LIST OF EXERCISES

Sets, functions, sequences, sums, cardinality
(Rosen - Chapter 2)

Required reading for this list: *Discrete Mathematics and Its Applications* (Rosen, 7^{th} Edition):

- Chapter 2.1: *Sets*
- Chapter 2.2: *Set Operations*
- Chapter 2.3: *Functions*
- Chapter 2.4: *Sequences and Summations*
- Chapter 2.5: *Cardinality of Sets*

Note: The exercises are classified into difficulty levels: easy, medium, and hard. This classification, however, is only indicative. Different people may disagree about the difficulty level of the same exercise. Do not be discouraged if you see a difficult exercise–you may find that it is actually easy, by discovering a simpler way to solve it!

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1. `[Easy]` (Rosen 2.1-5) Determine whether each of the following pairs of sets are equal or not.
   (a) $\{1, 3, 3, 3, 5, 5, 5, 5, 5\}$ and $\{5, 3, 1\}$
   (b) $\{\{1\}\}$ and $\{1, \{1\}\}$
   (c) $\varnothing$ and $\{\varnothing\}$

2. `[Easy]` (Rosen 2.1-9) Determine whether each of the following statements is true or false.
   (a) $0 \in \varnothing$
   (b) $\varnothing \in \{0\}$
   (c) $\{0\} \subset \varnothing$
   (d) $\varnothing \subset \{0\}$
   (e) $\{0\} \in \{0\}$
   (f) $\{0\} \subset \{0\}$
   (g) $\{0\} \subseteq \{0\}$

3. `[Easy]` (Rosen 2.1-11) Determine whether each of the following statements is true or false.
   (a) $x \in \{x\}$
   (b) $\{x\} \subseteq \{x\}$
   (c) $\{x\} \in \{x\}$
   (d) $\{x\} \subseteq \{\{x\}\}$
   (e) $\varnothing \subseteq \{x\}$
   (f) $\varnothing \in \{x\}$

4. `[Easy]` (Rosen 2.1-21) Find the power set of the following sets, where a and b are distinct elements.
   (a) $\{a\}$
   (b) $\{a, b\}$
   (c) $\{\varnothing, \{\varnothing\}\}$

5. `[Easy]` (Rosen 2.1-27) Let $A = \{a, b, c, d\}$ and $B = \{x, y\}$. Find

(a) $A \times B$

(b) $B \times A$

6. `[Easy]` (Rosen 2.1-30) Suppose $A \times B = \varnothing$, where A and B are sets. What can you conclude?

7. `[Easy]` (Rosen 2.1-39) Explain why $A \times B \times C$ and $(A \times B) \times C$ are not the same set.

8. `[Hard]` (Rosen 2.1-45) The defining property of an ordered pair is that two ordered pairs are equal if and only if their first elements are equal and their second elements are equal. Surprisingly, instead of taking the ordered pair as a primitive concept, we can construct ordered pairs using basic notions from set theory. Show that if we define the ordered pair (a, b) to be $\{\{a\}, \{a, b\}\}$, then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. [Hint: First show that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ if and only if $a = c$ and $b = d$.]

9. (Rosen 2.2-15) Show that if A and B are sets, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$: (This is one of De Morgan's laws.)

   (a) `[Medium]` By showing that each side is a subset of the other.

   (b) `[Easy]` Using a membership table.

   (c) $A - B \subseteq A$

   (d) $A \cup (B - A) = (A \cup B)$

10. `[Medium]` (Rosen 2.2-18) Let A, B, and C be sets. Using logical connective manipulation, show that:

    (a) $(A \cup B) \subseteq (A \cup B \cup C)$

    (b) $(A - B) - C \subseteq A - C$

    (c) $(B - A) \cup (C - A) = (B \cup C) - A$

11. `[Easy]` (Rosen 2.2-29) What can you say about sets A and B if you know that:

    (a) $A \cup B = A$?

    (b) $A \cap B = A$?

    (c) $A - B = A$?

    (d) $A \cap B = B \cap A$?

    (e) $A - B = B - A$?

12. `[Medium]` (Rosen 2.2-50) Determine $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ for each A_i below:

    (a) $A_i = \{i, i + 1, i + 2, \ldots\}$.

    (b) $A_i = \{0, i\}$.

    (c) $A_i = (0, i)$.

    (d) $A_i = (i, \infty)$.

13. `[Easy]` (Rosen 2.3-1) Why is f not a function from $\mathbb{R}$ to $\mathbb{R}$ if

    (a) $f(x) = 1/x$?

    (b) $f(x) = \sqrt{x}$?

    (c) $f(x) = \pm\sqrt{(x^2 + 1)}$?

14. `[Easy]` (Rosen 2.3-4) Find the domain and range of the functions below. Note that, in each case, to find the domain you should identify the set of elements to which the function assigns a value.

    (a) the function that assigns to each non-negative integer its last digit;

    (b) the function that assigns the next largest integer to a positive integer;

    (c) The function that assigns to each binary string the number of 1's it contains.

    (d) The function that assigns to each binary string its length (total number of bits).

15. `[Easy]` (Rosen 2.3-9) Find the value of:

(a) $\lceil \frac{3}{4} \rceil$
(b) $\lfloor \frac{7}{8} \rfloor$
(c) $\lceil -\frac{3}{4} \rceil$
(d) $\lfloor -\frac{7}{8} \rfloor$
(e) $\lceil 3 \rceil$
(f) $\lfloor -1 \rfloor$
(g) $\lceil \frac{1}{2} + \lceil \frac{3}{2} \rceil \rceil$
(h) $\lfloor \frac{1}{2} \cdot \lfloor \frac{5}{2} \rfloor \rfloor$

16. `[Easy]` (Rosen 2.3-12) Determine which of the following functions from $\mathbb{Z}$ to $\mathbb{Z}$ are injective, surjective, and bijective.

    (a) $f(n) = n - 1$
    (b) $f(n) = n^2 + 1$
    (c) $f(n) = n^3$
    (d) $f(n) = \lceil n/2 \rceil$

17. `[Easy]` (Rosen 2.3-33) Let g be a function from A to B and f a function from B to C.

    (a) Show that if f and g are both injective, then $f \circ g$ is also injective.
    (b) Show that if f and g are both surjective, then $f \circ g$ is also surjective.

18. (Rosen 2.3-42) Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ defined as $f(x) = x^2$. Find

    (a) `[Easy]` $f^{-1}(\{1\})$
    (b) `[Medium]` $f^{-1}(\{x \mid 0 < x < 1\})$
    (c) `[Medium]` $f^{-1}(\{x \mid x > 4\})$

19. `[Hard]` (Rosen 2.3-54) Prove that if x is a real number, then $\lfloor -x \rfloor = -\lceil x \rceil$ and $\lceil -x \rceil = -\lfloor x \rfloor$.

20. (Rosen 2.3-70) Prove or disprove each of the following statements:

    (a) `[Medium]` $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil \quad \forall x \in \mathbb{R}$.
    (b) `[Medium]` $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor \quad \forall x, y \in \mathbb{R}$.
    (c) `[Hard]` $\lceil \lceil \frac{x}{2} \rceil / 2 \rceil = \lceil \frac{x}{4} \rceil \quad \forall x \in \mathbb{R}$.
    (d) `[Medium]` $\lfloor \sqrt{\lceil x \rceil} \rfloor = \lfloor \sqrt{x} \rfloor \quad \forall x \in \mathbb{R}^+$.
    (e) `[Hard]` $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor \quad \forall x, y \in \mathbb{R}$.

21. (Rosen 2.3-79)

    a) `[Easy]` Show that if a set S has cardinality m, where m is a positive integer, then there is a one-to-one correspondence between S and the set $\{1, 2, \ldots, m\}$.
    b) `[Medium]` Show that if S and T are two sets each with m elements, where m is a positive integer, then there is a one-to-one correspondence between S and T.

22. `[Easy]` (Rosen 2.4-3) What are the terms a_0, a_1, a_2 and a_3 of the sequence $\{a_n\}$ where a_n is given by

    (a) $2^n + 1$
    (b) $(n+1)^{n+1}$
    (c) $\lfloor \frac{n}{2} \rfloor$
    (d) $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil$

23. `[Easy]` (Rosen 2.4-5) List the first 10 terms of these sequences.

    (a) the sequence starting with 2 where each term is 3 more than the previous one;
    (b) the sequence listing each positive integer three times in increasing order;
    (c) the sequence listing all positive odd integers in increasing order, listing each odd twice;

(d) the sequence whose n-th term is $n! - 2^n$;

(e) the sequence starting with 3 where each subsequent term is twice the previous term;

(f) the sequence whose first term is 2, the second is 4, and each next term is the sum of the previous two terms;

(g) the sequence whose n-th term is the number of bits in the binary representation of n;

(h) the sequence whose n-th term is the number of letters in the Portuguese word for n.

24. (Rosen 2.4-26) For each of the integer lists below, give a simple formula or rule that generates the terms of a sequence of integers that starts with the given list. Assuming your formula is correct, give the next three elements of the sequence.

    (a) `[Easy]` $3, 6, 11, 18, 27, 38, 51, 66, 83, 102, ...$

    (b) `[Easy]` $7, 11, 15, 19, 23, 27, 31, 35, 39, 43, ...$

    (c) `[Easy]` $1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, ...$

    (d) `[Easy]` $1, 2, 2, 2, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5, ...$

    (e) `[Hard]` $0, 2, 8, 26, 80, 242, 728, 2\,186, 6\,560, 19\,682, ...$

    (f) `[Hard]` $1, 3, 15, 105, 945, 10\,395, 135\,135, 2\,027\,025, 34\,459\,425, ...$

    (g) `[Medium]` $1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, ...$

25. (Rosen 2.4-43) What are the values of the following products:

    (a) `[Easy]` $\prod_{i=0}^{13} i$

    (b) `[Easy]` $\prod_{i=5}^{10} i$

    (c) `[Medium]` $\prod_{i=0}^{99} (-1)^i$

    (d) `[Medium]` $\prod_{i=0}^{11} 2$

26. `[Easy]` (Rosen 2.5-1) Determine if each of the sets below is finite, countably infinite, or uncountable. For countably infinite sets, list the first 10 elements in an enumeration.

    (a) the negative integers

    (b) the even integers

    (c) the integers less than 100

    (d) the real numbers between 0 and $1/2$

    (e) the positive integers less than $1\,000\,000\,000$

    (f) the integers that are multiples of 7

27. `[Easy]` (Rosen 2.5-2) Determine if each of the sets below is countable or uncountable. For countable sets, list the first 10 elements in an enumeration.

    (a) the integers greater than 10;

    (b) the negative odd integers;

    (c) the real numbers between 0 and 2;

    (d) the integers that are multiples of 10.

28. `[Medium]` (Rosen 2.5-11) Give an example of two uncountable sets A and B such that $A \cap B$ is

    (a) finite

    (b) countably infinite

    (c) uncountable

29. (Rosen 2.5-34) Determine if each of the sets below is countable or not. For countable sets, list the first 10 elements in an enumeration.

    (a) `[Easy]` the integers not divisible by 3.

    (b) `[Easy]` the integers divisible by 5 but not by 7.

    (c) `[Medium]` the real numbers with decimal representation containing only 1s.

    (d) `[Medium]` the real numbers with decimal representation containing only 1s or 9s.

30. `[Hard]` (Rosen 2.5-28) Show that the set $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.