Discrete Mathematics - COMP0147

2025 - Term 1
Prof. Elaine Pimentel

LIST OF EXERCISES

First-order Logic (FOL)
(Rosen - Chapter 1)

Required reading for this assignment: *Discrete Mathematics and Its Applications* (Rosen, 7^{th} Edition):

- Chapter 1.1: *Propositional Logic*
- Chapter 1.2: *Applications of Propositional Logic*
- Chapter 1.3: *Propositional Equivalences*
- Chapter 1.4: *Predicates and Quantifiers*
- Chapter 1.5: *Nested Quantifiers*
- Chapter 1.6: *Introduction to Proofs*
- Chapter 1.8: *Proof Methods and Strategy*

Note: The exercises are classified into difficulty levels: easy, medium, and hard. This classification, however, is only indicative. Different people may disagree about the difficulty level of the same exercise. Do not be discouraged if you see a difficult exercise–you may find that it is actually easy, by discovering a simpler way to solve it!

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1. `[Easy]` (Rosen 1.1.1) Which of the sentences below are propositions? What is the truth value of those that are propositions?

   (a) Boston is the capital of Massachusetts.
   (b) Miami is the capital of Florida.
   (c) $2 + 3 = 5$
   (d) $5 + 7 = 10$
   (e) $x + 2 = 11$
   (f) Answer this question.

2. `[Easy]` (Rosen 1.1.9) Let p and q be the propositions *"...swimming is allowed on the coast of New Jersey"* and *"Sharks have been sighted near the coast"*, respectively. Express each of the compound propositions below in a natural language sentence.

   (a) $\neg p \vee q$
   (b) $p \to \neg q$
   (c) $p \leftrightarrow \neg q$
   (d) $\neg p \wedge (p \vee \neg q)$

3. `[Medium]` (Rosen 1.1.15) Let p, q, and r be the following propositions:

   p : Brown bears have been sighted in the area.
   q : ...it is safe to walk on the trail.
   r : There are ripe fruits along the trail.

   Write the following propositions using p, q, r, and logical connectives.

(a) There are ripe fruits along the trail, but brown bears have not been sighted in the area.

(b) Brown bears have not been sighted in the area and it is safe to walk on the trail, but there are ripe fruits along the trail.

(c) If there are ripe fruits along the trail, walking is safe if and only if brown bears have not been sighted in the area.

(d) It is not safe to walk on the trail, but brown bears have not been sighted in the area and there are ripe fruits along the trail.

(e) For walking on the trail to be safe, it is necessary but not sufficient that there are no ripe fruits along the trail and brown bears have not been sighted in the area.

(f) Walking on the trail is not safe whenever brown bears have been sighted in the area and there are ripe fruits along the trail.

4. `[Medium]` (Rosen 1.1.23) Write each of the propositions below in the form *"if p, then q"*.

   (a) It snows whenever the wind blows from the northeast.

   (b) ...it is necessary to walk 8 miles to reach the top of Long's Peak.

   (c) To be appointed as a professor, it is sufficient to be world-famous.

   (d) Your warranty is valid only if you purchased your CD player less than 90 days ago.

   (e) Jan will swim unless the water is very cold.

5. `[Medium]` (Rosen 1.1.27) Give the converse, contrapositive, and inverse of each of the conditional propositions below.

   (a) If it snows today, I will ski tomorrow.

   (b) I go to class whenever it is exam day.

6. `[Hard]` (Rosen 1.2.17) Three professors are sitting in a restaurant, and the waitress asks them: "Does everyone want coffee?". The first professor says: "I don't know." The second professor says: "I don't know." Finally, the third professor says: "No, not everyone wants coffee." The waitress then brings coffee to the professors who wanted it. How did she deduce who wanted coffee?

7. `[Easy]` (Rosen 1.3.13) Use truth tables to verify the absorption law.

   (a) $p \vee (p \wedge q) \equiv p$

   (b) $p \wedge (p \vee q) \equiv p$

8. Prove the following problems through the manipulation of logical connectives. (In other words, do not use truth tables, but rather the equivalence axioms given in class.)

   (a) `[Medium]` (Rosen 1.3.20) $\neg(p \oplus q)$ and $p \leftrightarrow q$ are equivalent.

   (b) `[Easy]` (Rosen 1.3.25) $(p \to r) \vee (q \to r)$ and $(p \wedge q) \to r$ are equivalent.

9. `[Hard]` (Rosen 1.3.29) Show that $(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology.
   Obs: In this question, did you find it easier to use the truth table or to manipulate logical connectives?

10. `[Hard]` (Rosen 1.3.42) Suppose a truth table in n propositional variables is given. Show that a compound proposition from this truth table can be formed by taking the disjunction of conjunctions of variables or their negations, where one conjunction is included for each combination of values for which the compound proposition has the value true. The resulting compound proposition is said to be in **disjunctive normal form**.

11. `[Medium]` (Rosen 1.3.43) A collection of logical operators is called **functionally complete** if every compound proposition is logically equivalent to a compound proposition involving only these operators. Show that $\neg$, $\vee$ and $\wedge$ form a functionally complete collection of operators. (Hint: use the fact that every compound proposition is logically equivalent to another proposition in disjunctive normal form.)

12. `[Easy]` (Rosen 1.7.13) Prove that if x is irrational, then $1/x$ is irrational.

13. `[Easy]` (Rosen 1.7.17) Prove that if n is an integer and $n^3 + 5$ is odd, then n is even using
    (a) a proof by contraposition, and
    (b) a proof by contradiction.

14. `[Easy]` (Rosen 1.7.18) Prove that if n is an integer and $3n + 2$ is even, then n is even using
    (a) a proof by contraposition, and
    (b) a proof by contradiction.

15. (Rosen 1.4.7) Translate the expressions below into natural language, knowing that $C(x)$ is *"x is a comedian"*, $F(x)$ is *"x is funny"*, and the universe of discourse is the set of all people.
    (a) `[Medium]` $\forall x.(C(x) \to F(x))$
    (b) `[Easy]` $\forall x.(C(x) \wedge F(x))$
    (c) `[Medium]` $\exists x.(C(x) \to F(x))$
    (d) `[Easy]` $\exists x.(C(x) \wedge F(x))$

16. (Rosen 1.4.15) Determine the truth value of the following sentences, knowing that the domain of the variables consists of integers.
    (a) `[Easy]` $\forall n.n^2 \geq 0$
    (b) `[Easy]` $\exists n.n^2 = 2$
    (c) `[Easy]` $\forall n.n^2 \geq n$
    (d) `[Easy]` $\exists n.n^2 < 0$

17. `[Easy]` (Rosen 1.4.50) Show that $\forall x.P(x) \vee \forall x.Q(x)$ and $\forall x.(P(x) \vee Q(x))$ are not logically equivalent.

18. `[Medium]` (Rosen 1.5.1) Translate the following sentences into natural language, using the domain of real numbers.
    (a) $\forall x.\exists y.(x < y)$
    (b) $\forall x.\forall y.(((x \geq 0) \wedge (y \geq 0)) \to (xy \geq 0))$
    (c) $\forall x.\forall y.\exists z.(xy = z)$

19. `[Medium]` (Rosen 1.5.2) Translate the following sentences into natural language, using the domain of real numbers.
    (a) $\exists x.\forall y.(xy = y)$
    (b) $\forall x.\forall y.(((x \geq 0) \wedge (y < 0)) \to (x - y > 0))$
    (c) $\forall x.\forall y.\exists z.(x = y + z)$

20. (Rosen 1.5.9) Let $L(x, y)$ be the predicate *"x loves y"*, where the domain consists of all people in the world. Use quantifiers to express each of the following statements:
    (a) `[Easy]` Everyone loves Jerry.

(b) `[Easy]` Everyone loves someone.

(c) `[Easy]` There exists someone who is loved by everyone.

(d) `[Easy]` No one loves everyone.

(e) `[Easy]` There is someone whom Lydia does not love.

(f) `[Easy]` There exists someone who is not loved by anyone.

(g) `[Hard]` There is exactly one person who is loved by everyone.

(h) `[Hard]` There exist exactly two people that Lynn loves.

(i) `[Easy]` Everyone loves themselves.

(j) `[Medium]` There exists someone who loves no one except themselves.