Accelerated first-order methods for a class of semidefinite programs

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1. What structure? . . . k-exactness
2. How to use k-exactness? . . . Strongly convex reformulation
3. Algorithms
4. Conclusion

Standard SDP algorithms are expensive

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  - Simply writing down Y requires $O((n+k)^2)$ memory

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- Quadratic matrix program

Related: Beck [2007], Beck et al. [2012], Wang and Kılınç-Karzan [2020]

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$$(\textbf{QMP}) \geq \inf_{Y\in\mathbb{S}^{n+k}} \left\{ \langle M_0, Y\rangle : \begin{array}{l} \langle M_i, Y\rangle = 0,\ \forall i \in [m] \\ Y = \begin{pmatrix} * & * \\ * & I_k \end{pmatrix} \succeq 0 \end{array} \right\} = (\textbf{SDP})$$

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Structural assumptions on SDP

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- Have reduced SDP $\longrightarrow$ QMP:

$$Y^* = \begin{pmatrix} X^*(X^*)^\intercal & X^* \\ (X^*)^\intercal & I_k \end{pmatrix}$$

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Lagrangian reformulation

- Take the Lagrangian to get minimax problem without constraints

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&= \arg\min_{X \in \mathbb{R}^{n\times k}} \sup_{\gamma \in \mathbb{R}^m} \; q_0(X) + \sum_{i=1}^m \gamma_i q_i(X)
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$$A(\gamma) = \text{top-left block of } M(\gamma)$$

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- **Thought experiment**: strong duality + strict complementarity implies

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Suppose $\gamma^* \in \mathcal{C} \subseteq \mathbb{R}^m$ and $A(\gamma) \succ 0$ for all $\gamma \in \mathcal{C}$, then

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- **Algorithm**:
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  - Solve strongly convex quadratic matrix minimax problem (QMMP)

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Algorithms: CautiousAGD

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Related: Nesterov [2005], Devolder et al. [2013, 2014], Nesterov [2018]

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Theorem (CautiousAGD)

CautiousAGD produces iterates X_t such that

$$\max_{\gamma\in\mathcal{C}} q(\gamma, X_t) \le \min_X \max_{\gamma\in\mathcal{C}} q(\gamma, X) + \epsilon$$

after $O\left(\log(\epsilon^{-1})\right)$ iterations, $O(m\epsilon^{-1/2})$ matrix-vector products per iteration

Related: Nesterov [2005], Devolder et al. [2013, 2014], Nesterov [2018]

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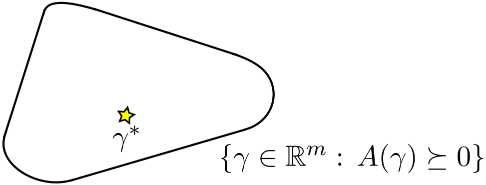

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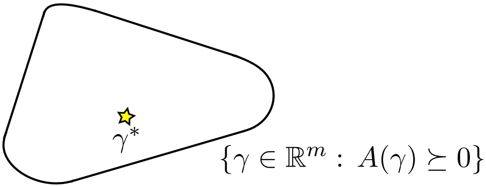

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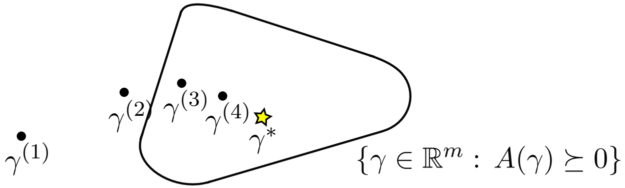

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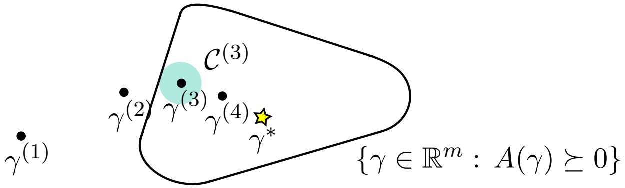

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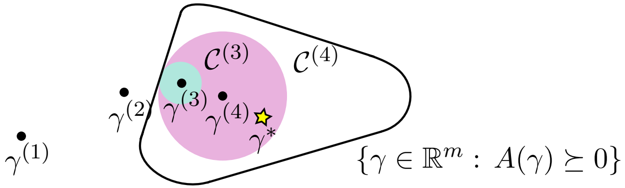

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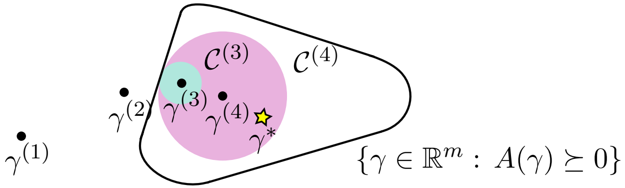

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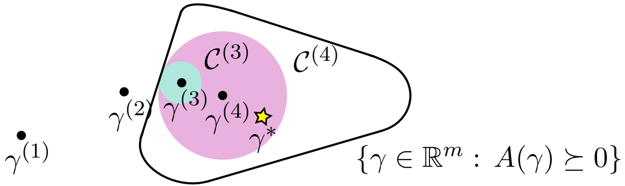

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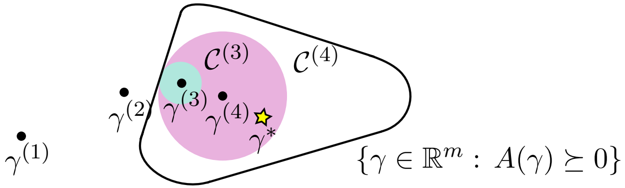

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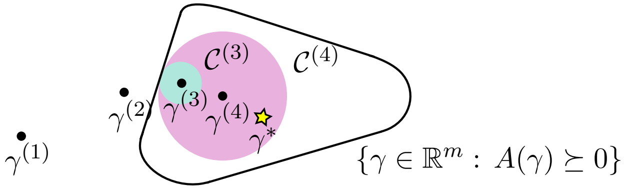

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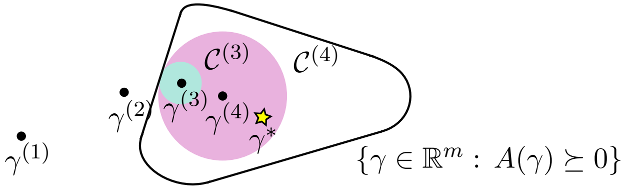
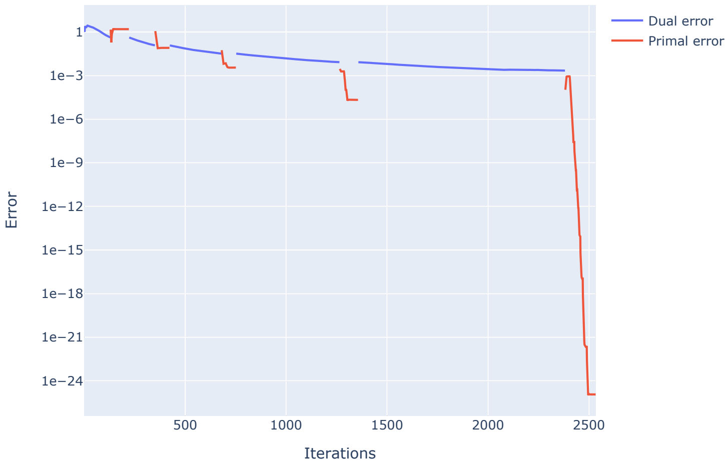
    - If $\gamma^* \in \mathcal{C}^{(i)}$ then CautiousAGD converges to X^* rapidly
    - Monitor convergence!
  - $\longrightarrow$ CertSDP



CertSDP convergence behavior



Theorem (CertSDP)

CertSDP produces iterates X_t such that

$$\left\langle M_0, \begin{pmatrix} X_t X_t^\intercal & X_t \\ X_t^\intercal & I_k \end{pmatrix} \right\rangle \leq \mathrm{Opt}_{\mathsf{SDP}} + \epsilon \qquad \left\| \left(\left\langle M_i, \begin{pmatrix} X_t X_t^\intercal & X_t \\ X_t^\intercal & I_k \end{pmatrix} \right\rangle + d_i \right)_i \right\|_2 \leq \epsilon$$

Related: Ding et al. [2021], Yurtsever et al. [2021], Friedlander and Macêdo [2016], Shinde et al. [2021]

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- **Storage**: $O(m + nk)$ storage

Related: Ding et al. [2021], Yurtsever et al. [2021], Friedlander and Macêdo [2016], Shinde et al. [2021]

1. What structure? . . . k-exactness
2. How to use k-exactness? . . . Strongly convex reformulation
3. Algorithms
4. Conclusion

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Thank you! Questions?

`https://arxiv.org/abs/2206.00224`

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