PS 150B/305B
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Derivation of Least Squares Estimators

We have the model

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

and we seek as estimators of β_0 and β_1 those values that minimize the sum of the squared errors (the residual sum of squares, or RSS)

$$RSS = \sum \hat{\epsilon}_i^2 = \sum \left(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i\right)^2,$$

where all summations in the remainder of this document are over $i = 1, \ldots, n$. Begin by differentiating with respect to $\hat{\beta}_0$ and $\hat{\beta}_1$:

$$\frac{\partial RSS}{\partial \hat{\beta}_0} = -2 \sum (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i), \tag{1}$$

$$\frac{\partial RSS}{\partial \hat{\beta}_1} = -2 \sum (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) X_i. \tag{2}$$

We set these first derivatives to zero and solve for the unknown parameters. First, for $\hat{\beta}_0$ we obtain

$$\begin{aligned}
-2 \sum (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) &= 0 \\
\sum (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) &= 0 \\
n\bar{Y} - n\hat{\beta}_0 - n\hat{\beta}_1 \bar{X} &= 0 \\
n\hat{\beta}_0 &= n\bar{Y} - n\hat{\beta}_1 \bar{X} \\
\hat{\beta}_0 &= \bar{Y} - \hat{\beta}_1 \bar{X}.
\end{aligned}$$

Substituting into the first-order condition for $\hat{\beta}_1$,

$$\begin{aligned}
-2 \sum (Y_i - \bar{Y} + \hat{\beta}_1 \bar{X} - \hat{\beta}_1 X_i) X_i &= 0 \\
\sum Y_i X_i - n\bar{Y}\bar{X} + n\hat{\beta}_1 \bar{X}^2 - \hat{\beta}_1 \sum X_i^2 &= 0 \\
\hat{\beta}_1 (\sum X_i^2 - n\bar{X}^2) &= \sum Y_i X_i - n\bar{Y}\bar{X} \\
\hat{\beta}_1 &= \frac{\sum Y_i X_i - n\bar{Y}\bar{X}}{\sum X_i^2 - n\bar{X}^2}. \qquad (3)
\end{aligned}$$

The denominator in equation 3 simplifies as follows:

$$\begin{aligned}\sum X_i^2 - n\bar{X}^2 &= \sum X_i^2 - 2n\bar{X}^2 + n\bar{X}^2 \\ &= \sum X_i^2 - 2n\bar{X}\bar{X} + \sum \bar{X}^2 \\ &= \sum X_i^2 - 2\sum X_i\bar{X} + \sum \bar{X}^2 \\ &= \sum (X_i - \bar{X})^2.\end{aligned}$$

The numerator of equation 3 also simplifies:

$$\begin{aligned}\sum Y_i X_i - n\bar{Y}\bar{X} &= \sum Y_i X_i - 2n\bar{Y}\bar{X} + n\bar{Y}\bar{X} \\ &= \sum Y_i X_i - n\bar{Y}\bar{X} - n\bar{Y}\bar{X} + n\bar{Y}\bar{X} \\ &= \sum Y_i X_i - \sum Y_i\bar{X} - \sum X_i\bar{Y} + \sum \bar{X}\bar{Y} \\ &= \sum (Y_i - \bar{Y})(X_i - \bar{X}).\end{aligned}$$

And so

$$\hat{\beta}_1 = \frac{\sum (Y_i - \bar{Y})(X_i - \bar{X})}{\sum (X_i - \bar{X})^2}.$$

Exercise for reader: verify that the least squares estimates actually do minimize the sum of the squared residuals.