Properties of Projections in Hilbert Spaces

BST 257: Theory and Methods for Causality II

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1 Recap of projection lemma

Let $(H, \langle \cdot\,, \cdot \rangle)$ be any inner product space. Let $v \in H$, then a projection of v onto for now an arbitrary set U, when it exists, is a vector $u_0 \in U$ satisfying:

(a) $u_0 \in U$, and

(b) $v - u_0 \perp U$.

Note that this implies uniqueness as if $u, u' \in U$ satisfy (b), then

$$u_0 - u_0' = (v - u_0') - (v - u_0) \perp U,$$

so $u_0 - u_0' \in U \cap U^{\perp} \subseteq \{0_H\}$, meaning $u_0 = u_0'$. We now show that when U is a linear subspace (not necessarily closed), properties (a) and (b) are equivalent to the characterization

$$u_0 = \arg\min_{u \in U} \|v - u\|. \tag{1}$$

To see this, first assume (a) and (b), then for any $u \in U$

$$\|v - u\|^2 = \|v - u_0\|^2 + \|u_0 - u\|^2 \geq \|v - u_0\|^2,$$

by the Pythagorean theorem (i.e., $u - u_0 \in U$, $v - u_0 \in U^{\perp}$), so (1) is established. Notice also that equality holds if and only if $\|u_0 - u\|^2 = 0 \iff u_0 = u$, so u_0 is the unique minimizer. Conversely, if (1) holds, then (a) holds by construction, and assume by way of contradiction that (b) is false: let $u_1 \in U$ satisfy $\langle v - u_0, u_1 \rangle = \delta \neq 0$. Then considering $u_1^* = u_0 + \delta \frac{u_1}{\|u_1\|^2} \in U$, observe that

$$\|v - u_1^*\|^2 = \|v - u_0\|^2 - 2\frac{\delta}{\|u_1\|^2}\langle v - u_0, u_1 \rangle + \left(\frac{\delta}{\|u_1\|}\right)^2 = \|v - u_0\|^2 - \left(\frac{\delta}{\|u_1\|}\right)^2 < \|v - u_0\|^2,$$

contradicting (1), thus establishing (b).

Assume the inner product space (or pre-Hilbert space) H is actually a Hilbert space, i.e., it is complete with respect to the norm induced by its inner product (that being $v \mapsto \sqrt{\langle v, v \rangle}$), and let $U \subseteq H$ be a closed linear subspace. The essence of the projection lemma is that it establishes that projections onto U exist. As a consequence of existence and uniqueness, the operator $\Pi_U : H \to U$ is well-defined, yielding *the* projection of elements in H onto U.

Recall also the notation and principal example from the course: the Hilbert space $L_2(P)$, consisting of all measurable real valued functions g of $O \sim P$, such that $\mathbb{E}_P(g(O)^2) < \infty$ (formally, the objects in $L_2(P)$ are equivalence classes of random functions that are almost surely equal). In this case, for any $g, h \in L_2(P)$, $\langle g, h \rangle = \mathbb{E}_P(g(O)h(O))$ and $\|g\|^2 = \langle g, g \rangle = \mathbb{E}_P(g(O)^2)$. For a random vector $\boldsymbol{T} = (t_1, \ldots, t_p) \in L_2(P)^p$, with linear span $\mathcal{T} = \left\{\alpha^T \boldsymbol{T} \mid \alpha \in \mathbb{R}^p\right\}$, in the course notes we write $\Pi[\,\cdot\,|\mathcal{T}]$ or $\Pi[\,\cdot\,|\boldsymbol{T}]$ in place of $\Pi_{\mathcal{T}}$. The function $\Pi[\,\cdot\,|\mathcal{M}]$, for an arbitrary closed linear subspace $\mathcal{M} \subseteq L_2(P)$ is, also overloaded to stack projections component-wise when the input is a vector.

2 Establishing basic properties of Π_U

We now verify some essential properties of the projection operator that follow from the definition. In the following exercises, let H be a Hilbert space, and U a closed linear subspace of H. In almost all cases, it is most convenient to use characterizing properties (a) and (b).

Exercise 1 – Linearity

Show that for any $a, b \in \mathbb{R}$, $v_1, v_2 \in H$,

$$\Pi_U(av_1 + bv_2) = a\Pi_U(v_1) + b\Pi_U(v_2).$$

Exercise 2 – Idempotence

Show that $\Pi_U \circ \Pi_U = \Pi_U$: for any $v \in H$,

$$\Pi_U(\Pi_U(v)) = \Pi_U(v).$$

Hint: show that for any $u \in U$, $\Pi_U(u) = u$.

Exercise 3 – Self-adjointness

Show that for any $v_1, v_2 \in H$,

$$\langle v_1, \Pi_U(v_2)\rangle = \langle \Pi_U(v_1), v_2\rangle.$$

Exercise 4 – Projecting in steps

Show that if $U_1 \subseteq U$ is a closed linear subspace, $\Pi_{U_1} \circ \Pi_U = \Pi_U \circ \Pi_{U_1} = \Pi_{U_1}$.

Exercise 5 – Projecting onto orthogonal direct sum

Show that if $U_1, U_2 \subseteq H$ are closed linear subspaces such that $U_1 \perp U_2$, $\Pi_{U_1 \oplus U_2} = \Pi_{U_1} + \Pi_{U_2}$.

3 Projections onto more general sums of subspaces

In Exercise 5, we showed how to project onto sums of orthgonal subspaces. More general sums are harder to deal with. Our strategy will be to orthgonalize one space with respect to the other, such that the sum remains the same – this will look a lot like a Gram-Schmidt procedure if you have seen it. Projection will then reduce to Exercise 5. We start with the following fact:

Fact 1 – Projecting onto relative orthgonal complement

Suppose $U_1 \subseteq U$ is a closed linear subspace. Then $\Pi_{U_1^\perp \cap U} = \Pi_U - \Pi_{U_1}$. In particular, when $U = H$, $\Pi_{U_1^\perp} = I_H - \Pi_{U_1}$, where $I_H \equiv \Pi_H$ is the identity map on H.

Proof. It is sufficient to show, for arbitrary $v \in H$,

(i) $\Pi_U(v) - \Pi_{U_1}(v) \in U_1^\perp \cap U$, and

(ii) $v - (\Pi_U(v) - \Pi_{U_1}(v)) \perp U_1^\perp \cap U$.

For (i), clearly $\Pi_U(v) - \Pi_{U_1}(v) \in U$, since $U_1 \subseteq U$ and U is a subspace. For arbitrary $w \in U_1$,

$$\langle \Pi_U(v) - \Pi_{U_1}(v), w\rangle = \langle \Pi_U(v), w\rangle - \langle \Pi_{U_1}(v), w\rangle = \langle v, \Pi_U(w)\rangle - \langle v, \Pi_{U_1}(w)\rangle = \langle v, w\rangle - \langle v, w\rangle = 0,$$

so $\Pi_U(v) - \Pi_{U_1}(v) \in U_1^\perp \implies \Pi_U(v) - \Pi_{U_1}(v) \in U_1^\perp \cap U$.

Next, for (ii) take $w \in U_1^\perp \cap U$ arbitrary. Then

$$\langle v - (\Pi_U(v) - \Pi_{U_1}(v)), w\rangle = \langle \underbrace{v - \Pi_U(v)}_{\in U^\perp}, w\rangle + \langle \underbrace{\Pi_{U_1}(v)}_{\in U_1}, w\rangle = 0,$$

since $w \in U$ and $w \in U_1^\perp$. Therefore, $v - (\Pi_U(v) - \Pi_{U_1}(v)) \perp U_1^\perp \cap U$. □

Fact 2 – Gram-Schmidt orthogonalization

Suppose the closed linear subspace U can be decomposed as $U = U_1 + U_2$, where $U_1 \subseteq H$ is arbitrary and $U_2 \subseteq H$ is a closed linear subspace. Then $U = U_1^* \oplus U_2$, where

$$U_1^* = \Pi_{U_2^\perp}(U_1) = \{\Pi_{U_2^\perp}(u) : u \in U_1\} = \{u - \Pi_{U_2}(u) : u \in U_1\}.$$

Proof. Treating U as a Hilbert space in itself (under the inner product of H, restricted to U),

$$U = (U_2^\perp \cap U) \oplus U_2,$$

since $U_2 \subseteq U$ — note that the orthogonal complement of U_2 in U is the set of elements of U that are orthogonal to all of U_2, i.e., $U_2^\perp \cap U$. By Fact 1 we have $\Pi_{U_2^\perp \cap U} = \Pi_U - \Pi_{U_2}$, so

$$\begin{aligned} U &= (U_2^\perp \cap U) \oplus U_2, \\ &= \Pi_{U_2^\perp \cap U}(U) \oplus U_2, \\ &= (\Pi_U - \Pi_{U_2})(U) \oplus U_2, \\ &= (I_H - \Pi_{U_2})(U_1 + U_2) \oplus U_2, \ \text{as } \Pi_U(u) = u = I_H(u) \text{ for all } u \in U, \\ &= \Pi_{U_2^\perp}(U_1) \oplus U_2, \ \text{as } \Pi_{U_2^\perp}(u) = (I_H - \Pi_{U_2})(u) = 0, \ \text{for all } u \in U_2. \end{aligned}$$

□

As a corollary, suppose in the setting of the above result, $U_1 = [\{v_1, \ldots, v_k\}]$, where $v_1, \ldots, v_k \in H$ and $[B]$ denotes linear span for any $B \subseteq H$. Then

$$\Pi_{U_2^\perp}(U_1) = [\{v_1 - \Pi_{U_2}(v_1), \ldots, v_k - \Pi_{U_2}(v_k)\}] = U_1^*.$$

Specializing to $L_2(P)$, given $\boldsymbol{T} = \begin{bmatrix} \boldsymbol{T}_1 \\ \boldsymbol{T}_2 \end{bmatrix} \in L_2(P)^p$, and letting $\mathcal{T}_j$ be the linear span of the components of $\boldsymbol{T}_j$, for $j = 1, 2$, this says

$$\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 = \mathcal{R}_1 \oplus \mathcal{T}_2,$$

where $\mathcal{R}_1$ is the linear span of the components of $\boldsymbol{R}_1 = \boldsymbol{T}_1 - \Pi[\boldsymbol{T}_1 \mid \boldsymbol{T}_2]$ — this is the general property discussed in section 7.1.1 of the course notes! By Exercise 5, $\Pi[\,\cdot\,|\boldsymbol{T}] = \Pi[\,\cdot\,|\boldsymbol{R}_1] + \Pi[\,\cdot\,|\boldsymbol{T}_2]$, and from this we can see that the $\boldsymbol{T}_1$ components of the population least squares parameter for projecting some vector $\boldsymbol{W} \in L_2(P)^k$ onto $\mathcal{T}$ can be obtained by projecting $\boldsymbol{W}$ onto $\mathcal{R}_1$.