A linear-time algorithm for the generalized TRS based on a convex quadratic reformulation

Alex L. Wang

Based on joint work with Fatma Kılınç-Karzan

1 Introduction

2 Convex hull result

3 Convex quadratic reformulation of the GTRS and algorithms

The Generalized Trust Region Subproblem (GTRS)

$$\mathrm{Opt} := \inf_{x\in\mathbb{R}^n} \{q_0(x) \,|\, q_1(x) \leq 0\}$$

- q_0 and q_1 are nonconvex quadratic functions

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- q_0 and q_1 are nonconvex quadratic functions
- Applications: nonconvex quadratic integer programs, signal processing, compressed sensing, robust optimization, trust-region methods

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  - Special instance of QCQP

Results/Outline

Under "mild assumptions"

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- New linear-time algorithm for approximating the GTRS

Suffices to optimize over convex hull of epigraph

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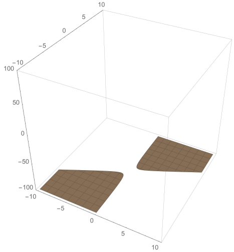

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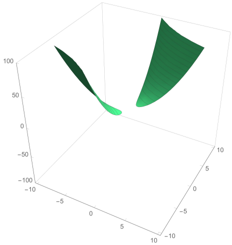

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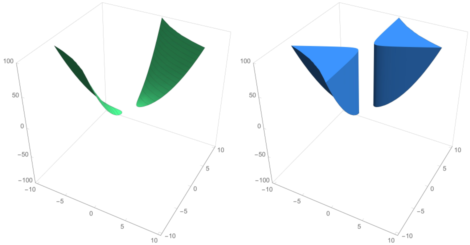

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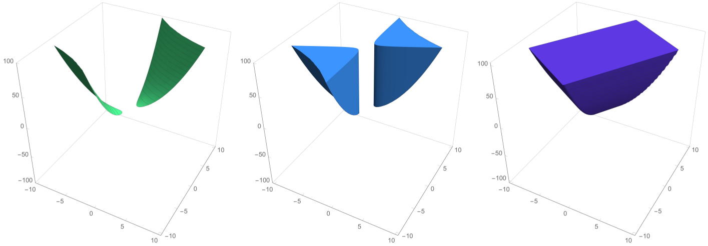

A pencil of quadratics

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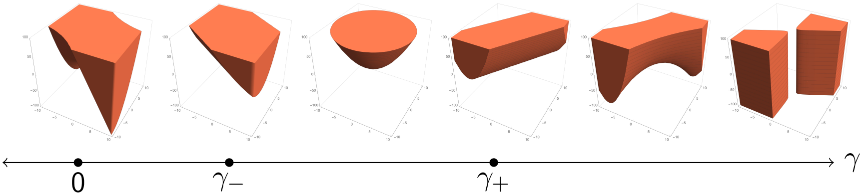

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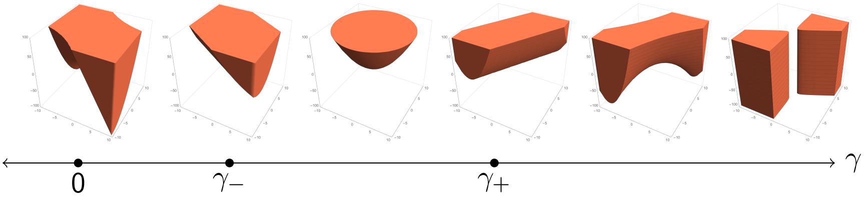

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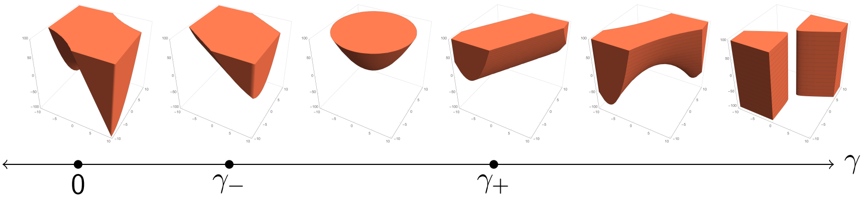

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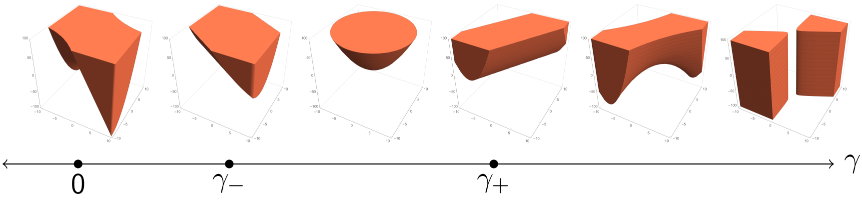

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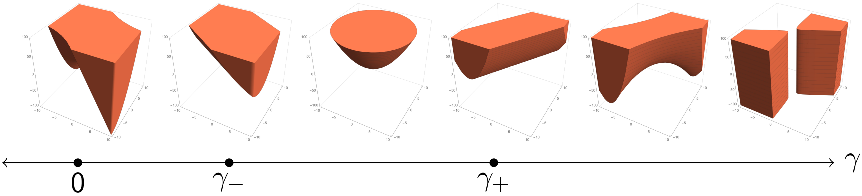

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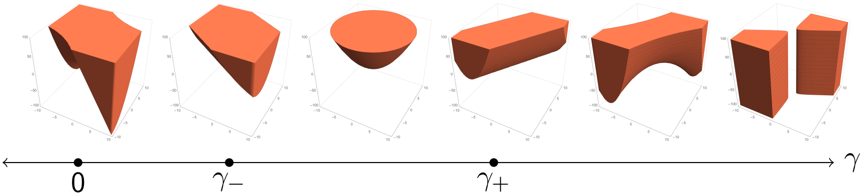

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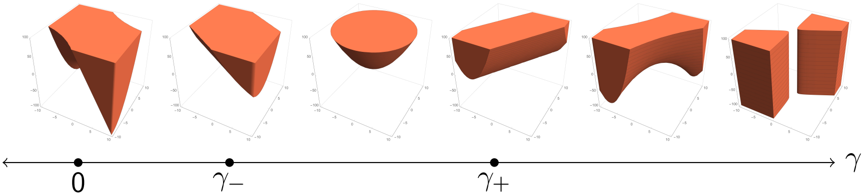

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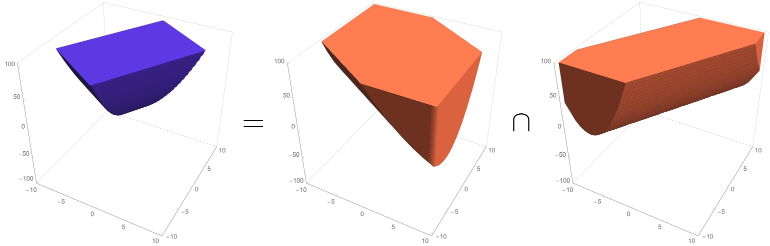
  - Define $\Gamma \coloneqq \{\gamma \geq 0 \mid \mathcal{S}(\gamma) \text{ is convex}\} \eqqcolon [\gamma_-, \gamma_+]$
  - $\overline{\text{conv}}(\mathcal{S}) = \mathcal{S}(\gamma_-) \cap \mathcal{S}(\gamma_+)$

Convex hull result

Theorem

Suppose q_0 and q_1 are both nonconvex and Γ is nonempty. Then Γ can be written $\Gamma = [\gamma_-, \gamma_+]$ and

$$\overline{\operatorname{conv}}(\mathcal{S}) = \mathcal{S}(\gamma_-) \cap \mathcal{S}(\gamma_+) = \left\{ (x, t) \;\middle|\; \begin{array}{l} q(\gamma_-, x) \leq t \\ q(\gamma_+, x) \leq t \end{array} \right\}$$



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Convex quadratic reformulation/algorithm

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Theorem ("informally")

There exists an algorithm, ALG, such that if q_0 and q_1 satisfy some "mild assumptions," then ALG outputs an ϵ-approximate optimizer to the GTRS with probability $\geq 1 - p$. ALG runs in time $\approx \tilde{O}\left(\frac{N}{\sqrt{\epsilon}}\right)$.

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- Gave an explicit description of $\overline{\text{conv}}(\mathcal{S})$
- Convex quadratic reformulation!
- Gave a linear (in N) time algorithm for solving the GTRS

Questions?