New York City Interscholastic Mathematics League

Senior A Division   Contest Number 1   Fall 2018

PART I   FALL 2018   CONTEST 1   TIME: 10 MINUTES

F18SA01 An integer is divided by 3, 4, 5, and 6, giving remainders 2, 3, 4, and 5, respectively. What are all the possible remainders when this integer is divided by 240? (Note: your answer(s) should be in the interval $[0, 240)$, and can be given in any order.)

F18SA02 If $\cos((2x+5)^\circ) = \sin((3x-5)^\circ)$ and $0^\circ < x^\circ < 180^\circ$, find all possible values of x. (Note: your answer(s) can be given in any order.)

PART II   FALL 2018   CONTEST 1   TIME: 10 MINUTES

F18SA03 For how many ordered pairs of integers (a, b) with $a, b \in \{1, 2, \ldots, 15\}$ does the equation $ax^2 + bx + a = 0$ have rational roots?

F18SA04 A line in the xy-coordinate plane intersects the x- and y-axes at A and B, respectively. Let C be the point on the y-axis equidistant from both A and B. If $A = (4, 0)$ and $B = (0, 3)$, find the area of triangle ABC. Express your answer as a fraction in simplest form.

PART III   FALL 2018   CONTEST 1   TIME: 10 MINUTES

F18SA05 Three faces of a rectangular box have areas with ratio $13 : 14 : 15$. Find the length of the shortest edge of the box divided by the length of the longest edge of the box. Express your answer as a fraction in simplest form.

F18SA06 Let the complex conjugate of a complex number z be denoted by $\overline{z}$. Let X be the set of all points z in the complex plane so that

$$(z + \overline{z}) - i(z - \overline{z}) \geq 0,$$
$$(z + 2\overline{z})(2z + \overline{z}) \leq 9.$$

The area of X can be written as $a\pi$ for some number a. Compute a. Express your answer as a fraction in simplest form.

New York City Interscholastic Mathematics League

Senior A Division   Contest Number 2   Fall 2018

PART I   FALL 2018   CONTEST 2   TIME: 10 MINUTES

F18SA07 Points A, B, P, C, D are chosen on a line in that order. If $AB = 3$, $BC = 4$, $CD = 7$, and $\frac{BP}{PC} = \frac{AP}{PD}$, find BP. Express your answer as a fraction in simplest form.

F18SA08 Suppose the following infinite fraction converges to a unique real value. Then express its value in the form $\frac{a+\sqrt{b}}{c}$, where a, b, and c are positive relatively prime integers:

$$1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}}.$$

PART II   FALL 2018   CONTEST 2   TIME: 10 MINUTES

F18SA09 Let a and b be positive integers. Find the smallest possible value of $a + b$ if each of $x^2 + ax + 30b = 0$ and $x^2 + 2bx + a = 0$ has distinct real roots.

F18SA10 An ordered triple of digits (a, b, c) is *interesting* if $0 < a \leq b \leq c$ and the six 2-digit integers $\overline{ab}, \overline{ac}, \overline{bc}, \overline{ba}, \overline{cb}, \overline{ca}$ are all prime. Find all distinct interesting triples.

PART III   FALL 2018   CONTEST 2   TIME: 10 MINUTES

F18SA11 Cities A and B are 50 miles apart. On the straight road from city A to city B, starting one mile from city A and ending one mile from city B, milestones are placed at 1 mile intervals (there are no milestones in either city). On every milestone the distance to city A and distance to city B are written. What is the sum of the digits on all of the milestones?

F18SA12 In a regular decagon all diagonals are drawn. Two distinct diagonals are chosen uniformly at random. What is the probability that the two diagonals do not intersect inside or on the decagon? Express your answer as a fraction in simplest form.

New York City Interscholastic Mathematics League

Senior A Division Contest Number 3 Fall 2018

PART I FALL 2018 CONTEST 3 TIME: 10 MINUTES

F18SA13 Let a be a real number so that $3x^2 + ax + 7 = 0$ has real roots r and s. Find the least possible value of $|r + s|$. Express your answer in simplest radical form.

F18SA14 If p and q are distinct prime numbers and pq evenly divides $p^q + q^p + 36p + 26q$, find pq.

PART II FALL 2018 CONTEST 3 TIME: 10 MINUTES

F18SA15 A biased coin is twice as likely to land heads as tails. If the coin is tossed independently n times, the probability that there will be a heads followed directly by a tails is greater than $\frac{9}{10}$. Find the least possible value of n.

F18SA16 Circle C_1 passes through the center of circle C_2 and intersects it at points A and B. If $AB = 8$ and the radius of C_2 is 5, find the radius of C_1. Express your answer as a fraction in simplest form.

PART III FALL 2018 CONTEST 3 TIME: 10 MINUTES

F18SA17 How many 3-digit positive integers with nonzero leading digit have repeated digits?

F18SA18 A line with slope 5 partitions the square with vertices $(0, 0)$, $(0, 2)$, $(2, 2)$, and $(2, 0)$ into two regions with equal areas. This line divides the triangle ΔABC with $A = (1, 10)$, $B = (3, 2)$, and $C = (11, 15)$ into a triangle and quadrilateral. Let the triangle have area P and the quadrilateral Q. Find $\frac{P}{Q}$ and express your answer as a fraction in simplest form.

New York City Interscholastic Mathematics League

Senior A Division   Contest Number 4   Fall 2018

PART I   FALL 2018   CONTEST 4   TIME: 10 MINUTES

F18SA19 How many digits are there in the decimal expansion of 5^{20}?

F18SA20 Two right triangles ABD and ACD share hypotenuse $\overline{AD}$, and B, C lie on the same side of $\overleftrightarrow{AD}$. If $\overline{AC}$ is the angle bisector of $\angle BAD$, $AC = 12$, and $CD = 5$, find AB. Express your answer as a fraction in simplest form.

PART II   FALL 2018   CONTEST 4   TIME: 10 MINUTES

F18SA21 Find the 20th digit after the decimal point in $\frac{1}{9998}$.

F18SA22 Find the ordered quadruple of real numbers (a, b, c, d) so that

$$a^2 + b^2 + c^2 + d^2 - ab - bc - cd - d + \frac{2}{5} = 0.$$

Express all fractions in simplest form.

PART III   FALL 2018   CONTEST 4   TIME: 10 MINUTES

F18SA23 In trapezoid $ABCD$, $m\angle A = m\angle D = 90°$. If $AB = 5$, $CD = 7$, $AD = 4$, and M is the midpoint of $\overline{BC}$, find AM. Express your answer in simplest radical form.

F18SA24 Find all 5-digit palindromes that can be expressed as a sum of two 4-digit palindromes. Consider only palindromes with nonzero leading digits.

New York City Interscholastic Mathematics League

Senior A Division   Contest Number 5   Fall 2018

PART I   FALL 2018   CONTEST 5   TIME: 10 MINUTES

F18SA25 Find the length of the hypotenuse of a right triangle with integer side lengths and one leg of length 17.

F18SA26 There is a unique unordered quadruple of positive integers (a, b, c, d) that satisfies the equation

$$a^5 + b^5 + c^5 + d^5 = 144^5.$$

Find the units digit of $a + b + c + d$.

PART II   FALL 2018   CONTEST 5   TIME: 10 MINUTES

F18SA27 Lines ℓ_1 and ℓ_2 intersect at a 45° angle at point P. Let O be the origin, Q be the intersection of ℓ_1 and the x-axis, and R be the intersection of ℓ_1 and the y-axis. Suppose that ℓ_2 intersects the y-axis at S and $OQ = OR = OS$. If the x-coordinate of Q is 5, find the area of ΔPRS.

F18SA28 In a bag there are 4 black and 5 white marbles. Every second, a marble is drawn from the bag uniformly and at random without replacement. The process ends when either all black or all white marbles are removed from the bag. What is the probability that the last marble removed from the bag is white? Express your answer as a fraction in simplest form.

PART III   FALL 2018   CONTEST 5   TIME: 10 MINUTES

F18SA29 Compute the smallest prime factor of 89951.

F18SA30 In rectangle $ABCD$, $CD = 4$ and $m\angle CBD = 15°$. The image of $ABCD$ under reflection through line $\overleftrightarrow{BD}$ is $A'B'C'D'$ where A' is the reflection of A', and likewise for the other points. Find the area of $ABA'CDC'$. Express your answer in simplified radical form.

New York City Interscholastic Mathematics League

Senior A Division   Contest Number 1   Spring 2019

PART I   SPRING 2019   CONTEST 1   TIME: 10 MINUTES

S19SA01 The cubic $x^3 - 3x^2 + 7x - 5$ has three distinct roots, α, β, and γ. The cubic $x^3 + rx^2 + sx + t$ has roots $\alpha + 1$, $\beta + 1$, and $\gamma + 1$. Find the ordered triple (r, s, t).

S19SA02 Larry repeatedly rolls two fair six-sided dice simultaneously, and after each roll he compares the two numbers on top of each die. If one of the numbers is a multiple of the other, Larry stops. Compute the probability that when Larry stops, at least one of the dice shows a factor of 6. Express your answer as a fraction in simplest form.

PART II   SPRING 2019   CONTEST 1   TIME: 10 MINUTES

S19SA03 Let A and B be digits and suppose the six-digit number $\overline{57AB79}$ is divisible by both 7 and 13. Find all possible ordered pairs (A, B).

S19SA04 A circle whose center is in the first quadrant contains the points $(1, 2)$ and $(3, 6)$ and is tangent to the x-axis. The radius of the circle can be expressed in the form $a - \frac{\sqrt{b}}{c}$, where a, b, and c are positive integers and b is square-free. Find (a, b, c).

PART III   SPRING 2019   CONTEST 1   TIME: 10 MINUTES

S19SA05 Suppose a, b, c (in that order) form a geometric sequence so that

$$\log \sqrt{ac} = \sqrt{\log(ac)}.$$

Find all possible values of b. (Note: here log is the base 10 logarithm.)

S19SA06 In unit square $ABCD$, let P be the midpoint of $\overline{AD}$, Q the projection of C onto $\overleftrightarrow{PB}$, R the projection of D onto $\overleftrightarrow{CQ}$, and S the projection of P onto $\overleftrightarrow{DR}$. Find the area of $PQRS$, expressing your answer as a fraction in simplest form.

New York City Interscholastic Mathematics League

Senior A Division   Contest Number 2   Spring 2019

PART I   SPRING 2019   CONTEST 2   TIME: 10 MINUTES

S19SA07 In a convex quadrilateral $ABCD$, $AB = AC = AD$ and $m\angle CDB = 30^\circ$. Find the degree measure of $\angle BAC$.

S19SA08 How many ways are there to distribute 30 identical truffles between Valerie, Milan, and Hannah so that everyone gets at least three truffles, Milan gets at least 3 more truffles than Hannah, and Hannah gets at least 5 more truffles than Valerie?

PART II   SPRING 2019   CONTEST 2   TIME: 10 MINUTES

S19SA09 Serina walks at 4 mph and rides a bike at 30 mph. Her brother William walks at 3 mph and rides a bike at 20 mph. They wish to travel 20 miles along the straight road from city A to city B, but they only have one bike. They decide to leave at the same time. Serina will take the bike first, leave the bike after riding some part of the distance, and walk the rest of the way. William will walk until he reaches the bike, at which point he will bike the rest of the way to city B. If Serina and William reach city B simultaneously, how many hours did the whole trip take? Express your answer as a fraction in simplest form.

S19SA10 Let x and y be real numbers so that $x^2 + y^2 = 1$ and $3x^2y - y^3 = \frac{1}{2}$. Find all possible values of $x^3 - 3xy^2$, expressing all your answers in simplest radical form.

PART III ***SPRING 2019*** ***CONTEST 2*** ***TIME: 10 MINUTES***

S19SA11 Three numbers are chosen uniformly at random, without replacement, from the set of $\{1, 2, \ldots, 7\}$. What is the probability that it is possible to form a triangle with those numbers as side lengths? Express your answer as a fraction in simplest form.

S19SA12 $\overline{AB}$ and $\overline{CD}$ are diameters of a circle with center O. E is on the circle so that $\overline{AE} \perp \overline{CD}$. Let $\overline{OB}$ intersect $\overline{ED}$ at N and suppose $\frac{AN}{NB} = \frac{7}{3}$. Find $\frac{EN}{ND}$, expressing your answer as a fraction in simplest form.

New York City Interscholastic Mathematics League

Senior A Division   Contest Number 3   Spring 2019

PART I   **SPRING 2019**   **CONTEST 3**   **TIME: 10 MINUTES**

S19SA13 A fair six-sided die with faces labeled $1, 2, 3, 4, 5, 6$ is rolled 10 times. What is the probability that the sum of all the rolls is divisible by 2?

S19SA14 The midpoints of the sides of a regular tetrahedron with edge length 1 are connected to form a regular octahedron. Find the surface area of the octahedron. Express your answer in simplified radical form.

PART II   **SPRING 2019**   **CONTEST 3**   **TIME: 10 MINUTES**

S19SA15 Let r, s, t be the distinct roots of $p(x) = x^3 - 3x^2 + 5x - 7$. Compute

$$\frac{r}{s} + \frac{s}{r} + \frac{r}{t} + \frac{t}{r} + \frac{s}{t} + \frac{t}{s}.$$

Express your answer as a fraction in simplest form.

S19SA16 In a deck of 12 cards, there are 6 identical red cards and 6 identical blue cards. The order of the cards in the deck is completely random, so all orderings of the 12 cards are equally likely. The deck is then split into two piles of 6, namely the first 6 cards and the last 6 cards. What is the probability that the two cards on top of one pile are both the same color and the two cards on top of the other pile are opposite colors?

PART III   **SPRING 2019**   **CONTEST 3**   **TIME: 10 MINUTES**

S19SA17 Compute $\binom{10}{1} + 3\binom{10}{3} + 5\binom{10}{5} + 7\binom{10}{7} + 9\binom{10}{9}$.

S19SA18 Suppose that a polyhedron has triangular faces, and that there are 5 edges incident to each vertex of the polyhedron. Compute the number of edges of the polyhedron.

New York City Interscholastic Mathematics League

Senior A Division   Contest Number 4   Spring 2019

PART I   SPRING 2019   CONTEST 4   TIME: 10 MINUTES

S19SA19 Mario and Mathew begin reciting numbers out loud. Mario says 17 and Mathew says 2019 at the same time; every minute, Mario says the number 6 greater than the number he said the previous minute, and Mathew says the number 7 fewer than the number he said the previous minute. After some time, Mario and Mathew will say the same number out loud. Compute that number.

S19SA20 Hana wants to vacuum her 10-by-$5\sqrt{3}$ meter room. There is a single socket in one corner of the room into which Hana must plug the vacuum cleaner. If the cord for the vacuum cleaner is 10 meters long, what fraction of the room will Hana be unable to vacuum? Express your answer in simplified radical form in terms of π.

PART II   SPRING 2019   CONTEST 4   TIME: 10 MINUTES

S19SA21 A broken change machine accepts only pennies. It randomly exchanges a penny for one of the following three combinations of coins, each with probability $\frac{1}{3}$: a penny and a nickel, a penny and a dime, or a nickel and a dime. A nickel is worth 5 cents and a dime 10. A penny can be exchanged multiple times. What is the expected value of a penny, in cents?

S19SA22 Kadir randomly rearranges the digits $1, 2, \ldots, 9$ to from a 9-digit number (each digit must be used exactly once). What is the probability that the resulting number is divisible by 4? Express your answer as a fraction in simplest form.

New York City Interscholastic Mathematics League

Senior A Division   Contest Number 4   Spring 2019

PART III   SPRING 2019   CONTEST 4   TIME: 10 MINUTES

S19SA23 Compute the unique 3-digit prime factor of $3^{35} - 2^{35}$.

S19SA24 How many ways are there to color the faces of a regular tetrahedron (only one color is allowed per face, and all faces must be colored) using 5 colors? Two colorings are equivalent if one can be obtained from the other through a rotation in 3D space.

New York City Interscholastic Mathematics League

Senior A Division   Contest Number 5   Spring 2019

PART I   SPRING 2019   CONTEST 5   TIME: 10 MINUTES

S19SA25 A fair, two-sided coin is tossed 6 times. What is the probability that there is at least one heads in each pair of consecutive tosses? Express your answer as a fraction in simplest form.

S19SA26 Quadilateral $ABCD$ is cyclic and has $AB = CD$. If $BC = 10$, $AD = 14$, and the area of $ABCD$ is 60, find the perimeter of $ABCD$. Express your answer in simplified radical form.

PART II   SPRING 2019   CONTEST 5   TIME: 10 MINUTES

S19SA27 A palindrome is an integer that reads the same forwards and backwards. For example, 123321 is a 6-digit palindrome and 86568 is a 5-digit palindrome, but 100011 is not a palindrome. Find the smallest possible value of n so that the number of n-digit palindromes containing only the digits 1 and 2 is greater than 2019.

S19SA28 In square $ABCD$, let P, Q, R, and S be the midpoints of sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$, respectively. Point X is chosen inside $ABCD$ so that the areas of $APXS$, $BQXP$, and $CRXQ$ are $\sqrt{3}$, 2, and $\sqrt{5}$, respectively. Find the area of $ABCD$. Express your answer in simplified radical form.

PART III   SPRING 2019   CONTEST 5   TIME: 10 MINUTES

S19SA29 Coco wants to buy a \$12 sandwich using only nickels (worth 5 cents) and dimes (worth 10 cents). In how many ways can he do that if he wants the total number of coins to be a perfect square?

S19SA30 Five people are playing laser tag. At a signal, each person selects one of the four others at random and shoots them. What is the probability that no two people shoot each other? Express your answer as a fraction in simplest form.

New York City Interscholastic Mathematics League

Senior A Division — Contest Number 1 Solutions — Fall 2018

F18SA01. **59, 119, 179, 239.** Let the unknown number be x. $x \equiv -1 \pmod{3, 4, 5, 6}$ exactly when $x \equiv -1$ modulo $\operatorname{lcm}(3, 4, 5, 6)$; in other words, $x = 60k - 1$ for some integer k (one way of seeing this is to note that $x + 1$ is a multiple of 3, 4, 5, and 6 exactly when it is a multiple of their least common multiple as well). We thus want all possible values of $60k - 1 \pmod{240}$ for integer k. Note that 59, 119, 179, and 239 are all possible values; they correspond to $k = 1$, 2, 3, and 4, respectively. Notice also that because $60 \cdot 4 = 240$, $60(k + 4) - 1 \equiv 60k - 1 \pmod{240}$, so whenever $k > 4$ or $k < 1$, the remainder we obtain is still one of the four above remainders. Therefore, the only possible remainders are **59**, **119**, **179**, and **239**, and these all satisfy the problem conditions, so we are done.

F18SA02. **18, 90, 100, 162.** For the entirety of this solution, we assume that sin and cos take degree values as arguments. Because $\sin(\theta) = \cos(90 - \theta)$, we can rewrite our equation as

$$\cos(2x + 5) = \cos(95 - 3x),$$

and since $\cos\alpha = \cos\beta$ exactly when $\alpha \equiv \pm\beta \pmod{360}$, we have

$$2x + 5 \equiv \pm(95 - 3x) \pmod{360}.$$

If $2x + 5 \equiv 95 - 3x \pmod{360}$, this is equivalent to $5x \equiv 90 \pmod{360}$; in other words, $5x = 360k + 90$ for some natural number k, which simplifies to $x = 72k + 18$. With the restriction that $0 < x < 180$, this gives us $x = 18, 90, 162$. On the other hand, if $2x + 5 \equiv 3x - 95 \pmod{360}$, this occurs exactly if $x \equiv 100 \pmod{360}$, so $x = 100$. Therefore our solution set is $x = \{\mathbf{18}, \mathbf{90}, \mathbf{100}, \mathbf{162}\}$.

F18SA03. **5.** We know we can compute the roots of such a quadratic directly using the quadratic formula; if a and b are both integers, the roots will be rational if and only if the discriminant $b^2 - 4a^2$ is a perfect square; to ensure that the equation is actually quadratic, we must also have $a \neq 0$. If $b^2 - 4a^2$ is a perfect square, then $2a$, $\sqrt{b^2 - 4a^2}$, and b form a Pythagorean triple with hypotenuse b. We can thus list all the Pythagorean triples with hypotenuse at most 15 and determine all possible values of (a, b):

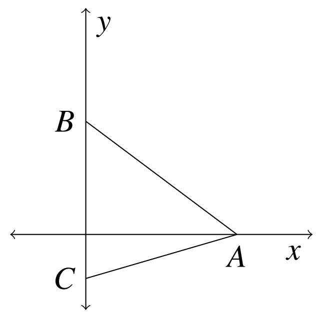
Triple	b	$2a$	a	# of pairs
$3 - 4 - 5$	5	4	2	1
$6 - 8 - 10$	10	6, 8	3, 4	2
$5 - 12 - 13$	13	12	6	1
$9 - 12 - 15$	15	12	6	1

We can ascertain that these are exactly the Pythagorean triples with hypotenuse at most 15 by either testing all the others (ditto) or recalling that all Pythagorean triples are of the form $k(m^2 - n^2), k(2mn), k(m^2 + n^2)$ for $m > n, k$ positive with m, n relatively prime and not both odd. We want $k(m^2 + n^2) \leq 15$. If $k = 1$, our only possible pairs of m, n are $(2, 1)$ and $(3, 2)$; $n = 1, m \geq 4$, $n = 2, m \geq 5$, and $n \geq 3, m \geq 4$ have $m^2 + n^2 > 15$. Thus $k = 1$ gives us the triples $(3, 4, 5)$ and $(5, 12, 13)$. If $k \geq 2$, the only possible m, n pair is $(2, 1)$, since $n = 1, m \geq 4$ and $n \geq 2, m \geq 3$ have $m^2 + n^2 > 8$. In this case, $m^2 + n^2 = 5$, so the only possible values of k are $2, 3$, and we get the triples $(6, 8, 10)$ and $(9, 12, 15)$.
So, in total there are **5** such ordered pairs.

F18SA04. $\mathbf{\frac{25}{3}}$.

Solution 1

Let C have coordinates $(0, c)$. We know that C is equidistant from A and B, so we must have $\sqrt{c^2 + 16} = AC = BC = \sqrt{(c-3)^2}$. Squaring both sides and solving, we see that $c^2 + 16 = (c-3)^2 = c^2 - 6c + 9$, so $6c = -7$, $c = -\frac{7}{6}$ and $BC = \frac{25}{6}$. Therefore, the area of triangle ABC is $\frac{1}{2} \cdot 4 \cdot \frac{25}{6} = \mathbf{\frac{25}{3}}$.



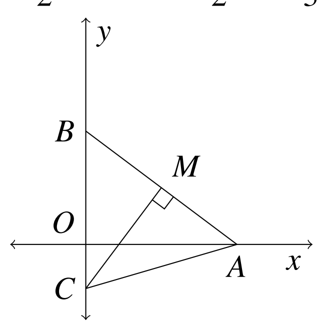
Solution 2

Let M be the midpoint of $\overline{AB}$ and O the origin. Since C is on the perpendicular bisector of $\overline{AB}$, we know $\overline{CM} \perp \overline{AB}$. Notice that $m\angle CMB = 90° = m\angle AOB$, so $\Delta ABO \sim \Delta CBM$ since $\angle B$ is shared by both triangles. ΔABO is a 3-4-5 right triangle and we know that $MB = \frac{1}{2}AB = \frac{5}{2}$. Therefore,

$$\frac{CM}{MB} = \frac{AO}{OB} \implies CM = \frac{AO}{OB} \cdot MB = \frac{4}{3} \cdot \frac{5}{2} = \frac{10}{3}$$

and

$$[ABC] = \frac{1}{2}AB \cdot CM = \frac{1}{2} \cdot 5 \cdot \frac{10}{3} = \mathbf{\frac{25}{3}}.$$



F18SA05. $\mathbf{\frac{13}{15}}$. Because the areas of the faces given are all different, we know that no two of these three faces can be congruent, and thus the height, length, and width of the solid are all distinct. Without loss of generality let $h < l < w$ with h the height, l the length, and w the width; the areas of the faces are hl, lw, and hw, and we must have $hl < hw < lw$. Therefore, $hl : hw : lw = 13 : 14 : 15$ and $\frac{h}{w} = \frac{hl}{lw} = \mathbf{\frac{13}{15}}$.

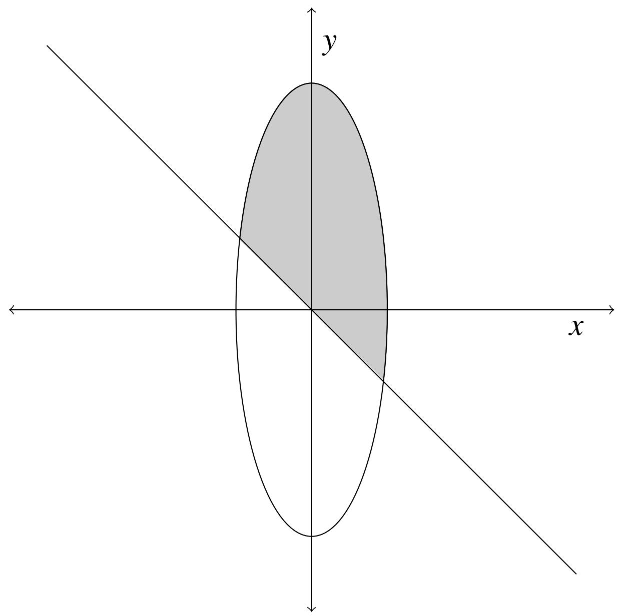
F18SA06. $\mathbf{\frac{3}{2}}$**.** Let $z = x + yi$ with x, y real; then our first inequality becomes

$$(x + yi + x - yi) - i(x + yi - x + yi) = 2x + 2y \geq 0 \Leftrightarrow x + y \geq 0,$$

which represents the region on or above the line $y = -x$. Similarly, from the second inequality we get

$$(x + yi + 2x - 2yi)(2x + 2yi + x - yi) = (3x - yi)(3x + yi) = 9x^2 + y^2 \leq 9 \Leftrightarrow x^2 + \frac{y^2}{9} \leq 1,$$

which represents the region on or inside the ellipse centered at the origin with major axis of length 6 along the y-axis and minor axis of length 2 along the x-axis. Graphing both of these on the xy-plane produces



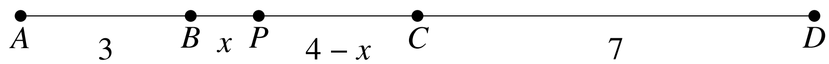
Notice that the line $y = -x$ cuts the ellipse in half because the line passes through the origin, which is the center of symmetry of the ellipse and the line; therefore, by symmetry, we can match each point above the line to its reflection about the origin, which will be below the line, and vice versa. Therefore, the shaded area is exactly half that of the ellipse's. The area of an ellipse with major axis of length $2a$ and minor axis of length $2b$ is $ab\pi$, so our ellipse has area 3π and X has area $\frac{3\pi}{2}$. Thus $a = \mathbf{\frac{3}{2}}$.

New York City Interscholastic Mathematics League

Senior A Division Contest Number 2 Solutions Fall 2018

F18SA07. $\mathbf{\frac{6}{5}}$**.** Let $BP = x$. Then $AP = x + 3$, $PC = 4 - x$, and $PD = 11 - x$ so

$$\frac{BP}{PC} = \frac{AP}{PD} \Rightarrow \frac{x}{4-x} = \frac{x+3}{11-x} \Rightarrow 11x - x^2 = -x^2 + x + 12 \Rightarrow x = \mathbf{\frac{6}{5}}.$$



F18SA08. $\mathbf{\frac{1+\sqrt{3}}{2}}$**.**
Let x be be the infinite fraction. Notice that

$$x = 1 + \frac{1}{2 + \frac{1}{x}},$$

and we can eliminate this nested fraction by multiplying both sides by $2 + \frac{1}{x}$ to get

$$2x + 1 = 2 + \frac{1}{x} + 1 \implies 2x - 2 - \frac{1}{x} = 0;$$

multiplying both sides by x, we get $2x^2 - 2x - 1 = 0$. This is a quadratic, so we can solve it using the quadratic equation to obtain

$$x = \frac{2 \pm \sqrt{12}}{4} = \frac{1 \pm \sqrt{3}}{2}.$$

However, the root $\frac{1-\sqrt{3}}{2}$ is negative while our original value for x was positive, so we discard it and obtain $\mathbf{\frac{1+\sqrt{3}}{2}}$.

F18SA09. **33.** A quadratic has distinct real roots iff its discriminant is positive. We thus must have

$$\begin{cases} a^2 - 120b > 0 \\ 4b^2 - 4a > 0 \end{cases} \Leftrightarrow \begin{cases} a^2 > 120b \\ b^2 > a. \end{cases}$$

Therefore, $b^4 > a^2 > 120b$, so $b^3 > 120$. Then $b \geq 5$. If $b = 5$, then $a^2 > 120b = 600$ and $a \geq 25$, but this contradicts $b^2 = 25 > a$. So we must have $b \geq 6$. If $b = 6$, then $a^2 > 120b = 720$ so $a \geq 27$; note that $a = 27, b = 6$ satisfy the above inequalities. Higher values of b would also result in at least as large values for a to satisfy the inequalities, so the minimum value of $a + b$ is $27 + 6 = \mathbf{33}$.

F18SA10. $(\mathbf{1},\mathbf{1},\mathbf{1}),(\mathbf{1},\mathbf{1},\mathbf{3}),(\mathbf{1},\mathbf{1},\mathbf{7}),(\mathbf{1},\mathbf{3},\mathbf{7})$**.** Note that every 2-digit prime ends in 1,3,7, or 9. Thus, $a,b,c \in \{1,3,7,9\}$. We now have two cases: either all three digits are distinct, or not. In the first case, we merely have to check the four possible triples. $(1,3,7)$ is interesting while $(1,3,9)$, $(1,7,9)$, and $(3,7,9)$ are not (91 and 93 are not prime). In the second, notice that if two digits are the same, one of the six 2-digit integers will be divisible by 11; if it is also prime, it must be 11, so we must have $a = b = 1$. Our possible triples are then $(1,1,1)$, $(1,1,3)$, $(1,1,7)$, and $(1,1,9)$; of these, only $(1,1,9)$ is not interesting because 91 is composite. Therefore, the only interesting triples are $(\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{1},\mathbf{3})$, $(\mathbf{1},\mathbf{1},\mathbf{7})$, and $(\mathbf{1},\mathbf{3},\mathbf{7})$.

F18SA11. **650.** Each of the numbers $1,2,\ldots,49$ appears precisely twice (once as a distance from A and once as a distance from B). In the sum of their digits, 1, 2, 3, and 4 appear in the tens position a total of 20 times each and $1,2,\ldots,9$ appear in the units position a total of 10 times each. (We may ignore 0s, since they do not affect the digit sum.) Hence, the total sum of the digits is

$$(1+2+3+4)\cdot 20 + (1+2+\ldots+9)\cdot 10 = \mathbf{650}.$$

F18SA12. $\mathbf{\frac{5}{17}}$**.** We use complementary counting by checking how many pairs of diagonals do intersect. First, we find the total number of diagonal pairs: there are $\frac{10\cdot 7}{2} = 35$ diagonals (choosing one endpoint, of which there are 10, then the other endpoint, which excludes the first endpoint and the two immediately adjacent, leaving 7, then dividing by 2 because order of choosing doesn't matter), so there are $\binom{35}{2} = 595$ pairs of diagonals. We now compute how many pairs of diagonals intersect. Two diagonals may either intersect inside the decagon or at an endpoint. If they intersect at an endpoint, then there are $10\cdot\binom{7}{2} = 210$ ways to select those two diagonals (first selecting the common endpoint, then choosing the other two endpoints for the diagonals). If they intersect inside the decagon, notice that finding these two diagonals is equivalent to selecting four distinct vertices: there is exactly one way to draw two diagonals between these vertices that intersect inside the decagon and any two such diagonals determine four vertices of the decagon. The number of ways to select 4 vertices is $\binom{10}{4} = 210$. Therefore, there are 420 pairs of diagonals that intersect either on or inside the decagon, so there are $595 - 420 = 175$ pairs that don't intersect. So the probability this occurs is $\frac{175}{595} = \mathbf{\frac{5}{17}}$.

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F18SA13. $\mathbf{\frac{2\sqrt{21}}{3}}$**.** By Vieta's formulas, the sum of the roots of $3x^2 + ax + 7$ is $\frac{-a}{3}$, so $|r+s| = \frac{|a|}{3}$. Note that in order for $3x^2 + ax + 7$ to have real roots, we need the discriminant to be nonnegative. The discriminant is $a^2 - 84$, and $a^2 - 84 \geq 0 \iff |a| \geq 2\sqrt{21}$. Therefore the minimum value of $|r+s|$ is $\frac{2\sqrt{21}}{3}$; this can indeed be achieved by taking $a = 2\sqrt{21}$ and $r, s = -\frac{\sqrt{21}}{3}$.

F18SA14. **111.** Because pq divides $p^q + q^p + 36p + 26q$, p and q each individually divide this expression as well. By Fermat's Little Theorem, $p^q \equiv p \pmod{q}$ and $q^p \equiv q \pmod{p}$. Therefore, we have

$$p^q + q^p + 36p + 26q \equiv q + 26q \equiv 27q \pmod{p};$$

in other words, $p|27q$. However, q is prime and $q \neq p$, so we know $p\nmid q$; thus $p|27$ and $p = 3$. Likewise, we see that

$$p^q + q^p + 36p + 26q \equiv p + 36p \equiv 37p \pmod{q};$$

here $q|37p$. Because $q\nmid p$, we must have $q|37$ and $q = 37$. Indeed, 3 and 37 satisfy the given conditions:

$$3^{37} + 37^3 + 36 \cdot 3 + 26 \cdot 37 \equiv 37 + 26 \cdot 37 \equiv 27 \cdot 37 \equiv 0 \pmod{3}$$
$$3^{37} + 37^3 + 36 \cdot 3 + 26 \cdot 37 \equiv 3 + 36 \cdot 3 \equiv 37 \cdot 3 \equiv 0 \pmod{37}$$

so $3 \cdot 37 | 3^{37} + 37^3 + 36 \cdot 3 + 26 \cdot 37$. Therefore, $pq = 3 \cdot 37 = \mathbf{111}$.

F18SA15. **8.** We use complementary counting and compute the smallest value of n so that the probability that there will be no head-tail sequences is less than $1 - \frac{9}{10} = \frac{1}{10}$. If there are no heads followed by tails, our flips must consist of a sequence of only tails followed by a sequence of only heads (each of these sequences can be of length 0). The probability of flipping heads is $\frac{2}{3}$ and the probability of flipping tails is $\frac{1}{3}$; thus the probability of getting a sequence of $n-k$ tails followed by k heads is $\frac{2^k}{3^n}$. k, the length of the heads sequence, can be anything from 0 to n, and k completely determines the length of the tails sequence as well; thus we can obtain the probability of having a run of tails followed by a run of heads by summing over all values of k:

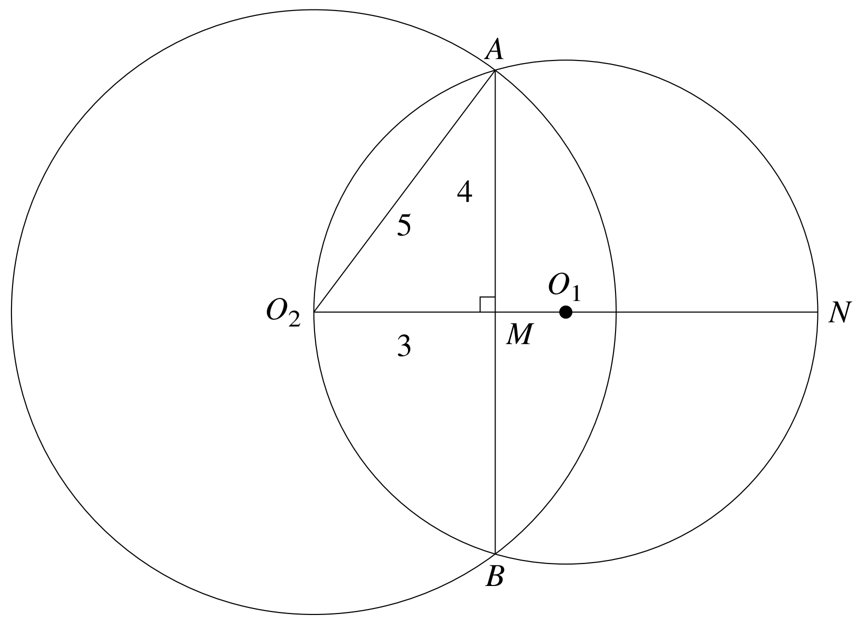
$$\sum_{k=0}^{n} \frac{2^k}{3^n} = \frac{\sum_{k=0}^{n} 2^k}{3^n} = \frac{1 + 2 + \cdots + 2^n}{3^n} = \frac{2^{n+1} - 1}{3^n}.$$

We want the least positive integer n for which

$$\frac{2^{n+1} - 1}{3^n} < \frac{1}{10};$$

we discover that for $n = 1, 2, \ldots, 7$, our fraction is greater than $\frac{1}{10}$ (e.g. when $n = 7$, $\frac{255}{2187} > \frac{1}{10}$), but for $n = 8$ our probability is $\frac{511}{6561} < \frac{1}{10}$, so the least positive integer value of n for which our probability is less than $\frac{1}{10}$ is **8**.

F18SA16. $\mathbf{\frac{25}{6}}$.

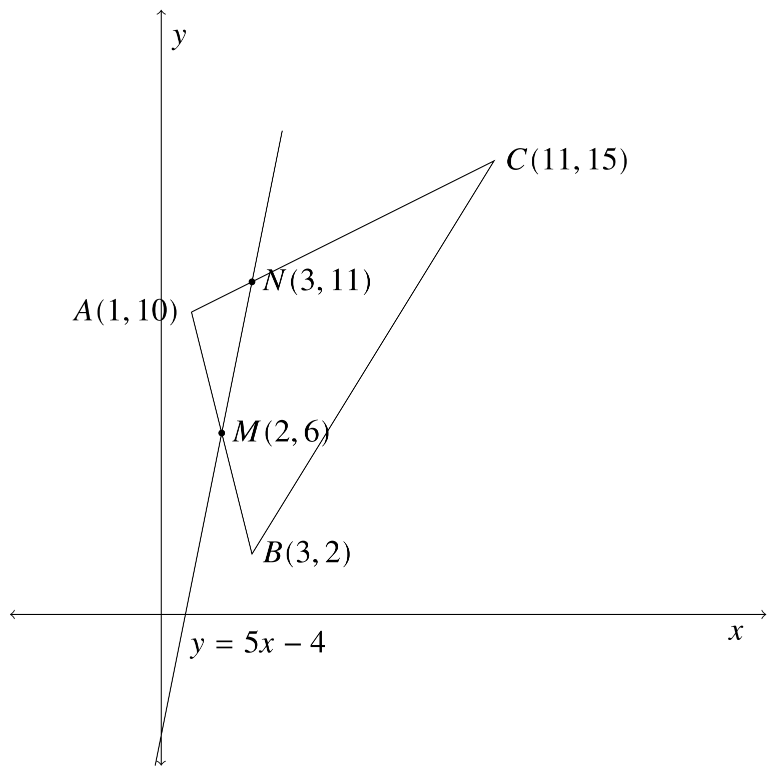


Let O_1, O_2 be the centers of C_1, C_2 respectively and let M be the intersection of $\overline{AB}$ and $\overline{O_1O_2}$. $\overleftrightarrow{AB}$ is the radical axis of C_1, C_2, so it must be perpendicular to $\overleftrightarrow{O_1O_2}$, the line of centers. But $\overline{O_1O_2}$ is also the radius of C_1 and $\overline{AB}$ a perpendicular chord. So $\overleftrightarrow{O_1O_2}$ must be the perpendicular bisector of $\overline{AB}$ and intersects $\overline{AB}$ at its midpoint, which must be M. Then $AM = MB = 4$. Using the Pythagorean Theorem, $O_2M = \sqrt{5^2 - 4^2} = 3$. Let $\overleftrightarrow{O_1O_2}$ intersect circle C_1 at N (see diagram). Then, by the chord-chord theorem, $O_2M \cdot MN = AM \cdot MB$, but we know that $AM \cdot MB = 4 \cdot 4 = 16$. Let r be the radius of circle C_1. Then $MN = 2r - 3$, so

$$3(2r-3) = 16 \iff 2r - 3 = \frac{16}{3} \iff r = \mathbf{\frac{25}{6}}.$$

F18SA17. **252.** We use complementary counting to obtain the number of 3-digit positive integers with nonzero leading digit and no repeated digits. In total, there are $9 \cdot 10 \cdot 10 = 900$ 3-digit numbers with nonzero leading digit: there are 9 ways to choose the first digit because it must be nonzero, and no restrictions on the other two digits. If no digits can be repeated, we only have 9 choices for a second digit distinct from the first, and 8 for a third digit distinct from the preceding two; thus there are $9 \cdot 9 \cdot 8 = 648$ such numbers. Therefore, the number of 3-digit positive integers with nonzero leading digit and repeated digits is $900 - 648 = \mathbf{252}$.

F18SA18. $\mathbf{\frac{1}{9}}$.



In order for this line with slope 5 to halve the area of this square, it must pass through its center, which is $(1, 1)$. Thus the line has equation $y = 5x - 4$. We now find where the line intersects ΔABC. $\overline{AB}$ has slope -4 and thus equation $y = -4x + 14$; it intersects $y = 5x - 4$ when $5x - 4 = -4x + 14$, which occurs at $(x, y) = (2, 6)$. Call this point M. Likewise, $\overline{AC}$ has slope 0.5 and equation $y = \frac{1}{2}x + \frac{19}{2}$; this line intersects $y = 5x - 4$ when $5x - 4 = \frac{1}{2}x + \frac{19}{2}$, which occurs at $(x, y) = (3, 11)$. Call this point N. M is the midpoint of AB because its x-coordinate is the average of those of A and B; likewise, N is one-fifth of the way from A to C. Let $[\cdot]$ denote area; we wish to find $\frac{[AMN]}{[MNBC]}$. Note that ΔAMN and ΔABC are two triangles sharing an angle, so

$$\frac{[AMN]}{[ABC]} = \frac{AM}{AB} \cdot \frac{AN}{AC} = \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}.$$

Therefore, $\frac{[MNBC]}{[ABC]} = 1 - \frac{1}{10} = \frac{9}{10}$ and

$$\frac{P}{Q} = \frac{[AMN]}{[MNBC]} = \mathbf{\frac{1}{9}}.$$

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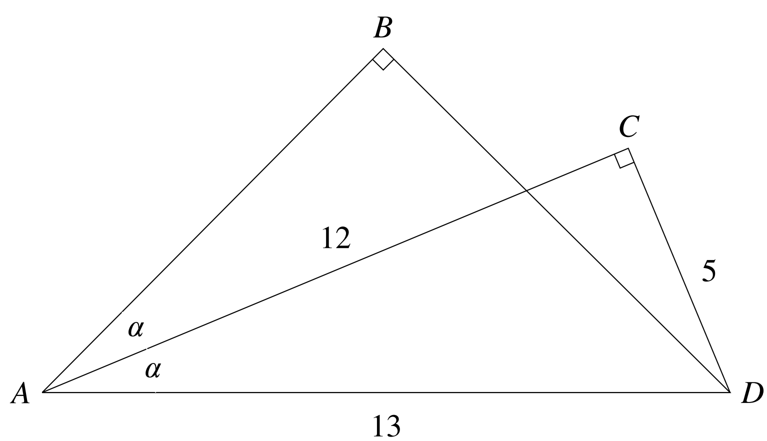
F18SA19. **14.** To find how many digits there are in the decimal expansion of 5^{20}, we seek to bound it between two consecutive powers of 10. Note that $5^{20} = \frac{10^{20}}{2^{20}}$ and $2^{20} = 1048576$ (we can compute $2^{10} = 1024$ easily and square this to obtain 2^{20}), so $10^6 < 2^{20} < 10^7$. Therefore,

$$\frac{10^{20}}{10^6} > \frac{10^{20}}{2^{20}} > \frac{10^{20}}{10^7} \implies 10^{14} > 5^{20} > 10^{13}.$$

Thus 5^{20} has **14** digits.

F18SA20. $\mathbf{\frac{119}{13}}$**.** By the Pythagorean theorem, $AD = \sqrt{5^2 + 12^2} = 13$. Let $m\angle CAD = \alpha$. Then $\sin\alpha = \frac{CD}{AD} = \frac{5}{13}$ and likewise $\cos\alpha = \frac{AC}{AD} = \frac{12}{13}$.

$$AB = AD\cos(\angle BAD) = 13\cos 2\alpha = 13(\cos^2\alpha - \sin^2\alpha) = 13\left(\left(\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2\right) = \mathbf{\frac{119}{13}}.$$



F18SA21. **6.** Notice that

$$\frac{1}{9998} = \frac{1}{10000-2} = \frac{\frac{1}{10000}}{1-\frac{2}{10000}}$$

and this is equivalent to summing a geometric series with first term $\frac{1}{10000} = 0.0001$ and common ratio $\frac{2}{10000} = 0.0002$:

$$\begin{aligned}\frac{\frac{1}{10000}}{1-\frac{2}{10000}} &= 0.0001 + 0.00000002 + 0.000000000004 + 0.0000000000000008 + \cdots \\ &= 0.00010002000400080016032\ldots\end{aligned}$$

Therefore, the twentieth digit after the decimal point is **6**.

We should verify the observation that the twentieth digit is not affected by any terms after the 5th term of the geometric series; however, it's then enough to verify that

$$\frac{1}{10000}\left(\left(\frac{2}{10000}\right)^5 + \left(\frac{2}{10000}\right)^6 + \left(\frac{2}{10000}\right)^7 + \cdots\right) < 10^{-20}.$$

But using the closed form expression for the sum of a geometric series with common ratio in $(-1, 1)$, the left side is

$$\frac{1}{10000} \cdot \frac{\left(\frac{2}{10000}\right)^5}{1-\frac{2}{10000}} = \left(\frac{1}{10000}\right)^5 \frac{32}{9998}$$

and

$$\left(\frac{1}{10000}\right)^5 \frac{32}{9998} < \left(\frac{1}{10000}\right)^5 = 10^{-20}$$

as desired.

F18SA22. $\left(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right)$**.** We complete the squares in the given expression to obtain

$$\left(a^2 - ab + \frac{1}{4}b^2\right) + \left(\frac{3}{4}b^2 - bc + \frac{1}{3}c^2\right) + \left(\frac{2}{3}c^2 - cd + \frac{3}{8}d^2\right) + \left(\frac{5}{8}d^2 - d + \frac{2}{5}\right) = 0;$$

that is,

$$\left(a - \frac{1}{2}b\right)^2 + \left(\frac{\sqrt{3}}{2}b - \frac{\sqrt{3}}{3}c\right)^2 + \left(\frac{\sqrt{6}}{3}c - \frac{\sqrt{6}}{4}d\right)^2 + \left(\frac{\sqrt{10}}{4}d - \frac{\sqrt{10}}{5}\right)^2 = 0.$$

Because squares of real numbers are always nonnegative, we can only achieve equality with 0 when every square in this expression is equal to 0. Thus

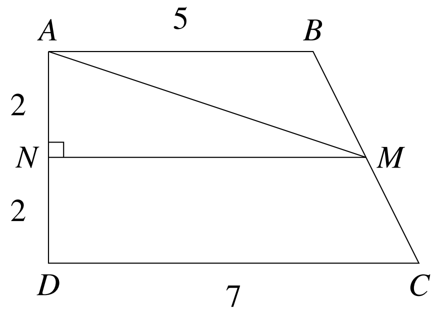
$$d = \frac{4}{5} \implies c = \frac{3}{5} \implies b = \frac{2}{5} \implies a = \frac{1}{5}$$

and our ordered quadruple is $\left(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right)$.

F18SA23. $\mathbf{2\sqrt{10}}$.

Solution 1

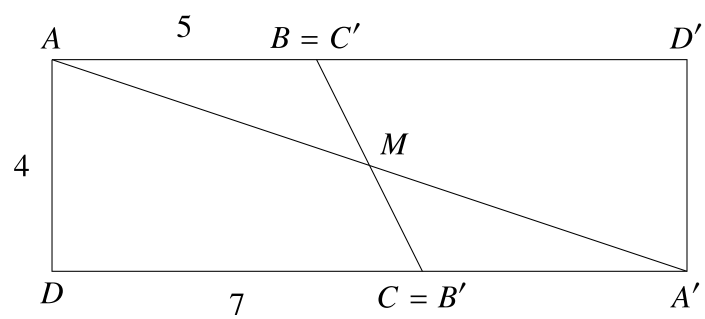
Let N be the midpoint of AD; thus $\overline{NM}$, being the midline of the trapezoid, is parallel to $\overline{AB}$ and $\overline{CD}$ and has length $\frac{AB+CD}{2}$. Therefore, $\angle ANM$ is right and $NM = 6$. Therefore, by the Pythagorean theorem on ΔANM, $AM = \sqrt{AN^2 + MN^2} = \mathbf{2\sqrt{10}}$.



Solution 2

Rotate $ABCD$ by $180°$ about M to produce trapezoid $A'B'C'D'$. Since $BM = MC$, $C' = B, B' = C$. Because $ABCD$ is a trapezoid, $m\angle ABC + m\angle BCD = 180°$; then $m\angle AC'D' = m\angle DB'A' = 180°$. Therefore, the triples A, B, D' and D, C, A' are in fact triples of collinear points. Thus $m\angle DAD' = m\angle DAB = 90°$ and likewise $m\angle A'DA$, $m\angle AD'A'$, and $m\angle DA'D' = 90°$, so $ADA'D'$ is a rectangle. By construction M is the center of symmetry of $\{A, A'\}$, so M is the midpoint of AA'. Thus AM is half the length of the diagonal AA'; by the Pythagorean theorem, $AA' = \sqrt{AD^2 + A'D^2}$. We know $AD = 4$, and $A'D = DB' + B'A' = DC + AB = 12$. Thus

$$AM = \frac{1}{2}AA' = \frac{1}{2}\sqrt{AD^2 + A'D^2} = \frac{1}{2}\sqrt{4^2 + 12^2} = \mathbf{2\sqrt{10}}.$$



F18SA24. **11011, 12221.** Let a 5-digit palindrome expressible as such a sum be $\overline{abcba}$ where a, b, c are digits and $a \geq 1$. Because the sum of two 4-digit numbers cannot exceed 19998, $a = 1$. Suppose the 4-digit palindromes in the sum are $\overline{pqqp}$, $\overline{rssr}$ with p, q, r, s digits and $p, r \geq 1$. Then

$$\overline{1bcb1} = \overline{pqqp} + \overline{rssr} = 1001(p + r) + 110(q + s).$$

Therefore $1001(p + r) + 110(q + s) \equiv 1 \pmod{10}$; then $p + r \equiv 1 \pmod{10}$. Moreover, $1 \leq p, r \leq 9$, so $2 \leq p + r \leq 18$ and $p + r = 11$. Therefore,

$$\overline{1bcb1} = 11011 + 110(q + s) \iff 110(q + s) = 100c + 1010(b - 1).$$

Since $q + s \leq 18$, $110(q + s) \leq 1980$. Thus $b - 1 \leq 1$.

If $b - 1 = 1$, then $100c = 110(q + s) - 1010$. Dividing by 10, we know $10c = 11(q + s) - 101$, so

$$11(q + s) - 101 \equiv 0 \pmod{10} \iff (q + s) \equiv 1 \pmod{10}.$$

But $0 \leq q, s \leq 9$, so $0 \leq q + s \leq 18$ and $q + s = 1, 11$. If $q + s = 1$, then $c < 0$; thus this equation is only satisfied when $q + s = 11$, so $c = 2$. Therefore, $\overline{abcba} = 12221$ and $12221 = 9999 + 2222$, so this is valid.

If $b - 1 = 0$, then $100c = 110(q + s)$. So $100c$ must be a multiple of 11; however, 11 is prime and does not divide 100, so it must divide c. As $0 \leq c \leq 9$, c must be 0. In that case $q + s = 0$ also, so $q = s = 0$. Thus $\overline{abcba} = 11011$ and $11011 = 9009 + 2002$, so this is also a valid palindrome.

Therefore, there are two such palindromes, **11011** and **12221**.

F18SA25. **145.** Let the other leg have length a and the hypotenuse have length c. By the Pythagorean theorem, $17^2 = c^2 - a^2 = (c-a)(c+a)$. We know $0 < c - a < c + a$ and 17^2 has only 3 divisors, 1, 17, and 289. Therefore, $c - a$ cannot be 17 as then $c - a = c + a$. Then the only possible values for $c - a$ and $c + a$ are 1 and 289, respectively; this yields $c = 145, a = 144$, and we can verify that $17^2 + 144^2$ is indeed 145^2. So the length of the hypotenuse is **145**.

F18SA26. **4.**

Solution 1

By Fermat's Little Theorem, $n^5 \equiv n \pmod 5$. Also, $n^5 \equiv n \pmod 2$ because n^5 and n will have the same parity. Thus, $n^5 - n \equiv 0$ modulo both 2 and 5. If $n^5 - n$ is a multiple of both 2 and 5, it is also a multiple of 10 since 2 and 5 are relatively prime. So, $n^5 - n \equiv 0 \pmod{10}$ and therefore $n^5 \equiv n \pmod{10}$. Then

$$a^5 + b^5 + c^5 + d^5 \equiv a + b + c + d \pmod{10}, 144^5 \equiv 4^5 \equiv 4 \pmod{10}.$$

Then $a + b + c + d \equiv 4 \pmod{10}$ and the units digit of $a + b + c + d$ is **4**.

Solution 2

Since any integer modulo 10 is congruent to its last digit, we only need to consider $n^5 \pmod{10}$ for each digit. Note that

$$0^5 \equiv 0, 1^5 \equiv 1, 2^5 = 32 \equiv 2, 3^5 = 243 \equiv 3, 4^5 = 1024 \equiv 4 \pmod{10},$$

and we know 5^5 ends in a 5. Because $n^5 \equiv -(-n)^5 \pmod{10}$, we see that

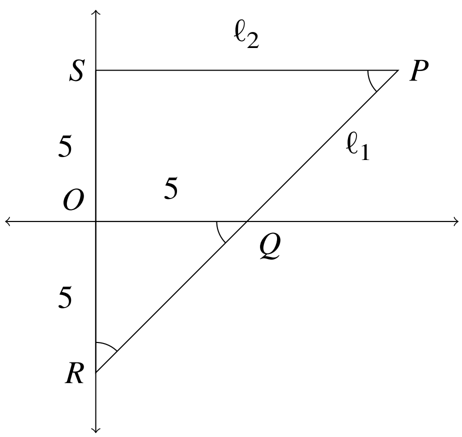
$$6^5 \equiv -(-6)^5 \equiv -4^5 \equiv -4 \equiv 6 \pmod{10}$$

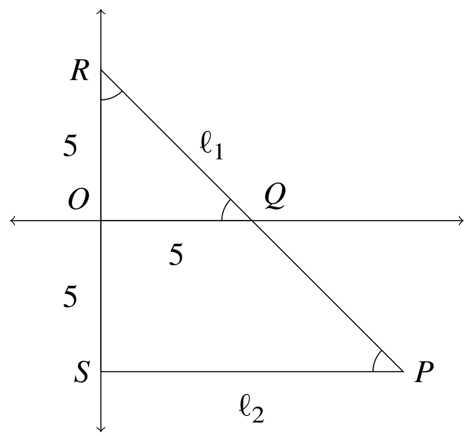
and similarly $7^5 \equiv 7$, $8^5 \equiv 8$, $9^5 \equiv 9 \pmod{10}$. Therefore, we see that $n^5 \equiv n \pmod{10}$ for all n. We then use the same logic as in Solution 1 to obtain that $a + b + c + d \equiv 4 \pmod{10}$.

F18SA27. **50.** First, note $OQ = 5$ since Q is on the x-axis and has x-coordinate 5, and O is the origin. Then $OQ = OR = OS = 5$. Because $OQ = OR$ and $\overline{OQ} \perp \overline{OR}$, we must have $m\angle OQR = 45^\circ = m\angle P$. Then ℓ_2 must be parallel to the x-axis. Then it makes a right angle with the y-axis, in particular, $\overline{SR}$, and ΔPRS is a $45 - 45 - 90$ right triangle. Let $[PRS]$ be the area of ΔPRS. Then

$$PS = SR = OS + OR = 10 \implies [PRS] = \frac{1}{2}(PS)(SR) = \mathbf{50}.$$

The diagrams below illustrate the cases when P lies in quadrants I or IV, respectively.





F18SA28. $\mathbf{\frac{4}{9}}$**.** If the last marble we draw from the bag is white, then we draw the last white marble and the remaining marbles in the bag are all black. In other words, if we were to continue drawing regardless of the problem's stopping conditions, the last marble we draw from the bag would be black. Thus the probability we want is the same as the probability that if the 9 marbles are all drawn from the bag, the last marble drawn is black. Note that all drawing orders are equally likely and there are a total of $\binom{9}{4}$ ways to arrange the 4 black marbles, with the 5 white marbles taking the remaining positions. If we fix the last position to be black, there are $\binom{8}{3}$ ways to arrange the remaining marbles. Therefore, the desired probability is

$$\frac{\binom{8}{3}}{\binom{9}{4}} = \frac{\frac{8!}{3!5!}}{\frac{9!}{5!4!}} = \mathbf{\frac{4}{9}}.$$

F18SA29. **293.** Notice that $89951 = 90000 - 49$, which we can rewrite as $300^2 - 7^2$. Therefore,

$$89951 = 300^2 - 7^2 = (300 - 7)(300 + 7) = 293 \cdot 307.$$

We can verify that both of these are prime by checking their divisibility with all the primes less than their square root. Because $19^2 = 361$, which is larger than both numbers, we only need to consider primes below 19, and we find that none of $\{2, 3, 5, 7, 11, 13, 17\}$ divide either number. Therefore, the smallest prime divisor of 89951 is the smaller of the two, **293**.

F18SA30. $\mathbf{56 + 32\sqrt{3}}$**.** Let $\overline{BC}$ and $\overline{A'D'}$ intersect at M, and $\overline{AD}$ and $\overline{B'C'}$ at N. Note that $B = B'$ and $D = D'$. Then

$$m\angle DBC' = m\angle DBC = 15^\circ \implies m\angle C'BC = 30^\circ \implies m\angle ABC' = 90^\circ - m\angle C'BC = 60^\circ.$$

Therefore, ΔABN is a $30-60-90$ right triangle. Since $\overline{AB}$ is the leg adjacent to the 60° angle, $AB = 4$; $\overline{AN}$, the other leg, has length $4\sqrt{3}$; and $\overline{BN}$, the hypotenuse, has length 8. Similarly, $\Delta C'DN$, $\Delta A'BM$, and ΔCDM are also $30-60-90$ triangles, and all have shorter leg of length 4. Therefore, all 4 triangles are congruent with shorter leg 4, longer leg $4\sqrt{3}$, and hypotenuse 8. In particular, note

$$BC = BM + MC = 8 + 4\sqrt{3}.$$

Using brackets to denote area, note

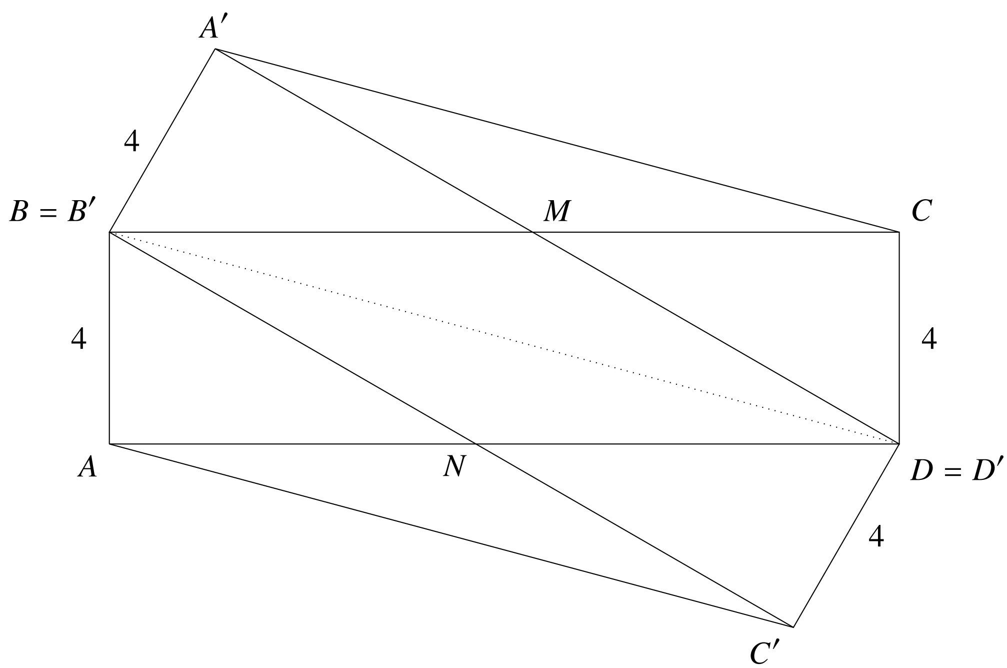
$$[A'BM] = \frac{1}{2}\cdot 4\cdot 4\sqrt{3} = 8\sqrt{3} \implies [A'BC] = \frac{BC}{BM}[A'BM] = \frac{8+4\sqrt{3}}{8}\cdot 8\sqrt{3} = 12 + 8\sqrt{3}.$$

Similarly, $[AC'D] = 12 + 8\sqrt{3}$. Also,

$$[ABCD] = BC\cdot CD = (8+4\sqrt{3})(4) = 32 + 16\sqrt{3}.$$

Therefore,

$$[ABA'CDC'] = [A'BC] + [AC'D] + [ABCD] = \mathbf{56 + 32\sqrt{3}}.$$



Senior A Division — Contest Number 1 Solutions — Spring 2019

S19SA01. **$(-6, 16, -16)$.**

Solution 1

Let $p(x) = x^3 - 3x^2 + 7x - 5$. Then, $p(\alpha) = p(\beta) = p(\gamma) = 0$. We want to find a polynomial q with leading coefficient 1 so that $q(\alpha+1) = q(\beta+1) = q(\gamma+1) = 0$. Note that if we let $q(x) = p(x-1)$, then this condition is satisfied, and we expand to get

$$q(x) = p(x-1) = (x-1)^3 - 3(x-1)^2 + 7(x-1) - 5 = x^3 - 6x^2 + 16x - 16$$

so $r = -6$, $s = 16$, and $t = -16$. Our desired ordered triple is then **$(-6, 16, -16)$**.

Solution 2

Let $p(x)$ be as in the above solution and note that $p(1) = 0$, so 1 is a root of p. We can thus compute the other roots of p by factoring p as $(x-1)(x^2 - 2x + 5)$ and using the quadratic equation to obtain the other two roots, $1 \pm 2i$. Therefore, q must be the polynomial with leading coefficient 1 and roots $2, 2 \pm 2i$. Expanding, we get

$$q(x) = (x-2)(x-(2-2i))(x-(2+2i)) = (x-2)(x^2-4x+8) = x^3 - 6x^2 + 16x - 16,$$

so we get the same ordered triple as above.

S19SA02. $\mathbf{\frac{10}{11}}$**.** There are 36 possible outcomes for each roll of the two dice, all equally likely; let's represent those by ordered pairs like $(1, 2)$. We want to compute the probability that at least one of the dice shows a factor of 6 – $1, 2, 3, 6$ – given that one of the numbers is a multiple of the other. There are 22 ways for the roll to occur so that one number is a multiple of the other, which we can count by considering the roll on the first die. If it is 1, the second die can show any number; if it is 2, the second die can show 1, 2, 4, or 6; if it is 3, the second die can show 1, 3, or 6; if it is 4, the second die can show 1, 2, or 4; if it is 5, the second die can show 1 or 5; and if it is 6, the second die can show 1, 2, 3, or 6. Totalling these, we get $6 + 4 + 3 + 3 + 2 + 4 = 22$. Thus the probability that one of the numbers is a multiple of the other is $\frac{22}{36}$. Of those rolls, only two do not contain a factor of 6: $(5, 5)$ and $(4, 4)$. Therefore, the probability of at least one die showing a factor of 6 and one of the numbers being a multiple of the other is $\frac{20}{36}$. So the conditional probability we want is

$$\frac{\frac{20}{36}}{\frac{22}{36}} = \mathbf{\frac{10}{11}}.$$

S19SA03. **$(0, 4)$, $(9, 5)$.** We first locate a valid (A, B) by noting that $1001 = 7 \cdot 11 \cdot 13$, so if $1001 | \overline{57AB79}$ then $7, 13 | \overline{57AB79}$; 579579 is a multiple of 1001, so $(9, 5)$ is one such pair for (A, B). Being divisible by both 7 and 13 is equivalent to being divisible by 91 because both 7 and 13 are prime, so any other valid $\overline{57AB79}$ differs from 579579 by a multiple of 91. So for some integer k,

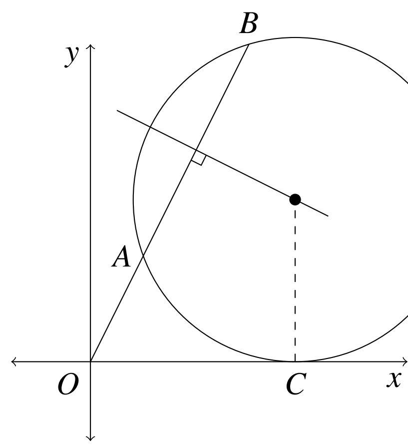
$$\overline{57AB79} = 579579 - 91k \iff 91k = 100(95 - \overline{AB}).$$

Because 91 and 100 are relatively prime, we must have $91 | (95 - \overline{AB})$. $\overline{AB}$ must be nonnegative and at most 99, so $95 - \overline{AB} = 0, 91$. Thus $\overline{AB} = 95, 04$ and we can verify that both possibilities imply $91 | \overline{57AB79}$. So the only possible ordered pairs are **$(0, 4)$** and **$(9, 5)$**.

S19SA04. **(5, 15, 2).** Let $A = (1, 2)$ and $B = (3, 6)$. Then $\overleftrightarrow{AB}$ has equation $y = 2x$ and passes through the origin. Consider the power of the origin O with respect to the circle. Let C be the point of tangency of the circle to the x-axis; $\overline{OC}$ is then tangent to the circle. Using the Pythagorean Theorem, we can compute that $OA = \sqrt{5}$ and $OB = 3\sqrt{5}$, so by Power of a Point

$$OA \cdot OB = OC^2 \implies 15 = OC^2 \iff OC = \sqrt{15}.$$

Thus the center has x-coordinate $\sqrt{15}$. Note that $\overline{AB}$ is a chord of the circle, so its perpendicular bisector passes through the center, and we can compute the equation of its perpendicular bisector. The line has slope $-\frac{1}{2}$ and passes through $(2, 4)$, the midpoint of $\overline{AB}$, so it has equation $y = -\frac{1}{2}x + 5$. The (unique) point on this line with x-coordinate $\sqrt{15}$ then has y-coordinate $5 - \frac{\sqrt{15}}{2}$. But the circle is tangent to the x-axis, so its radius is just the y-coordinate of the center, which we have computed above. So the desired ordered triple is **(5, 15, 2)**. Note that this ordered triple is actually unique, though we leave proving it as an exercise.



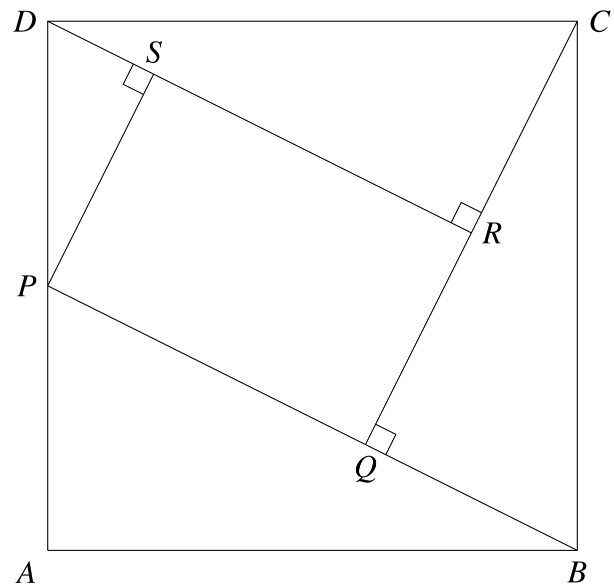
S19SA05. **±1**, **±100.** Rewrite the given equation as

$$\frac{1}{2}\log ac = \sqrt{\log(ac)} \implies (\log ac)^2 = 4\log ac.$$

We now have a quadratic equation in $\log ac$ with roots $0, 4$ (and we can verify both of those satisfy the given equation). Therefore, $ac = 1, 10^4$. Because a, b, c are in a geometric sequence, we know $b^2 = ac$, so $b =$ **±1**, **±100**.

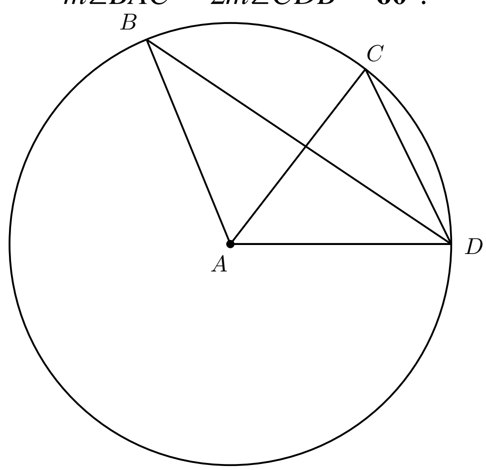
S19SA06. $\mathbf{\frac{3}{10}}$. Note that all the projections create right angles as shown in the diagram. We first angle chase a little. Let $m\angle APB = \alpha$. $\angle PAB$ is right, so $m\angle ABP = 90^\circ - \alpha$. $\angle ABC$ is right, as well, so $m\angle QBC = \alpha$. Looking at ΔCQB, we find that $m\angle BCQ = 90^\circ - \alpha$; this implies $m\angle DCR = \alpha$ and $m\angle CDR = 90^\circ - \alpha$. Therefore ΔABP, ΔQCB, and ΔRDC are similar triangles by AAA. Moreover, $BC = CD = 1$, so by ASA, $\Delta QCB \cong \Delta RDC$. We know $AP = \frac{1}{2}$ and $AB = 1$, so by the Pythagorean Theorem $PB = \frac{\sqrt{5}}{2}$. We also know that by similar triangles $QB : CQ : BC = \frac{1}{2} : 1 : \frac{\sqrt{5}}{2}$, and $CB = 1$, so we find that $QB = RC = \frac{\sqrt{5}}{5}$ and $CQ = DR = \frac{2\sqrt{5}}{5}$. So $PQ = PB - QB = \frac{3\sqrt{5}}{10}$ and $QR = CQ - RC = \frac{\sqrt{5}}{5}$. $PQRS$ is a rectangle because it has four right angles, so its area can be computed as

$$PQ \cdot QR = \frac{3\sqrt{5}}{10} \cdot \frac{\sqrt{5}}{5} = \mathbf{\frac{3}{10}}.$$



S19SA07. **60** or **60°.** Because $AB = AC = AD$, the points B, C, and D lie on a circle of radius AB centered at A (see diagram). Since $\angle BAC$ is a central angle for the arc that $\angle CDB$ is inscribed in,

$$m\angle BAC = 2m\angle CDB = \mathbf{60}°.$$



S19SA08. **10.** Suppose Valerie gets v truffles, Milan m, and Hannah h. We have the following constraints:

- $v + m + h = 30$
- $v, m, h \geq 3$
- $m - h \geq 3$, $h - v \geq 5$.

We now casework and count the number of triples (v, m, h) for each value of v. When $v = 3$, we know $h \geq 8$, $m \geq h + 3$, and $m + h = 27$; therefore the possible values of h range from 8 to 12 (inclusive). When $v = 4$, we know $h \geq 9$, $m \geq h + 3$, and $m + h = 26$, so the possible values for h range from 9 to 11. When $v = 5$, we know $h \geq 10$, $m \geq h + 3$, and $m + h = 25$, so the possible values of h are 10 and 11. When $v \geq 6$, note that $h \geq 11$, so $m \geq 14$, but then $v + m + h \geq 31$, contradiction. So Valerie can get at most 5 truffles. Counting, we obtain **10** possible arrangements. All the arrangements are listed in the below table:

v	(h, m)
3	$(8, 19)$, $(9, 18)$, $(10, 17)$, $(11, 16)$, $(12, 15)$
4	$(9, 17)$, $(10, 16)$, $(11, 15)$
5	$(10, 15)$, $(11, 14)$

S19SA09. $\mathbf{\frac{49}{15}}$. Suppose Serina leaves the bike x miles from city A. Then she spends $\frac{x}{30}$ hours on the bike and $\frac{20-x}{4}$ hours walking, while William spends $\frac{x}{3}$ hours walking and $\frac{20-x}{20}$ hours biking. We know they arrived at the same time, so we have

$$\begin{aligned} & \frac{x}{30}+\frac{20-x}{4}=\frac{x}{3}+\frac{20-x}{20} \\ \Longleftrightarrow \quad & 2x+15(20-x)=20x+3(20-x) \\ \Longleftrightarrow \quad & 30x=240 \\ \Longleftrightarrow \quad & x=8. \end{aligned}$$

Therefore, the total time spent on the trip is

$$\frac{8}{30}+\frac{12}{4}=\mathbf{\frac{49}{15}}.$$

S19SA10. $\pm\mathbf{\frac{\sqrt{3}}{2}}$.

Solution 1

Adding $(3x^2y-y^3)^2$ and $(x^3-3xy^2)^2$ and using the fact that $x^2+y^2=1$ yields

$$\begin{aligned} (3x^2y-y^3)^2+(x^3-3xy^2)^2 &= 9x^4y^2-6x^2y^4+y^6+x^6-6x^4y^2+9x^2y^4 \\ &= 3x^4y^2+3x^2y^4+x^6+y^6 \\ &= (x^2+y^2)^3 \\ &= 1. \end{aligned}$$

Therefore,

$$(x^3-3xy^2)^2=1-(3x^2y-y^3)^2=\frac{3}{4} \implies x^3-3xy^2=\pm\mathbf{\frac{\sqrt{3}}{2}}.$$

Solution 2

Because $x^2+y^2=1$, we can let $x=\cos\theta$ and $y=\sin\theta$ for some $\theta\in[0,2\pi)$. Note that

$$\begin{aligned} \sin 3\theta=\sin(2\theta+\theta) &= \sin 2\theta\cos\theta+\sin\theta\cos 2\theta \\ &= 2\sin\theta\cos^2\theta+\sin\theta(\cos^2\theta-\sin^2\theta) \\ &= 3\cos^2\theta\sin\theta-\sin^3\theta \\ &= 3x^2y-y^3=\frac{1}{2}. \end{aligned}$$

We can similarly compute that $\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta$. Therefore,

$$x^3-3xy^2=\cos 3\theta=\pm\sqrt{1-\sin^2 3\theta}=\pm\mathbf{\frac{\sqrt{3}}{2}}.$$

For both solutions, to prove that both these values are achievable, we want to find x and y so that $x^3-3xy^2=\pm\frac{\sqrt{3}}{2}$. But by the reasoning in the second solution, when $x=\cos 10^\circ$ and $y=\sin 10^\circ$, then $x^3-3xy^2=\cos 30^\circ=\frac{\sqrt{3}}{2}$; and when $x=\cos 50^\circ$ and $y=\sin 50^\circ$, $x^3-3xy^2=\cos 150^\circ=-\frac{\sqrt{3}}{2}$. In both cases by the Pythagorean theorem, $x^2+y^2=1$ and $3x^2y-y^3=\sin 30^\circ=\sin 150^\circ=\frac{1}{2}$, so all the necessary conditions are satisfied.

S19SA11. $\mathbf{\frac{13}{35}}$**.** Let the three numbers chosen be a, b, c; WLOG $a < b < c$. We want these three numbers to satisfy the triangle inequality. By assumption $b+c > a$ and $a+c > b$, so we only need to calculate the probability that $a+b > c$. We can make a table to determine which triples do so:

c	(b, a)
7	$(6,2)$, $(6,3)$, $(6,4)$, $(6,5)$, $(5,3)$, $(5,4)$
6	$(5,2)$, $(5,3)$, $(5,4)$, $(4,3)$
5	$(4,2)$, $(4,3)$
4	$(3,2)$

Thus there are 13 triples that satisfy the triangle inequality out of $\binom{7}{3} = 35$ total triples. Therefore, the probability of selecting a triple which can be the side lengths of a triangle is $\mathbf{\frac{13}{35}}$.

S19SA12. $\mathbf{\frac{3}{2}}$**.** See the diagram below. Let $m\angle CDE = \alpha$; $\angle CDE$ is inscribed in the circle so $\sin\alpha \neq 0$. Note that A is closer to C than to D; otherwise, $\overline{OB}$ cannot intersect $\overline{ED}$. ΔEOD is isosceles because $\overline{EO}$ and $\overline{OD}$ are radii, so $m\angle OED = \alpha$ as well. $\angle COE$ is the central angle of the arc that $\angle CDE$ subtends, so $m\angle COE = 2\alpha$. Let $\overline{CD}$, a diameter, intersect chord $\overline{AE}$ at M; the two lines are perpendicular, so M is the midpoint of $\overline{AE}$ and $AM = ME$. $OA = OE$ because they are both radii and $OM = OM$, so by SSS $\Delta EMO \cong \Delta AMO$. Then $m\angle AOM = m\angle EOM = 2\alpha$, and $\angle AOM$ and $\angle DON$ are vertical angles, so $\angle DON = 2\alpha$ as well. By the Law of Sines on ΔDON,

$$\frac{\sin\alpha}{ON} = \frac{\sin 2\alpha}{DN} = \frac{2\sin\alpha\cos\alpha}{DN} \implies \frac{1}{ON} = \frac{2\cos\alpha}{DN}.$$

Notice that $m\angle DEC = 90°$ because it subtends a diameter, so ΔDEC is right. Let $CD = d$; then $\cos\alpha = \frac{DE}{CD} = \frac{DE}{d}$. Substituting into the above equation,

$$\frac{1}{ON} = \frac{2DE}{d \cdot DN} \iff \frac{d}{2 \cdot ON} = \frac{DE}{DN}.$$

We know that $\frac{AN}{NB} = \frac{7}{3}$, so $\frac{NB}{AB} = \frac{3}{10}$. O is the midpoint of $\overline{AB}$, so $\frac{OB}{AB} = \frac{1}{2}$. We can also compute that $\frac{ON}{AB} = \frac{OB}{AB} - \frac{NB}{AB} = \frac{1}{5}$; $AB = d$, so $ON = \frac{d}{5}$. Then

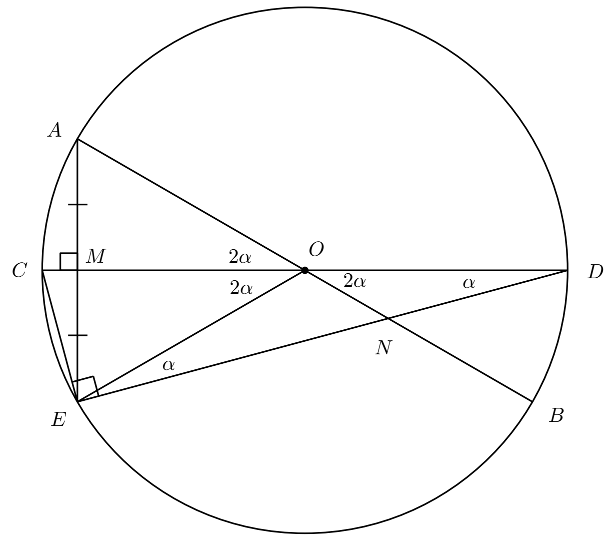
$$\frac{DE}{DN} = \frac{d}{2 \cdot \frac{d}{5}} = \frac{5}{2}.$$

Seeing as $EN + ND = ED$, we know

$$\frac{EN}{ED} + \frac{ND}{ED} = 1 \implies \frac{EN}{ED} = \frac{3}{5},$$

so

$$\frac{EN}{ND} = \frac{\frac{EN}{ED}}{\frac{ND}{ED}} = \mathbf{\frac{3}{2}}.$$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New York City Interscholastic Mathematics League

Senior A Division — Contest Number 3 Solutions — Spring 2019

S19SA13. $\frac{1}{2}$**.** The sum of the 10 rolls is even if and only if there are an even number of odd rolls. Then, we want the probability that 0, 2, 4, 6, 8, or 10 of the rolls are even. Each roll of the die yields an even number with probability $\frac{1}{2}$. Therefore, the probability that k of the rolls are even is $\frac{1}{2^{10}} \cdot \binom{10}{k}$, where we are choosing which of the 10 rolls we would like to be even. So the desired probability is

$$\frac{1}{2^{10}}\left(\binom{10}{0}+\binom{10}{2}+\binom{10}{4}+\binom{10}{6}+\binom{10}{8}+\binom{10}{10}\right).$$

We claim that the binomial sum is 2^9, which we will prove below. Then this expression is just

$$\frac{2^9}{2^{10}} = \mathbf{\frac{1}{2}}.$$

To see that the binomial sum is 2^9, consider the polynomial $p(x) = (x-1)^{10}$; this has x^n coefficient $(-1)^{10-n}\binom{10}{10-n}$. Therefore,

$$0 = p(1) = \sum_{n=0}^{10}(-1)^{10-n}\binom{10}{10-n}.$$

In this sum, the terms corresponding to $10-n$ even are positive and the terms corresponding to $10-n$ odd are negative. But $10-n$ is even if and only if n is even, and $10-n$ is odd if and only if n is odd, so

$$\binom{10}{0}+\binom{10}{2}+\binom{10}{4}+\binom{10}{6}+\binom{10}{8}+\binom{10}{10} = \binom{10}{1}+\binom{10}{3}+\binom{10}{5}+\binom{10}{7}+\binom{10}{9}.$$

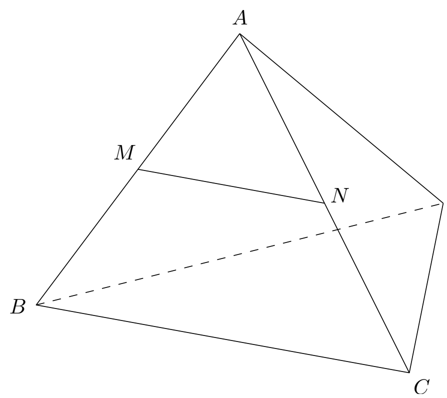
It's well known that $\sum_{n=0}^{10}\binom{10}{n} = 2^{10}$, so the sum over even n is exactly half that, which is 2^9.

S19SA14. $\frac{\sqrt{3}}{2}$**.** We know the octahedron is regular, so it suffices to compute the area of one of the faces and multiply by 8.
Label the vertices of the tetrahedron as in the diagram below: let ΔABC be one of the faces of the tetrahedron, let M be the midpoint of $\overline{AB}$, and let N be the midpoint of $\overline{AC}$. Then $\overline{MN}$ is a midline of ΔABC parallel to $\overline{BC}$, so $MN = \frac{1}{2}BC = \frac{1}{2}$. Similarly, because all the faces of the tetrahedron are equilateral triangles of side length 1, the other sides of the octahedron also have length $\frac{1}{2}$. Therefore, one face of the octahedron is an equilateral triangle with side length $\frac{1}{2}$. The area of an equilateral triangle with side length s with $\frac{s^2\sqrt{3}}{4}$, so the area of one of the octahedron's faces is

$$\frac{\frac{1}{4}\sqrt{3}}{4} = \frac{\sqrt{3}}{16}$$

and the surface area of the octahedron is

$$8 \cdot \frac{\sqrt{3}}{16} = \mathbf{\frac{\sqrt{3}}{2}}.$$



S19SA15. $-\frac{\mathbf{6}}{\mathbf{7}}$**.** By Vieta's formulas, we know that $r+s+t=3$, $rs+st+rt=5$, and $rst=7$. We have

$$\begin{aligned}\frac{r}{s}+\frac{s}{r}+\frac{r}{t}+\frac{t}{r}+\frac{s}{t}+\frac{t}{s} &= \frac{r^2t+s^2t+r^2s+st^2+rs^2+rt^2}{rst}\\ &= \frac{(r+s+t)(rs+rt+st)-3rst}{rst}\\ &= \frac{3\cdot 5-21}{7}\\ &= -\frac{\mathbf{6}}{\mathbf{7}}.\end{aligned}$$

S19SA16. $\frac{\mathbf{16}}{\mathbf{33}}$**.** We can represent red cards as *R*s and blue cards as *B*s, so an ordering of the deck corresponds to a string of 6 *R*s and 6 *B*s. The first pile's top two cards correspond to the first and second letters, while the seventh and eighth letters correspond to the top two cards of the second pile. We will compute how many orderings of the deck exist that satisfy the problem conditions. If two cards are of the same color, then they are either *RR* or *BB*. We have 2 choices also for the colors of two cards of different colors, namely *RB* and *BR*. So there are $2\cdot 2\cdot 2=8$ desired letter arrangements for the first two cards of each pile (2 to choose which pile will have cards of the same color, and 2 each for color arrangements). Note that in these four cards, we will have 3 red and 1 blue or vice versa, so the other 8 cards will consist of 3 of one color and 5 of the other. Then there are $\binom{8}{3}$ ways to arrange them, making $8\cdot\binom{8}{3}$ arrangements total satisfying the problem conditions. There are $\binom{12}{6}$ total possible deck arrangements (we choose where the red cards will be). Therefore, the probability we want is

$$\frac{8\binom{8}{3}}{\binom{12}{6}} = \frac{8\cdot 56}{\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{6!}} = \frac{\mathbf{16}}{\mathbf{33}}.$$

S19SA17. **2560.** Note that

$$k\binom{10}{k} = k\cdot\frac{10!}{k!(10-k)!} = 10\cdot\frac{9!}{(k-1)!(10-k)!} = 10\binom{9}{k-1}.$$

Therefore,

$$\binom{10}{1}+3\binom{10}{3}+5\binom{10}{5}+7\binom{10}{7}+9\binom{10}{9} = 10\left(\binom{9}{0}+\binom{9}{2}+\binom{9}{4}+\binom{9}{6}+\binom{9}{8}\right).$$

By a similar argument as in problem 13, the binomial sum in the parentheses is 2^8, so the sum is $10\cdot 2^8=\mathbf{2560}$.

S19SA18. **30.** Let the polyhedron have F faces, V vertices, and E edges. At every vertex, we know that 5 edges meet, but every edge has two vertices, so there are $\frac{5}{2}V$ edges total and $V = \frac{2}{5}E$. Because all the faces are triangular, we have $3F$ edges, but each edge is counted twice because every edge is part of two faces. Therefore, $E = \frac{3}{2}F$ and $F = \frac{2}{3}E$. Euler's Formula tells us that $V - E + F = 2$. Substituting our expressions for V and F in terms of E, we get

$$\frac{2}{5}E - E + \frac{2}{3}E = 2 \iff E = \mathbf{30}.$$

Senior A Division

S19SA19. **941.** Every minute, the difference between Mario's spoken number and Mathew's spoken number decreases by $6 + 7 = 13$. When the difference is 0, they will say the same number; this occurs after $\frac{2019-17}{13} = 154$ minutes. Note that because the difference between their numbers is strictly decreasing, they will never say the same number again. The number they speak will be $17 + 6 \cdot 154 = \mathbf{941}$, and we can confirm that $2019 - 7 \cdot 154 = 941$ also.

S19SA20. $\mathbf{\frac{27-4\pi\sqrt{3}}{36}}$**.** WLOG, let the socket be located in the lower right corner of the room as in the diagram. Hana cannot vacuum points which are more than 10 away from this corner, which are points inside the room but outside the circle of radius 10 centered at the lower right corner; this is represented by the shaded area on the diagram. Let the corners of the rectangle be A, E, D, and C in clockwise order from the top left, as labeled in the diagram. Then the vacuum cleaner is plugged in at corner D. Let the circle of radius 10 centered at D intersect side AC at B.

To compute the area of the shaded region, we will compute the areas of ΔBCD and the circle sector BDE. Note that ΔBCD is a right triangle with right angle at C and $BD = 10$, $CD = 5\sqrt{3}$ so $BC = 5$ and ΔBCD is a 30-60-90 right triangle. Thus $m\angle BDC = 30°$ and $m\angle BDE = 60°$. So we can compute the area of ΔBDC as

$$\frac{5 \cdot 5\sqrt{3}}{2} = \frac{25\sqrt{3}}{2}$$

and the area of the sector BDE as

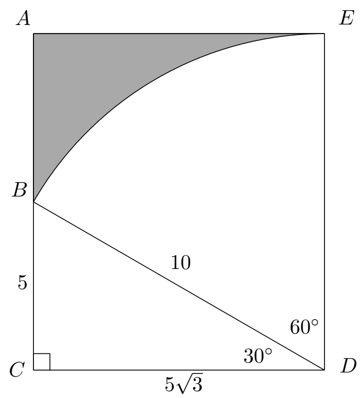
$$\frac{60°}{360°}\pi \cdot 10^2 = \frac{50\pi}{3}.$$

The total area of Hana's room is $10 \cdot 5\sqrt{3} = 50\sqrt{3}$, so the area of the shaded region is then

$$50\sqrt{3} - \frac{50\pi}{3} - \frac{25\sqrt{3}}{2}$$

and the fraction of the room Hana cannot vacuum is

$$\frac{50\sqrt{3} - \frac{50\pi}{3} - \frac{25\sqrt{3}}{2}}{50\sqrt{3}} = \mathbf{\frac{27 - 4\pi\sqrt{3}}{36}}.$$



S19SA21. **30.** Let the expected value of a penny be p. Then the expected value of a penny and a nickel is $p + 5$, the expected value of a penny and a dime is $p + 10$, and the expected value of a nickel and a dime is 15. Therefore,

$$p = \frac{1}{3}(p+5) + \frac{1}{3}(p+10) + \frac{1}{3} \cdot 15.$$

Simplifying, we get

$$\frac{1}{3}p = \frac{5}{3} + \frac{10}{3} + 5,$$

so $p = \mathbf{30}$.

S19SA22. $\mathbf{\frac{2}{9}}$**.** Recall that divisibility by 4 is determined entirely by whether or not the last 2 digits of a number are divisible by 4; and that all 2-digit numbers with distinct nonzero digits are equally likely to appear in the last 2 digits of a number Kadir makes, because if we fix the last two digits there are always 7! numbers with those last two digits. Therefore, it suffices to compute the fraction of 2-digit numbers with distinct nonzero digits which are divisible by 4. There are $9 \cdot 8 = 72$ such two-digit numbers in total (9 for the first digit and 8 for the second digit which cannot be the same as the first). To compute how many are divisible by 4, note that if we fix the first digit, we always have two choices for the second digit unless the first and second digits are the same (in the case of 44 and 88). If the first digit is odd, the possible second digits are 2 and 6; if the first digit is even, the possible second digits are 4 and 8 because 0 is not a valid digit. Thus there are $9 \cdot 2 - 2 = 16$ possible 2-digit numbers of this form divisible by 4. Then the probability that Kadir's number is divisible by 4 is

$$\frac{16}{72} = \mathbf{\frac{2}{9}}.$$

S19SA23. **211.** Both 3^{35} and 2^{35} are 7th powers, so we can factor

$$3^{35} - 2^{35} = (3^5)^7 - (2^5)^7 = (3^5 - 2^5)(3^{30} + 3^{25}2^5 + \cdots + 2^{30}).$$

Notice that $3^5 - 2^5 = 211$, which is prime, so the 3-digit prime factor we are looking for is **211**.

S19SA24. **75.** First, note that all faces of the tetrahedron are pairwise adjacent, so the orientation of the tetrahedron is determined entirely by any two faces and the edge they share. We split into cases based on how many colors we use in the coloring.

If there is only one color, then there is exactly one way to color the tetrahedron by setting all faces to be the same color.

If there are two colors, say C and M, then either each color is on 2 faces, or there are 3 faces of one color and 1 of another. In the first case, the 2 faces of color C will determine the orientation of the tetrahedron entirely, and there is only one way to color the remaining two faces (as both M), so there is one coloring. In the second case, we first choose the color on only one face; say it's C. Note that we can always rotate the tetrahedron so the face of color C is at the bottom. So a coloring of the tetrahedron in this case corresponds to a choice of the color that will be on only one face, and there are 2 colorings. In total, then, there are 3 colorings of the regular tetrahedron using 2 colors.

If there are 3 colors, then there are 2 faces of one color and 1 face each for the other two colors. The two faces of different colors will determine the orientation of the tetrahedron entirely, and the remaining two faces will be the same color, so a coloring of the tetrahedron in this case corresponds to choosing which color will go on two faces: 3 colorings total.

If there are 4 colors, we must use each color on exactly one face. Suppose our colors are C, Y, M, and K. Then the two faces of color C and Y determine the orientation of the tetrahedron, and then we must choose which of the remaining faces will be color M and which will be color K. These two colorings cannot be obtained by a rotation, since if we fix the positions of the C and Y faces, the M and K faces will be in different positions in each case. So there is a total of 2 colorings if we use 4 colors.

If we use n colors for $1 \leq n \leq 4$, there are $\binom{5}{n}$ ways to choose which colors to use. So in all, the total number of colorings is

$$1\binom{5}{1} + 3\binom{5}{2} + 3\binom{5}{3} + 2\binom{5}{4} = \mathbf{75}.$$

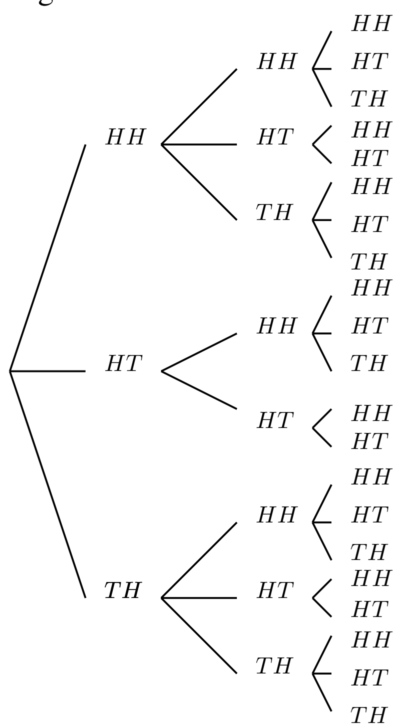
New York City Interscholastic Mathematics League

Senior A Division Contest Number 5 Solutions Spring 2019

S19SA25. $\mathbf{\frac{21}{64}}$. Represent the heads tosses as H and the tails as T; then a sequence of 6 tosses corresponds to a string of length 6 containing only H and T. There are 64 possible such strings, and the coin is fair so all are equally likely. The condition in the problem is equivalent to TT never occurring in the string; call a string with no TTs *valid*.

Method 1

We can brute force this by forming a tree to calculate the total number of favorable outcomes.



Adding these up, we have 21 favorable outcomes.

Method 2

Alternately, we can write a recurrence relation. Let H_n represent the number of valid strings of length n ending in H and T_n represent the number of valid strings of length n ending in T. We want $H_6 + T_6$.

We know $H_1 = T_1 = 1$ because both H and T are valid. Given a valid string of length n, the first $n-1$ tosses still form a valid string, so it ends either in H or T; and our length n valid string must end in H or T. Therefore, we can construct all valid strings of length n by appending H or T to valid strings of length $n-1$. We can append either H or T to strings ending in H, but only H to strings ending in T. Therefore, we have $H_n = H_{n-1} + T_{n-1}$ and $T_n = H_{n-1}$. It is easy to see that from $H_1 = T_1 = 1$ and this recurrence relation, we must have $H_n = F_{n+1}$ and $T_n = F_n$, and $H_n + T_n = F_{n+2}$ (where F_n is the nth Fibonacci number and we set $F_1 = F_2 = 1$). So $H_6 + T_6 = F_8 = 21$.

Therefore, the probability that every pair of consecutive tosses has at least one heads is $\mathbf{\frac{21}{64}}$.

S19SA26. $\mathbf{24+2\sqrt{29}}$. Because $AB = CD$, angles $\angle DAC$ and $\angle ACB$ subtend congruent chords and $m\angle DAC = m\angle ACB$. Therefore, $\overline{AB} \| \overline{CD}$ and $ABCD$ is an isosceles trapezoid. We can compute its height h because we know the area and the length of the two bases, so

$$\frac{1}{2}(10+14)h = 60 \iff h = 5.$$

As in the diagram, let M and N be the bases of altitudes of $ABCD$ from B and C to $\overline{AD}$. Therefore $m\angle BMA = m\angle CND = 90°$, $AB = CD$ and $BM = CN = 5$, so by HL congruence, $\Delta ABM \cong \Delta DCN$. Therefore, $AM = ND$. Moreover, $BCNM$ is a rectangle (all of its angles are right) so $BC = MN$. Then

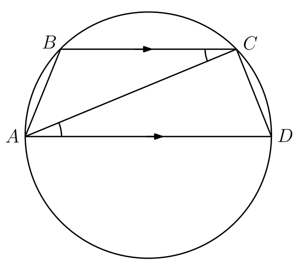
$$14 = AD = AM + MN + ND = 2AM + BC = 2AM + 10 \iff AM = 2.$$

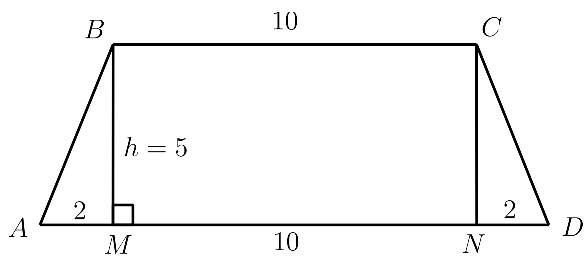
By the Pythagorean theorem,

$$AB = \sqrt{AM^2 + BM^2} = \sqrt{29}$$

so the perimeter of $ABCD$ is

$$AB + BC + CD + AD = \mathbf{24} + \mathbf{2\sqrt{29}}.$$





S19SA27. **21.** Note that the first $\lceil \frac{n}{2} \rceil$ digits of an n-digit palindrome determine the palindrome and are all independent, where $\lceil n \rceil$, the ceiling of n, is the smallest integer at least n. For n even, we can write any n-digit palindrome as $\overline{AA^*}$ where A has $\frac{n}{2}$ digits and A^* is the reverse of A; and for n odd, we can write any n-digit palindrome as $\overline{AxA^*}$ where A has $\frac{n-1}{2}$ digits and x is a digit. Thus when n is even, we pick an A; and when n is odd, we pick an A and an additional digit x. So in all, we pick $\lceil \frac{n}{2} \rceil$ digits, all of which are independent, to determine the palindrome, and there are

$$2^{\lceil \frac{n}{2} \rceil}$$

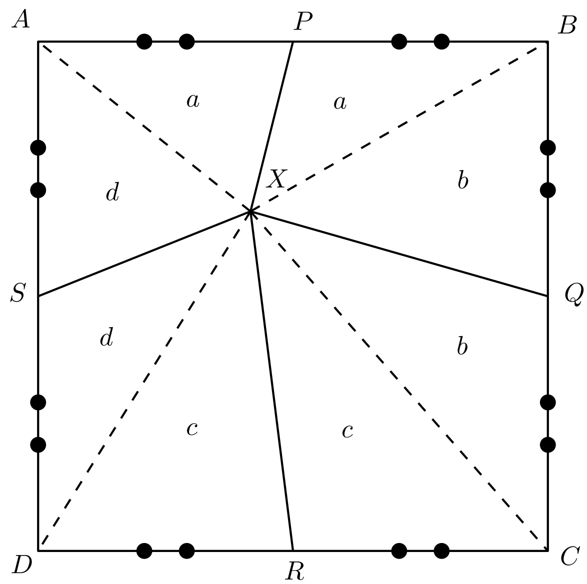
n-digit palindromes containing only 1 and 2 as digits. This is at least 2019 when $\lceil \frac{n}{2} \rceil \geq 11$, which is equivalent to $n \geq 21$ for integral n. So the smallest possible n is **21**.

S19SA28. $\mathbf{2\sqrt{3} + 2\sqrt{5}}$. Draw $\overline{XA}$, $\overline{XB}$, $\overline{XC}$, and $\overline{XD}$ as in the diagram. Now note that we have 4 pairs of triangles: ΔAPX and ΔBPX; ΔBQX and ΔCQX; ΔCRX and ΔDRX; and ΔDSX and ΔASX. Each pair shares an altitude, and because P, Q, R, S are midpoints of their respective sides, they also have bases of equal length. Therefore, the two triangles of each pair have equal areas. Let $[K]$ represent the area of a polygon K. Then

$$\begin{aligned}[APXS] + [CRXQ] &= [APX] + [ASX] + [CRX] + [CQX] \\ &= [BPX] + [DSX] + [DRX] + [BQX] \\ &= [BQXP] + [DSXR]\end{aligned}$$

and

$$[ABCD] = [APXS] + [CRXQ] + [BQXP] + [DSXR] = 2(\sqrt{3} + \sqrt{5}) = \mathbf{2\sqrt{3} + 2\sqrt{5}}.$$



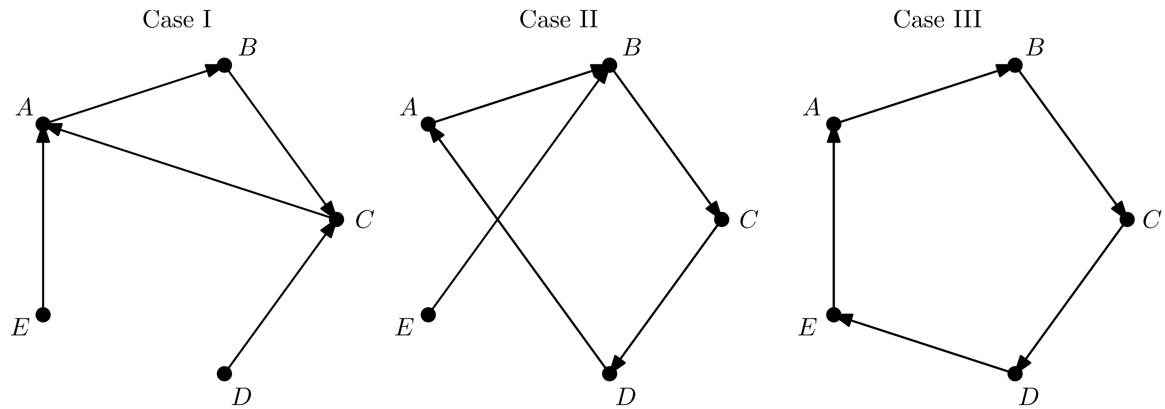
S19SA29. **5.** \$12 is 1200 cents, so if we wish to pay with only dimes, we'll use 120 dimes; this is the fewest number of coins we could use if we can only pay in nickels and dimes. The greatest number of coins we could use is 240, if we only used nickels. Note that we can replace a dime with two nickels; after such an exchange, the number of coins increases by 1 while the total value of the coins remains constant. Therefore, we can attain every number of coins between 120 and 240 inclusive, and no other number of coins outside those bounds. We then want to compute the number of squares between 120 and 240, inclusive; the smallest square in this range is $11^2 = 121$ and the largest is $15^2 = 225$. Therefore, there are $15 - 11 + 1 = \mathbf{5}$ ways to break \$12 into nickels and dimes so that the total number of coins is a perfect square.

S19SA30. $\mathbf{\frac{111}{256}}$. Label the players A, B, C, D, E. Now we do casework to compute the number of ways the shootout could occur by looking at cycles. There are no 2-cycles by assumption, but there must be a cycle. To see this, suppose we have no cycles; then WLOG, suppose that A shoots B and B shoots C (B cannot shoot A because no two players can shoot each other). C cannot shoot A or B because we have no cycles, so WLOG C shoots D; this requires D to shoot E. Now regardless of whom E shoots, E will create a cycle, contradiction.

Therefore, there is either a 3-cycle, a 4-cycle, or a 5-cycle. Note that there can be exactly one cycle with length at least 3 because each player can be part of at most one cycle (otherwise someone will have to shoot two people), and there are only 5 people. See the diagram for an example of each case.

- Suppose there is a 3-cycle: there are $\binom{5}{3} \cdot 2$ ways to choose such a cycle because we first choose the 3 members of the cycle, then we choose a direction for the cycle (equivalent to specifying whom the first member shoots). WLOG let A, B, C form the 3-cycle. D and E each have 4 choices for whom to shoot but cannot shoot each other, so there are $4 \cdot 4 - 1$ ways to specify whom D and E shoot. So there are
$$\binom{5}{3} \cdot 2 \cdot (4 \cdot 4 - 1) = 300$$
ways for this case to occur.
- Suppose there is a 4-cycle: there are $\binom{5}{4} \cdot 3!$ ways to choose such a cycle (we first choose the 4 members of our cycle, select an arbitrary "start" to the cycle, and order the 3 members to orient the cycle). The last person not in the cycle can shoot any of the other 4 players. So there are
$$\binom{5}{4} \cdot 3! \cdot 4 = 120$$
ways for this case to occur.
- There is only one possible 5-cycle, and there are 4! ways to choose it (we arbitrarily choose a "start" to the cycle and order the rest of the players).

In total, we have $300 + 120 + 24 = 444$ ways for the shootout to occur as specified. There are $4^5 = 1024$ total ways for the shootout because each person can shoot any of the other four players. So the probability that the shootout will occur with no two people shooting each other is
$$\frac{444}{1024} = \mathbf{\frac{111}{256}}.$$


Case I
A
B
C
D
E
Case II
A
B
C
D
E
Case III
A
B
C
D
E