Hardy-Muckenhoupt Bounds for Laplacian Eigenvalues

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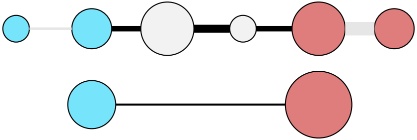
APPROX 2019



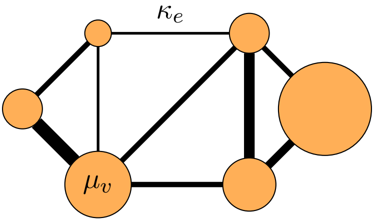
1. Introduction: Laplacians, eigenvalues, why do we care?

2. Bounding λ_2 in terms of graph structure

3. What is Ψ_2?

4. Summary

Graphs and the Laplacian



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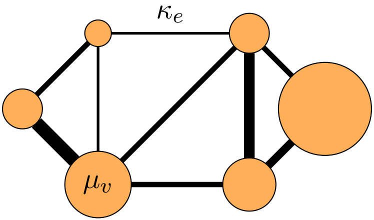

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- Mass, $\mu \in \mathbb{R}^V_{>0}$

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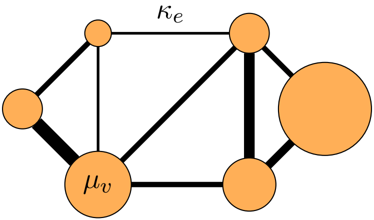

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- Mass, $\mu \in \mathbb{R}^V_{>0}$
- Edge weights, $\kappa \in \mathbb{R}^E_{>0}$

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Eigenvalues associated with L

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Eigenvalues associated with L

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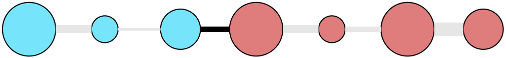
- Numerical linear algebra?
- Goal: give good bounds for λ_2 in terms of graph structure

Cuts and Cheeger's inequality

- Sparsest cut, Φ

$$\Phi(G) = \min_{A} \left\{ \frac{\sum_{e \in E(A,\bar{A})} \kappa_e}{\min(\mu(A), \mu(\bar{A}))} \,\middle|\, A, \bar{A} \neq \varnothing \right\}$$



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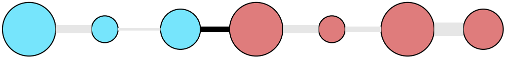


Theorem (Cheeger's Inequality)

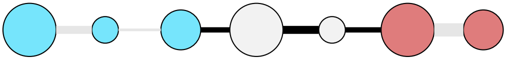
If $\mu_v = d_v$,

$$\frac{\Phi^2}{2} \leq \lambda_2 \leq 2\Phi.$$

Our work

- Neumann content, Ψ_2

$$\Psi_2(G) = \min_{A,B} \left\{ \frac{\kappa_{\text{eff}}(A,B)}{\left(\mu(A)^{-1} + \mu(B)^{-1}\right)^{-1}} \;\middle|\; A, B \neq \varnothing,\ A \cap B = \varnothing \right\}$$



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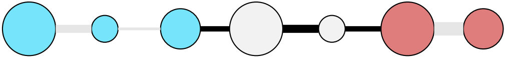

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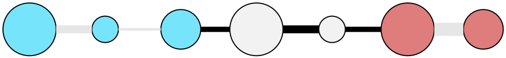

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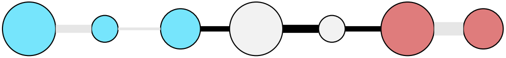

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Theorem ([MWW19])

Let G be a weighted connected graph. Then

$$\frac{\Psi_2}{4} \leq \lambda_2 \leq \Psi_2.$$

Furthermore, all constants are tight.

A first example

- Let G be a path graph on $2n$ vertices with $\kappa = \mu = 1$

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- $\Phi \approx 1/n$
- Cheeger's inequality $\implies \frac{1}{n^2} \lesssim \lambda_2 \lesssim \frac{1}{n}$

Related work

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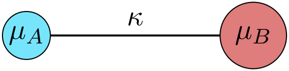
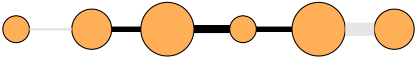
Graphs with two vertices



- Can compute

$$\lambda_2 = \frac{\kappa}{\left(\mu_A^{-1} + \mu_B^{-1}\right)^{-1}} = \Psi_2.$$

Graphs with more vertices



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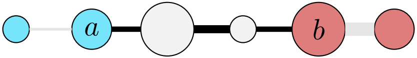

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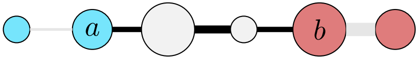

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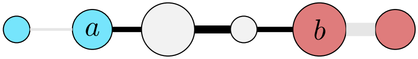

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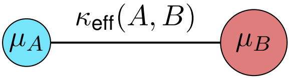

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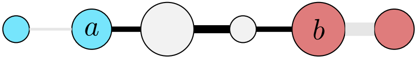

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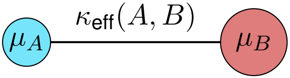

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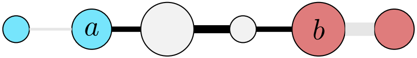

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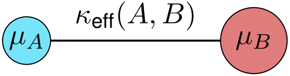

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- Then,

$$\Psi_2 = \min_{a<b} \frac{\kappa_{\text{eff}}(A, B)}{\left(\mu_A^{-1} + \mu_B^{-1}\right)^{-1}}$$

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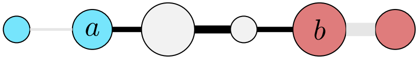

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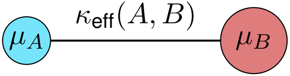

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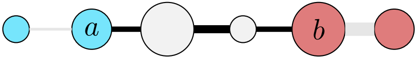

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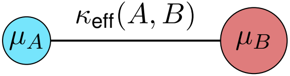

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Ψ_2 is λ_2 for the best two-vertex approximation of G

Summary

- Neumann content

$$\Psi_2 = \min_{A,B} \left\{ \frac{\kappa_{\text{eff}}(A,B)}{\left(\mu(A)^{-1} + \mu(B)^{-1}\right)^{-1}} \;\middle|\; A, B \neq \varnothing,\ A \cap B = \varnothing \right\}$$

- Showed

$$\frac{\Psi_2}{4} \leq \lambda_2 \leq \Psi_2$$

- Generalizations
- Future work: approximation algorithms?

Thanks for listening!

Questions?

Gary L. Miller, Noel J. Walkington, and Alex L. Wang. "Hardy-Muckenhoupt Bounds for Laplacian Eigenvalues". In: *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2019)*. Vol. 145. 2019, 8:1–8:19

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Aaron Schild. "A Schur Complement Cheeger Inequality". In: *arXiv preprint arXiv:1811.10834* (2018).