Discrete Mathematics - COMP0147

2025 - Term 1
Prof. Elaine Pimentel

LIST OF EXERCISES

Mathematical induction
(Rosen - Chapter 5)

Required reading for this list: *Discrete Mathematics and Its Applications* (Rosen, 7th Edition):

- Chapter 5.1: *Mathematical Induction*
- Chapter 5.2: *Strong Induction and Well-Ordering*

Note: The exercises are classified into difficulty levels: easy, medium, and hard. This classification, however, is only indicative. Different people may disagree about the difficulty level of the same exercise. Do not be discouraged if you see a difficult exercise–you may find that it is actually easy, by discovering a simpler way to solve it!

---

1. (Rosen 5.1-3) Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for the positive integer n.
   (a) `[Easy]` What is the statement $P(1)$?
   (b) `[Easy]` Show that $P(1)$ is true by completing the base case.
   (c) `[Easy]` What is the induction hypothesis?
   (d) `[Easy]` What do you need to prove in the inductive step?
   (e) `[Medium]` Complete the inductive step.
   (f) `[Easy]` Explain why the above steps show the formula is true for all positive integers n.
2. `[Medium]` (Rosen 5.1-6) Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$, for $n \geq 1$.
3. `[Medium]` (Rosen 5.1-10) Find a formula for $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$ by examining small values of n and prove that the formula is correct.
4. `[Medium]` (Rosen 5.1-11) Find a formula for $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$ by examining small values of n and prove that the formula is correct.
5. `[Medium]` (Rosen 5.1-21) Prove that $2^n > n^2$ for $n \geq 5$, n integer.
6. `[Medium]` (Rosen 5.1-33) Prove that 5 divides $n^5 - n$ whenever n is a non-negative integer.
7. `[Medium]` (Rosen 5.1-60) Prove that $\neg(p_1 \vee p_2 \vee \ldots \vee p_n) \equiv \neg p_1 \wedge \neg p_2 \wedge \ldots \wedge \neg p_n$, for all $n \geq 1$. (Hint: use De Morgan's law $\neg(p \vee q) \equiv \neg p \wedge \neg q$.)
8. `[Medium]` What is wrong with this argument by induction:

   "I am going to prove that everyone's eyes are the same color. Ready?

   If there is only one person, then it's obviously true; this person's eyes are the same color that this person's eyes.

   Suppose it is established that $n-1$ persons must have the same eye color. Consider n persons: the $n-1$ first have the same eye color, and the (n-1) last have the same eye color. Since the two overlap, everyone has the same eye color.

My initialization is verified, and so is my induction. Since I have brown eyes, everyone has brown eyes. Wait a minute, what?"

9. (Rosen 5.2-3) Let $P(n)$ be the proposition "a postage of n cents can be formed using only 3-cent and 5-cent stamps". This exercise illustrates a strong induction proof that $P(n)$ is true for $n \geq 8$.

   (a) [Easy] Show that the propositions $P(8)$, $P(9)$, and $P(10)$ are true, completing the base case.

   (b) [Easy] What is the inductive hypothesis?

   (c) [Easy] What do you need to prove in the inductive step?

   (d) [Hard] Complete the inductive step for $k \geq 10$.

   (e) [Medium] Explain why these steps show that the proposition is true for all $n \geq 8$.

10. (Rosen 5.2-4) Let $P(n)$ be the proposition "a postage of n cents can be formed using only 4-cent and 7-cent stamps". This exercise illustrates a strong induction proof that $P(n)$ is true for $n \geq 18$.

    (a) [Easy] Show that the propositions $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true, completing the base case.

    (b) [Easy] What is the inductive hypothesis?

    (c) [Easy] What do you need to prove in the inductive step?

    (d) [Hard] Complete the inductive step for $k \geq 21$.

    (e) [Medium] Explain why these steps show that the proposition is true for all $n \geq 18$.

11. [Medium] Consider a chocolate bar made of a single row of n squares as shown below.

1	2	3	$\cdots$	$n-1$	n

    Suppose you want to separate all the squares of the bar into individual squares. Assume you can only break the bar between two consecutive squares (i.e., you cannot split a square in half, only separate squares from each other).

    Using strong induction, prove that for any bar of n squares, exactly $n-1$ breaks are required to separate all squares.

12. [Hard] (Rosen 5.2-12) Use strong induction to show that every positive integer n can be written as a sum of distinct powers of 2, i.e., as a sum of a subset of $2^0, 2^1, 2^2, \ldots$. (Hint: in the inductive step, consider separately the cases $k+1$ odd or even. Note that $\frac{k+1}{2}$ is an integer when $k+1$ is even.)

13. [Medium] The *Fibonacci numbers*, $f_0, f_1, \ldots$ are defined by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots$. Use strong induction to show that

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

    for $n = 0, 1, 2, \ldots$