New York City Interscholastic Mathematics League

Soph/Frosh Division Contest Number 1 Fall 2018

PART I FALL 2018 CONTEST 1 TIME: 10 MINUTES

F18SF01 Compute the value of $789 \cdot 37 + 2018 \cdot 63 + 1229 \cdot 15 + 1229 \cdot 22$.

F18SF02 Suppose that the number N is expressed as 401 in base $b - 1$ and as 276 in base $b + 1$. Compute $N + b$ and express the result in base 10.

PART II FALL 2018 CONTEST 1 TIME: 10 MINUTES

F18SF03 Suppose that lines $\overleftrightarrow{NY}$ and $\overleftrightarrow{LA}$ are drawn such that $\overleftrightarrow{NY} \parallel \overleftrightarrow{LA}$ and point C is in between $\overleftrightarrow{NY}$ and $\overleftrightarrow{LA}$ such that $m\angle YNC = 42^\circ$ and $m\angle CLA = 56^\circ$. Compute $m\angle NCL$ in degrees.

F18SF04 Compute the number of positive integers less than or equal to 500 that are multiples of at least one of 5, 7, or 11.

PART III FALL 2018 CONTEST 1 TIME: 10 MINUTES

F18SF05 There are fifteen nonempty subsets of the set $\{2, 0, 1, 8\}$. For each of these subsets, consider the sum of its elements. Compute the number of different sums.

F18SF06 In quadrilateral $QUAD$, $\overline{QU} \perp \overline{UA}$ and $\overline{DA} \perp \overline{UA}$, $QU = 2$, $QD = 20$, and $DA = 18$. Compute the volume of the solid obtained by revolving $QUAD$ 360° around $\overline{QU}$.

New York City Interscholastic Mathematics League

Soph/Frosh Division — Contest Number 2 — Fall 2018

PART I ***FALL 2018*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F18SF07 Compute the greatest common factor of 1014, 1092, and 2301.

F18SF08 Let N be the least positive integer whose digits sum to 2018. Compute the sum of the digits of $N + 1$.

PART II ***FALL 2018*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F18SF09 In the expansion of $(a+b)^n$, where n is an integer, like terms are collected, and the sum of the coefficients of the terms is 128^2. Find the coefficient of the term in which the exponents of a and b are equal.

F18SF10 Given regular dodecagon $HOUSEWARMING$, $\triangle HEM$ has area $36\sqrt{3}$. Compute the area of $HOUSEWARMING$.

PART III ***FALL 2018*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F18SF11 Given that $83\underline{A}37 \cdot 243 = 2022\underline{B}591$, where A and B are digits, compute $A + B$.

F18SF12 If you ask for a ride through the Meeber app, a car will arrive at a predetermined location at a uniformly random time between noon and 12:30 pm. If the Meeber driver waits more than 5 minutes and you don't show up, the driver will leave and you will miss your ride. Jimmy asks for a ride through the Meeber app and arrives at the predetermined location at a uniformly random time (independent of the driver's arrival time) between noon and 12:30 pm. Jimmy will wait until 12:30 pm regardless of when he arrives at he pickup location. Compute the probability that Jimmy catches a ride with a Meeber car. Express your answer as a fraction in simplest form.

New York City Interscholastic Mathematics League

Soph/Frosh Division   Contest Number 3   Fall 2018

PART I   FALL 2018   CONTEST 3   TIME: 10 MINUTES

F18SF13 Suppose that 40% of A plus 20% of B equals 100% of B. Suppose also that 20% of A plus 40% of B equals k% of B. Compute k.

F18SF14 In quadrilateral $QUAD$, $m\angle Q : m\angle U : m\angle A : m\angle D = 3 : 2 : 3 : 4$. Given that $UQ = UA = 6$, compute DU. Express your answer in simplest radical form.

PART II   FALL 2018   CONTEST 3   TIME: 10 MINUTES

F18SF15 Compute the greatest prime divisor of $6! + 9!$.

F18SF16 The sum of the lengths of the diagonals of rhombus $RHOM$ is 20. Given that $RH = 9$, compute the area of $RHOM$.

PART III   FALL 2018   CONTEST 3   TIME: 10 MINUTES

F18SF17 Compute the number of integers x that satisfy $x^2 - 6x - 12 < 0$.

F18SF18 In a long hallway are 100 boxes numbered 1 through 100 (in that order). Three people walk through the hallway dropping rocks into boxes as follows. Adam drops a rock in Box 1 and every other box after that. Beth drops a rock in Box 1 and every third box after that. Carl drops a rock in Box 1 and every fifth box after that. When all rocks have been dropped, how many boxes have exactly two rocks?

New York City Interscholastic Mathematics League

Soph/Frosh Division Contest Number 1 Spring 2019

PART I ***Spring 2019*** ***CONTEST 1*** ***TIME: 10 MINUTES***

S19SF01 Larry flips a fair coin 4 times (with the flips independent of each other). What is the probability that he gets at most 3 heads? Express your answer as a fraction in simplest form.

S19SF02 Compute the greatest prime factor of $5^8 - 3^8$.

PART II ***Spring 2019*** ***CONTEST 1*** ***TIME: 10 MINUTES***

S19SF03 Given that the two of the three lines $3x + 4y = 5$, $5x + By = 13$, and $7x = 2425$ are perpendicular, compute B. Express your answer as a fraction in simplest form.

S19SF04 Compute the greatest root of $x^4 - x^3 - 19x^2 + 4x + 60 = 0$. Express your answer in simplified radical form.

PART III ***Spring 2019*** ***CONTEST 1*** ***TIME: 10 MINUTES***

S19SF05 A palindrome is a number that reads the same forwards and backwards (e.g. 121 and 3003). Compute the number of four-digit palindromes (with a nonzero leading digit) that are less than 2019.

S19SF06 Regular octagon $ABCDEFGH$ has side length 2. Points W, X, Y, and Z are the midpoints of $\overline{AB}$, $\overline{CD}$, $\overline{EF}$, and $\overline{GH}$, respectively. If a point is chosen uniformly at random from the interior of $ABCDEFGH$, compute the probability that the point is in the interior of $WXYZ$. Express your answer in simplified radical form.

New York City Interscholastic Mathematics League

Soph/Frosh Division — Contest Number 2 — Spring 2019

PART I ***Spring 2019*** ***CONTEST 2*** ***TIME: 10 MINUTES***

S19SF07 Compute the greatest two-digit integer that is exactly halfway between a positive integer and its square.

S19SF08 The lengths of the sides of an isoceles trapezoid are 10, 10, 13, and 19. Compute the length of one diagonal of the isoceles trapezoid. Express your answer in simplified radical form.

PART II ***Spring 2019*** ***CONTEST 2*** ***TIME: 10 MINUTES***

S19SF09 One of the roots of $x^2 - bx + 5$, where b is a real number, is the cube of the other root. Compute the sum of the squares of the roots. Express your answer in simplified radical form.

S19SF10 Compute the number of three-digit integers (from 100 to 999 inclusive) that have at least one digit that is a 4.

PART III ***Spring 2019*** ***CONTEST 2*** ***TIME: 10 MINUTES***

S19SF11 Three pirates split a treasure. RedBeard takes 70% of the treasure, and WhiteBeard and BlueBeard split the rest in a 4:3 ratio. Compute WhiteBeard's part of the treasure as a simplified fraction of the total.

S19SF12 A triangle is chosen at random from the set of all non-congruent triangles with integer side lengths, two of which are 9 and 12. Compute the probability that the chosen triangle is obtuse. Express your answer as a fraction in simplified form.

New York City Interscholastic Mathematics League

Soph/Frosh Division — Contest Number 3 — Spring 2019

PART I ***Spring 2019*** ***CONTEST 3*** ***TIME: 10 MINUTES***

S19SF13 Compute all values of x such that $x - 1$ is the reciprocal of $x + 1$. Express your answer in simplified radical form.

S19SF14 $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Compute the value of $x > 0$ such that $x\lfloor x \rfloor + 3x = 20$. Express your answer as a fraction in simplified form.

PART II ***Spring 2019*** ***CONTEST 3*** ***TIME: 10 MINUTES***

S19SF15 Eight students are lining up in a row for the finale of the school musical. They may do so in any way they choose so long as Adam is somewhere in front of Brett and Carol is somewhere in front of Dana. In how many ways can the eight students line up?

S19SF16 Points P_1, P_2, P_3, and so on are placed on the circumference of a circle with center O such that $m\angle OP_iP_{i+1} = 40^\circ$ and $P_i \neq P_{i+2}$ for all positive integers i. Compute the least positive $i > 1$ such that $P_i = P_1$.

PART III ***Spring 2019*** ***CONTEST 3*** ***TIME: 10 MINUTES***

S19SF17 Compute $23^3 - 3 \cdot 529 \cdot 19 + 3 \cdot 23 \cdot 361 - 19^3$.

S19SF18 Compute the sum of all four-digit numbers $\underline{A}\ \underline{B}\ \underline{C}\ \underline{D}$ (where A, B, C, and D are digits, not necessarily distinct) such that $\underline{A}\ \underline{B}\ \underline{C}\ \underline{D} + \underline{A}\ \underline{B} + \underline{C}\ \underline{D} + \underline{A} + \underline{B} + \underline{C} + \underline{D} = 2019$.

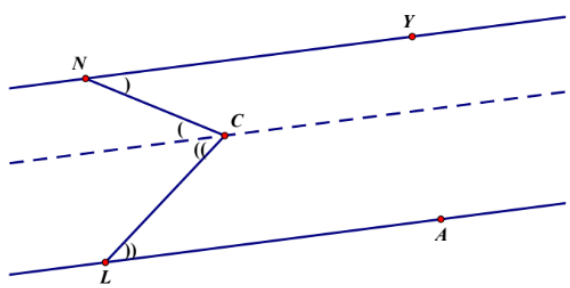
F18SF01. **201800**. The value of the given expression is $789 \cdot 37 + 2018 \cdot 63 + 1229 \cdot 37 = 2018 \cdot 63 + 2018 \cdot 37 = 2018 \cdot 100 = \mathbf{201800}$.

F18SF02. **335**. Solve

$$\begin{aligned} 4(b-1)^2 + 1 &= 2(b+1)^2 + 7(b+1) + 6 \\ 4b^2 - 8b + 5 &= 2b^2 + 11b + 15 \\ 2b^2 - 19b - 10 &= 0 \\ (2b+1)(b-10) &= 0 \end{aligned}$$

and this has only one integer solution: $b = 10$. This means that $N = 4 \cdot 9^2 + 1 = 325$ and the desired quantity is $325 + 10 = \mathbf{335}$.

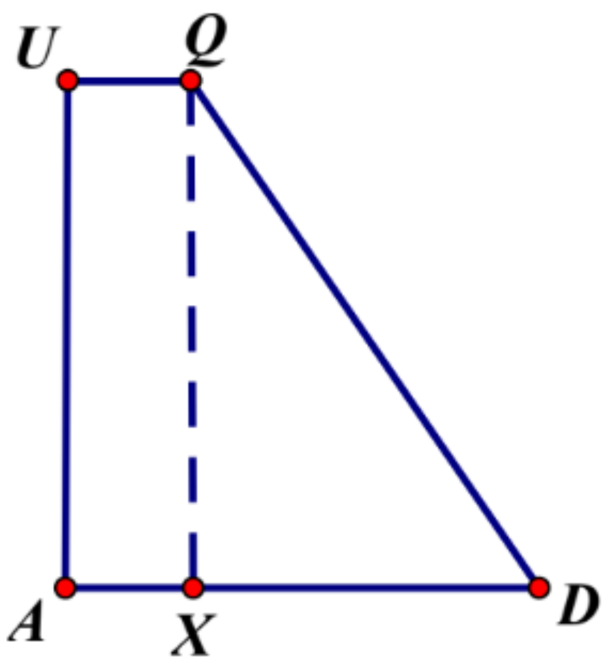
F18SF03. **98°**. Draw a line parallel to the parallel lines through C: see diagram. Note that this creates pairs of congruent angles as marked in the diagram. Then $m\angle NCL$ is the sum of the measures of two angles each of which is alternate interior to one of the given angles. Thus the answer is $42^\circ + 56^\circ = \mathbf{98}^\circ$.



F18SF04. **188**. First notice that there are $\lfloor \frac{500}{5} \rfloor = 100$ multiples of 5, $\lfloor \frac{500}{7} \rfloor = 71$ multiples of 7, and $\lfloor \frac{500}{11} \rfloor = 45$ multiples of 11 in the first 500 positive integers. Of these, $\lfloor \frac{500}{35} \rfloor = 14$ numbers are multiples of 5 and 7, $\lfloor \frac{500}{77} \rfloor = 6$ are multiples of 7 and 11, and $\lfloor \frac{500}{55} \rfloor = 9$ are multiples of 5 and 11. Similarly, there is $\lfloor \frac{500}{5 \cdot 7 \cdot 11} \rfloor = \lfloor \frac{500}{385} \rfloor = 1$ positive integer less than or equal to 500 that is a multiple of 5, 7, and 11. By the Principle of Inclusion and Exclusion, the answer is $100 + 71 + 45 - 14 - 9 - 6 + 1 = \mathbf{188}$.

F18SF05. **8**. Note that the 7 nonempty subsets of $\{0, 1, 2\}$ generate sums of 0, 1, 2, and 3. If 8 is an element of the subset, then we need to consider 8 alone or in union with one of these 7 other subsets, and the possible sums are 8, 9, 10, and 11. Thus, the total number of different sums is **8**.

F18SF06. **1824π**. Because $\overline{QU} \perp \overline{UA}$ and $\overline{DA} \perp \overline{UA}$, $QUAD$ is a trapezoid with parallel bases $\overline{QU}$ and $\overline{DA}$. Drop a perpendicular from Q to $\overline{DA}$ with foot X; then $DX = 16$ and $XA = 2$. By the Pythagorean Theorem, $QX = \sqrt{20^2 - 16^2} = 12$. Revolving the quadrilateral about $\overline{QU}$ results in a cylinder with a cone removed. The cylinder has height $DA = 18$ and base radius $UA = QX = 12$, so the cylinder has volume $\pi \cdot 12^2 \cdot 18 = 2592\pi$. The cone has height with length equal to $DX = 16$ and base radius with length equal to $QX = 12$, so the volume of the cone is $\frac{\pi}{3} \cdot 12^2 \cdot 16 = 768\pi$. The difference is the volume of the solid, namely **1824π**.

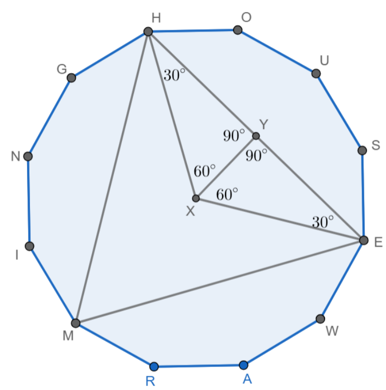


F18SF07. **39**. The greatest common factor of the three numbers divides each of the numbers but also the differences between any two of the numbers. Thus the GCF must divide $1092 - 1014 = 78 = 2 \cdot 39$. Because 2301 is odd, the GCF must divide 39. It can be confirmed that $1014 = 39 \cdot 26$, $1092 = 39 \cdot 28$, and $2301 = 39 \cdot 59$, so the greatest common factor of 1014, 1092, and 2301 is **39**.

F18SF08. **3**. The least integer whose digits sum to 2018 will have as many $9's$ as possible in places other than the greatest place value. Because $2018 = 224 \cdot 9 + 2$, the number N is a 2 followed by 224 $9's$. Adding 1 to N results in a number that has a 3 followed by 224 $0's$. The sum of the digits of $N + 1$ is **3**.

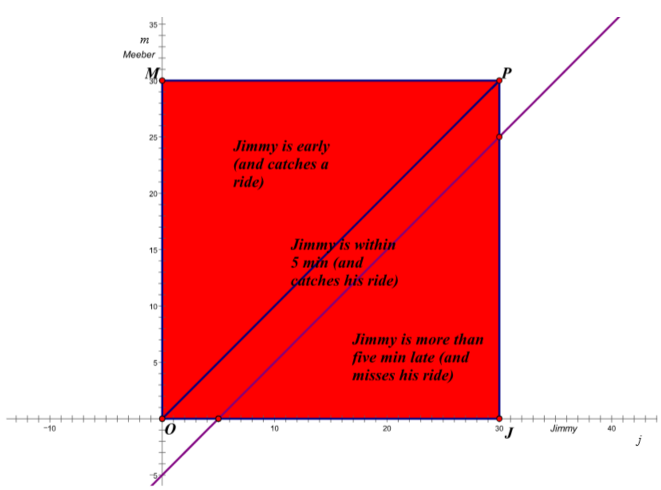
F18SF09. **3432**. Note that the sum of the coefficients of the binomial expansion $(a+b)^n$ is 2^n. (To see why, note that all terms in the expansion are of the form $a^k \cdot b^{n-k}$, where k is an integer such that $0 \leq k \leq n$, because every term is the product of either a or b from each of the n instances of $(a + b)$, thus substituting $a = b = 1$ yields the desired sum of the coefficients.) Since $128^2 = (2^7)^2 = 2^{14}$, we have $n = 14$. The term in question has the form $\binom{14}{7} \cdot a^7 \cdot b^7$ (there are $\binom{14}{7}$ ways to choose 7 a's and 7 b's from each of the n instances of $(a + b)$, thus the coefficient is $\binom{14}{7} = \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} =$ **3432**.

F18SF10. **144**. Since $HOUSEWARMING$ is regular, $HE = EM = MH$, so $\triangle HEM$ is equilateral, so $\frac{(HE)^2\sqrt{3}}{4} = 36\sqrt{3}$, so $HE = 12$. Now let X be the center of the dodecagon and consider $\triangle HXE$. Since X is the center of $\triangle HEM$, HX and EX bisect $\angle EHM$ and $\angle HEM$, respectively. So, dropping a perpendicular from X to HE forms two 30-60-90 triangles. These triangles are congruent by AAS: the two 30° angles, the right angles, and shared side XY. So we have that $HY = 6$. So $HX = HY \cdot \frac{2}{\sqrt{3}} = 6 \cdot \frac{2}{\sqrt{3}} = 4\sqrt{3}$. Alternatively we can use the Law of Cosines on $\triangle HXE$ and $\angle HXE$ to get $HX = 4\sqrt{3}$. Now consider $\triangle HXO$. Since $HOUSEWARMING$ is regular, we have that $HX = OX$ and $m\angle HXO = \frac{360^\circ}{12} = 30^\circ$. So its area is $\frac{1}{2} \cdot HX \cdot OX \cdot \sin(m\angle HXO) = \frac{1}{2} \cdot 4\sqrt{3} \cdot 4\sqrt{3} \cdot \sin 30^\circ = 12$. Connecting X to all the vertices of the dodecagon gives us 12 congruent triangles, so the area of $HOUSEWARMING$ is $12 \cdot 12 =$ **144**.



F18SF11. **8**. Notice that $2022B591$ is divisible by 9 because $243 = 3^5$ is divisible by 9. Thus $2+0+2+2+B+5+9+1 = 21+B$ is divisible by 9, so $B = 6$. Now notice that $2-0+2-2+6-5+9-1 = 11$ is divisible by 11, so 20226591 is divisible by 11. But 243 is not divisible by 11, so $83A37$ is divisible by 11. Thus $8-3+A-3+7 = 9+A$ is divisible by 11, so $A = 2$, and $A + B = 2 + 6 = \mathbf{8}$.

F18SF12. $\mathbf{\frac{47}{72}}$. Consider the diagram below, where Jimmy's arrival time j is on the horizontal axis and the Meeber driver's arrival time m is on the vertical axis (both in minutes past noon). Square $JOMP$ (which contains all points such that $0 \le j \le 30$ and $0 \le m \le 30$) contains all the possible scenarios (j, m). Also, the line $m = j - 5$ is plotted; this is the line that determines the latest the car will wait for Jimmy. The upper left triangle contains all scenarios where $j < m$, so Jimmy will arrive before his Meeber car. In the second region (between $m = j$ and $m = j - 5$), the Meeber car as arrived first but Jimmy is within five minutes, so he gets a ride then, too. In the third region, Jimmy misses the Meeber driver. Thus the desired probability is $1 - \frac{1}{2} \cdot \frac{25}{30} \cdot \frac{25}{30} = 1 - \frac{1}{2} \cdot \frac{5}{6} \cdot \frac{5}{6} = \mathbf{\frac{47}{72}}$.



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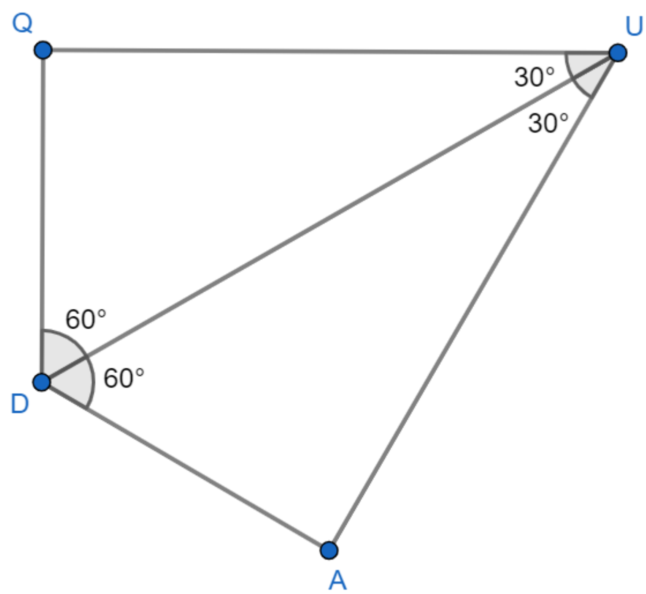
Soph/Frosh Division Contest Number 3 Solutions Fall 2018

F18SF13. 80. The first sentence of the problem implies $0.4A + 0.2B = B$, which implies $0.4A = 0.8B$, so $A = 2B$. Then the second sentence of the problem implies $0.2(2B) + 0.4B = kB/100$, which solves to give us $k = \mathbf{80}$.

F18SF14. $\mathbf{4\sqrt{3}}$. Using the given ratios and the fact that the angles in any quadrilateral sum to 360°, we know there exists an x such that $3x + 2x + 3x + 4x = 360°$. Solving, we get that $x = 30°$. So, the angles of $QUAD$ measure 90°, 60°, 90°, and 120°. Because $\triangle QUD$ and $\triangle AUD$ are right triangles, and we have $UQ = UA$ and shared side $\overline{DU}$, the two triangles are congruent by hypotenuse-leg. Since $m\angle QDA = 120°$, and $m\angle QUD = m\angle ADU$, $m\angle QUD = m\angle ADU = 60°$. So we have that $\triangle QUD$ and $\triangle AUD$ are 30-60-90 triangles. Because $\overline{QU}$ is opposite a 60° angle in $\triangle QUD$, $DU = \frac{6}{\sqrt{3}} \cdot 2 = \mathbf{4\sqrt{3}}$.



F18SF15. 101. The value of $6! + 9!$ is $6!\,(1 + 7 \cdot 8 \cdot 9) = 6! \cdot 505$. Notice that $6!$ has only 2, 3, 5, and 7 as prime factors, and $505 = 5 \cdot 101$. 101 is prime. so the greatest prime factor of $6! + 9$ is **101**.

F18SF16. 19. Let the lengths of the diagonals of the rhombus be d_1 and d_2. Then $d_1 + d_2 = 20$, and squaring both sides gives us $d_1^2 + 2d_1d_2 + d_2^2 = 400$. Also, since the diagonals of a rhombus are perpendicular bisectors of each other, $(\frac{d_1}{2})^2 + (\frac{d_2}{2})^2 = 9^2$, so $(d_1)^2 + (d_2)^2 = 324$. Subtracting the two equations gives us $2d_1d_2 = 76$. The area of $RHOM$ is $\frac{1}{2}d_1d_2 = 76 = \mathbf{19}$.

F18SF17. **9**. The roots of the function $f(x) = x^2 - 6x - 12$ are $\frac{6 \pm \sqrt{36-(4)(1)(-12)}}{2} = \frac{6 \pm \sqrt{84}}{2} = 3 \pm \sqrt{21}$. Since the coefficient of x^2 in $f(x)$ is positive, the graph of $f(x)$ is an upward facing parabola, so $f(x)$ is negative for values of x in between the roots. Since $4 < \sqrt{21} < 5$, $-2 < 3 - \sqrt{21} < -1$ and $7 < 3 + \sqrt{21} < 8$, so all integers between -1 and 7, inclusive, satisfy the condition. These are the integers -1, 0, 1, 2, 3, 4, 5, 6, and 7, for a total of **9** integers.

F18SF18. **22**. There are $\binom{3}{2} = 3$ different ways to choose exactly 2 of Adam, Beth, and Carl. Every 6 boxes, Adam and Beth drop a rock into the same box, since the least common multiple of 2 and 3 is 6. So they both drop rocks in Boxes $1, 7, 13, ..., 97$, totaling to 17 boxes. Every 10 boxes, Adam and Carl drop a rock into the same box, since the least common multiple of 2 and 5 is 10. So they both drop rocks in Boxes $1, 11, 21, ..., 91$, totaling to 10 boxes. Every 15 boxes, Beth and Carl drop a rock into the same box, since the least common multiple of 3 and 5 is 15. So they both drop rocks into Boxes $1, 16, 31, ..., 91$, totaling to 7 boxes. Boxes 1, 31, 61, and 91 all have 3 rocks in them, since the least common multiple of 2, 3, and 5 is 30, so each of these boxes were counted 3 times, which means we must subtract $3 \cdot 4 = 12$ from our sum. This gives us $17 + 10 + 7 - 12 =$ **22** boxes.

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S19SF01. $\mathbf{\frac{15}{16}}$. Using complementary counting, we get that the probability that Larry gets at most 3 heads is $1-$ (the probability that all 4 flips are tails) $= 1 - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \mathbf{\frac{15}{16}}$.

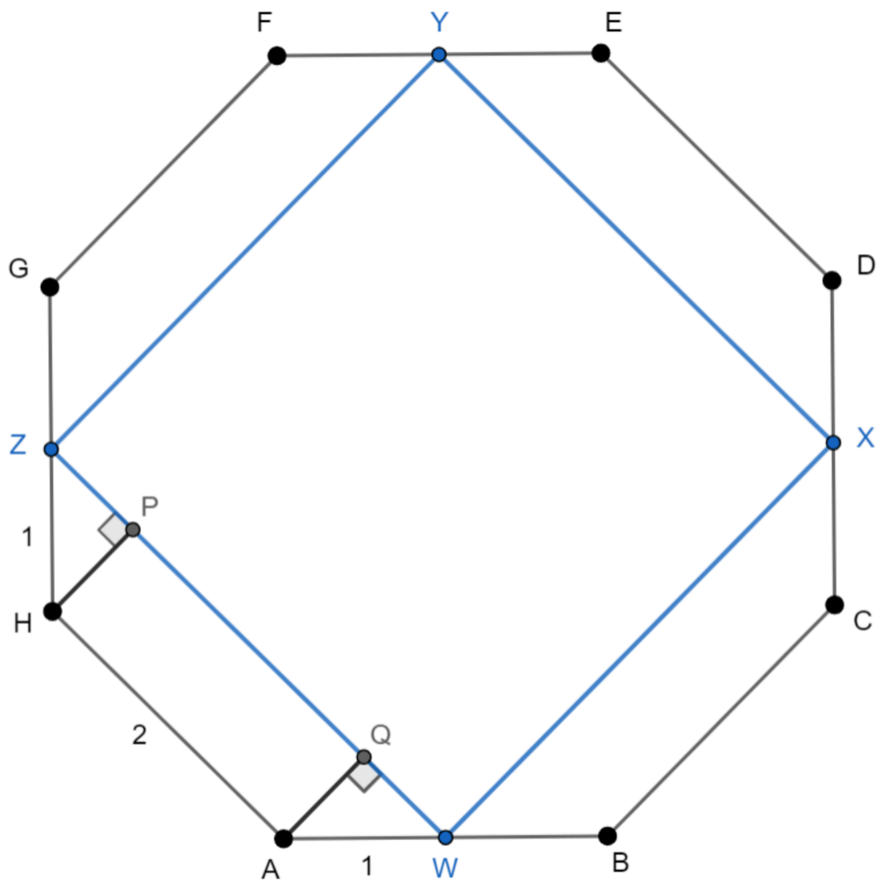
S19SF02. $\mathbf{353}$. Factor: $5^8 - 3^8 = (5^4 - 3^4)(5^4 + 3^4) = (5^2 - 3^2)(5^2 + 3^2)(5^4 + 3^4) = 16 \cdot 34 \cdot 706 = 2^6 \cdot 17 \cdot 353$. Since 2, 17, and 353 are prime (353 is not divisible by any prime up to 19 - since $19 = \sqrt{361} > \sqrt{353}$, if 353 is divisible by some $x > 19$, then it is also divisible by $\frac{353}{x} < 19$), the largest prime factor of $5^8 - 3^8$ is $\mathbf{353}$.

S19SF03. $-\mathbf{\frac{15}{4}}$. Since $7x = 2425$ is a vertical line, $3x + 4y = 5$ is neither horizontal nor vertical, and $5x + By = 13$ cannot be a horizontal line, it must be that $3x + 4y = 5$ and $5x + By = 13$ are perpendicular. The slopes of the lines $3x + 4y = 5$ and $5x + By = 13$ are $-\frac{3}{4}$ and $-\frac{5}{B}$, respectively. Because the slopes of perpendicular lines multiply to -1, solve $-\frac{3}{4} \cdot -\frac{5}{B} = -1$ to obtain $15 = -4B$ so $B = -\mathbf{\frac{15}{4}}$.

S19SF04. $\mathbf{\frac{1+\sqrt{61}}{2}}$. Factor by rewriting $x^4 - x^3 - 19x^2 + 4x + 60 = 0$ as $x^4 - 19x^2 + 60 + (-x^3 + 4x) = (x^2 - 15)(x^2 - 4) - x(x^2 - 4) = (x^2 - 4)(x^2 - x - 15) = 0$. The first factor has roots ± 2, and the second has roots $\frac{1 \pm \sqrt{61}}{2}$. Since $7 < \sqrt{61} < 8$, $-4 < \frac{1-\sqrt{61}}{2} < -3$ and $4 < \frac{1+\sqrt{61}}{2} < 5$. So $\frac{1-\sqrt{61}}{2} < -2 < 2 < \frac{1+\sqrt{61}}{2}$, so the greatest of these four roots is $\mathbf{\frac{1+\sqrt{61}}{2}}$.

S19SF05. $\mathbf{11}$. Since we are looking for four-digit palindromes less than 2019, they must either begin with 1 or 2. If a 4-digit palindrome begins with 1, it is of the form $1aa1$, where a can be any digit from 0 to 9, and all 10 of these possibilities yields a number less than 2019. Similarly if a 4-digit palindrome begins with 2, it is of the form $2bb2$, where b can be any digit from 0 to 9; for such a palindrome to be less than 2019, b can only be 0 (since otherwise the palindrome is 2112 or greater), giving us 1 more solution. So there are a total of $\mathbf{11}$ palindromes.

S19SF06. The probability that the point in is $WXYZ$ is $\frac{[WXYZ]}{[ABCDEFGH]}$, where $[R]$ denotes the area of polygon R. By symmetry, quadrilateral $WXYZ$ has equal sides and equal interior angles, and thus is a square. Let P and Q by the feet of the perpendiculars from H and A to $\overline{WZ}$, respectively. By symmetry, $m\angle AWQ = m\angle BWX$, and since these must sum to $90°$ (given $m\angle XWZ = 90°$), they are each $45°$. Thus, $\triangle AQW$ (and similarly, $\triangle HPZ$) is a 45-45-90 triangle with a hypotenuse of length 1 and legs of length $\frac{\sqrt{2}}{2}$. Now, note that each interior angle of an octagon is $\frac{180°(8-2)}{8} = 135°$, so $m\angle QAH = (135-45)° = 90°$, so $AHPQ$ is a rectangle with $AH = PQ = 2$. Now, we have $WZ = ZP+PQ+QW = \frac{\sqrt{2}}{2}+2+\frac{\sqrt{2}}{2} = 2+\sqrt{2}$, so $[WXYZ] = (2+\sqrt{2})^2 = 6+4\sqrt{2}$. We also have $[AWZH] = [AHPQ]+[HPZ]+[AQW] = 2\cdot\frac{\sqrt{2}}{2}+\frac{1}{2}(\frac{\sqrt{2}}{2})^2+\frac{1}{2}(\frac{\sqrt{2}}{2})^2 = \frac{1}{2}+\sqrt{2}$. Thus, by symmetry, $[ABCDEFGH] = [WXYZ]+4[AWZH] = (6+4\sqrt{2})+4(\frac{1}{2}+\sqrt{2}) = 8+8\sqrt{2}$. So the desired probability is $\frac{6+4\sqrt{2}}{8+8\sqrt{2}} = \frac{(6+4\sqrt{2})(8-8\sqrt{2})}{(8+8\sqrt{2})(8-8\sqrt{2})} = \frac{-16-16\sqrt{2}}{-64} = \mathbf{\frac{\sqrt{2}+1}{4}}$.



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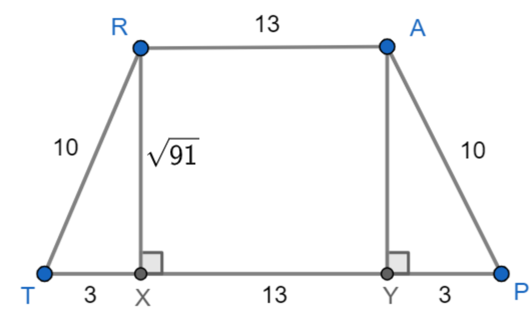
Soph/Frosh Division Contest Number 2 Solutions Spring 2019

S19SF07. **91**. The goal is to find a number H that is exactly halfway between N and N^2 for some N. Solve $\frac{N+N^2}{2} = H$ for the largest two-digit solution for H. This is equivalent to $N(N+1) = 2H$. If $N = 13$, then $H = 91$. If $N = 14$, then $H = 105$. Since $\frac{N(N+1)}{2}$ increases as N increases, the greatest two-digit integer halfway between a number and its square is **91**.

S19SF08. $\boldsymbol{\sqrt{347}}$. Let the trapezoid be called $TRAP$ with $TR = AP = 10$, $RA = 13$, and $TP = 19$. Drop perpendiculars from R and A to $\overline{TP}$ with feet X and Y, respectively. $RX = AY$ since both represent the distance between parallel lines $\overleftrightarrow{AR}$ and $\overleftrightarrow{TP}$, so $\triangle RXT$ is congruent to $\triangle AYP$ by hypotenuse-leg. Since $m\angle RXP = m\angle AYP$, $RX \parallel AY$, which means that RX and AY are parallel and congruent, so $ARXY$ is a parallelogram, so $AR = XY$. So we have $TX = PY = \frac{19-XY}{2} = \frac{19-AR}{2} = \frac{19-13}{2} = 3$. So $RX = \sqrt{10^2 - 3^2} = \sqrt{91}$. Using the Pythagorean Theorem on $\triangle RXP$, we get that the length of a diagonal $RP = AT = \sqrt{91 + 16^2} = \sqrt{91 + 256} = \boldsymbol{\sqrt{347}}$.



S19SF09. $\boldsymbol{6\sqrt{5}}$. let the roots be r and r^3. Then $x^2 - bx + 5$ can be expressed as $(x-r)(x-r^3) = x^2 - (r+r^3)x + r^4$. Thus $r^4 = 5$, so $r^2 = \pm\sqrt{5}$. If $r^2 = -\sqrt{5}$, then r is not real, and $r + r^3 = r(1+r^2) = -4r$ would not be real, contradicting the given that b is real. So $r^2 = 5$, and the sum of the squares of the roots is $r^2 + r^6 = r^2(1+r^4) = r^2(1+5) = 6r^2 = \boldsymbol{6\sqrt{5}}$.

S19SF10. **252**. We will first compute the number of 3-digit integers that do not contain any 4's. There are 8 choices for the hundreds digit (1-9 excluding 4), and 9 choices each for the tens and units digits (0-9 excluding 4). Each of these 3 choices are independent, so there are $8 \cdot 9 \cdot 9 = 648$ 3-digit integers not containing any 4's. Since there are $999 - 100 + 1 = 900$ total 3-digit numbers, there are $900 - 648 = \mathbf{252}$ 3-digit numbers that have at least one digit that is a 4.

S19SF11. $\boldsymbol{\frac{6}{35}}$. Redbeard took $\frac{7}{10}$ of the treasure, leaving $\frac{3}{10}$ left for the other two to split. WhiteBeard took $\frac{4}{7}$ of that treasure, so WhiteBeard's part is $\frac{4}{7} \cdot \frac{3}{10} = \boldsymbol{\frac{6}{35}}$.

S19SF12. $\frac{\mathbf{9}}{\mathbf{17}}$. The length of the third side must be an integer strictly between $12-9=3$ and $12+9=21$, so the third side is an integer in $[4,20]$ - 17 possible integers. Because 9-12-15 is a Pythagorean triple, if the third side is greater than 15, the triangle will be obtuse. Also, if the third side is less than $\sqrt{12^2-9^2}=\sqrt{63}$ (i.e. the 3 sides form a right triangle with hypotenuse 12), the triangle will be obtuse. Thus the triangle is obtuse if the third side is 4, 5, 6, 7, or 16, 17, 18, 19, 20. This adds up to 9 possibilities, so the probability that the chosen triangle is obtuse is $\frac{\mathbf{9}}{\mathbf{17}}$.

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S19SF13. $\pm\sqrt{\mathbf{2}}$. Since $x-1$ and $x+1$ are reciprocals, we have that $\frac{1}{x-1} = x+1$, or $(x-1)(x+1) = 1$. This gives us $x^2 - 1 = 1$, or $x^2 = 2$, so our solutions are $\pm\sqrt{\mathbf{2}}$.

S19SF14. $\mathbf{\frac{10}{3}}$. Rewrite $x\lfloor x\rfloor + 3x = 20$ as $x(\lfloor x\rfloor + 3) = 20$. If $x \geq 4$, then $x(\lfloor x\rfloor + 3) \geq 28$, which is a contradiction. If $x < 3$, then $x(\lfloor x\rfloor + 3) < 18$, which is a contradiction. Thus $3 \leq x < 4$, so $\lfloor x\rfloor = 3$. Thus $\lfloor x\rfloor + 3 = 6$, and $x = \frac{20}{6} = \mathbf{\frac{10}{3}}$.

S19SF15. $\mathbf{10080}$. Eight students can arrange themselves in $8! = 40320$ ways. In half of these, Aaron is in front of Brett, leaving 20160 ways. In half of these, Carol is in front of Dana, leaving $\mathbf{10080}$ ways.

S19SF16. $\mathbf{19}$. Every triangle $\triangle OP_iP_{i+1}$ is isoceles with vertex angle (and central angle for the circle) $(180-40-40)^\circ = 100^\circ$. Since $P_i \neq P_{i+2}$, the P_i''s progress around the circle without changing direction. Therefore, $P_i = P_{i+1}$ if and only if 360 divides $100(i-1)$, which happens if and only if 18 divides $5(i-1)$ (canceling out a factor of 20). Since 5 and 18 are relatively prime, $i-1$ must be a multiple of 18 for this condition to be met, so the smallest $i > 1$ is $i - 1 = 18$, or $i = \mathbf{19}$.

S19SF17. $\mathbf{64}$. This fits the model $a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3$, where $a = 23$ and $b = 19$. So the expression is equivalent to $(23-19)^3 = 4^3 = \mathbf{64}$.

S19SF18. $\mathbf{7660}$. The given equation implies that $1011A + 102B + 21C + 3D = 2019$. If $A = 2$, then $1011A = 2022 > 2019$, so $A < 2$. If $A = 0$, then $102B + 21C + 3D \leq 102\cdot 9 + 21\cdot 9 + 3\cdot 9 = 1134 < 2019$, so $A > 0$. Since A is a digit, this gives us $A = 1$. So $102B + 21C + 3D = 1008$. Because B, C and D are each digits, and because $21\cdot 9 + 3\cdot 9 = 216$, $102B \geq 1008 – 216 = 792$ so $B \geq 8$. If $B = 9$, then $21C + 3D = 1008 – 102\cdot 9 = 90$. This has integer solutions $C = 4, D = 2$ and $C = 3, D = 9$ ($C > 4$ would correspond to a negative value of D, and $C < 3$ would correspond to a value of D greater than 9, but D is a digit). If $B = 8$, then $21C + 3D = 1008 – 102\cdot 8 = 192$. This integer solutions $C = 9, D = 1$ and $C = 8, D = 8$. ($C > 9$ would mean that C isn't a digit, and $C < 8$ would correspond to a value of D greater than 9, but D is a digit). The four numbers are 1942, 1939, 1891, and 1888, which sum to $\mathbf{7660}$.