall see later, the dependence relations between the components of $\left(N, \underline{X}\right)$ can, intuitively, be hierarchically structured. Mainly, we first establish the dependence structure between the component N and $\underline{X}$ of the sequence $\left(N, \underline{X}\right)$. Given that dependence relation, we then establish the dependence construction between the components of $\underline{X}$.

2.2. Specific structures of dependence

We present in this section specific risk models with dependence chosen from the general class $\aleph$ of collective risk models. We aim to later compare them in different settings. Note that the symbols $\perp$ and $+$ are used to indicate independence and comonotonicity respectively.

Let $\aleph^{(\perp,\,)} \subset \aleph$ be the subclass of $\aleph$ and $\left(N^{\perp}, \underline{X}^{(\perp,\,)}\right) \in \aleph^{(\perp,\,)}$ a sequence which defines a collective risk model where the sequence $\underline{X}^{(\perp,\,)}$ and $N^{\perp}$ are independent. Evidently, we have $N^{\perp} \sim N$. For $(N^{\perp}, \underline{X}^{(\perp,\,)}) \in \aleph^{(\perp,\,)}$ we define the aggregate claim amount rv by $S^{(\perp,\,)} = \sum_{i=1}^{N^{\perp}} X_i^{(\perp,\,)}$. Examples of collective risk models defined by sequences $\left(N^{\perp}, \underline{X}^{(\perp,\,)}\right) \in \aleph^{(\perp,\,)}$ can be found in, e.g., Section 4 of Cossette et al. (2017) and Section 2.4 of Willmot and Woo (2015). Also, the class of risk models defined in Section 3 of Albrecher et al. (2014) over a fixed time interval $(0, t]$ belong to the subset $\aleph^{(\perp,\,)}$. For a sequence $\left(N^{\perp}, \underline{X}^{(\perp,\,)}\right) \in \aleph^{(\perp,\,)}$ (3) becomes

$$F_{S^{(\perp,\,)}}(x) = \Pr\left(N^{\perp}=0\right) + \sum_{n=1}^{\infty} \Pr\left(N^{\perp}=n\right) \times \Pr\left(\sum_{i=1}^{n} X_i^{(\perp,\,)} \leq x\right), \; x \geq 0, \tag{7}$$

and the expectation of the aggregate claim amount rv in (5) becomes

$$E[S^{(\perp,\,)}] = E[N^{\perp}] \times E[X^{(\perp,\,)}] = E[N] \times E[X], \tag{8}$$

which means that the expected aggregate claim amount corresponds to the product of the expected claim count and the expected claim amount. Also, the expression of the LST in (6) becomes

$$\mathcal{L}_{S^{(\perp,\,)}}(t) = \Pr\left(N^{\perp}=0\right) + \sum_{n=1}^{\infty} \Pr\left(N^{\perp}=n\right) \times \mathcal{L}_{X_1,\ldots,X_n}(t, \ldots, t), \; t \geq 0. \tag{9}$$

We consider the following specific dependence structures.

Model 1 is the collective risk model defined by the sequence $\left(N^{\perp},\underline{X}^{(\perp,+)}\right)\in\aleph^{(\perp,\ )}$ with a sequence of comonotonic individual claim amounts, independent of the claim number $N^{\perp}$. Let $U\sim Unif(0,1)$ be independent of $N^{\perp}$. This means that, for any $k\in\{2,3,\ldots\}$, the representation

$$X_1^{(\perp,+)}=F_X^{-1}(U),\ldots,X_k^{(\perp,+)}=F_X^{-1}(U) \tag{10}$$

holds, where F_X^{-1} is the generalized inverse of F_X. Recall that the generalized inverse of the cdf of a given rv Y with cdf F_Y is defined as $F_Y^{-1}(\kappa)=\inf\{x\in\mathbb{R},F_Y(x)\geq\kappa\}$. Using the representation in (10), the corresponding aggregate claim amount rv within Model 1 is defined by

$$S^{(\perp,+)}=\sum_{i=1}^{\infty}X_i^{(\perp,+)}\times 1_{\{N^{\perp}\geq i\}}=N^{\perp}\times X^{(\perp,+)}. \tag{11}$$

Clearly, for the sequence $(N^{\perp},\underline{X}^{(\perp,+)})\in\aleph^{(\perp,\ )}$, we have

$$F_{S^{(\perp,+)}}(x)=\Pr(N=0)+\sum_{n=1}^{\infty}\Pr(N=n)\times\Pr\left(X\leq\frac{x}{n}\right),x\geq 0.$$

Also, from (11), the expectation of $S^{(\perp,+)}$ is

$$E[S^{(\perp,+)}]=E[N^{\perp}]\times E[X^{(\perp,+)}]=E[N]\times E[X].$$

Model 2, a special case of $\aleph^{(\perp,\ )}\subset\aleph$, is the classical collective risk model which is defined by the sequence $\left(N^{\perp},\underline{X}^{(\perp,\perp)}\right)\in\aleph^{(\perp,\ )}$ where $\underline{X}^{(\perp,\perp)}=\left\{X_i^{(\perp,\perp)},i\in\mathbb{N}\right\}$ forms a sequence of iid rvs (with $X_i^{(\perp,\perp)}\sim X^{(\perp,\perp)}\sim X$, for $i\in\mathbb{N}$) also independent of the rv $N^{\perp}$ (with $N^{\perp}\sim N$). Within Model 2, we denote the corresponding aggregate claim amount rv by

$$S^{(\perp,\perp)}=\sum_{i=1}^{N^{\perp}}X_i^{(\perp,\perp)}=\sum_{i=1}^{\infty}X_i^{(\perp,\perp)}\times 1_{\{N^{\perp}\geq i\}}.$$

The properties of $S^{(\perp,\perp)}$ within Model 2 (classical risk model) have been thoroughly investigated in the actuarial literature (see, e.g., Rolski et al. (1999) and Klugman et al. (2009)). It is well known that the cdf of $S^{(\perp,\perp)}$ is as given in (7), and its expectation is given by

$$E[S^{(\perp,\perp)}]=E[N^{\perp}]\times E[X^{(\perp,\perp)}]=E[N]\times E[X], \tag{12}$$

which corresponds to (8). Also, from (9), we obtain the following classical result:

$$\mathcal{L}_{S^{(\perp,\perp)}}(t)=P_{N^{\perp}}\left(L_{X^{(\perp,\perp)}}(t)\right)=P_N\left(L_X(t)\right),t\geq 0,$$

where $P_N(s)=E\left[s^N\right]$, $s\in[0,1]$, corresponds to the pgf of the positive discrete rv N.

Model 3 is the collective risk model defined by the sequence $\left(N^{+},\ \underline{X}^{(+,+)}\right)\in\aleph$, where $N^{+},X_1^{(+,+)},X_2^{(+,+)},\ldots$ forms a sequence of comonotonic rvs (with $N^{+}\sim N$ and $X_i^{(+,+)}\sim X^{(+,+)}\sim X$, for $i\in\mathbb{N}$). This means that there exists a rv $U\sim Unif(0,1)$, such that, for any $k\in\mathbb{N}$, the representation

$$N^{+}=F_N^{-1}(U),X_1^{(+,+)}=F_X^{-1}(U),\ldots,X_k^{(+,+)}=F_X^{-1}(U) \tag{13}$$

holds, where F_N^{-1} and F_X^{-1} are the generalized inverse of the cdfs F_N and F_X respectively. Within Model 3, the corresponding aggregate claim amount rv defined in (1) and (2) is here denoted by

$$S^{(+,+)}=\sum_{i=1}^{N^{+}}X_i^{(+,+)}=\sum_{i=1}^{\infty}X_i^{(+,+)}\times 1_{\{N^{+}\geq i\}}. \tag{14}$$

Clearly, due to the representation in (13), it can be shown that (14) can also be defined as

$$S^{(+,+)}=N^{+}\times X^{(+,+)}. \tag{15}$$

See Liu and Wang (2017) for the proof of (15) and for additional properties of the collective risk model defined by the sequence $\left(N^{+},\underline{X}^{(+,+)}\right)$. Using the representation in (15), Liu and Wang (2017) show that

$$E\left[S^{(+,+)}\right]=E\left[N^{+}\times X^{(+,+)}\right]=E\left[F_N^{-1}(U)\times F_X^{-1}(U)\right], \tag{16}$$

assuming that the expectation exists. Note that even though the definitions of the aggregate claim amount in (15) and (11) appear similar, the one in (15) is the product of two comonotonic rvs, while in (11), the aggregate claim amount rv is the product of two independent rvs. A sufficient condition for $E[S^{(+,+)}]<\infty$ is that both $E[N^2]<\infty$ and $E[X^2]<\infty$.

Model 4 is the collective risk model defined by the sequence $\left(N^{-},\ \underline{X}^{(-,+)}\right)\in\aleph$, where $N^{-},X_1^{(-,+)},X_2^{(-,+)},\ldots$ form a sequence of rvs such that for any $k\in\mathbb{N}$, we have the following representation:

$$N^{-}=F_N^{-1}(U),X_1^{(-,+)}=F_X^{-1}(1-U),\ldots,X_k^{(-,+)}=F_X^{-1}(1-U), \tag{17}$$

in terms of the rv $U\sim Unif(0,1)$. Given (17), it implies that $\left(N^{-},X_i^{(-,+)}\right)$ forms a pair of counter-monotonic rvs, for $i\in\mathbb{N}$. Also, $X_1^{(-,+)},X_2^{(-,+)},\ldots$ is a sequence of comonotonic rvs. Within Model 4, the corresponding aggregate claim amount rv defined in (1) and (2) becomes

$$S^{(-,+)}=\sum_{i=1}^{N^{-}}X_i^{(-,+)}=\sum_{i=1}^{\infty}X_i^{(-,+)}\times 1_{\{N^{-}\geq i\}}. \tag{18}$$

From (17), the aggregate claim amount rv $S^{(-,+)}$ in (18) can also be represented as

$$S^{(-,+)}=F_N^{-1}(U)\times F_X^{-1}(1-U),$$

which implies that

$$E\left[S^{(-,+)}\right]=E\left[F_N^{-1}(U)\times F_X^{-1}(1-U)\right],$$

assuming that the expectation exists.

To sum up, the relation in (8) remains unchanged for a collective risk model assuming independence between the claim number rv and the sequence of (independent or dependent) claim amounts. However, as we have seen in (16) from the general expression in (5), the relation in (8) is no longer valid for a collective risk model defined by a sequence $(N,\underline{X})\in\aleph\backslash\aleph^{(\perp,\ )}$, i.e., for a collective risk model with a given dependence structure between the claim number rv and the sequence of claim amount rvs. In this case, one may observe either

$$E[S]<E[N]\times E[X]\text{ or }E[S]>E[N]\times E[X]. \tag{19}$$

In this paper, we consider a variety of collective risk models with dependence defined by sequences $(N,\underline{X})\in\aleph\backslash\aleph^{(\perp,\ )}$ and we examine the computation of F_S, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$.

3. Risk assessment and ordering of collective risk models

3.1. Risk assessment

Risk measures can be used to make the risk assessment of an insurance portfolio. The popular risk measures Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR) are used namely to determine the capital of an insurance portfolio. The VaR at a confidence level $\kappa \in (0, 1)$ of the rv S is defined as

$$VaR_{\kappa}(S) = F_S^{-1}(\kappa). \tag{20}$$

Note that, given the representation of $S^{(+,+)}$ in (15), Liu and Wang (2017) show that

$$VaR_{\kappa}(S^{(+,+)}) = VaR_{\kappa}(N^{+}) \times VaR_{\kappa}(X^{(+,+)}),$$

for $\kappa \in (0, 1)$.

Assuming $E[S] < \infty$, the TVaR at the confidence level $\kappa \in (0, 1)$ of the aggregate claim rv S is given by

$$TVaR_{\kappa}(S) = \frac{1}{1-\kappa}\int_{\kappa}^{1} VaR_u(S)\, du. \tag{21}$$

As shown in, e.g., Denuit et al. (2006), the following alternative expression for $TVaR_{\kappa}(S)$ can be found from (21)

$$TVaR_{\kappa}(S) = VaR_u(S) + \frac{1}{1-\kappa}\pi_S\left(VaR_u(S)\right),$$

where

$$\pi_S(x) = E\left[\max(S - x; 0)\right],$$

for $x \in \mathbb{R}$. See, e.g., McNeil et al. (2015) for properties and applications of these two risk measures. These quantities are crucial for actuaries and risk managers. Generally, numerical optimization methods need to be used to perform the inversion in (20). In the context of the present paper,

$$E\left[\max(S - x; 0)\right] = \sum_{n=1}^{\infty} \Pr(N = n) \times E\left[\max(X_1 + \cdots + X_n - x; 0)\,|N = n\right]$$

for $x \geq 0$.

Stochastic orders are used to compare risks according to how risky and dangerous they are. They have many applications in, e.g., actuarial science, applied probability, reliability, and economics. See Müller and Stoyan (2002), Denuit et al. (2006), and Shaked and Shanthikumar (2007) for a review on stochastic orders. In the following section, we use stochastic orders to compare the riskiness of two collective risk models defined with two different sequences belonging to $\aleph$.

3.2. Impact of dependence

We aim to examine the impact of the dependence relation between the components of the sequence $(N, \underline{X}) \in \aleph$ on the random sum defined in (2). To achieve this, we use the (increasing) convex order and the supermodular order. We recall their definitions.

Definition 1. Let X and X^* be two rvs with finite expectations. Then, X is said to be smaller than X^* according to the convex order (increasing convex order), denoted $X \preceq_{cx} X^*$ ($X \preceq_{icx} X^*$), if $E[\phi(X)] \leq E[\phi(X^*)]$ for all (increasing) convex function ϕ, when the expectations exist.

The convex and increasing convex orders are variability orders. Note that, if $X \preceq_{icx} X^*$ and $E[X] = E[X^*]$, then $X \preceq_{cx} X^*$. Proposition 3.4.8 of Denuit et al. (2006) provides an important result about the increasing convex order and the TVaR for applications in actuarial science and quantitative risk management:

$$X \preceq_{icx} X^* \text{ if and only if } TVaR_{\kappa}(X) \leq TVaR_{\kappa}(X^*),\ \forall\, \kappa \in (0, 1). \tag{22}$$

Aiming to compare two random vectors, we use the supermodular dependence order. Let us first recall the definition of a supermodular function.

Definition 2. Let $f : \mathbb{R}^k \to \mathbb{R}$. The function f is supermodular if the following inequality holds:

$$\begin{aligned} &f(x_1, \ldots, x_i + \epsilon, \ldots, x_j + \delta, \ldots, x_k) \\ &\quad - f(x_1, \ldots, x_i + \epsilon, \ldots, x_j, \ldots, x_k) \\ &\geq f(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_k) - f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_k), \end{aligned}$$

$\forall (x_1, \ldots, x_k) \in \mathbb{R}^k$, $\forall \epsilon, \delta > 0$, and $1 \leq i \leq j \leq k$.

Definition 3. Let $(X_1, \ldots, X_k)$ and $\left(X_1^*, \ldots, X_k^*\right)$ be two random vectors such that, for $i = 1, \ldots, k$, X_i and X_i^* have the same marginal distribution. Then, $\left(X_1^*, \ldots, X_k^*\right)$ is greater than $(X_1, \ldots, X_k)$ according to the supermodular order, denoted $(X_1, \ldots, X_k) \preceq_{sm} \left(X_1^*, \ldots, X_k^*\right)$, if $E\left[f(X_1, \ldots, X_k)\right] \leq E\left[f\left(X_1^*, \ldots, X_k^*\right)\right]$, for any supermodular function f, when the expectations exist.

For more properties and details on the convex, increasing convex, and supermodular orders, see e.g., Müller and Stoyan (2002), Denuit et al. (2006), and Shaked and Shanthikumar (2007).

It is shown (see, e.g., Bäuerle and Müller (2006)) that, for any $k \in \mathbb{N}$,

$$\text{if } (X_1, \ldots, X_k) \preceq_{sm} \left(X_1^*, \ldots, X_k^*\right) \text{ then } \sum_{i=1}^{k} X_i \preceq_{cx} \sum_{i=1}^{k} X_i^*. \tag{23}$$

For any $k \in \{2, 3, \ldots\}$, the result in (23) means that, for a finite sum, since the marginals are not modified, the dependence relation between the components of $(X_1, \ldots, X_k)$ does not have an impact on the expectation of their sum, i.e., for any $k \in \mathbb{N}$, we have

$$E\left[X_1 + \cdots + X_k\right] = E\left[X_1^* + \cdots + X_k^*\right]. \tag{24}$$

For any $k \in \mathbb{N}$, if $E\left[X_i^2\right] < \infty$ and $E\left[X_i^2\right] < \infty$, for $i \in (1, 2, \ldots, k)$, it follows from (23), that

$$Var(X_1) + \cdots + Var(X_k) \leq Var\left(X_1^*\right) + \cdots + Var\left(X_k^*\right). \tag{25}$$

Moreover, combining (23) with (22), we find

$$TVaR_{\kappa}(X_1 + \cdots +)\, TVaR_{\kappa}(X_k) \leq TVaR_{\kappa}\left(X_1^*\right) + \cdots + TVaR_{\kappa}\left(X_k^*\right), \tag{26}$$

for $\kappa \in (0, 1)$.

In the context of collective risk models with dependence, we are not looking only at the impact of the dependence between the claim amount rvs but rather at the impact of the dependence between all the components of the sequence $(N, \underline{X}) \in \aleph$ on the random sum defined in (2). The result in (26) can be generalized to the random sum (2) as shown in the next proposition. This is not possible however for the results given in (24) and (25).

Proposition 4. *We consider two collective risk models respectively defined by the sequences $(N, \underline{X}) \in \aleph$ and $(N^*, \underline{X}^*) \in \aleph$ such that*

$$(N, X_1, \ldots, X_k) \preceq_{sm} \left(N^*, X_1^*, \ldots, X_k^*\right),\ \forall k \in \mathbb{N}, \tag{27}$$

and the expectations of $S = \sum_{i=1}^{N} X_i$ and $S^* = \sum_{i=1}^{N^*} X_i^*$ exist. Then, we have

$$S \preceq_{icx} S^*. \tag{28}$$

By (22), we conclude

$$TVaR_{\kappa}(S) \leq TVaR_{\kappa}\left(S^*\right), \tag{29}$$

for $\kappa \in (0, 1)$. In particular, we have

$$E[S] \leq E\left[S^*\right]. \tag{30}$$

Proof. We introduce the rv $S_{1:k}$, defined by

$$S_{1:k} = \sum_{i=1}^{k} X_i \times 1_{\{N \geq i\}} = \phi_{1:k}(N, X_1, \ldots, X_k),$$

where

$$\phi_{1:k}(x_0, x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i \times 1_{\{x_0 \geq i\}} \tag{31}$$

is a supermodular function. Since, for any $k \in \mathbb{N}$, we have $(N, X_1, \ldots, X_k) \preceq_{sm} (N^*, X_1^*, \ldots, X_k^*)$ and given that the function $\phi_{1:k}$ defined in (31) is supermodular, it implies that

$$S_{1:k} \preceq_{icx} S_{1:k}^*, \tag{32}$$

for all $k \in \mathbb{N}$ (see Theorem 9.A.16 on page 399 of Shaked and Shanthikumar (2007) for more details). Letting $k \to \infty$ in (32), we obtain the result in (28). ■

In general, due to (5), we cannot conclude that

$$S = \sum_{i=1}^{N} X_i \preceq_{cx} \sum_{i=1}^{N^*} X_i^* = S^* \tag{33}$$

for two collective risk models respectively defined by the sequences $(N, \underline{X}) \in \aleph$ and $(N^*, \underline{X}^*) \in \aleph$ satisfying the condition in (27). It means that, for any $k \in \mathbb{N}$, if $E\left[X_i^2\right] < \infty$ and $E\left[X_i^{*2}\right] < \infty$, for $i \in (1, 2, \ldots, k)$, we cannot conclude from (28) that $Var\,(S) \leq Var\,(S^*)$.

As shown in the following result, the conclusion in (33) can be reached for collective risk models within the subclass $\aleph^{(\perp,)}$.

Corollary 5. *We consider two collective risk models respectively defined by the sequences* $(N^{\perp}, \underline{X}^{(\perp,)}) \in \aleph^{(\perp,)}$ *and* $(N^{\perp *}, \underline{X}^{(\perp,)*}) \in \aleph^{(\perp,)}$ *such that*

$$\left(N^{\perp}, X_1^{(\perp,)}, \ldots, X_k^{(\perp,)}\right) \preceq_{sm} \left(N^{\perp *}, X_1^{(\perp,)*}, \ldots, X_k^{(\perp,)*}\right), \ \forall k \in \mathbb{N},$$

and the expectations of $S^{(\perp,)} = \sum_{i=1}^{N^{\perp}} X_i^{(\perp,)}$ *and* $S^{(\perp,)*} = \sum_{i=1}^{N^{\perp *}} X_i^{(\perp,)*}$ *exist. Then, we have*

$$S^{(\perp,)} \preceq_{cx} S^{(\perp,)*}. \tag{34}$$

Proof. The result in (34) follows from (28) of Proposition 4 and (8). ■

In particular, under the conditions of Corollary 5, for any $k \in \mathbb{N}$, if $E\left[X_i^2\right] < \infty$ and $E\left[X_i^2\right] < \infty$, for $i \in (1, 2, \ldots, k)$, it follows from (34), that $Var\left(S^{(\perp,)}\right) \leq Var\left(S^{(\perp,)*}\right)$. In Sections 4 and 5, we present and investigate different families of collective risk models. Within these contexts, we provide additional conditions such that the result in (33) can also be established for two collective risk models respectively defined by the sequences $(N, \underline{X}) \in \aleph \backslash \aleph^{(\perp,)}$ and $(N^*, \underline{X}^*) \in \aleph \backslash \aleph^{(\perp,)}$.

Proposition 4 is also useful to compare any collective risk model defined with $(N, \underline{X}) \in \aleph$ to Model 3 defined with $(N^+, \underline{X}^{(+,+)}) \in \aleph$. For any sequence $(N, \underline{X}) \in \aleph$, we know that, for any $k \in \mathbb{N}$, $(N, X_1, \ldots, X_k) \preceq_{sm} (N^+, X_1^+, \ldots, X_k^+)$ (see, e.g., Theorem 9.A.21 of Shaked and Shanthikumar (2007)). Then, the next corollary follows from Proposition 4.

Corollary 6. *Consider a collective risk model defined by a sequence* $(N, \underline{X}) \in \aleph$ *and Model 3 defined with* $(N^+, \underline{X}^{(+,+)}) \in \aleph$*, such that the expectations of* S *and* $S^{(+,+)}$ *exist. Then,* $S \preceq_{icx} S^{(+,+)}$*. Also, by* (22),

$$TVaR_{\kappa}(S) \leq TVaR_{\kappa}\left(S^{(+,+)}\right), \forall \kappa \in (0, 1). \tag{35}$$

Remark 7. According to Corollary 6, the sequence $(N^+, \underline{X}^{(+,+)})$ is the extremal element of $\aleph$ under the supermodular order.

From (35), it is clear that $E[S^{(+,+)}] \geq E[S]$ for any collective risk model defined by a sequence $(N, \underline{X}) \in \aleph$. Note that Liu and Wang (2017) used another approach to obtain the result (35) in Corollary 6.

4. Three collective risk models with dependence

In this section, we investigate three collective risk models with dependence.

4.1. Collective risk model with multivariate mixed Erlang distributions

Inspired from Section 2.4 of Willmot and Woo (2015), we examine a collective risk model defined with multivariate mixed Erlang distributions. Univariate and multivariate mixed Erlang distributions are very flexible and can be applied in various contexts in actuarial science (see, e.g., Willmot and Lin (2011), Lee and Lin (2012), Cossette et al. (2015), and Willmot and Woo (2015) for details).

We first recall a few definitions related to mixed Erlang distributions. Let the rv X follow a univariate mixed Erlang distribution with

$$F_X(x) = \sum_{j=1}^{\infty} \gamma_J(j) H(x; j, \beta), \ \text{for } \beta > 0 \text{ and } x \geq 0, \tag{36}$$

where γ_J is the pmf of a strictly positive rv J, with pgf $\mathcal{P}_J(s) = \sum_{j=1}^{\infty} \gamma_J(j) s^j$ ($|s| \leq 1$), and $H(x; j, \beta)$ corresponds to the cdf of the Erlang distribution (with parameters $j \in N$ and $\beta > 0$), with

$$H(x; j, \beta) = 1 - e^{-\beta x} \sum_{l=0}^{j-1} \frac{(\beta x)^l}{l!}, \text{ for } x \geq 0.$$

The mixed Erlang distribution may be viewed as the compound distribution of a rv X defined by a random sum of iid rvs whose common distribution is exponential with mean $\frac{1}{\beta}$, i.e. $X = \sum_{j=1}^{J} C_j$, where $C_j \sim C$ is an exponential rv with mean $\frac{1}{\beta}$ and LST $\mathcal{L}_C(t) = \frac{\beta}{\beta + t}$, for $j \in \mathbb{N}$. The LST of X is hence given by

$$\mathcal{L}_X(t) = \mathcal{P}_J(\mathcal{L}_C(t)) = \mathcal{P}_J\left(\frac{\beta}{\beta + t}\right), \text{ for } t > 0.$$

Note that a mixed Erlang distribution is also called "mixture of Erlangs" distribution or "hyper-Erlang distribution", see, e.g., Fang (2001) and Lee and Lin (2010). See also, e.g., Lee and Lin (2010) and Willmot and Lin (2011) for details on univariate mixed Erlang distributions with applications in actuarial science.

For any $k \in \{2, 3, \ldots\}$, let $(X_1, \ldots, X_k)$ be a vector of identically distributed rvs which follows a multivariate mixed Erlang distribution with

$$F_{X_1,\ldots,X_k}(x_1, \ldots, x_k) = \sum_{j_1=1}^{\infty} \cdots \sum_{j_k=1}^{\infty} \gamma_{J_1,\ldots,J_k}(j_1, \ldots, j_k) \times H(x_1; j_1, \beta) \cdots H(x_k; j_k, \beta),$$

for $\beta > 0$, $x_1 \geq 0$, …, $x_k \geq 0$, $k \in \mathbb{N}$. Also, for any $k \in \{2, 3, \ldots\}$, $\gamma_{J_1,\ldots,J_k}(j_1, \ldots, j_k) \in [0, 1]$ corresponds to the joint pmf of a vector of identically distributed discrete rvs $(J_1, \ldots, J_k)$,

$$\gamma_{J_1,\ldots,J_k}(j_1, \ldots, j_k) = \Pr(J_1 = j_1, \ldots, J_k = j_k),$$

for $j_i \in \mathbb{N}$, $i = 1, 2, \ldots, k$, and with $\sum_{j_1=1}^{\infty} \cdots \sum_{j_k=1}^{\infty} \gamma_{J_1,\ldots,J_k}(j_1, \ldots, j_k) = 1$. See, e.g., Lee and Lin (2010) and Willmot and

Woo (2015) for details on the class of multivariate mixed Erlang distributions.

In Section 2.4 of Willmot and Woo (2015), the authors study the properties of a collective risk model with multivariate Erlang distribution which belongs to $\aleph^{(\perp,)} \subset \aleph$. In this section, we examine an extension of that class, relaxing the assumption of independence between the claim number rv N and the sequence of claim amounts $\underline{X}$. We consider a collective risk model defined by $\left(N, \underline{X}\right) \in \aleph$ such that, for any $k \in \mathbb{N}$, the random vector $(N, X_1, \ldots, X_k)$ follows a multivariate distribution with

$$E\left[1_{\{N=n, X_1 \leq x_1, \ldots, X_k \leq x_k\}}\right] = \sum_{j_1=1}^{\infty} \cdots \sum_{j_k=1}^{\infty} \gamma_{N,J_1,\ldots,J_k}(n, j_1, \ldots, j_k) \times H(x_1; j_1, \beta) \cdots H(x_k; j_k, \beta), \tag{37}$$

for $\beta > 0$, $n \in \mathbb{N}_0$, $x_1 \geq 0, \ldots, x_k \geq 0$. Note from (37) that for any $k \in \mathbb{N}$, the multivariate distribution of $(N, X_1, \ldots, X_k)$ has a discrete component (associated to the rv N) and a (multivariate) continuous component (associated to the $(X_1, \ldots, X_k)$). For any $k \in \mathbb{N}$, the joint pmf of $(N, J_1, \ldots, J_k)$ corresponds to $\gamma_{N,J_1,\ldots,J_k}(n, j_1, \ldots, j_k)$ ($n \in \mathbb{N}_0$ and $j_1, \ldots, j_k \in \mathbb{N}$) with

$$\sum_{n=0}^{\infty}\sum_{j_1=1}^{\infty} \cdots \sum_{j_k=1}^{\infty} \gamma_{N,J_1,\ldots,J_k}(n, j_1, \ldots, j_k) = 1.$$

Then, for such a collective risk model defined by $\left(N, \underline{X}\right) \in \aleph$, it implies that, for any $k \in \mathbb{N}$, the joint cdf of $(N, X_1, \ldots, X_k)$ is

$$F_{N,X_1,\ldots,X_k}(n, x_1, \ldots, x_k) = \sum_{j_0=0}^{n}\sum_{j_1=1}^{\infty} \ldots \sum_{j_k=1}^{\infty} \gamma_{N,J_1,\ldots,J_k}(j_0, j_1, \ldots, j_k) H(x_1; j_1, \beta) \ldots H(x_k; j_k, \beta), \tag{38}$$

for $\beta > 0$, $n \in \mathbb{N}_0$, $x_1 \geq 0, \ldots, x_k \geq 0$.

Assume that if, for any $k \in \mathbb{N}$, we have

$$\gamma_{N,J_1,\ldots,J_k}(n, j_1, \ldots, j_k) = \gamma_N(n)\, \gamma_{J_1,\ldots,J_k}(j_1, \ldots, j_k) \quad \text{(for all } n \in \mathbb{N}_0 \text{ and for all } (j_1, \ldots, j_k) \in \mathbb{N}^k). \tag{39}$$

Then, (38) becomes

$$F_{N,X_1,\ldots,X_k}(n, x_1, \ldots, x_k) = F_N(n) \times \sum_{j_1=1}^{\infty} \cdots \sum_{j_k=1}^{\infty} \gamma_{J_1,\ldots,J_k}(j_1, \ldots, j_k) H(x_1; j_1, \beta) \cdots H(x_k; j_k, \beta), \tag{40}$$

for $\beta > 0$, $n \in \mathbb{N}_0$, $x_1 \geq 0, \ldots, x_k \geq 0$. It follows that the rv N and the sequence $\underline{X}$ are independent, meaning that the corresponding collective risk models are those defined with the sequence belonging to $\aleph^{(\perp,)}$. Note that Models 1 and 3 with mixed Erlang marginals for the claim amount cannot be obtained as special cases of the collective risk model with multivariate mixed Erlang distributions. Model 2 corresponds to a special case where, for any $k \in \{2, 3, \ldots\}$, we have

$$\gamma_{J_1,\ldots,J_k}(j_1, \ldots, j_k) = \gamma_{J_1}(j_1) \times \cdots \times \gamma_{J_k}(j_k)$$

(for all $(j_1, \ldots, j_k) \in \mathbb{N}^k$) in (39), which implies that (40) becomes

$$F_{N,X_1,\ldots,X_k}(n, x_1, \ldots, x_k) = F_N(n) \times \prod_{i=1}^{k}\sum_{j_i=1}^{\infty} \gamma_{J_i}(j_i) H(x_i; j_i, \beta) = F_N(n) \times \prod_{i=1}^{k} F_{X_i}(x_i),$$

for $\beta > 0$, $n \in \mathbb{N}_0$, $x_1 \geq 0, \ldots, x_k \geq 0$.

Let $L = \sum_{i=1}^{N} J_i$ be a non-negative discrete rv with pmf

$$\Pr(L = l) = \zeta_l = \begin{cases} \Pr(N=0) & , \quad l = 0 \\ \sum_{i=1}^{l} \Pr(N = i, J_1 + \cdots + J_i = l) & , \quad l \in \mathbb{N} \end{cases} = \begin{cases} \Pr(N=0) & , \quad l = 0 \\ \sum_{i=1}^{l} \sum_{j_1+\cdots+j_i=l} \gamma_{N,J_1,\ldots,J_{l-i+1}}(i, j_1, \ldots, j_i) & , \quad l \in \mathbb{N} \end{cases} \tag{41}$$

and pgf $\mathcal{P}_L(s) = \sum_{l=0}^{\infty} \zeta_l s^l$, $|s| \leq 1$. Then, the rv S follows a mixed Erlang distribution with

$$F_S(x) = \zeta_0 + \sum_{l=1}^{\infty} \zeta_l H(x; l, \beta), \; x \geq 0, \tag{42}$$

and

$$\mathcal{L}_S(t) = \mathcal{P}_L\left(\frac{\beta}{\beta + t}\right), \text{ for } t \geq 0.$$

From (42), the $TVaR_\kappa(S)$ is

$$TVaR_\kappa(S) = \frac{1}{1-\kappa}\sum_{l=1}^{\infty} \zeta_l \frac{l}{\beta} \overline{H}(VaR_\kappa(S); l+1, \beta), \text{ for } \kappa \in (0, 1),$$

where $\overline{H}(x; l, \beta) = 1 - H(x; l, \beta)$, $VaR_\kappa(S) = 0$, for $\kappa \in (0, \zeta_0]$, and $VaR_\kappa(S)$ for $\kappa \in (\zeta_0, 1)$, is found with numerical optimization tools.

In the following corollary, we use Proposition 4 to examine the impact of dependence within the collective risk model defined with a multivariate mixed Erlang distribution.

Corollary 8. *We consider two collective risk models with sequences* $(N, \underline{X}) \in \aleph$ *and* $(N^*, \underline{X}^*) \in \aleph$ *defined with multivariate mixed Erlang distributions, where, for any* $k \in \mathbb{N}$, *the joint cdf of* $(N, X_1, \ldots, X_k)$ *(and* $\left(N^*, X_1^*, \ldots, X_k^*\right)$*) is of the form given in (38). For* $i \in \mathbb{N}$, *define* $X_i = \sum_{j=1}^{J_i} C_{i,j}$, *where* $C_{i,1}, C_{i,2}, \ldots$ *are iid (with* $C_{i,l} \sim Exp(\beta)$*) and independent of* J_i. *Similarly, for* $i \in \mathbb{N}$, *define also* $X_i^* = \sum_{j=1}^{J_i^*} C_{i,j}^*$, *where* $C_{i,1}^*, C_{i,2}^*, \ldots$ *are iid (with* $C_{i,l}^* \sim Exp(\beta)$*) and independent of* J_i^*. *For any* $k \in \mathbb{N}$, *let the vectors* $(N, J_1, \ldots, J_k)$ *and* $(N, J_1^*, \ldots, J_k^*)$ *be such that*

$$(N, J_1, \ldots, J_k) \preceq_{sm} \left(N^*, J_1^*, \ldots, J_k^*\right). \tag{43}$$

Then, for any $k \in \mathbb{N}$, *it follows that*

$$(N, X_1, \ldots, X_k) \preceq_{sm} \left(N^*, X_1^*, \ldots, X_k^*\right), \tag{44}$$

$$S = \sum_{i=1}^{N} X_i \preceq_{icx} \sum_{i=1}^{N^*} X_i^* = S^*, \tag{45}$$

and

$$TVaR_\kappa(S) \leq TVaR_\kappa\left(S^*\right), \text{ for } \kappa \in (0, 1). \tag{46}$$

Proof. Using Proposition 2 of Denuit et al. (2002) with (43) and letting $N = \sum_{i=1}^{N} 1$, we obtain (44). Then, (45) and (46) follow from Proposition 4. ■

4.2. Collective risk model with mixing

In this section, we consider a collective risk model defined with sequences $\left(N, \underline{X}\right) \in \aleph$, in which we use bivariate copulas to account for the dependence between the components of $(N, \underline{X})$. Our approach differs from what has been proposed by Czado et al. (2012) and Krämer et al. (2013) who used bivariate copulas to model the dependence between the claim count and the average claim amount. In our approach, the dependence structure is a two-level hierarchical structure. At the first level, we introduce a strictly positive discrete common mixing rv $\mathcal{V}$ and we assume a dependence relation between the claim counting rv N and the mixing rv $\mathcal{V}$ with pmf $\gamma_{\mathcal{V}}$. Such a dependence structure can be easily modeled using a bivariate copula. The bivariate pmf of the pair of discrete rvs $(N, \mathcal{V})$ is denoted by

$$\gamma_{N,\mathcal{V}}(n, \nu) = \Pr(N = n, \mathcal{V} = \nu), \text{ for } n \in \mathbb{N}_0 \text{ and } \nu \in \mathbb{N}.$$

At the second level, the dependence structure between the claim rvs $\underline{X}$ is assumed to be induced via the common mixing rv $\mathcal{V}$. Given $\mathcal{V} = \nu$ for $\nu \in \mathbb{N}$, $(X_1|\mathcal{V} = \nu), (X_2|\mathcal{V} = \nu), \ldots$ are assumed to be conditionally independent. For any $i \in \mathbb{N}$, we can construct the mixed distribution of the rv X_i either by using the conditional cdf of $(X_i|\mathcal{V} = \nu)$, for $\nu \in \mathbb{N}$, or by using its conditional survival function, i.e.,

$$F_{X_i}(x) = \sum_{\nu=1}^{\infty} \gamma_{\mathcal{V}}(\nu) F_{X_i|\mathcal{V}=\nu}(x), x \geq 0, \tag{47}$$

or

$$\overline{F}_{X_i}(x) = \sum_{\nu=1}^{\infty} \gamma_{\mathcal{V}}(\nu) \overline{F}_{X_i|\mathcal{V}=\nu}(x), x \geq 0. \tag{48}$$

If the distribution of the X_i's is defined using (47), then, for any $k \in \{2, 3, \ldots\}$, the joint cdf of $(X_1, \ldots, X_k)$ is given by

$$F_{X_1,\ldots,X_k}(x_1, \ldots, x_k) = \sum_{\nu=1}^{\infty} \gamma_{\mathcal{V}}(\nu) \prod_{i=1}^{k} F_{X_i|\mathcal{V}=\nu}(x_i), x_i \geq 0. \tag{49}$$

Also, for any $k \in \mathbb{N}$, the joint cdf of $(N, X_1, \ldots, X_k)$ is given by

$$F_{N,X_1,\ldots,X_k}(n, x_1, \ldots, x_k) = \sum_{l=0}^{n} \sum_{\nu=1}^{\infty} \gamma_{N,\mathcal{V}}(l, \nu) \prod_{i=1}^{k} F_{X_i|\mathcal{V}=\nu}(x_i), \quad n \in \mathbb{N}_0, x_1, \ldots, x_k \geq 0. \tag{50}$$

When the distribution of the X_i's is defined using (48), then, for any $k \in \{2, 3, \ldots\}$, the joint survival function of $(X_1, \ldots, X_k)$ is given by

$$\overline{F}_{X_1,\ldots,X_k}(x_1, \ldots, x_k) = \sum_{\nu=1}^{\infty} \gamma_{\mathcal{V}}(\nu) \prod_{i=1}^{k} \overline{F}_{X_i|\mathcal{V}=\nu}(x_i), x_i \geq 0. \tag{51}$$

Also, for any $k \in \mathbb{N}$, the joint survival function of $(N, X_1, \ldots, X_k)$ is given by

$$\overline{F}_{N,X_1,\ldots,X_k}(n, x_1, \ldots, x_k) = \sum_{l=n+1}^{\infty} \sum_{\nu=1}^{\infty} \gamma_{N,\mathcal{V}}(l, \nu) \prod_{i=1}^{k} \overline{F}_{X_i|\mathcal{V}=\nu}(x_i), \tag{52}$$

or $n \in \mathbb{N}_0, x_1, \ldots, x_k \geq 0$.

In both cases, i.e., with (50) or , the cdf of S in (3) and the LST of S in (6) respectively become

$$F_S(x) = \Pr(N = 0) + \sum_{n=1}^{\infty} \sum_{\nu=1}^{\infty} \gamma_{N,\mathcal{V}}(n, \nu) F_{X_1+\cdots+X_n|\mathcal{V}=\nu}(x), x \geq 0, \tag{53}$$

and

$$\mathcal{L}_S(t) = \Pr(N = 0) + \sum_{n=1}^{\infty} \sum_{\nu=1}^{\infty} \gamma_{N,\mathcal{V}}(n, \nu) \left(\mathcal{L}_{X|\mathcal{V}=\nu}(t)\right)^n, t \geq 0. \tag{54}$$

Let us consider specifically a collective risk model by mixing defined with sequences $\left(N, \underline{X}\right) \in \aleph$, where each X_i $(i \in \mathbb{N})$ follows the same univariate mixed Erlang distribution. The cdf of X_i $(i \in \mathbb{N})$ given in (47) becomes

$$F_{X_i}(x) = \sum_{\nu=1}^{\infty} \gamma_{\mathcal{V}}(\nu) H(x; \nu, \beta), x \geq 0, \tag{55}$$

and, for any $k \in \{2, 3, \ldots\}$, the joint cdf of $(X_1, \ldots, X_k)$ given in (49) becomes

$$F_{X_1,\ldots,X_k}(x_1, \ldots, x_k) = \sum_{\nu=1}^{\infty} \gamma_{\mathcal{V}}(\nu) H(x_1; \nu, \beta) \cdots H(x_k; \nu, \beta), x_1, \ldots, x_k \geq 0, \tag{56}$$

with $\beta > 0$. Note that in this case, the mixing occurs over the shape parameter. Also, if for any $k \in \{2, 3, \ldots\}$, $J_1, \ldots, J_k$ in Section 4.1 are comonotonic rvs with $J_i = F_{\mathcal{V}}^{-1}(U)$, where $U \sim Unif(0, 1)$, then

$$\gamma_{J_1,\ldots,J_k}(j_1, \ldots, j_k) = \begin{cases} \gamma_{\mathcal{V}}(j), & j_1 = j_2 = \cdots = j_k = j \\ 0, & \text{otherwise.} \end{cases}$$

Then, the resulting collective risk model with mixing is a special case of the collective risk model based on the multivariate mixed Erlang distribution, presented in Section 4.1. For any $k \in \mathbb{N}$, the joint cdf of $(N, X_1, \ldots, X_k)$ in (50) and (38) becomes

$$F_{N,X_1,\ldots,X_k}(n, x_1, \ldots, x_k) = \sum_{l=0}^{n} \sum_{\nu=1}^{\infty} \gamma_{N,\mathcal{V}}(l, \nu) \times H(x_1; \nu, \beta) \cdots H(x_k; \nu, \beta), n \in \mathbb{N}_0, x_1, \ldots, x_k \geq 0, \tag{57}$$

which leads to the following cdf of S:

$$F_S(x) = \Pr(N = 0) + \sum_{n=1}^{\infty} \sum_{\nu=1}^{\infty} \gamma_{N,\mathcal{V}}(n, \nu) H(x; n\nu, \beta), x \geq 0. \tag{58}$$

The $TVaR_\kappa(S)$ is given by

$$TVaR_\kappa(S) = \frac{1}{1-\kappa} \sum_{n=1}^{\infty} \sum_{\nu=1}^{\infty} \gamma_{N,\mathcal{V}}(n, \nu) \frac{n\nu}{\beta} \times \overline{H}(VaR_\kappa(S); n\nu + 1, \beta), \text{ for } \kappa \in (0, 1), \tag{59}$$

where $VaR_\kappa(S) = 0$, for $\kappa \in (0, \zeta_0]$, and, when $\kappa \in (\zeta_0, 1)$, $VaR_\kappa(S)$ is found with (58) and using numerical optimization tools. As in Willmot and Woo (2007), the rv S follows a mixed Erlang distribution where its cdf in (58) reduces to (42) with probabilities

$$\zeta_l = \begin{cases} \Pr(N = 0) & , \quad l = 0 \\ \sum_{n=1}^{l} \gamma_{N,\mathcal{V}}\left(n, \frac{l}{n}\right) \times 1_{\left\{\frac{l}{n} = \left[\frac{l}{n}\right]\right\}} & , \quad l \in \mathbb{N} \end{cases}.$$

Also, the expectation of the aggregate claim amount rv S is

$$E[S] = \sum_{n=1}^{\infty} \sum_{\nu=1}^{\infty} \gamma_{N,\mathcal{V}}(n, \nu) \frac{n\nu}{\beta} = \sum_{l=1}^{\infty} \zeta_l \frac{l}{\beta}. \tag{60}$$

From (60), we observe that $E[S] \neq E[N] \times E[X]$, if N and $\mathcal{V}$ are not independent. Model 2 can be obtained as a special case in this collective risk model when $N^{\perp} \sim N$ and $\mathcal{V}^{\perp} \sim \mathcal{V}$ are independent, where

$$\gamma_{N^{\perp},\mathcal{V}^{\perp}}(n, \nu) = \gamma_{N^{\perp}}(n) \times \gamma_{\mathcal{V}^{\perp}}(\nu) = \gamma_N(n) \times \gamma_{\mathcal{V}}(\nu), \tag{61}$$

for $n \in \mathbb{N}_0$ and $\nu \in \mathbb{N}$.

As expected, it follows from (61) that

$$E\left[S^{(\perp,\perp)}\right] = \sum_{n=1}^{\infty} \sum_{\nu=1}^{\infty} \gamma_N(n) \times \gamma_{\mathcal{V}}(\nu) \frac{n\nu}{\beta} = E[N] \times E[X].$$

Let us consider another collective risk model with mixing, in which, for any $k \in \mathbb{N}$, $(X_1, \ldots, X_k)$ follows a multivariate Moran–Downton exponential distribution. The latter is constructed as follows. Assume in (55)–(57) that the mixing rv $\mathcal{V}$ follows a geometric distribution with $\gamma_{\mathcal{V}}(\nu) = (1-\theta)\theta^{\nu-1}$ ($\nu \in \mathbb{N}$ and $\theta \in [0,1)$) and the parameter β associated to the Erlang components is $\beta = \frac{\lambda}{1-\theta}$ ($\lambda > 0$). As it is explained in, e.g., Kotz et al. (2004) and Cossette et al. (2015), these two additional assumptions imply that $X_i \sim Exp(\lambda)$, $i \in \mathbb{N}$, and that, for any $k \in \{2, 3, \ldots\}$, $(X_1, \ldots, X_k)$ follows a Moran–Downton's multivariate exponential distribution with dependence parameter $\theta \in [0, 1)$. For any $k \in \{2, 3, \ldots\}$, the Pearson correlation coefficient is $\rho_P(X_i, X_{i'}) = \theta$, for $i \neq i' \in \{1, 2, \ldots, k\}$. For any $k \in \{2, 3, \ldots\}$, $\theta = 0$ corresponds to the case where the components of $(X_1, \ldots, X_k)$ are independent while $\theta \to 1$ corresponds to the case where the components of $(X_1, \ldots, X_k)$ are comonotonic. Finally, we define the joint cdf of the pair of rvs $(N, \mathcal{V})$ by

$$F_{N,\mathcal{V}}(n, \nu) = C(F_N(n), F_{\mathcal{V}}(\nu)), \text{ for } n \in \mathbb{N}_0 \text{ and } \nu \in \mathbb{N},$$

where C is a copula with dependence parameter α. The cdf and the TVaR of S are respectively given by (58) and (59) where $\beta = \frac{\lambda}{1-\theta}$ and

$$\begin{aligned}\gamma_{N,\mathcal{V}}(n, \nu) &= C(F_N(n), F_{\mathcal{V}}(\nu)) + C(F_N(n-1), F_{\mathcal{V}}(\nu-1)) \\ &\quad - C(F_N(n-1), F_{\mathcal{V}}(\nu)) - C(F_N(n), F_{\mathcal{V}}(\nu-1)),\end{aligned}$$

for $(n, \nu) \in \mathbb{N} \times \mathbb{N}$ and $C(F_N(n), F_{\mathcal{V}}(\nu)) = 0$ when $\nu = 0$. If C is a complete copula (e.g., Frank copula), the collective risk model defined with the multivariate Moran–Downton exponential distribution, includes Models 1, 2, 3 and 4 with exponentially distributed claim amounts as special cases.

We provide below a numerical illustration in which we have recourse to the Gumbel and Frank bivariate copulas for the joint distribution of $(N, \mathcal{V})$.

Example 9. We consider a collective risk model with mixing, in which N follows a Poisson distribution with mean 5 and, for any $k \in \mathbb{N}$, $(X_1, \ldots, X_k)$ follows a multivariate Moran–Downton exponential distribution (with $\lambda = 1$). We consider two different copulas C for the bivariate distribution of $(N, \mathcal{V})$. First, a Gumbel copula C with $\alpha \in \{1, 1.5, 4\}$, and $\theta \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$. When $\alpha = 1$, N and $\underline{X}$ are independent, which means that the corresponding collective risk model is defined with a sequence $\left(N^{\perp}, \underline{X}^{(\perp,)}\right) \in \aleph^{(\perp,)}$ and the aggregate claim amount rv is denoted by $S^{(\perp,)}$. Model 1 with exponentially distributed claim amounts is found as a special case when $\alpha = 1$ and $\theta \to 1$. Model 2 with exponentially distributed claim amounts is obtained as a special case when $\alpha = 1$ and $\theta = 0$. Model 3 with exponentially distributed claim amounts is obtained as a special case when $\alpha \to \infty$ and $\theta \to 1$. In Tables 1 and 2, we provide the values of the expectation, the variance, the VaR and the TVaR of the rvs S, $S^{(\perp,\perp)}$, and $S^{(+,+)}$. The values of their cdfs are shown in Fig. 1. We observe the following with the Gumbel copula:

- for a fixed value of θ, the values of $E[S]$ and $TVaR_\kappa(S)$ increase as the dependence parameter α of the copula C increases;
- for a fixed value of α, the values of $E[S]$ and $TVaR_\kappa(S)$ increase, when the dependence parameter θ increases.

Second, we want to investigate the impact of a negative dependence relation between the counting rv N and the mixing rv $\mathcal{V}$. For that reason, we choose a Frank copula C with $\alpha \in \{-10, -5, 5, 10\}$ and $\theta = \frac{1}{3}$. When $\alpha \to 0$, N and $\underline{X}$ are independent, which means that the corresponding collective risk model is defined with a sequence $\left(N^{\perp}, \underline{X}^{(\perp,)}\right) \in \aleph^{(\perp,)}$ and the aggregate claim amount rv is denoted by $S^{(\perp,)}$. Model 1 with exponentially distributed claim amounts is found as a special case when $\alpha \to 0$ and $\theta \to 1$. Model 2 with exponentially distributed claim amounts is obtained as a special case when $\alpha \to 0$ and $\theta = 0$. Model 3 with exponentially distributed claim amounts is obtained as a special case when $\alpha \to \infty$, and $\theta \to 1$. We give the values of the expectation, the variance, the VaR and the TVaR of the rvs S, $S^{(\perp,\perp)}$, and $S^{(+,+)}$ in Table 3. The values of their cdfs are shown in Fig. 2. Based on those results using the Frank copula, one observes the following:

- for a fixed value of θ and for $\alpha > 0$, the values of $E[S]$ and $TVaR_\kappa(S)$ increase as the dependence parameter α of the Frank copula C increases; also, $E[S] \geq E\left[S^{(\perp,\perp)}\right]$;
- for a fixed value of θ and for $\alpha < 0$, the values of $E[S]$ and $TVaR_\kappa(S)$ decrease as the dependence parameter α of the Frank copula C decreases; also, $E[S] \leq E\left[S^{(\perp,\perp)}\right]$.

Note that $\alpha > 0$ ($\alpha < 0$) corresponds to a positive (negative) dependence between the claim number rv N and the mixing rv $\mathcal{V}$, which induces a positive (negative) dependence relation between N and the amount of the ith claim, for $i \in N$.

4.3. Collective risk models with multivariate copulas

Another strategy to model the dependence between the components of $(N, \underline{X}) \in \aleph$ is to consider multivariate copulas. Since

Table 1
Collective risk model defined with the Gumbel copula with different values of the dependence parameter α, and the Moran–Downton's multivariate exponential distribution with dependence parameter $\theta = \frac{1}{3}$.

	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 4$	$(N^{\perp}, X^{(\perp,\perp)})$	$(N^{+}, X^{(+,+)})$
$E[S]$	5	5.64	6.03	5	7.09
$Var(S)$	18.33	39.87	51.15	10	108.99
$VaR_{0.9}(S)$	10.16	12.34	14.04	9.28	18.42068
$VaR_{0.99}(S)$	20.98	31.24	34.77	14.40	50.66
$VaR_{0.999}(S)$	33.44	56.54	60.93	18.85	89.80
$TVaR_{0.9}(S)$	14.81	20.30	22.85	11.55	31.65
$TVaR_{0.99}(S)$	26.34	42.05	45.97	16.35	67.39
$TVaR_{0.999}(S)$	39.43	69.56	74.20	20.64	111.76

Table 2
Collective risk model defined with the Gumbel copula with different values of the dependence parameter α, and the Moran–Downton's multivariate exponential distribution with dependence parameter $\theta = \frac{2}{3}$.

	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 4$	$(N^{\perp}, X^{(\perp,\perp)})$	$(N^{+}, X^{(+,+)})$
$E[S]$	5	5.95	6.58	5	7.09
$Var(S)$	26.67	62.01	79.54	10	108.98
$VaR_{0.9}(S)$	11.19	14.07	16.28	9.28	18.42
$VaR_{0.99}(S)$	24.88	38.43	42.98	14.40	50.66
$VaR_{0.999}(S)$	40.89	71.56	77.20	18.85	89.80
$TVaR_{0.9}(S)$	17.05	24.30	27.58	11.55	31.65
$TVaR_{0.99}(S)$	31.77	52.57	57.63	16.35	67.39
$TVaR_{0.999}(S)$	48.67	88.73	94.78	20.64	111.76

Table 3
Collective risk model defined with the Frank copula with different values of the dependence parameter α, and the Moran–Downton's multivariate exponential distribution with dependence parameter $\theta = \frac{1}{3}$.

	$\alpha = -10$	$\alpha = -5$	$\alpha = 5$	$\alpha = 10$	$(N^{\perp}, X^{(\perp,\perp)})$	$(N^{+}, X^{(+,+)})$
$E[S]$	4.23	4.42	5.62	5.84	5	7.09
$Var(S)$	5.39	7.85	32.35	38.89	10	108.98
$VaR_{0.9}(S)$	7.26	7.89	12.90	13.90	9.28	18.42
$VaR_{0.99}(S)$	11.28	13.69	26.96	29.44	14.40	50.66
$VaR_{0.999}(S)$	15.88	21.42	41.53	44.99	18.85	89.80
$TVaR_{0.9}(S)$	9.04	10.39	19.00	20.62	11.55	31.65
$TVaR_{0.99}(S)$	13.26	16.99	33.27	36.19	16.35	67.39
$TVaR_{0.999}(S)$	18.46	25.60	48.22	51.99	20.64	111.76

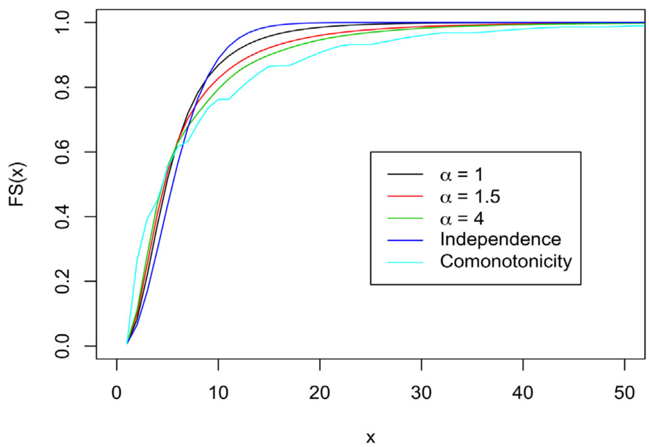

Fig. 1. Cdf of S for different dependence structures as defined in Example 9: Gumbel copula C, for different values of α, comonotonicity, and independence.

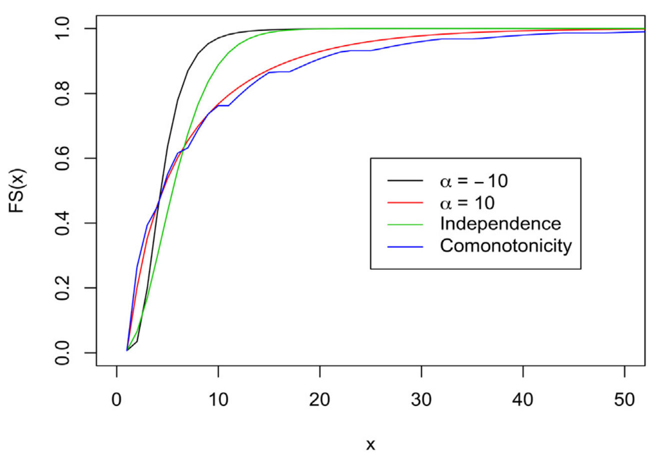

Fig. 2. Cdf of S for different dependence structures as defined in Example 9: Frank copula C, for different values of α, comonotonicity, and independence.

we aim to propose dependence models for which the quantification of the aggregate claim amount is feasible, we consider, for the remainder of the paper, multivariate copulas that naturally extend to any dimension $k \in \{2, 3, \ldots\}$.

We consider a collective risk model defined by a sequence $(N, \underline{X}) \in \aleph$ where, for any $k \in \mathbb{N}$, the multivariate cdf of $(N, X_1, \ldots, X_k)$ (or its multivariate survival function) is defined with a $(k+1)$-dimensional copula C and univariate cdfs $F_N, F_{X_1}, \ldots, F_{X_k}$ (or the univariate survival functions $\overline{F}_N, \overline{F}_{X_1}, \ldots, \overline{F}_{X_k}$), i.e., for $k \in \mathbb{N}$, either

$$F_{N,X_1,\ldots,X_k}(n, x_1, \ldots, x_k) = C\left(F_N(n), F_{X_1}(x_1), \ldots, F_{X_k}(x_k)\right), \tag{62}$$

or

$$\overline{F}_{N,X_1,\ldots,X_k}(n, x_1, \ldots, x_k) = C\left(\overline{F}_N(n), \overline{F}_{X_1}(x_1), \ldots, \overline{F}_{X_k}(x_k)\right), \tag{63}$$

where $n \in \mathbb{N}_0$ and $(x_1, \ldots, x_k) \in [0, \infty)^k$.

For any $k \in \mathbb{N}$, let $(U_0, U_1, \ldots, U_k)$ be a vector of rvs, with $U_i \sim Unif\ (0, 1)$ $(i = 0, 1, \ldots, k)$, for which the dependence structure is induced via a $(k+1)$-dimensional copula C. Let $\left(N, \underline{X}\right) \in \aleph$ such that, for any $k \in \mathbb{N}$, the discrete rv $N = F_N^{-1}(U_0)$, and the continuous rv $X_i = F_{X_i}^{-1}(U_i)$, for $i = 1, 2, \ldots, k$. For any $k \in \mathbb{N}$, it implies that the joint cdf of $(N, X_1, \ldots, X_k)$ is given in (62). Note that, for any $k \in \mathbb{N}$, the multivariate distribution of $(N, X_1, \ldots, X_k)$ is not absolutely continuous, where the joint probability function of $(N, X_1, \ldots, X_k)$ is given by

$$\begin{aligned} & f_{N,X_1,\ldots,X_k}(n, x_1, \ldots, x_k) \\ &= \frac{\partial^k}{\partial x_1 \ldots \partial x_k} C\left(F_N(n), F_{X_1}(x_1), \ldots, F_{X_k}(x_k)\right) \\ &\quad - \frac{\partial^k}{\partial x_1 \cdots \partial x_k} C\left(F_N(n-1), F_{X_1}(x_1), \ldots, F_{X_k}(x_k)\right) \end{aligned} \tag{64}$$

for $n \in N_0$ and $(x_1, \ldots, x_k) \in [0, \infty)^k$.

A well known and flexible multivariate copula that can be generalized to any dimension $k \in \mathbb{N}$ is the Eyraud–Farlie–Gumbel–Morgenstern (EFGM) copula. The EFGM copula is generated as a perturbation of the independence copula allowing for more flexibility in the dependence structure. See e.g., Durante and Sempi (2010) and references therein for more details. For any $k \in \mathbb{N}$, let $(U_0, U_1, \ldots, U_k)$ be a vector of rvs, with $U_i \sim Unif\ (0, 1)$ $(i = 0, 1, \ldots, k)$, for which the dependence structure is induced via the following $(k+1)$-dimensional copula C:

$$\begin{aligned} C(u_0, u_1, \ldots, u_k) &= u_0 u_1, \ldots, u_k \\ &\quad \times \left(1 + \alpha \sum_{i=1}^{k} (1-u_0)(1-u_i)\right) \\ &= u_0 u_1 \ldots u_k + \alpha u_0 u_1 \ldots u_k \\ &\quad \times \sum_{i=1}^{k} (1-u_0)(1-u_i), \end{aligned} \tag{65}$$

where the dependence parameter $\alpha \in [-1, 1]$ and for $(u_0, u_1, \ldots, u_k) \in [0, 1]^{k+1}$. The copula C defined in (65) is a special case of the generalized EFGM copula family presented in Durante and Sempi (2010). For any $k \in \mathbb{N}$, such a copula allows a dependence structure for the $k+1$ rvs $U_0, U_1, \ldots, U_k$, in which there is a dependence relationship between U_0 and each U_i, for $i = 1, 2, \ldots, k$, meaning, there is only a dependence between U_0 and U_1, U_0 and U_2, etc. Note that the copula C defined in (65) becomes the independence copula when $\alpha = 0$, inducing Model 2 as a special case. The joint probability function of $(N, X_1, \ldots, X_k)$ in (64) becomes

$$\begin{aligned} & f_{N,X_1,\ldots,X_k}(n, x_1, \ldots, x_k) \\ &= F_N(n) \prod_{i=1}^{k} f_{X_i}(x_i) + \alpha F_N(n)(1 - F_N(n)) \\ &\times \prod_{i=1}^{k} f_{X_i}(x_i) \sum_{i=1}^{k} \left(1 - 2F_{X_i}(x_i)\right) \\ &- F_N(n-1) \prod_{i=1}^{k} f_{X_i}(x_i) - \alpha F_N(n-1)(1 - F_N(n-1)) \\ &\times \prod_{i=1}^{k} f_{X_i}(x_i) \sum_{i=1}^{k} \left(1 - 2F_{X_i}(x_i)\right), \end{aligned}$$

where $n \in \mathbb{N}_0$, $(x_1, \ldots, x_k) \in [0, \infty)^k$, and $F_N(-1) = 0$.

If we further assume that $X_i \sim Exp(\beta), i = 1, 2, \ldots, k$, for $k \in \mathbb{N}$, we can obtain the following explicit formula for the cdf of the aggregate loss S

$$\begin{aligned} F_S(x) &= \Pr(N = 0) + \sum_{k=1}^{\infty} \Pr(N = k, S_k \le x) \\ &= \Pr(N = 0) + \sum_{k=1}^{\infty} \Pr(N = k)\, H(x; k, \beta) \\ &\quad + \alpha F_N(1)(1 - F_N(1))\, (H(x; 1, 2\beta) - H(x; 1, \beta)) \\ &\quad - \alpha F_N(0)(1 - F_N(0))\, (H(x; 1, 2\beta) - H(x; 1, \beta)) \end{aligned}$$

$$
\begin{aligned}
&+\alpha\sum_{k=2}^{\infty}(k-1)F_N(k)\overline{F}_N(k)\\
&\times\left\{\sum_{j=0}^{\infty}p_{1j}H(x;k+j,2\beta)\right.\\
&\left.-\ 2\sum_{l=0}^{\infty}p_{2l}H(x;k+l,2\beta)+H(x;k,\beta)\right\}\\
&-\alpha\sum_{k=2}^{\infty}(k-1)F_N(k-1)\overline{F}_N(k-1)\\
&\times\left\{\sum_{j=0}^{\infty}p_{1j}H(x;k+j,2\beta)\right.\\
&\left.-\ 2\sum_{l=0}^{\infty}p_{2l}H(x;k+l,2\beta)+H(x;k,\beta)\right\}\\
=\ &\Pr(N=0)+\sum_{k=1}^{\infty}\Pr(N=k)\,H(x;k,\beta)\\
&+\alpha\,(H(x;1,2\beta)-H(x;1,\beta))\left(F_N(1)\overline{F}_N(1)-F_N(0)\overline{F}_N(0)\right)\\
&+\alpha\sum_{k=2}^{\infty}(k-1)\left\{\sum_{j=0}^{\infty}p_{1j}H(x;k+j,2\beta)\right.\\
&\left.-\ 2\sum_{l=0}^{\infty}p_{2l}H(x;k+l,2\beta)+H(x;k,\beta)\right\}\\
&\times\left(F_N(k)\overline{F}_N(k)-F_N(k-1)\overline{F}_N(k-1)\right),
\end{aligned}
\tag{66}
$$

where $p_{1j}=0.5^{k-2}\,\xi_j$, for $j\in\mathbb{N}_0$, with

$$\nu_j=\frac{k-2}{j}0.5^j,$$

and

$$\xi_0=1,\ \ \xi_j=\frac{k-2}{j}\sum_{i=1}^{j}0.5^i\,\xi_{j-i},$$

for $j\in\mathbb{N}$. Also, $p_{2l}=0.5^{k-1}\,\xi_l$, for $l\in\mathbb{N}_0$, with

$$\nu_l=\frac{k-1}{l}0.5^l,$$

and

$$\xi_0=1,\ \ \xi_l=\frac{k-1}{l}\sum_{i=1}^{l}0.5^i\,\xi_{l-i},$$

for $j\in\mathbb{N}$. Using $E[S]=\int_0^{\infty}(1-F_S(x))\,\mathrm{d}x$ with (66), we find that the expectation of the rv S is

$$
\begin{aligned}
E[X]=\ &\sum_{k=1}^{\infty}\Pr(N=k)\frac{k}{\beta}+\alpha F_N(1)\overline{F}_N(1)\left(\frac{1}{2\beta}-\frac{1}{\beta}\right)\\
&-\alpha F_N(0)\overline{F}_N(0)\left(\frac{1}{2\beta}-\frac{1}{\beta}\right)\\
&+\alpha\sum_{k=2}^{\infty}(k-1)F_N(k)\overline{F}_N(k)\\
&\times\left\{\sum_{j=0}^{\infty}p_{1j}\frac{k+j}{2\beta}-2\sum_{l=0}^{\infty}p_{2l}\frac{k+l}{2\beta}+\frac{k}{\beta}\right\}\\
&-\alpha\sum_{k=2}^{\infty}(k-1)F_N(k-1)\overline{F}_N(k-1)\\
&\times\left\{\sum_{j=0}^{\infty}p_{1j}\frac{k+j}{2\beta}-2\sum_{l=0}^{\infty}p_{2l}\frac{k+l}{2\beta}+\frac{k}{\beta}\right\}\\
=\ &\sum_{k=1}^{\infty}\Pr(N=k)\frac{k}{\beta}+\alpha\left(\frac{1}{2\beta}-\frac{1}{\beta}\right)\\
&\times\left(F_N(1)\overline{F}_N(1)-F_N(0)\overline{F}_N(0)\right)\\
&+\alpha\sum_{k=2}^{\infty}(k-1)\left\{\sum_{j=0}^{\infty}p_{1j}\frac{k+j}{2\beta}-2\sum_{l=0}^{\infty}p_{2l}\frac{k+l}{2\beta}+\frac{k}{\beta}\right\}\\
&\times\left(F_N(k)\overline{F}_N(k)-F_N(k-1)\overline{F}_N(k-1)\right).
\end{aligned}
\tag{67}
$$

Table 4
Numerical results of Example 10 about the collective risk model defined with the FGM copula.

Dependence parameter α	$E[S]$ for $\lambda=0.1$	$E[S]$ for $\lambda=1$
−1 (negative dependence)	0.059276	0.980937
0 (independence)	0.1	1
1 (positive dependence)	0.140726	1.019063

Example 10. In the collective risk model defined with multivariate FGM copula C (dependence parameter α) given in (65), we assume that N follows a Poisson distribution with parameter $\lambda=0.1$ and 1 (such that $E[N]=1$) and the claim amount X is exponentially distributed with $\beta=1$ (such that $E[X]=1$). The values of $E[S]$ (see Table 4) are computed with (67), $\alpha=-1$, 0 (independence) and 1. If the number of claims is negatively (positively) related to the amounts of claim X_i ($i\in\mathbb{N}$), the expected value of the aggregated claim amount is lower (greater) than the expectation of the aggregate claim amount $S^{(\perp,\perp)}$, $E\left[S^{(\perp,\perp)}\right]=\frac{\lambda}{\beta}=1$. Observe also the impact of the dependence parameter α on $E[S]$ is more important for $\lambda=0.1$ than for $\lambda=1$.

Another family of multivariate copulas that can be used to account for the dependence between the number of claims and the individual claim amounts are the well known hierarchical Archimedean copulas. Due to their great flexibility, simple construction procedure, multivariate generalization, and their ability to capture different tail dependencies, much attention has been devoted to Archimedean copulas and their different hierarchical extensions in the last few years. These copulas are very good candidates to model the dependence structure within the sequence of rvs $(N,\underline{X})\in\aleph$. In Section 5, we treat in detail this class of copulas and discuss a computation methodology for the aggregate claim amount S under such a dependence construction.

5. Collective risk model with hierarchical Archimedean copulas

5.1. Definitions and basic relations

In this section, we examine a collective risk model where the dependence structure for the components of the sequence $(N,\underline{X})\in\aleph$ is defined with either an Archimedean or a hierarchical Archimedean copula. Archimedean copulas are good candidates to model such a dependence structure. However, the inherent exchangeability in Archimedean copulas implies that the dependence between the number of claims N and the claim amounts X_i for $i\in N$ is the same as the dependence between the components of $\underline{X}$. In practice, this exchangeability property is a very strong assumption. A more realistic dependence structure

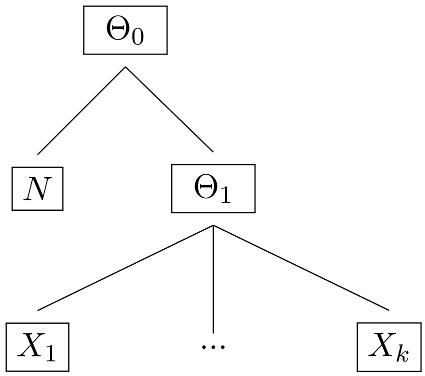

Fig. 3. One level hierarchical tree structure.

could be a hierarchical one. For example, we can consider a one level hierarchical Archimedean copula allowing to have different dependency relationships between the counting rv N and the components of the sequence of claim amounts $\underline{X}$ and also between all the components of $\underline{X}$. Such a dependence structure can be illustrated with a tree representation as shown in Fig. 3. In this section, we consider nested Archimedean copulas (introduced by Joe (1997)) and hierarchical Archimedean copulas through compounding proposed in Cossette et al. (2017) to model such a dependence structure.

A copula C is said to be a nested Archimedean copula if at least one of its arguments is an Archimedean copula. For instance, for any $k \in \mathbb{N}$, a one level $(k+1)$-dimensional nested Archimedean copula, that can fit the dependence structure depicted in Fig. 3, is given by

$$\begin{aligned} C(u_0, u_1, \ldots, u_k) &= C(u_0, C(u_1, \ldots, u_k)) \\ &= \psi_0\left(\psi_0^{-1}(u_0) + \psi_0^{-1} \circ \psi_1\left(\sum_{i=1}^{k} \psi_1^{-1}(u_i)\right)\right), \end{aligned} \tag{68}$$

where ψ_0 and ψ_1 are respectively the generators of the outer copula (also named the mother copula) and the inner copula (or the child copula). Only nested Archimedean copulas of the form (68) for which ψ_0, ψ_1, and $\psi_{0,1}(t;\theta) = \exp\left\{-\theta\, \psi_0^{-1} \circ \psi_1(t)\right\}$ are LSTs of positive rvs Θ_0, Θ_1, and $\Theta_{0,1}$ respectively will be considered. This insures that the copula is properly defined (see e.g., Joe (2014) for details).

To bypass the constraints related to the nesting condition of nested Archimedean copulas, Cossette et al. (2017), among others, proposed a hierarchical copula obtained from a multivariate mixed exponential distribution and using multivariate compound distributions. To fit the dependence model presented in this paper, these hierarchical Archimedean copulas are given by

$$C(u_0, u_1, \ldots, u_k) = \mathcal{L}_M\left(\mathcal{L}_M^{-1}(u_0) - \ln\left(\mathcal{L}_B\left(\sum_{i=1}^{k} \mathcal{L}_{\Theta_1}^{-1}(u_i)\right)\right)\right), \tag{69}$$

with $\Theta_1 = \sum_{i=1}^{M} B_i$, where M is a strictly positive discrete rv, $\underline{B} = \{B_i, i = 1, 2, \ldots\}$ forms a sequence of iid and strictly positive rvs (convention: $B_i \sim B$, $i \in \mathbb{N}$), and $\underline{B}$ is independent of the rv M. Letting $\Theta_0 = M$ and $\Theta_1 = \sum_{i=1}^{M} B_i$, with $\psi_0(t) = \mathcal{L}_M(t)$ and

$$\mathcal{L}_{\Theta_1}(t) = \mathcal{P}_M(\mathcal{L}_B(t)) = \mathcal{L}_M(-\ln(\mathcal{L}_B(t))) = \psi_1(t), \text{ for } t \geq 0,$$

the representation of the $(k+1)$ dimensional copula C in (69) is equivalent to the representation provided in (68). Note that

$$\psi_0^{-1} \circ \psi_1(t) = -\ln(\mathcal{L}_B(t)), \text{ for } t \geq 0.$$

We treat the impact of the dependence within both the collective risk models with Archimedean copulas and hierarchical Archimedean copulas within the next two corollaries.

Corollary 11. *We consider two collective risk models with sequences $(N, \underline{X}) \in \aleph$ and $(N^*, \underline{X}^*) \in \aleph$ where, for any $k \in N$, the multivariate distribution of $(N, X_1, \ldots, X_k)$ (and $\left(N^*, X_1^*, \ldots, X_k^*\right)$) is defined with either (62) or (63), where C is a $(k+1)$-dimensional Archimedean copula with dependence parameters α and α^* respectively. If $\alpha \leq \alpha^*$, then, for any $k \in \mathbb{N}$, we have*

$$C_\alpha \preceq_{sm} C_{\alpha^*}$$

and

$$(N, X_1, \ldots, X_k) \preceq_{sm} \left(N^*, X_1^*, \ldots, X_k^*\right).$$

Then, it follows that

$$S = \sum_{i=1}^{N} X_i \preceq_{icx} \sum_{i=1}^{N^*} X_i^* = S^*, \tag{70}$$

and

$$TVaR_\kappa(S) \leq TVaR_\kappa\left(S^*\right), \text{ for } \kappa \in (0,1). \tag{71}$$

Proof. To show $C_\alpha \preceq_{sm} C_{\alpha^*}$, see Wei and Hu (2002). Using the property of closure under all increasing (or decreasing) transforms of the supermodular order (see, e.g. Theorem 9.A.9.(a) of Shaked and Shanthikumar (2007)), we can conclude that, for all $k \in \mathbb{N}$, $(N, Y_1, \ldots, Y_k) \preceq_{sm} \left(N^*, Y_1^*, \ldots, Y_k^*\right)$. Then, (70) and (71) follow from Proposition 4. ■

The collective risk model with hierarchical Archimedean copulas allows to examine the impact of two dependence relations: firstly, we examine the impact of the dependence between the rv N and the sequence of the claim amounts $\underline{X}$ induced by the outer copula and secondly, for a given level of dependence induced by the outer copula, the impact of the dependence relation within the claim amounts $\underline{X}$ induced by the inner copula.

Corollary 12. *We consider two collective risk models with sequences $(N, \underline{X}) \in \aleph$ and $(N^*, \underline{X}^*) \in \aleph$ where, for any $k \in N$, the multivariate distribution of $(N, X_1, \ldots, X_k)$ (and $\left(N^*, X_1^*, \ldots, X_k^*\right)$) is defined with either (62) or (63), where C is a $(k+1)$-dimensional one level hierarchical Archimedean copula as illustrated in Fig. 3, with parameters α_0, α_1 and α_0^*, α_1^* respectively.*

1. *If $\alpha_0 \leq \alpha_0^*$ and $\alpha_1 = \alpha_1^*$, then, for any $k \in \mathbb{N}$, $C_{\alpha_0,\alpha_1} \preceq_{sm} C_{\alpha_0^*,\alpha_1}$;*
2. *If $\alpha_0 = \alpha_0^*$ and $\alpha_1 \leq \alpha_1^*$, then, for any $k \in \mathbb{N}$, $C_{\alpha_0,\alpha_1} \preceq_{sm} C_{\alpha_0,\alpha_1^*}$;*
3. *If $\alpha_0 \leq \alpha_0^*$ and $\alpha_1 \leq \alpha_1^*$, then, for any $k \in \mathbb{N}$, $C_{\alpha_0,\alpha_1} \preceq_{sm} C_{\alpha_0^*,\alpha_1^*}$.*

Therefore, for all three cases and for any $k \in \mathbb{N}$, we have

$$(N, X_1, \ldots, X_k) \preceq_{sm} \left(N^*, X_1^*, \ldots, X_k^*\right). \tag{72}$$

Then, it follows that

$$S = \sum_{i=1}^{N} X_i \preceq_{icx} \sum_{i=1}^{N^*} X_i^* = S^*, \tag{73}$$

and

$$TVaR_\kappa(S) \leq TVaR_\kappa\left(S^*\right), \text{ for } \kappa \in (0,1). \tag{74}$$

Proof. The proofs of 12.1. and 12.2. can be found in Wei and Hu (2002) for nested Archimedean copulas. Given the links between nested Archimedean copulas and hierarchical Archimedean copulas through compounding (see Cossette et al. (2017) for details), this result also holds here for the one-level hierarchical Archimedean copula defined in (69). If $\alpha_0 \leq \alpha_0^*$ and $\alpha_1 \leq \alpha_1^*$, then, using 12.1. and 12.2. and for any $k \in \mathbb{N}$, we have $C_{\alpha_0,\alpha_1} \preceq_{sm} C_{\alpha_0^*,\alpha_1} \preceq_{sm} C_{\alpha_0^*,\alpha_1^*}$. Once again the property of closure under all increasing (or decreasing) transforms of the supermodular order can be used to show that, for any $k \in \mathbb{N}$, we have (72). Finally, (70) and (71) are obtained using Proposition 4. ■

In the case of Corollary 12.2, the dependence parameters of the outer copula are identical ($\alpha_0 = \alpha_0^*$) and the dependence parameters of the inner copula are fixed such that $\alpha_1 \leq \alpha_1^*$. It implies that $E[S] = E[S^*]$. Then, the result in (73) becomes $S \preceq_{cx} S^*$, from which we conclude, in addition to (74), that $Var(S) \leq Var(S^*)$.

5.2. Sampling algorithm

In this section, we propose an efficient algorithm to generate samples of S. Inspired by the sampling algorithms of both nested Archimedean copulas and hierarchical copulas through compounding (see e.g., Marshall and Olkin (1988), Hofert (2008), and Cossette et al. (2017)), we propose a general sampling algorithm that generates samples of a random sum S incorporating a dependence relationship between the number and amounts of claims using hierarchical Archimedean copulas.

Let Θ_0 and Θ_1 be two strictly positive rvs. Given $\Theta_0 = \theta_0$, we denote by $\Theta_{0,1}$ the conditional rv $(\Theta_1|\Theta_0 = \theta_0)$. Note that Θ_0 and $\Theta_{0,1}$ represent the mixing rvs such that given $\Theta_0 = \theta_0$ and $\Theta_{0,1} = \theta_{0,1}$, $\left(X_1|\Theta_0=\theta_0, \Theta_{0,1}=\theta_{0,1}\right), \ldots, \left(X_k|\Theta_0=\theta_0, \Theta_{0,1}=\theta_{0,1}\right)$ are conditionally iid and independent of $(N|\Theta_0 = \theta_0)$. Note that if C is a nested Archimedean copula as in (68), Θ_0 and Θ_1 represent the mixing rvs related to the outer and inner copulas respectively, and $\Theta_{0,1}$ is such that $\mathcal{L}_{\Theta_{0,1}}(t;\theta) = \exp\left\{-\theta\mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_1}(t)\right\}$. As for the case where C is defined as in (69), the rv Θ_0 plays the same role as the rv M, $\Theta_1 = \sum_{j=1}^{M} B_j$, and, given $\Theta_0 = \theta_0$, $\Theta_{0,1} = \sum_{j=1}^{\theta_0} B_j$, for $\theta_0 \in \mathbb{N}$.

Algorithm 13. Let C be a one level hierarchical Archimedean copula with generators $\mathcal{L}_{\Theta_0}$ and $\mathcal{L}_{\Theta_{0,1}}$ allowing a dependence structure as depicted in Fig. 3.

1. Sample Θ_0;
2. Sample $R \sim Exp(1)$;
3. Calculate $U = \mathcal{L}_{\Theta_0}\left(\frac{R}{\Theta_0}\right)$;
4. Return $N = F_N^{-1}(U)$;
5. If $N = 0$ return $S = 0$; else
    5.1. Sample $\Theta_{0,1}$;
    5.2. Sample $R_i \sim Exp(1)$ for $i = 1, \ldots, N$;
    5.3. Calculate $U_i = \mathcal{L}_{\Theta_1}\left(\frac{R_i}{\Theta_{0,1}}\right)$ for $i = 1, \ldots, N$;
    5.4. Calculate $X_i = F_X^{-1}(U_i)$ for $i = 1, \ldots, N$;
    5.5. Return $S = \sum_{k=1}^{N} X_i$;
6. Return S.

In the following example, we aim to apply Algorithm 13 and to illustrate the results of Corollary 12.

Example 14. We consider a collective risk model defined with the sequence $(N, \underline{X})$, in which, for any $k \in \mathbb{N}$, the joint distribution of $(N, X_1, \ldots, X_k)$ is assumed to be defined with a hierarchical Archimedean copula through compounding C_{α_0,α_1} as given in (69), with $M \sim Logarithmic(q = 1 - e^{-\alpha_0})$ $(\mathcal{L}_M(t) = \frac{-1}{\alpha_0}\ln(1-(1-e^{-\alpha_0})e^{-t})$, $t \geq 0)$ and $B_i \sim B \sim Gamma(1/\alpha_1, 1)$ $(\mathcal{L}_B(t) = \left(\frac{1}{1+t}\right)^{\frac{1}{\alpha_1}}$, $t \geq 0)$. We assume that $N \sim Poisson(2)$ and $X_i \sim Pareto(3, 100)$, for $i \in \mathbb{N}$. To analyze the impact of the dependence relations induced by the outer copula and the inner copula, we consider the values $-\ln(0.5)$ and $-\ln(0.1)$ for α_0, and the values 5 and 25 for α_1. Approximative values (using Algorithm 13, with 10 million simulations) of F_S, $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$ are given in Table 5. For comparison purposes, the same quantities are also provided for $S^{(\perp,+)}$, $S^{(\perp,\perp)}$ and for $S^{(+,+)}$. We mention that a limit case of the copula C_{α_0,α_1} when both $\alpha_0 \to \infty$ and $\alpha_1 \to \infty$ is the upper Frechet–Hoeffding bound copula, which corresponds to the dependence structure of Model 3. Also, the independence copula corresponds to a limit case of the copula C_{α_0,α_1} when both $\alpha_0 \to 0$ and $\alpha_1 \to 0$ and the corresponding special case is Model 2. The dependence structure of Model 1 is obtained when $\alpha_0 \to 0$ and $\alpha_1 \to \infty$. Thus, Models 1, 2 and 3 (where the univariate marginal distribution of a claim amount is Pareto) are special cases of the collective risk model defined in the present example. We have the following comments: when the inner dependence parameter α_1 remains fixed and the outer dependence parameter α_0 increases, $E[S]$ and $TVaR_\kappa(S)$ increase. Moreover, when the outer dependence parameter α_0 remains fixed and the inner dependence parameter α_1 increases, $Var(S)$ and $TVaR_\kappa(S)$ increase while $E[S]$ does not change.

Table 5
Values of the expectation, variance, VaR and TVaR of S, $S^{(\perp,\perp)}$, $S^{(\perp,+)}$, and $S^{(+,+)}$, as defined in Example 14.

(α_0, α_1)	(0.7, 5)	(2.3, 5)	(2.3, 25)	$(\perp,\perp)$	$(\perp,+)$	$(+,+)$
$E[S]$	108.04	125.44	125.44	100	100	191.88
$Var(S)$	34426.71	44913.80	52347.69	20000	52235.55	394838.90
$VaR_{0.9}(S)$	293.24	348.58	347.41	243.32	245.55	462.37
$VaR_{0.99}(S)$	852.98	966.30	1070.83	611.78	907.57	2197.46
$VaR_{0.999}(S)$	1664.45	1837.66	2048.95	1288.62	2393.54	7252.53
$TVaR_{0.9}(S)$	534.52	616.37	658.37	405.19	532.71	1196.14
$TVaR_{0.99}(S)$	1211.57	1351.95	1500.56	911.11	1555.41	4187.68
$TVaR_{0.999}(S)$	2301.54	2540.50	2748.07	1950.60	3865.14	11950.60

5.3. Computational methodology

Algorithm 13 is used to obtain approximated values of the cdf of S via Monte Carlo simulations as illustrated in Example 14. This approach is efficient and very practical especially in the case of continuous mixing rvs and/or continuous marginals. Based on Cossette et al. (2018), we derive, in this section, another computational methodology allowing to compute exact values of the cdf of S for discrete rvs X_i, $i \in \mathbb{N}$, even in high dimensions. Here, we only present the case of discrete mixing rvs and discrete marginals, whilst the case of continuous marginals or continuous mixing rvs can be treated in a similar fashion as in Cossette et al. (2018) where discretization methods are used to derive upper and lower bounds for the exact values of the cdf of S.

The collective risk model with dependence presented in this section is an extension of the one treated in Section 4 of Cossette et al. (2018). We adapt here its proposed computational strategy to our context. The idea behind it is to use the conditional independence assumption to identify the conditional distributions of N and X_i, $i \in \mathbb{N}$. While this computational strategy works naturally for discrete rvs X_i, discretization methods can be used to approximate continuous rvs X_i, $i \in \mathbb{N}$ (see details in Cossette et al. (2018); see also e.g. Müller and Stoyan (2002) and Bargès et al. (2009) for a review of different discretization methods).

In this section, we assume discrete marginals for X_i with $X_i \in \{0, 1h, 2h, \dots\}$, $i = 1, 2, \dots$, and discrete mixing rvs Θ_0 and $\Theta_{0,1}$ with respective LSTs $\mathcal{L}_{\Theta_0}$ and $\mathcal{L}_{\Theta_{0,1}}$, respective pmfs $f_{\Theta_0}(\theta_0) = \Pr(\Theta_0 = \theta_0)$ and $f_{\Theta_{0,1}}\left(\theta_{0,1}\right) = \Pr\left(\Theta_{0,1} = \theta_{0,1}\right)$, and respective cdfs $F_{\Theta_0}(\theta_0) = \Pr(\Theta_0 \le \theta_0) = \sum_{j=1}^{\theta_0} f_{\Theta_0}(j)$ and $F_{\Theta_{0,1}}\left(\theta_{0,1}\right) = \Pr\left(\Theta_{0,1} \le \theta_{0,1}\right) = \sum_{j=1}^{\theta_{0,1}} f_{\Theta_{0,1}}(j)$, for $\theta_0, \theta_{0,1} \in \mathbb{N}$.

For any $k \in \mathbb{N}$, the multivariate cdf of $(N, X_1, \dots, X_k)$ is given by

$$
\begin{aligned}
&F_{N,X_1,\dots,X_k}(n, m_1h, \dots, m_kh) \\
&= \sum_{\theta_0=1}^{\infty} F_{N|\Theta_0=\theta_0}(n) \\
&\quad \times \sum_{\theta_{0,1}=1}^{\infty} \prod_{i=1}^{k} F_{X_i|\Theta_0=\theta_0,\Theta_{0,1}=\theta_{0,1}}(m_ih) f_{\Theta_{0,1}}\left(\theta_{0,1}\right) f_{\Theta_0}(\theta_0) \\
&= \sum_{\theta_0=1}^{\infty} \mathrm{e}^{-\theta_0 \mathcal{L}_{\Theta_0}^{-1}(F_N(n))} \\
&\quad \times \sum_{\theta_{0,1}=1}^{\infty} \prod_{i=1}^{k} \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_1}^{-1}\left(F_{X_i}(m_ih)\right)} f_{\Theta_{0,1}}\left(\theta_{0,1}\right) f_{\Theta_0}(\theta_0),
\end{aligned}
\tag{75}
$$

where $n, m_1, \dots, m_k \in \mathbb{N}_0$, $F_{N|\Theta_0=\theta_0}(n) = \mathrm{e}^{-\theta_0\mathcal{L}_{\Theta_0}^{-1}(F_N(n))}$, $\forall n \in \mathbb{N}_0$, and

$$
F_{X_i|\Theta_0=\theta_0,\Theta_{0,1}=\theta_{0,1}}(m_ih) = \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_1}^{-1}\left(F_{X_i}(m_ih)\right)},
\tag{76}
$$

for $m_i \in \mathbb{N}_0$, $i = 1, 2, \dots, k$, and $\theta_0, \theta_{0,1} \in \mathbb{N}$.

Let N_{θ_0} and $X_{i,\theta_0,\theta_{0,1}}$ denote, respectively, the conditional rvs $(N|\Theta_0 = \theta_0)$ and $\left(X_i|\Theta_0 = \theta_0, \Theta_{0,1} = \theta_{0,1}\right)$, for $i \in \mathbb{N}$. The pmf and pgf of N_{θ_0} are respectively given by

$$
\Pr\left(N_{\theta_0} = n\right) = \begin{cases} \mathrm{e}^{-\theta_0\mathcal{L}_{\Theta_0}^{-1}(F_N(0))} & , n = 0 \\ \mathrm{e}^{-\theta_0\mathcal{L}_{\Theta_0}^{-1}(F_N(n))} - \mathrm{e}^{-\theta_0\mathcal{L}_{\Theta_0}^{-1}(F_N(n-1))} & , n \in \mathbb{N} \end{cases},
\tag{77}
$$

and

$$
\mathcal{P}_{N_{\theta_0}}(t) = E\left[t^{N_{\theta_0}}\right] = \sum_{n=0}^{\infty} t^n \Pr\left(N_{\theta_0} = n\right).
\tag{78}
$$

For $i \in \mathbb{N}$ and for $\theta_0, \theta_{0,1} \in \mathbb{N}$, we can find the values of $f_{X_{i,\theta_0,\theta_{0,1}}}(m_ih)$ with

$$
\begin{aligned}
&f_{X_{i,\theta_0,\theta_{0,1}}}(m_ih) \\
&= \begin{cases} \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_1}^{-1}\left(F_{X_i}(0)\right)} & , \quad m_i = 0 \\ \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_1}^{-1}\left(F_{X_i}(m_ih)\right)} - \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_1}^{-1}\left(F_{X_i}((m_i-1)h)\right)} & , \quad m_i \in \mathbb{N} \end{cases}.
\end{aligned}
\tag{79}
$$

Similar results are obtained when, for any $k \in \mathbb{N}$, the dependence structure of $(N, X_1, \dots, X_k)$ is induced via a copula C and survival functions as in (63). For any $k \in \mathbb{N}$, the survival function of $(N, X_1, \dots, X_k)$ is given by

$$
\begin{aligned}
&\overline{F}_{N,X_1,\dots,X_k}(n, m_1h, \dots, m_kh) \\
&= \sum_{\theta_0=1}^{\infty} \mathrm{e}^{-\theta_0\mathcal{L}_{\Theta_0}^{-1}(\overline{F}_N(n))} \\
&\quad \times \sum_{\theta_{0,1}=1}^{\infty} \prod_{i=1}^{k} \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_1}^{-1}\left(\overline{F}_{X_i}(m_ih)\right)} f_{\Theta_{0,1}}\left(\theta_{0,1}\right) f_{\Theta_0}(\theta_0),
\end{aligned}
\tag{80}
$$

where $n, m_1, \dots, m_k \in \mathbb{N}_0$. In this case, the pmf of N_{θ_0} is

$$
\Pr\left(N_{\theta_0} = n\right) = \begin{cases} 1 - \mathrm{e}^{-\theta_0\mathcal{L}_{\Theta_0}^{-1}(\overline{F}_N(0))} & , n = 0 \\ \mathrm{e}^{-\theta_0\mathcal{L}_{\Theta_0}^{-1}(\overline{F}_N(n-1))} - \mathrm{e}^{-\theta_0\mathcal{L}_{\Theta_0}^{-1}(\overline{F}_N(n))} & , n \in \mathbb{N} \end{cases},
\tag{81}
$$

and its pgf is obtained with (78) and (81).

As for $X_{i,\theta_0,\theta_{0,1}}$, we have

$$
\overline{F}_{X_{i,\theta_0,\theta_{0,1}}}(m_ih) = \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_1}^{-1}\left(\overline{F}_{X_i}(m_ih)\right)},
\tag{82}
$$

for $m_i \in \mathbb{N}_0$ and $i, \theta_0, \theta_{0,1} \in \mathbb{N}$. For $i \in \mathbb{N}$ and for $\theta_0, \theta_{0,1} \in \mathbb{N}$, we can find the values of $f_{X_{i,\theta_0,\theta_{0,1}}}(m_ih)$ with

$$
\begin{aligned}
&f_{X_{i,\theta_0,\theta_{0,1}}}(m_ih) \\
&= \begin{cases} 1 - \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_1}^{-1}\left(\overline{F}_{X_i}(0)\right)} & , \quad m_i = 0 \\ \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_1}^{-1}\left(\overline{F}_{X_i}((m_i-1)h)\right)} - \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_1}^{-1}\left(\overline{F}_{X_i}(m_ih)\right)} & , \quad m_i \in \mathbb{N} \end{cases}.
\end{aligned}
\tag{83}
$$

For any $k \in \mathbb{N}$, the expression for $f_{N,X_1,\dots,X_k}(n, m_1h, \dots, m_kh)$ in this case is given by

$$
\begin{aligned}
f_{N,X_1,\dots,X_k}(n, m_1h, \dots, m_kh) &= \sum_{\theta_0=1}^{\infty} \Pr\left(N_{\theta_0} = n\right) \\
&\times \left\{ \sum_{\theta_{0,1}=1}^{\infty} \prod_{i=1}^{k} f_{X_{i,\theta_0,\theta_{0,1}}}(m_ih) f_{\Theta_{0,1}}\left(\theta_{0,1}\right) \right\} f_{\Theta_0}(\theta_0),
\end{aligned}
\tag{84}
$$

where $\Pr\left(N_{\theta_0} = n\right)$ is given in (77) or (81), and $f_{X_{i,\theta_0,\theta_{0,1}}}(m_ih)$ is provided in (79) or (83). Let $S_{\theta_0,\theta_{0,1}} = \sum_{i=1}^{N_{\theta_0}} X_{i,\theta_0,\theta_{0,1}}$ be the sum of conditionally independent rvs and $f_{S_{\theta_0,\theta_{0,1}}}$ be its corresponding pmf. Let $\mathcal{L}_{X_{\theta_0,\theta_{0,1}}}$ be the LST of $X_{\theta_0,\theta_{0,1}}$, where $X_{i,\theta_0,\theta_{0,1}} \sim X_{\theta_0,\theta_{0,1}}$, for $i \in \mathbb{N}$. Then, the LST of $S_{\theta_0,\theta_{0,1}}$ is given by

$$
\mathcal{L}_{S_{\theta_0,\theta_{0,1}}}(t) = \mathcal{P}_{N_{\theta_0}}\left(\mathcal{L}_{X_{\theta_0,\theta_{0,1}}}(t)\right).
\tag{85}
$$

Using (85) and the Fast Fourier Transform (FFT), it is easy to compute the exact values of $f_{S_{\theta_0,\theta_{0,1}}}$ for each θ_0 and $\theta_{0,1}$. Finally, due to the representation of $f_{N,X_1,\dots,X_k}$ in (84), the unconditional pmf of S can be computed using

$$
f_S(mh) = \sum_{\theta_0=1}^{\infty} \sum_{\theta_{0,1}=1}^{\infty} f_{S_{\theta_0,\theta_{0,1}}}(mh) f_{\Theta_{0,1}}\left(\theta_{0,1}\right) f_{\Theta_0}(\theta_0), \quad m \in \mathbb{N}_0.
\tag{86}
$$

The computational methodology used to find the exact values of f_S is summarized in the following algorithm.

Algorithm 15 (*Computation of the values of f_S*).

1. Fix $\theta_0 = 1$;
2. Fix $\theta_{0,1} = 1$;
3. Calculate either $F_{X_{i,\theta_0,\theta_{0,1}}}(m_ih)$ with (76) or $\overline{F}_{X_{i,\theta_0,\theta_{0,1}}}(m_ih)$ with (82), for $m_i \in \mathbb{N}_0$;
4. Calculate $f_{X_{i,\theta_0,\theta_{0,1}}}(m_ih)$, for $m_i \in \mathbb{N}_0$;
5. Use FFT to return the vector $\tilde{\underline{f}}_{X_{i,\theta_0,\theta_{0,1}}}$, where $\tilde{\underline{f}}$ denotes the vector of values of the characteristic function, also known as the Fourier transform (see e.g., Klugman et al. (2009));
6. Calculate $\Pr\left(N_{\theta_0} = n\right)$ using (77) (or (81)) for $n = 1, 2, \dots, n^*$, such that $F_{N_{\theta_0}}(n^*) \le 1 - \varepsilon$ where ε is fixed as small as desired (e.g., $\varepsilon = 10^{-10}$);

7. Use the pgf of N_{θ_0} given in (78) to calculate $\tilde{\underline{f}}_{S_{\theta_0,\theta_{0,1}}} = \mathcal{P}_{N_{\theta_0}}\left(\tilde{\underline{f}}_{X_{\theta_0,\theta_{0,1}}}\right)$;
8. Use FFT (inverse) to compute $f_{S_{\theta_0,\theta_{0,1}}}(mh)$ for $m \in \mathbb{N}_0$;
9. Repeat steps 3–8 for $\theta_{0,1} = 2, \ldots, \theta^*_{0,1}$ where $\theta^*_{0,1}$ is chosen such that $F_{\Theta_{0,1}}\left(\theta^*_{0,1}\right) \leq 1-\varepsilon$ where ε is fixed as small as desired (e.g., $\varepsilon = 10^{-10}$);
10. Compute $f_{S|\Theta_0=\theta_0}(mh) = \sum_{\theta_{0,1}=1}^{\theta^*_{0,1}} f_{S_{\theta_0,\theta_{0,1}}}(mh) f_{\Theta_{0,1}}\left(\theta_{0,1}\right)$, for $m \in \mathbb{N}_0$;
11. Repeat steps 2–10 for $\theta_0 = 2, \ldots, \theta^*_0$ where θ^*_0 is chosen such that $F_{\Theta_0}\left(\theta^*_0\right) \leq 1-\varepsilon$ where ε is fixed as small as desired (e.g., $\varepsilon = 10^{-10}$);
12. Compute $f_S(mh) = \sum_{\theta_0=1}^{\theta^*_0} f_{S|\Theta_0=\theta_0}(mh) f_{\Theta_0}(\theta_0)$, for $m \in \mathbb{N}_0$.

In the next example, we provide an illustration of Algorithm 15.

Example 16. Let $N \sim Poisson(2)$ and $X_i \sim Gamma\,(\alpha = 2, \beta = \frac{1}{50})$, for $i \in \mathbb{N}$. Also, assume that the joint distribution of $(N, X_1, \ldots, X_k)$ is defined as in (62) with a nested Ali–Mikhail–Haq (AMH) copula with outer parameter $\alpha_0 = 0.1$ and inner parameter $\alpha_1 = 0.2$, i.e., $\Theta_0 \sim Geometric(1-\alpha_0)$, $\Theta_1 \sim Geometric\,(1-\alpha_1)$, and $\Theta_{0,1} \sim Shifted\ Negative\ Binomial\left(\alpha_0, \frac{1-\alpha_1}{1-\alpha_0}\right)$. We use Algorithm 15 to compute two approximations of the values of f_S allowing to derive the exact values of $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$, applying the lower and the upper discretization methods as explained in e.g. Müller and Stoyan (2002) and Bargès et al. (2009). The resulting approximations are provided in Table 6. We also compute approximations of the same quantities based on 1,000,000 simulations sampled with Algorithm 13. We can see that the approximated values obtained with both Algorithms 13 and 15 are close, with comparable computation times. The corresponding measures of $S^{(+,+)}$ are also computed by Monte-Carlo simulations. We observe that $TVaR_\kappa(S) \leq TVaR_\kappa(S^{(+,+)})$ for all values of κ, providing an illustration of Corollary 6.

Remark 17. Algorithm 15 is simple to apply. As mentioned in Abate and Whitt (1992), one needs to be careful with eventual numerical anomalies while using FFT (but we did not encounter them for this paper). Computation time increases as the value of n^* becomes large.

If C is an Archimedean copula with mixing rv Θ, meaning that $N, X_1, X_2, \ldots$ are conditionally independent given $\Theta = \theta$, the computation procedure is nearly the same. One has only to replace $\Theta_{0,1}$ and Θ_0 by Θ and Algorithm 15 becomes Algorithm 18.

Algorithm 18 (*Computation of the values of f_S*).
1. Fix $\theta = 1$;
2. Calculate either $F_{X|\Theta=\theta}(m_i h) = e^{-\theta\mathcal{L}^{-1}_{\Theta}(F_X(m_i h))}$ or $\overline{F}_{X|\Theta=\theta}(m_i h) = e^{-\theta\mathcal{L}^{-1}_{\Theta}(\overline{F}_X(m_i h))}$, for $m_i \in \mathbb{N}_0$;
3. Deduce $f_{X|\Theta=\theta}(m_i h)$ from step 2, for $m_i \in \mathbb{N}_0$;
4. Use FFT to return the vector $\tilde{\underline{f}}_{X|\Theta=\theta}$;
5. Calculate $\Pr\left(N_{\theta_0} = n\right)$ using (77) (or (81)) for $n = 1, 2, \ldots, n^*$, such that $F_{N_{\theta_0}}(n^*) \leq 1-\varepsilon$ where ε is fixed as small as desired (e.g., $\varepsilon = 10^{-10}$);
6. Use the pgf of N_θ given in (78) to calculate $\tilde{\underline{f}}_{S|\Theta=\theta} = \mathcal{P}_{N_\theta}\left(\tilde{\underline{f}}_{X|\Theta=\theta}\right)$;
7. Use FFT (inverse) to compute $f_{S|\Theta=\theta}(mh)$, for $m \in \mathbb{N}_0$;
8. Repeat steps 2–7 for $\theta = 2, \ldots, \theta^*$ where θ^* is chosen such that $F_\Theta(\theta^*) \leq 1-\varepsilon$ where ε is fixed as small as desired (e.g., $\varepsilon = 10^{-10}$);
9. Compute $f_S(mh) = \sum_{\theta=1}^{\theta^*} f_{S|\Theta=\theta}(mh) f_\Theta(\theta)$, for $m \in \mathbb{N}_0$.

5.4. Collective risk model with hierarchical Archimedean copulas and with multivariate mixed Erlang distributions

We aim to provide an example in which we do not have to recourse to an approximation method to evaluate quantities of interest in regard to the distribution of the rv S. For this reason, we consider a collective risk model with a multivariate mixed Erlang distribution as presented in Section 4.1. For any $k \in \mathbb{N}$, let $(N, X_1, \ldots, X_k)$ follow a mixed multivariate distribution where $N \in \{0, 1, 2, 3\}$ and X_i follows a univariate mixed Erlang distribution, for $i \in \mathbb{N}$. For any $k \in \mathbb{N}$, we consider the joint distribution of $(N, J_1, \ldots, J_k)$ to be defined as in (62) with a hierarchical Archimedean copula as in (68) or in (69), with dependence parameters α_0 and α_1. Note that the pmf of L is as given in (88), and, for any $k \in \mathbb{N}$, the joint pmf of $(N, J_1, \ldots, J_k)$ can be derived from its joint cdf as follows

$$\gamma_{N,J_1,\ldots,J_k}(n, j_1, \ldots, j_k) = \sum_{i_0=0,1}\sum_{i_1=0,1}\cdots\sum_{i_k=0,1}(-1)^{i_1+\cdots+i_k} \times F_{N,J_1,\ldots,J_k}\left((n-i_0), (j_1-i_k), \ldots, (j_k-i_k)\right), \tag{87}$$

for $n \in \{0, 1, 2, 3\}$ and $j_i \in \mathbb{N}$, for $i = 1, \ldots, k$.

We provide below a numerical illustration in which we consider a collective risk model with mixed Erlang distributions within the context just discussed. The assumptions have been chosen such that we can derive exact results without using an approximation method.

Example 19. Let $N \sim Binomial(3, 0.1)$ and X_i follow a univariate mixed Erlang distribution, for $i = 1, 2, 3$ with cdf as given in (36), where

$$\gamma_J(j) = \begin{cases} 0.2281946, & j = 2 \\ 0.5429590, & j = 9 \\ 0.2288464, & j = 14 \\ 0, & \text{otherwise,} \end{cases}$$

and $\beta = 1.960312$. Such a distribution is an approximation of a mixture of two gamma distributions obtained in Lee and Lin (2010) (see Table 15). For any $k \in \{0, 1, 2, 3\}$, the joint distribution of $(N, J_1, \ldots, J_k)$ is assumed to be defined with a hierarchical Archimedean copula through compounding C_{α_0,α_1} as given in (69), with $M \sim Logarithmic(q = 1 - e^{-\alpha_0})$ $(\mathcal{L}_M(t) = \frac{-1}{\alpha_0}\ln(1-(1-e^{-\alpha_0})e^{-t})$, $t \geq 0)$ and $B_i \sim B \sim Gamma(1/\alpha_1, 1)$ $(\mathcal{L}_B(t) = \left(\frac{1}{1+t}\right)^{\frac{1}{\alpha_1}}$, $t \geq 0)$. As explained in Section 4.1, the rv S follows a mixed Erlang distribution with cdf as given in (42).

Since the support of the rv N is $\{0, 1, 2, 3\}$, the probabilities ζ_l $(l \in \mathbb{N}_0)$ in (41) are computed as follows:

$$\zeta_l = \begin{cases} \Pr(N=0) & , \quad l = 0 \\ \gamma_{N,J_1}(1,1) & , \quad l = 1 \\ \gamma_{N,J_1}(1,2) + \gamma_{N,J_1,J_2}(2,1,1) & , \quad l = 2 \\ \gamma_{N,J_1}(1,l) + \sum_{j=1}^{l}\gamma_{N,J_1,J_2}(2,j,l-j) & \\ \quad + \sum_{j=1}^{l}\sum_{i=1}^{l-j}\gamma_{N,J_1,J_2,J_3}(3,j,i,l-i-j) & , \quad l \geq 3 \end{cases}. \tag{88}$$

To analyze the impact of the dependence relations induced by the outer copula and the inner copula, we consider the values $-\ln(0.5)$ and $-\ln(0.1)$ for α_0, and the values 5 and 25 for α_1. Resulting values of F_S, $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$ are given in Table 7. For comparison purposes, the same quantities are also provided for $S^{(+,+)}$ and for $S^{(\perp,\perp)}$. We mention that a limit

Table 6
Values of the expectation, variance, VaR and TVaR of S as defined in Example 16.

	Upper (Algorithm 15)	Lower (Algorithm 15)	1M MC simulations (Algorithm 13)	Comonotonicity
$E[S]$	40.09	41.09	40.59	59.44
$Var(S)$	1304.65	1346.13	1324.56	6832.09
$VaR_{0.9}(S)$	89.00	91.00	89.83	155.68
$VaR_{0.99}(S)$	156.00	158.50	157.02	398.28
$VaR_{0.999}(S)$	217.00	220.00	218.54	739.367
$TVaR_{0.9}(S)$	118.12	119.86	119.45	252.06
$TVaR_{0.99}(S)$	181.36	183.91	184.14	520.69
$TVaR_{0.999}(S)$	241.91344	245.11326	244.04378	863.67

Table 7
Values of the cdf, expectation, variance, VaR and TVaR of S as defined in Example 19.

(α_0, α_1)	$(0.7, 5)$	$(2.3, 5)$	$(2.3, 25)$	$(\perp, \perp)$	$(+, +)$
$E[S]$	1.39	1.50	1.57	1.31	2.36
$Var(S)$	7.99	9.44	9.48	7.10	21.18
$VaR_{0.9}(S)$	5.75	6.31	6.30	5.51	7.71
$VaR_{0.99}(S)$	11.76	12.79	12.87	10.93	21.54
$VaR_{0.999}(S)$	17.54	18.47	18.71	15.98	26.32
$TVaR_{0.9}(S)$	8.36	9.01	9.02	7.92	12.22
$TVaR_{0.99}(S)$	14.37	15.40	15.57	13.15	25.06
$TVaR_{0.999}(S)$	19.62	20.58	20.88	17.85	42.23

Table 8
Values of the cdf, expectation, variance, VaR and TVaR of S as defined in Example 20.

(α_0, α_1)	$(0.7, 5)$	$(2.3, 5)$	$(2.3, 25)$	$(\perp, \perp)$
$E[S]$	1.22	1.04	1.04	1.31
$Var(S)$	6.62	5.16	5.21	7.10
$VaR_{0.9}(S)$	5.19	4.54	4.53	5.51
$VaR_{0.99}(S)$	10.81	9.75	9.80	10.93
$VaR_{0.999}(S)$	16.39	14.71	15.09	15.98
$TVaR_{0.9}(S)$	7.68	6.85	6.88	7.92
$TVaR_{0.99}(S)$	13.25	11.90	12.06	13.15
$TVaR_{0.999}(S)$	18.44	16.67	17.19	17.85

case of the copula C_{α_0,α_1} when both $\alpha_0 \to \infty$ and $\alpha_1 \to \infty$ is the upper Frechet–Hoeffding bound copula (also called comonotonicity copula). Also, the independence copula corresponds to a limit case of the copula C_{α_0,α_1} when both $\alpha_0 \to 0$ and $\alpha_1 \to 0$ and the corresponding special case is Model 2. Note that Model 1 and Model 3 cannot be obtained as special cases of the collective risk model defined in the present example. We have the following comments:

- For a fixed α_0, we observe that the expectation of the aggregate loss rv S remains unchanged as the value of α_1 increases while its variance and its TVaR increase.
- For a fixed α_1, we observe that the expectation, the variance and the TVaR of the aggregate loss rv S increase as the value of α_1 increases.
- As expected, the greatest values for the expectation and the TVaR of S are reached when the components of the sequence $\left(N, \underline{X}\right)$ are comonotonic, i.e., $\left(N, \underline{X}\right) = \left(N^+, \underline{X}^{(+,+)}\right)$. As mentioned in Remark 7 , $\left(N^+, \underline{X}^{(+,+)}\right)$ is the extremal element under the supermodular order.

The chosen hierarchical structure for $\left(N, \underline{X}\right)$ in Example 19 introduces a positive dependence relation between the counting rv N and the components of the random vector of claim amounts $\underline{X}$ (level 1) and a positive dependence relation within the components of $\underline{X}$ (level 2). We modify slightly this hierarchical structure in order to introduce a negative dependence relation between the counting rv N and the components of $\underline{X}$. This will allow us to investigate the impact of such a dependence on the rv S, notably on the expectation of S.

Let C be a $(k+1)$-dimensional Archimedean or hierarchical Archimedean copula and $(U_0, U_1, \ldots, U_k)$ a vector of uniformly distributed rvs such that

$$F_{U_0,U_1,\ldots,U_k}(u_0, u_1, \ldots, u_k) = C(u_0, u_1, \ldots, u_k),$$

where $k \in \mathbb{N}$. We also suppose that $N = F_N^{-1}(1-U_0)$, and $X_i = F_{X_i}^{-1}(U_i)$, for $i = 1, \ldots, k$. Then, for any $k \in \mathbb{N}$, the joint cdf of $(N, X_1, \ldots, X_k)$ can be written as

$$\begin{aligned}
& F_{N,X_1,\ldots,X_k}(n, x_1, \ldots, x_k) \\
&= \Pr\left(N \le n, X_1 \le x_1, \ldots, X_k \le x_k\right) \\
&= \Pr\left(F_N^{-1}(1-U_0) \le n, F_{X_1}^{-1}(U_1) \le x_1, \ldots, F_{X_k}^{-1}(U_k) \le x_k\right) \\
&= \Pr\left(U_0 > 1 - F_N(n), U_1 \le F_{X_1}(x_1), \ldots, U_k \le F_{X_k}(x_k)\right) \\
&= \Pr\left(U_1 \le F_{X_1}(x_1), \ldots, U_k \le F_{X_k}(x_k)\right) \\
&\quad - \Pr\left(U_0 \le 1 - F_N(n), U_1 \le F_{X_1}(x_1), \ldots, U_k \le F_{X_k}(x_k)\right) \\
&= C\left(F_{X_1}(x_1), \ldots, F_{X_k}(x_k)\right) \\
&\quad - C\left(1 - F_N(n), F_{X_1}(x_1), \ldots, F_{X_k}(x_k)\right). \qquad (89)
\end{aligned}$$

Example 20. We maintain the same setting as the one in Example 19, except that, for any $k \in \{1, 2, 3\}$, the joint cdf of $(N, X_1, \ldots, X_k)$ is defined as in (89). As in Example 19, the independence copula is a limit case of the copula C_{α_0,α_1} when both $\alpha_0 \to 0$ and $\alpha_1 \to 0$, and the corresponding special case is Model 2. Note that Model 4 cannot be obtained as a special case of the collective risk model defined in the present example. The results are provided in Table 8. We can see that the expectation of the aggregate loss S under this dependence structure is smaller than the one in the classical risk model, i.e., $E[S] \le E[N] \times E[X] = 1.307986$. Also, in this case, and contrarily to the result of Example 19, we can see that when the outer dependence parameter α_0 increases, the expectation, the variance, and the TVaR decrease. Recall that the parameter α_0 captures the degree of dependence between N and the components of $\underline{X}$. Finally, for a fixed α_0, we observe that the variance and the TVaR increase as the inner dependence parameter α_1 becomes larger, since the latter parameter only affects the strength of the dependence between the claim amounts.

Acknowledgments

The authors thank two anonymous referees whose suggestions led to improvements of the paper. This work was partially supported by the Natural Sciences and Engineering Research Council of Canada (Cossette: 054993; Marceau: 053934) and by the Chaire en actuariat de l'Université Laval (Cossette, Marceau, and Mtalai: FO502320).

References

Abate, J., Whitt, W., 1992. The Fourier-series method for inverting transforms of probability distributions. Queueing Syst. 10 (1–2), 5–87.

Albrecher, H., Boxma, O.J., Ivanovs, J., 2014. On simple ruin expressions in dependent sparre Andersen risk models. J. Appl. Probab. 51 (1), 293–296.

Bargès, M., Cossette, H., Marceau, E., 2009. Tvar-based capital allocation with copulas. Insurance Math. Econom. 45 (3), 348–361.

Bäuerle, N., Müller, A., 2006. Stochastic orders and risk measures: Consistency and bounds. Insurance Math. Econom. 38 (1), 132–148.

Boudreault, M., Cossette, H., Landriault, D., Marceau, E., 2006. On a risk model with dependence between interclaim arrivals and claim sizes. Scand. Actuar. J. 2006 (5), 265–285.

Cossette, H., Gadoury, S.-P., Marceau, É., Mtalai, I., 2017. Hierarchical Archimedean copulas through multivariate compound distributions. Insurance Math. Econom. 76, 1–13.

Cossette, H., Marceau, E., Marri, F., 2008. On the compound Poisson risk model with dependence based on a generalized Farlie–Gumbel–Morgenstern copula. Insurance Math. Econom. 43 (3), 444–455.

Cossette, H., Marceau, E., Mtalai, I., Veilleux, D., 2018. Dependent risk models with Archimedean copulas: A computational strategy based on common mixtures and applications. Insurance Math. Econom. 78, 53–71.

Cossette, H., Marceau, E., Perreault, S., 2015. On two families of bivariate distributions with exponential marginals: Aggregation and capital allocation. Insurance Math. Econom. 64, 214–224.

Czado, C., Kastenmeier, R., Brechmann, E.C., Min, A., 2012. A mixed copula model for insurance claims and claim sizes. Scand. Actuar. J. 2012 (4), 278–305.

Denuit, M., Dhaene, J., Goovaerts, M., Kaas, R., 2006. Actuarial Theory for Dependent Risks: Measures, Orders and Models. John Wiley & Sons.

Denuit, M., Genest, C., Marceau, É., 2002. Criteria for the stochastic ordering of random sums, with actuarial applications. Scand. Actuar. J. 2002 (1), 3–16.

Durante, F., Sempi, C., 2010. Copula theory: An introduction. In: Copula Theory and Its Applications. Springer, pp. 3–31.

Fang, Y., 2001. Hyper-Erlang distribution model and its application in wireless mobile networks. Wirel. Netw. 7 (3), 211–219.

Frees, E.W., Gao, J., Rosenberg, M.A., 2011. Predicting the frequency and amount of health care expenditures. N. Am. Actuar. J. 15 (3), 377–392.

Garrido, J., Genest, C., Schulz, J., 2016. Generalized linear models for dependent frequency and severity of insurance claims. Insurance Math. Econom. 70, 205–215.

Gschlossl, S., Czado, C., 2007. Spatial modelling of claim frequency and claim size in non-life insurance. Scand. Actuar. J. 2007 (3), 202–225.

Hofert, M., 2008. Sampling Archimedean copulas. Comput. Statist. Data Anal. 52 (12), 5163–5174.

Joe, H., 1997. Multivariate Models and Multivariate Dependence Concepts. CRC Press.

Joe, H., 2014. Dependence Modeling with Copulas. CRC Press.

Klugman, S.A., Panjer, H.H., Willmot, G.E., Venter, G., 2009. Loss models: From data to decisions. Ann. Actuar. Sci. 4 (2), 343.

Kotz, S., Balakrishnan, N., Johnson, N.L., 2004. Continuous multivariate distributions. In: Models and Applications, vol. 1. John wiley & sons.

Kousky, C., Cooke, R.M., 2009. The unholy trinity: Fat tails, tail dependence, and micro-correlations. Resources for the future. Discuss. Pap. 09 (36), 1–36.

Krämer, N., Brechmann, E.C., Silvestrini, D., Czado, C., 2013. Total loss estimation using copula-based regression models. Insurance Math. Econom. 53 (3), 829–839.

Landriault, D., Lee, W.Y., Willmot, G.E., Woo, J.-K., 2014. A note on deficit analysis in dependency models involving coxian claim amounts. Scand. Actuar. J. 2014 (5), 405–423.

Lee, S.C., Lin, X.S., 2010. Modeling and evaluating insurance losses via mixtures of Erlang distributions. N. Am. Actuar. J. 14 (1), 107–130.

Lee, S.C., Lin, X.S., 2012. Modeling dependent risks with multivariate Erlang mixtures. ASTIN Bull. : J. IAA 42 (1), 153–180.

Liu, H., Wang, R., 2017. Collective risk models with dependence uncertainty. ASTIN Bull. : J. IAA 47 (2), 361–389.

Marshall, A.W., Olkin, I., 1988. Families of multivariate distributions. J. Amer. Statist. Assoc. 83 (403), 834–841.

McNeil, A.J., Frey, R., Embrechts, P., 2015. Quantitative Risk Management: Concepts, Techniques and Tools. Princeton university press.

Müller, A., Stoyan, D., 2002. Comparison Methods for Stochastic Models and Risks, vol .389. Wiley.

Rolski, T., Schmidli, H., Schmidt, V., Teugels, J.L., 1999. Stochastic Processes for Insurance and Finance. John Wiley & Sons.

Shaked, M., Shanthikumar, J.G., 2007. Stochastic orders. In: Science & Business Media. Springer.

Wei, G., Hu, T., 2002. Supermodular dependence ordering on a class of multivariate copulas. Stat. Probab. Lett. 57 (4), 375–385.

Willmot, G.E., Lin, X.S., 2011. Risk modelling with the mixed Erlang distribution. Appl. Stoch. Models Bus. Ind. 27 (1), 2–16.

Willmot, G.E., Woo, J.-K., 2007. On the class of Erlang mixtures with risk theoretic applications. N. Am. Actuar. J. 11 (2), 99–115.

Willmot, G.E., Woo, J.-K., 2015. On some properties of a class of multivariate Erlang mixtures with insurance applications. ASTIN Bull. : J. IAA 45 (1), 151–173.