Accurately and efficiently solving nonconvex optimization problems

Alex L. Wang
Carnegie Mellon University

These slides are publicly available at `cs.cmu.edu/~alw1`

Collaborators

Carnegie Mellon University (OR, Math, CS),
Northwestern University, Fudan University,
Peking University

Convex optimization

- Convex optimization is influential in many different fields

Engineering
Controller stability, power allocation, truss design, +

Statistics
(Linear) Regression, parameter estimation, +

Finance
Portfolio optimization, risk analysis, +

- Convex optimization is accurate and efficient

Convex optimization, meet nonconvex problems

- Unfortunately, many practical optimization problems are nonconvex
- Example: Low-rank matrix completion (**Netflix problem**)

$$\begin{array}{cc} & \text{Movies} \\ \text{Users} & \begin{pmatrix} 5 & 4 & ? & ? & 4 \\ 3 & ? & ? & 3 & ? \\ ? & 2 & 4 & 1 & 1 \\ ? & 3 & ? & ? & 4 \end{pmatrix} \end{array} \qquad \min_{X \in \mathbb{R}^{n \times k}} \{\mathrm{rank}(X) : X \text{ agrees with revealed entries}\}$$

- Rank constraints, binary constraints, sparsity constraints
- Generally hard, **but not always!**
- Some nonconvex problems can be solved using convex optimization

Research goal

Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them "well" using **convex optimization**

- **Completed work**:
  - Nonconvex problems: quadratically constrained quadratic programs (QCQPs)
  - Convex relaxations: semidefinite programs (SDPs)

Today's questions

Understand **structures within QCQPs** that enable us to solve them exactly and efficiently **using SDPs**

Today's outline

- **Preliminaries**
  QCQPs and their applications, the SDP relaxation
- **Understand structures within QCQPs that enable us to solve them**. . .
  - **exactly** **[IPCO 20]**, **[Math. Prog. 21]**, **[Math. Prog. *under review*]**
    Objective value, convex hull exactness, applications
  - **efficiently** **[Math. Prog. 20]**, **[SIAM J. Optim. *under review*]**, **[Ongoing]**
    The generalized trust-region subproblem and regular QCQPs
- **Conclusion and future directions**

1 Preliminaries

2 Objective value exactness, convex hull exactness, applications

3 Efficient algorithms for structured QCQPs

4 Conclusion and future directions

Quadratically constrained quadratic programs (QCQPs)

- $q_{\text{obj}}, q_1, \ldots, q_m : \mathbb{R}^n \to \mathbb{R}$ quadratic (possibly nonconvex!)

$$q_i(x) = x^\intercal A_i x + 2b_i^\intercal x + c_i$$

$$\text{Opt} := \inf_{x \in \mathbb{R}^n} \left\{ q_{\text{obj}}(x) : \; q_i(x) \leq 0, \, \forall i \in [m] \right\}$$

- Highly expressive:
  - MAX-CUT, MAX-CLIQUE, pooling, truss design, facility location, production planning
  - binary program $\quad x_1(1 - x_1) = 0$
  - polynomial optimization problems $\quad x_1 x_2 = z_{12}$
- NP-hard in general

The QCQP epigraph

- QCQP epigraph $\mathcal{D} := \left\{ (x,t) \in \mathbb{R}^{n+1} : \begin{array}{l} q_{\text{obj}}(x) \le t \\ q_i(x) \le 0,\ \forall i \in [m] \end{array} \right\}$

- How can we derive ~~convex~~ relaxations of $\mathcal{D}$?
- If $\gamma \in \mathbb{R}^m_+$, then $\forall (x,t) \in \mathcal{D}, \quad \underbrace{q_{\text{obj}}(x) + \sum_{i=1}^{m} \gamma_i q_i(x)}_{=:\, q(\gamma, x)} \le t$

The SDP relaxation

- SDP relaxation = impose all convex aggregated inequalities!

$\mathcal{D}_{\text{SDP}}$

- Formally,

$$\Gamma := \{\gamma \in \mathbb{R}^m_+ : q(\gamma, x) \text{ is convex in } x\} = \left\{\gamma \in \mathbb{R}^m_+ : A_{\text{obj}} + \sum_{i=1}^m \gamma_i A_i \succeq 0\right\}$$

$$\mathcal{D}_{\text{SDP}} := \bigcap_{\gamma \in \Gamma} \{(x,t) : q(\gamma, x) \le t\} = \left\{(x,t) \in \mathbb{R}^{n+1} : \sup_{\gamma \in \Gamma} q(\gamma, x) \le t\right\}$$

$$\text{Opt}_{\text{SDP}} := \inf_{x \in \mathbb{R}^n} \sup_{\gamma \in \Gamma} q(\gamma, x)$$

The usual SDP relaxation

$$
\begin{aligned}
&\inf_{x\in\mathbb{R}^n} \{q_{\text{obj}}(x) : q_i(x) \le 0, \forall i \in [m]\} \\
&= \inf_{x\in\mathbb{R}^n, X\in\mathbb{S}^n} \left\{ \langle A_{\text{obj}}, X\rangle + 2b_{\text{obj}}^\top x + c_{\text{obj}} : \begin{array}{l} X = xx^\top \\ \langle A_i, X\rangle + 2b_i^\top x + c_i \le 0, \forall i \in [m] \end{array} \right\} \\
&\ge \inf_{x\in\mathbb{R}^n, X\in\mathbb{S}^n} \left\{ \langle A_{\text{obj}}, X\rangle + 2b_{\text{obj}}^\top x + c_{\text{obj}} : \begin{array}{l} X - xx^\top \succeq 0 \\ \langle A_i, X\rangle + 2b_i^\top x + c_i \le 0, \forall i \in [m] \end{array} \right\} \\
&= \inf_{x\in\mathbb{R}^n} \inf_{X\in\mathbb{S}^n} \dots \\
&= \inf_{x\in\mathbb{R}^n} \sup_{\gamma\in\Gamma} q(\gamma, x)
\end{aligned}
$$

Preliminaries recap

- Main objects of interest

	nonconvex QCQP	convex SDP
Optimum value	Opt	$\mathrm{Opt}_{\mathsf{SDP}}$
Epigraph	$\mathcal{D}$	$\mathcal{D}_{\mathsf{SDP}}$

- Useful for analysis:
  - $q(\gamma, x) =$ Lagrangian function
  - $\Gamma =$ aggregation weights giving convex $q(\gamma, x)$

1 Preliminaries

2 Objective value exactness, convex hull exactness, applications

3 Efficient algorithms for structured QCQPs

4 Conclusion and future directions

Forms of exactness

- What does exactness mean?
  - Objective value exactness: $\mathrm{Opt} = \mathrm{Opt}_{\mathsf{SDP}}$
  - Convex hull exactness: $\mathrm{conv}(\mathcal{D}) = \mathcal{D}_{\mathsf{SDP}}$ $\leftarrow$ convexification of substructures

Obj. val. ex. ✗

Conv. hull ex. ✗

Obj. val. ex. ✓

Conv. hull ex. ✗

Obj. val. ex. ✓

Conv. hull ex. ✓

Convex hull exactness

- $\mathrm{conv}(\mathcal{D}) \stackrel{?}{=} \mathcal{D}_{\mathsf{SDP}}$

$\mathrm{conv}(\mathcal{D}) = \mathcal{D}_{\mathsf{SDP}} \iff$ "Given any point in $\mathcal{D}_{\mathsf{SDP}} \setminus \mathcal{D}$, exists direction such that can move forward and backward inside $\mathcal{D}_{\mathsf{SDP}}$"

Sufficient conditions for exactness

- Can carry out this idea for QCQPs!
- Leads to sufficient conditions based on abstract properties

Theorem ([Math. Prog. 21])

Suppose Γ polyhedral. If for every semidefinite face $\mathcal{F} \trianglelefteq \Gamma$,

$$\operatorname{aff}\left(\operatorname{Proj}_{\mathcal{V}(\mathcal{F})}\{b(\gamma) : \gamma \in \mathcal{F}\}\right) \neq \mathcal{V}(\mathcal{F}),$$

then $\operatorname{conv}(\mathcal{D}) = \mathcal{D}_{\text{SDP}}$.

Theorem ([Math. Prog. *under review*])

If for every $(x,t) \in \mathcal{D}_{\text{SDP}} \setminus \mathcal{D}$,

$$\left\{(x',t') \in \mathbb{R}^{n+1} : \begin{array}{l} x' \in \ker(A(f)) \\ \langle A(\eta)x + b(\eta), x'\rangle - t' = 0,\ \forall (1,\eta) \in \mathcal{G}^{\perp} \end{array}\right\} \neq \{0\},$$

then $\operatorname{conv}(\mathcal{D}) = \mathcal{D}_{\text{SDP}}$.

Based on: **[IPCO 19]**, **[Math. Prog. 21]**, **[Math. Prog. *under review*]**

Example: the trust-region subproblem

- Convex hull exactness in the case of single ball constraint

$$\mathrm{Opt} = \inf_{x \in \mathbb{R}^n} \left\{ q_{\mathsf{obj}}(x) : \|x\|^2 \leq 1 \right\}$$

- **Applications**:
  - Nonlinear minimization (trust-region methods), combinatorial optimization, robust optimization

Based on: **[IPCO 19]**, **[Math. Prog. 20]**
Related: Yakubovich [1971], Yıldıran [2009], Ho-Nguyen and Kılınç-Karzan [2017]

Example: QCQPs with symmetry

- Convex hull exactness in the case of "highly symmetric" QCQPs
- Suppose $A_{\text{obj}} = I_k \otimes \mathbb{A}_{\text{obj}}$, $A_i = I_k \otimes \mathbb{A}_i$ for all $i \in [m]$

$$I_k \otimes \mathbb{A} = \begin{pmatrix} \mathbb{A} & & & \\ & \mathbb{A} & & \\ & & \ddots & \\ & & & \mathbb{A} \end{pmatrix}$$

  and $k \geq m$
- **Applications**:
  - Robust least squares, sphere packing, QCQPs with spherical constraints, orthogonal Procrustes problem

Based on: **[Math. Prog. *under review*]**
Related: Beck [2007], Beck et al. [2012]

Example: Random underconstrained quadratic systems

- Obj. val. exactness in the case of random underconstrained quadratic systems
- Solve $$\inf_{x \in \mathbb{R}^n} \left\{ \|x\|^2 : q_i(x) = 0, \forall i \in [m] \right\}$$

- Fix m, let $n \to \infty$, if data generated "as Gaussians", then objective value exactness w.p. $1 - o(1)$

Based on: **[Math. Prog. *under review*]**
Related: Burer and Ye [2019], Locatelli [2020]

Summary of Part 1

- Sufficient conditions for convex hull exactness
- **Necessary and sufficient** if Γ is polyhedral (dual facially exposed)
- Sufficient conditions for objective value exactness
- Rank-one-generated (ROG) cones: strengthening of convex hull exactness
- **Applications**:
  - Diagonal QCQPs with sign-definite linear terms, semi-random QCQPs, ratios of quadratic functions

Exactness	**ROG Cones**
[IPCO 20], [Math. Prog. 21], [Math. Prog. *under review*]	[Tut. Oper. Res. 21], [Math. Oper. Res. 21]

Take-home message

Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them "well" using **convex optimization**

SDPs provide exact reformulations for broad classes of QCQPs!

1 Preliminaries

2 Objective value exactness, convex hull exactness, applications

3 Efficient algorithms for structured QCQPs

4 Conclusion and future directions

Revisiting the SDP relaxation

- Usual SDP relaxation in $x \in \mathbb{R}^n$ and $X \in \mathbb{S}^n$ $\implies \approx n^2$ variables
- Our view: $$\mathrm{Opt}_{\mathsf{SDP}} := \inf_{x \in \mathbb{R}^n} \left(\sup_{\gamma \in \Gamma} q(\gamma, x) \right)$$

  is a minimization problem in the original space $\implies n$ variables

The generalized trust-region subproblem (GTRS)

- Special setting with single constraint ($\leq$ or $=$)

$$\mathrm{Opt} := \inf_{x \in \mathbb{R}^n} \{q_{\mathsf{obj}}(x) : q_1(x) \leq 0\}$$

- **TRS Applications**:
  - nonlinear programming (trust-region methods), combinatorial optimization, robust optimization

- **GTRS Applications**:
  - minimizing quartics of the form $q(x, p(x))$

$$\inf_{x \in \mathbb{R}^n, \alpha} \{q(x, \alpha) : \alpha = p(x)\}$$

(source localization, constrained rank-one approximation), iterative QCQP solvers

Linear-time algorithm for the GTRS

- Convex hull exactness holds $\quad \mathrm{conv}(\mathcal{D}) = \mathcal{D}_{\mathsf{SDP}} = \left\{ (x,t) : \sup_{\gamma \in \Gamma} q(\gamma, x) \leq t \right\}$

- Recall $\quad \Gamma = \{\gamma \in \mathbb{R}_+ : q(\gamma, x) \text{ is convex in } x\}$

$q(\gamma, x)$

$\gamma = 0 \qquad \gamma_- \qquad \gamma_+$

Based on: **[Math. Prog. 20]**
Related: Hazan and Koren [2016], Ho-Nguyen and Kılınç-Karzan [2017], Jiang and Li [2019, 2020]

Linear-time algorithm for the GTRS

- $\Gamma = [\gamma_-, \gamma_+] \implies \mathrm{Opt} = \mathrm{Opt}_{\mathsf{SDP}} = \inf_{x \in \mathbb{R}^n} \max_{\gamma \in \{\gamma_-, \gamma_+\}} q(\gamma, x)$

- **Algorithmic idea**
  - Compute γ_- and γ_+ to some accuracy
  - Apply accelerated gradient descent

$$\implies \tilde{O}\left(\frac{N}{\sqrt{\epsilon}} \log\left(\frac{n}{p}\right) \log\left(\frac{1}{\epsilon}\right)\right) \approx \frac{1}{\sqrt{\epsilon}}$$

Based on: **[Math. Prog. 20]**
Related: Hazan and Koren [2016], Ho-Nguyen and Kılınç-Karzan [2017], Jiang and Li [2019, 2020]

Is this running time optimal?

- GTRS:

$$\inf_{x \in \mathbb{R}^n} \{q_{\text{obj}}(x) : q_1(x) \leq 0\} \qquad \tilde{O}\left(\frac{N}{\sqrt{\epsilon}} \log\left(\frac{n}{p}\right) \log\left(\frac{1}{\epsilon}\right)\right)$$

- Minimum eigenvalue:

$$\inf_{x \in \mathbb{R}^n} \left\{x^\intercal A x : \|x\|^2 = 1\right\} \qquad O\left(\frac{N}{\sqrt{\epsilon}} \log\left(\frac{n}{p}\right)\right)$$

- Smooth convex quadratic:

$$\inf_{x \in \mathbb{R}^n} q_{\text{obj}}(x) \qquad O\left(\frac{N}{\sqrt{\epsilon}}\right)$$

- But, these are hardest problems within the GTRS!
- Can do better when regularity $\mu^* > 0$

Based on: **[Math. Prog. 20]**
Related: Kuczynski and Wozniakowski [1992], Nesterov [2018]

Linear convergence for regular GTRS

- $\mu^* > 0$ holds for most GTRS
- Dual problem

$$\mathrm{Opt}_{\mathsf{SDP}} := \inf_{x \in \mathbb{R}^n} \sup_{\gamma \in \Gamma} q(\gamma, x) = \sup_{\gamma \in \Gamma} \inf_{x \in \mathbb{R}^n} q(\gamma, x)$$

Definition

Let γ^* be dual optimizer. Define $\mu^* := \lambda_{\min}(A_{\mathsf{obj}} + \gamma^* A_1)$. GTRS instance is regular if $\mu^* > 0$.

- Minimum eigenvalue problem is not regular
- Some instances of smooth quadratic minimization are not regular

Based on: **[SIAM J. Optim. *under review*]**
Related: Carmon and Duchi [2018]

Linear convergence for regular GTRS

- Suppose $\mu^* > 0$

$$\max_{\gamma \in \{\tilde{\gamma}_-, \tilde{\gamma}_+\}} q(\gamma, x)$$

- Suppose $\gamma^* \in [\tilde{\gamma}_-, \tilde{\gamma}_+] \subseteq \Gamma \implies \mathrm{Opt} = \inf_{x \in \mathbb{R}^n} \max_{\gamma \in \{\tilde{\gamma}_-, \tilde{\gamma}_+\}} q(\gamma, x)$
- Suffices to estimate γ^* roughly and can exploit strong convexity

$$\tilde{O}\left(\frac{N}{\sqrt{\mu^*}} \log\left(\frac{1}{\mu^*}\right) \log\left(\frac{n}{p}\right) \log\left(\frac{1}{\epsilon}\right)\right) \approx \log\left(\frac{1}{\epsilon}\right)$$

- Linear in N and $\log(1/\epsilon)$

Based on: **[SIAM J. Optim. *under review*]**
Related: Carmon and Duchi [2018]

Numerical experiments, n = 1,000

Numerical experiments, n = 1,000

Numerical experiments, n = 10,000

Numerical experiments, n = 100,000

Efficient algorithms for regular QCQPs

- Regularity can also be defined for general QCQPs!

Definition

Let γ^* be dual optimizer. Define $\mu^* := \lambda_{\min}\left(A_{\text{obj}} + \sum_{i=1}^m \gamma_i^* A_i\right)$. QCQP instance is regular if $\mu^* > 0$.

- $\mu^* > 0 \implies$ objective value exactness
- $\mu^* > 0$ in a number of statistical recovery problems
- If $\gamma^* \in B(\tilde{\gamma}, \delta) \subseteq \Gamma$

$$\mathrm{Opt} = \mathrm{Opt}_{\text{SDP}} = \inf_{x \in \mathbb{R}^n} \sup_{\gamma \in \Gamma} q(\gamma, x) = \inf_{x \in \mathbb{R}^n} \sup_{\gamma \in B(\tilde{\gamma}, \delta)} q(\gamma, x)$$

- **Applications**:
  - TRS, GTRS, random QCQPs, phase-retrieval, clustering

Based on: **[Ongoing]**

Summary of Part 2

- Efficient algorithms for the GTRS
- Efficient algorithms for regular QCQPs
- Algorithms for diagonalizing QCQPs

GTRS

[Math. Prog. 20],
[SIAM J. Optim. *under review*]

Regular QCQPs

[Ongoing]

Diagonalizing QCQPs

[Math. Prog. *under review*]

Take-home message

Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them "well" using **convex optimization**

Some nonconvex problems can be solved efficiently via first-order methods!

1 Preliminaries

2 Objective value exactness, convex hull exactness, applications

3 Efficient algorithms for structured QCQPs

4 Conclusion and future directions

Future work

Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them "well" using **convex optimization**

- **Solving nonconvex problems accurately**
  - **Completed**: SDPs provide exact reformulations for broad classes of QCQPs!
  - **Future**:
    - Can we understand approximation quality systematically within general framework?
    - Can we understand exactness/approximation for more powerful convex relaxations?

Future work

Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them "well" using **convex optimization**

- **Solving nonconvex problems efficiently**
  - **Completed**: Some nonconvex problems can be solved efficiently via first-order methods!
  - **Future**:
    - Exactness $\approx$ efficiency?
    - Can we develop efficient algorithms for semidefinite programs with low-rank solutions
    - Can we approximate "expensive" tools (e.g., SDPs) with cheap tools (e.g., linear programs, second-order cone programs)

Summary of my research

Exactness

[IPCO 20], [Math. Prog. 21], [Math. Prog. *under review*]

ROG Cones

[Tut. Oper. Res. 21], [Math. Oper. Res. 21]

GTRS

[Math. Prog. 20], [SIAM J. Optim. *under review*]

Regular QCQPs

[Ongoing]

Diagonalizing QCQPs

[Math. Prog. *under review*]

Long-term research goal

Understand **structures within nonconvex problems** that enable us to solve them "well" using **convex optimization**

Thank you! Questions?

References I

Argue, C., Kılınç-Karzan, F., and Wang, A. L. (2020). Necessary and sufficient conditions for rank-one generated cones. *arXiv preprint*, 2007.07433.

Beck, A. (2007). Quadratic matrix programming. *SIAM J. Optim.*, 17(4):1224–1238.

Beck, A., Drori, Y., and Teboulle, M. (2012). A new semidefinite programming relaxation scheme for a class of quadratic matrix problems. *Oper. Res. Lett.*, 40(4):298–302.

Burer, S. and Ye, Y. (2019). Exact semidefinite formulations for a class of (random and non-random) nonconvex quadratic programs. *Math. Program.*, 181:1–17.

Carmon, Y. and Duchi, J. C. (2018). Analysis of Krylov subspace solutions of regularized nonconvex quadratic problems. pages 10728–10738.

Dutta, A., Vijayaraghavan, A., and Wang, A. L. (2017). Clustering stable instances of Euclidean k-means. In *Advances in Neural Information Processing Systems*, pages 6500–6509.

Hazan, E. and Koren, T. (2016). A linear-time algorithm for trust region problems. *Math. Program.*, 158:363–381.

Ho-Nguyen, N. and Kılınç-Karzan, F. (2017). A second-order cone based approach for solving the Trust Region Subproblem and its variants. *SIAM J. Optim.*, 27(3):1485–1512.

References II

Jiang, R. and Li, D. (2019). Novel reformulations and efficient algorithms for the Generalized Trust Region Subproblem. *SIAM J. Optim.*, 29(2):1603–1633.

Jiang, R. and Li, D. (2020). A linear-time algorithm for generalized trust region problems. *SIAM J. Optim.*, 30(1):915–932.

Kılınç-Karzan, F. and Wang, A. L. (2021). Exactness in SDP relaxations of QCQPs: Theory and applications. Tut. in Oper. Res. INFORMS.

Kuczynski, J. and Wozniakowski, H. (1992). Estimating the largest eigenvalue by the power and Lanczos algorithms with a random start. *SIAM J. Matrix Anal. Appl.*, 13(4):1094–1122.

Locatelli, M. (2020). KKT-based primal-dual exactness conditions for the Shor relaxation. *arXiv preprint*, 2011.05033.

Miller, G. L., Walkington, N. J., and Wang, A. L. (2019). Hardy-Muckenhoupt bounds for Laplacian eigenvalues. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2019)*, pages 8:1–8:19.

Nesterov, Y. (2018). *Lectures on convex optimization*, volume 137 of *Springer Optimization and Its Applications*. Springer.

References III

Wang, A. L. and Jiang, R. (2021). New notions of simultaneous diagonalizability of quadratic forms with applications to QCQPs. *arXiv preprint*, 2101.12141.

Wang, A. L. and Kılınç-Karzan, F. (2020a). The generalized trust region subproblem: solution complexity and convex hull results. *Math. Program.* Forthcoming.

Wang, A. L. and Kılınç-Karzan, F. (2020b). A geometric view of SDP exactness in QCQPs and its applications. *arXiv preprint*, 2011.07155.

Wang, A. L. and Kılınç-Karzan, F. (2020c). On convex hulls of epigraphs of QCQPs. In *Integer Programming and Combinatorial Optimization (IPCO 2020)*, pages 419–432. Springer.

Wang, A. L. and Kılınç-Karzan, F. (2021). On the tightness of SDP relaxations of QCQPs. *Math. Program.* Forthcoming.

Yakubovich, V. A. (1971). S-procedure in nonlinear control theory. *Vestnik Leningrad Univ. Math.*, pages 62–77.

Yıldıran, U. (2009). Convex hull of two quadratic constraints is an LMI set. *IMA J. Math. Control Inform.*, 26(4):417–450.

Rank-one-generated cones

Definition

Cone $\mathcal{S} \subseteq \mathbb{S}^n_+$ is rank-one-generated (ROG) if $\mathcal{S} = \operatorname{conv}\left(\mathcal{S} \cap \left\{xx^\top\right\}\right)$.

Compare: $P \subseteq [0,1]^n$ is integral if $P = \operatorname{conv}(P \cap \{0,1\}^n)$

- Given QCQP, if constraints correspond to ROG cone, then objective value exactness and convex hull exactness regardless of objective function
- Suppose $\mathcal{S} = \left\{X \in \mathbb{S}^n_+ : \langle M, X\rangle \leq 0, \forall M \in \mathcal{M}\right\}$

Goal

What properties of $\mathcal{M} = \{M_1, \ldots, M_k\}$ imply $\mathcal{S}$ is ROG?

Based on: **[Math. Oper. Res. 21]**, **[Tut. Oper. Res. 21]**

Rank-one-generated cones

Theorem (Sufficient conditions)

$\mathcal{S}$ is ROG if

- for all $i \neq j$, there exists $(\alpha, \beta) \neq (0, 0)$ such that $\alpha M_i + \beta M_j \succeq 0$, or
- there exists $a \in \mathbb{R}^n$ such that $M_i = ab_i^\top + b_i a^\top$.

Theorem (Characterization of ROG for $|\mathcal{M}| = 2$)

Suppose $\mathcal{M} = \{M_1, M_2\}$. Then sufficient condition above is also necessary.

Based on: **[Math. Oper. Res. 21]**, **[Tut. Oper. Res. 21]**

Diagonalizing QCQPs

Definition

$\{A_i\} \subseteq \mathbb{S}^n$ is simultaneously diagonalizable via congruence (SDC) if there exists invertible $P \in \mathbb{R}^{n \times n}$ such that $P^\top A_i P$ is diagonal $\forall i$.

- Nice property because: SDP relaxation of diagonal QCQP is SOCP (faster), Γ is polyhedral (better understanding of exactness)

Goal

Most sets of matrices are not SDC, can we find other computationally variants of SDC and understand such properties?

Based on: **[Math. Prog. *under review*]**

Diagonalizing QCQPs: variants of SDC

Definition

$\{A_i\} \subseteq \mathbb{S}^n$ is almost SDC (ASDC) if for all $\epsilon > 0$, there exists $\|A_i' - A_i\| \leq \epsilon$ such that $\{A_i'\}$ is SDC.

- "Limit of SDC sets"

Definition

$\{A_i\} \subseteq \mathbb{S}^n$ is d-restricted SDC (d-RSDC) if there exists $A_i' = \left(\begin{smallmatrix} A_i & * \\ * & * \end{smallmatrix}\right) \in \mathbb{S}^{n+d}$ such that $\{A_i'\}$ is SDC.

- "Restriction of SDC sets"

Based on: **[Math. Prog. *under review*]**

Diagonalizing QCQPs: ASDC

Theorem

Let $\{A, B\} \subseteq \mathbb{S}^n$ and suppose A invertible. Then $\{A, B\}$ is ASDC if and only if $A^{-1}B$ has real spectrum. (+ construction)

Theorem

Let $\{A, B\} \subseteq \mathbb{S}^n$. If $\{A, B\}$ is singular, then it is ASDC. (+ construction)

Theorem

Let $\{A, B, C\} \subseteq \mathbb{S}^n$ and suppose A invertible. Then $\{A, B, C\}$ is ASDC if and only if $\left\{A^{-1}B, A^{-1}C\right\}$ commute and have real spectrum. (+ construction)

Based on: **[Math. Prog. *under review*]**

Diagonalizing QCQPs: d-RSDC

Theorem

Let $\{A, B\} \subseteq \mathbb{S}^n$. If A is invertible and $A^{-1}B$ has simple eigenvalues, then $\{A, B\}$ is 1-RSDC. (+ construction)

- Condition holds generically

Based on: **[Math. Prog. *under review*]**

“Fuzzy” spectral partitioning

- Connected graph $G = (V, E)$
- Vertex masses $\mu : V \to \mathbb{R}_{++}$ and edge weights $\kappa : E \to \mathbb{R}_{++}$
- Laplacian $L = D - A$ w.r.t. κ

Theorem (Cheeger’s inequality)

If $\mu_v = d_v$, then
$$\frac{\Phi^2}{2} \leq \lambda_2(L, M) \leq 2\Phi$$

- $\lambda_2(L, M)$ is first nontrivial generalized eigenvalue
- Φ is sparsest cut

Based on: **[APPROX 19]**

“Fuzzy” spectral partitioning

- We define “Fuzzy cuts”

Definition

$$\Psi \approx \min_{A,B} \left\{ \frac{\kappa_{\text{eff}}(A,B)}{\min\left(\mu(A), \mu(B)\right)},\ A, B \neq \varnothing,\ A \cap B = \varnothing \right\}$$

- Φ must partition, Ψ may leave out. $\Psi = \Phi$ if A, B is a partition.

Theorem

$$\frac{\Psi}{4} \leq \lambda_2(L, M) \leq \Psi$$

Based on: **[APPROX 19]**

Stable Euclidean k-means

- k-means clustering: $\{x_1, \ldots, x_n\} \subseteq \mathbb{R}^d$
- Suppose there exist true clustering that is unique optimum even if for all i, $x_i \mapsto x_i' \in B(x_i, \epsilon)$

Theorem

Two clusters. There exists $c \geq 1$ such that for any fixed $\epsilon > 0$, we can recover true clustering in time $d \cdot n^{O(\epsilon^{-c})}$.

Additional results for ≥ 3 clusters given an additional "separation" assumption

Based on: **[NeurIPS 17]**