COMP0147 – Discrete Mathematics

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Introduction to Mathematical Reasoning

I. Introduction

- Mathematical properties are formulated in "normal language" / plain English, but they may look ambiguous or unclear to the uninitiated.
- Even if we understand a property, we may not understand how to approach proving it.
- Solution: understanding that mathematical properties can be formulated in the precise language of logic.
- It gives us a methodical approach to understanding mathematical properties and proving them.

I. Introduction

- In this lecture, we will:
- Understand the architecture of reasoning.
- Study how mathematical properties are formulated.
- Examine their underlying logic and the relevant symbolic language.
- Study tools to prove mathematical properties.

I. Introduction

- In this lecture, we will NOT:

- Study logic and proofs formally (to be seen in other modules)

- Study advanced mathematical properties (to be seen in other modules)

- Instead, we focus on the logical structure of reasoning, mathematical properties and proof strategies.

- This is essential for upcoming Maths and CS modules.

Outline

- Basic logical symbols: $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\Leftrightarrow$
- Quantifiers: $\exists, \forall$
- Proving and disproving mathematical statements.

II. The Language of Mathematics

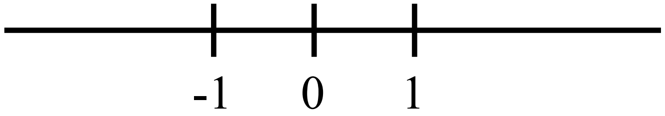
- Prerequisites: sets of numbers.

- Definitions:

  $\mathbb{N}$ is the set of natural numbers: $0,1,2,3,4\ldots$

  $\mathbb{Z}$ is the set of integers: $\ldots -3,-2,-1,0,1,2,3\ldots$

  $\mathbb{R}$ is the set of real numbers.



II. The Language of Mathematics

- Mathematical logic is a symboling language that models reasoning.
- It allows to express and connect properties.
- There exist many different varieties of logic: Classical Logic, Intuitionistic Logic, Modal Logic…(to be seen in upcoming modules)
- Here, we use classical first-order logic.

(Aka predicate logic)

II. The Language of Mathematics

- **Boolean operations: Conjunction, Disjunction, Negation**

  - Conjunction: binary connective AND, symbol $\wedge$

The formula $P \wedge Q$ is true when both P and Q are true.

Examples:
n is an even number and a multiple of 3.
f is differentiable and f' is continuous.

II. The Language of Mathematics

- Boolean operations: Conjunction, Disjunction, Negation

  - Disjunction: binary connective OR, symbol $\vee$

The formula $P \vee Q$ is true when P or Q is true (or if they are both true)

Examples:
n is a multiple of 2 or a multiple of 6.
$(f \times g)(x) = 0$ if $f(x) = 0$ or $g(x) = 0$.

II. The Language of Mathematics

- Boolean operations: Conjunction, Disjunction, Negation
  - Negation: unary connective NOT, symbol $\neg$

The formula $\neg \mathrm{P}$ is true when P is not true.

Examples:
f is not continuous.
x is not an integer.

II. The Language of Mathematics

- The interactions of these connectives are described by De Morgan's Laws.

- What is the negation of a conjunction? Of a disjunction?

In what cases are they false?

$\neg(A \wedge B) =$

$\neg(A \vee B) =$

II. The Language of Mathematics

- The interactions of these connectives are described by De Morgan's Laws.
- What is the negation of a conjunction? Of a disjunction?

$$\neg(A \wedge B) = \neg A \vee \neg B$$

$$\neg(A \vee B) = \neg A \wedge \neg B$$

II. The Language of Mathematics

- **Example:**
  The weather forecast for tomorrow states:

  Weather will be cold and rainy.

- What is its negation? In what cases can we say the forecast was wrong?

II. The Language of Mathematics

- **Example:**
  The weather forecast for tomorrow states:

  Weather will be cold and rainy.

- What is its negation? In what cases can we say the forecast was wrong?

  Weather is not cold, or not rainy.

II. The Language of Mathematics

- Example:
  Consider the following mathematical statement:

$$f(x) = 0 \text{ or } g(x) = 0.$$

- What is its negation?

II. The Language of Mathematics

- **Example:**
  Consider the following mathematical statement:

  $f(x) = 0$ or $g(x) = 0$.

- What is its negation?

  $f(x) \neq 0$ and $g(x) \neq 0$.

II. The Language of Mathematics

- Implication: binary connective IF…THEN, symbol $\Rightarrow$

II. The Language of Mathematics

- Implication: binary connective IF…THEN, symbol $\Rightarrow$

The formula $P \Rightarrow Q$ can be expressed in many ways:

- P implies Q.
- Whenever P is true, Q must be true.
- P is a sufficient condition for Q.
- Q is a necessary condition for P.

II. The Language of Mathematics

- Implication: binary connective IF…THEN, symbol $\Rightarrow$

The formula $P \Rightarrow Q$ can be expressed in many ways:

- P implies Q.
- Whenever P is true, Q must be true.
- P is a sufficient condition for Q.
- Q is a necessary condition for P. -> We cannot have P without having Q

II. The Language of Mathematics

- Implication: binary connective IF…THEN, symbol $\Rightarrow$

The implication is more subtle than it seems, and can be easily misunderstood.

Example of misunderstanding: see the Wason Selection Task. (In class if we have time)

Let us study it more precisely by giving its truth table.

II. The Language of Mathematics

- **Implication:** binary connective IF…THEN, symbol $\Rightarrow$

$P \Rightarrow Q$ is:

	Q true	Q false
P true	TRUE	FALSE
P false	TRUE	TRUE

II. The Language of Mathematics

- Implication: binary connective IF…THEN, symbol $\Rightarrow$

$P \Rightarrow Q$ is:

	Q true	Q false
P true	TRUE	FALSE
P false	TRUE	TRUE

This row is easy:
when P is true,
Q must be true

The 2nd row is less intuitive:
when P is false, the implication is considered true regardless of Q

- Implication: binary connective IF…THEN, symbol $\Rightarrow$

$P \Rightarrow Q$ is:

	Q true	Q false
P true	TRUE	FALSE
P false	TRUE	TRUE

Only case where the implication is false: P is true, but Q is false.

Again, we cannot have P without having Q

II. The Language of Mathematics

- Implication: binary connective IF…THEN, symbol $\Rightarrow$

- What is the negation of the implication?
  $\neg(P \Rightarrow Q) =$?

- In what cases is the implication false?

What is a property that is true exactly when the implication is false?

II. The Language of Mathematics

- Implication: binary connective IF…THEN, symbol $\Rightarrow$

- What is the negation of the implication?
  $\neg(P \Rightarrow Q) = ?$

- In what cases is the implication false?
  P true and Q false.

- This means that $\boxed{\neg(P \Rightarrow Q) = P \wedge \neg Q}$

-> Implication is only false if we have P without having Q

II. The Language of Mathematics

- Implication: binary connective IF…THEN, symbol $\Rightarrow$

- One well-known trick: Consider the truth table.

- Among the 4 cases, we can gather the 3 cases where $P \Rightarrow Q$ is true as follows:

	Q true	Q false
P true	TRUE	FALSE
P false	TRUE	TRUE

II. The Language of Mathematics

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Cases where Q is true
Cases where P is false

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(They overlap)

II. The Language of Mathematics

- Implication: binary connective IF…THEN, symbol $\Rightarrow$

- One well-known trick: Consider the truth table.

- Among the 4 cases, we can gather the 3 cases where $P \Rightarrow Q$ is true as follows:

  Cases where Q is true
  Cases where P is false

	Q true	Q false
P true	TRUE	FALSE
P false	TRUE	TRUE

- Hence $P \Rightarrow Q$ can be reformulated as: Q is true, or P is false.

(They overlap)

- In other words: $\boxed{P \Rightarrow Q = Q \vee \neg P}$

- **Example:**
  Consider the following mathematical statement:

  If n is even, then $P(n) = 0$.

  In what case is this implication false?

- What is its negation?

II. The Language of Mathematics

- **Example:**
  Consider the following mathematical statement:

  If n is even, then $P(n) = 0$.

In what case is this implication false?

- What is its negation?

  n is even and $P(n) \neq 0$.

II. The Language of Mathematics

- Another trick: contraposition.
- We know that $P \Rightarrow Q = Q \vee \neg P$
- This can be rewritten as $\neg P \vee Q$ or even $\neg P \vee \neg(\neg Q)$

II. The Language of Mathematics

- Another trick: contraposition.
- We know that $P \Rightarrow Q = Q \vee \neg P$
- This can be rewritten as $\neg P \vee Q$ or even $\boxed{\neg P} \vee \neg\boxed{(\neg Q)}$
- $P \Rightarrow Q$ is therefore equivalent to $\boxed{\neg Q} \Rightarrow \boxed{\neg P}$

This is called the contrapositive form

- This makes sense:
  Assume $P \Rightarrow Q$. If we don't have Q, we cannot have P.

II. The Language of Mathematics

- Equivalence: binary connective with symbol $\Leftrightarrow$

- $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
- In plain English, we write "P if and only if Q" for $P \Leftrightarrow Q$
- $P \Leftrightarrow Q$ means that the properties P and Q always coincide.
- If we have P, we have Q. If we have Q, we have P.

$$\begin{tikzcd} P \arrow[r, bend left] & Q \arrow[l, bend left] \end{tikzcd}$$

This part is called the converse

II. The Language of Mathematics

- **Equivalence:** binary connective with symbol $\Leftrightarrow$

- **Example:**

- A matrix M is invertible if and only if its determinant is not equal to zero.

Each of these 2 properties implies the other

II. The Language of Mathematics

- Equivalence: binary connective with symbol $\Leftrightarrow$

- Main difficulty: not all implications are equivalences.
- If $x = 3$, then $x^2 = 9$
- $x = 3$ if and only if $x^2 = 9$

II. The Language of Mathematics

- Equivalence: binary connective with symbol $\Leftrightarrow$

- Main difficulty: not all implications are equivalences.
- If $x = 3$, then $x^2 = 9$ is true.
- $x = 3$ if and only if $x^2 = 9$ is false: we could have $x = -3$.
- The implication $x^2 = 9 \Rightarrow x = 3$ (the converse) is false.

$$x = 3 \Rightarrow x^2 = 9$$

$$x = 3 \cancel{\Leftrightarrow} x^2 = 9$$

II. The Language of Mathematics

- Quantifiers

- Logical connectives that allow to reason about variables.

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- Quantifiers
- Logical connectives that allow to reason about variables.
- Universal Quantifier: FOR ALL, symbol $\forall$
- We write For all x, $P(x)$ or $\forall x, P(x)$ where $P(x)$ is a proposition involving x.
- $\forall x, P(x)$ is true if $P(x)$ is true for every value of x.

II. The Language of Mathematics

- <u>Quantifiers</u>

- Logical connectives that allow to reason about variables.

- $\boxed{\text{\underline{Universal Quantifier:} FOR ALL, symbol } \forall}$

- We write For all x, $P(x)$ or $\forall x, P(x)$ where $P(x)$ is a proposition involving x.

- $\forall x, P(x)$ is true if $P(x)$ is true for every value of x.

- In Mathematics, we will often write $\forall x \in S, P(x)$ for a set S.
  Technically, it is is a shortcut for $\forall x, x \in S \Rightarrow P(x)$.

II. The Language of Mathematics

- Examples:
- For all real number x, $x^2 \geq 0$.
- If for all $x \in [0,1]$, $f(x) \geq 0$, then $\int_0^1 f(x)dx \geq 0$.
- $\forall n \in \mathbb{N}, n \geq 0$

II. The Language of Mathematics

- **Existential Quantifier: THERE EXISTS, symbol $\exists$**
- We write There exists x *such that* $P(x)$ or $\exists x, P(x)$ where $P(x)$ is a proposition involving x.
- $\exists x, P(x)$ is true if there exists a x for which $P(x)$ is true.
- In Mathematics, we will often write $\exists x \in S, P(x)$ for a set S.
Technically, it is is a shortcut for $\exists x, x \in S \wedge P(x)$.
- There can be many different x for which $P(x)$ is true. What matters is that there is at least one.

II. The Language of Mathematics

- **Examples:**

- $\exists n \in \mathbb{N}, n^2 = 25.$

- If f is continuous and $f(-1) < 0$ and $f(1) > 0$,
  then there exists x such that $f(x) = 0$.

- If P is a polynomial of degree 3, then $\exists x \in \mathbb{R}, P(x) = 0.$

Are these true?

II. The Language of Mathematics

- **Examples:**
- $\exists n \in \mathbb{N}, n^2 = 25.$ True: n=5 works.
- If f is continuous and $f(-1) < 0$ and $f(1) > 0$, then there exists x such that $f(x) = 0$. By intermediate value theorem
- If P is a polynomial of degree 3, then $\exists x \in \mathbb{R}, P(x) = 0$.

explained in in COMP0011

II. The Language of Mathematics

- Negation of Quantifiers

- What is the negation of $\forall x, P(x)$? For instance, $\forall x, f(x) \geq 0$.

In what case is this proposition false?

II. The Language of Mathematics

- Negation of Quantifiers

- What is the negation of $\forall x, P(x)$? For instance, $\forall x, f(x) \geq 0$.

- It is false when there exists a certain x for which $f(x) < 0$.

- We have: $\boxed{\neg(\forall x, P(x)) = \exists x, \neg P(x)}$

The contrary of "for all x, P" is "there exists an x such that NOT P"

- Negation of Quantifiers
- What is the negation of $\exists x, P(x)$? For instance, $\exists x, f(x) = g(x)$.

In what case is this proposition false?

II. The Language of Mathematics

- Negation of Quantifiers
- What is the negation of $\exists x, P(x)$? For instance, $\exists x, f(x) = g(x)$.
- It is false when for all values of x we have $f(x) \neq g(x)$.
- We have $\boxed{\neg(\exists x, P(x)) = \forall x, \neg P(x)}$

The contrary of "there exists an x such that P" is "for all x we have NOT P",
Or "whatever the value of x, we never have P"

II. The Language of Mathematics

- Examples: What is the negation of the following statements?
- For all $x \in \mathbb{R}$, $f(x) > 1$.
- There exists $n \in S$ such that n^2 is an even number.
- Every student has passed the exam.

II. The Language of Mathematics

- Examples: What is the negation of the following statements?

- For all $x \in \mathbb{R}, f(x) > 1$.
  There exists $x \in \mathbb{R}, f(x) \leq 1$

- There exists $n \in S$ such that n^2 is an even number.
  For all $n \in S$, n^2 is an odd number.

- Every student has passed the exam.
  There exists a student who has failed the exam.

II. The Language of Mathematics

- **Remarks:**

- It is possible to quantify over several variables:
  We write $\forall x, y$ as a shortcut for $\forall x, \forall y$.

- We will see in future modules some combinations of quantifiers. Their order matters! $\forall x, \exists y, P(x, y)$ is not equivalent to $\exists y, \forall x, P(x, y)$.

- **Example:**
  $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x < y$ vs $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x < y$

TBC in problem class

III. Proving Mathematical Propositions

- Now that we have identified the logical backbone of mathematical properties, we study:

- How to prove properties?

- How to disprove properties? (How to prove they are false)

III. Proving Mathematical Propositions

- Proving Conjunction, Disjunction
- How to prove $P \wedge Q$?
- We must prove both P and Q.
- How to prove $P \vee Q$?
- We must prove P or Q, we don't necessarily know which one is true.

III. Proving Mathematical Propositions

- Disproving Conjunction, Disjunction

- How to disprove $P \wedge Q$?

- How to disprove $P \vee Q$?

III. Proving Mathematical Propositions

- Disproving Conjunction, Disjunction
- How to disprove $P \wedge Q$?
- We must prove $\boxed{\neg(P \wedge Q) = \neg P \vee \neg Q}$
- How to disprove $P \vee Q$?
- We must prove $\boxed{\neg(P \vee Q) = \neg P \wedge \neg Q}$

Use De Morgan's Laws

III. Proving Mathematical Propositions

- **Example:**

  Consider the property:
  $P(x)=$ *"x is an even number, and it is a multiple of 3."*

  How to disprove P? (How to prove that P is false?)

III. Proving Mathematical Propositions

- **Example:**

  Consider the property:
  P(x)= "x is an even number, and it is a multiple of 3."

  How to disprove P? (How to prove that P is false?)

- We must prove that *x is not an even number*, or that *x is not a multiple of 3*.

III. Proving Mathematical Propositions

- Proving implication

- How to prove $P \Rightarrow Q$?

III. Proving Mathematical Propositions

- Proving implication
- How to prove $P \Rightarrow Q$?
- $\boxed{\text{Assume } P\text{, show } Q.}$
- Example: Prove that "if n is an even number, then n^2 is also even"

III. Proving Mathematical Propositions

- Proving implication
- How to prove $P \Rightarrow Q$?
- Assume P, show Q.
- Example: Prove that "if n is an even number, then n^2 is also even"

Assume that n is even,
therefore it can be written as $n = 2k$, and we have $n^2 = 2k \times 2k = 2(2k^2)$ which is even.

III. Proving Mathematical Propositions

- Proving equivalence
- How to prove $P \Leftrightarrow Q$?
- Treat it as $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$: both implications need to be proved.

III. Proving Mathematical Propositions

- Disproving implication, equivalence
- How to disprove $P \Rightarrow Q$?
- Recall that $\neg(P \Rightarrow Q) = P \wedge \neg Q$
- To prove that an implication $P \Rightarrow Q$ is false, we must show that P is true and Q is false.
- How to disprove $P \Leftrightarrow Q$?
- We must prove that either $P \Rightarrow Q$ is false, or $Q \Rightarrow P$ is false (or both).

III. Proving Mathematical Propositions

- **Proving Quantifiers**

- **How to prove $\forall x \in S, P(x)$?**

III. Proving Mathematical Propositions

- Proving Quantifiers
- How to prove $\forall x \in S, P(x)$?
- We must consider a certain x within the set S with unspecified value, and prove $P(x)$.
- Formally, we write: "Let $x \in S$," and prove the property $P(x)$.
- Because we never specify the value of x, we end up proving $P(x)$ for all x.
- It is crucial to start such proofs with "Let $x \in S$".

III. Proving Mathematical Propositions

- Remarks:

- Many mathematical properties are formulated with a universal quantifiers.

- It is the meaning of words like “every” or “any”.

- When proving such a property, an example is not enough: we need it to be verified for all values of the considered variable.

III. Proving Mathematical Propositions

- **Proving Quantifiers**
- **How to prove $\exists x \in S, P(x)$?**

III. Proving Mathematical Propositions

- Proving Quantifiers
- How to prove $\exists x \in S, P(x)$?
- Here we must show that there exists a value of x such that $P(x)$ holds.
- In most cases, we must find such a value: an example of x.
- In some cases, we do not exhibit the value, but use a separate theorem that guarantees the existence of a value of x such that $P(x)$ holds.

III. Proving Mathematical Propositions

- **Examples**

- **Prove the following statement:**

There exists an integer k such that $k^2 + 1 > 10$.

III. Proving Mathematical Propositions

- Examples

- Prove the following statement:

There exists an integer k such that $k^2 + 1 > 10$.

We have to prove $\exists k \in \mathbb{N}, k^2 + 1 > 10$, hence we must find an example of such a k.

$k = 4$ verifies the property, since $4^2 + 1 = 17 > 10$. Therefore the statement is proven.

III. Proving Mathematical Propositions

- Examples

- **Prove the following statement:**

For any integer n, we have $2n + 1 + n^2 \geq 0$.

III. Proving Mathematical Propositions

- Examples

- **Prove the following statement:**

For any integer n, we have $2n + 1 + n^2 \geq 0$.

Universal quantifier

We can try some examples for n, but that will not constitute a proof.

III. Proving Mathematical Propositions

- Examples

- Prove the following statement:

For any integer n, we have $2n + 1 + n^2 \geq 0$.

We must prove $\forall n \in \mathbb{Z}, 2n + 1 + n^2 \geq 0$.

Let $n \in \mathbb{Z}$. We have $2n + 1 + n^2 = (n + 1)^2 \geq 0$, therefore the statement is proven.

III. Proving Mathematical Propositions

- Disproving Quantifiers

- How to disprove $\forall x \in S, P(x)$?

III. Proving Mathematical Propositions

- Disproving Quantifiers

- How to disprove $\forall x \in S, P(x)$?

- We want to show that the statement "for all x in S, $P(x)$" is false:
  we have to show that for a certain $x \in S$, the property $P(x)$ isn't verified.

- This amounts to: $\neg(\forall x \in S, P(x)) = \exists x \in S, \neg P(x)$.

- To disprove a universal quantifier, we only have to find a counterexample.

III. Proving Mathematical Propositions

- Disproving Quantifiers
- How to disprove $\exists x \in S, P(x)$?

III. Proving Mathematical Propositions

- Disproving Quantifiers
- How to disprove $\exists x \in S, P(x)$?
- We want to show that there exists no $x \in S$ such that $P(x)$ is verified.
- We must $\boxed{\text{show that for all } x \in S,\ P(x) \text{ is false.}}$
- This corresponds to $\neg(\exists x \in S, P(x)) = \forall x \in S, \neg P(x)$.

III. Proving Mathematical Propositions

- **Examples:**
  Disprove the following statement. (Prove that it is false)

All even numbers are multiples of 4.

III. Proving Mathematical Propositions

- Examples:
  Disprove the following statement. (Prove that it is false)

All even numbers are multiples of 4.

- It is a universal quantifier: to disprove it, we must find a counterexample.

- For instance, 6 is a counterexample: it is even, but not a multiple of 4.

III. Proving Mathematical Propositions

- **Examples:**
  Disprove the following statement. (Prove that it is false)

There exists a number n such that $n^2 + 4n + 4 < 0$

III. Proving Mathematical Propositions

- Examples:
  Disprove the following statement. (Prove that it is false)

There exists a number n such that $n^2 + 4n + 4 < 0$

- It is an existential quantifier: we must show that for all n, we have $n^2 + 4n + 4 \geq 0$
- We can see that $n^2 + 4n + 4 = (n + 2)^2 \geq 0$, therefore the statement has been disproven.
- There exists no number n such that $n^2 + 4n + 4 < 0$.

Conclusion

Things to remember:

- Basic logical symbols: $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\Leftrightarrow$
- Quantifiers: $\exists, \forall$
- Reformulating statements
- Proving and disproving statements