NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Junior Division   CONTEST NUMBER 1

PART I   FALL 2018   CONTEST 1   TIME: 10 MINUTES

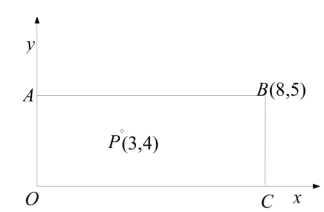
F18J01 Compute all values of a such that the points $P\ (0,-5), Q\ (a,-3)$ and $R\ (3,a)$ are collinear.

F18J02 Compute all points that are equidistant from both $P\ (0,2)$ and the line $y = -2$, and that are also 3 units from the origin.

PART II   FALL 2018   CONTEST 1   TIME: 10 MINUTES

F18J03 Let n be a positive integer such that there are 503 integers between n^2 and $(n+2)^2$. Compute n.

F18J04 Given $ABCO$ be a rectangle with $A = (0,5), B = (8,5)$, $C = (8,0)$, $O = (0,0)$. Compute the slope of the unique line that passes through $P(3,4)$, crosses the horizontal sides $\overline{AB}$ and $\overline{OC}$ and divides the rectangle into two regions of equal area.



PART III   FALL 2018   CONTEST 1   TIME: 10 MINUTES

F18J05 Compute all values of x, such that $0° \leq x < 360°$, which satisfy $\sin(60° + x) = 2\sin x$. Specify the answer in degrees.

F18J06 Compute the number of pairs (x, z) of positive integers such that $x^2 + 144^3 = z^2$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Junior Division   CONTEST NUMBER 2

PART I ***FALL 2018*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F18J07 Convert the base 9 repeating decimal $0.\overline{5}_9$ to a base 4 decimal, e.g. 0.123_4.

F18J08 The quadratic equation $x^2 - 2x - 2 = 0$ has two roots, r and s. Compute $r^4 + 3(sr^3 + rs^3) + s^4$.

PART II ***FALL 2018*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F18J09 Let $a = \log 2$, $b = \log 3$, and $f(x) = 4^x + 2^x$. In terms of only a and b and not f, compute $f^{-1}(156) - 2$.

F18J10 Compute all complex and real roots of the equation

$$x^4 - 4x^2 - 16x + 32 = 0.$$

PART III ***FALL 2018*** ***CONTEST 2*** ***TIME: 10 MINUTES***

F18J11 Find all 4-digit whole numbers of the form $\underline{x01y}$ which are divisible by 28.

F18J12 Compute the number of 3-digit numbers such that the first and last digits are greater than 4, the last digit is odd, and at least two of the digits are the same.

New York City Interscholastic Mathematics League

Junior Division   Contest Number 3   Fall 2018

PART I   FALL 2018   CONTEST 3   TIME: 10 MINUTES

F18J13 Compute the remainder when $1^3 + 2^3 + \cdots + 6^3 + 7^3$ is divided by 7.

F18J14 Compute $a + b$ if a and b are positive integers such that $\max(a, b) = (a - b)^3$ and $\min(a, b) = 99\gcd(a, b)$.

PART II   FALL 2018   CONTEST 3   TIME: 10 MINUTES

F18J15 What is the probability that, when 3 fair dice are rolled, the sum of the numbers shown is divisible by 6?

F18J16 $\triangle ABC$ has incenter I and is circumscribed by circle ω. Point D is on the midpoint of minor arc $\overset{\frown}{BC}$. If $m\angle CID = 50^\circ$ and $m\angle BAD < 45^\circ$, compute the measure of $\angle IBC$.

PART III   FALL 2018   CONTEST 3   TIME: 10 MINUTES

F18J17 If each point of the circle $x^2 + y^2 = 9$ is reflected across the point $P(3, 2)$, the image satisfies the equation $x^2 + y^2 + Dx + Ey + F = 0$. Compute the ordered triple of real numbers (D, E, F).

F18J18 Isosceles triangle ABC has vertex A. Circle ω has diameter $\overline{BC}$ and intersects $\overline{AC}$ at D. Point X is on ω such that $DX \perp BC$ and DX intersects BC at E. Point F is on $\overline{AB}$ such that $\overline{DF} \perp \overline{AB}$. If $DC = 4$ and $EX = 2\sqrt{3}$, compute the ratio $\frac{DE}{DF}$.

New York City Interscholastic Mathematics League

Junior Division   Contest Number 1   Spring 2019

PART I **Spring 2019** **CONTEST 1** **TIME: 10 MINUTES**

S19J01 Compute the greatest common factor of 77^{777} and 777^{77}. Express your answer in the form a^b, where a and b are positive integers, with a being prime.

S19J02 The polynomial $x^2 - 6x + 3$ has two roots, r and s. Compute the value of

$$\frac{r+1}{s} + \frac{s+1}{r}.$$

PART II **Spring 2019** **CONTEST 1** **TIME: 10 MINUTES**

S19J03 A bag has 3 red marbles and 8 green marbles. If 2 marbles are picked uniformly at random without replacement, what is the probability that the marbles are of different colors? Express your answer as a fraction in simplest form.

S19J04 Rhombus $ABCD$ has $AB = 13$ and $BD = 10$. Compute the radius of the circumcircle of $\triangle ABC$. Express your answer as a fraction in simplest form.

PART III **Spring 2019** **CONTEST 1** **TIME: 10 MINUTES**

S19J05 Find all real solutions to $x^3 + 3x^2 + 3x + 2 = 0$.

S19J06 6 people are sitting at a round table. They each roll a fair, six-sided die with faces labeled 1 through 6. How many possible outcomes are there such that no two people that sit directly next to each other roll the same number?

New York City Interscholastic Mathematics League

Junior Division   Contest Number 2   Spring 2019

PART I   **Spring 2019**   **CONTEST 2**   **TIME: 10 MINUTES**

S19J07 For some positive integer $b > 3$, $21_4 \cdot 13_5 = 132_b$. Compute the base-10 representation of 11_b. Note: x_b denotes x in base b. For example, $101_2 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 5$.

S19J08 Square $ABCD$ has sidelength 3. X is on $\overline{AD}$ such that $AX : XD = 2 : 1$, and Y is on $\overline{BX}$ such that $BY : YX = 2 : 1$. If Z is the midpoint of $\overline{CD}$, compute the area of $\triangle CYZ$. Express your answer as a fraction in simplest form.

PART II   **Spring 2019**   **CONTEST 2**   **TIME: 10 MINUTES**

S19J09 Compute all integer solutions x to $\lfloor \frac{5x}{3} \rfloor - 2 = x$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. For example, $\lfloor \frac{7}{3} \rfloor = 2$.

S19J10 If two numbers m and n satisfy $m + n = 6$ and $m^3 + n^3 = 504$, compute the value of $\frac{1}{m^2} + \frac{1}{n^2}$. Express your answer as a fraction in simplest form.

PART III   **Spring 2019**   **CONTEST 2**   **TIME: 10 MINUTES**

S19J11 The integers $1, 2, 3, \ldots, 9$ are placed in sequence in a random order, with each possible permutation equally likely. What is the probability that 1 and 7 appear consecutively (in either order) in the sequence? Express your answer as a fraction in simplest form.

S19J12 Compute the number of ordered pairs of positive integers (a, b) such that $\gcd(a, b) + \text{lcm}(a, b) = 421$. Note that $\gcd(a, b)$ denotes the greatest common factor of a and b, and $\text{lcm}(a, b)$ denotes the least common multiple of a and b.

New York City Interscholastic Mathematics League

Junior Division   Contest Number 3   Spring 2019

PART I   **Spring 2019**   **CONTEST 3**   **TIME: 10 MINUTES**

S19J13 Compute the value of $\frac{1!}{3!}+\frac{2!}{4!}+\frac{3!}{5!}+\cdots+\frac{7!}{9!}+\frac{8!}{10!}$. Express your answer as a fraction in simplest form.

S19J14 If $p^3 - q^3$ is prime for prime numbers p and q, find the ordered pair (p, q).

PART II   **Spring 2019**   **CONTEST 3**   **TIME: 10 MINUTES**

S19J15 Compute the product of all real solutions x to $|x|^2 + 2|x| - 8 = 0$.

S19J16 Square $ABCD$ has sidelength 2. M is the midpoint of $\overline{CD}$, and E is on $\overline{BC}$ such that $\overline{AE}$ bisects $\angle BAM$. Compute the length of $\overline{BE}$. Express your answer in simplest radical form.

PART III   **Spring 2019**   **CONTEST 3**   **TIME: 10 MINUTES**

S19J17 How many 4 digit numbers made up of only the digits 1 and 7 contain no more than three 1s and no more than two 7s?

S19J18 For positive integers x, let $f(x)$ be the result of moving the last digit of x to the front of x. For example, $f(1234) = 4123$. Compute the smallest positive integer a such that $a = \frac{2}{3}f(a)$.

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

JUNIOR DIVISION

CONTEST NUMBER 1 SOLUTIONS

F18J01. $\boldsymbol{a = -6, 1}$.

The three points being collinear implies that the slopes of the line segments PQ, PR and QR are all equal. Equating the first two gives $\frac{2}{a} = \frac{a+5}{3}$, which gives the quadratic equation $a^2 + 5a - 6 = 0 \to (a+6)(a-1) = 0 \to \boldsymbol{a = -6, 1}$.

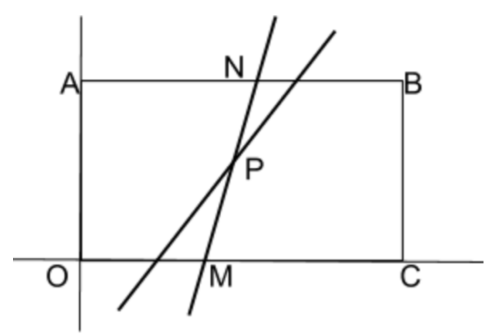
F18J02. $\left(\boldsymbol{\pm 2\sqrt{2}, 1}\right)$.

The locus of points equidistant from $P\ (2, 0)$ and the line $y = -2$ is the parabola $x^2 = 8y$; the locus of points 3 units from the origin is the circle $x^2 + y^2 = 9$.
From here, plug the parabola equation into the circle equation and solve for y:
$8y + y^2 = 9 \Longrightarrow y^2 + 8y - 9 = 0 \Longrightarrow (y+9)(y-1) = 0$, which yields $y = -9, 1$. Since the point must be distance exactly 3, $y = -9$ is an extraneous solution. Therefore, the points are $\left(\boldsymbol{\pm 2\sqrt{2}, 1}\right)$.

F18J03. $\boldsymbol{n = 125}$.

Consider the set $\{n^2, n^2+1, n^2+2, n^2+3, \ldots, n^2+k, (n+2)^2 = n^2+4n+4\}$.
In this set, there are $(n+2)^2 - (n^2) - 1 = 4n + 3$ integers, not including n^2 and $(n+2)^2$ themselves.
$\to k = 4n + 3 = 503 \to \boldsymbol{n = 125}$.

F18J04. $\boldsymbol{m = \frac{-3}{2}}$.



Consider a line of arbitrary slope that goes through P; its equation is $y = m(x-3) + 4$.
The line crosses the horizontal line $y = 0$ when $x = 3 - \frac{4}{m}$, and crosses $y = 5$ when $x = 3 + \frac{1}{m}$.
Let these points be M and N respectively, keeping in mind that these intercepts must satisfy $0 \leq x \leq 8$.
The area of the rectangle is 40, so we set the area of the trapezoid $OMNA$ equal to 20:
$20 = [OMNA] = \left(\frac{1}{2} * 5\right)\left(3 - \frac{4}{m} + 3 + \frac{1}{m}\right) \to 6 - \frac{3}{m} = 8 \to \boldsymbol{m = \frac{-3}{2}}$.
To check if this works, we compute the x- coordinates of M:
$x = 3 - \frac{4}{m} = 3 + \frac{8}{3} = 5\frac{2}{3}$. This falls in the range $0 \leq x \leq 8$, so it is valid.

F18J05. $\boldsymbol{x = 30°,\ 210°}$.

$\sin(60° + x) = 2 sinx \to sin60° cosx + \cos 60° \sin x = 2\sin x$

$$\to \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = 2\sin x \to \sqrt{3}\cos x + \sin x = 4\sin x \to \sqrt{3}\cos x = 3\sin x$$

$\to \tan x = \frac{\sqrt{3}}{3} \to x = \boldsymbol{30°,\ 210°}$.

F18J06. **38.**

The equation $x^2 + 144^3 = z^2$can be rewritten as $z^2 - x^2 = 144^3 \Rightarrow$
$(z + x)(z - x) = 144^3$.

Since x and z are both integers, solutions can be obtained by factoring 144^3into a product of two integers $a \cdot b$, and assigning a to $z + x$ and b to $z - x$.

We know $144^3 = 2^{12} * 3^6$, so there are $(6 + 1)(12 + 1) = 91$ possible factorizations.

An assignment of this type gives $z = \frac{a+b}{2}$ and $x = \frac{a-b}{2}$.

x and z are also positive integers, so such a factorization must satisfy $a > b$ and $2|(a + b)$. Note, in particular, that $a \neq b$, so we first take away one possibility: $(12^3)(12^3)$, leaving 90 possibilities.

Of these, exactly half of them satisfy $a > b$: note the symmetry between the pairs $(a, b)(b, a)$. This leaves $\frac{90}{2} = 45$ possibilities.

Finally, observe that the pairs for which only one of the two values is odd violate the condition $2|(a + b)$. The only such pairs are those where
$b = 3^0, 3^1, 3^2, \dots, 3^6$, a total of 7 pairs. Taking these away, we are left with
38 pairs.

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

JUNIOR DIVISION

CONTEST NUMBER 2 SOLUTIONS

F18J07. $\mathbf{0.22_4}$.

Base 9 to Base 10:

Easy way: $0.\overline{5}_9 = \frac{5}{9-1} = \frac{5}{8}$ (just as we recognize that is base 10, $\frac{x}{9} = 0.\overline{x}$)

or let n denote the required fraction in base 10.

base 10 base 9

$$9n = 5.\overline{5} \quad \rightarrow 8n_{(10)} = 5_{(9)} \rightarrow n_{(10)} = \frac{5_{(9)}}{8_{(10)}}$$

$$n = 0.\overline{5}$$

But 5 in base 9 and 5 in base 10 are equivalent.

Now we must convert 5/8 to a repeating decimal in base 4.

$\frac{5}{8} = \frac{1}{2} + \frac{1}{8} = \frac{2}{4} + \frac{2}{16} = 2(4^{-1}) = 2(4^{-2}) = \mathbf{0.22_{(4)}}$

F18J08. **8.**

To calculate the roots, use the quadratic formula.

$$r = \frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-(-2)+\sqrt{(-2)^2-4(1)(-2)}}{2(1)} = \frac{2+\sqrt{12}}{2} = \frac{2+2\sqrt{3}}{2} = 1+\sqrt{3}.$$

$$s = \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-(-2)-\sqrt{(-2)^2-4(1)(-2)}}{2(1)} = \frac{2-\sqrt{12}}{2} = \frac{2-2\sqrt{3}}{2} = 1-\sqrt{3}.$$

Substituting $1+\sqrt{3}$ for r and $1-\sqrt{3}$ for s, you obtain

$r^4 + 3(sr^3 + rs^3) + s^4 = \mathbf{8.}$

F18J09. $\boldsymbol{\frac{b}{a}}$.

The equation $f^{-1}(156) = x$ is equivalent to $156 = f(x) = 4^x + 2^x$. Substitute $y = 2^x$ to obtain a quadratic equation in y.

$$156 = y^2 + y \Rightarrow (y-12)(y+13) = 0.$$

This gives $y = 12$ as the only positive solution; we reject -13 because $f(x)$ is always positive. Let $12 = 2^x$ and take logs of both sides. We obtain $x = \frac{\log 12}{\log 2} = \frac{\log 4 + \log 3}{\log 2} = 2 + \boldsymbol{\frac{b}{a}}$.

Therefore, $f^{-1}(156) - 2$ simplifies to $\boldsymbol{\frac{b}{a}}$.

F18J10. $\mathbf{2, -2 \pm 2}\boldsymbol{i}$**.**

$x^4 - 4x^2 - 16x + 32 = 0$

$\Rightarrow x^2(x^2 - 4) - 16(x - 2) = 0$

$\Rightarrow x^2(x - 2)(x + 2) - 16(x - 2) = 0$

$\Rightarrow x(x - 2)(x^2(x + 2) - 16) = 0$

$\Rightarrow (x - 2)(x^3 + 2x^2 - 16) = 0$

Note that since $x = 2$ makes the second factor zero as well, so we can pull out another factor of $(x - 2)$, which factors into $(x - 2)(x^2 + 4x + 8)$. Therefore, $x = 2$ is a double root. To find the other roots, solve $(x^2 + 4x + 8) = 0$. It helps to complete the square.

$(x^2 + 4x + 4) = -4$

$\Rightarrow (x + 2)^2 = -4$

$\Rightarrow x = -2 \pm 2i.$

In total, the solutions are $\mathbf{2, -2 \pm 2}\boldsymbol{i}$**.**

F18J11. $\mathbf{2016, 5012, 9016}$**.**

The four-digit number $\underline{x01y}$ represents the integer $1000x + 10 + y$.

Since $28 = 4 \cdot 7$ and $4|1000$, 4 must divide $(10 + y)$. Hence $y = 2$ or 6.

In either of these cases, $1000x + 10 + y$ must be $0 \; mod \; 7$.

Case 1: $(y = 2)$. Reduced mod 7, we have $6x + 5 \equiv 0(mod\ 7)$, which gives $x = 5$ as the only one-digit solution.

Case 2: $(y = 6)$. Reduced mod 7, we have $6x + 2 \equiv 0(mod\ 7)$, which gives $x = 2,9$.

In total, there are three solutions: $\mathbf{5012, 9016}$ and $\mathbf{2016}$**.**

F18J12. **54.**

We proceed by counting three-digit numbers satisfying the first two conditions, then taking away all of those that fail the third.

There are five choices $\{5, 6, 7, 8, 9\}$ for the first digit, ten choices $\{0, \dots ,10\}$ for the second digit and three choices $\{5, 7, 9\}$ for the last digit. In total, there are $5 \cdot 10 \cdot 3 = 150$ three-digit numbers that satisfy just the first two conditions.

Notice that the opposite of "at least two of the digits are equal" is the statement "all three digits are distinct". Imagine choosing the digits in this order: ones, hundredths, tenths.

There are three choices for the ones digit, four remaining choices for the hundredths digit, then eight remaining choices for the tens digit. In total, there are 96 such three-digit numbers. Therefore, there are $150 = 96 = 54$ three digit numbers that satisfy all three conditions.

NEW YORK CITY INTERSCHOLASTIC MATH LEAGUE

JUNIOR DIVISION CONTEST NUMBER 2 SOLUTIONS

F18J07. $\mathbf{0.22_4}$.

Base 9 to Base 10:

Easy way: $0.\bar{5}_9 = \frac{5}{9-1} = \frac{5}{8}$ (just as we recognize that is base 10, $\frac{x}{9} = 0.\bar{x}$)

or let n denote the required fraction in base 10.

base 10 base 9

$$9n = 5.\bar{5} \quad \rightarrow 8n_{(10)} = 5_{(9)} \rightarrow n_{(10)} = \frac{5_{(9)}}{8_{(10)}}$$

$$n = 0.\bar{5}$$

But 5 in base 9 and 5 in base 10 are equivalent.

Now we must convert 5/8 to a repeating decimal in base 4.

$\frac{5}{8} = \frac{1}{2} + \frac{1}{8} = \frac{2}{4} + \frac{2}{16} = 2(4^{-1}) = 2(4^{-2}) = \mathbf{0.22}_{(4)}$

F18J08. **8.**

To calculate the roots, use the quadratic formula.

$$r = \frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-(-2)+\sqrt{(-2)^2-4(1)(-2)}}{2(1)} = \frac{2+\sqrt{12}}{2} = \frac{2+2\sqrt{3}}{2} = 1+\sqrt{3}.$$

$$s = \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-(-2)-\sqrt{(-2)^2-4(1)(-2)}}{2(1)} = \frac{2-\sqrt{12}}{2} = \frac{2-2\sqrt{3}}{2} = 1-\sqrt{3}.$$

We can alter $r^4 + 3(sr^3 + rs^3) + s^4$ as follows:

$r^4 + 3(sr^3 + rs^3) + s^4$

$= r^4 + 4sr^3 + 6s^2r^2 + 4rs^3 + s^4 - sr^3 - rs^3 - 6s^2r^2$

$= (r+s)^4 - rs(r^2+s^2) - 6(rs)^2$

By Vietes, $r + s = 2, rs = -2$.

$(r+s)^2 = r^2 + s^2 + 2rs$

$r^2 + s^2 = (r+s)^2 - 2rs = 4 - 2(-2) = 8.$

Hence, the whole expression will yield,

$(r+s)^4 - rs(r^2+s^2) - 6(rs)^2 = (2)^4 - (-2)(8) - 6(-2)^2 = 16 + 16 - 24 = \mathbf{8}.$

F18J09. $\boldsymbol{\frac{b}{a}}$.

The equation $f^{-1}(156) = x$ is equivalent to $156 = f(x) = 4^x + 2^x$. Substitute $y = 2^x$ to obtain a quadratic equation in y.

$$156 = y^2 + y \Rightarrow (y-12)(y+13) = 0.$$

This gives $y = 12$ as the only positive solution; we reject -13 because $f(x)$ is always positive. Let $12 = 2^x$ and take logs of both sides. We obtain $x = \frac{\log 12}{\log 2} =$

$\frac{\log 4+\log 3}{\log 2} = 2 + \frac{\boldsymbol{b}}{\boldsymbol{a}}$.

Therefore, $f^{-1}(156) - 2$ simplifies to $\frac{\boldsymbol{b}}{\boldsymbol{a}}$.

F18J10. $\mathbf{2, -2 \pm 2}\boldsymbol{i}$.

$x^4 - 4x^2 - 16x + 32 = 0$

$\Longrightarrow x^2(x^2 - 4) - 16(x - 2) = 0$

$\Longrightarrow x^2(x - 2)(x + 2) - 16(x - 2) = 0$

$\Longrightarrow x(x - 2)(x^2(x + 2) - 16) = 0$

$\Longrightarrow (x - 2)(x^3 + 2x^2 - 16) = 0$

Note that since $x = 2$ makes the second factor zero as well, so we can pull out another factor of $(x - 2)$, which factors into $(x - 2)(x^2 + 4x + 8)$. Therefore, $x = 2$ is a double root. To find the other roots, solve $(x^2 + 4x + 8) = 0$. It helps to complete the square.

$(x^2 + 4x + 4) = -4$

$\Longrightarrow (x + 2)^2 = -4$

$\Longrightarrow x = -2 \pm 2i$.

In total, the solutions are $\mathbf{2, -2 \pm 2}\boldsymbol{i}$.

F18J11. $\mathbf{2016, 5012, 9016}$.

The four-digit number $\underline{x01y}$ represents the integer $1000x + 10 + y$.

Since $28 = 4 \cdot 7$ and $4|1000$, 4 must divide $(10 + y)$. Hence $y = 2$ or 6.

In either of these cases, $1000x + 10 + y$ must be $0 \; mod \; 7$.

Case 1: $(y = 2)$. Reduced mod 7, we have $6x + 5 \equiv 0(mod\ 7)$, which gives $x = 5$ as the only one-digit solution.

Case 2: $(y = 6)$. Reduced mod 7, we have $6x + 2 \equiv 0(mod\ 7)$, which gives $x = 2{,}9$.

In total, there are three solutions: $\mathbf{5012, 9016}$ and $\mathbf{2016}$.

F18J12. **54.**

We proceed by counting three-digit numbers satisfying the first two conditions, then taking away all of those that fail the third.

There are five choices $\{5, 6, 7, 8, 9\}$ for the first digit, ten choices $\{0, \ldots ,10\}$ for the second digit and three choices $\{5, 7, 9\}$ for the last digit. In total, there are $5 \cdot 10 \cdot 3 = 150$ three-digit numbers that satisfy just the first two conditions.

Notice that the opposite of "at least two of the digits are equal" is the statement "all three digits are distinct". Imagine choosing the digits in this order: ones, hundredths, tenths.

There are three choices for the ones digit, four remaining choices for the hundredths digit, then eight remaining choices for the tens digit. In total, there are

96 such three-digit numbers. Therefore, there are $150 = 96 = 54$ three digit numbers that satisfy all three conditions.

New York City Interscholastic Mathematics League

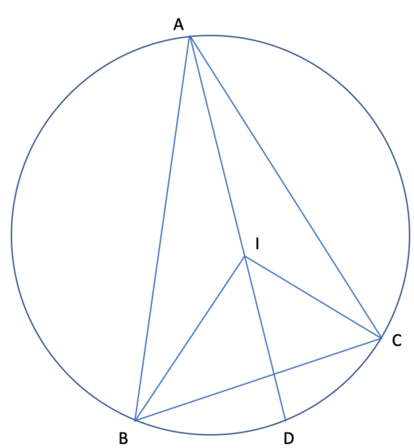
Junior Division — Contest Number 3 Solutions — Fall 2018

F18J13. **0**. Since $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$, and $(a^2 - ab + b^2)$ is an integer if a and b are integers, we have that $1^3 + 6^3 = (1 + 6)(M_1)$, $2^3 + 5^3 = (2 + 5)(M_2)$, and $3^3 + 4^3 = (3 + 4)(M_3)$, for some integers M_1, M_2, M_3. Note that these expressions, along with 7^3, all have factors of 7, so the remainder is **0**.

F18J14. **1990**. Since a and b are positive integers and $\max(a, b) = (a - b)^3$, $a > b$ to make $(a - b)^3$ positive. Thus, $\max(a, b) = a$ and $\min(a, b) = b$. We have $a = (a - b)^3$, so let $a = k^3$ for some integer k, and we get $k^3 = (k^3 - b)^3$, so $b = k^3 - k$. Note that $\gcd(a, b) = \gcd(k^3, k^3 - k) = \gcd(k^3, (k^3 - k) - k^3) = \gcd(k^3, -k) = k$. The second equation gives $b = 99 \gcd(a, b)$, and substituting yields $k^3 - k = 99k$. k must be positive since a is positive, and solving gives $k = 10$, so $a = 1000$, $b = 990$ and $a + b = \mathbf{1990}$.

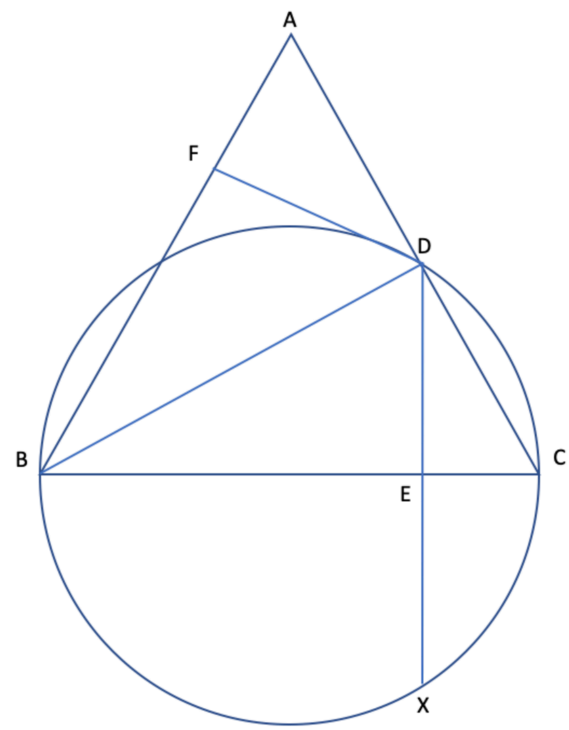
F18J15. $\mathbf{\frac{1}{6}}$. After two rolls, the sum will be some number S which is $0, 1, 2, 3, 4$, or 5 mod 6. Each of these values requires the next roll to be $6 - S$ for the rolls to sum up to a multiple of 6, i.e. 0 needs a 6, 1 needs a 5, and so on. Since we need 1 specific value out of 6 possible values, the answer is $\mathbf{\frac{1}{6}}$.
Note: The answer is $\frac{1}{6}$ no matter how many dice we roll

F18J16. **40°**. Since D is the midpoint of $\widehat{BC}$, $\widehat{BD} = \widehat{DC}$, so $m\angle BAD = m\angle DAC$ by inscribed angle theorem. Thus, I must be on AD, as AD is the angle bisector of $\angle BAC$. Since I is the incenter, AI, BI,and CI are all angle bisectors. Let $m\angle IBC = \theta$ and $m\angle ICB = \phi$. Since BI and CI are angle bisectors, $m\angle ABI = \theta$ and $m\angle ACI = \phi$. Since the sum of the angles in a triangle is 180°, $m\angle BAC = 180 - 2\theta - 2\phi$, and since AI is an angle bisector, $m\angle BAD = 90 - \theta - \phi$. Since $\angle BID$ is an exterior angle for $\triangle ABI$, $m\angle BID = 90 - \phi$. Summing up the angles in $\triangle IBC$, we get that $\theta + 140° = 180° \to \theta = \mathbf{40}°$



F18J17. $(\mathbf{-12}, \mathbf{-8}, \mathbf{43})$. The circle will remain a circle after the reflection, with new center $(6, 4)$, and new equation $(x - 6)^2 + (y - 4)^2 = 9 \Rightarrow x^2 + y^2 - 12x - 8y + 43 = 0$, giving the ordered triple $(\mathbf{-12}, \mathbf{-8}, \mathbf{43})$.

F18J18. **1**. Draw in BD. Note that $BD \perp AC$, as $\angle BDC$ subtends a semicircle. Since DE is the altitude to a hypotenuse, we have $\triangle DBE \sim \triangle DEC$. By similarity, we have $\frac{BE}{DE} = \frac{DE}{EC} \rightarrow DE^2 = (BE)(EC)$. By power of a point, $(BE)(EC) = 2\sqrt{3}(DE)$, and substituting gives us $DE^2 = 2\sqrt{3}(DE) \rightarrow DE = 2\sqrt{3}$. The Pythagorean theorem on $\triangle DEC$ gives $EC = 2$, and substituting back into the power of a point equation gives $BC = 6$. The Pythagorean theorem on $\triangle DBC$ gives us $DB = 4\sqrt{3}$, which tells us that $\triangle DBC$ is $30-60-90$. Since $\triangle ABC$ is isosceles and $m\angle DCB = 60°$, $m\angle FBD = 30°$, and $\triangle BDF \sim \triangle DEC$. We have that $\frac{FD}{4\sqrt{3}} = \frac{2\sqrt{3}}{4\sqrt{3}} \rightarrow FD = 2\sqrt{3}$, so $\frac{DE}{DF} = \mathbf{1}$.



New York City Interscholastic Mathematics League

Junior Division   Contest Number 1   Spring 2019

Solutions

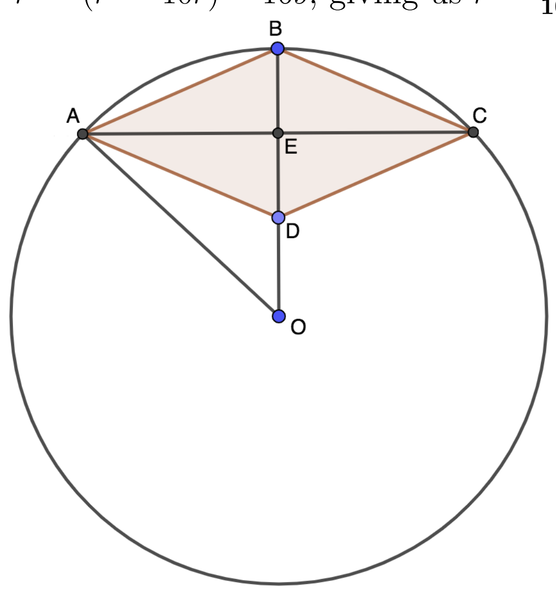
S19J01. $\mathbf{7^{77}}$. The prime factorization of 77 is $7 \cdot 11$, and the prime factorization of 777 is $3 \cdot 7 \cdot 37$, so the prime factorization of 77^{777} is $7^{777} \cdot 11^{777}$ and the prime factorization of 777^{77} is $3^{77} \cdot 7^{77} \cdot 37^{77}$. Therefore the greatest common factor is $\mathbf{7^{77}}$ and 7 is prime.

S19J02. $\mathbf{12}$. Multiplying the top and bottom by rs yields $\frac{r^2+s^2+r+s}{rs}$. Also, $r^2 + s^2 = (r + s)^2 - 2rs$. By Vieta's formulas, $r + s = 6$, and $rs = 3$. Plugging these values in yields $\frac{6^2-2\cdot3+6}{3} = \mathbf{12}$.

S19J03. $\mathbf{\frac{24}{55}}$. **Solution 1:** There are $3 \cdot 8 = 24$ ways to pick 1 marble of each color and $\frac{10\cdot11}{2} = 55$ ways to pick any 2 marbles, so the answer is $\mathbf{\frac{24}{55}}$.

Solution 2: We will find the probability that two marbles of the same color are picked, and subtract the value from 1. The probability we pick 2 red marbles is $\frac{3}{11} \cdot \frac{2}{10} = \frac{3}{55}$, and the probability for picking 2 green marbles is $\frac{8}{11} \cdot \frac{7}{10} = \frac{28}{55}$. Our answer is thus $1 - \frac{28}{55} - \frac{3}{55} = \mathbf{\frac{24}{55}}$.

S19J04. $\mathbf{\frac{169}{10}}$. Draw in AC, and let E be the intersection of the diagonals of $ABCD$. The diagonals of a rhombus perpendicularly bisect each other, so $BE = 5$. Using the Pythagorean theorem on $\triangle ABE$, we have $AE = 12$. Let O be the center of the circumcircle. Since the center of a circle lies on the perpendicular bisectors of all chords of the circle, O lies on $\overleftrightarrow{BD}$, so E, D, and O are colinear. The Pythagorean theorem on $\triangle AEO$ gives
$12^2 + (r - 5)^2 = r^2 \to r^2 - (r^2 - 10r) = 169$, giving us $r = \mathbf{\frac{169}{10}}$.



S19J05. $-\mathbf{2}$. Let $P(x) = x^3 + 3x^2 + 3x + 2$. We can see that $P(x) = (x+1)^3 + 1$. Using the sum of cubes factoring identity, we get

$$P(x) = (x+2)((x+1)^2 - (x+1) + 1) = (x+2)(x^2 + x + 1)$$

Since the discriminant of $x^2 + x + 1$ is $-3 < 0$, it has no real roots. $x + 2$ has one real root, giving us the solution $-\mathbf{2}$.

S19J06. 15630. **Solution 1:** Label the people A through F. We will do casework on the rolls of A, C, and E, who do not sit next to each other.
Case 1: A, C, and E roll the same number, and there are 6 choices for this number. The other 3 people each have 5 valid choices for what they can roll. This gives us $6 \cdot 5^3 = 750$ cases.
Case 2: A, C, and E roll two different numbers. There are $6 \cdot 5$ ways to choose the two numbers rolled, and 3 ways to choose the person who rolled a different number. Among the other three people, one person has 5 valid rolls, while the other two have 4 valid rolls, giving $6 \cdot 5 \cdot 3 \cdot 5 \cdot 4^2 = 7200$ cases.
Case 3: A, C, and E roll three different numbers. There are $6 \cdot 5 \cdot 4$ ways to choose these numbers, and the other three people each have 4 valid choices for their roll, giving $6 \cdot 5 \cdot 4 \cdot 4^3 = 7680$ cases.
The total number of cases is thus $750 + 7200 + 7680 = \mathbf{15630}$.

Solution 2: Let the people be labeled 1 through 6, and let the "desired property" refer to no two people sitting directly next to each other rolling the same number. For integers $k > 1$, consider people 1 through k sitting at a straight table satisfying the desired property. Let Q_k be the number of such possibilities with persons 1 and k rolling a different number, and R_k be the number of such possibilities with persons 1 and k rolling the same number. Note that $Q_2 = 6 \cdot 5 = 30$ since there are 6 possibilities for person 1's roll and 5 ways for person 2 to roll a different number than person 1. Also, $R_2 = 0$ because it is not possible to have the desired property if persons 1 and 2 roll the same number.

Now for $k > 2$, $Q_k = 4 \cdot Q_{k-1} + 5 \cdot R_{k-1}$, since if persons 1 and $k-1$ rolled different numbers then there are 4 possibilities for what person k can roll, and if persons 1 and $k-1$ rolled the same number then there are 5 possibilities for what person k can roll. Similarly, $R_k = Q_{k-1}$ since if persons 1 and $k-1$ rolled a different number then person k can roll the same as person 1, but if persons 1 and $k-1$ rolled the same number person k cannot roll the same as person 1 while preserving the desired property. Thus,

$$Q_2 = 30, R_2 = 0$$

$$Q_3 = 4 \cdot 30 + 5 \cdot 0 = 120, R_3 = 30$$

$$Q_4 = 4 \cdot 120 + 5 \cdot 30 = 630, R_4 = 120$$

$$Q_5 = 4 \cdot 630 + 5 \cdot 120 = 3120, R_5 = 630$$

$$Q_6 = 4 \cdot 3120 + 5 \cdot 630 = 15360, R_6 = 3120$$

If persons 1 through 6 sit at a circular table, and have the desired property, persons 1 and 6 must roll different numbers, so the answer is $Q_6 = \mathbf{15630}$.

Solution 3: Label the people A through F, and let the "desired property" refer to no two people sitting directly next to each other rolling the same number. There are 6 possibilities for A's roll, then 5 possibilities for B's roll distinct from A. Similarly there are 5 possibilities for C's roll distinct from B, and so on, leading to $6 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 18750$ possibilities (*) that satisfy the desired property except that we need to exclude (subtract) those possibilities in which A and F roll the same number.

Similarly, there are $6 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 3750$ possibilities (**) in which A through E have the desired property, which includes possibilities in which A and E rolled the same number. If A and E rolled different numbers, then we are free to assign the same roll for F as for A (these are the possibilities we want to subtract from *), but if A and E rolled the same number then F must be different from A to preserve the desired property (these are possibilities we do not want to subtract from *). So we need to subtract 3750 from 18750 and add back the latter.

To consider those cases in which A and E rolled the same number, we consider that there are $6 \cdot 5 \cdot 5 \cdot 5 = 750$ possibilities in which A through D have the desired property, except that this includes those possibilities in which A through D rolled the same number, which similar to above we need to exclude from adding back.

Continuing, there are $6 \cdot 5 \cdot 5 = 150$ possibilities in which A through C have the desired property, except that this includes those possibilities in which A and C rolled the same number. And there are $6 \cdot 5 = 30$ possibilities in which A through B have the desired property, all of which involve A and B rolling different numbers.

Thus, the desired answer is $18750 - 3750 + 750 - 150 + 30 = \mathbf{15630}$.

Problems tend to have many solutions, and it is encouraged to solve problems in multiple ways and discuss solutions after the contest.

New York City Interscholastic Mathematics League

Junior Division

Contest Number 2 **Spring 2019**

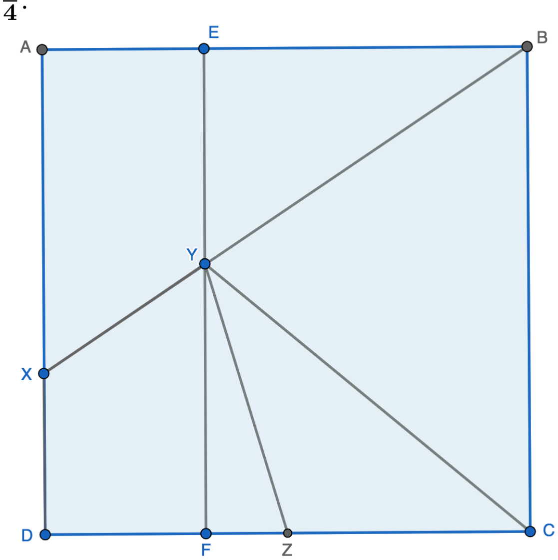
Solutions

S19J07. 8. $21_4 = 9_{10}$ and $13_5 = 8_{10}$, so we have $9_{10} \cdot 8_{10} = 72_{10} = 132_b$. Converting 132_b to base 10 yields $b^2 + 3b + 2$, so we have

$$72 = b^2 + 3b + 2$$
$$b^2 + 3b - 70 = 0$$
$$(b - 7)(b + 10) = 0$$
$$b = 7, -10.$$

Since $b > 3$, we must have $b = 7$. Converting 11_b to base 10 yields $1 \cdot 7 + 1 = \mathbf{8}$.

S19J08. $\mathbf{\frac{5}{4}}$. Let E be the foot of the perpendicular from Y to $\overline{AB}$. Since $\overleftrightarrow{AX} || \overleftrightarrow{EY}$, $\angle AXB = \angle EYB$, and $\angle BAX$ and $\angle BEY$ are both right angles, so we have $\triangle AXB \sim \triangle EYB$ by angle-angle similarity. The ratio of similitude is $3 : 2$, since $BX = BY + YX$, and $BY : YX = 2 : 1$, so we have $BX : BY = 2 : 3$, and $AX = 2$, since $AX : XD = 2 : 1$ and the sidelength of the square is 3. By similarity, $EY = \frac{3}{2} AX = \frac{4}{3}$. Extend EY through Y to hit DC at point F. $\overline{YF} \perp \overline{DC}$ since $\overleftrightarrow{YF} \perp \overline{AB}$ and $\overleftrightarrow{AB} || \overleftrightarrow{DC}$. Since $AEFD$ forms a rectangle, $EF = AD = 3$. We can then see that $\overline{YF}$, the altitude of $\triangle CYZ$, has length $3 - \frac{4}{3} = \frac{5}{3}$, and the base $ZC = \frac{3}{2}$, since Z is the midpoint of $\overline{DC}$, so the area of $\triangle CYZ$ is $\frac{1}{2} \cdot \frac{5}{3} \cdot \frac{3}{2} = \mathbf{\frac{5}{4}}$.



S19J09. $\mathbf{3}, \mathbf{4}$. Since x is an integer,

$$\lfloor \frac{5x}{3} \rfloor = \lfloor x + \frac{2x}{3} \rfloor = x + \lfloor \frac{2x}{3} \rfloor$$

Our equation becomes $x + \lfloor \frac{2x}{3} \rfloor - 2 = x \to \lfloor \frac{2x}{3} \rfloor = 2$. Thus, we need $\frac{2x}{3} \in [2, 3) \to x \in [3, \frac{9}{2})$. The only two integers in this range are $\mathbf{3}, \mathbf{4}$.

S19J10. $\mathbf{\frac{17}{64}}$. Factoring m^3+n^3 yields $(m+n)(m^2-mn+n^2) = 504$. Dividing by $m+n=6$ on both sides yields

$$\begin{aligned} m^2 - mn + n^2 &= 84 \\ (m+n)^2 - 3mn &= 84 \\ 36 - 3mn &= 84 \\ 3mn &= -48 \\ mn &= -16 \end{aligned}$$

Since $\frac{1}{m^2} + \frac{1}{n^2} = \frac{n^2}{m^2n^2} + \frac{m^2}{m^2n^2} = \frac{(m+n)^2-2mn}{m^2n^2} = (\frac{m+n}{mn})^2 - \frac{2}{mn}$, we can substitute in the known values for $m+n$ and mn to get $(\frac{6}{-16})^2 + \frac{2}{16} = \mathbf{\frac{17}{64}}$.

S19J11. $\mathbf{\frac{2}{9}}$.If we treat 1 and 7 and one unit, there are 8! ways to permute the numbers, then 2 possible orders for 1 and 7. The total number of permutations of the numbers without treating 1 and 7 as one unit is 9!, which gives a probability of $\frac{2\cdot 8!}{9!} = \mathbf{\frac{2}{9}}$.

S19J12. **16**. Let $g = \gcd(a,b)$ and $l = \operatorname{lcm}(a,b)$. Note that the left hand side of the equation is divisible by g, since both g and l are divisible by g. (l is divisible by g since $g|a$ and $a|l$.) Thus, 421, a prime number, must also be divisible by g, so $g = 1$ or $g = 421$. If $g = 421$, since $l \geq 1$, the left hand is greater than 421, so there are no solutions. Thus, $g = 1$, and $l = 420$. Prime factorizing, we get $420 = 2^2 \cdot 3 \cdot 5 \cdot 7$. To find the number of ordered pairs (a,b) that satisfy $l = 420$, we just need to count the number of factors of 420. Thus, we have $a = 2^i 3^j 5^m 7^n$ and $b = \frac{420}{a}$, where $i = 0$ or 2, and $j,m,n = 0$ or 1. (i cannot be 1 as we would then have $\gcd(a,b) = 2$, but we know that $g = 1$.) There are 2 possible values for each of i, j, m, and n, giving a total of $2^4 = \mathbf{16}$ solutions.

New York City Interscholastic Mathematics League

Junior Division Contest Number 3 Spring 2019

Solutions

S19J13. $\mathbf{\frac{2}{5}}$. Note that the expression is equivalent to $\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\cdots+\frac{1}{9\cdot10}$. Also, $\frac{1}{2\cdot3}=\frac{1}{2}-\frac{1}{3}$, and in general, $\frac{1}{n(n-1)}=\frac{1}{n-1}-\frac{1}{n}$, as we have $\frac{1}{n-1}-\frac{1}{n}=\frac{n}{n(n-1)}-\frac{n-1}{n(n-1)}=\frac{1}{n(n-1)}$. This will cause the series to telescope, yielding $(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\cdots+(\frac{1}{9}-\frac{1}{10})=\frac{1}{2}-\frac{1}{10}$. Thus, our solution is $\frac{1}{2}-\frac{1}{10}=\mathbf{\frac{2}{5}}$.

S19J14. $(\mathbf{3},\mathbf{2})$. Factoring yields $p^3-q^3=(p-q)(p^2+pq+q^2)$. Since this is a prime number, we must have $p-q=1$ or $p^2+pq+q^2=1$. Since p and q are positive, we must have $p-q=1$. The only two primes separated by 1 are 3 and 2, and checking gives $3^3-2^3=19$, which is prime. Thus, $(p,q)=(\mathbf{3},\mathbf{2})$.

S19J15. $-\mathbf{4}$. Factoring the expression yields $(|x|+4)(|x|-2)=0$, which gives $|x|=-4,2$. However, $|x|$ cannot be negative, so $|x|=2\rightarrow x=2,-2$. The product of these solutions is $-\mathbf{4}$.

S19J16. $\sqrt{\mathbf{5}}-\mathbf{1}$. **Solution 1:** Extend $\overline{AM}$ to hit $\overleftrightarrow{BC}$ at point F. Since $\overleftrightarrow{AB}||\overleftrightarrow{MC}, \angle BAF \cong \angle CMF$, and since both $\angle B$ and $\angle MCF$ are right, $\triangle ABF \sim \triangle MCF$ by AA similarity. $AB = 2$ and $MC = 1$, so the ratio of similitude is $2:1$. Using the Pythagorean theorem on $\triangle ADM$ yields $AM = \sqrt{AD^2+DM^2} = \sqrt{2^2+1^2} = \sqrt{5}$, and using similarity yields $AF = 2MF \to AM + MF = 2MF \to MF = AM = \sqrt{5}$, so $AF = 2MF = 2\sqrt{5}$. Similarly, $BF = 2CF \to BF = 2BC = 4$. Let $BE = x$, and $EF = 4-x$. Using the angle bisector theorem on $\triangle ABF$ yields

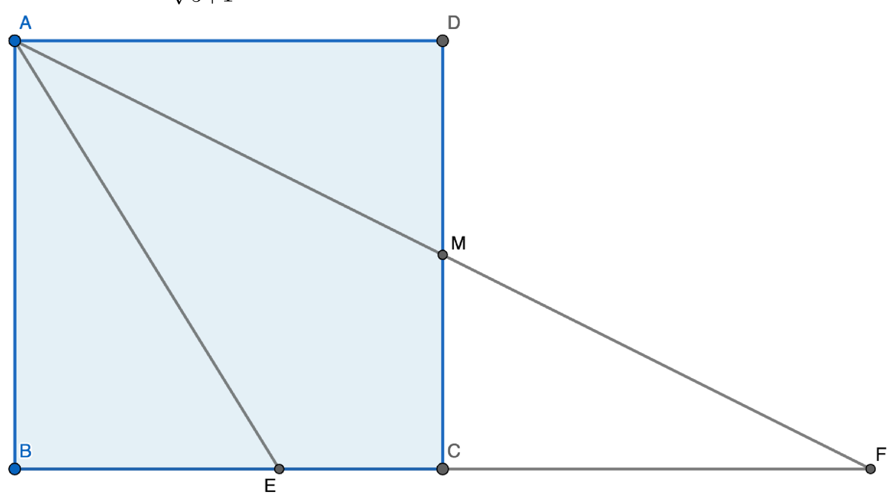
$$\frac{x}{2} = \frac{4-x}{2\sqrt{5}}$$

$$2x\sqrt{5} = 8-2x$$

$$2x\sqrt{5}+2x = 8$$

$$x = \frac{8}{2\sqrt{5}+2} = \frac{4}{\sqrt{5}+1}$$

Our answer is thus $\frac{4}{\sqrt{5}+1}$, and rationalizing the denominator yield $x = \sqrt{\mathbf{5}}-\mathbf{1}$.



Solution 2 (Trig): Using the Pythagorean theorem on $\triangle ADM$ yields $AM = \sqrt{AD^2+DM^2} = \sqrt{2^2+1^2} = \sqrt{5}$. Let $m\angle MAD = \alpha$. Then, $m\angle EAB = \frac{1}{2}(90^\circ - \alpha) = 45^\circ - \frac{\alpha}{2}$. By the tangent half-angle identity, $\tan\frac{\alpha}{2} = \frac{1-\frac{AD}{AM}}{\frac{DM}{AM}} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\frac{2}{\sqrt{5}}}{\frac{1}{\sqrt{5}}} = \sqrt{5}-2$. By the tangent subtraction identity, we have $\tan\angle EAB = \tan(45^\circ - \frac{\alpha}{2}) = \frac{\tan 45^\circ - tan\frac{\alpha}{2}}{1+\tan 45^\circ \tan\frac{\alpha}{2}} = \frac{1-\tan\frac{\alpha}{2}}{1+\tan\frac{\alpha}{2}} = \frac{3-\sqrt{5}}{\sqrt{5}-1} = \frac{(3-\sqrt{5})(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)} = \frac{2\sqrt{5}-2}{4} = \frac{\sqrt{5}-1}{2}$. Thus, $EB = AB \cdot \tan\angle EAB = 2 \cdot \frac{\sqrt{5}-1}{2} = \sqrt{\mathbf{5}}-\mathbf{1}$.

S19J17. **10**. There are two cases, either the number contains three 1s and one 7, or the number has two 1s and two 7s. There are $\binom{4}{1} = \frac{4!}{3!1!} = 4$ numbers in the first case, as we just need to chose where the 7 goes, and the 1s fill in the rest of the digits. In the second case, there are $\binom{4}{2} = \frac{4!}{2!2!} = 6$ cases, as we fill in the 7s, and the 1s fill in the remaining spots, giving a total of **10** solutions.

S19J18. 285714 Let $a = 10A+B$, where B is the unit's digit of a, and $A = \frac{a-B}{10}$ (i.e. a with the last digit removed). We can see that $f(a) = f(10A+B) = 10^k B + A$, where k is 1 more than the number of digits of a (as it turns out, the actual value of k does not matter, it only matters that k is an integer). Thus, to satisfy the given equation, we need

$$3(10A+B) = 2(10^k B + A)$$

$$30A + 3B = 2 \cdot 10^k B + 2A$$

$$28A = B(2 \cdot 10^k - 3)$$

$2 \cdot 10^k - 3$ is odd, since it is the difference of an even number and an odd number. Thus, since $28A$ is divisible by 4 and B is a digit, $B = 4, 8$. There are now two cases, if $B = 4$ or $B = 8$.

Case 1: $B = 4$

Substituting in $B = 4$ and dividing yields $7A = 2 \cdot 10^k - 3$. Since the left hand side is divisible by 7, $2 \cdot 10^k - 3$ is a multiple of 7.

$$2 \cdot 10^k - 3 \equiv 0 \mod 7$$

$$2 \cdot 10^k \equiv 3 \mod 7$$

$$5 \cdot 2 \cdot 10^k \equiv 3 \cdot 5 \mod 7$$

$$10^{k+1} \equiv 1 \mod 7$$

By Fermat's Little Theorem, $k = 5$ is a solution. Any smaller solution must be a multiple of $k + 1 = 6$, but checking $k + 1 = 1, 2, 3$ yields no solutions. Substituting for k, we have $7A = 10^5 - 3$, which gives $A = 28571$, and since $B = 4$, we have $a = 10A + B = 285714$.

Case 2: $B = 8$

In order for B to be 8, we must have $8 \mid 28A \to 2 \mid 7A$, so A must be even (Note: $a \mid b$ means a divides b). Let $A = 2A'$. Substituting and dividing yields $7A' = 2 \cdot 10^k - 3$. This is the same equation as the last case, so we know $k = 5$ gives the smallest solution, and substituting yields $A' = 28571 \to A = 57142$, and since $B = 8$, we have $a = 10A + B = 571428$.

The smaller of the two solution is **285714**.