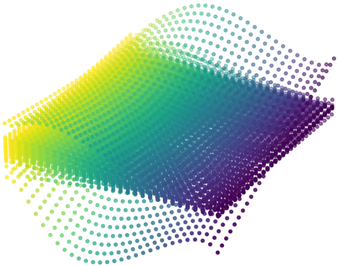
New notions of simultaneous diagonalizability of quadratic forms with applications to QCQPs

Alex L. Wang, CMU Theory Lunch, Apr. 21



Joint work with Rujun Jiang, Fudan University

Quadratically constrained quadratic programs (QCQPs)

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- QCQPs where $\{A_i\}$ are diagonal matrices

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- Better understanding of exactness of relaxations[2]
- Black-box global solvers seem to perform better

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Definition

Such sets $\{A_i\} \subseteq \mathbb{S}^n$ are simultaneously diagonalizable via congruence (SDC)

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SDC: What is known?

Definition

$\{A_i\} \subseteq \mathbb{S}^n$ is SDC if there exists invertible $P \in \mathbb{R}^{n\times n}$:

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[Horn, Johnson 12]

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Theorem

If A is invertible. Then

$\{A, B\}$ SDC $\iff$ $A^{-1}B$ diagonalizable, real spectrum

[Horn, Johnson 12]

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eigenvalues of $A^{-1}B = \{\pm i\} \implies \{A, B\}$ not SDC

SDC: Revisited

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d-RSDC

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$\{A_i\} \subseteq \mathbb{S}^n$ is d-Restricted SDC if there exists $\{\overline{A}_i\} \subseteq \mathbb{S}^{n+d}$ SDC

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Theorem ([W and Jiang 21])

Let $\{A, B\} \subseteq \mathbb{S}^n$. Suppose $A^{-1}B$ has only simple eigenvalues. Then $\{A, B\}$ is 1-RSDC.

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Let $\{A, B\} \subseteq \mathbb{S}^n$. Suppose $A^{-1}B$ has only simple eigenvalues. Then $\{A, B\}$ is 1-RSDC.

- Tools: canonical form for pairs of symmetric matrices[2]

[2] [Uhlig 76], [Lancaster, Rodman 05]

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  $$= \ldots$$

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- $$\begin{aligned} &\det\left(\overline{A}^{-1}\overline{B} - zI\right) \\ &\quad = \dots \\ &\quad = \gamma(z^2+1) + (2\alpha\beta)z + (\beta^2 - \alpha^2)1 - z(z^2+1) \end{aligned}$$

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Basis for deg. 2 polynomials in z

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Basis for deg. 2 polynomials in z

Surjective function

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- Pick $\{\lambda_1, \lambda_2, \lambda_3\} \subseteq \mathbb{R}$

Main Idea

- $$\overline{A} = \left(\begin{array}{cc|c} & 1 & \\ 1 & & \\ \hline & & 1 \end{array}\right), \qquad \overline{B} = \left(\begin{array}{cc|c} 1 & & \alpha \\ & -1 & \beta \\ \hline \alpha & \beta & \gamma \end{array}\right)$$
- $$\begin{aligned} &\det\left(\overline{A}^{-1}\overline{B} - zI\right) \\ &\quad = \dots \\ &\quad = \gamma(z^2+1) + (2\alpha\beta)z + (\beta^2-\alpha^2)1 - z(z^2+1) \end{aligned}$$
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$$\gamma(\lambda_1^2+1) + (2\alpha\beta)\lambda_1 + (\beta^2-\alpha^2)1 = \lambda_1(\lambda_1^2+1)$$

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$$\gamma(2) + (2\alpha\beta)(-1) + (\beta^2-\alpha^2)1 = -2$$

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- Similar calculations generalize to (almost every) pair $\{A, B\} \subseteq \mathbb{S}^n$

Outline

Setup

- $$\begin{aligned} \inf_{x \in \mathbb{R}^n} \quad & x^\intercal A_1 x \\ \text{s.t.} \quad & x^\intercal A_2 x \leq 0 \\ & Lx \leq 1 \end{aligned}$$

Note: Slightly different setup than in paper

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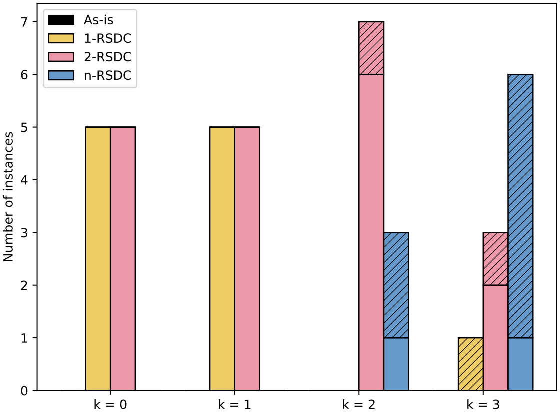
- Tested: As-is, 1-RSDC, 2-RSDC, n-RSDC

Note: Slightly different setup than in paper

Results for $n = 15$



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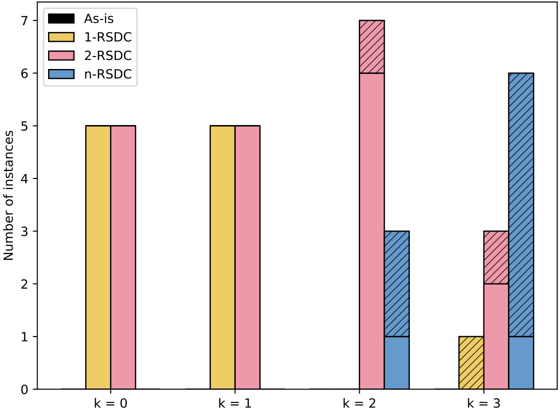

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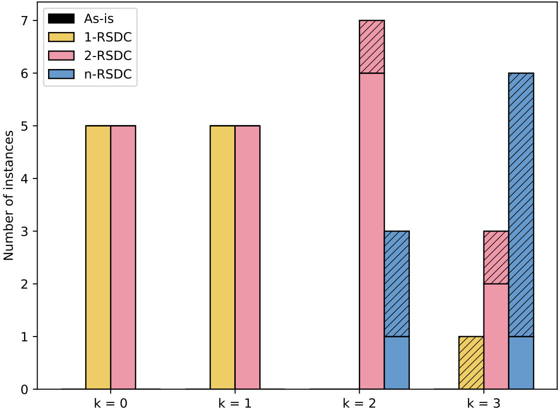

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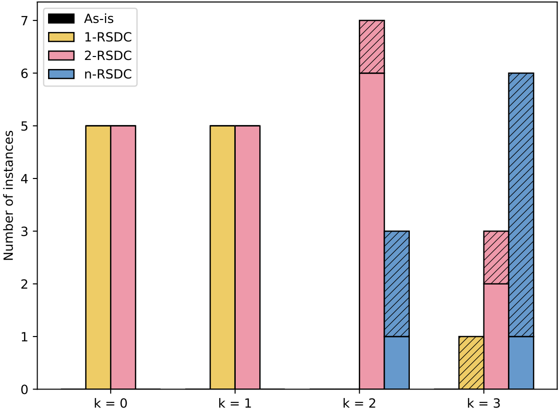

- $*$-RSDC outperforms As-is on every instance
- Condition number blows up with k
  - $k = 3$: 1-RSDC ($\sim 10^3$), 2-RSDC ($\sim 10^2$), n-RSDC (1)

Outline

1. Introduction: QCQPs and diagonalization
2. Prior work: SDC, first examples
   - When is $\{A, B\}$ SDC?
3. New notions of simultaneous diagonalizability
   - d-Restricted SDC
   - When is $\{A, B\}$ 1-RSDC? Almost everywhere!
4. Experiments
5. Conclusion: additional work, future directions

Summary, future directions

- New notions: d-RSDC

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  - Parameterized constructions of d-RSDC?
- Thank you. Questions?

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Additional results

Definition

$\{A_i\}$ is almost SDC (ASDC) if for all $\epsilon > 0$, there exists $\{A_i'\}$ SDC, $\max_i \|A_i - A_i'\| \leq \epsilon$

Theorem ([W and Jiang 21])

Let $\{A, B\} \subseteq \mathbb{S}^n$

- If A invertible, then

$$\{A, B\} \text{ ASDC} \iff A^{-1}B \text{ has real spectrum}$$

- If $\mathrm{span}(\{A, B\})$ does not contain invertible matrix, then

$$\{A, B\} \text{ ASDC}$$

Related: [O'meara, Vinsonhaler 06]

Additional results

Theorem

$A \in \mathbb{S}^n$ invertible. Then,

$$\{A, B, C\} \text{ ASDC} \quad \iff \quad \{A^{-1}B, A^{-1}C\} \text{ commute, real spectrum}$$

Theorem

$\{A = I_n, B, C\} \subseteq \mathbb{S}^n$. If $d \leq \operatorname{rank}([B, C])/2$, then

- $\{A, B, C\}$ is not d-RSDC
- $\left\{ \begin{pmatrix} A & \\ & 0_d \end{pmatrix}, \begin{pmatrix} B & \\ & 0_d \end{pmatrix}, \begin{pmatrix} C & \\ & 0_d \end{pmatrix} \right\}$ is not ASDC

[W and Jiang 21]

Additional results

Theorem

There exists $\{A_1, \ldots, A_7\} \subseteq \mathbb{S}^6$ such that

- A_1 invertible,
- $\{A_1^{-1}A_2, \ldots, A_1^{-1}A_7\}$ commute, real spectrum,
- not ASDC

Theorem

There exists $\{A_1, \ldots, A_5\} \subseteq \mathbb{H}^4$ such that

- A_1 invertible,
- $\{A_1^{-1}A_2, \ldots, A_1^{-1}A_5\}$ commute, real spectrum,
- not ASDC

[W and Jiang 21]