NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 1   Fall 2018

PART I   ***Fall 2018***   ***CONTEST 1***   ***TIME: 10 MINUTES***

F18B01   Compute the value of $2018 \cdot 35 - 10 \cdot 30 - 10 \cdot 5 - 8 \cdot 30 - 8 \cdot 5$.

F18B02   Consider all of the 4-digit positive integers with a non-zero leading digit that can be formed by using the digits 2, 0, 1, and 8 exactly once each. These are numbers like 2018 and 8210 but not 0218. Compute the sum of all of these four-digit positive integers.

PART II   ***Fall 2018***   ***CONTEST 1***   ***TIME: 10 MINUTES***

F18B03   Mike, Dustin, and Lucas are going to distribute 11 identical candies amongst themselves such that each of them gets at least one. Compute the number of ways this can be done.

F18B04   Isosceles trapezoid $ABCD$ is inscribed in a circle such that $\overline{CD}$ is a diameter. Given that $AB = 4$ and $AD = 7$, compute the radius of the circle. Express your answer in simplified radical form.

PART III   ***Fall 2018***   ***CONTEST 1***   ***TIME: 10 MINUTES***

F18B05   In an arithmetic sequence, the sum of the second term and the sixth term is 92, and the seventh term is 85. Compute the first term.

F18B06   In convex pentagon $ABCDE$, $AB = BC = CD = DE = 3$ and $\cos\angle ABC = \cos\angle BCD = \cos\angle CDE = -\frac{1}{3}$. Compute AE^2. Express your answer as a fraction in simplest form.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Senior B Division   CONTEST NUMBER 2 Fall 2018

PART I   ***Fall 2018***   ***CONTEST 2***   ***TIME: 10 MINUTES***

F18B07 In ΔABC, $AB = BC$. Side $\overline{AC}$ is extended past C to D such that $AB = CD$. Given that $m\angle ABC = 128°$, compute $m\angle BDC$ in degrees.

F18B08 A fair coin is tossed five times with all tosses being independent. Compute the probability that the coin comes up heads at least three times in a row.

PART II   ***Fall 2018***   ***CONTEST 2***   ***TIME: 10 MINUTES***

F18B09 A rectangular prism is placed in the xyz-coordinate system such that two vertices of the prism are at $(2, 3, 8)$ and $(4, 8, 1)$. Each face of the prism is parallel to one of the xy-, xz-, or yz-planes. Compute the volume of the prism in cubic units.

F18B10 The roots of the function $f(x) = ax^4 + bx^3 + cx^2 + dx + 2018$ include, but are not limited to, $x = -2$ and $x = 2$. Compute $4a + c$. Express your answer as a fraction in simplest form.

PART III   ***Fall 2018***   ***CONTEST 2***   ***TIME: 10 MINUTES***

F18B11 Compute the remainder when 20180^2 is divided by 201. The remainder is an integer between 0 and 200.

F18B12 Compute the distance between the graphs of $x^2 + y^2 = 25$ and $4x + 3y = 300$. The distance between two graphs is the length of the shortest possible segment joining a point on one graph and a point on the other graph.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 3   Fall 2018

PART I | ***Fall 2018*** | ***CONTEST 3*** | ***TIME: 10 MINUTES***

F18B13 In simplest form, the infinitely repeating decimal number $0.20\overline{18}$ is equal to the fraction $\frac{p}{q}$ where p and q are relatively prime positive integers. Compute $p + q$.

F18B14 Compute the number of positive perfect cubes that divide 12!.

PART II | ***Fall 2018*** | ***CONTEST 3*** | ***TIME: 10 MINUTES***

F18B15 Given that $i = \sqrt{-1}$, compute all real x such that $(x - 3i)(20 + (x + 4)i)$ is a real number.

F18B16 Given ΔARC with $AC = 10$, $AR = 8$, and $CR = 6$, let M be the midpoint of $\overline{AC}$. Circles C_1 and C_2 are the incircles of ΔCRM and ΔARM respectively. That is, C_1 is internally tangent to each side of ΔCRM and C_2 is internally tangent to each side of ΔARM. Compute the sum of the areas of C_1 and C_2. Express your answer as a fractional multiple of π in simplest form.

PART III | ***Fall 2018*** | ***CONTEST 3*** | ***TIME: 10 MINUTES***

F18B17 Compute the value of x such that $\log_3(4x) - \log_3(x - 2) = 2$. Express your answer as a fraction in simplest form.

F18B18 The vertices of pentagon $BRONX$ are $B(3, 7)$, $R(0, 3)$, $O(0, 0)$, $N(4, 0)$, and $X(4, 7)$. A point A is chosen uniformly and at random from the interior of $BRONX$. Compute the probability that $\angle BAR$ is obtuse. Express your answer as a fractional multiple of π in simplest form.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 4   Fall 2018

PART I   ***Fall 2018***   ***CONTEST 4***   ***TIME: 10 MINUTES***

F18B19 Compute the remainder when 3^{2018} is divided by 13. The remainder is an integer between 0 and 12.

F18B20 Compute the value of $(2018^3 - 8)/(2 \cdot 2018^2 + 8080)$.

PART II   ***Fall 2018***   ***CONTEST 4***   ***TIME: 10 MINUTES***

F18B21 Order 2^{32}, 3^{20}, 15! from least to greatest. Give your answer in the form $A < B < C$.

F18B22 In acute ΔTRG, $\sin T = 3/5$ and $\cos R = 5/13$. Compute $\tan G$ as a fraction in simplest form.

PART III   ***Fall 2018***   ***CONTEST 4***   ***TIME: 10 MINUTES***

F18B23 Let A and B be independent events. The probability that both events A and B happen is 0.4. The probability that event A happens but event B does not happen is 0.4. Compute the probability that neither event A nor B happens as a fraction in simplest form.

F18B24 Two circles C_2 and C_3 of radius 2 and 3 respectively have centers in the first quadrant and are externally tangent to each other and to the x-axis. A third circle C is externally tangent to each of C_2 and C_3 and also to the x-axis. Compute the radius of C as an expression of the form $r - s\sqrt{t}$, where r, s, and t are positive integers and t does not have any factors that are perfect squares greater than 1.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 5   Fall 2018

PART I   ***Fall 2018***   ***CONTEST 5***   ***TIME: 10 MINUTES***

F18B25   Given that the number $\overline{A03307A6}$ is divisible by 44, compute the digit A.

F18B26   Compute all real values of x such that $2^{2x} = 2^x + 56$.

PART II   ***Fall 2018***   ***CONTEST 5***   ***TIME: 10 MINUTES***

F18B27   Regular hexagon $NYCIML$ and equilateral triangle GEO have equal perimeters. Given that the area of $NYCIML$ is 36, compute the area of GEO.

F18B28   Suppose that n is a natural number such that $n + 2n + 3n + \cdots + 2018n$ has exactly 36 distinct positive integer factors. Compute the smallest possible value of n.

PART III   ***Fall 2018***   ***CONTEST 5***   ***TIME: 10 MINUTES***

F18B29   Given that $2^A \cdot 3^B \cdot 5^{13} = 20^D \cdot 18^{12}$, where A, B, and D are positive integers, compute $A + B + D$.

F18B30   The terms u_i of a sequence $u_1 = 2018$, $u_2 = 1738$, $u_3 = 1818$, $u_4 = 2018$, ... are generated by adding corresponding terms of an arithmetic sequence a_i and a geometric sequence g_i. Compute u_5.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE
Senior B Division CONTEST NUMBER 1 Spring 2019

PART I ***Spring 2019*** ***CONTEST 1*** ***TIME: 10 MINUTES***

S19B01 Compute the least integer N such that the line passing through $(20, 19)$ and $(39, N)$ has a negative y-intercept.

S19B02 Given that $m : n : p = 7 : 1 : 8$ and $m + 2n + 3p = 2442$, compute $3m + 2n + p$.

PART II ***Spring 2019*** ***CONTEST 1*** ***TIME: 10 MINUTES***

S19B03 Let P and Q both represent prime numbers such that $5P + 7Q = 2019$. Compute P.

S19B04 Compute the value of $\frac{2019^2-14141}{2019^2+2013\cdot 2019-16168}$. Express your answer as a fraction in simplest form.

PART III ***Spring 2019*** ***CONTEST 1*** ***TIME: 10 MINUTES***

S19B05 Let $g(x) = ax + b$ for all real x, where a and b are integers and a is positive, and let $g(g(x)) = 16x + 30$ for all real x. Compute the ordered pair (a, b).

S19B06 Suppose that, in rectangle $RECT$ with point X on $\overline{EC}$ such that $m\angle ERX$ is twice $m\angle EXR$, $\overline{XT}$ is an angle bisector of $\angle RXC$. Given that $RE = 1$, compute the area of $RECT$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 2 Spring 2019

PART I	Spring 2019	CONTEST 2	TIME: 10 MINUTES

S19B07 An elevator of a New York City skyscraper goes from Floor 1 to Floor 4 in 12 seconds. Compute the number of seconds it should take the elevator to go from Floor 14 to Floor 38. Note that floor numbers are all distinct and are the natural numbers from 1 to 38, the elevator travels at a constant rate without stops, and all floors are of the same height.

S19B08 The numbers $a_1, a_2, a_3, \cdots$ form an arithmetic sequence. Given that $a_6 + a_7 - a_5 = 20$ and $a_{13} + a_{14} - a_{11} = 52$, compute $a_{20} + a_{19}$.

PART II	Spring 2019	CONTEST 2	TIME: 10 MINUTES

S19B09 Nine fair coins are independently flipped. Compute the probability that an odd number of them come up heads. Express your answer as a fraction in simplest form.

S19B10 A woodworker has a cubical block of wood that measures 8 cm on a side. The woodworker drills a cylindrical hole from the center of one face of the cube to the center of the opposite face. The hole is 2 cm in diameter. Compute the surface area of the resulting solid in square cm. Express your answer in terms of π.

PART III	Spring 2019	CONTEST 2	TIME: 10 MINUTES

S19B11 Rectangle $RECT$ has a perimeter of P inches and an area of P square inches, and all of its side lengths are integers. Given that $RECT$ is not a square, compute P.

S19B12 Compute the number of distinct positive integers N such that when 2019 is divided by N, the remainder is 12.

New York City Interscholastic Mathematics League

Senior B Division Contest Number 3 Spring 2019

Part I ***Spring 2019*** ***Contest 3*** ***Time: 10 Minutes***

S19B13 The sum of 27 consecutive odd integers is $3^9 = 19683$. Compute the least of these consecutive odd integers.

S19B14 Solve for real x: $2^{3x+2} - 2^{3x} = 48$. Express your answer as a fraction in simplest form.

Part II ***Spring 2019*** ***Contest 3*** ***Time: 10 Minutes***

S19B15 At a party, the ratio of girls to boys is $5 : 4$. Then, 18 boys join the party, and the ratio of boys to girls is $5 : 4$. Compute the number of people originally at the party.

S19B16 Compute the coordinates of all points (x, y) that are on both hyperbolas $x^2 + 3xy - y^2 = 48$ and $x^2 - 5xy + y^2 = -48$.

Part III ***Spring 2019*** ***Contest 3*** ***Time: 10 Minutes***

S19B17 Compute the number of all distinct integers x for which $|20x - 19| \leq 2019$.

S19B18 Given equilateral triangle TRI with $TR = 8$. Point X is in the interior of ΔTRI such that the distance from X to $\overline{TR}$ is 2 and the distance from X to $\overline{RI}$ is 4. Compute the distance from X to $\overline{TI}$. Express your answer in simplified radical form.

New York City Interscholastic Mathematics League

Senior B Division   Contest Number 4   Spring 2019

Part I   ***Spring 2019***   ***Contest 4***   ***Time: 10 Minutes***

S19B19   Compute the greatest prime factor of 9951.

S19B20   One diagonal of a cube has length 12. Compute the surface area of the cube.

Part II   ***Spring 2019***   ***Contest 4***   ***Time: 10 Minutes***

S19B21   Parallelogram $MATH$ has a perimeter of 60. The altitude from A to $\overline{MH}$ meets $\overline{MH}$ at a point X and $AX = 12$. Given that $MX = 5$, compute the area of $MATH$.

S19B22   Compute the number of distinct ordered pairs of positive integers (x, y) for which $3x + 4y = 500$.

Part III   ***Spring 2019***   ***Contest 4***   ***Time: 10 Minutes***

S19B23   There are nine non-negative integer elements in the collection $\{2, 0, 1, 9, 201, 9, 20, 19, x\}$. Compute x such that the difference between the mean and the median of the collection is 245.

S19B24   Factor $a^4 + a^3 - a^2b + ab^2 + a - b^4 - b^3 + b + 1$ into the product of two polynomials in a and b each with constant term 1, where one polynomial has degree 3 and the other has degree 1.

New York City Interscholastic Mathematics League
Senior B Division   Contest Number 5   Spring 2019

Part I   ***Spring 2019***   ***Contest 5***   ***Time: 10 Minutes***

S19B25 Compute the least positive integer that does not divide 2019!.

S19B26 Suppose that a, b, and c are positive numbers such that $a^2 + b^2 + c^2 = 83$ and $ab + ac + bc = 71$. Compute $a + b + c$.

Part II   ***Spring 2019***   ***Contest 5***   ***Time: 10 Minutes***

S19B27 The line $20x + 19y = 2019$ is graphed in the coordinate plane. Some of the points on the line are *lattice points*, points both of whose coordinates are integers. How many first-quadrant lattice points are on the line $20x + 19y = 2019$?

S19B28 Compute the positive solution for x: $(3x)^{\log 3} = (7x)^{\log 7}$. Express your answer as a fraction in simplest form.

Part III   ***Spring 2019***   ***Contest 5***   ***Time: 10 Minutes***

S19B29 Quadrilateral $KITE$ is inscribed in a circle. Given that $KI = KE = 5$ and $IT = ET = 12$, compute IE. Express your answer as a fraction in simplest form.

S19B30 The sequence w, x, y, z is arithmetic. The sequence $w, x, y + 3, z + 10$ is geometric. Compute z.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 1 SOLUTIONS Fall 2018

F18B01. $\mathbf{70000}$. The given expression is equivalent to
$2018 \cdot 35 - (10 + 8)(30 + 5) = 2018 \cdot 35 - 18 \cdot 35 = 2000 \cdot 35$, or $\mathbf{70000}$.

F18B02. $\mathbf{70884}$. First, consider all 4-digit integers, even those with 0 as the leading digit, that can be formed using each of 2,0,1,8 exactly once. Because these digits are all distinct, there are $4! = 24$ such integers. Across these 24 integers, each of the 4 digits appears in the thousands position $3! = 6$ times because there are 3! ways to arrange the other 3 digits. Similarly, the digits 2,0,1,8 each appear $3! = 6$ times in each of the hundreds, tens, and ones positions. Thus the sum of these 24 integers is $(2 + 0 + 1 + 8) \cdot 6666 = 73326$.

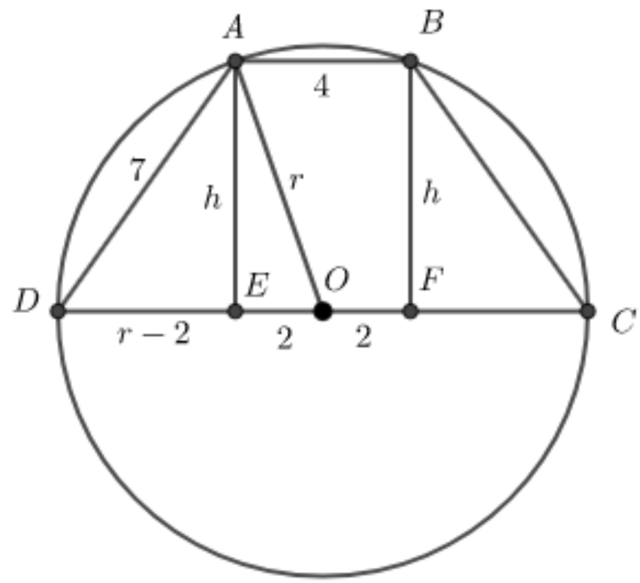
Now consider the 4-digit integers that can be formed using 2,0,1,8 exactly once each where 0 is the leading digit. There are $3! = 6$ such integers (because there are 3! ways to arrange the other 3 digits). Similar to the above, the digits 2,1,8 each appear $2! = 2$ times in each of the hundreds, tens, and ones positions. Thus the sum of these 6 integers is
$(2 + 1 + 8) \cdot 222 = 2442$.

Therefore the sum of the integers of interest (i.e. those without 0 as the leading digit) is
$73326 - 2442 = \mathbf{70884}$.

F18B03. $\mathbf{45}$. Three of the candies are given to the three boys right away, so the question asks how to split the other eight. Consider a string of eight identical stars representing the eight candies and two identical bars that divide the string into three possibly empty parts. There are ten objects to be arranged, two of which are bars. Thus, by "stars and bars", the answer is
$\binom{10}{2} = \frac{10\cdot 9}{2\cdot 1} = \mathbf{45}$ ways to split the candies.

F18B04. $\mathbf{\frac{2+\sqrt{102}}{2}}$ **or** $\mathbf{1 + \frac{\sqrt{102}}{2}}$. Since $ABCD$ is an isosceles trapezoid, either $\overline{AB} \,||\, \overline{CD}$ and $AD = BC$, or $\overline{BC} \,||\, \overline{AD}$ and $AB = CD$. But since $\overline{CD}$ is a diameter, for $AB = CD$, $\overline{AB}$ would also have to be a diameter (in a circle, the only chords that are as long as a diameter are other diameters) and therefore $\overline{AB}$ and $\overline{CD}$ would intersect (at the center of the circle) and $ABCD$ would not be a convex trapezoid. Therefore, bases $\overline{AB}$ and $\overline{CD}$ are parallel. Since $ABCD$ is an isosceles trapezoid, and O is the center of diameter/base $\overline{CD}$, then the trapezoid is symmetric about the line through O and perpendicular to CD, and therefore this line passes through the center of base $\overline{AB}$. Let E and F be the feet of the altitudes from A and B to $\overline{CD}$, respectively. By construction, rectangle $ABFE$ is also symmetric about this line.

Since $AB = EF = 4$, $OE = EF/2 = 2$. If we let the radius of the circle be r, then E splits $\overline{DO}$ into segments of length $r-2$ and 2. The height $AE = BF = h$ can be found in two ways: $(r-2)^2 + h^2 = 49$ and $4 + h^2 = r^2$. This implies that $r^2 - 4 = 49 - (r-2)^2$, which implies $r^2 - 4 = 45 - r^2 + 4r \leftrightarrow 2r^2 - 4r - 49 = 0$, which has roots $r = \frac{4 \pm \sqrt{16 - 4 \cdot 2 \cdot (-49)}}{4} = \frac{4 \pm \sqrt{408}}{4} = \frac{2 \pm \sqrt{102}}{2}$. $r > 0$, so we reject the negative root to obtain $r = \mathbf{\frac{2+\sqrt{102}}{2} = 1 + \frac{\sqrt{102}}{2}}$.



F18B05. **7**. Let a be the value of the first term and d be the common difference. The sum of the second and sixth terms is $(a+d) + (a+5d) = 2a + 6d$. The seventh term is $a + 6d$, so subtracting yields $a = (2a+6d) - (a+6d) = 92 - 85 = \mathbf{7}$.

F18B06. $\mathbf{\frac{32}{3}}$. By the Law of Cosines on ΔABC,

$$AC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos\angle ABC = 3^2 + 3^2 - 2(3)(3)(-\tfrac{1}{3}) = 24.$$

Similarly, the law of Cosines applied to triangle ΔCDE implies $CE^2 = 24$. Since ΔABC and ΔDCE are both isosceles triangles with $AB = BC$ and $CD = DE$,

$$m\angle ACB = m\angle CAB = \tfrac{1}{2}(180° - m\angle ABC) = 90° - \tfrac{1}{2}(\cos^{-1}(-\tfrac{1}{3})) \text{ and}$$
$$m\angle CED = m\angle ECD = \tfrac{1}{2}(180° - m\angle CDE) = 90° - \tfrac{1}{2}(\cos^{-1}(-\tfrac{1}{3})).$$

Then

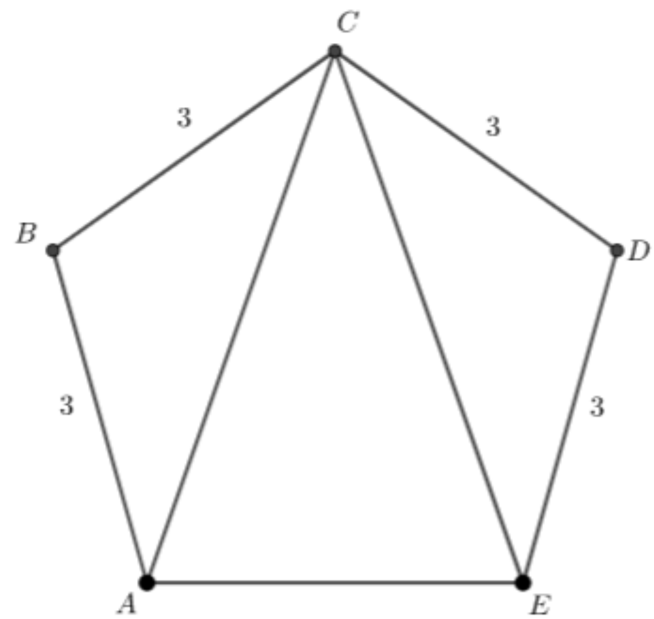
$$\begin{aligned} m\angle ACE &= m\angle BCD - m\angle ACB - m\angle ECD \\ &= \cos^{-1}(-\tfrac{1}{3}) - (90° - \tfrac{1}{2}\cos^{-1}(-\tfrac{1}{3})) - (90° - \tfrac{1}{2}\cos^{-1}(-\tfrac{1}{3})) \\ &= 2\cos^{-1}(-\tfrac{1}{3}) - 180°. \end{aligned}$$

Using the cosine of the difference of two angles,

$$\cos\left(2\cos^{-1}(-\tfrac{1}{3}) - 180°\right) = \cos\left(2\cos^{-1}(-\tfrac{1}{3})\right)\cos 180° + \sin\left(2\cos^{-1}(-\tfrac{1}{3})\right)\sin 180°$$
$$= -\cos(2\cos^{-1}(-\tfrac{1}{3})).$$

By the double-angle formula, this is equal to $1 - 2\cos^2(\cos^{-1}(-\frac{1}{3})) = 1 - 2(-\frac{1}{3})^2 = \frac{7}{9}$. Now apply the Law of Cosines to ΔACE to obtain

$$AE^2 = AC^2 + CE^2 - 2 \cdot AC \cdot CE \cdot \cos\angle ACE \ = 24 + 24 - 2 \cdot \sqrt{24} \cdot \sqrt{24} \cdot \frac{7}{9} = \mathbf{\frac{32}{3}}.$$

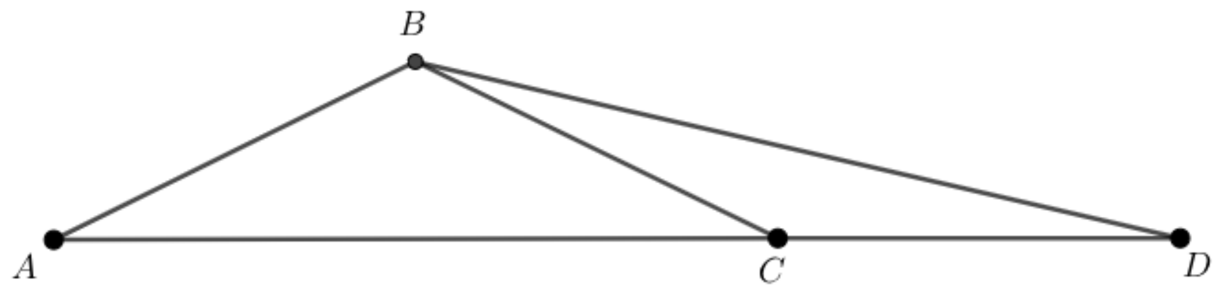


NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 2 SOLUTIONS Fall 2018

F18B07. **13 or 13°**. Because ΔABC is isosceles, $m\angle BCA = \frac{180°-128°}{2} = 26°$. Since, $AB = BC$ and $AB = CD$, $BC = CD$, so ΔBCD is also isosceles. Therefore,

$$m\angle BDC = \frac{1}{2}\cdot(180 - m\angle BCD) = \frac{1}{2}\cdot\left(180° - (180° - m\angle BCA)\right) = \frac{1}{2}\cdot m\angle BCA = \mathbf{13°}.$$



F18B08. $\frac{\mathbf{1}}{\mathbf{4}}$. There are $2^5 = 32$ five-letter strings of H's and T's. Of these, some have at least three H's in a row. There is one string with five H's. The strings with exactly four H's in a row are of the form $HHHHx$ or $xHHHH$, where x is either an H or a T. Since $x = H$ would yield a string with five H's in a row, $x = T$ for both forms, which yields 2 strings in total. Similarly, the strings with exactly three H's in a row are of the form $HHHxy$, $xHHHy$, or $yxHHH$, where x and y are each either an H or a T. The first case and the third case are symmetric, so we can handle both at the same time. In each, $x = H$ would give a string with at least 4 H's in a row, so $x = T$. y has no restriction, so these two cases give a total of $2 + 2 = 4$ strings. For the second case, $x = H$ or $y = H$ would result in strings with at least 4 H's in a row, so $x = y = T$, which gives us 1 string. The desired probability is $\frac{1+2+5}{32} = \frac{8}{32} = \frac{\mathbf{1}}{\mathbf{4}}$.

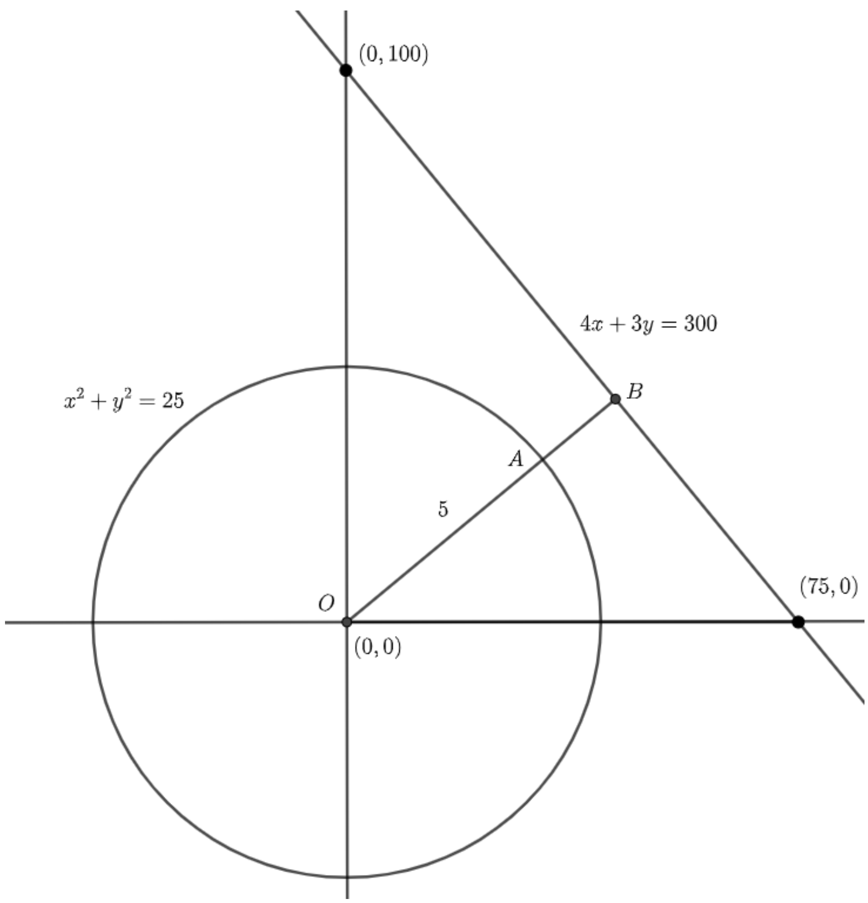
F18B09. $\mathbf{70}$. The volume of the prism is the product of the differences in x-, y-, and z-values. Thus the volume is $(4-2)(8-3)(8-1) = 2\cdot 5\cdot 7 = \mathbf{70}$.

F18B10. $-\frac{\mathbf{1009}}{\mathbf{2}}$. Note that $f(2) = 16a + 8b + 4c + 2d + 2018 = 0$ and $f(-2) = 16a - 8b + 4c - 2d + 2018 = 0$. Adding yields $32a + 8c + 4036 = 0$. Dividing by 8 and isolating gives us $4a + c = -\frac{\mathbf{1009}}{\mathbf{2}}$.

F18B11. $\mathbf{169}$. Notice that 20180^2 will have the same remainder when dividing by 201 as $(20180 - 201\cdot 100)^2 = (20180 - 20100)^2 = 80^2$. By arithmetic, $80^2 = 6400 = 201\cdot 31 + 169$, so the remainder is **169**.

F18B12. $\mathbf{55}$. The graph $x^2 + y^2 = 25$ represents a circle with radius 5 centered at the origin, while the graph $4x + 3y = 300$ represents a straight line. We can compute the distance between the origin and the line by considering the altitude to the hypotenuse of the triangle formed by the two axes and the line. To get the coordinates of the two points that make up the hypotenuse, we can set $x = 0$ in the line equation to get the point $(0, 100)$ and set $y = 0$ in the line equation to get the point $(75, 0)$. The area of this triangle is $\frac{100 \cdot 75}{2} = 3750$ and the length of the hypotenuse is $\sqrt{100^2 + 75^2} = 25\sqrt{4^2 + 3^2} = 25 \cdot 5 = 125$, so the length of the altitude to the hypotenuse is equal to $\frac{3750}{125 \cdot \frac{1}{2}} = 60$. This is greater than the length of the radius of the circle, so the two graphs do not intersect.

If A is a point on the circle, B is a point on the line, and O is the origin, then by the Triangle Inequality, $AB \geq BO - AO \geq 60 - 5 = 55$. If B is the foot of the altitude referenced above, and A is the intersection of the segment OB and the circle, then $AB = BO - AO = 55$. Therefore, the length of the shortest possible segment joining a point on the circle and a point on the line (i.e. the distance between the two given graphs) is $\mathbf{55}$.



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NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

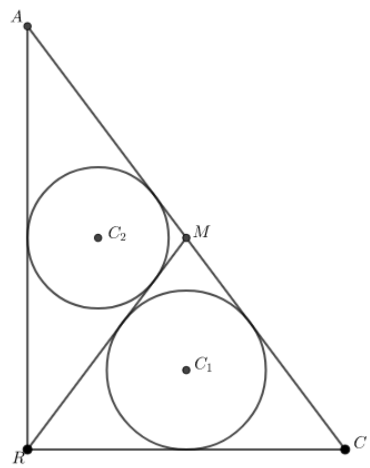
Senior B Division   CONTEST NUMBER 3 SOLUTIONS   Fall 2018

F18B13. $\mathbf{661.}$ Let $N = 0.20181818\ldots$. Then $100N = 20.18181818\ldots$. Subtracting, $99N = 19.98$, so $N = \frac{19.98}{99} = \frac{1998}{9900} = \frac{111}{550}$. Since the prime factorization of 111 is $3 \cdot 37$ and the prime factorization of 550 is $2 \cdot 5^2 \cdot 11$, 111 and 550 are relatively prime, so $p + q = 111 + 550 = \mathbf{661}$.

F18B14. $\mathbf{8.}$ Express $12!$ as $2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$. The cube in question needs to be of the form $2^a 3^b 5^c 7^d 11^e$, where a, b, c, d, e are nonnegative multiples of 3 and $a \leq 10, b \leq 3, c \leq 2, d \leq 1, e \leq 1$. Therefore, the (independent) choices of a are 0, 3, 6, 9 and of b are 0, 3; the only choice for c, d, and e are 0. Thus there are $4 \cdot 2 = \mathbf{8}$ factors of $12!$ that are perfect cubes.

F18B15. $\mathbf{-10}$ **and** $\mathbf{6.}$ Expand the expression to obtain $20x + 3(x + 4) + (x(x + 4) - 60)i$. $(x - 3i)(20 + (x + 4)i)$ is real if and only if its imaginary part is 0. Then $x^2 + 4x - 60 = 0 \leftrightarrow (x + 10)(x - 6) = 0 \leftrightarrow x = \mathbf{-10}$ **and** $\mathbf{6}$.

F18B16. $\mathbf{\frac{145\pi}{36}}$**.** Note that the circumcircle of ΔACR has center M because $\angle ARC$ is a right angle, which implies that $\overline{AC}$ is a diameter of the circumcircle. Since the radius of this circumcircle is $\frac{10}{2} = 5$, $CM = AM = RM = 5$. Note that ΔAMR and ΔMRC have the same area because they have altitudes (from R) of the same height and equal bases $AM = MC$. If we let the radius of circle C_1 be r_1, then $[CRM] = [CRC_1] + [RMC_1] + [CMC_1] = \frac{6r_1}{2} + \frac{5r_1}{2} + \frac{5r_1}{2} = 8r_1$. Since the areas of ΔAMR and ΔMRC sum to the area of ΔARC, which has area $\left(\frac{1}{2}\right) \cdot 6 \cdot 8 = 24$, ΔAMR and ΔCRM each have area $\frac{24}{2} = 12$, so $r_1 = \frac{12}{8} = \frac{3}{2}$. If we let the radius of circle C_2 be r_2, then we can use a similar approach on ΔARM to get $\frac{8r_2}{2} + \frac{5r_2}{2} + \frac{5r_2}{2} = \frac{18r_2}{2} = 12$, so $r_2 = \frac{4}{3}$. Therefore, the sum of the areas of the two circles is equal to $\left(\frac{3}{2}\right)^2 \pi + \left(\frac{4}{3}\right)^2 \pi = \frac{9\pi}{4} + \frac{16\pi}{9} = \mathbf{\frac{145\pi}{36}}$.

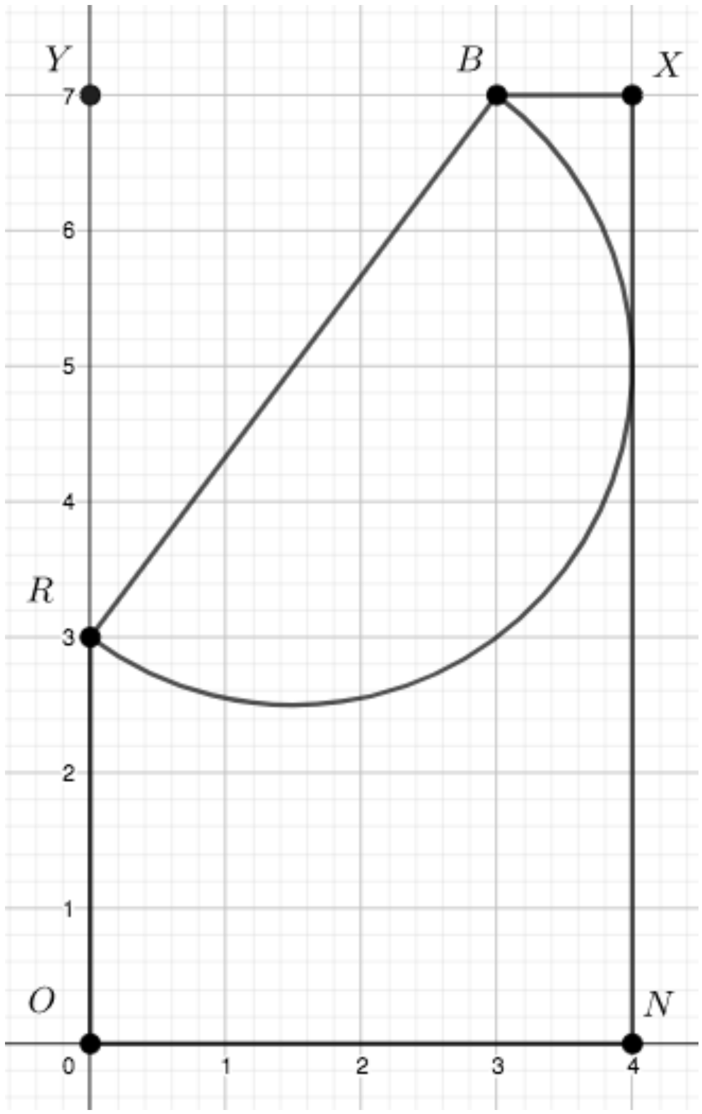


F18B17. $\frac{\mathbf{18}}{\mathbf{5}}$**.** By the quotient property of logarithms, this is equivalent to $\log_3 \frac{4x}{x-2} = 2$, which implies $\frac{4x}{x-2} = 9$. Cross-multiply to obtain $4x = 9x - 18$, which solves to obtain $x = \frac{\mathbf{18}}{\mathbf{5}}$.

F18B18. $\frac{\mathbf{25\pi}}{\mathbf{176}}$**.** Consider the semicircle with diameter $\overline{BR}$ below the line segment $\overline{BR}$. By the distance formula, the length of $\overline{BR}$ is equal to $\sqrt{(3-0)^2 + (7-3)^2} = 5$, so the radius of the semicircle is $\frac{5}{2}$. Note that the midpoint of $\overline{BR}$, $(\frac{3}{2}, 5)$, is $\frac{5}{2}$ units away from $\overline{XN}$ and 5 units away from $\overline{ON}$, so the semicircle is completely contained by $BRONX$.

For all of the points A on the semicircle with diameter $\overline{BR}$ below the line segment $\overline{BR}$, $\angle BAR$ will be right. For all points A inside that semicircle, $\angle BAR$ will be obtuse. For all points A outside that semicircle and inside $BRONX$, $\angle BAR$ will be acute. Therefore, to find the desired probability we divide the area of the interior of the semicircle, which is equal to the area of the semicircle, by the area of the interior of $BRONX$, which is equal to the area of $BRONX$.

The area of the semicircle is $\frac{1}{2} \cdot \pi \cdot (\frac{5}{2})^2 = \frac{25\pi}{8}$. If we let $Y = (0, 7)$ one can see that $YXNO$ is a 4 by 7 rectangle based on the coordinates of its vertices. Also, one can see that ΔYBR is a right triangle with legs $YB = 3$ and $YR = 4$. Now note that the pentagon $BRONX$ can be obtained by removing ΔYBR from rectangle $YXNO$, thus $BRONX$ has area $(4 \cdot 7) - \left(\frac{1}{2} \cdot 3 \cdot 4\right) = 28 - 6 = 22$. Therefore, the desired probability is $\frac{\frac{25\pi}{8}}{22} = \frac{\mathbf{25\pi}}{\mathbf{176}}$.



Y
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6
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0
1
2
3
4

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 4 SOLUTIONS   Fall 2018

F18B19. 9. Notice that $27 = 3^3 \equiv 1\ (mod\ 13)$, so $3^{2018} = 3^{2016} \cdot 3^2 = (3^3)^{672} \cdot 3^2 \equiv 1 \cdot 3^2$ (mod 13), so the remainder is **9.**

F18B20. 1008. Let $x = 2018$. Then the given expression is equivalent to $\frac{x^3-8}{2x^2+4x+8} = \frac{(x-2)(x^2+2x+4)}{2(x^2+2x+4)} = \frac{x-2}{2}$. Substituting $x = 2018$ yields a value of $\frac{2018-2}{2} = \mathbf{1008}$.

F18B21. $\mathbf{3^{20} < 2^{32}} < 15!$. First, establish that $3^{20} < 2^{32}$ because $243^4 = (3^5)^4 = 3^{20} < 2^{32} = (2^8)^4 = 256^4$. Since

$$15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 > 8^8 = 2^{24},$$
$$7 \cdot 6 \cdot 5 \cdot 4 > 4^4 = 2^8,$$
$$3 \cdot 2 \cdot 1 > 2^2,$$

$15! = (15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8) \cdot (7 \cdot 6 \cdot 5 \cdot 4) \cdot (3 \cdot 2 \cdot 1) > 2^{24+8+2} = 2^{34} > 2^{32}$, so the correct order is $\mathbf{3^{20} < 2^{32} <}$ **15!.**

F18B22. 63/16. The value of tan G is tan $(180° - (T + R))$, so first find tan $(T + R)$ by the formula $\tan(T+R) = \frac{\tan T + \tan R}{1 - \tan T \tan R}$. This formula does not hold if $m\angle T + m\angle R = 90°$, since sin T and cos R are not equal, $\angle T$ and $\angle R$ are not complements and do not sum to 90°, so we can use this addition formula. Since ΔTRG is acute, each angle has a degree measure less than 90°, so $\cos T > 0$ and $\sin R > 0$. Then $\tan T = \sin T/\cos T = \frac{\frac{3}{5}}{+\sqrt{1-\left(\frac{3}{5}\right)^2}} = \frac{3}{4}$ and

$\tan R = \sin R/\cos R = \frac{\sqrt{1-\left(\frac{5}{13}\right)^2}}{\frac{5}{13}} = \frac{12}{5}$. Substituting these values into the previous tangent addition formula yields $\tan(T+R) = \frac{\frac{3}{4}+\frac{12}{5}}{1-\frac{3}{4}\cdot\frac{12}{5}} = \frac{-63}{16}$, so

$\tan G = \tan(180° - (T+R)) = \frac{\tan 180 - \tan(T+R)}{1 + \tan 180 \tan(T+R)} = \frac{0-\frac{-63}{16}}{1+0\cdot\frac{-63}{16}} = \mathbf{\frac{63}{16}}$. Note that one can solve for tan G by using the law of tangents, which states that $\tan T \tan R \tan G = \tan T + \tan R + \tan G$ for any ΔTRG.

F18B23. 1/10. Let $P(A)$ be the probability that an event A occurs and let $P(A') = 1 - P(A)$ be its complement. Then the probability that both events A and B happen is $P(A \cap B)$ and the probability that event A happens but event B does not happen is $P(A \cap B')$. Since $P(A \cap B) = P(A)P(B)$ for independent events A and B, $P(A \cap B) = P(A)P(B) = 0.4$ and $P(A \cap B') = P(A)P(B') = 0.4$. Since $P(A)P(B) = 0.4 = P(A)P(B')$, $P(A)\big(P(B) - P(B')\big) = 0$. $P(A)P(B) = 0.4$, so $P(A)$ is not equal to 0 which means that $P(B) - P(B') = 0$. $P(B') = 1 - P(B)$, so

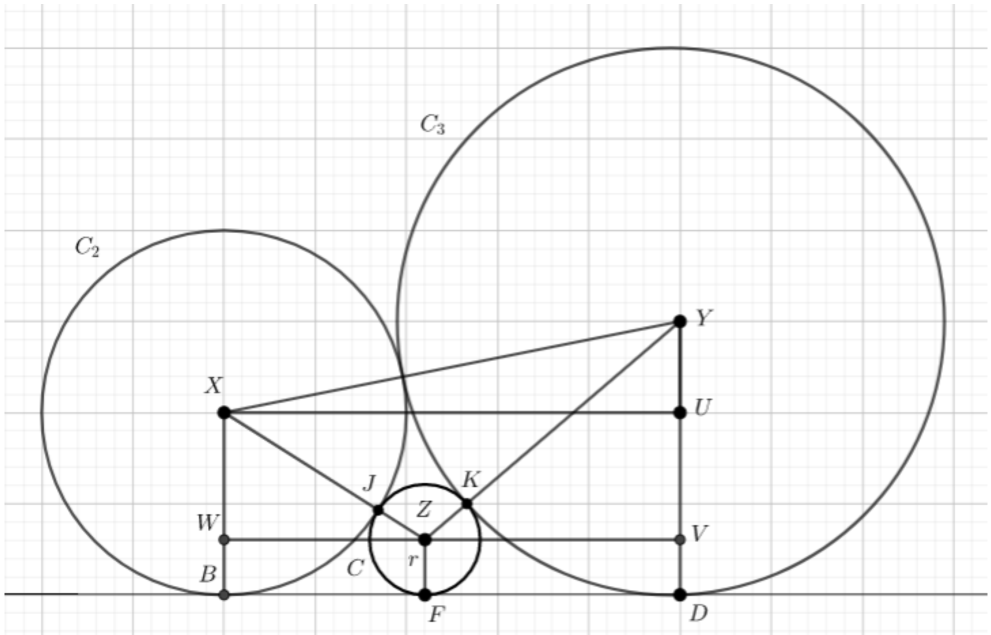
$P(B) - (1 - P(B)) = 2P(B) - 1 = 0 \leftrightarrow P(B) = \frac{1}{2}$. Substituting this into $P(A)P(B) = 0.4$ yields $P(A) = \frac{0.4}{0.5} = 0.8$. Then $P(A') = 1 - P(A) = .2$ and $P(B) = .5$, so $P(A' \cap B') = P(A')P(B') = .2 \cdot .5 = .1 = \mathbf{1/10.}$

F18B24. $\mathbf{30 - 12\sqrt{6}}$. Let X, Y, and Z be the centers of C_2, C_3, and C respectively and let r be the radius of C. Also define points B, D, F, J, K, W, V and U such that

- B, D, F are the points of tangency of C_2, C_3, and C, respectively, with the x-axis.
- J is the point of tangency of C_2 and C.
- K is the point of tangency of C_3 and C.
- W is the point on $\overrightarrow{XB}$ such that $\overline{WZ}$ is parallel to the x-axis
- V is the point on $\overrightarrow{YD}$ such that $\overline{VZ}$ is parallel to the x-axis
- U is the point on $\overrightarrow{YD}$ such that $\overline{XU}$ is parallel to the x-axis

Note that because $\overline{XB}$, $\overline{YD}$, and $\overline{ZF}$ are radii to points of tangency with the x-axis, they form right angles with the x-axis (and hence are vertical line segments). Also because $\overline{WZ}$, $\overline{VZ}$, and $\overline{XU}$ are parallel to the x-axis, $\overline{WZ}$ is perpendicular to $\overline{XB}$, and $\overline{VZ}$ and $\overline{XU}$ are each perpendicular to $\overline{YD}$. Hence, ΔXUY, ΔYWZ, ΔYVW are right triangles, and $WBZF$, $ZFDV$, $XUVW$ are rectangles. Furthermore, $WZ \parallel ZV$ implies W, Z, V are collinear.

Then $XW = XB - WB = XB - ZF = |2 - r|$,
$XZ = XJ + JZ = 2 + r$, $YV = YD - DV = YD - ZF = 3 - r$, $YZ = YK + KZ = 3 + r$,
$YU = YD - UD = YD - XB = 3 - 2 = 1$, and $XY = 2 + 3 = 5$. Since $\overline{YU}$ is vertical and $\overline{UX}$ is horizontal, ΔYUX is a right triangle. Similarly, ΔXWZ and ΔYVZ are also right triangles. Thus by the Pythagorean Theorem on ΔYUX, $XU = WV = \sqrt{5^2 - 1^2} = \sqrt{24}$. Notice that $WZ + VZ = \sqrt{24}$ and by the Pythagorean Theorem on ΔXWZ and ΔYVZ,
$WZ = \sqrt{XZ^2 - WX^2}$ and $VZ = \sqrt{YZ^2 - VY^2}$ so
$\sqrt{(2+r)^2 - (|2-r|)^2} + \sqrt{(3+r)^2 - (3-r)^2} = \sqrt{24}$. This implies $\sqrt{8r} + \sqrt{12r} = \sqrt{24}$ so
$r = \frac{24}{(\sqrt{8}+\sqrt{12})^2} = \frac{24}{20+8\sqrt{6}} = \frac{6}{5+2\sqrt{6}} = \mathbf{30 - 12\sqrt{6}}$.



C_3
C_2
Y
X
U
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V
r
B
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F
D

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division CONTEST NUMBER 5 SOLUTIONS Fall 2018

F18B25. **1**. Because $\overline{A03307A6}$ is divisible by 44, it is divisible by 4 and 11. The latter implies $A - 0 + 3 - 3 + 0 - 7 + A - 6 = 2A - 13$ is divisible by 11. Since A is a digit, $0 \le A \le 9$, implying $-13 \le 2A - 13 \le 5$. The only multiples of 11 in this range are -11 and 0. If $2A - 13 = -11$, then $A = 1$. If $2A - 13 = 0$ then A is not an integer. Since any integer is divisible by 4 if and only if the number formed by the last two digits is divisible by 4, $A = 1$ ensures that the original number is divisible by 4, so the answer is **1**.

F18B26. **3**. Rewrite the equation as $(2^x)^2 - 2^x - 56 = 0$, which is equivalent to $(2^x - 8)(2^x + 7) = 0$. The first factor equals 0 when $x = 3$, and the second factor is never equal to 0 for all real values of x, so the only real value of x that satisfies the equation is $x = \mathbf{3}$.

F18B27. **24**. Because $NYCIML$ is a regular hexagon, diagonals $\overline{NI}$, $\overline{YM}$, $\overline{CL}$ intersect at a common point X and divide $NYCIML$ into 6 congruent equilateral triangles. The area of one of these triangles, e.g. ΔMIX, is $\frac{(MI^2)\sqrt{3}}{4} = \frac{36}{6} = 6$. Now, the perimeter of $NYCIML$ is $6 \cdot MI$ and the perimeter of GEO is $3 \cdot GO$, so $6 \cdot MI = 3 \cdot GO \leftrightarrow 2 \cdot MI = GO$. The area of ΔGEO is $\frac{(GO^2)\sqrt{3}}{4} = \frac{4(MI^2)\sqrt{3}}{4} = 4 \cdot 6 = \mathbf{24}$.

F18B28. **12**. By the formula for the sum of an arithmetic sequence, the sum is $\frac{2018(2019n)}{2} = 1009 \cdot 673 \cdot 3 \cdot n$. Note that the number of positive integer factors of an integer with prime factorization $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is $(p_1 + 1)(p_2 + 2) \cdots (p_k + 1)$. Assume first that $3n$ is not a multiple of 1009 or 673. Since 1009 and 673 are both prime, $3n$ must have $\frac{36}{(1+1)\cdot(1+1)} = 9$ distinct positive integer factors if the sum is to have 36 distinct positive integer factors. As a product of one or more integers greater than 1, 9 can only be expressed as 9 or $3 \cdot 3$, which, after using the above formula, means that $3n$ is of the form p^8 for a prime p or of the form q^2r^2 for distinct primes q and r. Since $3n$ does not divide 2^8 for any positive integer n, the smallest possible value of n, where $3n$ is not a multiple of 1009 or 673, satisfies either $3n = 3^8$ or $3n = 2^2 3^2$. The latter case produces a smaller value for n, which is $n = \frac{36}{3} = 12$. If $3n$ is a multiple of 1009 or a multiple of 673, then n would be larger than 12, so therefore, the smallest possible value of n is **12**.

F18B29. **75**. We can prime factorize both sides of $2^A \cdot 3^B \cdot 5^{13} = 20^D \cdot 18^{12}$ to obtain $2^A \cdot 3^B \cdot 5^{13} = 5^D \cdot 2^{2D} \cdot 2^{12} \cdot 3^{24}$. Since the total exponent of each prime is an integer, we must have that each such total is identical on both sides of this equation. Then $B = 24$, $D = 13$, and $A = 2D + 12 = 2(13) + 12 = 38$, so $A + B + D = 38 + 24 + 13 = \mathbf{75}$.

F18B30. $\mathbf{2258}$. Let $a_i = a + d(i-1)$ and $g_i = gr^{i-1}$, where i is a positive integer. Note that because u_i is not monotonic, whereas arithmetic sequences are, g_i must not be a degenerate sequence, i.e. $g \neq 0$ and $r \neq 1$. Then we have:

$$\begin{aligned}
u_1 &= 2018 = a \qquad\quad + g, \text{(***)}\\
u_2 &= 1738 = a + \ \ d \ + gr,\\
u_3 &= 1818 = a + 2d \ + gr^2,\\
u_4 &= 2018 = a + 3d \ + gr^3,\\
u_5 &= ???? = a + 4d \ + gr^4.
\end{aligned}$$

Subtracting successive terms yields

$$\begin{aligned}
u_2 - u_1 &= -280 \ = d \ + \ g(r-1), \text{(**)}\\
u_3 - u_2 &= \quad 80 \quad = d \ + \ gr(r-1),\\
u_4 - u_3 &= \ \ 200 \ \ = d \ + \ gr^2(r-1),
\end{aligned}$$

and subtracting these equations yields

$$\begin{aligned}
80 + 280 = 360 &= g(r-1)(r-1) \text{ (*)},\\
200 - 80 = 120 &= gr(r-1)(r-1).
\end{aligned}$$

Dividing these equations yields $r = \frac{120}{360} = \frac{1}{3}$.

Now substituting back into (*) yields $360 = g(-\frac{2}{3})(-\frac{2}{3}) \leftrightarrow g = 810$. Also (**) implies $-280 = d + 810(-\frac{2}{3}) \leftrightarrow d = 260$. Further, (***) yields $2018 = a \ + \ 810 \leftrightarrow a = 1208$. Therefore, $u_5 = 1208 + 4 \cdot 260 + 810 \cdot (\frac{1}{3})^4 = 1208 + 1040 + 10 = \mathbf{2258}$.

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 1 SOLUTIONS   Spring 2019

S19B01. $\mathbf{38}$. Let the y-intercept of the line be b. The slope of the line is $\frac{N-19}{19}$ or $\frac{19-b}{20}$. Equating, $20N - 380 = 361 - 19b$. Thus $19b = 741 - 20N < 0$ (since $b < 0$), implying $N > \frac{741}{20} = 37.05$. The least integer N that satisfies the conditions of the problem is $N = \mathbf{38}$.

S19B02. $\mathbf{2294}$. Because $m : n : p = 7 : 1 : 8$, represent $m = 7n$ and $p = 8n$. Then the given equation is equivalent to $(7n) + 2(n) + 3(8n) = 2442 \leftrightarrow 33n = 2442 \leftrightarrow n = 74$. Then $3m + 2n + p = 3(7n) + 2(n) + (8n) = 31n = 31(74) = \mathbf{2294}$.

S19B03. $\mathbf{401}$. Suppose that neither P nor Q is 2; then $5P + 7Q$ is the sum of two odd numbers and therefore even. This is a contradiction because the sum must be 2019, so either P or Q is 2. If $P = 2$, then $Q = \frac{2019-5\cdot 2}{7} = \frac{2009}{7} = 287$ which factors as $7 \cdot 41$, which is not prime. Then $Q = 2$, and $P = \frac{2019-7\cdot 2}{5} = \frac{2005}{5} = \mathbf{401}$, which is prime.

S19B04. $\frac{\mathbf{1}}{\mathbf{2}}$. Since 16168 can be factored as $2021 \cdot 8$, $2021 - 8 = 2013$, and $14141 = 2019 \cdot 7 + 8$, we can rewrite the given expression as $\frac{2019^2-7\cdot 2019-8}{2019^2+(2021-8)\cdot 2019-2021\cdot 8}$, and then factor to obtain $\frac{(2019-8)(2019+1)}{(2019-8)(2019+2021)}$. After cancelling out the common factor of $2019 - 8$, this expression simplifies to $\frac{2020}{4040} = \frac{\mathbf{1}}{\mathbf{2}}$.

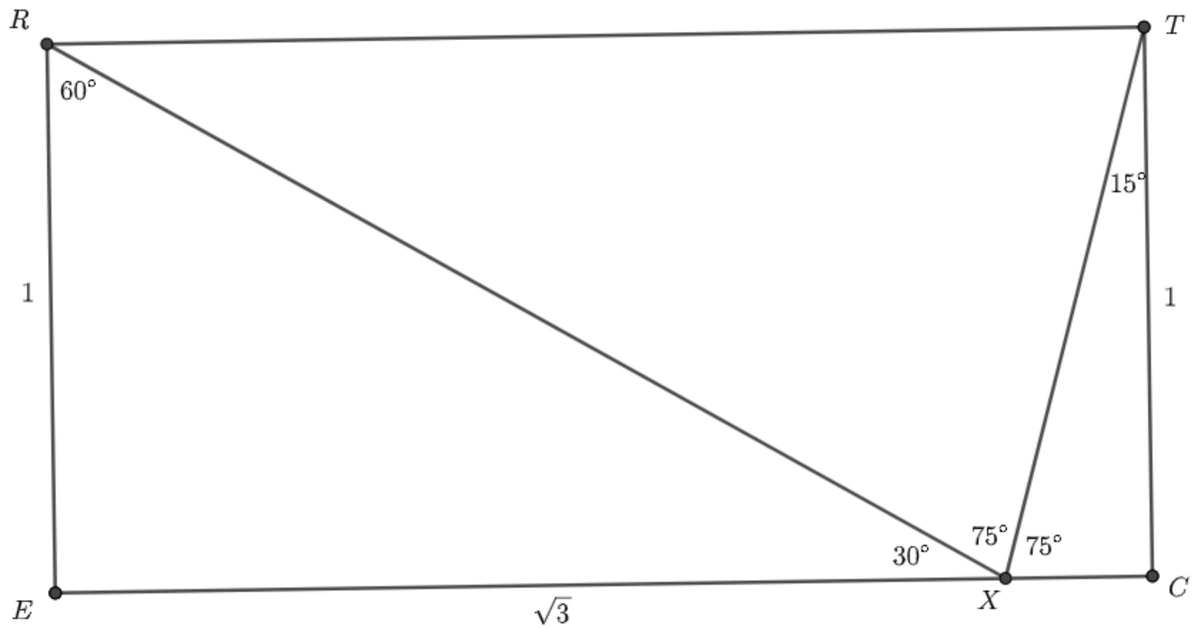
S19B05. $(\mathbf{4}, \mathbf{6})$. Notice that in general $g\big(g(x)\big) = a(ax + b) + b = a^2x + ab + b = 16x + 30$ for all real x. This implies that $a^2 = 16$ and $ab + b = 30$. Since a is positive, $a = 4$, and so $ab + b = b(a + 1) = 5b = 30$, so $b = 6$. Therefore, the ordered pair (a, b) is $(\mathbf{4}, \mathbf{6})$.

S19B06. **2**. **Solution 1.** Since $m\angle REX = 90°$, $m\angle ERX$ and $m\angle EXR$ sum to $90°$. The measure of $\angle ERX$ is twice the measure of $\angle EXR$, so the angles have measures of $60°$ and $30°$, respectively. Thus ΔREX is a 30-60-90 triangle, with $RE = 1$, $EX = \sqrt{3}$, and $RX = 2$. Also, $m\angle TXC = \frac{1}{2} \cdot m\angle RXC = \frac{1}{2}(180° - m\angle EXR) = \frac{1}{2} \cdot 150° = 75°$, so $m\angle XTC = 15°$. Since $\tan\angle XTC = \frac{XC}{TC} = \frac{XC}{1}$, $XC = \tan 15°$, so

$$EC = EX + XC = \sqrt{3} + \tan 15° = \tan 60° + \tan 15° = \frac{\sin 75°}{\cos 60° \cos 15°} = \frac{1}{\cos 60°} = 2,$$

so the area of $RECT$ is $RE \cdot EC = 1 \cdot 2 = \mathbf{2}$.

Solution 2. We have $RX = 2$ from Solution 1. Note that $m\angle RTX = m\angle TXC$ because they are alternating interior angles, and $m\angle TXC = m\angle RXT$ because $\overline{XT}$ is an angle bisector. Thus, $m\angle RTX = m\angle RXT$ and ΔXRT is isosceles with $RT = RX = 2$. The area of $RECT$ is $RE \cdot RT = 1 \cdot 2 = \mathbf{2}$.



NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

Senior B Division   CONTEST NUMBER 2 SOLUTIONS   Spring 2019

S19B07. $\mathbf{96}$. The elevator goes up three floors in 12 seconds for an average of four seconds per floor. The trip from Floor 14 to Floor 38 is a rise of 24 floors, which takes $24 \cdot 4 = \mathbf{96}$ seconds.

S19B08. $\mathbf{132}$. Let a be the first term of the arithmetic sequence and let d be the common difference of the sequence. Then $a_n = a + (n-1) \cdot d$ for all positive integers n. The two given equations imply $a + 5d + a + 6d - a - 4d = a + 7d = 20$, and
$a + 12d + a + 13d - a - 10d = a + 15d = 52$. Subtracting, $8d = 32$, so $d = 4$ and
$a = 52 - 15 \cdot 4 = -8$. Now, the desired quantity is
$a + 19d + a + 18d = 2a + 37d = -16 + 148 = \mathbf{132}$.

S19B09. $\frac{\mathbf{1}}{\mathbf{2}}$. Since the coin is fair, the probability that a coin will come up heads is equal to the probability that it will come up tails. Note that since each of fair coins comes up either heads or tails, the condition "exactly k fair coins come up tails" is equivalent to "exactly $9 - k$ fair coins come up heads", for each integer k from 0 to 9. Also, by symmetry, the probability that exactly k fair coins will come up heads equals the probability that exactly k fair coins will come up tails, for each integer k from 0 to 9. Therefore, the probability that exactly k fair coins will come up heads equals the probability that exactly $9 - k$ fair coins will come up heads, for each integer k from 0 to 9.

Since k and $9 - k$ have opposite parity for any integer k, this implies the total probability that an odd number of coins will come up heads is equal to the total probability that an even number of coins will come up heads. Since the number of coins that will come up heads must be either odd or even (i.e. the sum of these total probabilities is 1), the probability that an odd number of coins will come up heads is $\frac{\mathbf{1}}{\mathbf{2}}$.

S19B10. $\mathbf{384 + 14\pi}$. The cube begins with $6 \cdot 8 \cdot 8 = 384$ square cm of surface area. Then, subtract two circles of radius 1 cm, with a total area of $2 \cdot \pi \cdot 1^2 = 2\pi$ square cm. Now, add the area of the curved surface from the interior of the wood; it has area $2\pi \cdot 8 = 16\pi$ square cm. The surface area of the resulting solid is therefore $384 - 2\pi + 16\pi = \mathbf{384 + 14\pi}$ square cm.

S19B11. $\mathbf{18}$. Let l and w be the length and width of $RECT$, respectively. The problem requires that $2l + 2w = lw$. Solving for w yields $w = \frac{2l}{l-2} = 2 + \frac{4}{l-2}$. Since w is a positive integer, $l - 2$ evenly divides 4. Since l is a positive integer, $l - 2 = -1, 1, 2,$ or $4 \leftrightarrow l = 1, 3, 4,$ or 6.
$l = 1$ yields $w = 2 - 4 = -2$, $l = 3$ yields $w = 2 + 4 = 6$, $l = 4$ yields $w = 2 + 2 = 4$, and
$l = 6$ yields $w = 2 + 1 = 3$. The case where $l = 4$ and $w = 4$ results in $RECT$ being a square,

so we can discard this case. The case where $l = 1$ and $w = -2$ can be discarded because w must be positive, which leaves us with $w = 6$ when $l = 3$ and $w = 3$ when $l = 6$. $lw = 18$ for both of these cases, so $P = \mathbf{18}$.

S19B12. $\mathbf{3}$. 2019 leaves a remainder of 12 when divided by a positive integer N if and only if 2007 is a multiple of N and $N > 12$. Thus $3^2 \cdot 223$ must be a multiple of N. By the formula for the number of positive integer factors of a positive integer, 2007 has $(3 + 1)(2 + 1) = 6$ factors. Of these, 1, 3, and 9 are too small to leave a remainder of 12. The other three factors 223, 669, and 2007 are all large enough to leave remainders of 12, so the answer is $6 - 3 = \mathbf{3.}$

NEW YORK CITY INTERSCHOLASTIC MATHEMATICS LEAGUE

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S19B13. $\mathbf{703}$. The mean of these integers is $\frac{3^9}{27} = \frac{3^9}{3^3} = 3^6 = 729$. This will be the "middle" consecutive odd integer. There will be $\frac{27-1}{2} = 13$ odd integers on either side of 729, so the least of these is $729 - 13(2) = \mathbf{703}$.

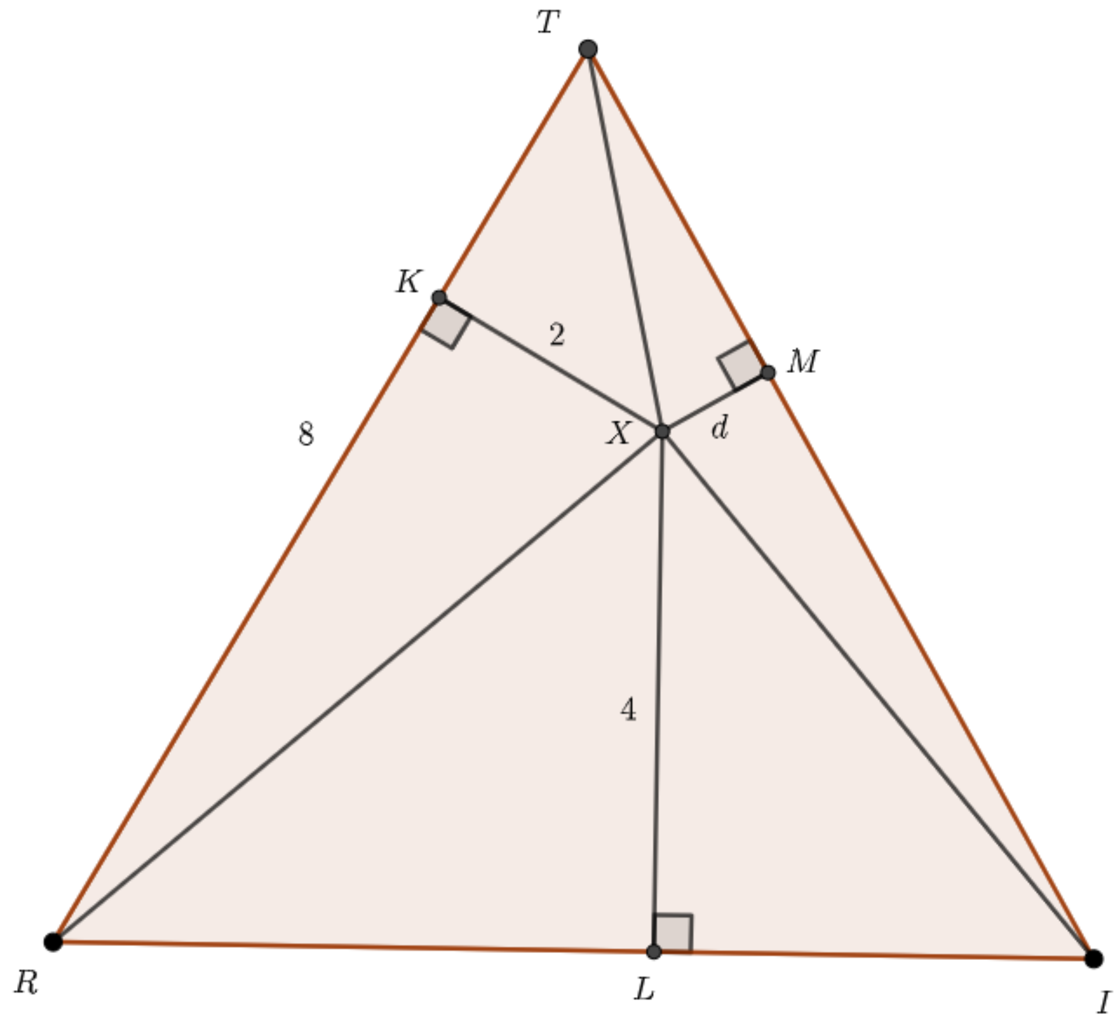
S19B14. $\mathbf{\frac{4}{3}}$. Rewrite $2^{3x+2} - 2^{3x} = 48$ as $2^{3x}(2^2 - 1) = 48 \leftrightarrow 2^{3x} = 16$, so $3x = 4$ and $x = \mathbf{\frac{4}{3}}$.

S19B15. $\mathbf{72}$. The original number of girls and boys are $5x$ and $4x$ for some integer x. Then, after 18 boys join, solve $\frac{4x+18}{5x} = \frac{5}{4}$ to obtain $16x + 72 = 25x$, which yields $x = 8$. The original number of people at the party is $5x + 4x = 9x = 9 \cdot 8 = \mathbf{72}$.

S19B16. $(\mathbf{4, 4})$ **and** $(\mathbf{-4, -4})$. Adding the two equations together yields $2x^2 - 2xy = 0 \leftrightarrow 2x(x - y) = 0$. If $x = 0$, there are no real solutions y. If $x - y = 0$, then $y = x$, and the first given equation becomes $x^2 + 3x^2 - x^2 = 48$, which solves to obtain $x = \pm 4$. In this case, the y-values are also ± 4, so the answers are $(\mathbf{4, 4})$ **and** $(\mathbf{-4, -4})$.

S19B17. $\mathbf{202}$. Rewrite $|20x - 19| \leq 2019$ as $-2019 \leq 20x - 19 \leq 2019$, so $-2000 < 20x \leq 2038 \leftrightarrow -100 \leq x \leq 101.9$. The integers in this range run from -100 to 101. There are $101 - (-100) + 1 = \mathbf{202}$ integers in this range.

S19B18. $\mathbf{4\sqrt{3} - 6}$. Let $[ABC]$ denote the area of ΔABC. Then $[TRI] = [TRX] + [TIX] + [RIX]$. An equilateral triangle with side length s has area $\frac{s^2\sqrt{3}}{4}$, so the area of ΔTRI is $\frac{8^2\sqrt{3}}{4} = 16\sqrt{3}$. Let K, L, and M be the points on lines $\overleftrightarrow{TR}$, $\overleftrightarrow{RI}$, and $\overleftrightarrow{IT}$, respectively, such that $\overline{XK} \perp \overleftrightarrow{TR}$, $\overline{XL} \perp \overleftrightarrow{RI}$, and $\overline{XM} \perp \overleftrightarrow{IT}$ (see the diagram). Since $\angle TRI$, $\angle RIT$, and $\angle ITR$ are acute angles, the angles $\angle XRT$, $\angle XTR$, $\angle XTI$, $\angle XIT$, $\angle XRI$, and $\angle XIR$ are all acute angles as well. This means that points K, L, and M lie on segments $\overline{TR}$, $\overline{RI}$, and $\overline{IT}$, respectively, and therefore XK, XL, and XM are distances from X to segments $\overline{TR}$, $\overline{RI}$, and $\overline{IT}$, respectively. Then $[XTR] = \frac{XK \cdot TR}{2} = \frac{2 \cdot 8}{2} = 8$, $[XRI] = \frac{XL \cdot RI}{2} = \frac{4 \cdot 8}{2} = 16$, and $[XIT] = \frac{XM \cdot IT}{2} = \frac{d \cdot 8}{2} = 4d$, where d is the desired distance. Now we have $[TRI] = [TRX] + [TIX] + [RIX] = 24 + 4d$, and setting this expression equal to $16\sqrt{3}$ yields $d = \mathbf{4\sqrt{3} - 6}$.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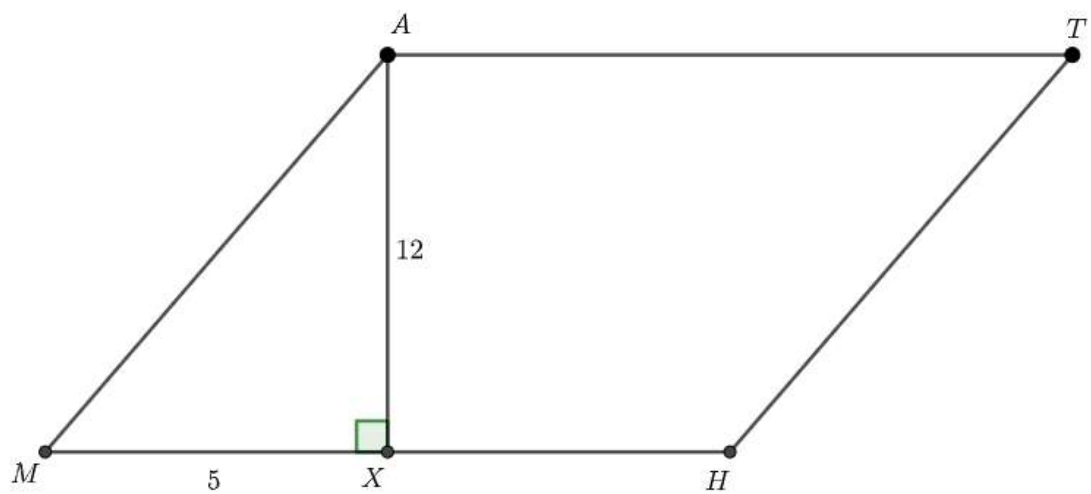
New York City Interscholastic Mathematics League

Senior B Division CONTEST NUMBER 4 SOLUTIONS Spring 2019

S19B19. Notice that $9951 = 100^2 - 7^2 = (100+7)(100-7) = 107 \cdot 93 = 107 \cdot 31 \cdot 3$. Since 107, 31, and 3 are prime, the answer is $\mathbf{107.}$

S19B20. $\mathbf{288}$. The length of a diagonal of a cube of side length s is $s\sqrt{3}$, so solve $s\sqrt{3} = 12 \leftrightarrow s = 4\sqrt{3}$. The surface area of the cube is $6s^2 = 6(4\sqrt{3})^2 = 6(16 \cdot 3) = \mathbf{288.}$

S19B21. $\mathbf{204}$. By the Pythagorean Theorem, $MA = \sqrt{MX^2 + AX^2} = \sqrt{5^2 + 12^2} = 13$. Since opposite sides of a parallelogram have equal lengths, the sum of two adjacent sides is equal to half the perimeter, which is equal to $\frac{1}{2} \cdot 60 = 30$. Using this, $AT + AM = 30$, so $AT = 30 - AM = 30 - 13 = 17$. Therefore, the area of the parallelogram is $AT \cdot AX = 17 \cdot 12 = \mathbf{204}$.



S19B22. $\mathbf{41}$. Since $3x = 500 - 4y = 4(125 - y)$, $3x$ must be divisible by 4. Since 3 is prime, 4 does not divide 3, so 4 must divide x. Then $x = 4k$, where k is a positive integer. Note that $3x = 12k = 500 - 4y \leq 500 - 4 = 496$. Then $3k \leq 124$, so $k \leq \frac{124}{3} = 41\frac{1}{3}$. Since k is an integer, $k \leq 41$. Note that for every integer k between 1 and 41 inclusive, we can take $x = 4k$, which yields $y = \frac{500-3x}{4} = \frac{500-12k}{4} = 125 - 3k \geq 125 - 3 \cdot 41 = 2 > 0$. This yields positive integers x and y satisfying $3x + 4y = 500$, and different values of k yield different ordered pairs (x, y) since $x = 4k$. Therefore, the number of distinct ordered pairs of positive integers (x, y) for which $3x + 4y = 500$ equals the number of integers between 1 and 41 inclusive, so the answer is $\mathbf{41}$.

S19B23. $\mathbf{2025}$. The elements of the collection, excluding x, placed in sorted order are $\{0, 1, 2, 9, 9, 19, 20, 201\}$. Note that if $x < 9$, then the median of the collection is 9 because 0, 1, 2, and x are all less than 9 and 9, 19, 20, and 201 are greater than or equal to 9. Similarly, we can show that the median is 9 when $x = 9$ and when $x > 9$, so the median of the collection is 9 regardless of the value of x. Since x is non-negative, the mean of the collection is non-negative with value $9 + 245 = 254$. The sum of the numbers in the collection is $2 + 0 + 1 + 9 + 201 + 9 + 20 + 19 + x = 261 + x$. For this collection to have a mean of 254, we must have $261 + x = 254 \cdot 9 = 2286$, which yields $x = 2286 - 261 = \mathbf{2025}$.

S19B24. $\boldsymbol{(a + b + 1)(a^3 - a^2b + ab^2 - b^3 + 1)}$ **or** $\boldsymbol{(a^3 - a^2b + ab^2 - b^3 + 1)(a + b + 1)}$ **or any reordering of the terms in each factor**. First, group the terms as follows: $(a^4 - b^4) + (a^3 - a^2b + ab^2 - b^3) + (a + b + 1)$. Note that by difference of squares, the polynomial in the first set of parentheses factors as $(a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a + b)(a - b) = (a^3 - a^2b + ab^2 - b^3)(a + b)$. Therefore, the sum of the first two expressions is equal to $(a^3 - a^2b + ab^2 - b^3)(a + b + 1)$. Notice that there is an $a + b + 1$ term at the end of the sum in the first sentence, so the entire given expression factors as $\boldsymbol{(a^3 - a^2b + ab^2 - b^3 + 1)(a + b + 1)}$. Note that $a^3 - a^2b + ab^2 - b^3 + 1$ is a polynomial with degree 3 and constant term 1, and $a + b + 1$ is a polynomial with degree 1 and constant term 1, as required.

Challenge: Try to prove that the required factorization is unique up to the order of factors and/or terms in each factor.

S19B25. $\mathbf{2027}$. Each of the positive integers from 1 through 2019 will divide 2019!. Consider integers just greater than 2019. We can factor these to obtain $2020 = 1010 \cdot 2$, $2021 = 43 \cdot 47$, $2022 = 1011 \cdot 2$, $2023 = 7 \cdot 289$, $2024 = 1012 \cdot 2$, $2025 = 3 \cdot 675$, and $2026 = 1013 \cdot 2$, all of which divide 2019!. The next integer to check is 2027. We claim that 2027 is prime. To see if 2027 is prime, it suffices to show that none of the primes less than or equal to $\sqrt{2027}$ divides 2027. Since $45 = \sqrt{2025} < \sqrt{2027} < \sqrt{2116} = 46$, we will check primes not greater than 45. The prime numbers not greater than 45 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, and since

$$
\begin{gathered}
2027 = 2 \cdot 1013 + 1, \\
2027 = 3 \cdot 675 + 2, \\
2027 = 5 \cdot 405 + 2, \\
2027 = 7 \cdot 289 + 4, \\
2027 = 11 \cdot 184 + 3, \\
2027 = 13 \cdot 155 + 12, \\
2027 = 17 \cdot 119 + 4, \\
2027 = 19 \cdot 106 + 13, \\
2027 = 23 \cdot 88 + 3, \\
2027 = 29 \cdot 69 + 26, \\
2027 = 31 \cdot 65 + 12, \\
2027 = 37 \cdot 54 + 29, \\
2027 = 41 \cdot 49 + 18, \\
2027 = 43 \cdot 47 + 6,
\end{gathered}
$$

2027 is prime. As a result, 2027 does not divide 2019! because $2019 < 2027$ and 2027 is prime. Therefore, the answer is $\mathbf{2027}$.

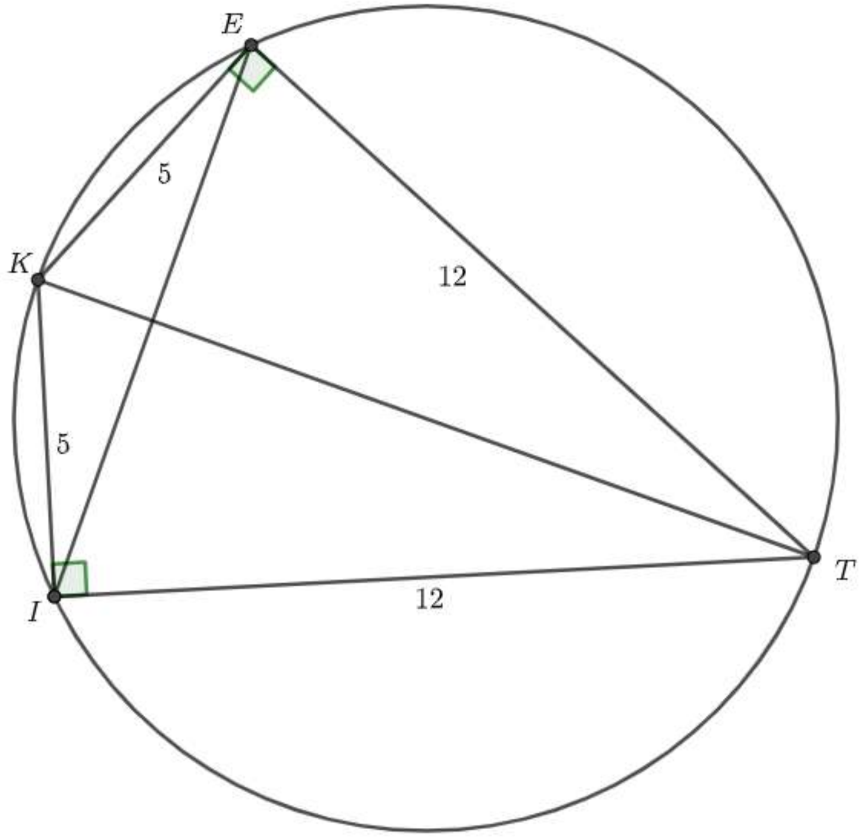
S19B26. $\mathbf{15}$. By algebra,
$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac = 83 + 2 \cdot 71 = 225.$
Since a, b, and c are positive, $a + b + c$ is positive, so we can reject the negative square root to get $a + b + c = \mathbf{15}$.

S19B27. $\mathbf{6}$. Note that for all first-quadrant points (x, y), $x > 0$ and $y > 0$. Since $20x = 2019 - 19y$, $2019 - 19y$ must be divisible by 20.
$2019 - 19y = (2000 - 20y) + (19 + y)$ and $2000 - 20y$ is divisible by 20, $19 + y$ must be divisible by 20. Note that $19y = 2019 - 20x \le 2019 - 20 = 1999$. Then $19y < 1999$, so $y < \frac{1999}{19} = 105\frac{4}{19}$. Since y is an integer, $y \le 105$. This means that

$19 < 19 + y \le 19 + 105 = 124$, so the multiples of 20 that $19 + y$ can be equal to are 20, 40, 60, 80, 100, and 120. This yields the ordered pairs (x, y): (100, 1), (81, 21), (62, 41), (43, 61), (24, 81), and (5, 101). All 6 ordered pairs are distinct and satisfy all required conditions, so there are a total of 6 first-quadrant lattice points on the line $20x + 19y = 2019$.

S19B28. $\mathbf{\frac{1}{21}}$. Take the log (base 10) of both sides to obtain $\log 3 \cdot \log(3x) = \log 7 \cdot \log(7x)$, which implies $\log 3(\log 3 + \log x) = \log 7(\log 7 + \log x)$. This simplifies to $(\log 3)^2 + \log 3 \log x = (\log 7)^2 + \log 7 \log x$. Rearranging and factoring yields $\log x\,(\log 3 - \log 7) = (\log 7)^2 - (\log 3)^2$. $\log 3 - \log 7 = \log\frac{3}{7} \neq 0$, so we can divide both sides by this expression. Factoring the difference of two squares and dividing both sides by $\log 3 - \log 7$ yields $\log x = -(\log 7 + \log 3) = -\log 21$, so $x = \mathbf{\frac{1}{21}}$.

S19B29. $\mathbf{\frac{120}{13}}$. ΔKIT and ΔKET are congruent triangles by SSS because $KI = KE$, $IT = ET$, and $KT = KT$. Then $m\angle KIT = m\angle KET$. Since $\angle KIT$ and $\angle KET$ inscribe opposite arcs of the circle, $\angle KIT + \angle KET = 180°$, so $m\angle KIT = m\angle KET = 90°$. Then $KT = \sqrt{KI^2 + IT^2} = \sqrt{5^2 + 12^2} = 13$ and the area of $KITE$ is $2 \cdot \frac{1}{2} \cdot 5 \cdot 12 = 60$. Since $KI = KE$ and $TI = TE$, $KITE$ four sides can be grouped into two pairs of equal-length sides that are adjacent to each other, making it a kite. The area of a kite is half the product of its diagonals, so $\frac{1}{2} \cdot 13 \cdot IE = 60$, which yields $IE = \mathbf{\frac{120}{13}}$.



S19B30. $\mathbf{54}$. The arithmetic sequence is w, $w + d$, $w + 2d$, $w + 3d$ and the geometric sequence is w, $w + d$, $w + 2d + 3$, $w + 3d + 10$. Notice that $(w + d)^2 = (w)(w + 2d + 3)$, which implies $w^2 + 2wd + d^2 = w^2 + 2wd + 3w$, so $d^2 = 3w$. Also, $(w)(w + 3d + 10) = (w + d)(w + 2d + 3)$, which implies $w^2 + 3wd + 10w = w^2 + 3wd + 2d^2 + 3w + 3d$, so $7w = 2d^2 + 3d$. Substituting $3w$ for d^2 in this last equation yields $w = 3d$. Then the arithmetic sequence is $3d$, $4d$, $5d$, $6d$ and the geometric sequence is $3d$, $3dr$, $3dr^2$, $3dr^3$. Equating $4d = 3dr$ yields $r = \frac{4}{3}$ or $d = 0$. If $r = \frac{4}{3}$, $5d + 3 = 3dr^2 \leftrightarrow 5d + 3 = \frac{16}{3}d \leftrightarrow d = 9$. Therefore, the value of z in this case is $z = 6 \cdot d = 6 \cdot 9 = 54$. If $d = 0$, then the arithmetic sequence is 0, 0, 0, 0 and the geometric sequence is 0, 0, 0, 0. This leads to $z = 0$ from the arithmetic sequence and $z + 10 = 0$ from the geometric sequence, which forms a contradiction. Therefore, the only possible value for z is $\mathbf{54}$.