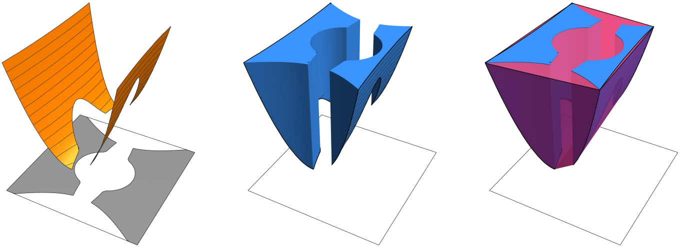
Exactness in SDP relaxations of QCQPs *now with 50% more pictures!

Alex L. Wang , PSE Seminar, Oct. 20



Supported in part by NSF grant CMMI 1454548 and ONR grant N00014-19-1-2321

1. Introduction: QCQPs and SDPs
2. SDP relaxations and convex Lagrange multipliers
3. Symmetries in quadratic forms
4. Some results
5. Application: robust least squares
6. Conclusion

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- More generally

  Binary programming, polynomial programming

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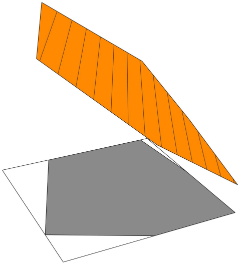

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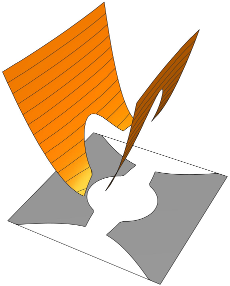

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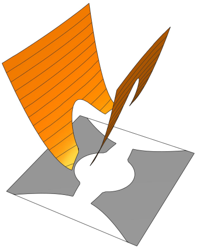

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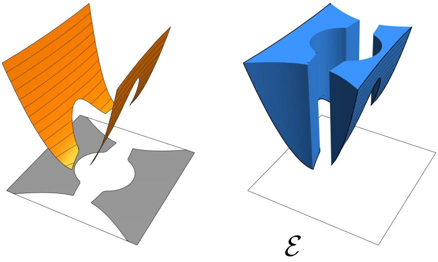

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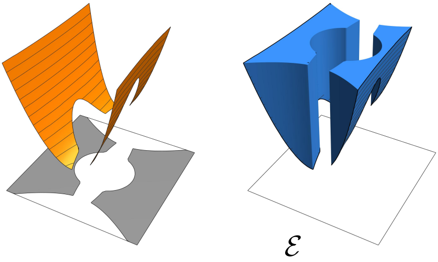

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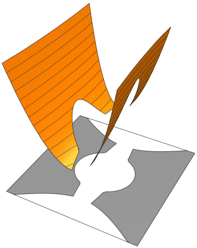
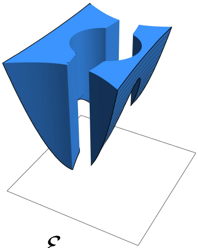
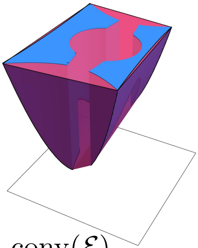
\end{aligned}$$





$\mathcal{E}$



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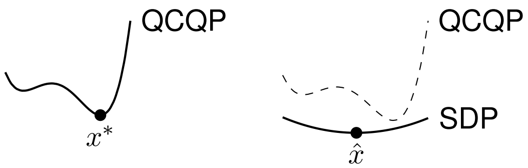
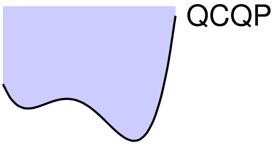


- SDP relaxation can be solved efficiently

Convex relaxations

- Properties you might want for a convex relaxation



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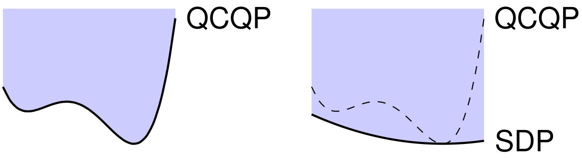

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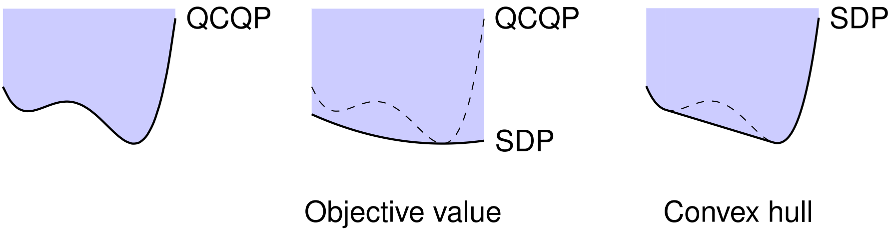

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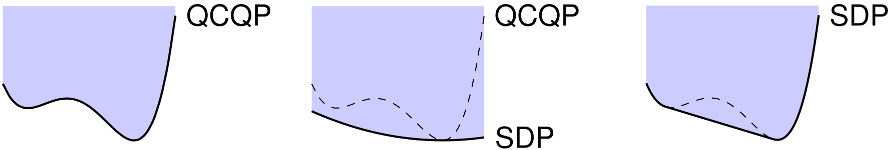

Objective value

Convex hull

- **Q:** When do these properties hold?

The standard SDP relaxation of QCQP

Standard semidefinite program (SDP) relaxation

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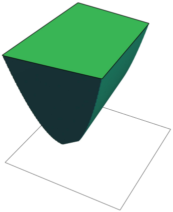

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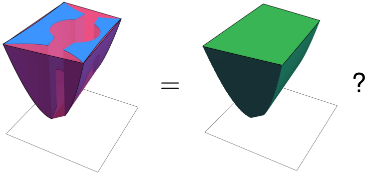

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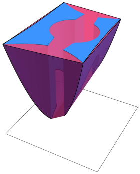

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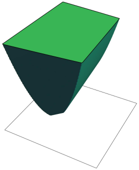

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  - Objective value exactness: $\mathrm{Opt} = \mathrm{Opt}_{\mathsf{SDP}}$?

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[**W** and Kılınç-Karzan 19]

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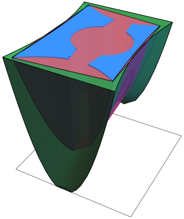

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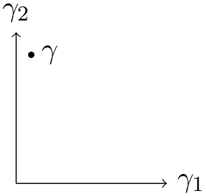

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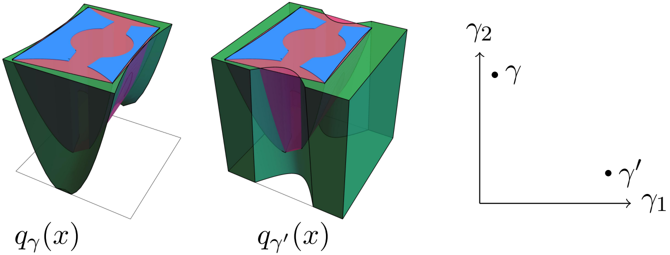

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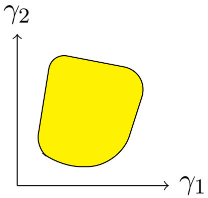

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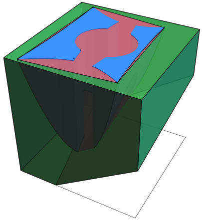

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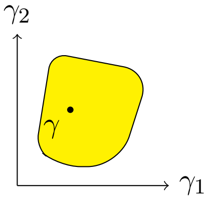

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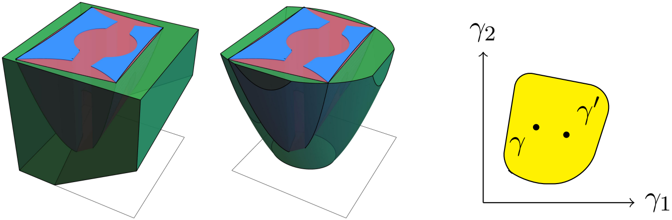


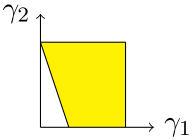
$q_\gamma(x)$ $q_{\gamma'}(x)$

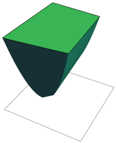
Rewriting the SDP in terms of Γ

Lemma

Suppose primal feasibility and dual strict feasibility, then

$$\mathcal{E}_{\mathsf{SDP}} = \left\{ (x,t) : \max_{\gamma \in \Gamma} q_\gamma(x) \leq t \right\}$$





Related: [Fujie and Kojima 97]

Rewriting the SDP in terms of Γ

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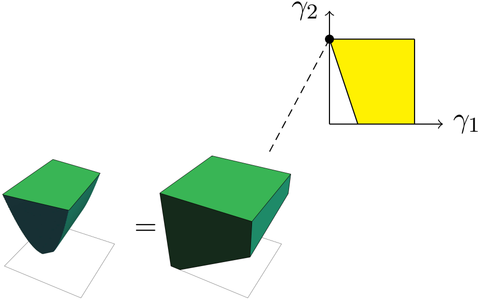

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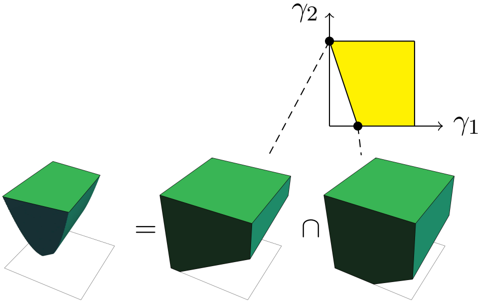

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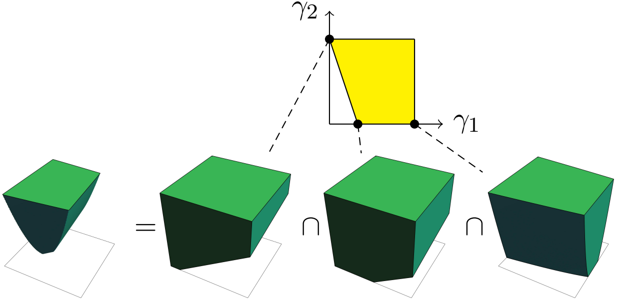

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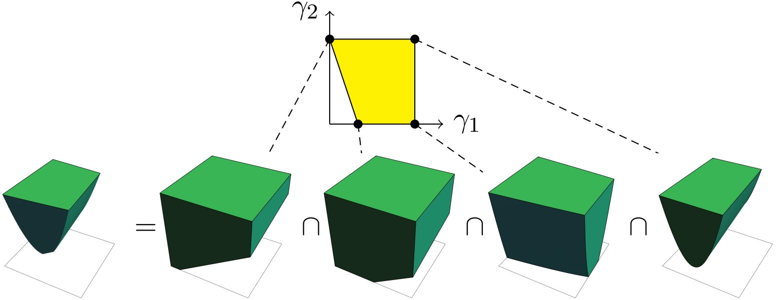

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What does Γ look like?

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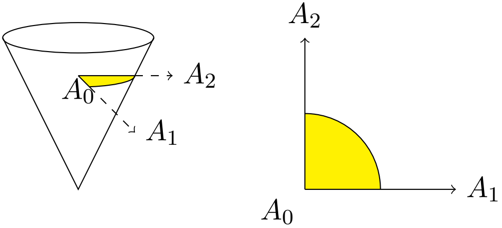

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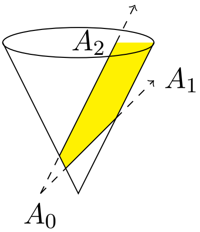

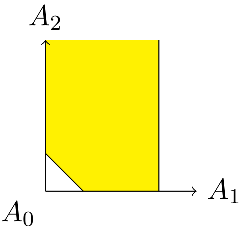

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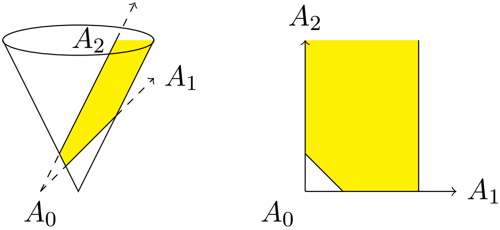

- When A_is diagonal $\implies$ Γ is polyhedral

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Quadratic eigenvalue multiplicity

Definition

Let $1 \le k \le n$ be the largest integer such that for each $i = 0, \dots, m$, the matrix $A_i \in \mathbb{S}^n$ has the following block form

$$\boxed{A_i = \hat{A}_i \otimes I_k} = \begin{pmatrix} \hat{A}_i & & & \\ & \hat{A}_i & & \\ & & \ddots & \\ & & & \hat{A}_i \end{pmatrix}$$

where $\hat{A}_i \in \mathbb{S}^{n/k}$

Quadratic eigenvalue multiplicity

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$(x_1 - x_2)^2 + (x_3 - x_4)^2$	$\begin{pmatrix} 1 & -1 & & \\ -1 & 1 & & \\ & & 1 & -1 \\ & & -1 & 1 \end{pmatrix}$	$k = 2$

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Corollary

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$$k \geq \min\left(m, \, \left|\{b_i \neq 0\}_{i=1}^m\right| + 1\right),$$

then
$$\operatorname{conv}(\mathcal{E}) = \mathcal{E}_{\mathsf{SDP}} \quad \text{and} \quad \mathrm{Opt} = \mathrm{Opt}_{\mathsf{SDP}}\,.$$

Related: [Yakubovich 79], [Burer and Ye 19]

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- $m = 1$
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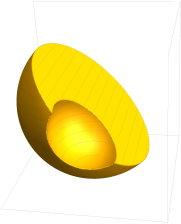
- $A_i = \alpha_i I_n$ for all i and $n \geq m$

Related: [Yakubovich 79], [Burer and Ye 19]

Example: Swiss cheese

- Minimizing distance to a piece of Swiss cheese

$$\min_{x\in\mathbb{R}^n} \left\{ \|x\|^2 : \begin{array}{l} \text{inside ball constraints} \\ \text{outside ball constraints} \\ \text{linear constraints} \end{array} \right\}$$



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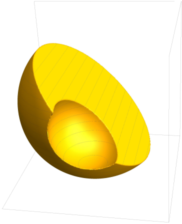

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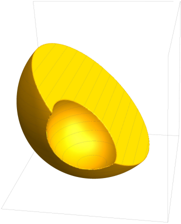

- inside ball $\mapsto I$, outside ball $\mapsto -I$, linear constraints $\mapsto 0$
- If nonempty and $n \geq m$, then the standard SDP relaxation is tight for this QCQP

Some results

Corollary

Suppose primal feasibility and dual strict feasibility.

- If $k \geq m + 2$, then $\mathrm{conv}(\mathcal{E}) = \mathcal{E}_{\mathsf{SDP}}$
- If $k \geq m + 1$, then $\mathrm{Opt} = \mathrm{Opt}_{\mathsf{SDP}}$

Related: [Beck 07]

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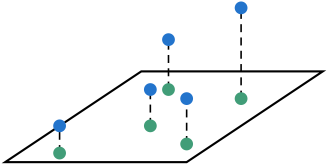
Least squares



- Input, output pairs

$$Y^* = \begin{pmatrix} (y_1^*)^\top \\ \vdots \\ (y_n^*)^\top \end{pmatrix} \in \mathbb{R}^{n\times k}, \qquad z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^n$$

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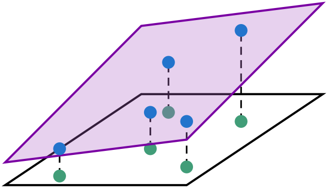

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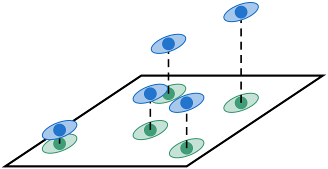
- Want $Y^*\ell \approx z$

$$\min_{\ell\in\mathbb{R}^k} \|Y^*\ell - z\|_2^2$$

Robust least squares



- Empirical measurement $\hat{Y}$ and uncertainty $\mathcal{U}$, i.e.,
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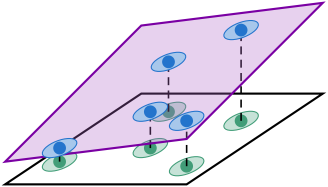
Robust least squares



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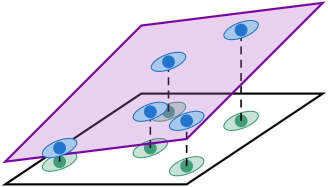

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- When $\mathcal{U}$ is defined by quadratics, can apply our theory!

Robust Least Squares

- Suppose

$$\mathcal{U} = \left\{ X \in \mathbb{R}^{n\times k} : \; \|L_i X\|_F^2 \leq c_i, \, \forall i \in [m] \right\}$$

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Recap, additional work, future work

- QCQPs are NP-hard

Related: [Argue, Kılınç-Karzan, and **W** 20]

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  - Conditions that only depend on the constraints?
  - Can approximation results be explained in this framework?

Related: [Argue, Kılınç-Karzan, and **W** 20]

Thank you. Questions?

Slides

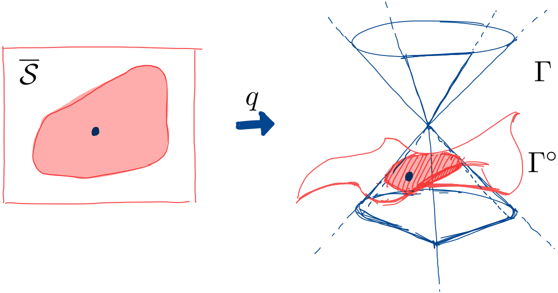
cs.cmu.edu/~alw1

Full version

A. L. Wang and F. Kılınç-Karzan. "On the tightness of SDP relaxations of QCQPs". In: *arXiv preprint* arXiv:1911.09195 (2019)

When Γ is non-polyhedral

- The geometry of Γ and Γ°



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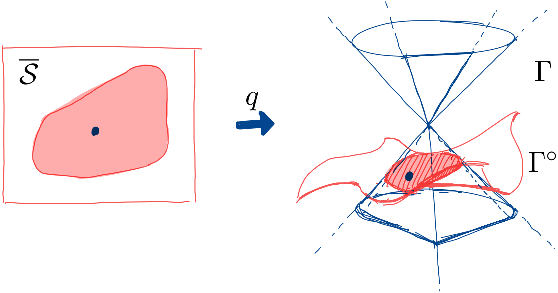

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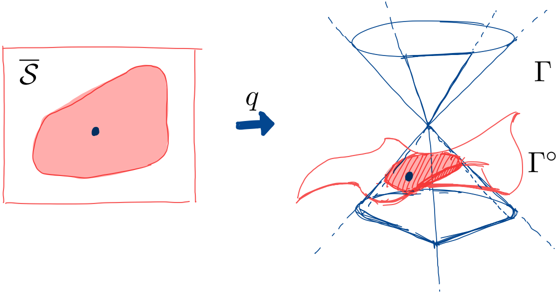

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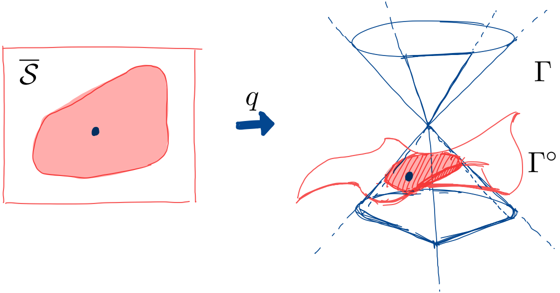

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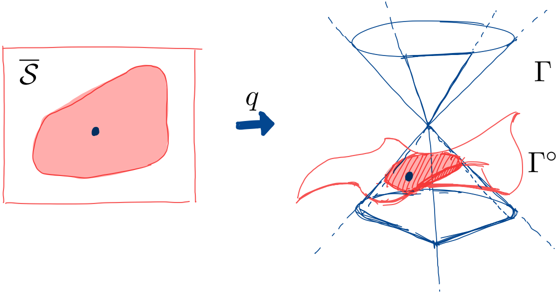
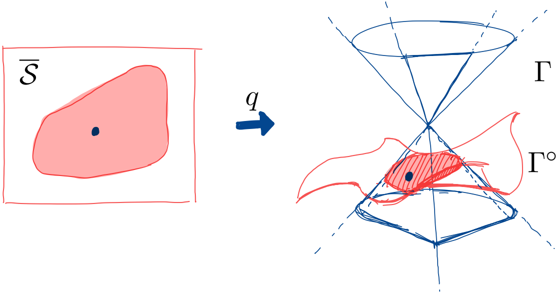
- Correct when Γ° is facially exposed



When Γ is non-polyhedral

- The geometry of Γ and Γ°
- Rounding scheme: Given $x \in \overline{\mathcal{S}}$,
  - Either output convex decomposition for $x \in \mathrm{conv}(\mathcal{S})$
  - Or, claim $x' \in \overline{\mathcal{S}} \setminus \mathrm{conv}(\mathcal{S})$.
- Correct when Γ° is facially exposed
- Extends framework to handle complementarity constraints $\longrightarrow$ sparse regression



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