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RESEARCH STATEMENT

Convex optimization has been influential in shaping data science and modern computing. This subfield of optimization has found numerous applications in a variety of domains (e.g., machine learning, statistics, signal processing and engineering). Unfortunately, a growing number of interesting problems encountered by data scientists, engineers, and the scientific community at large are by nature *highly nonconvex*. In response, the optimization community has begun to investigate more "high-powered" machinery (e.g., semidefinite programs or the sum-of-squares hierarchy) with the hope of addressing some of these nonconvex problems. While powerful, many of these techniques have yet to see widespread adoption in practice and are considered *impractical* in large-scale applications.

My research addresses this divide by developing foundational theory enabling the practical application of tools from convex optimization to interesting structured nonconvex problems. The goal of my research is to enable practitioners to **accurately** and **efficiently** solve the nonconvex problems that they face in practice.

Within this goal, my research has thus far centered on the ability of a particular tool from *convex* optimization, namely semidefinite programs (SDPs), to accurately and efficiently solve the fundamental class of *nonconvex* problems known as quadratically constrained quadratic programs (QCQPs). In the QCQP problem, we are asked to minimize a possibly nonconvex multivariate quadratic objective function subject to a number of possibly nonconvex multivariate quadratic constraints. While QCQPs are highly intractable even in simple settings, they also arise naturally in practice—for example, the facility location and production planning problems from operations research and the pooling and truss design problems in engineering can all be naturally formulated as QCQPs. In fact, any mixed binary integer program or polynomial optimization problem can be formulated as a QCQP. Due to their broad applicability, QCQPs are central to the advancement of optimization theory and demand theoretical attention. In this direction, SDPs constitute one of the most powerful tools we have for building convex relaxations of QCQPs.

There are two important questions that must be addressed if SDPs are to be of practical importance in this setting:

Nonconvex optimization theory: What structures within a nonconvex problem (e.g., a QCQP) ensure that its convex relaxation (e.g., the SDP relaxation) is *accurate*?

Convex optimization algorithms: What structures within a convex problem (e.g., the SDP relaxation of a QCQP, itself with some structure) allow it to be solved *efficiently*?

Below, I discuss the research that I have completed along these two complementary directions as well as my intended future work. While my research has thus far focused specifically on semidefinite programs and quadratically constrained quadratic programs, my research interests are broader and I am excited to pursue and collaborate on additional lines of work related to these two research directions, broadly construed.

COMPLETED WORK

Accurately solving nonconvex QCQPs using SDPs

I have completed a number of projects towards the goal of understanding *exactness* in SDP relaxations of QCQPs in terms of *abstract* properties. This complements the large body of important work on approximation results and guarantees for SDP relaxations of concrete classes of QCQPs. This line of work additionally extends the much smaller body of literature on exactness guarantees. These projects can be categorized under the general themes of sufficient conditions for SDP exactness and the rank-one-generated property.

Sufficient conditions for SDP exactness—[IPCO 2020], [Math. Prog. 2021], [*working paper* 2021]

In joint works [7, 10] with Fatma Kılınç-Karzan, I introduced a powerful framework for deriving sufficient conditions for *objective value exactness*—the condition that the optimal values of the nonconvex QCQP and the convex SDP relaxation coincide; and *convex hull exactness*—the condition that the convex hull of the nonconvex QCQP epigraph and the projected epigraph of the SDP relaxation coincide. These results develop a theoretical toolkit for convexification (a paradigm that has found much success in the mixed integer programming literature) in the context of general QCQPs.

Our framework is built on *abstract* structures in QCQPs that may hold across a broad range of problems. As an example, in [10], we apply our framework to prove statements of the form:

> Consider a QCQP where the set of convex Lagrange multipliers is polyhedral, and suppose the "natural symmetry parameter" is sufficiently large. Then, the SDP relaxation of this QCQP is exact.

Our framework unifies and significantly generalizes the proofs of a number of results on the trust-region subproblem, the generalized trust-region subproblem [8], simultaneously diagonalizable QCQPs, and quadratic matrix programs. Additionally, variants of our main result hold for a number of interesting applied problems. For example, it is possible to apply results derived via our framework to show that robust least squares with *nonconvex* uncertainty regions or certain sphere packing problems can be solved exactly using SDPs.

In follow-up work [9] with Fatma Kılınç-Karzan, I offer a significant generalization of the framework we first introduced in [7, 10]. One of the main assumptions made in the original framework was that the set of convex Lagrange multipliers is polyhedral. Although this assumption holds for several important classes of QCQPs (for example QCQPs where the quadratic forms are diagonal or simultaneously diagonalizable), it may fail for QCQPs involving complementarity constraints (also referred to as big-M constraints). This constraint is used to model binary "on-off" decisions for continuous variables and is ubiquitous in practical applications such as sparse regression, portfolio selection, or network design.

In [9], I show that results similar to those from [7, 10] hold even when the polyhedrality assumption is replaced by weaker geometric conditions, e.g., it suffices for the polar cone of the set of convex Lagrange multipliers to be a *nice* cone. In particular, this enables us to extend the framework of [7, 10] to QCQPs involving complementarity constraints.

Perhaps surprisingly, when analyzed using this generalized framework, QCQPs with complementarity constraints [9] share a number of striking similarities with random under-constrained QCQPs (current work). Indeed, in both cases the set of convex Lagrange multipliers coincides (approximately) with the second-order cone. Consequently, we are able to apply similar lines of reasoning to analyze these two seemingly disparate problems.

The abstract nature of the properties utilized in our generalized framework [4, 9] allow us to apply our results to a wide range of problems and we believe that this framework and our results will remain relevant even as the concrete "problems of today" change.

This research [10] was recognized with the ***INFORMS Optimization Society 2021 Student Paper Award***.

Rank-one-generated cones—[Tut. on Oper. Res. 2021], [Math. Oper. Res. (3rd round review after minor revision)]

In many practical applications, the objective function of a QCQP may not be known exactly (for example, it may only be known up to some noise or it may be evolving over time). In joint work [1] with C.J. Argue and Fatma Kılınç-Karzan, I study feasible domains of QCQPs that exhibit SDP exactness for *every* choice of objective function. Specifically, we study a property of matrix cones known as the rank-one-generated (ROG) property, which can be shown to imply both objective value and convex hull exactness regardless of the objective function. This property has additional connections to statistics and sum-of-squares programs. For example, in the context of positive semidefinite (PSD) matrix completion, one may show that the set of PSD matrices with a given sparsity pattern is ROG if and only if the sparsity pattern corresponds to a chordal graph. In turn, the ROG property can be used to prove that the obvious necessary condition for PSD completability is also sufficient if and only if the underlying sparsity pattern is chordal.

In [1], we develop a toolkit for analyzing the ROG property and offer new sufficient conditions for this property. Our new sufficient conditions handle additional conic constraints and can be applied to variants of the trust-region subproblem—a fundamental problem in the area of nonlinear programming. We additionally give a *complete characterization* of ROG cones defined by two quadratic constraints. This extends the only other setting (a single quadratic constraint and the celebrated S-lemma) for which an explicit characterization of the ROG property is known.

In follow-up joint work [4] with Fatma Kılınç-Karzan, I show additional connections between the ROG property, the problem of minimizing ratios of quadratic functions over quadratically constrained domains, and variants of the S-lemma—a fundamental result in control theory which finds applications in robust optimization analysis and the analysis of nonlinear control systems. In particular, we show how to recover a result on the regularized total least squares problem (a problem frequently appearing in signal recovery) as a simple consequence of the ROG toolkit.

We expect that our results on the ROG property will find further important applications in related domains.

Efficiently solving nonconvex QCQPs using SDPs

While semidefinite programs are extremely powerful and have been shown to address difficult problems, they have yet to see widespread adoption in practice. Notably, the task of solving an SDP is often considered to be too computationally burdensome or too slow to be of use in practice, e.g., in machine learning or large-data settings where the number of variables is impractically large or in robotics/planning applications where decisions need to be made almost instantaneously.

Towards this end, my second major research direction designs efficient algorithms for structured convex optimization problems, with a particular focus on those convex problems arising as reformulations of nonconvex problems. Projects that I have completed in this direction can be categorized under efficient algorithms for the generalized trust-region subproblem and simultaneous diagonalizability and its applications in solving QCQPs.

Efficient algorithms for the generalized trust-region subproblem—[Math. Prog. 2020], [*working paper* 2021], [*working paper* 2021]

A particularly simple class of QCQPs is the class of QCQPs with a single possibly nonconvex quadratic constraint. This class of problems, known as the generalized trust-region subproblem (GTRS), has applications in signal processing, compressed sensing, and engineering, where it can be used to model the so-called double-well potential functions (see [8] and references therein). As an example, the GTRS can be used to model Stackelberg prediction games with least-squares loss functions [3] (ongoing work). More broadly, iterative ADMM-based algorithms for general QCQPs using the GTRS as a subprocedure have shown exceptional numerical performance and outperform previous state-of-the-art approaches on a number of real world problems (e.g., multicast beamforming and phase retrieval). This application of the GTRS as a subprocedure within an iterative solver parallels the use of the trust-region subproblem (TRS), i.e., the special case where the quadratic constraint is convex, within iterative methods for general nonlinear programs. These methods, known as trust-region methods, are among the most empirically successful techniques for general nonlinear programs.

In joint work [8] with Fatma Kılınç-Karzan and follow-up work [11] with Yunlei Lu and Fatma Kılınç-Karzan, I developed two new first-order algorithms for the GTRS. In both algorithms, we take a given instance of the GTRS, show how to efficiently compute a convex reformulation of the problem, and solve the convex reformulation using efficient custom first-order methods. While the worst-case running time of both algorithms are the same, our second algorithm is capable of exploiting *implicit regularity* in the GTRS instances so that it may significantly outperform our first algorithm whenever this quantity is large. These algorithms solve the GTRS in time comparable to *or faster than* the time required for *a single sparse minimum eigenvalue call*. For comparison, note that the problem of computing a minimum eigenvalue is *itself* an instance of the GTRS. These running times *establish the state-of-the-art* among current algorithms for the GTRS. Additionally, our preliminary numerical results suggest that even simple implementations of both of our algorithms outperform prior approaches.

Constructing diagonalizable QCQPs—[Math. Prog. (under review)]

In contrast to the general setting, the SDP relaxation of a diagonalizable QCQP, i.e., a QCQP where the quadratic forms involved are *simultaneously diagonalizable via congruence* (SDC), is much more computationally tractable. Specifically, in this setting, the semidefinite constraint may be replaced by a simpler *second-order cone* constraint. Furthermore, in relation to the question of exactness in SDP relaxations of QCQPs, the exactness of the SDP relaxation of a *diagonalizable* QCQP is much better understood [4] than that of a general QCQP [9]. A natural question from a computational perspective, then, is whether a given QCQP may be rewritten as a diagonalizable QCQP.

With this motivation in mind, in joint work [6] with Rujun Jiang, I give a partial answer to this question by *completely characterizing* the pairs and triples of quadratic forms which are the limits of pairs and triples of SDC quadratic forms. Among other results, we show that surprisingly *every* singular pair of quadratic forms is the limit of pairs of SDC quadratic forms and that *almost every* pair of quadratic forms is the projection of a pair of SDC quadratic forms in *only one additional dimension*.

Concretely, our results can be used to rewrite instances of the GTRS with additional linear constraints (a problem with applications for example in portfolio deleveraging) as diagonalizable QCQPs in *only one additional dimension*. Our preliminary numerical results suggest that these constructions, applied as a preprocessing procedure, can significantly reduce the time needed for off-the-shelf optimization packages to solve these problems.

Other work

I have additionally worked on questions related to clustering and spectral graph theory.

Clustering and graph partitioning—[NeurIPS 2017], [APPROX 2019]

In joint work [2] with Aravindan Vijayaraghavan and Abhratanu Dutta, I showed that the Euclidean k-means clustering problem can be solved *exactly* using efficient algorithms under various *perturbation resilience* assumptions. These assumptions capture the idea that the optimal clustering should be tolerant to measurement errors or uncertainties.

In joint work [5] with Gary Miller and Noel Walkington, I show how to extend classical ideas in spectral graph theory to perform sparse *fuzzy* cuts in graphs. Specifically, we derive a variant of Cheeger's inequality to relate the sparsest fuzzy cut of a graph to its fundamental eigenvalue and show how to efficiently compute a constant-factor approximation of the sparsest fuzzy cut of a graph. Beyond applications in graph partitioning, our results can additionally be used as an analytical tool for bounding the fundamental eigenvalue of a graph, a key quantity in the analysis of random walks.

Current and future work

I am interested in developing foundational tools for both convex and nonconvex optimization. In the next five years, I plan to continue working broadly on both convex and nonconvex optimization, with a particular focus on questions arising in the practical application of convex optimization tools to nonconvex problems.

Nonconvex optimization theory

The framework that we introduced in [4, 7, 9, 10] has thus far offered clarity on many exactness results. As a significant next line of work, I plan to investigate the "accuracy" of SDP relaxations of QCQPs *beyond exactness* via this framework. Specifically:

Question. *How can we generalize the framework developed in [7, 9, 10] to understand approximation results? What new results can we derive from such a framework? Can such results guide theoretical and algorithmic developments in other fields facing structured nonconvexity?*

This is a challenging direction that I believe could be immensely rewarding. Indeed, SDP approximation results have already begun to play significant roles in optimization-adjacent fields such as computer science and statistics.

In ongoing work with Fatma Kılınç-Karzan, I have shown that SDP relaxations of random QCQPs are exact up to arbitrarily small additive errors whenever the dimension of the ambient space is large enough with respect to the number of constraints. Besides their parallels to random QCQPs that arise in statistical settings (e.g., in phase retrieval), random QCQPs have served as a testing ground for new ideas in SDP relaxations of QCQPs and it is promising that our framework explains this phenomenon so succinctly.

Convex optimization algorithms

In [6, 8, 11], we take first steps in asking when a given SDP relaxation of a QCQP can be efficiently solved without explicitly solving the SDP. There are a number of broader questions to ask in this direction, both long-term and short-term. In the long term, I plan to address:

Question. *How can we use practically tractable tools from convex optimization (e.g., linear programs or second-order cone programs) to approximate more powerful but less practical tools from convex optimization (e.g., semidefinite programs)?*

I am particularly excited about this question as positive results in this direction would bring powerful tools from the forefront of convex optimization research into the realm of what practitioners view as tractable.

I am currently exploring accelerated first-order algorithms for semidefinite programs *with exactness* (ongoing work with Fatma Kılınç-Karzan). In this direction, we hope to generalize some of the algorithmic ideas that we developed for the GTRS (specifically, *implicit regularity*) to the setting with multiple quadratic constraints. In particular, we have reason to believe that when a SDP possesses a particular form of exactness, that it can be solved efficiently in the original space using ideas from accelerated gradient descent and second-order cone programs in time similar to that of accelerated gradient descent. We are excited about this line of investigation and expect it to have widespread applications wherever SDP exactness can be proved, e.g., in phase retrieval, clustering, and any of the applications of quadratic matrix programs. Beyond exactness, we are also interested in understanding whether these implicit regularity assumptions may hold for general SDPs through Burer–Monteiro-type reformulations.

Funding, grants, and research collaborations

Throughout my Ph.D. work I was supported by grants through the National Science Foundation (NSF) and the Office of Naval Research (ONR). As I take my next steps in academia, I plan to be actively engaged in securing support through agencies and research departments such as the NSF, ONR, Army Research Office, Department of Energy, Amazon Research, Facebook Research, and Microsoft Research wherever appropriate and developing a wide network of collaborators. Towards this end, I am additionally interested in pursuing interdisciplinary research related to my core research directions.

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