SPH 202
ELECTRICITY AND MAGNETISM II

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Prerequisite Course or knowledge

In order to be able to complete this course, you must have successfully passed the magnitude Electricityand Magnetism 1 and you must have a clear understanding of the following notions:

Ohm's Law;

Kirchoff's laws;

Effect of an electric field on a charged particle

Effect of a magnetic field on a moving charged particle

Magnetic flux

Ampere's theorems

COURSE AIM

To enhance the learner to the one of the MOST interesting areas of Physics, which is Electricity and Magnetism.

UNIT AIM

To develop the capacity and competences of the learner to work in electronic industries and be able to repair electronic equipment such as radio, television, mobile phones, fridges, cars, etc., and faulty electricity supply systems.

COURSE OBJECTIVE

At the end of the course, the learner should be able to demonstrate an understanding of the Science of Physics and its different branches, and solve problems requiring concepts of Electricity and Magnetism.

UNIT OBJECTIVES

At the end of this course the leaner should be able to:

i. Describe charges as a quantity that, when in motion produces electricity and has a magnetic effect.
ii. Distinguish between electric and magnetic charges and their interactions.
iii. Replace calculations of forces and potential energy of charges with fields involved.
iv. Describe the properties and effects of alternating currents.
v. Distinguish between the electric and magnetic energy stored in the electric fields and magnetic fields respectively.
vi. Explain the electric and magnetic properties of matter.
vii. Derive Maxwell's Equations of motion for the electromagnetic field.

THE UNIT OUTLINE

In this Unit, the learner will be introduced to:

i. The electric field
ii. Steady current
iii. Magnetism
iv. Gauss Law
v. Electromagnetism
vi. Ampere's Law
vii. Electromagnetic induction
viii. Alternating currents
ix. Magnetic properties of matter
x. Maxwell's Equations

REFERENCE BOOKS

1) Physics by David Halliday & Robert Reshick , John Wiley & sons, Inc. Ny 1966/
2) Electricity and Magnetism by Kurrelmeyer B., D. Van nostrand Co. Inc. New York 1967
3) Electricity and Magnetism by W. J. Daffin, McGraw-Hill Publishing Co. Ltd New York 1965
4) Elements of Classical Physics by Martin C. Martin and Charles A. Hewett, Pergamon press Inc. New York 1975
5) Electricity and Magnetism by K. K. Tewari, S. Chad & Co. Ltd Ram Nagar, New Delhi 1999
6) Advanced Level Physics by M. Nelkon and P. Parker, Heineman Educational Books Ltd London 1979
7) A second course of Electricity by A. E. E. Mackenzie, Cambridge University Press London N. W. 1 1965

TIME TABLE http://timetabling.uonbi.ac.ke/

INTRODUCTION

What is Electricity?

It is energy transfer from one place to another due to the existence of charged particles, and there are two types of electricity, the Static electricity and Dynamic electricity.

- **Static electricity it is caused when non-conductive materials (e.g., rubber, plastic or glass) are rubbed together, causing a transfer of electrons, which then results in an imbalance of charges between the two materials/objects. This means that the objects will exhibit an attractive or repulsive force. It is also occurs when the negatively charged particles/electrons build up in a particular area (*stationary charge*), and then can be released suddenly (e.g. when you close a car door and get an unexpected static shock, this is as a result of charges building up, and then suddenly moving in or out of you when you touch the door. Also it happens due to friction, like when you rub a balloon against your hair and produce forces which cause your hair to stand on end). When these things happen, energy is transferring and forces are being applied.**

conti. INTRODUCTION

- **Dynamic electricity** is the flow of charged electrons (*moving charge*) around an electrical circuit. This flow is called a current, and is how all electrical devices are powered, including lights, switches, computers, and cell phones.
- The electrons essentially carry energy from the battery to the components in the circuit they are powering, and then return to the battery. They are repelled from the negative side of the battery, and attracted to the positive side of the battery. This repulsion and attraction happens due to a voltage (or difference in potential) between the two sides of the battery. This is why batteries and power supplies have a number of volts written on them - this is a way of representing how strong this attraction and repulsion is.

conti. INTRODUCTION

Sources of the electric charge

are *elementary particle*, an *electron* (has a negative charge), a *proton* (has a positive charge), an *ion*, or any *larger body that has an imbalance of positive and negative charge*. Positive and negative charges attracts each other (e.g., protons are attracted to electrons), while like charges repel each other (e.g., protons repel other protons and electrons repel other electrons).

Observed form of electricity

It is probably lightning. Lightning is a big spark that occurs when lots of electrons move from one place to another very quickly. There are three basic forms of lightning, *cloud to cloud*, *cloud to surface*, and *surface to cloud*. All are created when there is an unequal distribution of electrons.

conti. INTRODUCTION

What is magnetism?

It is defined as the physical phenomenon produced by moving electric charge or the property of magnetite attracting pieces of iron.

- **A magnetic field can induce charged particles to move, producing an electric current.**
- **It produces attraction and repulsion between objects.**
- **While electricity is based on positive and negative charges, there are no known magnetic monopoles. Any magnetic particle has a "north" and "south" pole, with the directions based on the orientation of the Earth's magnetic field. Like poles of a magnet repel each other (e.g., north repels north), while unlike/opposite poles attract one another (e.g., north and south attract).**

conti. INTRODUCTION

Examples of magnetism include

i. **a compass needle’s reaction to Earth's magnetic field,**
ii. **attraction and repulsion of bar magnets, and**
iii. **the field surrounding electromagnets.**

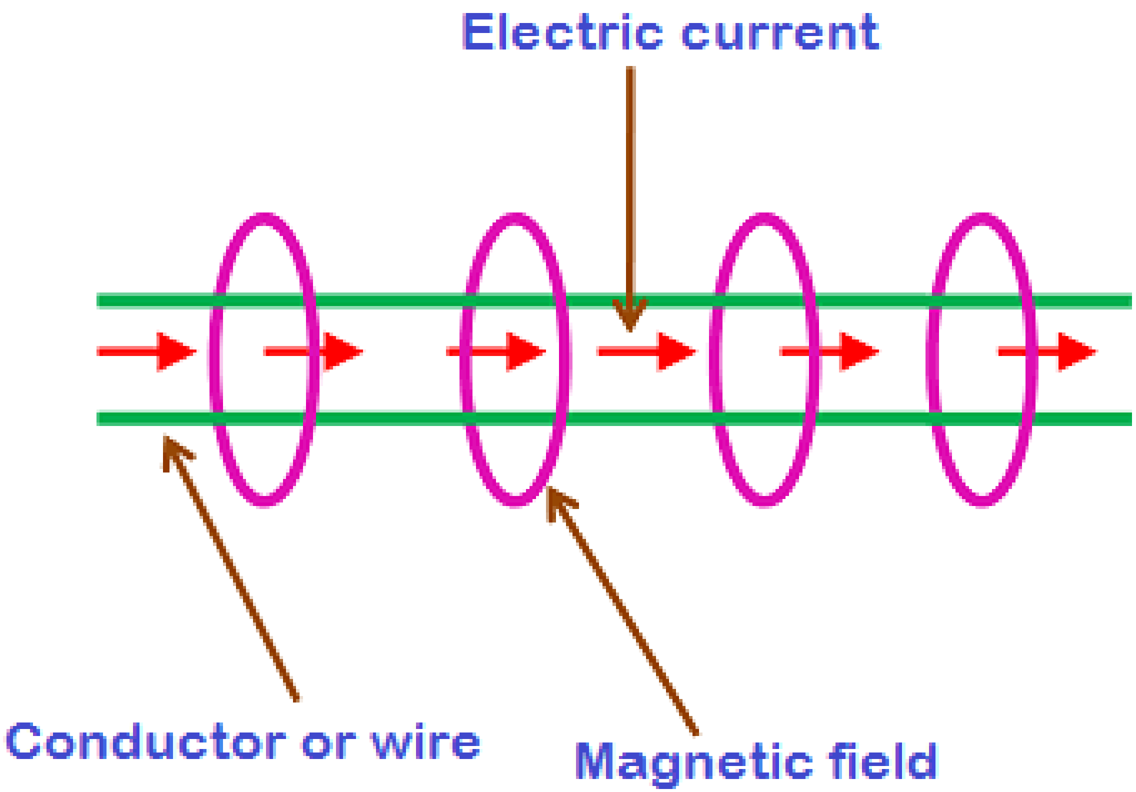
Since every moving electric charge has a magnetic field, so the orbiting electrons of atoms produce a magnetic field; there is a magnetic field associated with power lines; and hard discs and speakers rely on magnetic fields to function.

conti. INTRODUCTION

TAKE NOTE

- **Electricity and magnetism are two related phenomena produced by the electromagnetic force, thus together they form electromagnetism.**
- **A stationary point charge has an electric field (i.e. electricity present), but if the charge is set in motion, it also generates a magnetic field (i.e. magnetism present). Thus a moving electric charge generates a magnetic field.**
- **So electricity can exist without magnetism, but magnetism cannot exist without electricity.**
- **A magnetic field induces electric charge movement, producing an electric current.**
- **In an electromagnetic wave, the *electric field* and *magnetic field* are perpendicular to one another (see Figure below).**

conti. INTRODUCTION



conti. INTRODUCTION

Conti. TAKE NOTE

- **The protons are trapped inside the nucleus and can't escape the nucleus. As a result, it is the moving electrons that are primarily responsible for electricity.**

- **Understanding of electric current must begin with the nature of matter. All matter is composed of molecules. All molecules are made up of atoms, which are themselves made up of electrons, protons, and neutrons.**

conti. INTRODUCTION

- **Common SI units of electricity include the ampere (A) for current, coulomb (C) for electric charge, volt (V) for potential difference, ohm (Ω) for resistance, and watt (W) for power.**

- **Common SI units of magnetism include the tesla (T) for magnetic flux density, weber (Wb) for magnetic flux, ampere per meter (A/m) for magnetic field strength, and henry (H) for inductance.**

conti. INTRODUCTION

- **Electricity - Greeks ($\sim 600BC$) used a rubbed amber/elektron to attract pieces of straw/chaffs. Amber is rubbed with fur, it acquires *resinous electricity* $(-)$. Just as, when glass is rubbed with silk, it acquires *vitreous electricity* $(+)$. Electricity repels electricity of the same kind, but attracts electricity of the opposite kind.**
- **Magnetism – it dates at a time when naturally occurring stones/minerals called Lodestone/magnetite (Fe_3O_4) were observed to attract iron.**

➢ **In 1820 Hans Christian Oersted observed a connection between Electricity and Magnetism (both associated with electromagnetic force). That an electric current attracts/affect a magnetic compass, and this gave birth to electromagnetism. New science of electromagnetism was developed by many scientists, especially Michael Faraday**

➢ **The Laws of Electromagnetism (Biot-Savart, Ampere's, Force, and Faraday's laws) were derived by James Clerk Maxwell**

conti. INTRODUCTION

Maxwell's deductions:

- that the speed of light ≈ speed of various types of waves (310,740,000 meters per second), that depended on numbers that came from electric and magnetic measurements.
- that light is an electromagnetic wave (he unified Faraday's Law and Ampere's Law). He was the first to determine the speed of propagation of electromagnetic (EM) waves was the same as the speed of light - and hence to conclude that EM waves and visible light were really the same thing.

His work include *fundamental principles of all large-scale electromagnetic and optical devices* **e.g. motor, transformers, cyclotrons, electronic computers, radio, television, microwave radar, microscope and telescopes.**

conti. INTRODUCTION

Present interest in Electromagnetic (from Maxwell's view):

1) at the level of engineering applications (where Maxwell's equations are used to solve various practical problems).

2) at the level of foundations of the theory (where electromagnetic extends its scope to other areas of physics e.g. gravitation and modern quantum theory).

This course will enhance the electricity and magnetism knowledge already introduced to the learner (in 1st and 2nd year courses).

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LECTURE 1:
THE ELECTRIC FIELD (*E*)

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1.2 Electric forces

1.3 Electric field

1.4 Electric potential

LECTURE OBJECTIVES

At the end of this lecture, the learner should be able to:

a) **Identify the sign of a charge of any body using the law of force between charges**
b) **Define electrostatic unit of a charge**
c) **Describe the effects of forces in terms vector fields**
d) **Describe the effects of potential energy**
e) **Give examples of forces, fields and potentials**

1.2 ELECTRIC FORCES

Knowing the attractive and repulsive forces showed us that there must be two different types of quantities responsible for the force. *Gravity* had only one *type – mass* – but this new *electric force* must have two different types of "*mass*," which we refer to as *electric charge*. The units for this quantity are named after *coulombs* (*C*) who explored the force. When we play around with this electric force, we find that when two of the same kind of charge are brought together, the force is repulsive, while when we bring different types of charge together, the force is attractive. We summarize this phenomenon with the commonly-used saying: *unlike charges attract, while like charges repel*

The two elements of the electric force, as for gravity that was done in your previous course are:

i) By holding one charge fixed and varying the other,

We can determine the relationship between the amount of charge causing or experiencing the force and the strength of the force.

We find that the strength of the force (like gravity) is proportional to both the amount of charge *causing* the force and the amount of charge *feeling* the force. Recall for gravitation ($F_g = G\frac{m_1 m_2}{r^2}$) we had m_1 and m_2, and now for electric force we have q_1 and q_2 (where q is the electric charge).

The force (F)between two point charges is proportional to the product $q_1 q_2$: $\Rightarrow F \propto q_1 q_2$ 1.1

ii. **Test the dependence of the force between two charges on the separation (r) of those charges:**

We find that when the separation is doubled, the force goes down by a factor of $\frac{1}{4} = \frac{1}{2^2}$, and when it is tripled it goes down by a factor of $\frac{1}{9} = \frac{1}{3^2}$, exactly as it does for gravitation.

So the electric force also obeys an *inverse-square relation*:

$$F \propto \frac{1}{r^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 1.2$$

Putting these two Equations together with a constant (similar to G, but with different units) that exists to make our units work out correctly, we end up with a magnitude of the electric force that obeys:

$$\Rightarrow F_{q_1 \text{ on } q_2} = k\frac{q_1 q_2}{r^2} \qquad \ldots\ldots\ldots\ldots 1.3$$

All *that is missing is* our vector direction in $F_{q_1 \text{ on } q_2} = k\frac{q_1 q_2}{r^2}$), which was covered with the $-\hat{r}$ in the case of gravity.

So what do we do?

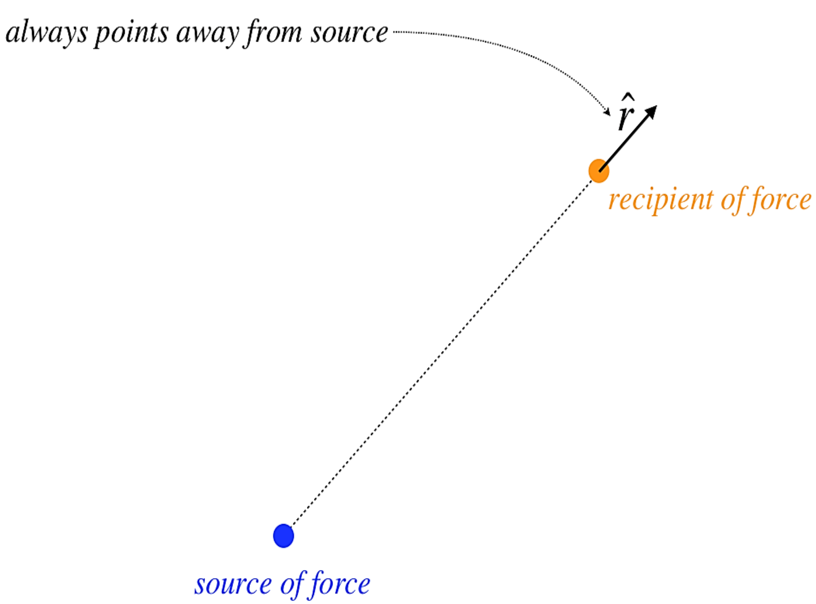
We append a little explanation, *"repulsive if both charges are the same type, attractive if they are not?"* It is not very satisfying mathematically, and it turns out there is a better solution:

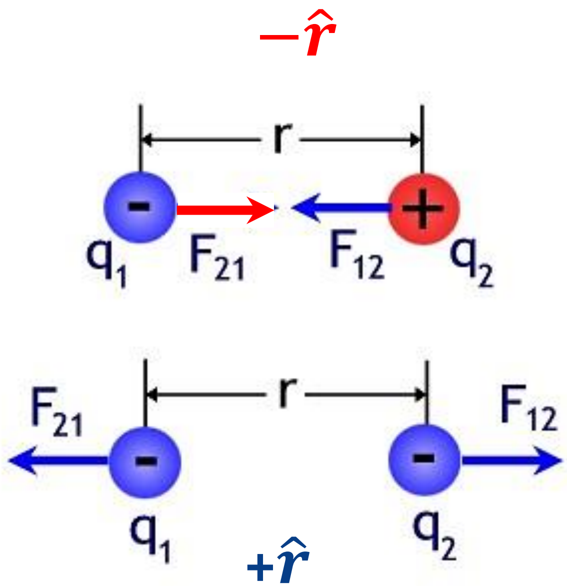
Let one of the types of charges be "positive," and the other "negative," and write the force vector as follows:

$$\vec{F}_{q_1 on\, q_2} = k\frac{q_1 q_2}{r^2}\hat{r} \;\ldots\ldots\ldots\ldots\ldots \; (1.4)$$

TAKE NOTE

$\hat{r}$ – is the unit vector that indicates the direction of this Newton's universal law of gravitation force. It is defined as pointing *away from the source of the force*. The *minus sign* indicates the *opposite direction, which is towards the source*. The force is acting on the other object, so since the direction of the force on it is towards the gravity source, gravity is said to be an *attractive force*.





conti. TAKE NOTE

The $\hat{r}$ means the same as it did before (it points away from the charge causing the force - see the above sketch). But notice what our mathematical definition of positive and negative charges accomplishes:

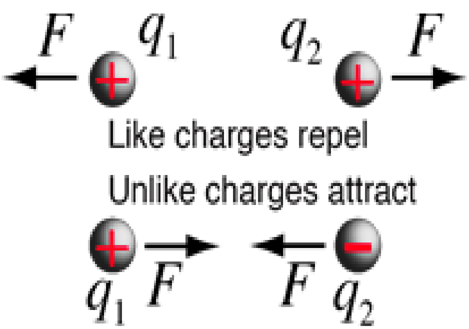
***If both charges are the same type,* then their product is positive, and the force points in the direction of $+\hat{r}$, which is repulsive (object is pushed away from the source). But *if the charges have opposite signs*, then their product is negative, and the force direction is $-\hat{r}$, resulting in an attractive force.**

The previous force law equation (Eqn. 1.3: $F_{q_1 \text{ on } q_2} = k\frac{q_1 q_2}{r^2}$) is called the *Coulomb's law*.

In summary:

conti. TAKE NOTE

Coulomb's Law states that **the magnitude of the electrostatic force of attraction or repulsion between two point charges is directly proportional to the product of the magnitudes of charges and inversely proportional to the square of the distance between them.**



$$F = \frac{kq_1q_2}{r^2} = \frac{q_1q_2}{4\pi\varepsilon_0 r^2} \quad \textit{Coulomb's Law} \quad \ldots\ldots\ldots\ldots \quad 1.5$$

where ε_0 is permittivity of free space = $\left(8.85 \times 10^{-12}\,(C^2/N.m^2) \text{ or } (Farad/m)\right)$

and $k = \dfrac{1}{4\pi\varepsilon_0} \approx 9x10^9\, N \cdot m^2/C^2 = \text{Coulomb's constant}$

EXAMPLE 1

Two point-like charges carrying charges of $+3 \times 10^{-9}\,C$ and $-5 \times 10^{-9}\,C$ are $2m$ apart. Determine the magnitude of the force between them and state whether it is attractive or repulsive.

SOLUTION

STEP 1 Determine what is required

We are required to determine the force between two point charges given the charges and the distance between them.

STEP 2 Determine how to approach the problem

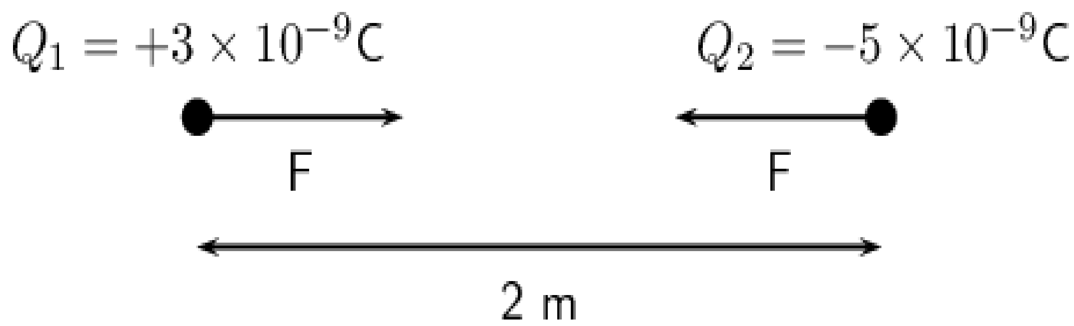
We can use Coulomb's law to calculate the magnitude of the force: $\Rightarrow \quad F = k\frac{q_1 q_2}{r^2}$

STEP 3 Determine what is given

We are given: q_1 = +3×10^{-9} C, q_2= −5×10^{-9} C and r = 2m

But k = 9.0×10^{9} N·m^{2}·C^{-2}

We can draw a diagram of the situation:



STEP 4 Check units

All quantities are in SI units (i.e., charge in Coulombs, distance in metres, and k in Newton Metre squared per Coulomb squared).

STEP 5 Determine the magnitude of the force

Using Coulomb's law:

$$\Rightarrow F = k\frac{Q_1 Q_2}{r^2}$$

$$= (9.0\times10^{9}\ \mathrm{N{\cdot}m^2{\cdot}C^{-2}})\ [(3\times10^{-9}\mathrm{C})(5\times10^{-9}\mathrm{C})] \div (2\mathrm{m})^2$$

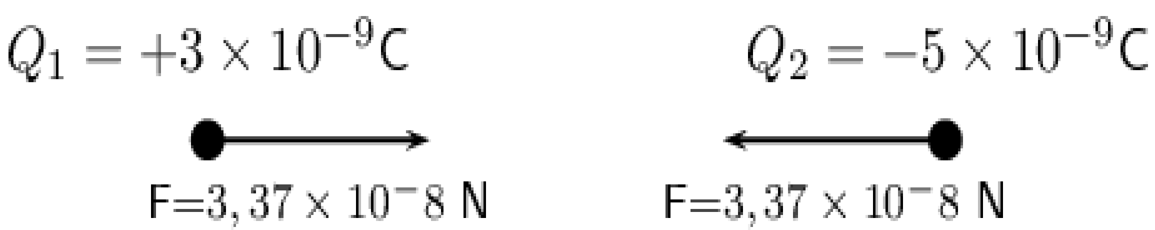
$$= 3.37\times10^{-8}\ \mathrm{N}$$

Thus the *magnitude* of the force is 3.37×10^{-8} N.

However since the point charges have opposite signs, the force will be attractive.

STEP 6 Free body diagram

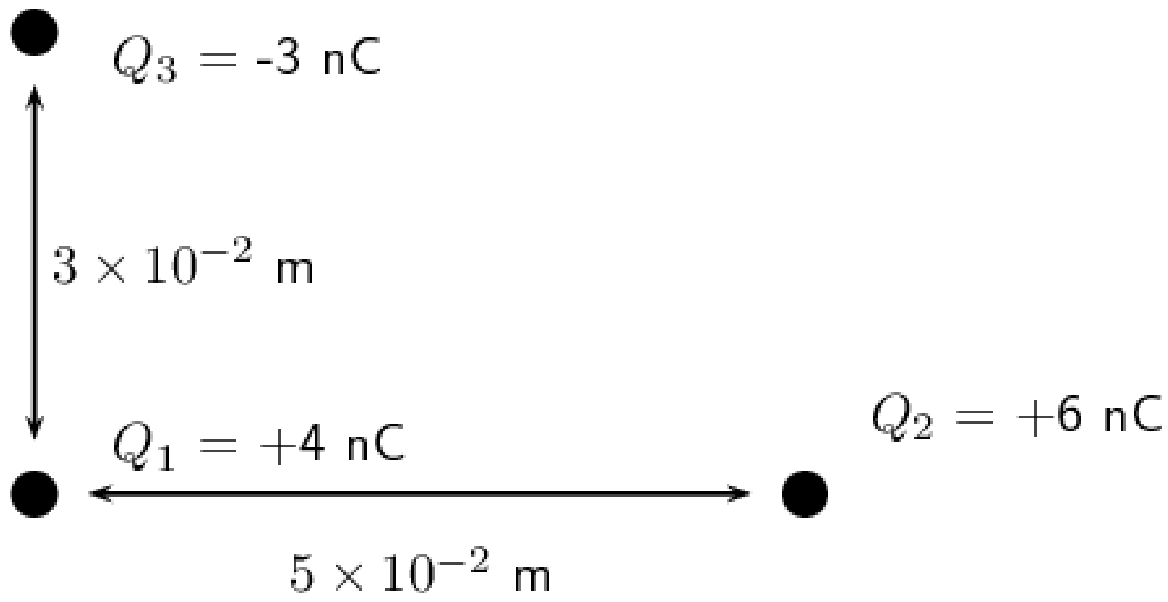
A free body diagram be drawn to show the forces. Each charge experiences a force with the same magnitude and the forces are attractive, so we have:



EXAMPLE 2

Three point charges form a right-angled triangle. Their charges are $Q_1 = 4 \times 10^{-9} C, Q_2 = 6 \times 10^{-9} C$ and $Q_3 = -3nC$. The distance between Q_1 and Q_2 is $5 \times 10^{-2} m$ and the distance between Q_1 and Q_3 is $3 \times 10^{-2} m$.

What is the net electrostatic force on Q_1 due to the other two charges if they are arranged as shown below?



SOLUTION

STEP 1 Determine what is required

We need to calculate the net force on Q_1

This force is the sum of the two electrostatic forces - the forces of Q_2 on Q_1 and Q_3 on Q_1

STEP 2 Determine how to approach the problem

We need to calculate, using Coulomb's law, the electrostatic force exerted on Q_1 by Q_2, and the electrostatic force exerted on Q_1 by Q_3.

We then need to add up the two forces using the rules for adding vector quantities, because force is a vector quantity.

STEP 3 Determine what is given

We are given all the charges and two of the distances.

STEP 4 **Calculate the magnitude of the forces**

The magnitude of the force exerted by Q_2 on Q_1, which we will let be F_2, is: $F = k\frac{Q_1Q_2}{r^2}$

$$\Rightarrow F_2 = k\frac{Q_1Q_2}{r^2}$$

$$\Rightarrow F_2 = (9.0\times10^{9}\mathrm{N\cdot m^2\cdot C^{-2}})\ [(4\times10^{-9}\mathrm{C})(6\times10^{-9}\mathrm{C})] \div (5\times10^{-2}\mathrm{m})^2$$

$$= \{(9.0\times10^{9})[(4\times10^{-9})(6\times10^{-9})]\div(25\times10^{-4})\}\mathrm{N}$$

$$= 8.630\times10^{-5}\ \mathrm{N}$$

The magnitude of the force exerted by Q_3 on Q_1, which we will let be F_3 is:

$$\Rightarrow F_3 = k\frac{Q_1 Q_3}{r^2}$$

$$=(9.0\times10^{9}\,\mathrm{N\cdot m^{2}\cdot C^{-2}})(4\times10^{-9}\,\mathrm{C})(-3\times10^{-9}\,\mathrm{C})\div(3\times10^{-2}\,\mathrm{m})^{2}$$

$$=\{(9.0\times10^{9})(4\times10^{-9})(-3\times10^{-9})\div(9\times10^{-4})\}\mathrm{N}$$

$$=-1.199\times10^{-4}\ \mathrm{N}$$

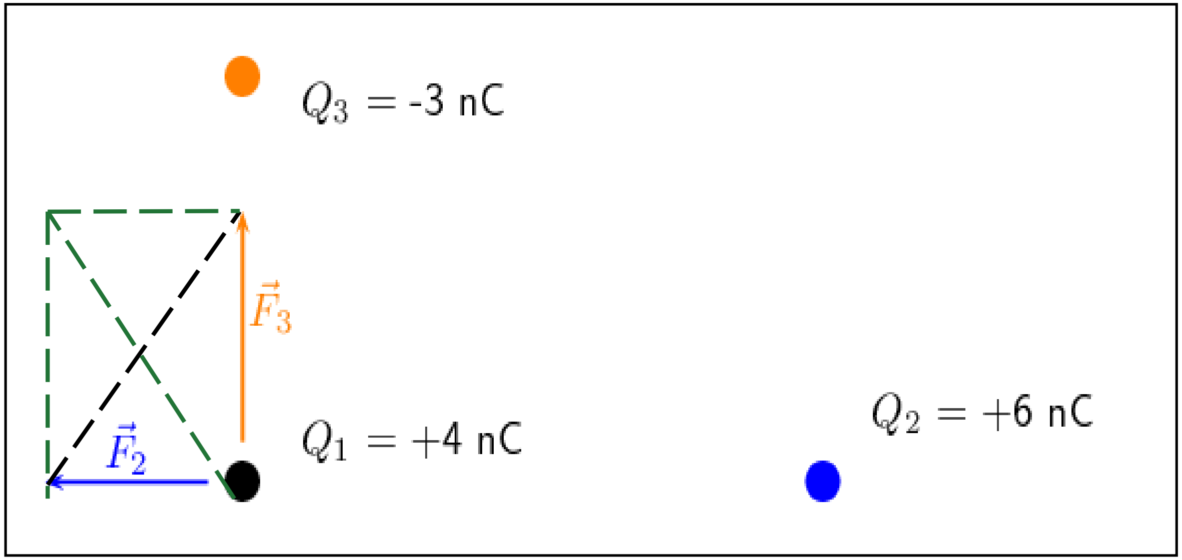
STEP 5 Vector addition of forces

This is a two-dimensional problem involving vectors. So we will use same procedure as taught in previous courses. That is determine the vectors on the Cartesian plane, break them into components in the $x-$ and $y-directions$, and then sum components in each direction to get the components of the resultant.

- **We choose the positive directions to be to the right (the positive x-direction) and up (the positive y-direction). We know the magnitudes of the forces but we need to use the signs of the charges to determine whether the forces are repulsive or attractive. Then we can use a diagram to determine the directions.**

- **The force between Q_1 and Q_2 is repulsive (like charges). This means that it pushes Q_1 to the left, or in the negative $x - direction$. The force between Q_1 and Q_3 is attractive (unlike charges) and pulls Q_1 in the positive $y - direction$.**

- **We can redraw the diagram as a free-body diagram illustrating the forces to make sure we can visualize the situation:**



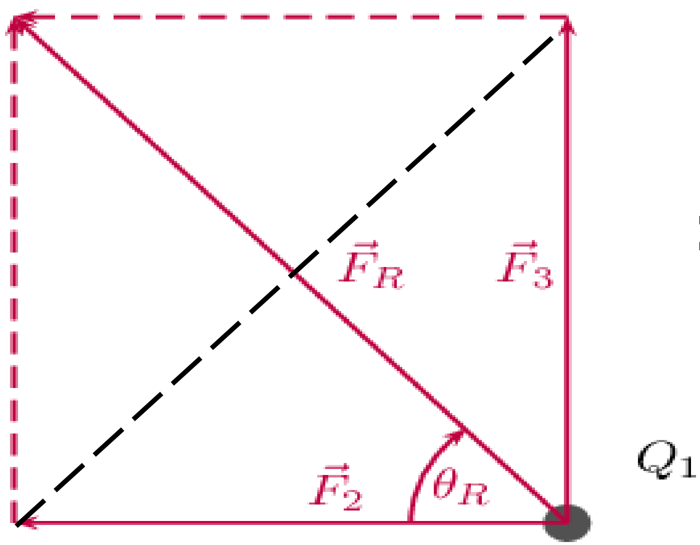
STEP 6 Resultant force

The magnitude of the resultant force (F_R)acting on Q_1 can be calculated from the forces using *Pythagoras' Theorem* because there are only two forces and they act in the $x-$ and $y-$ $directions$:

$$\Rightarrow F_R{}^2 = F_2{}^2 + F_3{}^2$$

$$F_R = \sqrt{(8.630 \times 10^{-5} N)^2 + (-1.199 \times 10^{-4} N)^2} = 1.48 \times 10^{-4} \text{ N}$$

The angle, θ_R made with the *x*-axis can be found using *trigonometry:*



$\Rightarrow$

$$\tan\theta_R = \frac{y - component}{x - component}$$

$$= \frac{1.199 \times 10^{-4}}{8.630 \times 10^{-5}}$$

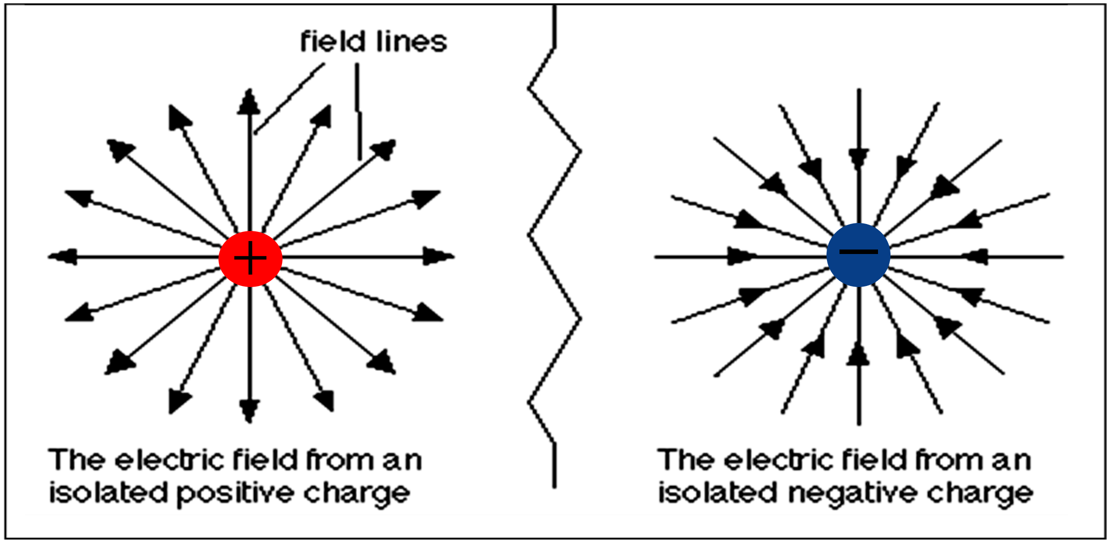
$$\theta_R = \tan^{-1}\left[\frac{1.199 \times 10^{-4}}{8.630 \times 10^{-5}}\right]$$

$$\approx 54.25^{o}$$

The final resultant force acting on Q_1 is $1.48{\times}10^{-4}$ N acting at an angle of 54.25° to the negative *x*-axis or (180° - 54.25°) = 125.75° to the positive *x*-axis.

1.3 ELECTRIC FIELD (E)

In Physics *the space surrounding an electric charge has a property called an electric field,* which exerts a force on other electrically charged bodies. This field was introduced by Michel Faraday. The electric field is also known as *electrostatic field intensity/strength*. The Figure below shows the directions of the flow of the electric fields/force from a positive charge and a negative charge.

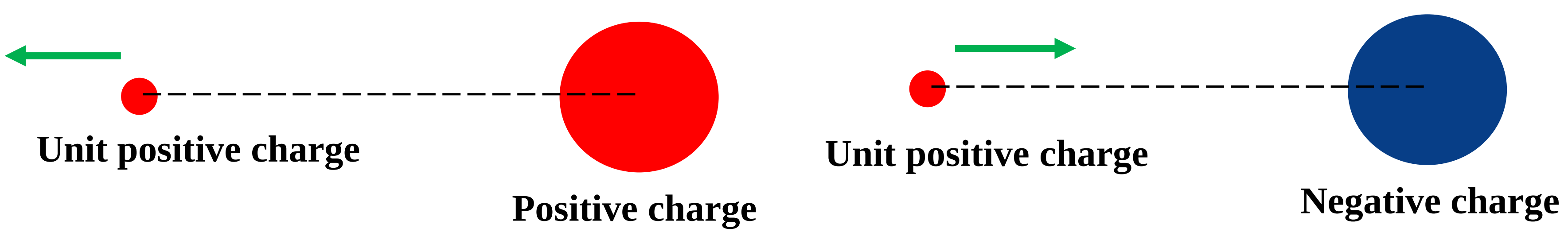


Electric field is defined as *electric force per unit charge*:

$$\vec{E} = \frac{\vec{F}}{Q} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 1.6$$

Direction of the electric field (due to a point/unit charge)

If a unit positive charge is placed near a positive charge, the unit positive charge will experience a repulsive force, due to which the unit positive charge will move away from the said charge. The imaginary line (*the green line*) through which the unit positive charge moves, is known as the line of force.



Similarly, if a unit positive charge placed in the field of negative electric charge, the unit positive charge will experience an attractive force, due to which the unit positive charge will come closer to the said negative charge. In that case, line through which the positive unit charge moves, is also known as the line of force.

So if a charge is positive charge, then the *lines of force come out of this charge*.

But for a negative charge, these *lines of force come into this charge* (as shown in previous Figure-slide 50).

Using Coulomb's law, the electric force experienced by the unit positive charge is: $\vec{F} = \frac{q_1 Q_2}{4\pi\varepsilon_0 r^2}$

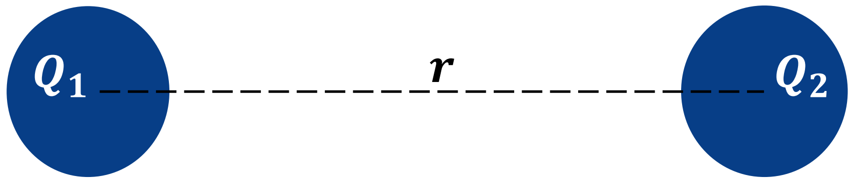
$$\vec{F} = \frac{q_1 Q_1}{4\pi\varepsilon_0 r^2} = \frac{1 \times Q_1}{4\pi\varepsilon_0 r^2} = \frac{Q_1}{4\pi\varepsilon_0 r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 1.6$$

$$\vec{F} = \frac{q_2 Q_2}{4\pi\varepsilon_0 r^2} = \frac{1 \times Q_2}{4\pi\varepsilon_0 r^2} = \frac{Q_2}{4\pi\varepsilon_0 r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 1.7$$

But $\vec{E} = \frac{\vec{F}}{Q} \Rightarrow \vec{E} = \frac{q_n Q_n}{4\pi\varepsilon_0 r^2 Q}$

Thus the force experienced by the unit positive charge is the measurement of electric field of Q_1 or Q_2 at the point where unit positive charge is placed, as shown in Equations 1.6 and 1.7 above.

Now replace a unit positive charge q_1 with a negative charge Q_1:



Since two positive charged particles repel each other, two negative charges repel each other and two oppositely charged particles attract each other with force. This attraction or repulsion force $(\vec{F})$ must be within the electric field$(\vec{E})$ that denotes how strongly an electric charge is repulsed or attracted by the charge which creates the electric field:

$$\vec{F} = \frac{Q_1 Q_2}{4\pi\varepsilon_o r^2} = \vec{E}Q \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 1.8$$

Calculation of electric fields (superposition principle)

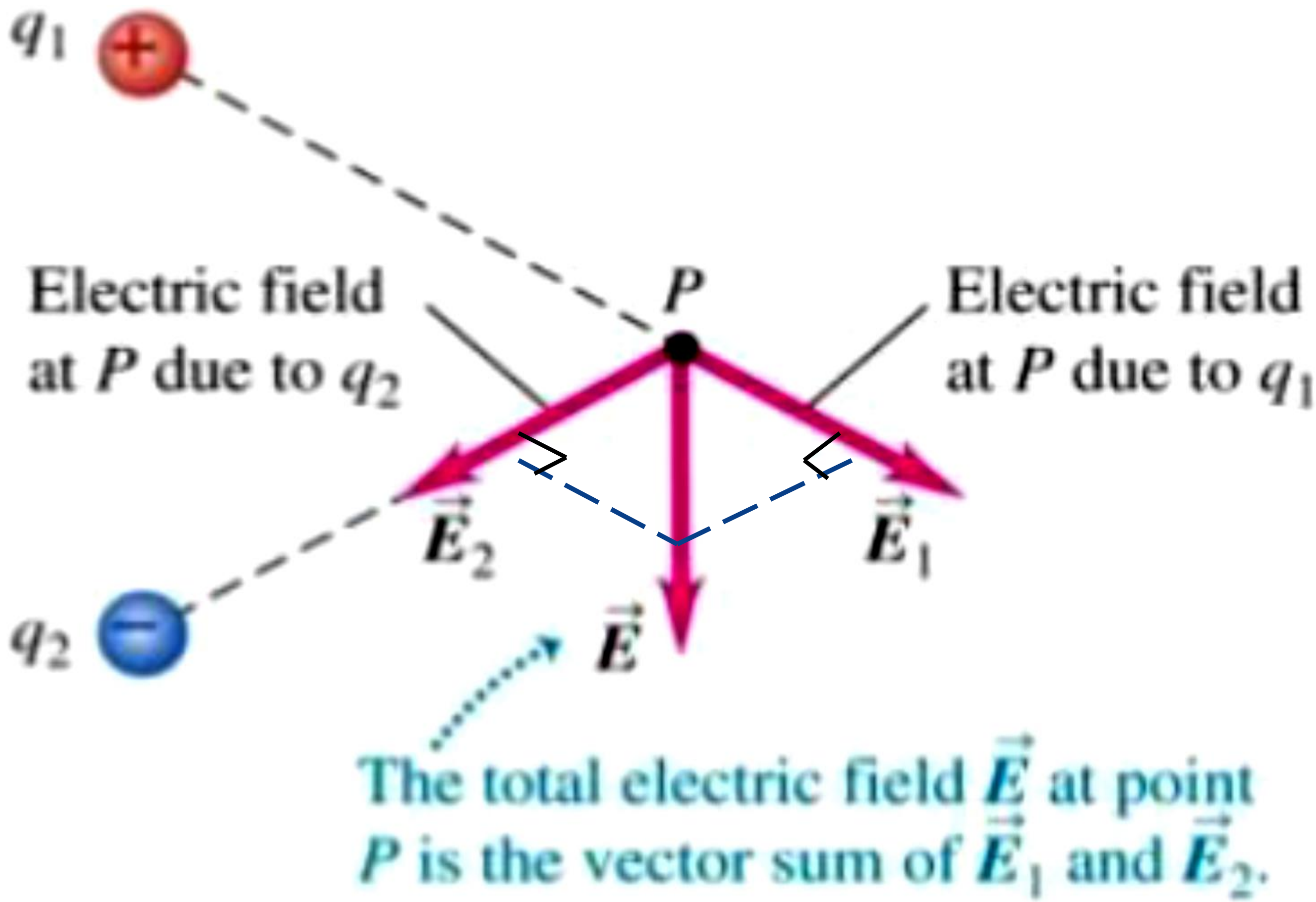
In real situations, we encounter charge that is distributed over space. To find the field caused by a distribution, we imagine the distribution to be made up of many point charges, $q_1, q_2, q_3 \ldots\ldots q_n$. At any given point P, each point charge produces its own electric field $\vec{E}_1, \vec{E}_2, \vec{E}_3, \ldots\ldots\ldots \vec{E}_n$, so a test charge q_0 placed at P experiences a force $\vec{F}_1 = q_0\vec{E}_1$ from charge q_1 and a force $\vec{F}_2 = q_0\vec{E}_2$ from charge q_2 and so on.
From the principle of superposition of forces, the total forces $\vec{F}_0$ that the charge distribution exerts on the q0 is the vector sum of these individual forces,

$$\vec{F}_0 = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots = q_0\vec{E}_1 + q_0\vec{E}_2 + q_0\vec{E}_3 + \cdots\cdots \quad 1.9$$

Then the total electric field at point P,

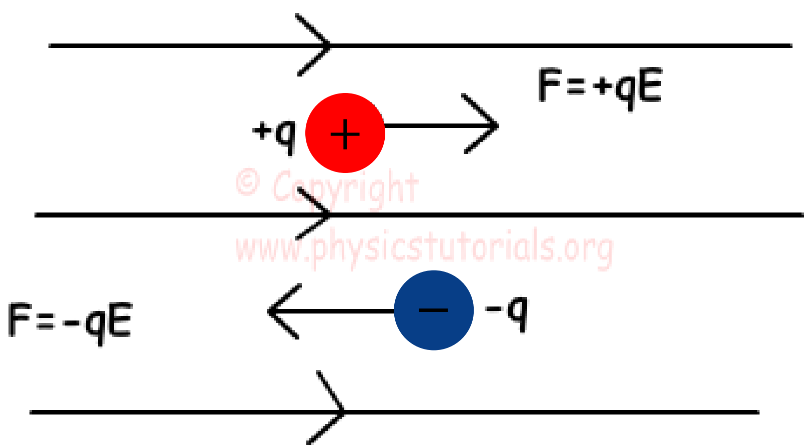
$$\vec{E} = \frac{\vec{F}_0}{q_0} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots\cdots\cdots\cdots \quad 1.10$$

Recall example 2 above: it means point *P* is positively charged



DIRECTION OF ELECTRIC FORCE $(\vec{F})$ AND ELECTRIC FIELD $(\vec{E})$

- **If q is positive $(+Q)$, the electric force and the electric field point in the same direction:** $\vec{E} = \dfrac{\vec{F}}{+Q} \Rightarrow \vec{F} = (+Q)\,\vec{E}$
- **If q is negative $(-Q)$, the electric force and the electric field point in opposite directions:** $\vec{E} = \dfrac{\vec{F}}{-Q} \Rightarrow \vec{F} = (-Q)\,\vec{E}$



Characteristics of electric field lines

i. Electric lines of force start from the positively charged surface of a body and end negatively charged surface of a body.
ii. Closeness of lines of forces symbolizes more strength of electric field and vice versa.
iii. Parallel lines indicate uniform field.
iv. Two lines of forces never intersect each other.
v. Are perpendicular to the surface of conductors.
vi. The tangential direction at any point on the lines of forces indicates the direction of the force acting on the positive charge at that point.
vii. Electric field strength is a vector.
viii. Electric field unit is Newton per Coulomb.

See the sketch below.

Electric Field Line Rules

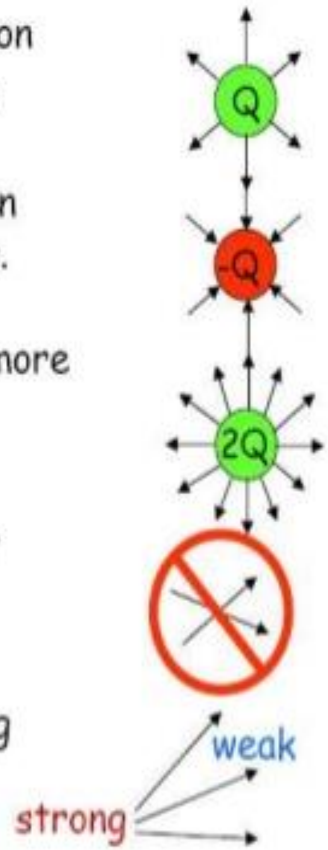
Field lines ***start*** on **positive** charges.

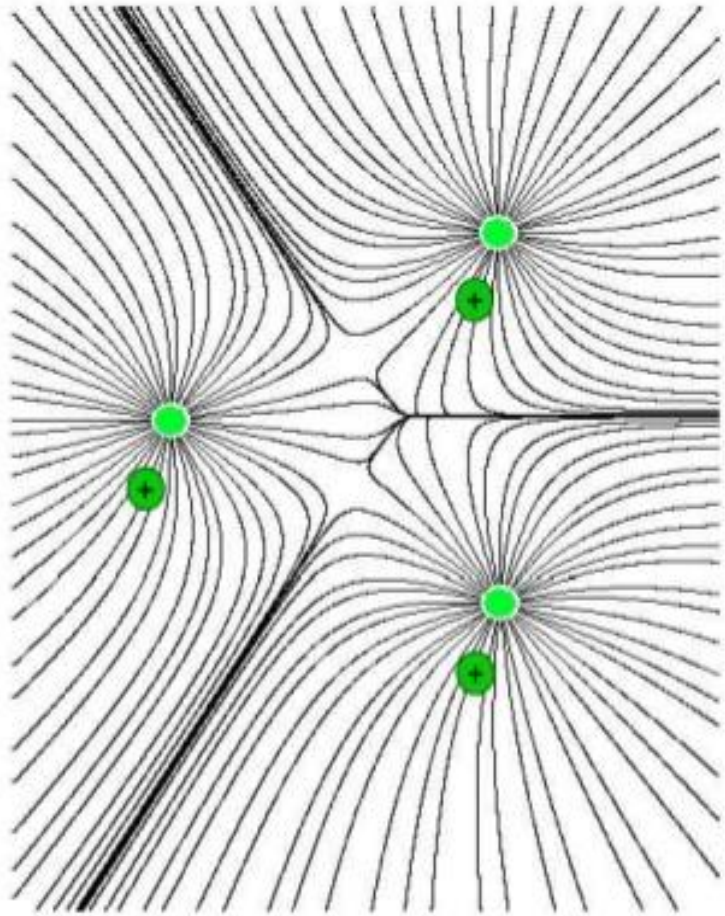
Field lines ***stop*** on **negative** charges.

More charge ⇒ more field lines.

Field lines never cross.

Field line spacing indicates field strength





Direction of E is tangent to the field line.

EXAMPLE 3

What is the force on 1C charge and on an electron when the field strength is 1000NC^{-1}?

(where charge of an electron$(1e) = 1.6 \times 10^{-19} C$**).**

SOLTUTION

i) Force on 1C charge:

$$F = EQ = 1000 \; NC^{-1} \times 1C = 1000 \; N$$

ii) Force on an electron:

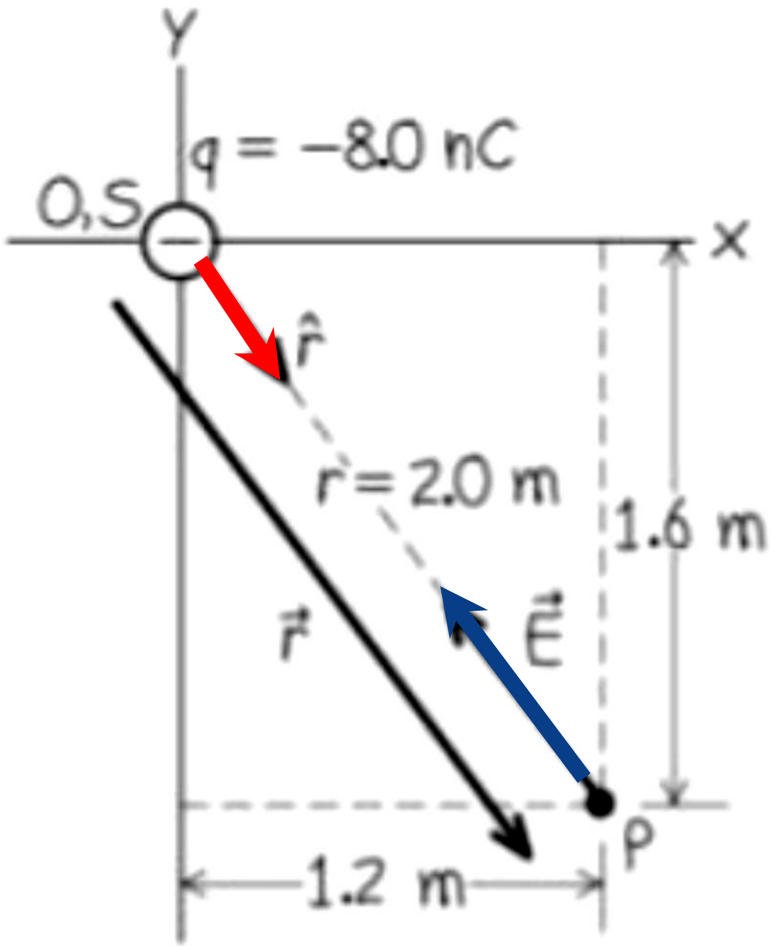
$$F = EQ = 1000 \; NC^{-1} \times (1.6 \times 10^{-19} C) = 1.6 \times 10^{-16} N$$

EXAMPLE 4

A point charge q = -8.0 nC is located at the origin. Find the electric-field vector at the field point $(\boldsymbol{P})\ \boldsymbol{X = 1.2\ m}$ y = -1.6 m?

SOLUTION

Pythagorean Theorem was used to determine the value of r



The vector of field point P is

$$\vec{r} = 1.2\hat{\imath} - 1.6\hat{\jmath}$$

The distance from the charge at point source, S to the field point, P is

$$r = \sqrt{x^2 + y^2} = \sqrt{1.2^2 + (-1.6)^2} = 2.0\boldsymbol{m}$$

The vector unit,

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{1.2\hat{\imath} - 1.6\hat{\jmath}}{2.0} = 0.6\hat{\imath} - 0.8\hat{\jmath}$$

Hence the electric-field vector is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \quad \text{or} \quad \vec{\boldsymbol{E}} = \frac{\mathbf{1}}{\mathbf{4}\boldsymbol{\pi\varepsilon_0}}\frac{\boldsymbol{q}}{\boldsymbol{r^2}}\frac{\vec{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}|} = \frac{\mathbf{1}}{\mathbf{4}\boldsymbol{\pi\varepsilon_0}}\frac{\boldsymbol{q}}{\boldsymbol{r^2}}\frac{\vec{\boldsymbol{r}}}{\boldsymbol{r}} = \frac{\mathbf{1}}{\mathbf{4}\boldsymbol{\pi\varepsilon_0}}\frac{\boldsymbol{q}}{\boldsymbol{r^3}}\vec{\boldsymbol{r}}$$

$$= (9.0 \times 10^9 N\, m^2 C^{-2})\frac{-8.0 \times 10^{-9}\, C}{(2.0)^2}(0.6\hat{\imath} - 0.8\hat{\jmath})$$

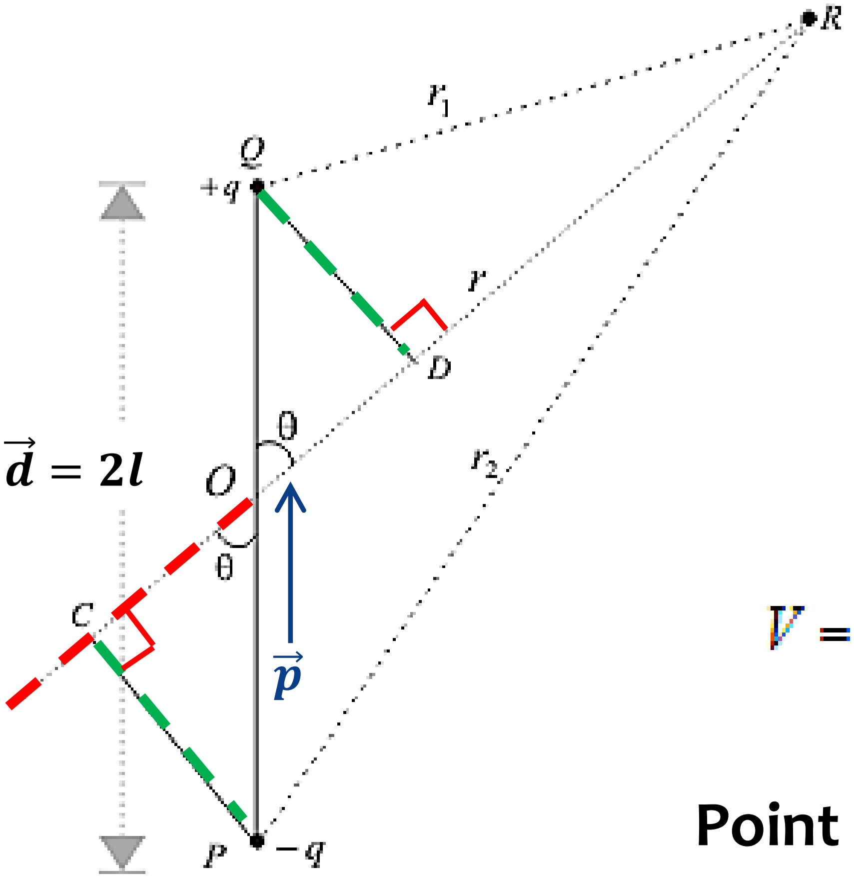
$$= -11\frac{N}{C}\hat{\imath} + 14\frac{N}{C}\hat{\jmath}$$

POTENTIAL DUE TO ELECTRIC DIPOLE

- **Electric dipole** is an arrangement that consists of two equal and opposite charges ($+q$ *and* $-q$) separated by a small distance of $2l$.
- **Dipole moment** ($\vec{p}$) is the exact measure of the strength associated with an electric dipole:

$$\vec{p} = q\vec{d} = q2\vec{l}$$

- **Electric dipole moment** is represented by a *vector* $\vec{p}$ of magnitude $2ql$ and this vector *points in direction from* $-q$ *to* $+q$.
- To find electric potential due to a dipole consider charge $-q$ is placed at point P and charge $+q$ is placed at point Q as shown in the Figure below.



Since electric potential obeys superposition principle, so *potential* (V) *due to electric dipole* $(\vec{p})$ as a whole would be sum of potentials due to both charges $+q$ and $-q$:

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1} - \frac{q}{r_2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \mathbf{1.11}$$

Point P: $\cos\theta$= [OC÷OP]=OC÷ l=[OC/l]

∴OC=$l\cos\theta$; similarly point Q: OD=$l\cos\theta$

Now: r_1 = QR≅RD = OR-OD = $r - l\cos\theta$

r_2 = PR≅RC = OR+OC = $r + l\cos\theta$

So Equation 1.11 becomes:

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r - l\cos\theta} - \frac{1}{r + l\cos\theta}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{2l\cos\theta}{r^2 - l^2\cos^2\theta}\right) \quad \text{......... 1.12}$$

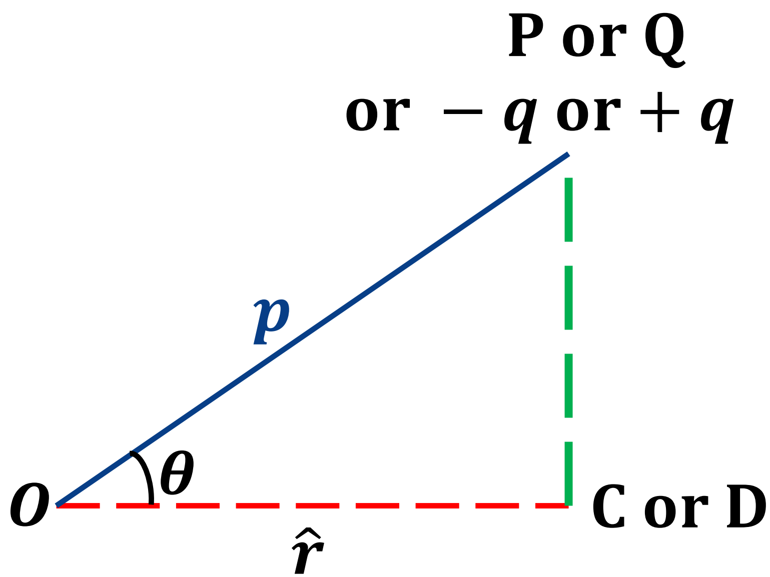
Since magnitude of dipole is $|\vec{p}|=p= 2ql$:

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{p\cos\theta}{r^2 - l^2\cos^2\theta}\right) \quad \text{.................................. 1.13}$$

If we consider the case where $r \gg l$ then, electric potential:

$$V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} \quad \text{.................................. 1.14}$$

Equation 1. 14 can be re-written as shown below:



$$\Rightarrow cos\theta = \frac{\hat{r}}{p} \Rightarrow p\, cos\theta = \ p \cdot \hat{r}\ ,$$

where $\hat{r}$ is the unit vector along the vector r. Thus the electric potential of a dipole (Equation 1.14) can be re-written as:

$$V = \frac{p\,.\hat{r}}{4\pi\varepsilon_0 r^2} \quad \text{.........................} \ 1.15$$

- **Equation 1.14** shows that the potential due to electric dipole does not only ***depends on r*** but also depends on angle $\boldsymbol{\theta}$ between position vector $\boldsymbol{r}$ and dipole moment $\boldsymbol{p}$.

- **Equation 1.15** shows that potential due to electric dipole is ***inversely proportional*** to $\boldsymbol{r^2}$ not $\boldsymbol{\frac{1}{r}}$ which is the case for potential due to single charge.

EXAMPLE 5

What is the dipole moment for a dipole having equal charges $-2C$ and $+2C$ separated with a distance of $2cm$.

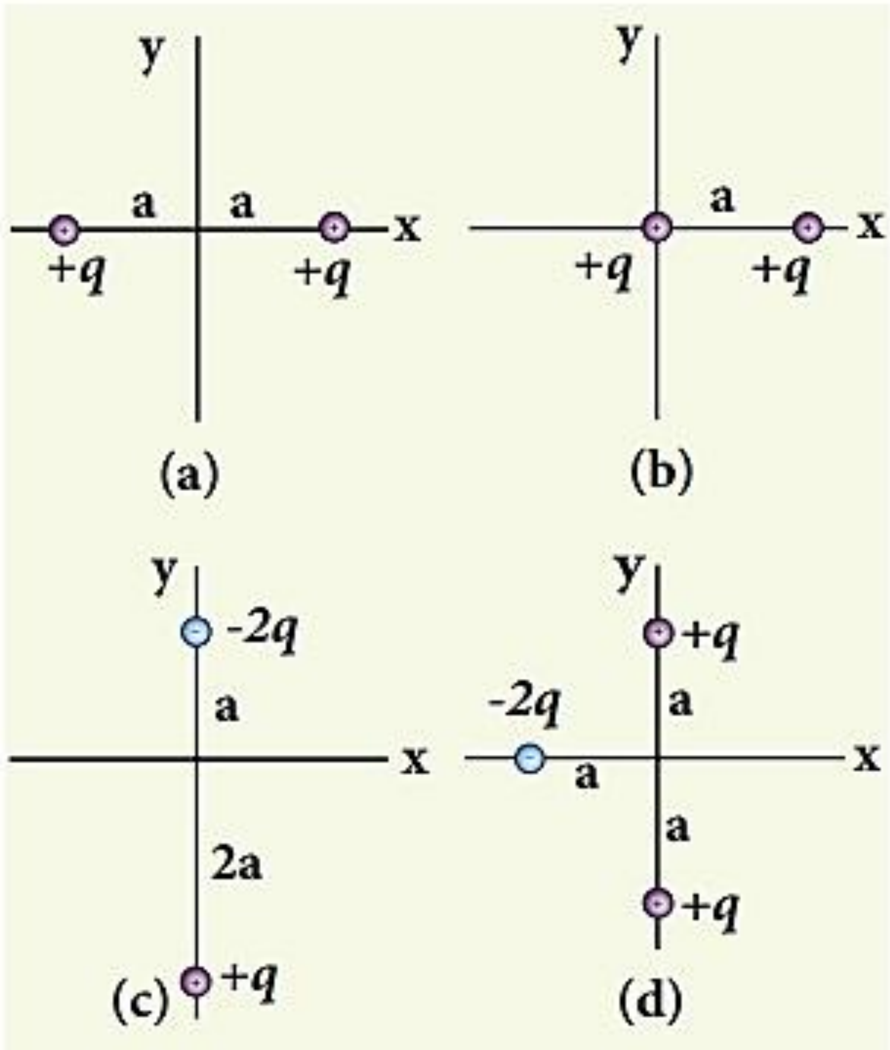
SOLUTION

The dipole moment for this condition is:

$$\vec{p} = q \times \vec{d} \Rightarrow p = 2C \times 0.02m = 0.04\,(Cm)$$

EXAMPLE 6

Calculate the electric dipole moment for the following charge configurations.



SOLUTION

(a) The position vector for the $+q$ on the positive $x-axis$ is $a\hat{\imath}$ and position vector for the $+q$ charge the negative $x-axis$ is $-a\,\hat{\imath}$. So the dipole moment is:

$$\vec{p} = (+q)(a\hat{i}) + (+q)(-a\hat{i}) = 0$$

(b) Here one charge is placed at the origin, so its position vector is zero. Hence only the second charge $+q$ with position vector $a\hat{\imath}$ contributes to the dipole moment: $\vec{p} = +qa\hat{\imath}$

(c) In this case $\vec{p}$ is directed from $-2q$ to $+q$:

$$\vec{p} = (-2q)a\hat{j} + q(2a)(-\hat{j}) = -4qa\hat{j}.$$

(d) $\vec{p} = -2qa(-\hat{i}) + qa\hat{j} + qa(-\hat{j}) = 2qa\hat{i}$

1.4 ELECTRIC POTENTIAL(V)

An electric potential is the amount of work needed to move an unit positive charge from a reference point to a specific point inside an electric field without producing acceleration.

The electric potential energy (U) is the potential energy that a charge has due to its position relative to charges $(Q_1, Q_2, Q_3, \dots \dots \dots ..)$.

In this section we will define the electric potential and discuss how we will calculate it for various arrangements of charged particles and objects; and how it is related to electric potential energy$(U)$and electric field$\left(\vec{E} = \frac{\vec{F}}{Q}\right)$.

The ***electric potential*** (V) at a specific position is a measure of the amount of electric potential energy per unit charge (q) would have at that position.

The electric potential (V):

- Is a ***scalar quantity*** (because there is no direction associated with U or charge).
- It can be negative or positive, because potential energy and charge have signs.
- Its SI unit is volt (V) which is $1\,\mathrm{Joule\ per\ Coulomb}\,(1\,\mathrm{J/C})$.

So, if a charge (q) has an electric potential energy (U), the electric potential (V) at the location of q is

$$V = \frac{U}{q} \tag{1.3.1}$$

We can compute the change in electric potential as:

$$\Delta V = \frac{\Delta U}{q} \tag{1.3.2}$$

Let the electric potential be zero at infinity $(V_\infty = 0)$ that is very far away from any electric charges. Then electric potential (V) at some point r just refers to the change in electric potential in moving the charge from infinity to point r.

$$\Delta V = V_r - V_\infty \rightarrow V_r \qquad 1.3.3$$

So, **work done** (W) by the electric field in moving an electric charge from *infinity* to point r is

$$W = -\Delta U = \underbrace{-q\Delta V}_{\text{Eqn.1.3.2}} = -q\underbrace{\left(V_r - V_\infty\right)}_{\text{Eqn.1.3.3}} = -qV_r \qquad 1.3.4$$

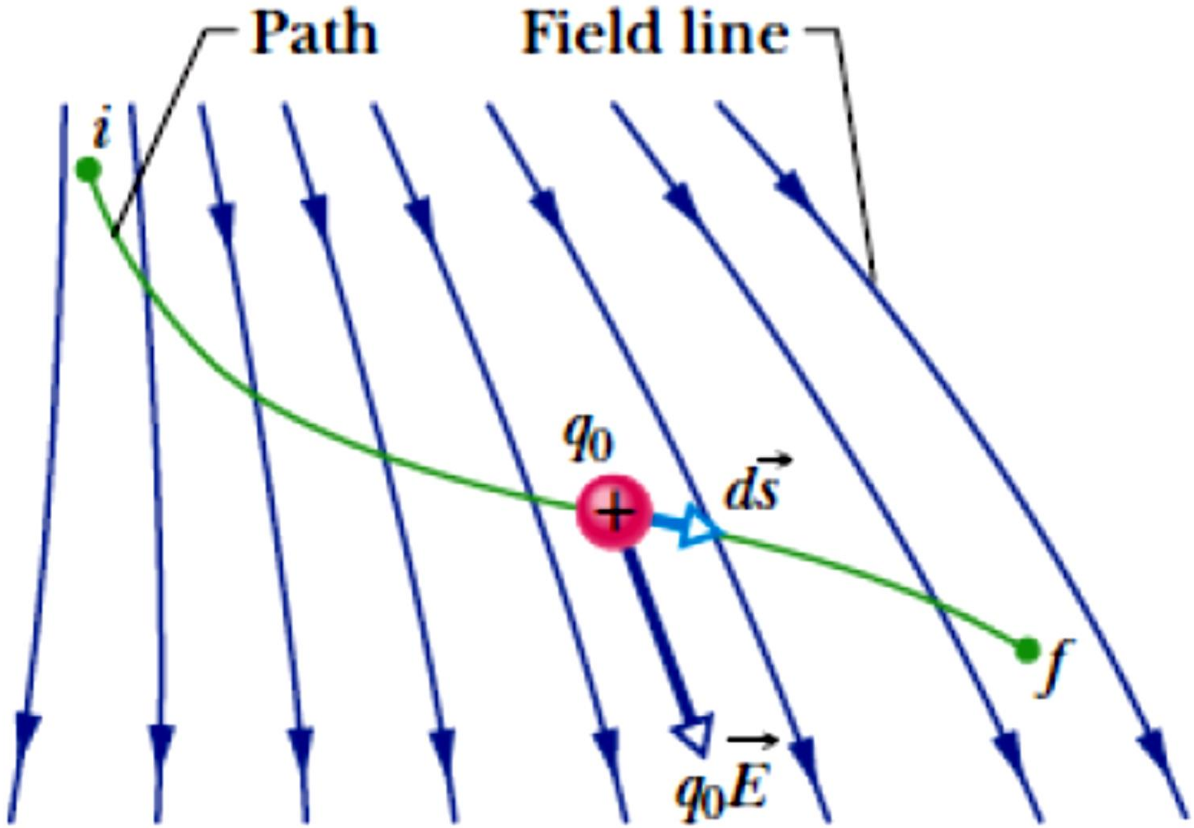
The electric field (E) ***can be used to derive the electric potential*** (V)

Consider ***work done*** by the electric field in moving a charge (q_0) a distance (ds):

$\left(\text{Recal Equation 1.2.1:} \quad E = \frac{F}{Q}\right)$ and $W = \text{Force}(F) \times \text{Distance}(s)$

$$dW = \vec{F}.\vec{ds} = q_0 \vec{E}.\vec{ds} \qquad 1.3.5$$

Therefore ***total work done*** by the electric field to move a charge from initial (i) point to final (f) point is given by line integral along the path (see Figure below):


Path
Field line
i
q_0
$d\vec{s}$
f
$q_0\vec{E}$

$$W = q_0 \int_i^f \vec{E}.d\vec{s} \qquad \Rightarrow \frac{W}{q_o} = \int_i^f \vec{E}.d\vec{s} \tag{1.3.6}$$

This ***work is related to the negative charge in potential energy*** (U) or electric potential (V) [see Equation 1.3.4 above $\left(W = -q\Delta V \ \Rightarrow \frac{W}{q} = -\Delta V \right)$]:

$$\Rightarrow \frac{W}{q_0} = \underbrace{-\Delta V}_{\text{Eqn.1.3.4}} = -\left(V_f - V_i\right) \tag{1.3.7}$$

Substitute the left hand side of Equation 1.3.7 with Equation 1.3.6:

$$\Rightarrow \int_i^f \vec{E}.d\vec{s} = -\Delta V = -\left(V_f - V_i\right) \Rightarrow \int_i^f \vec{E}.d\vec{s} = -\Delta V = -\left(V_f - V_i\right) \Rightarrow \Delta V = \left(V_f - V_i\right) = -\int_i^f \vec{E}.d\vec{s}$$

$$\Rightarrow \left(V_f - V_i\right) = \int_f^i \vec{E}.d\vec{s} \qquad 1.3.8a$$

Recall the ***dot product of vector*** $\left(\vec{E}.d\vec{s} = E \times ds\cos\theta\right)$:

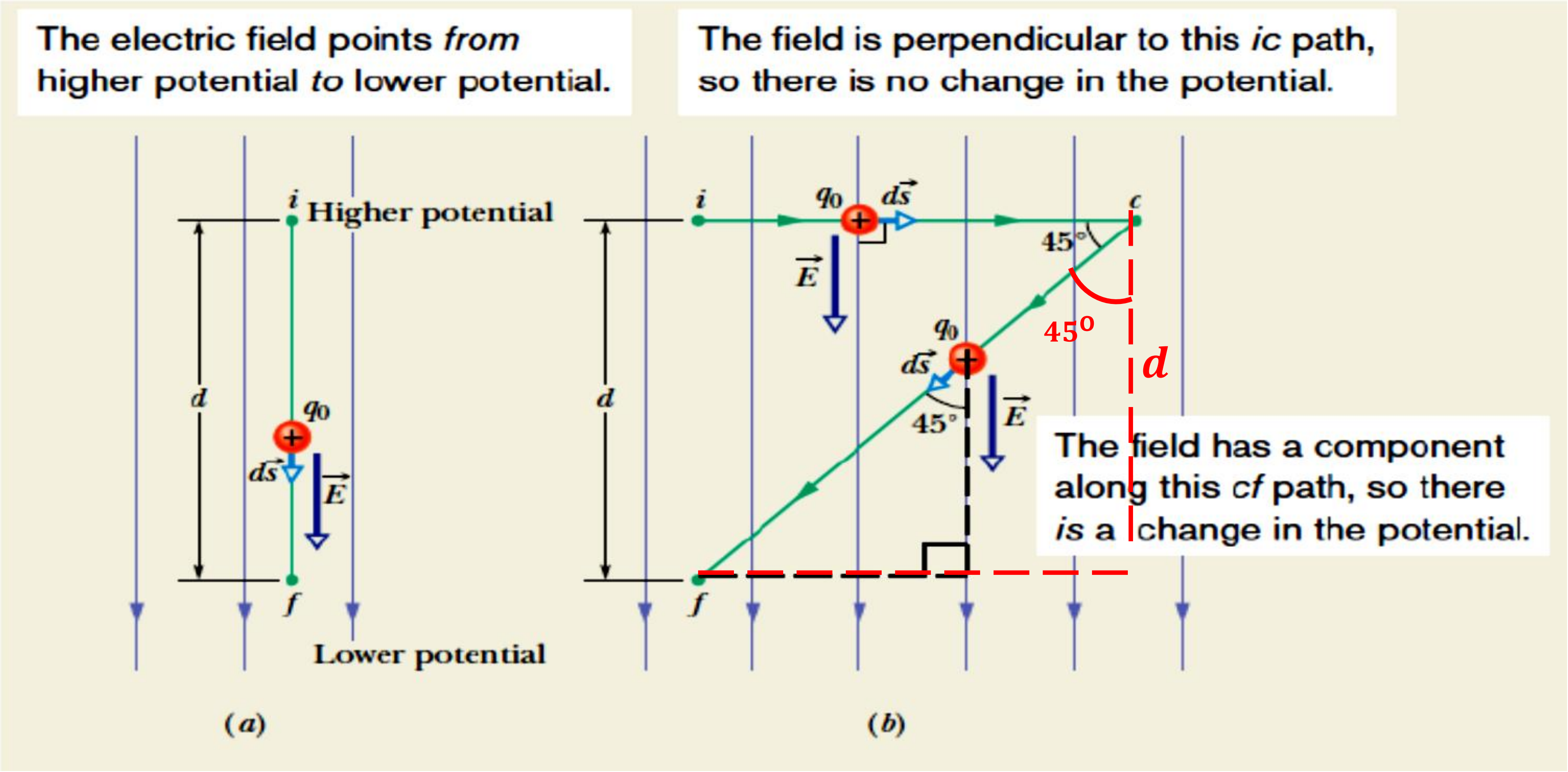
$$\Rightarrow \left(V_f - V_i\right) = \int_f^i E \times ds\ \cos\theta$$

1.3.8b

If $\theta = 90^\circ$, $\cos\theta = 0$ and there will be **NO CHANGE** in electric potential; that is $V_f = V_i$.

EXAMPLE

Find the potential difference by moving a positive test charge (q_o) along the paths in Figures (a) and (b) below in a uniform electric field $(\vec{E})$; and comment on the results.



SOLUTION

a) Using Equation 1.3.8a above

$$\Rightarrow (V_f - V_i) = \int_f^i \vec{E}.d\vec{s} = \int_f^i E \times ds \ \cos\theta$$

$$\Rightarrow (V_f - V_i) = E \times \cos 0^\circ \int_f^i ds \ = E \times \cos 0^\circ \times d = Ed$$

So the potential difference is $(V_f \text{-} V_i) = Ed$

b) The test charge (q_o) is moving from i to f through c. So the potential difference will be sum of movement from i to c and c to f.

$$(V_f - V_i) = \int_c^i \vec{E}.d\vec{s} + \int_f^c \vec{E}.d\vec{s} = \left[\int_c^i E \times ds \times \cos 90°\right] + \left[\int_f^c E \times ds \times \cos 45°\right]$$

$$\Rightarrow (V_f - V_i) = \left[E \times \cos 90° \times \int_c^i ds\right] + \left[E \times \cos 45° \times \int_f^c ds\right]$$

$$\Rightarrow (V_f - V_i) = [E \times 0 \times ds] + [E \times \cos 45° \times (\text{length of } cf)]$$

$$\Rightarrow (V_f - V_i) = [0] + \left[E \times \cos 45° \times \left(\frac{d}{\cos 45°}\right)\right]$$

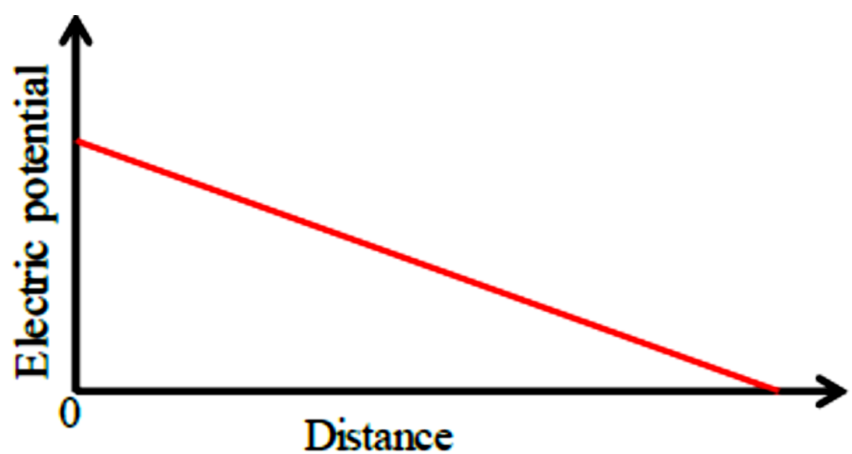
$$= [0] + [E \times d]$$

$$= Ed$$

The potential difference $(V_f\text{-}V_i)$ for Figure (a) is the same as for displacement cf i.e. Ed . Thus the potential difference between two points doesn't depend on the path connecting them.

For uniform field, Equation 1.3.8a becomes: $\Rightarrow \underbrace{(V_f - V_i) = \int_f^i \vec{E}.d\vec{s}}_{\text{Eqn.1.3.8a}} = E\Delta x$ 1.3.8c

The ***electric potential always decreases as one moves in the direction of the electric field***. In the graph below the <u>electric field is constant</u>; as a result the electric potential decreases uniformly with distance.



ELECTRIC POTENTIAL ENERGY (U)

TAKE NOTE

1) Objects in gravitational fields have ***potential energy*** (PE).

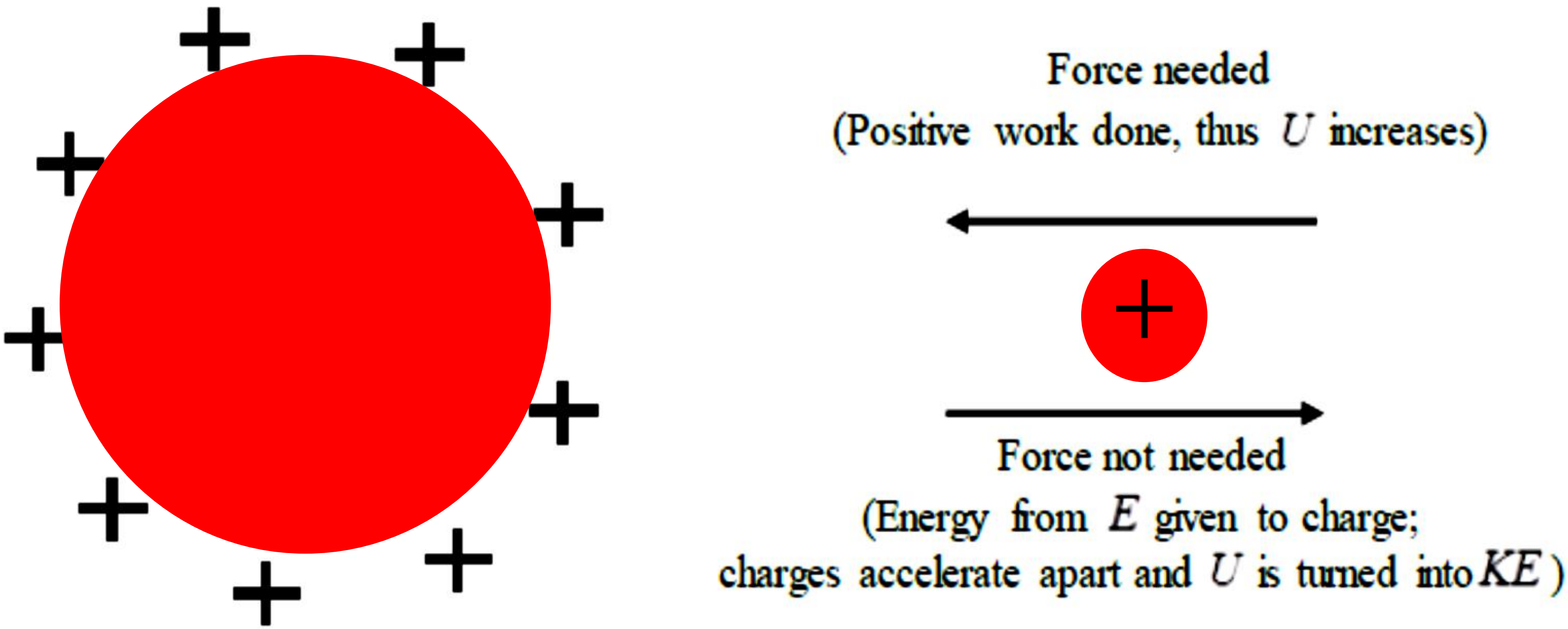
   Similarly, charges in electric fields (E) have ***electric potential energy*** (U).

2) To move an object in a gravitational field requires work. There is a ***change in the object's*** PE.

   Similarly, to move a charge in an electric field requires work. There is a ***change in the charge's electric potential energy*** (U).

3) In gravitational fields a force is required to move masses apart. That is, ***positive work is done*** (by an external agent - you).

   Similarly, in electric fields a force is required to push like charges together. That is ***positive work is done*** (by an external agent-you).



Force needed
(Positive work done, thus U increases)
+
Force not needed
(Energy from E given to charge;
charges accelerate apart and U is turned into KE)

We can compute the change in electric potential energy too from Equation 1.3.2 $\left(\Delta V = \dfrac{\Delta U}{q}\right)$:

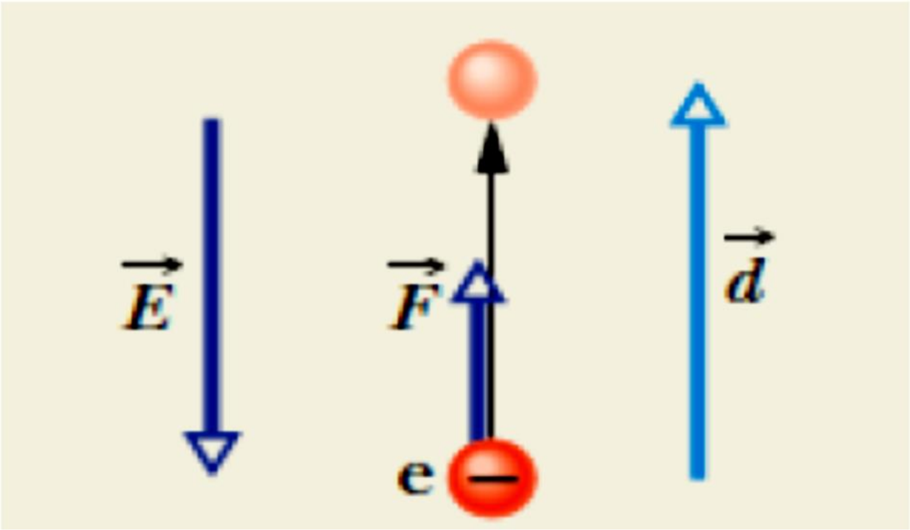
$$\Delta U = q\Delta V \qquad 1.4.1$$

The electric potential energy will have a new SI unit. To derive it using Equation 1.4.1 above, let us consider an electron crossing a potential difference of 1 volt:

$\Delta U = q\Delta V = e \times \Delta V = \left(1.6\times 10^{-19} C\right)\times 1V = 1.6\times 10^{-19} J = 1eV$ and this is a basic unit used to measure the tiny energies of subatomic particles like the electron.

EXAMPLE

Charged particle (electron in the atmosphere)



Above is a sketch of an electron in the atmosphere, it is moved upwards through a displacement $(\vec{d})$ by an electric force $(\vec{F})$ due to an electric field $(\vec{E})$ of $150 N/C$.

a) What is the change in electric potential energy (U) of the released electron, when electric force causes it to move vertically upward through a distance $(\vec{d})$ of $520m$?

b) Through what electric potential (V) change does the electron move?

SOLUTION

a) Work done on electron by electric field is $W = -\Delta U$

Work done by constant force on a particle undergoing displacement is $W = \vec{F}.\vec{d}$

Electric force and electric field are related by force equation $\vec{F} = q\vec{E}$

$$\Rightarrow W = \vec{F}.\vec{d} = q\vec{E}.\vec{d} = qEd\cos\theta$$

$$= (-1.6\times10^{-19}C)(150N/C)(520m)\underbrace{\cos180^\circ}_{-1} = 1.2\times10^{-14}J$$

where θ is the angle between directions of $\vec{E}$ and $\vec{d}$.

But $W = -\Delta U \Rightarrow \Delta U = -W = -1.2\times10^{-14}J$

It means that as the electron move up, electric potential energy of the electron decreases

by $-1.2\times10^{-14}J$.

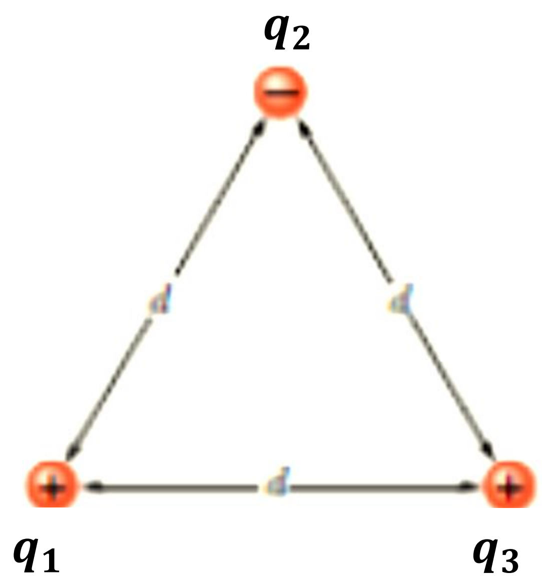
b) Recall Equation 1.3.2 $(\Delta U = q\Delta V)$

$$\Delta V = \frac{\Delta U}{q} = \frac{-1.2\times10^{-14}\,J}{-1.6\times10^{-19}\,C} = 4.5\times10^{4}\,V = 45kV$$

It tells us that the electric force does work to move the electron to a higher potential.

EXAMPLE

Three charges ($q_1, q_2, and\ q_3$) are fixed at the vertices of an equilateral triangle as shown below.
What is the electric potential energy of the system?



Total electric potential energy

$$(u) = u_{12} + u_{13} + u_{23}$$

SOLUTION

The potential energy U of the system is equal to the work we must do to assemble the system, bringing in each charge from an infinite distance.

Calculations: Let's mentally build the system starting with one of the charges, say q_1, in place and the others at infinity. Then we bring another one, say q_2, in from infinity and put it in place. From with d substituted for r, the potential energy U_{12} associated with the pair of charges q_1 and q_2 is

$$U_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{d}.$$

We then bring the last charge q_3 in from infinity and put it in place. The work that we must do in this last step is equal to the sum of the work we must do to bring q_3 near q_1 and the work we must do to bring it near q_2. , with d substituted for r, that sum is

$$W_{13} + W_{23} = U_{13} + U_{23} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_3}{d} + \frac{1}{4\pi\varepsilon_0}\frac{q_2q_3}{d}.$$

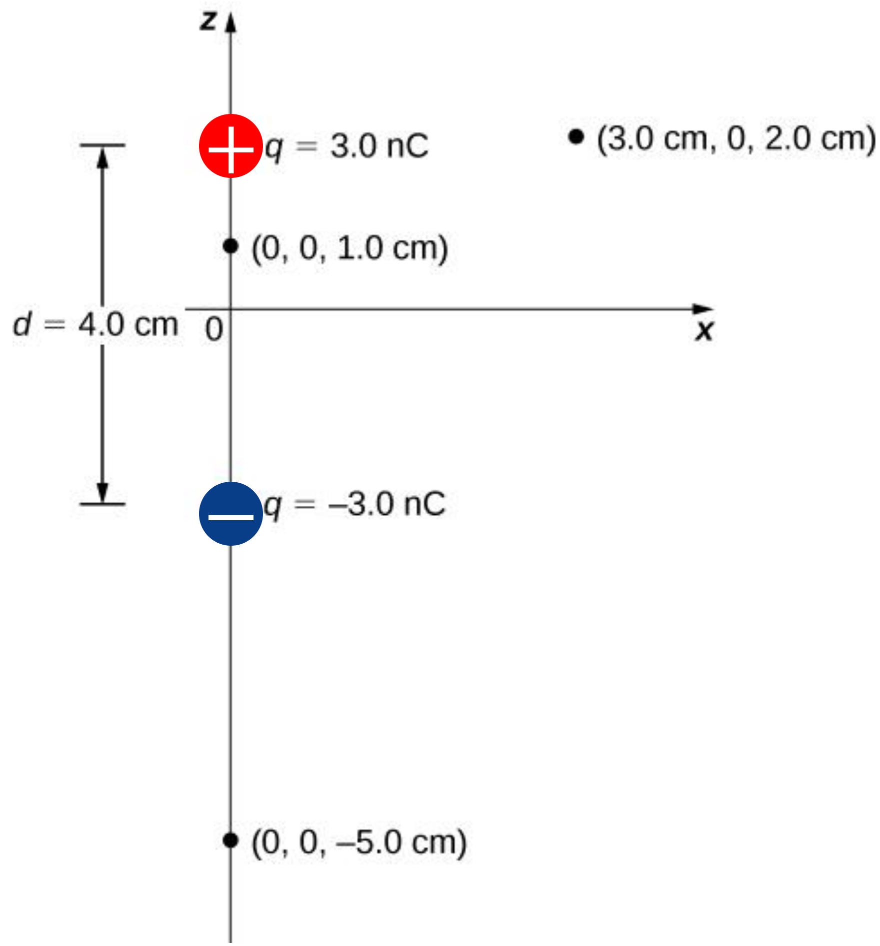
The total potential energy U of the three-charge system is the sum of the potential energies associated with the three pairs of charges. This sum (which is actually independent of the order in which the charges are brought together) is

$$
\begin{aligned}
U &= U_{12} + U_{13} + U_{23} \\
&= \frac{1}{4\pi\varepsilon_0}\left(\frac{(+q)(-4q)}{d} + \frac{(+q)(+2q)}{d} + \frac{(-4q)(+2q)}{d}\right) \\
&= -\frac{10q^2}{4\pi\varepsilon_0 d} \\
&= -\frac{(8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2})(10)(150 \times 10^{-9}\ \mathrm{C})^2}{0.12\ \mathrm{m}} \\
&= -1.7 \times 10^{-2}\ \mathrm{J} = -17\ \mathrm{mJ}. \qquad \text{(Answer)}
\end{aligned}
$$

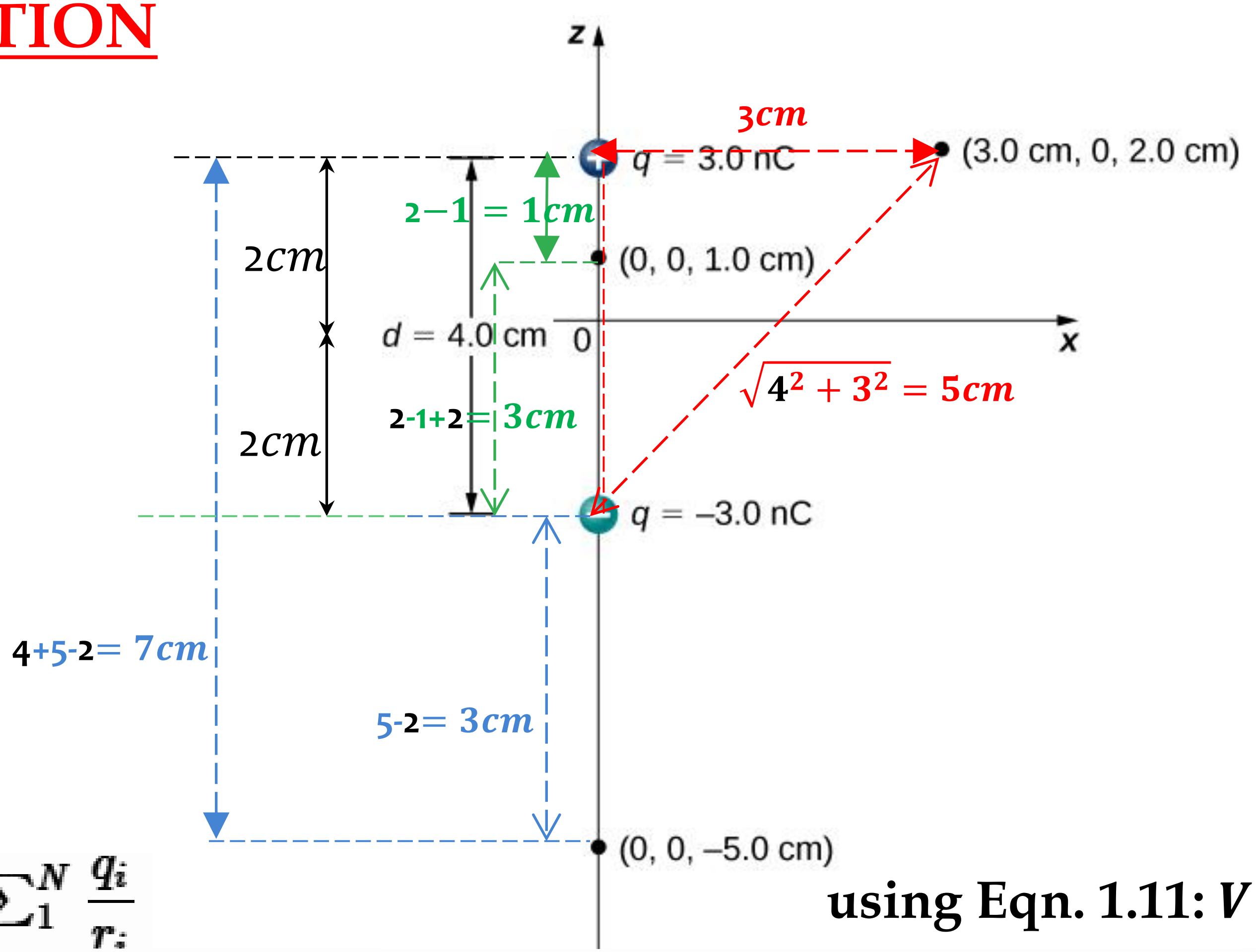
The lesson here is this: If you are given an assembly of charged particles, you can find the potential energy of the assembly by finding the potential of every possible pair of the particles and then summing the results.

EXAMPLE

Consider the dipole below with the charge magnitude of $q = 3.0\mu C$ and separation distance $d = 4.0cm$. What is the potential at the following locations in space? (a) (0, 0, 1.0 cm); (b) (0, 0, –5.0 cm); (c) (3.0 cm, 0, 2.0 cm).



SOLUTION



$$V_p = k\sum_1^N \frac{q_i}{r_i}$$

using Eqn. 1.11: $V = \frac{1}{4\pi\varepsilon_o}\left[\frac{+q}{r_1} + \frac{-q}{r_2}\right]$

a. $V_p = k\sum_1^N \frac{q_i}{r_i} = (9.0 \times 10^9\ N \cdot m^2/C^2)\left(\frac{3.0\ nC}{0.010\ m} - \frac{3.0\ nC}{0.030\ m}\right) = 1.8 \times 10^3\ V$ …..(green)

b. $V_p = k\sum_1^N \frac{q_i}{r_i} = (9.0 \times 10^9\ N \cdot m^2/C^2)\left(\frac{3.0\ nC}{0.070\ m} - \frac{3.0\ nC}{0.030\ m}\right) = -5.1 \times 10^2\ V$ …..(blue)

c. $V_p = k\sum_1^N \frac{q_i}{r_i} = (9.0 \times 10^9\ N \cdot m^2/C^2)\left(\frac{3.0\ nC}{0.030\ m} - \frac{3.0\ nC}{0.050\ m}\right) = 3.6 \times 10^2\ V$ …..(red)

LECTURE 2:
STEADY CURRENT(I)

Content

2.1 Introduction

2.2 Current and current density

2.3 Resistance, resistivity and conductivity

2.4 Ohm's Law

2.5 Kirchhoff's Laws

OBJECTIVES

At the end of this lecture, the learner should be able to

a) Make a transition from static electricity to current electricity.
b) Discuss flow of charge in conductors and the concept of resistance and electromotive force.
c) Break down complex electric circuits into simpler series and parallel combinations.
d) Apply two general principles known as Kirchhoff's laws in calculating the currents in completed electric circuits.

2.1 INTRODUCTION

In this lecture we will make a transition from static electricity to current electricity. We will discuss the flow of a charge in conductors and introduce the concepts of current, resistance, and electromotive force. Following this we will give a detailed discussion of direct current, electric circuits and some measuring instruments.

2.2 CURRENT AND CURRENT DENSITY

ELECTRIC CURRENT

It is the rate of flow of charge. If the electric charge flows through a conductor, then there is an electric current in the conductor (*a conductor is a material/substance that allows electricity/electrons to flow through it*).

Types of Current

- **Direct Current (DC): It travels towards the same direction at all points, although the instantaneous magnitude can differ. An example the current generated by an electrochemical cell.**
- **Alternating Current (AC): The flow of charge carriers is towards opposite direction periodically in an alternating current. The number of AC cycles per second is known as frequency and calculated in Hertz.**

Electric current Formula

If a charge Q flows through the cross-section of a conductor in time t, the current:

$$I = \frac{Q}{t} \Rightarrow I = \frac{dQ}{dt} \ldots\ldots\ldots\ldots\ldots \quad 2.1$$

- **The metre-kilogram-seconds(mks) unit or International System of Units (S.I) of current:**

S.I unit of charge is coulomb and for time is second, so measurement of electric current is in coulomb (C) per second(s) which is 'ampere'(A): $1A = 1C/s$

- **The centimetre–gram–second (Cgs) unit of electromagnetic unit (e.m.u) current:**

Is abampere (abA) or Biot (Bi):

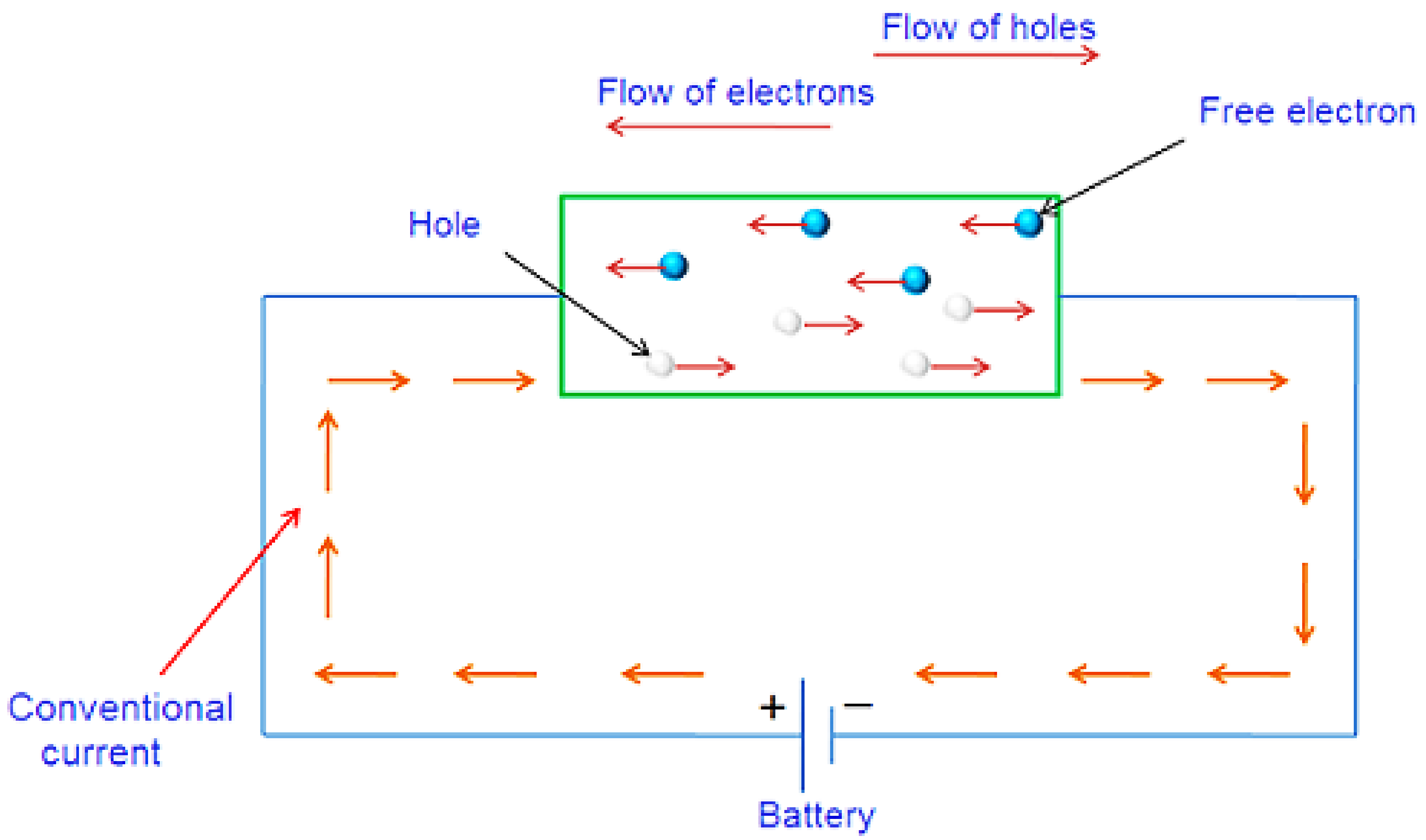
1abA=1 abcoulomb (1 abC) of charge carriers moving past a specific point in 1 second (1s) ⇒$1\ abA = 1abC/s = 10\ A$;

Therefore $1abA = 10C/s$.

Direction of flow of current in a circuit

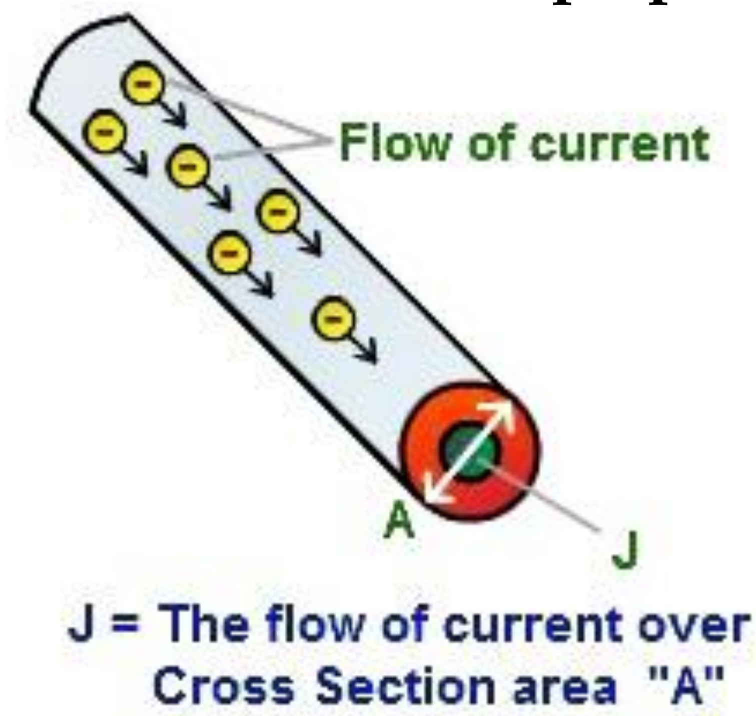
- **The *electron flow direction* of electric current flows from the negative terminal to the positive terminal of the cell as shown in the circuit below. This is due to the flow of electrons that are loosely bound to nucleus of an atom, unlike the flow of protons that are strongly bound to the nucleus. So negative electrons are charge carriers in most conductors.**

- **The *conventional direction* of electric current (that is the electric current (holes/protons) that flows from positive terminal to the negative terminal of battery) is taken as opposed to the direction of flow of electrons.**
- **TAKE NOTE: According to Physicists, *Current is considered to move from relatively positive to negative points*, and this is known as conventional current.**


Flow of holes
Flow of electrons
Free electron
Hole
Conventional current
+
−
Battery

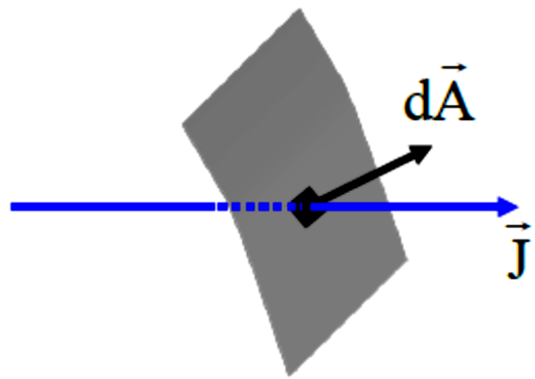
CURRENT DENSITY (j)

It is the amount of electric current (I) travelling per unit cross-section area (A) and expressed in amperes per square meter (A/m^2). It is a *vector quantity* having *both a direction and a scalar magnitude*. The electric current flowing through a solid having units of charge per unit time is calculated towards the direction perpendicular to the flow of direction.



$$\Rightarrow j = \frac{I}{A} \quad \ldots\ldots\ldots\ldots \quad 2.2$$

- current density is a **vector**
  (direction is direction of velocity of positive charge carriers)



- current density $\vec{J}$ flowing through infinitesimal area $d\vec{A}$ produces infinitesimal current $dI = \vec{J} \cdot d\vec{A}$

- total current passing through A is

$$I = \int_{\text{surface}} \vec{J} \cdot d\vec{A} \qquad \text{.....................} \quad 2.3$$

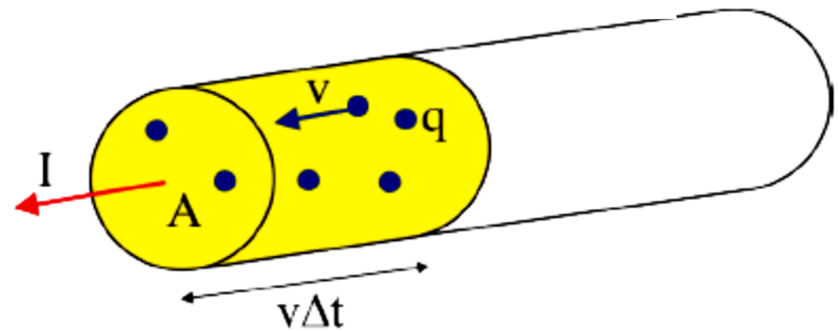
Only the component of $\vec{j}$ $\perp$ to the surface contributes to the flow of charge across the surface.

if $\vec{J}$ is uniform and parallel to $d\vec{A}$:

$$I = \int_{surface} \vec{J} \cdot d\vec{A} = J \int_{surface} dA = JA \Rightarrow \boxed{J = \frac{I}{A}}$$ ……Eqn. 2.2 confirmed

MICROSCOPIC VIEW OF ELECTRIC CURRENT:

- Carrier density (n (number of charge carriers per volume)
- carriers move with speed v



∴ Number of charges that pass through surface A in time Δt: Recall: $n = \frac{N}{volme}$

$$N = \left(\frac{number}{volume}\right) \times volume = n(\,v\Delta t\;A\,)$$ …………………………… 2.4a

v-velocity, t-time and A-area.

amount of charge passing through A in time Δt: $\Delta Q = q\; nv\Delta t\; A$ … 2.4b

divide by Δt to get the current… $I = \frac{\Delta Q}{\Delta t} = nqv\;A$……… 2.4c

…and by A to get the current density: $J = nqv$. …………… 2.4d

To account for the vector nature of the current density,

$$\vec{J} = nq\vec{v} \quad \text{..................} \quad \mathbf{2.4e}$$

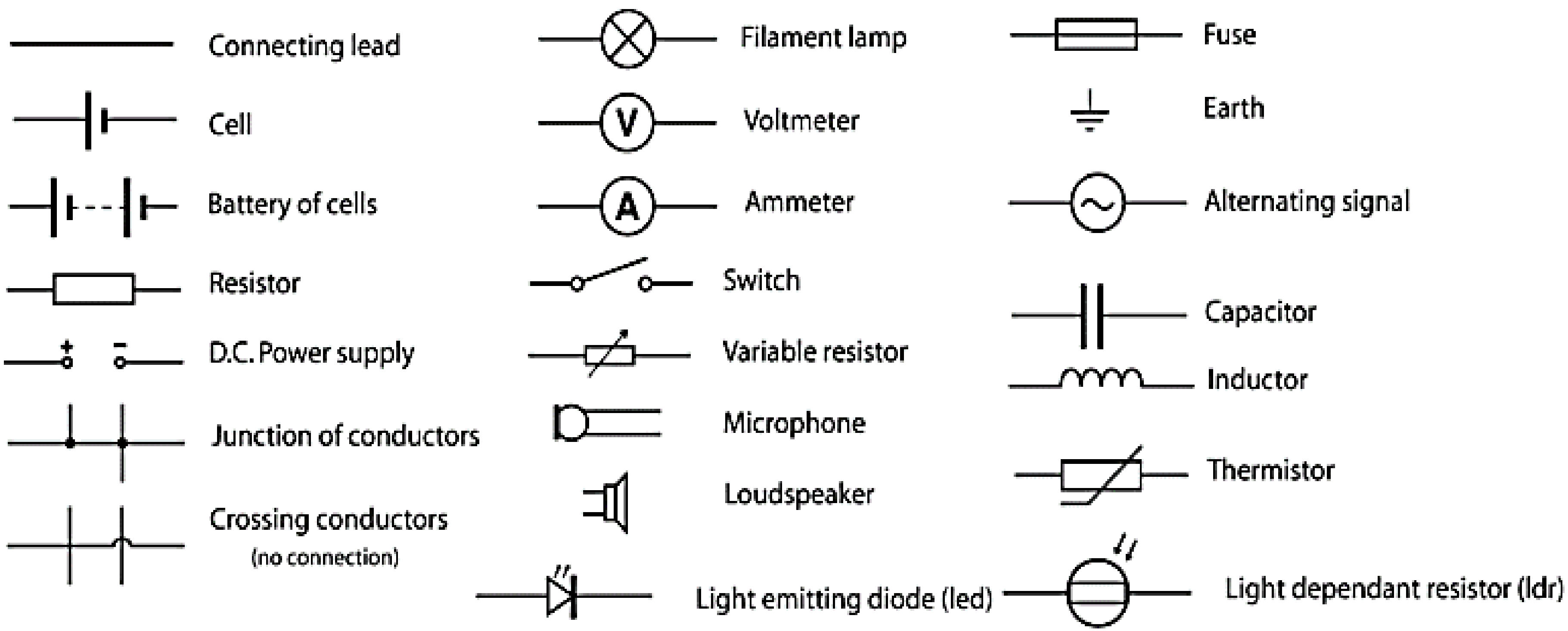
and if the charge carriers are electrons, q=-e so that

$$\vec{J}_e = -n\ e\ \vec{v} \quad \text{..............} \quad \mathbf{2.4f}$$

The – sign demonstrates that the velocity of the electrons is antiparallel to the conventional current direction.

TAKE NOTE

Electric Current and Circuit Diagram Elements



EXAMPLE 1

A current of 0.75A is drawn by the filament of an electric bulb for 10 minutes.

Find the amount of electric charge that flows through the circuit.

SOLUTION

Known values: I= 0.75A, t=10 minutes= 600 s

$$\Rightarrow Q = I \times t = 0.75A \times 600\ s$$

Therefore, Q= 450 As, but 1A = 1C/s

$$\therefore Q = 450\ C$$

EXAMPLE 2

Determine the current density when 40 Amperes of current is flowing through the battery in a given area of 10m^2.

SOLUTION

I = 40 *A*, Area = 10 m^2

The current density: $j = \frac{I}{A}$

$$\Rightarrow j = 40A / 10m^2$$

$$j = 4\ A/\mathrm{m}^2$$

EXAMPLE 3

Example: the 12-gauge copper wire in a home has a cross-sectional area of 3.31×10^{-6} m^2 and carries a current of 10 A. The conduction electron density in copper is 8.49×10^{28} electrons/m^3. Calculate the drift speed of the electrons.

SOLUTION

$A = 3.31\times10^{-6} m, I = 10A,$

carrier or electron density (n)=8.49×10^{28} electrons/m^3, $V_d = ?$

current density $(J) = \frac{I}{A}$ and $\vec{J} = nq\,\vec{v} \Rightarrow \frac{I}{A} = nq\,\vec{v}_d$

$$\Rightarrow \vec{v}_d = \frac{I}{nqA}$$

$$|v_d| = \frac{I}{neA}$$

$$|v_d| = \frac{10\,C/s}{(8.49\times10^{28}\,m^{-3})(1.60\times10^{-19}\,C)(3.31\times10^{-6}\,m^2)}$$

$$|v_d| = 2.22\times10^{-4}\,m/s$$

EXAMPLE 4

Find the velocity of copper wire whose cross-sectional area is $1mm^2$ when the wire carries a current of 10 A. Assume that each copper atom contributes one election to the electron gas.

Solution:

Given data: $^{A}_{29}Cu$

Current $I = 10\ A$

Electron density (n) $= 8.5 \times 10^{28}$ electrons/m^3

Area of cross-section A $= 1mm^2 = 10^{-6} m^2$

We know $J = neV_d$ or $J = \frac{I}{A}$

$$V_d = \frac{J}{ne}$$

$$J = \frac{I}{A} \Rightarrow \text{Current density } J = \frac{\text{Current (I)}}{\text{Area A cross section (A)}} = \frac{10}{10^{-6} m^2}$$

$$J = 10 \times 10^{+6} Am^{-2}$$

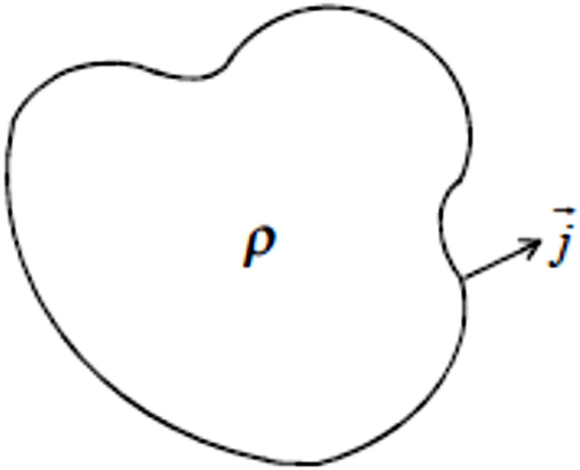
$$\therefore V_d = \frac{J}{ne} = \frac{10 \times 10^{+6} Am^{-2}}{8.5 \times 10^{28} \times 1.6 \times 10^{-19}}$$, but $1A = 1\frac{C}{s}$.

Drift velocity $V_d = 7.353 \times 10^{-4} ms^{-1}$

CONSERVATION OF CHARGE

The principle of conservation of charge states that charge can neither be created nor destroyed, although equal amounts of positive and negative charge may be simultaneously created, obtained by separation, destroyed or lost by recombination.

Now consider a **closed** surface S enclosing the volume V. Charges are flowing out of it. Therefore, there is current density $\vec{j}$ leaving the surface:



[Recall: $\vec{J} = nq\,\vec{v} = \frac{I}{A}$
ρ is volume density (to be covered in Gauss's Law)]

Total current flowing out of surface: $I = \iint_S \vec{j} \cdot d\vec{S}$ ………… 2.5a

This must equal the rate of decrease of total charge inside. Thus:

$$I = -\frac{dq}{dt} = \iint_S \vec{j} \cdot d\vec{S} \quad \ldots\ldots\ldots\ldots \ 2.5b$$

But $q = \iiint_v \rho \, dv$

Therefore, $\iiint_v \frac{\partial \rho}{\partial t} dv = -\iint_S \vec{j} \cdot d\vec{S} \quad \ldots\ldots\ldots\ldots \ 2.6a$

Now we apply the Divergence Theorem:

The Divergence Theorem relates volume integrals to surface integrals of vector fields.

Eqn. 2.5b becomes: $\iint_s \ (\vec{j}.d\vec{s}) = \iint_s \ (\vec{j}.\hat{n})ds = \iiint_v \ (div\,\vec{j})dv = \iiint_v \ \nabla.\vec{j}dv$...2.5c

Substitute Eqn. 2.5c in 2.6a: $\iiint_v \frac{\partial \rho}{\partial t} dv = -\iiint_v \nabla \cdot \vec{j} \, dv$

$$\boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{j}}$$ **Continuity Equation** ….. 2.6b

- **Continuity equation describes the fact that the charge is conserved. The decrease in value of charge inside a small volume must correspond to a flow of charge.**

- **A steady state is one in which all the derivatives in Equation 2.6 are zero or Equation 2.3 or Eqn. 2.5 is zero.**

2.3 RESISTANCE, RESISTIVITY AND CONDUCTIVITY

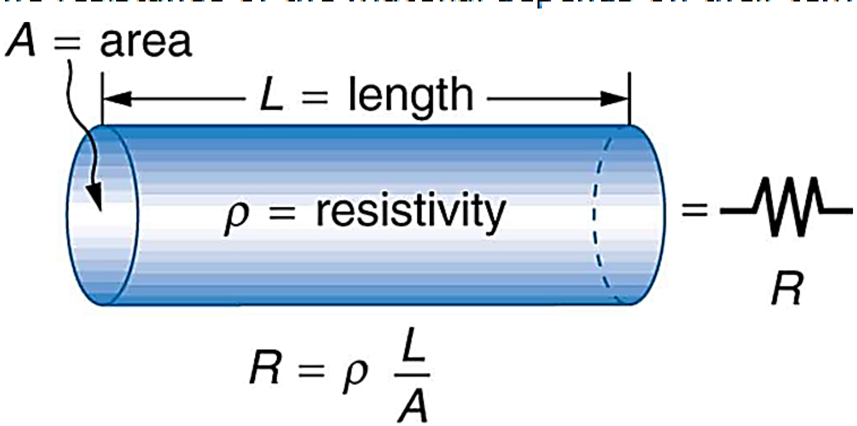
RESISTANCE (R)

Is the property of the material which creates an obstruction in the flow of the current. Thus, $R = \dfrac{V}{I}$ ………… **2.7a (Ohm's Law)**

Factors Affecting Resistance

The resistance of the wire depends on the following factors.

1. The resistance of the wire increases with the length of the conductor.
2. It is inversely proportional to the cross section area of the conductor.
3. It depends on the material of the wire.
4. The resistance of the material depends on their temperature.



$$\therefore R = \frac{\rho L}{A}$$

ρ = resistivity
L = length
A = cross sectional area

RESISTIVITY(ρ)

- It is the property of the material which defines the resistance of the material having specific dimension (i.e., dependent of its shape or size). Also known as specific resistance.

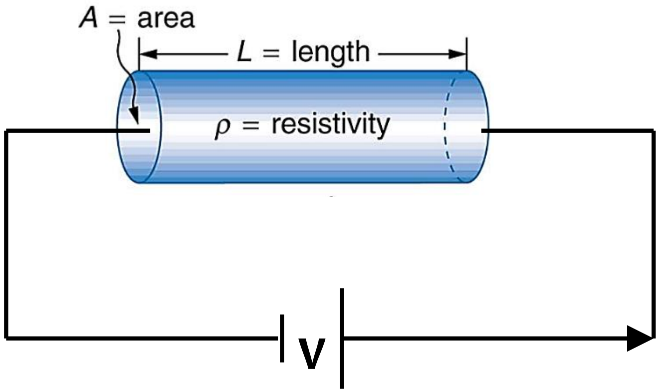
- It is defined in terms of the magnitude of the electric field($\boldsymbol{E}$) across it that gives a certain current density($\boldsymbol{j}$):

$$\Rightarrow \rho = \frac{E}{j} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 2.8$$

- Its SI unit is $(\frac{Vm^{-1}}{Am^{-2}}) = \Omega m$

DERIVING EQUATION FOR RESISTIVITY AND RESISTANCE

Consider a uniform cylindrical conductor of cross sectional area A, of length l carrying a current I. Let us apply a potential difference V between its ends. Then the electric field strength E and the current density j will be constant for all points in the cylinder. Thus,



$$\Rightarrow E = \frac{V}{l} \text{ and } j = \frac{I}{A}$$

$$\therefore\ \rho = \frac{E}{j} = \frac{V/l}{I/A} = \frac{V}{l}\frac{A}{I} = \frac{V}{l}\frac{A}{V/R} = \frac{AR}{l}$$

$$\Rightarrow \rho = \frac{RA}{l} \quad \text{.................................} \quad 2.9$$

$$\Rightarrow R = \rho\frac{l}{A} \quad \text{.................................} \quad 2.7b$$

CONDUCTIVITY(σ)

- **It is the measure of the ease at which an electric charge or heat can pass through a material or**
- **It is a material's ability to conduct electricity or heat.**

Conductivity is of many types: *electrical*, ionic, hydraulic, or thermal conductivities.

Electrical conductivity

It defines a material's ability to conduct electricity. The electric current flows easily through a material with high conductivity (e.g. a metal like copper has a high conductivity, while rubber has a very low conductivity).

- ***Conductivity is the reciprocal of resistivity :***

$$\text{Electrical conductivity } (\sigma) = \frac{1}{\rho} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 2.10$$

- **Factors such as temperature have a large effect on conductivity. So that a material with *high conductivity* has *low resistivity* and *high current density*, and a material with *low resistivity* has *low resistance* and is, by definition, a *good conductor*.**
- **Thus resistivity and conductivity are both properties of conductors.**
- **Also electrical conductivity is a ratio of the current density(j) to the electric field strength(E):**

$$\sigma = \frac{j}{E} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 2.11$$

- **Electrical conductivity is measured in *siemens per metre* $(S.m^{-1})$ or *mho per metre* $(\Omega m)^{-1}$.**

where $1S = 1mho$ **and** $1mho = \frac{1}{ohm} \Rightarrow 1S = \frac{1}{ohm} = \frac{1}{\Omega}$

$\Rightarrow 1(Sm^{-1}) = \frac{1}{\Omega m} = 1(\Omega m)^{-1}$ **(confirm using Equation 2.11 too)**

TAKE NOTE

- **Since $1\,siemens = [1\,mho$ or $\left(\frac{1}{\Omega}\right)]$; means conductance (measured in siemens or mho) is the reciprocal of resistance(measured in ohms).**

- **An electrical conductivity unit (the old unit) is a decimal unit of *electrical conductivity (EC),* which by definition is equal to 1 microsiemens per centimeter (μS/cm):**

$$1\,EC = 1\,\mu S/cm$$

Conti. TAKE NOTE

Comparison Chart

Basis For Comparison	Resistance	Resistivity
Definition	Property of substance due to which it opposes the flow of electrons.	It is defined as the resistance of material having specific dimensions.
Formula	$R = \rho \frac{l}{A}$	$\rho = \frac{R \times A}{l}$
SI Unit	Ohms	Ohms-meter
Symbol	R	ρ
Dependence	Length, cross-section area of conductor and temperature.	Temperature and nature of material

EXAMPLE 1

A car headlight filament is made of tungsten and has a cold resistance of $0.350\ \Omega$. If the filament is a cylinder 4.00 cm long (it may be coiled to save space), what is its diameter?

Solution

Where resistivity of Tungsten is $5.6 \times 10^{-8}\Omega m$

The cross-sectional area, found by rearranging the expression for the resistance of a cylinder given in $R = \frac{\rho L}{A}$, is

$$A = \frac{\rho L}{R}.$$

Substituting the given values, and taking ρ from yields

$$\begin{aligned} A &= \frac{(5.610^{-8}\ \Omega\cdot\text{m})(4.0010^{-2}\ \text{m})}{0.350\ \Omega} \\ &= 6.4010^{-9}\ \text{m}^2. \end{aligned}$$

The area of a circle is related to its diameter D by

$$A = \frac{\pi D^2}{4}.$$

Solving for the diameter D, and substituting the value found for A, gives

$$\begin{aligned} D &= 2\left(\frac{A}{p}\right)^{\frac{1}{2}} = 2\left(\frac{6.4010^{-9}\ \text{m}^2}{3.14}\right)^{\frac{1}{2}} \\ &= 9.010^{-5}\ \text{m}. \end{aligned}$$

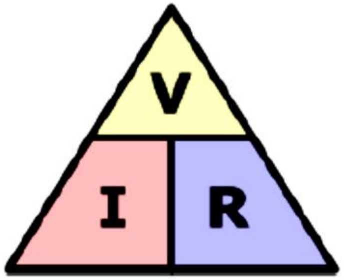
2.4 OHM'S LAW

The law was developed by Georg Ohm. He found out that at a constant temperature, the electrical current (I) flowing through a ***fixed linear resistance*** (R) is directly proportional to the voltage (V) applied across it, and also inversely proportional to the resistance (R).

This relationship between the Voltage, Current and Resistance forms the basis of **Ohm's Law** and is as shown below.

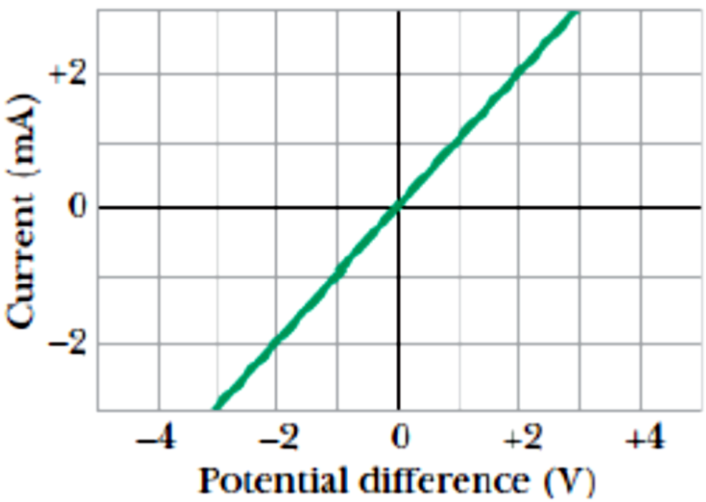
$$current(I) = \frac{voltage(V)}{\mathrm{Resistance}(R)} \quad (\text{in Amperes}(A)) \quad \ldots\ldots\ldots \quad 2.12$$

The Ohms Law Triangle

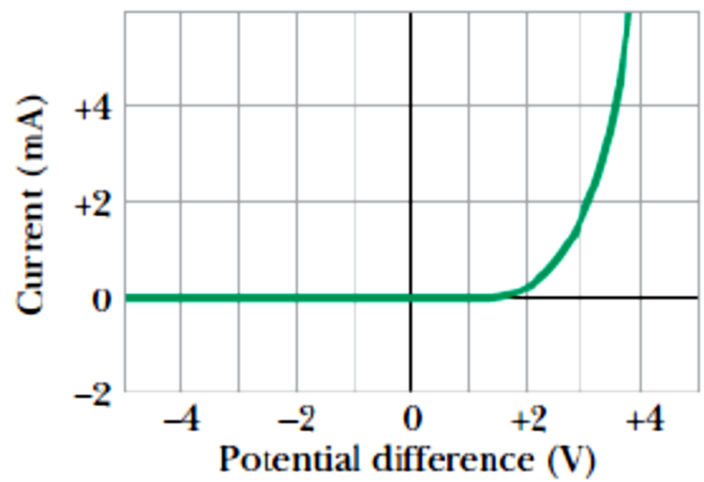


where I is the current through the conductor in units of ***amperes***, V is the potential difference measured across the conductor in units of ***volts***, and R is the resistance of the conductor in units of ***ohms***.

Obeys ohms law $(I \propto V)$(using 1000 ohm resistor)

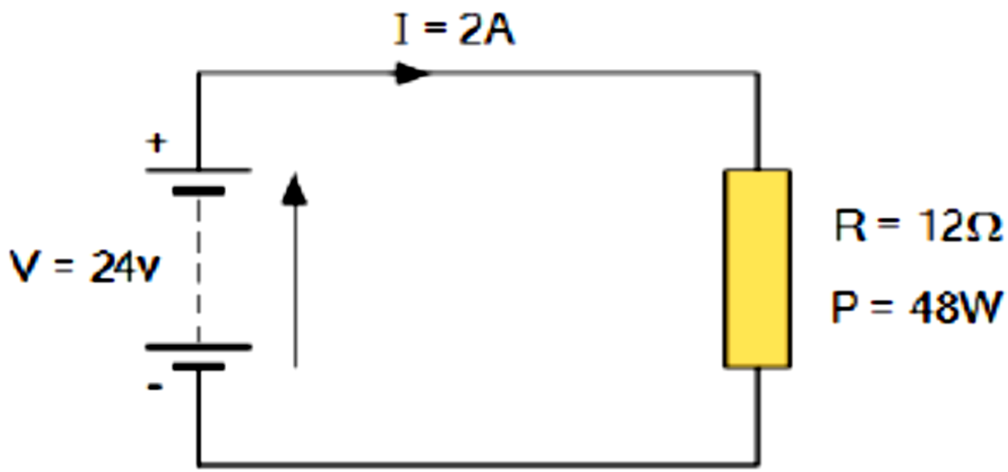


Doesn't obey ohms law (using a pn junction diode)



EXAMPLE 2

For the circuit shown below find the Voltage (V), Current (I) and Resistance (R).



SOLUTION

Voltage $[V = I \times R] = 2\,A \times 12\Omega = 24V$

Current $[I = V \div R] = 24V \div 12\Omega = 2A$

Resistance $[R = V \div I] = 24V \div 2A = 12\,\Omega$

RESISTORS

HOW TO CALCULATE THE TOTAL RESISTANCE IN A CIRCUIT

Resistance can be calculated using the:

1) Series circuit
2) Parallel circuit
3) Combination circuit
4) Formulas using power ($P = VI$), that is $\left(R_T = \frac{V_T}{I_T} = \frac{V_T^2}{P_T} = \frac{P_T}{I_T^2}\right)$ (not covered in this course).

SERIES CIRCUIT

A series circuit is a circuit in which resistors are arranged in a chain, so the

- *current is the same through each resistor.*
- *total resistance of the circuit is found by simply adding up the resistance values of the individual resistors.*

Equivalent resistance of resistors in series (R) = $R_1 + R_2 + R_3 + \ldots + R_n$ **2.13**

PARALLEL CIRCUIT

A parallel circuit is a circuit in which the resistors are arranged with their heads connected together, and their tails connected together. So the

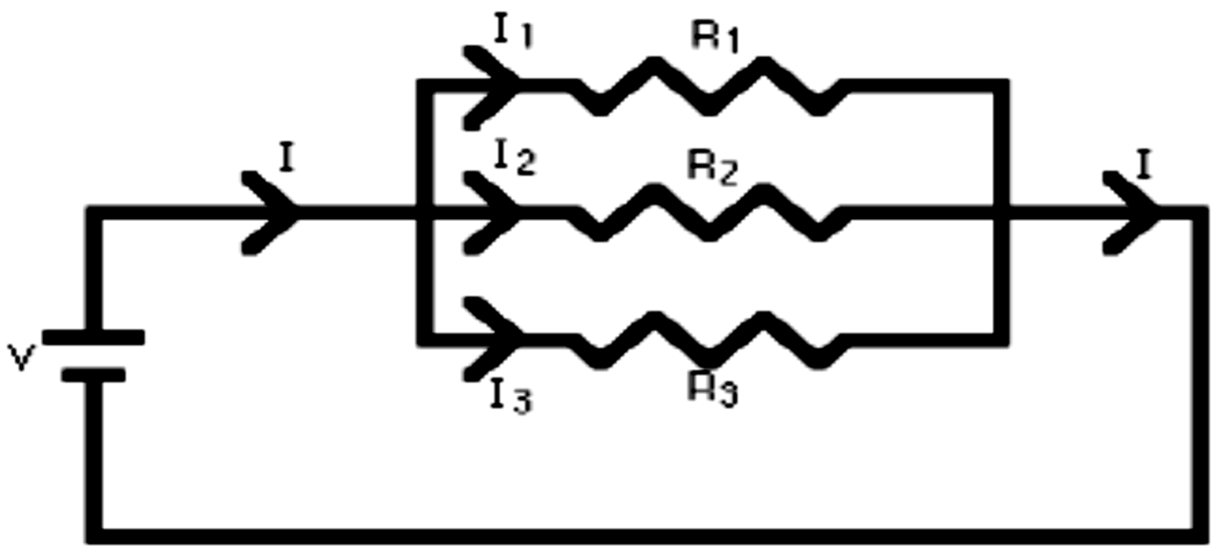
- *current in a parallel circuit breaks up.*
- *voltage across each resistor in parallel is the same.*
- *branches with zero resistance* $(R = 0\Omega)$ means that, that branch on the parallel circuit has no resistance; so all of the current in the circuit will flow through the branch. In practical applications, it means a resistor has failed or has been bypassed (short-circuited), and the high current could damage other parts of the circuit.
- *total resistance of a set of resistors in parallel is found by adding up the reciprocals of the resistances values, and then taking the reciprocal of the total.*

Equivalent resistance of resistors in parallel: $(1 / R) = (1 / R_1 + 1 / R_2 + 1 / R_3 + ... + 1/R_n)$ … 2.14

$$\therefore R = \frac{1}{(1/R_1)+(1/R_2)+(1/R_3)+ \ldots\ldots\ldots +(1/R_n)} \quad \ldots\ldots\ldots\ldots\ldots\ 2.14$$

EXAMPLE

Determine the values of I_1, I_2 and I_3 in the parallel circuit shown below. If the resistance for the resistors are $R_1 = 8\Omega$, $R_2 = 8\Omega$ and $R_3 = 4\Omega$ and a battery has a voltage of 10V.



SOLUTION

Total resistance $\left(\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)\left(\Rightarrow\frac{1}{R}=\frac{1}{8\Omega}+\frac{1}{8\Omega}+\frac{1}{4\Omega}=\frac{1}{2}\right)\ \Rightarrow R=2\Omega$

But $V=IR\Rightarrow I=\frac{V}{R}=\frac{10V}{2\Omega}=5A$

In parallel circuit the current supplied by the battery splits up $(I=I_1+I_2+I_3)$, and the amount of current going through each resistor depends on the resistance:

$$\Rightarrow I=\frac{V}{R}$$

$$I_1=\frac{V}{R_1}=\frac{10V}{8\Omega}=1.25A;\ I_2=\frac{V}{R_2}=\frac{10V}{8\Omega}=1.25A \text{ and } I_3=\frac{V}{R_3}=\frac{10V}{4\Omega}=2.5A$$

Confirming that $I=5A$:

$(I=I_1+I_2+I_3)\Rightarrow I=(1.25+1.25+2.5)A=5A$ confirmed.

CHARACTERISTICS OF PARALLEL - SERIES CIRCUITS (of resistors)

i. Resistors in series, the current is the same for each resistor.

ii. Resistors in parallel, the voltage is the same for each resistor.

iii. When resistors are in parallel, there are many different means to an end, so the total resistance is found by adding up the reciprocals of the resistances values, and then taking the reciprocal of the total.

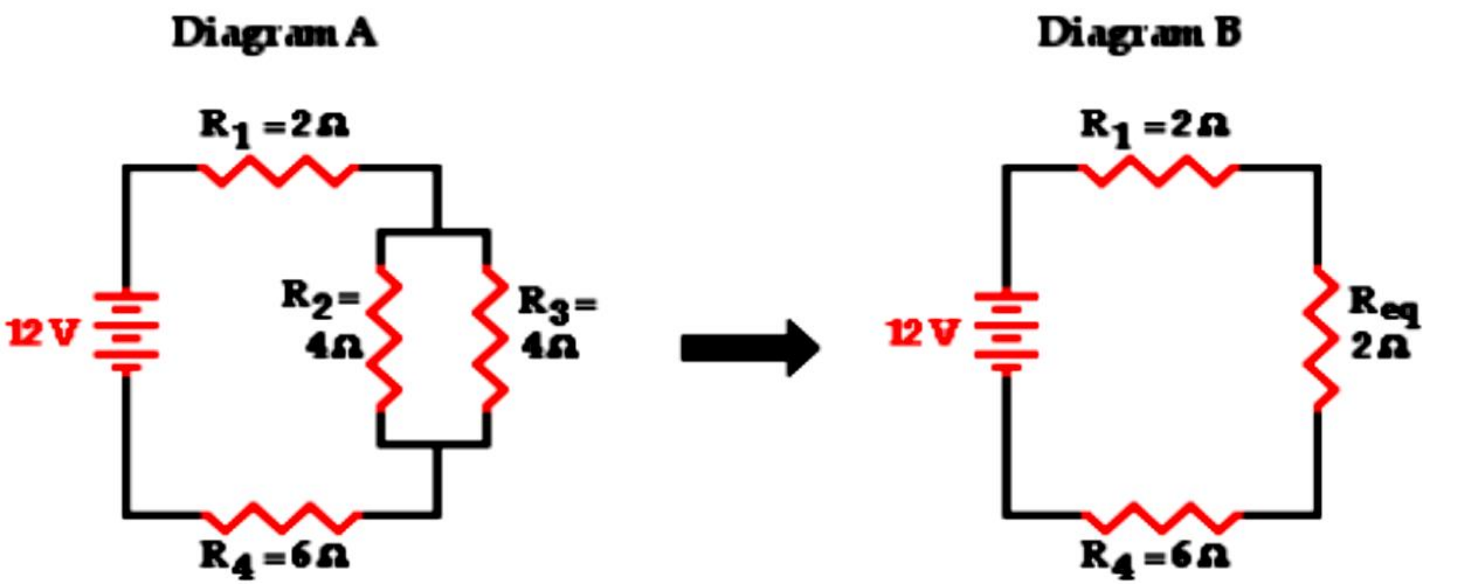
iv. When resistors are in series, the current will have to travel through each resistor, so the individual resistors will add to give the total resistance for the series.

v. Equivalent resistance (R_{eq} or R) is always smaller than the *smallest contributor* for a parallel circuit (see iii).

vi. Equivalent resistance (R_{eq} or R) is always greater than the *greatest contributor* for a series circuit (see iv).

COMBINATION CIRCUIT

A combination circuit is a circuit that has some components linked together in series (one after the other), and others in parallel (on different branches). Thus these circuits must be broken down into *series sections* and *parallel sections* (is the using of equivalent circuits). First find the total resistances of each parallel section then add it/them to the resistances for the series section. See the Figure below.



RULES FOR DOING THE REDUCTION PROCESS

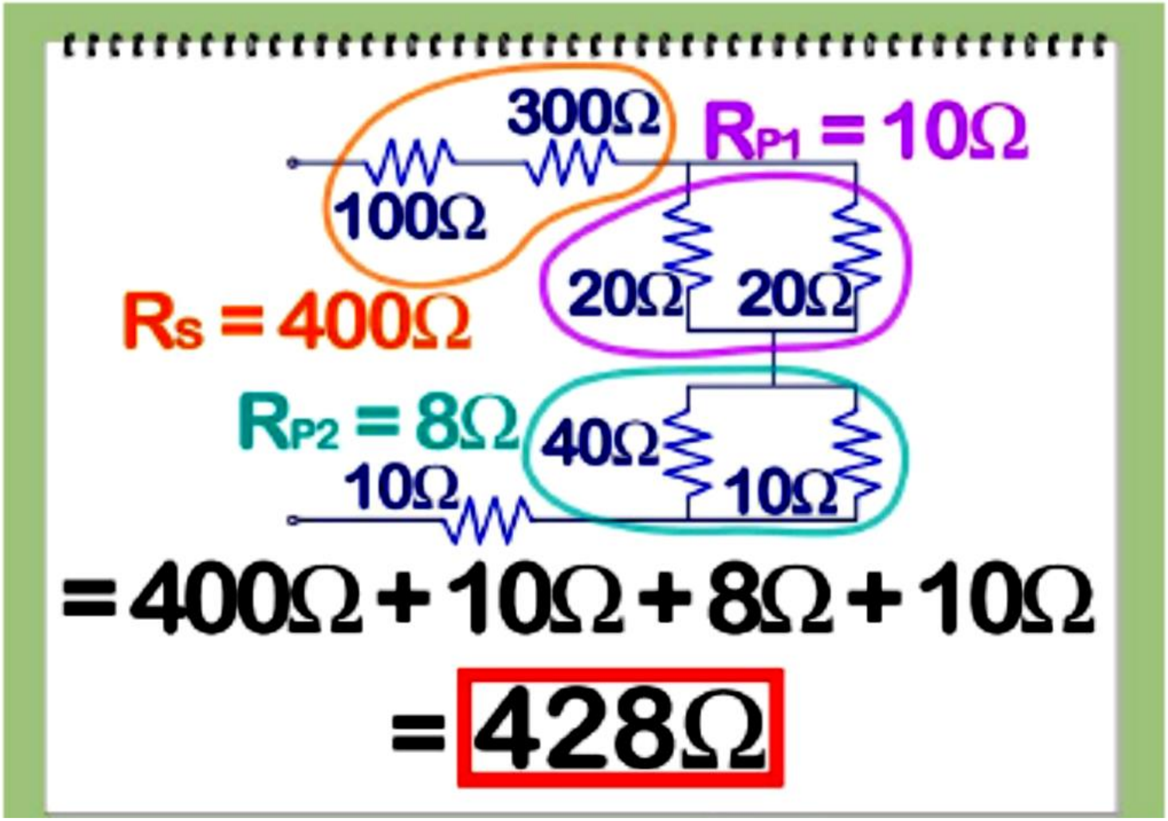
(for circuits that have a combination of series and parallel resistors)

1. Two (or more) resistors with their heads directly connected together and their tails directly connected together are in parallel, and they can be reduced to one resistor using the equivalent resistance equation for resistors in parallel
2. Two resistors connected together so that the tail of one is connected to the head of the next, with no other path for the current to take along the line connecting them, are in series and can be reduced to one equivalent resistor.

EXAMPLE

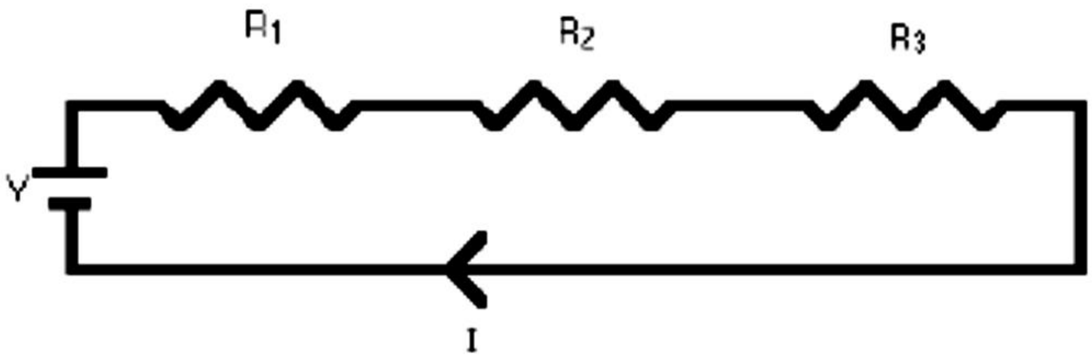
Calculate the total resistance for the circuit below.

SOLUTION: Total resistance is the sum the purple, orange and green color values from the figure below [i.e. (10+400+8) Ω = 428 Ω].



EXAMPLE

Calculate the current for a series circuit shown in the diagram below. If the resistance for the resistors are $R_1 = 8\Omega$, $R_2 = 8\Omega$ and $R_3 = 4\Omega$ and a battery has a voltage of 10V.



SOLUTION

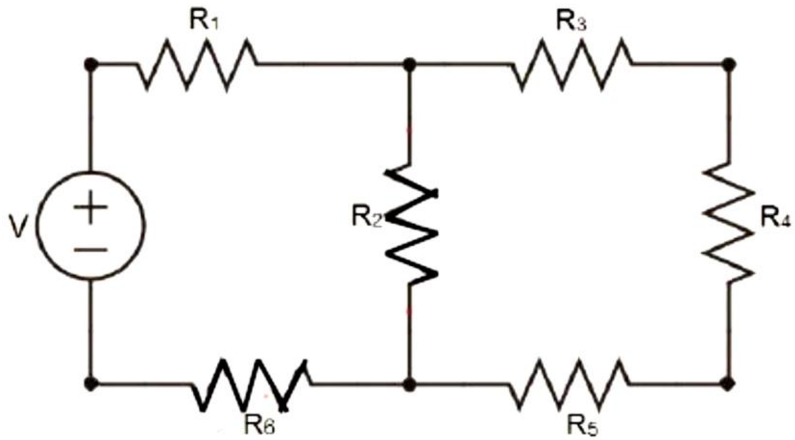
$I = V / R = V/(R_1 + R_2 + R_3) = 10V / (20)\Omega = 0.5A.$

The current through each resistor would be 0.5A.

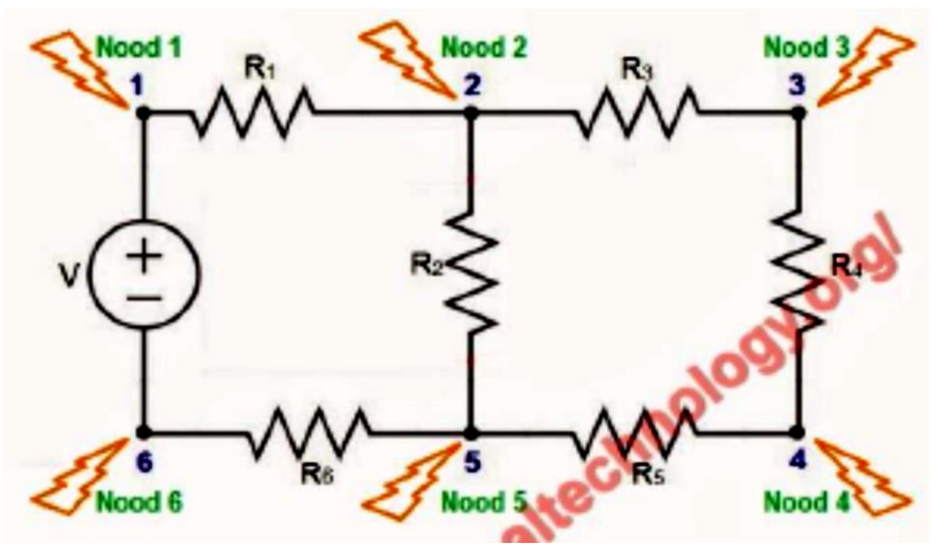
2.5 KIRCHHOFF’S LAWS

Common DC Circuit Theory Terms:

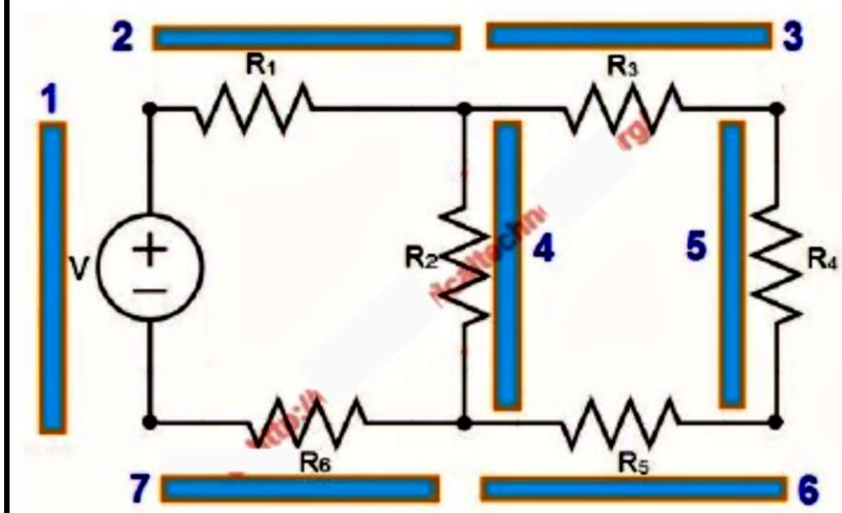
- **Circuit** – a circuit is a closed loop conducting path in which an electrical current flows.
- **Path** – a single line of connecting elements or sources.
- **Node** – A node is a junction, connection or terminal within a circuit where two or more circuit elements (resistor, capacitor, inductor etc.) are connected or joined together giving a connection point between two or more branches and it is indicated by a dot.
- **Branch** – a branch is a single or group of components such as resistors or a source which are connected between two nodes/junctions and they have two terminals.
- **Loop** – a loop is a simple closed path in a circuit in which no circuit element or node is encountered more than once. So it is a closed path in a circuit where more than two meshes can be occurred. But a mesh does not contain on one loop.
- **Mesh** – a mesh is a single open loop that does not have a closed path. There are no components inside a mesh. It can be a path which does not contain on other paths is called Mesh.



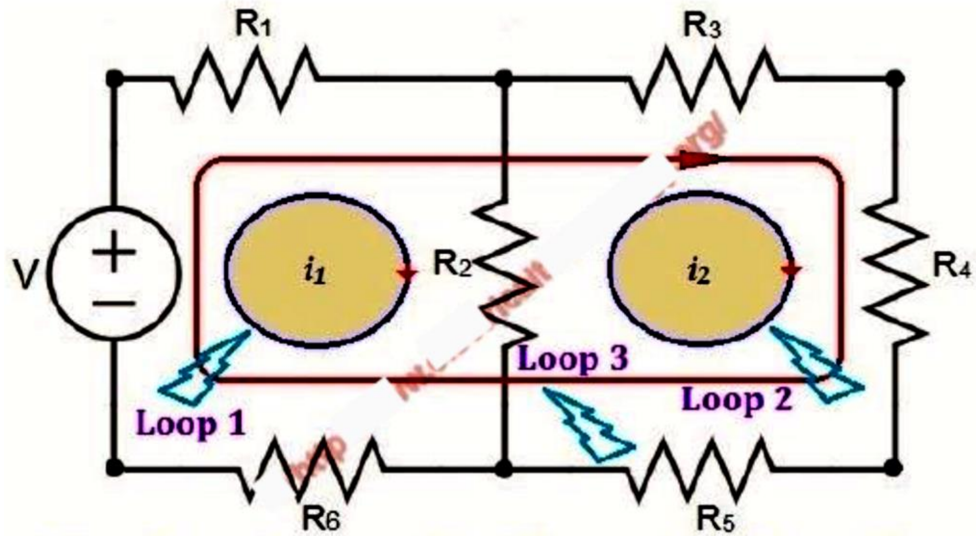
6 NODES



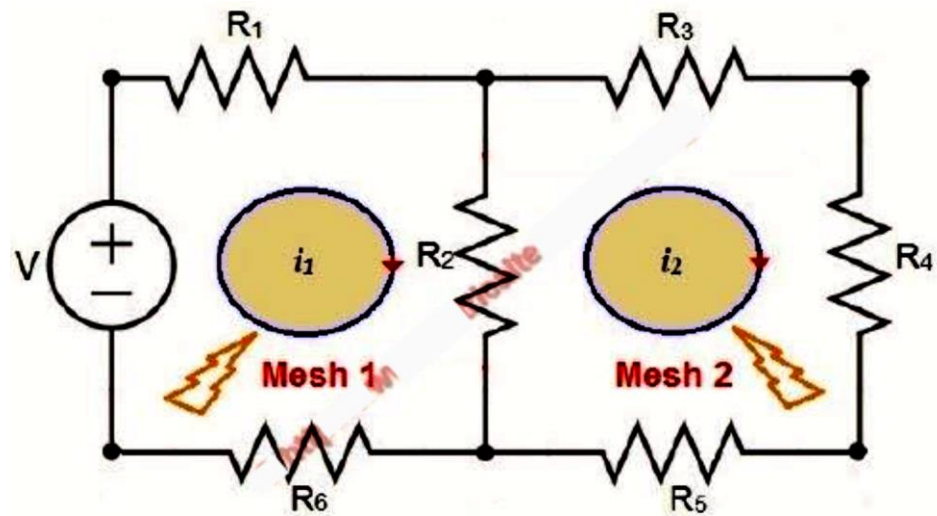
7 BRANCHES



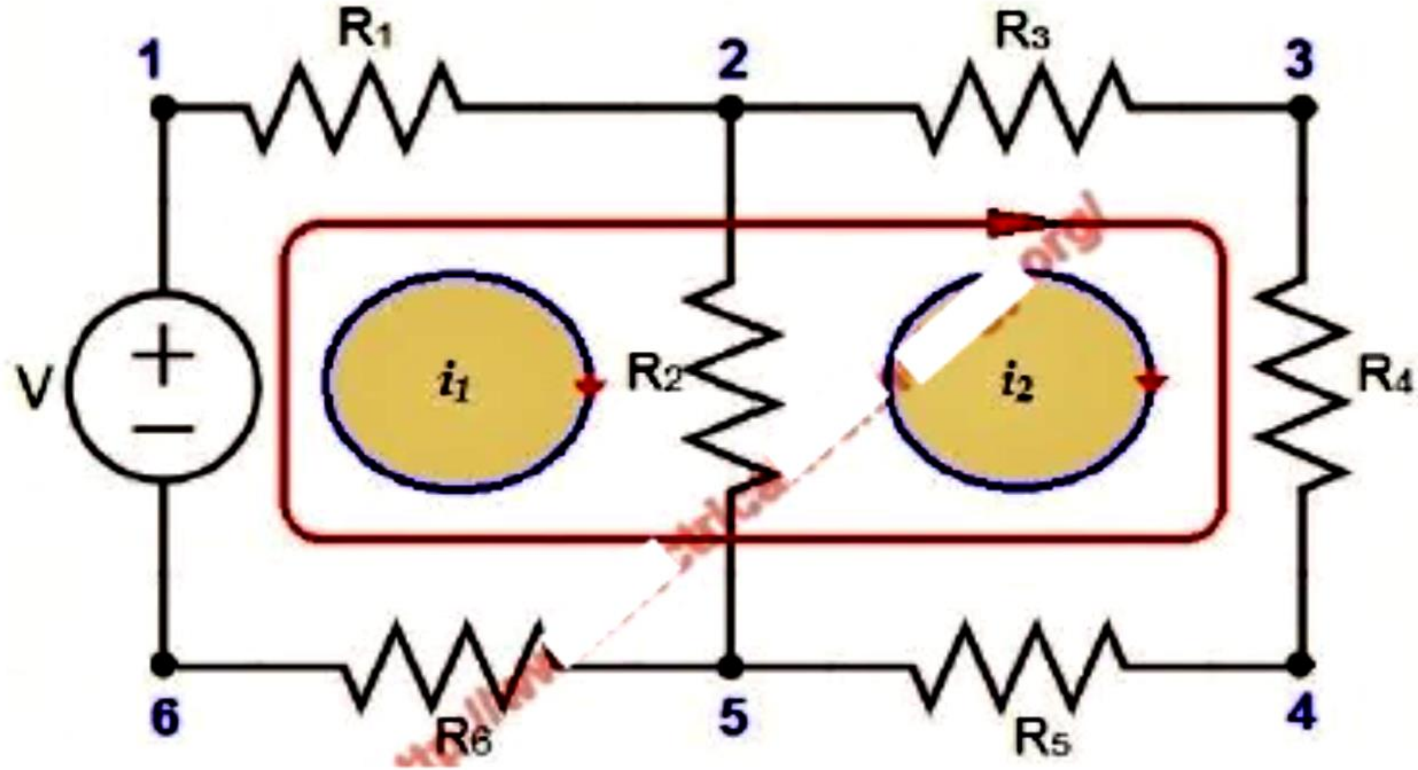
3 LOOPS



2 MESHES

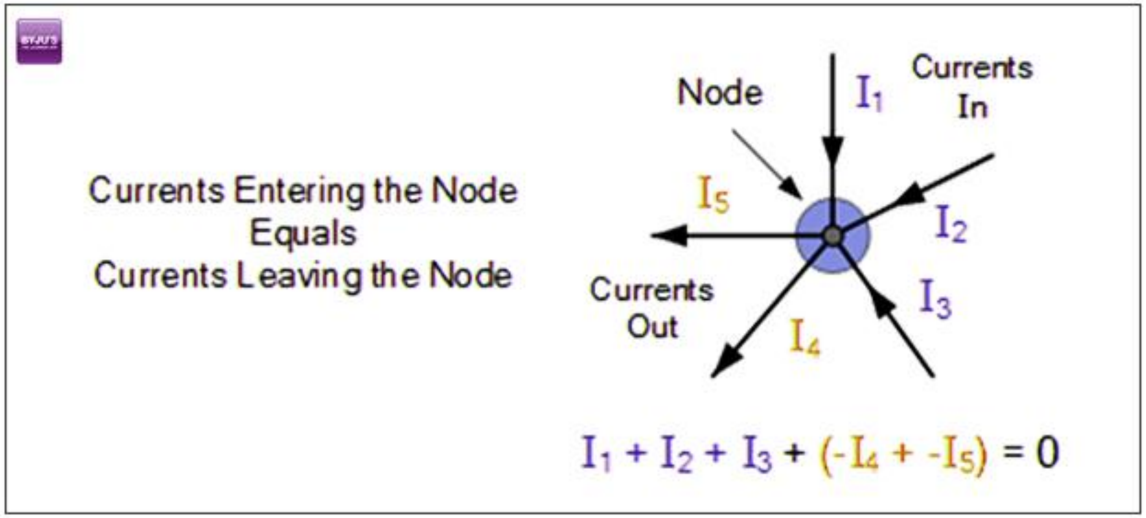


Thus the circuit has 6 Nodes, 7 Branches, 3 Loops and 2 Meshes, as shown below:



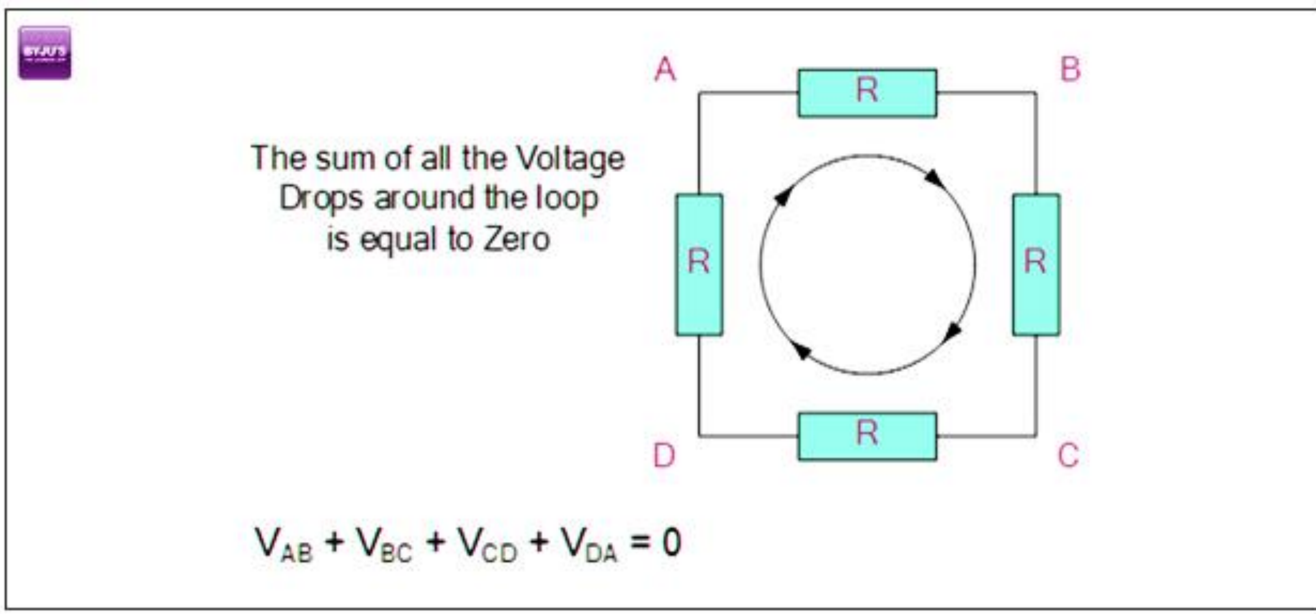
A German physicist Gustav Kirchhoff developed Kirchhoff's laws which are fundamental to circuit theory. They quantify how current flows through a circuit and how voltage varies around a loop in a circuit.

a) Kirchhoff's First Law (Kirchhoff's Current Law (KCL) /Junction Rule) states that the total current entering a junction or a node is equal to the charge leaving the node as no charge is lost. So the algebraic sum of every current entering and leaving the node has to be zero. This property of Kirchhoff law is commonly called as conservation of charge.



$$\Rightarrow \sum I_{in} = \sum I_{out}$$

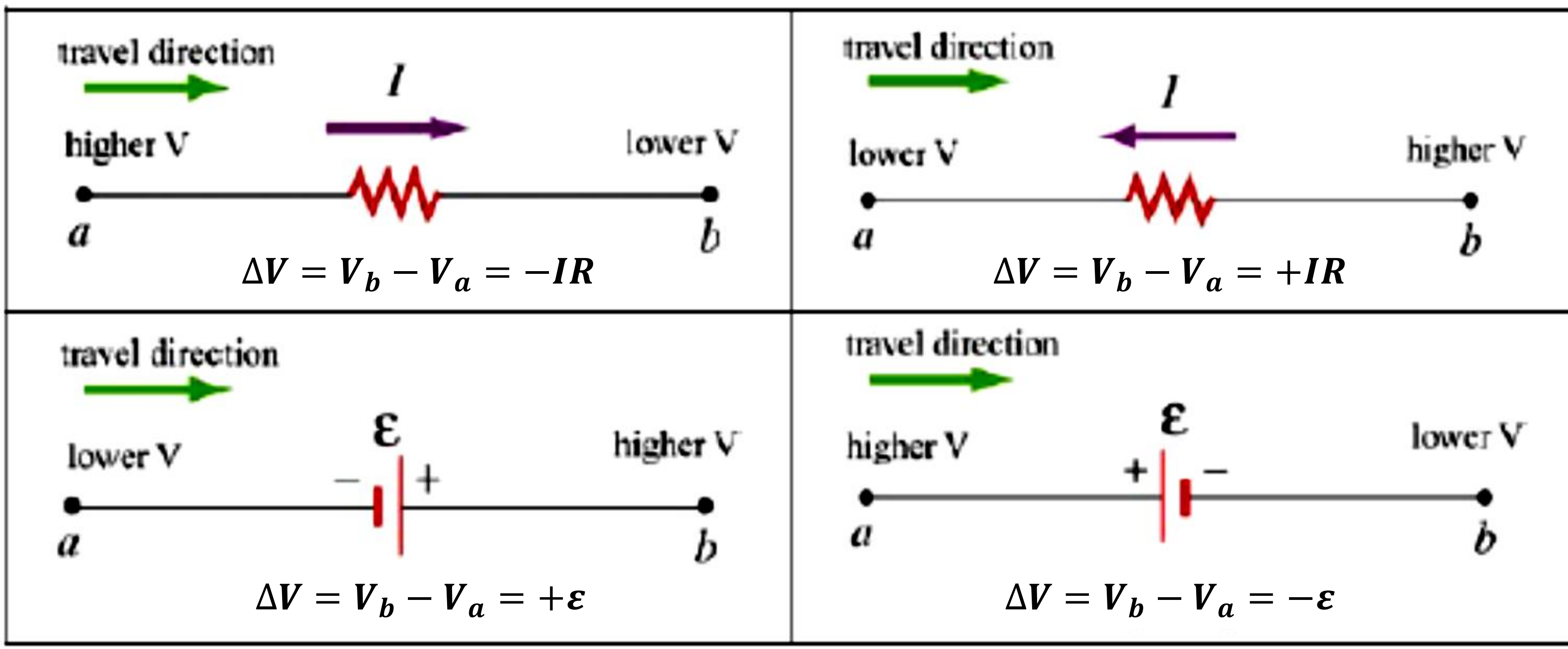
b) Kirchhoff’s Second Law (Kirchhoff’s Voltage Law (KVL)/Closed Loop Rule) states that the voltage around a loop equals to the sum of every voltage drop in the same loop for any closed network and also equals to zero and this property of Kirchhoff’s law is called as conservation of energy. You MUST maintain the direction either anticlockwise or clockwise; else the final voltage value will not be equal to zero.



$$\Rightarrow \sum \Delta V_{close\ loop} = 0$$

Conventions for KVL/Loop Rule

Conventions apply to determining the sign of ΔV across circuit elements (resistors and batteries). Travel direction is the direction we choose to proceed round the loop.

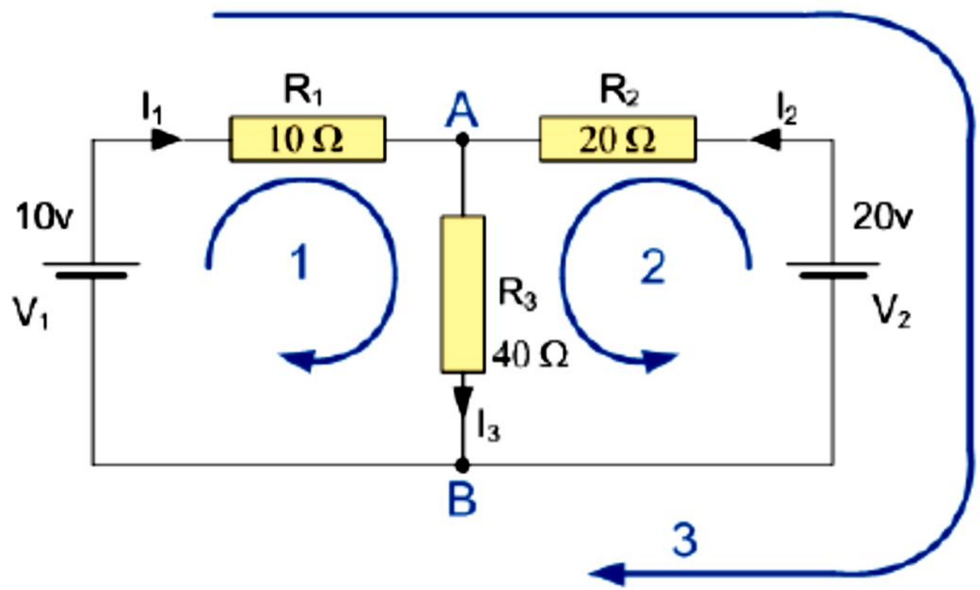


Procedure to be followed for using the Kirchhoff's Circuit Laws:

These two laws enable the Currents and Voltages in a circuit to be found. That is, the circuit is said to be "Analysed".

1. Assume all voltages and resistances are given (If not label them V_1, V_2,… R_1, R_2,….etc.).
2. Label each branch with a branch current. (I_1, I_2, I_3,…… etc.).
3. Find Kirchhoff's first law equations for each node.
4. Find Kirchhoff's second law equations for each of the independent loops of the circuit.
5. Use Linear simultaneous equations (The elimination method - see Example below) as required to find the unknown currents.

Example 3: Determine the values of the the current flowing through each of the resistors.



SOLUTION

The circuit has two nodes (at A and B). We have the choice of choosing only two of the three loops shown (blue). This is because only two of the loops are independent.

KCL: Node A $I_1 + I_2 = I_3$

Node B $I_3 = I_1 + I_2$

KVL: Loop 1 $10 - I_1 R_1 - I_3 R_3 = 0$

Loop 2 $20 - I_2 R_2 - I_3 R_3 = 0$

Use linear simultaneous equation by elimination method.

By substitution: $\boldsymbol{I_1 = -0.143A \ and \ I_2 = 0.429A}$

Using **Kirchhoff's Voltage Law (KVL)** the equations are given as:

$\Rightarrow$ Loop 1 is given as : $10V = R_1 I_1 + R_3 I_3 = 10\Omega \times I_1 + 40\Omega \times I_3$ (3)

Loop 2 is given as : $20V = R_2 I_2 + R_3 I_3 = 20\Omega \times I_2 + 40\Omega \times I_3$ (4)

Loop 3 is given as : $(10V - 20V) = 10\Omega \times I_1 - 20\Omega \times I_2$

Substitute Equation (1 or 2) in Equations (3 and 4), then obtain:

$\Rightarrow$ Eq. No 3 : $10V = 10\Omega I_1 + 40\Omega\ (I_1 + I_2) = 50\ \Omega \times I_1 + 40\Omega \times I_2$ (5)

Eq. No 4 : $20V = 20\Omega I_2 + 40\Omega\ (I_1 + I_2) = 40\ \Omega \times I_1 + 60\Omega \times I_2$ (6)

Thus Equations (5 and 6) are two "**Simultaneous Equations**" that can be reduced to give us the values of I_1 and I_2 .

Solve the following pair of simultaneous linear equations using the elimination method:

$\Rightarrow$ $10V = (50I_1 + 40I_2)\Omega$ (5)

$20V = (40I_1 + 60I_2)\Omega$ (6)

Step 1: Multiply each equation by a suitable number so that the two equations have the *same leading coefficient*. An easy choice is to multiply Equation 5 by 4, the coefficient of I_1 in Equation 5, and multiply Equation 6 by 5, the I_1 coefficient in Equation 6:

$4 \times$ Equation 5 ---> $4_\times (50I_1 + 40I_2 = 10)$ ---> $200I_1 + 160I_2 = 40$

$5 \times$ Equation 6 --> $5_\times (40I_1 + 60I_2 = 20)$ ---> $200I_1 + 300I_2 = 100$

Both equations now have the same leading coefficient = 200

Step 2: Subtract the second equation from the first.

$$
\begin{array}{r}
200\,I_1 + 160I_2 = 40 \\
-\quad 200\,I_1 + 300I_2 = 100 \\
\hline
-\,140I_2\,\Omega = -\,60V
\end{array}
$$

Step 3: Solve this new equation for I_2.

$$I_2 = -60V/-140\,\Omega = 0.4285714286\,V/\Omega = 0.429$$

Step 4: Substitute $I_2 = 0.4285714286\,V/\Omega = 0.429\,V/\Omega$ into either Equation 5 or Equation 6 above and solve for I_1. We'll use Equation 5.

Using Eqn.5: $50\,\Omega\, I_1 + 40\,\Omega\, I_2 = 10\,V$

$$50\,\Omega\, I_1 + 40\,\Omega\ (0.4285714286)\ V/\Omega = 10\ \ V$$

$$I_1 = -\ 7.142857144\,V/50\,\Omega\ = -\ 0.142857143\text{Amps}$$

Using Eqn.6: $40I_1 + 60I_2 = 20$

$$40I_1 + 60(0.4285714286) = 20$$

$$40I_1 = 20\text{-}25.71428572$$

$$= -\ 5.71428572/40$$

$$I_1 = -\ 0.142857143\text{Amps}$$

Thus $I_1 = -0.143$A and $I_2 = 0.429$A.

Using Equation (1or 2), the current flowing in the resistor R_3 is $(0.143 + 0.429) = 0.286$ Amps

and the voltage across the resistor R_3 is $(0.286 \times 40) = 11.44$ volts

NOTE: The negative sign for I_1 means that the direction of current flow initially chosen was wrong; but never the less still valid. In fact, the 20V battery is charging the 10V battery.

LECTURE 3: MAGNETISM

Content

3.1 Introduction

3.2 The inverse square law of force

3.3 Strength of magnetic field on a pole

3.4 Magnetic force due to a bar magnet

3.5 The deflection magnetometer

3.6 Couple exerted by magnetic field

3.7 Vibration of a magnet in a magnetic field

3.8 Vibration magnetometer

3.9 Determination of the horizontal component of earth's magnetic field.

3.1 INTRODUCTION

This lecture introduces the Coulomb's law of force between magnetic poles, the magnetic field strength concept of a couple exerted by a magnetic field as an analogue of an electric dipole moment. Couples are then used in measuring devices such as deflection magnetometer and vibration magnetometer. A specific measurement of the horizontal component of the earth's of magnetic field is described

3.1 Introduction

By 600BC the Greeks had known that a mineral called **lodesone** has the property of attracting pieces of iron. Lodestone, also called **magnetite** is a certain form of iron ore ($F_{e3}O_4$). The property of magnetite attracting pieces of iron is called **magnetism**. Oscillating lodestone in a horizontal plane comes to rest in a North-South direction. In the Middle Ages, crude navigation compasses were made by attaching pieces of lodestone to wooden splints floating on bowls of water. The splints always come to rest pointing in a north-south direction. The name **Lodestone** was given because it acted as a **leading stone** in navigation. **The word** magnetism comes from **mgnesia,** a district in Asia Minor where these stones were first found. Peter Peregrines in 1269 and William Gilbert in 1600 developed an elementary theory of magnetism. A magnetic stone posses a north-seeking pole at one end or north pole and south pole at the other. It is impossible to produce a north pole without simultaneously producing a south pole. Isolated poles cannot exist. *This means that magnetic monopoles do not exist.*

In lecture one, we considered electric forces between static charges. We must now investigate in a similar detail the magnetic forces. In 1985 Coulomb applied Newtonian mechanics to the phenomenon of magnetism and its analogy to electrostatic theory. He discovered an inverse square law, parallel definitions of magnetic pole strength and electric charge and strength of magnetic and electric fields.

Lecture Objectives

At the end of this lecture, the learner should be able to:

i) DefineThe law of force between magnetic poles

ii) Identify the magnetic field as having a magnetic moment analogous to the dipole moment of an electric field.

iii) Do simple experiments using deflection and vibration magnetometers using deflection and vibration magnetometers that can be made from bar magnets.

iv) Measure magnetic field strength of the earth's field at different points in any laboratory.

TAKE NOTE

TYPES OF MAGNETSS

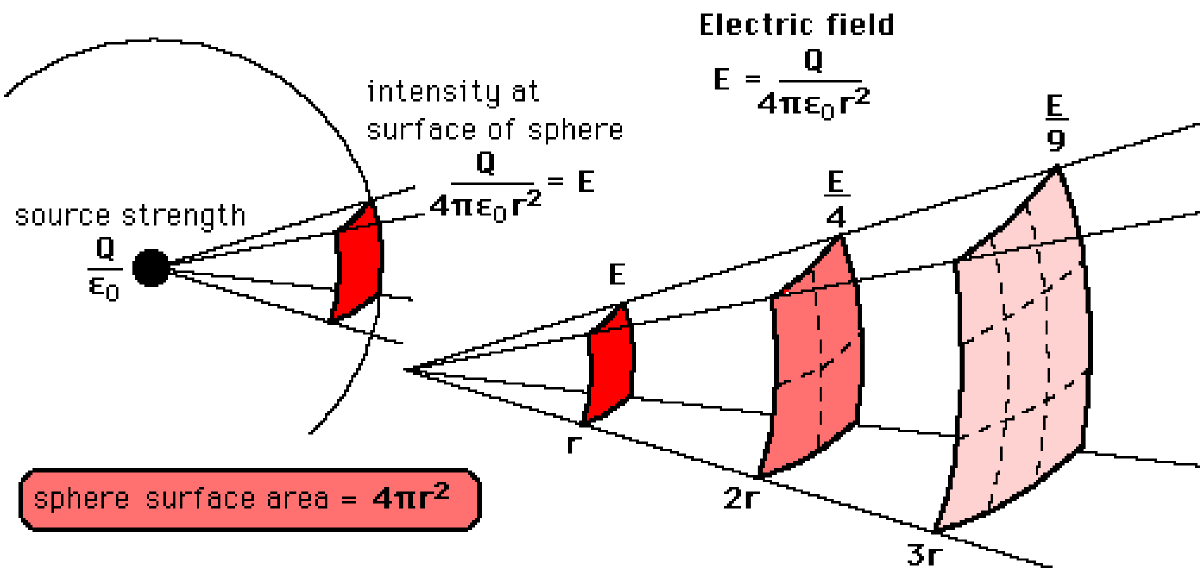
- Bar magnet

- Magnetic needle

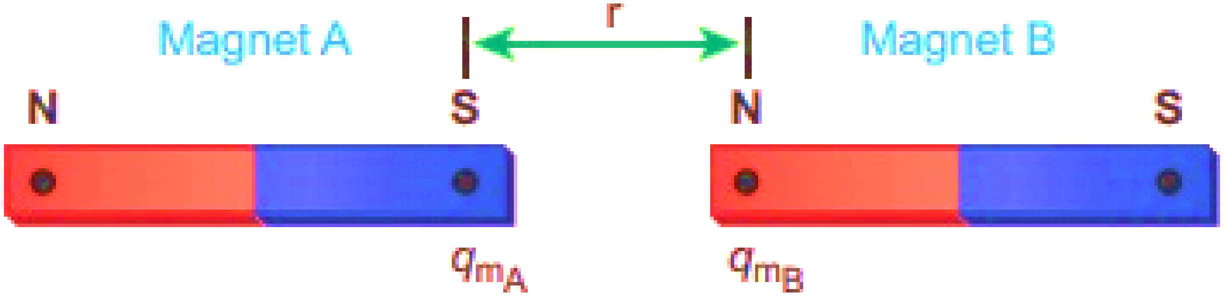
- Horse shoe magnet
- Ball ended magnet

3.2 The inverse square law of force

- **Magnetic fields are produced by magnetic dipoles, using either *permanent magnets* or *current-carrying loops of wire*. This is different from the usual method of producing an electric field, using electric charges (or "monopoles"). *For both monopoles and dipoles, the field strength decreases as the distance from the source increases.* See the Figure below.**



- **A bar magnet is an artificial magnet that is in the form of a rectangular or cylindrical bar. It has two poles each of *magnetic strength m*, but of opposite nature (the north pole (*m*) and the south pole (−*m*)). *Like poles repel and unlike poles attract each other*, so the ability of a pole to attract or repel another pole near it is called its *pole strength (m)* and is a *scalar quantity*.**



BASIC PROPERTIES OF MAGNETS

(i) When the magnet is dipped in iron filings, they cling to the ends of the magnet. The attraction is maximum at the two ends of the magnet. These ends are called poles of the magnet.

(ii) A freely suspended magnet always points along north-south direction. The pole pointing towards geographic north is called north pole N and the pole which points towards geographic south is called south pole S.

(iii) Magnetic poles always exist in pairs. So isolated magnetic pole does not exist.

(iv) The magnetic length of a magnet is always less than its geometric length, because the poles are situated a little inwards from the free ends of the magnet $(l \approx 0.83\ L)$. But for the purpose of calculation the geometric length is always taken as magnetic length.

(v) Like poles repel each other and unlike poles attract each other. North pole of a magnet when brought near north pole of another magnet, we can observe repulsion, but when the north pole of one magnet is brought near south pole of another magnet, we observe attraction.

(vi) The force of attraction or repulsion between two magnetic poles is given by Coulomb's inverse square law.

- The inverse square law of magnetism states that the force (F) of attraction or repulsion between two magnetic poles is directly proportional to the product of their pole strengths (m_A and m_B) and inversely proportional to the square of the distance (r) between them and acts along the line joining the two poles.

Mathematically (vector form): $\vec{F} \propto \frac{q_{m_A} q_{m_B}}{r^2}\hat{r} \Rightarrow \vec{F} = k\frac{q_{m_A} q_{m_B}}{r^2}\hat{r}$

In magnitude: $F = k\frac{q_{m_A} q_{m_B}}{r^2}$ or $F = k\frac{m_A \ m_B}{r^2} \ldots\ldots\ldots 3.1$

k is a constant of proportionality, and its value depends on the magnetic property of the medium surrounding the poles, called *magnetic permeability* (μ), and also on the system of units used for measuring the pole strength.

- In the MKS unit $k = \frac{1}{4\pi\mu}$

(where μ is magnetic permeability of the medium, which is the ability of a medium to allow magnetic field (B or H) lines to pass through it, or to allow itself to be influenced by a magnetic field)

m its SI unit is the ampere-metre($A.m$).

k's MKS unit is the *weber* (Wb).

F is Newton (N)

μ is Weber per ampere-metre ($\text{Wb}.(A.m)^{-1}$)

- In the cgs unit $k = 1$ and F is *dynes*

EXAMPLE

The repulsive force between two magnetic poles in air is 9×10^{-3} N. If the two poles are equal in strength and are separated by a distance of 10 cm, calculate the pole strength of each pole.

SOLUTION

The force between two poles are given by

$$\vec{F} = k\frac{q_{m_A} q_{m_B}}{r^2}\hat{r}$$

The magnitude of the force is

$$F = k\frac{q_{m_A} q_{m_B}}{r^2}$$

Given : $F = 9 \times 10^{-3}$N, $r = 10$ cm $= 10 \times 10^{-2}$ m

Therefore,

$$9\times10^{-3} = 10^{-7} \times \frac{q_m^2}{\left(10\times10^{-2}\right)^2} \Rightarrow q_m = 30\,NT^{-1}$$

3.3 Strength of a magnetic field

- **A magnetic field is characterized by magnetic field lines. In the SI system of units, the *expression for force between magnetic poles in free space* is given by the inverse square law: (where $k = \frac{\mu_o}{4\pi}$ for free space)**

$$\vec{F} = \frac{\mu_o}{4\pi}\frac{m_A m_B}{r^2} \quad \ldots\ldots\ldots\ldots\ldots \quad 3.2$$

where μ_0 is called the permeability of free space, and its value in the SI system is equal to *$4\pi \times 10^{-7}$ henry per metre*.

- **If the *two magnetic poles (m_A and m_B) are surrounded by a medium other than free space*, then the force(F) between the magnetic poles is given by: $\vec{F} = \frac{m_A m_B}{\mu r^2}$ ……… 3.3**

where μ_r is Relative permeability of the medium, and $\mu_r = 1$ for air or non-magnetic materials, the force for air. $\mu = \mu_r \mu_o$

- A *unit magnet pole* is defined as one that exerts a force of one dyne (10^{-7} newton) on another unit magnet pole when poles are in free space and separated from each other by a distance of one cm.

Field strength is then measured in units called oersteds. If a magnetic pole of strength m units experiences a force of F dynes at a point in a magnetic field the strength of the field at that point is

$$H = \frac{F}{m} \text{ oersteds} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 3.4$$

where H is the symbol used for magnetic field strength or intensity.

Substitute Eqn. 3.1 in 3.4: $H = k\,\frac{m_A m_B}{r^2}\,\frac{1}{m}$

So magnetic field strength for *force per unit pole*, $m = m_B$:

$$\Rightarrow H = k\,\frac{m_A}{r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 3.5$$

(similar to Equation 1.8: $E = F/Q = k\,q_1 / r^2$)

TAKE NOTE

- In electromagnetics, the term "magnetic field" is used for two distinct but closely related fields denoted by the symbols B and H.
- In the International System of Units:

➢ H, magnetic field strength, is measured in the SI base units of amperes per meter (A/m).

➢ B, magnetic flux density, is measured in teslas (in SI base units: kilogram per second squared per ampere ($kg/A.s^2$) or newton per meter per ampere ($N/A.m$).

- H and B differ in how they account for magnetization:

➢ In a vacuum, B and H are the same aside from units;

➢ In a magnetized material, $\frac{B}{\mu_0}$ and H differ by the magnetization M of the material at that point in the material: $B = \mu H = \mu_r \mu_0 H$ ($Weber/m^2$) or (Wb/m^2)

where μ_r-relative permeability of the medium and μ_0-permeability of free space

conti. TAKE NOTE

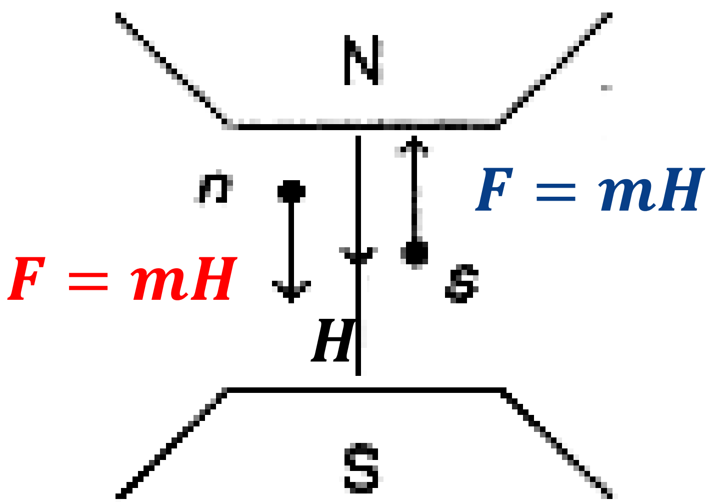
where $Wb = \frac{kg.m^2}{s^2.A} = V.s = H.A = T.m^2 = \frac{J}{A} 10Mx$

$V - volt, T - tesla, J - joule, A\ amprere, H - henry\ and\ Mx - Maxwell$

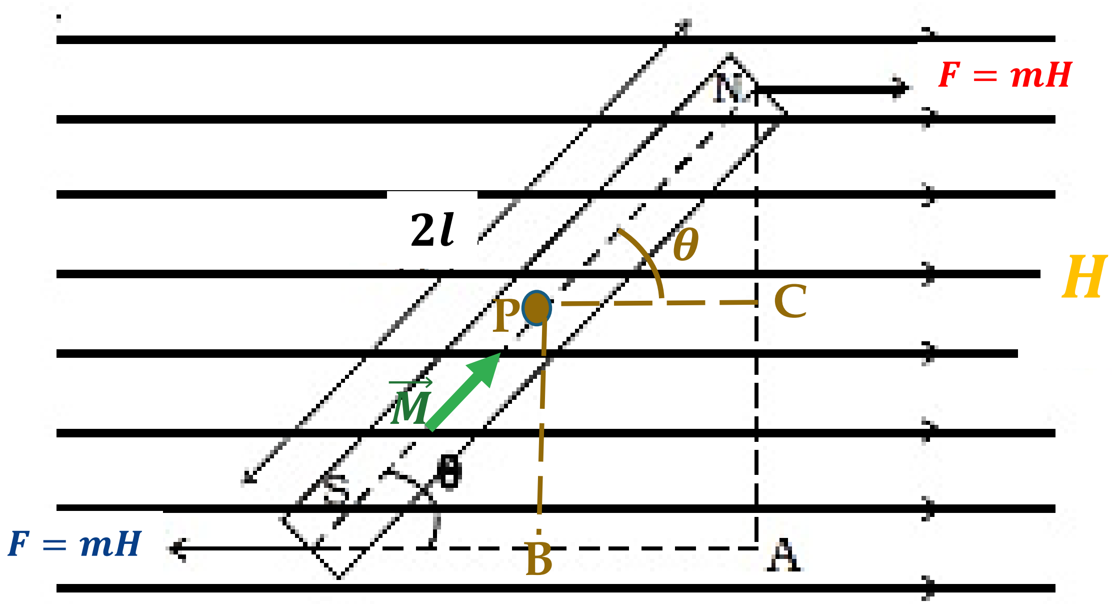
Force acting on a magnetic pole of strength m kept in a magnetic field H is

$$F = m\,H \quad \text{(as Eqn. 3.4)}$$

For north pole (n) direction of H and F are same and for south pole (s) the direction of H and F are opposite.



The bar magnet experiences torque (τ) when placed in a uniform magnetic field(H) as shown below.



Due to the magnetic field H, a force mH acts on the north pole (N) along the direction of the magnetic field and a force mH acts on the south pole (S) along the direction opposite to the magnetic field. These two forces are equal and opposite, hence constitute a couple/torque (τ), and they rotate the magnet so as it aligns along the field H.

So $sin\theta = \frac{\mathrm{NA}}{\vec{M}} = \frac{\mathrm{NA}}{2l}$ ……………………………………..3.6

Torque(τ) = Force $\times$ perpendicular distance between the forces

$$= mH \times \text{NA}$$

Substitute above NA using Eqn. 3.6:

$$\Rightarrow \tau = mH \times 2l\, sin\theta = m2lH\, sin\theta = MH\, sin\theta \ldots\ldots\ldots \quad 3.6$$

where $M = (2ml)$ is the magnetic dipole moment and its direction is from S-pole to N-pole, $2l$ is the distance between the poles (N-S) and it is less than the actual length of the magnet.

Vectorially: $\vec{\tau} = \overrightarrow{M} \times \overrightarrow{H}$

The direction of $\vec{\tau}$ is perpendicular to the plane containing $\overrightarrow{M}$ and $\overrightarrow{H}$. So,

a) If $H = 1$ and $\theta = 90^{o}$, then from equation (3.6),

$$\Rightarrow \tau = M \ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.7a$$

Hence, moment of the magnet M is equal to the torque necessary to keep the magnet at right angles to a magnetic field of unit magnetic induction.

b) If $\theta = 90^o$, then from equation (3.6) becomes:

$$\tau = 2mlH \ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.7b$$

$$\tau = MH \ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.7c$$

This torque can be considered to arise because of the opposite force exerted on the North and South poles due to the magnetic field. These forces form a couple of moment $F\frac{l}{2}$. If this is to equal torque given by

$$F\frac{l}{2} = 2mlH \Rightarrow Fl = mlH \quad \ldots\ldots\ldots\ldots \quad 3.8$$

And as before

$$F = mH \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 3.9 \quad \text{(same as Eqn. 3.4)}$$

The expression is similar to that for the force between two electric charges.

A magnetic field of strength 1 oersted is represented by 1 line of magnetic force per square centimeter. The field of H oersted is represented by h lines per square centimeter. According to this convention a unit pole must have 4π lines of force. a pole of strength m c.g.s units must have $4\pi m$ lines of force.

$$\text{Field strength} = H = \frac{km}{r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; 3.10$$

This force is experienced at all points on the sphere due to a pole of pole strength m.

Since the surface area of the sphere is $4\pi r^2$, total number of lines of force is

$$\text{Total lines of force} = k\frac{m}{r^2} \times 4\pi r^2 = k\,4\pi m \quad \ldots\ldots\ldots\ldots \; 3.11$$

In cgs $k = 1$ The unit is Maxwell

No of lines $= 4\pi m$ Maxwell's

In MKS, $K = \dfrac{1}{4\pi\mu}$ The unit is Weber $= 10^8$ lines

Weber $= 10^8$ Maxwell's.

$$\text{No of lines} = \frac{4\pi m}{4\pi\mu} = \frac{m}{\mu}$$

EXAMPLE

Find the force exerted in a pole of strength $m_1 = 8 \times 10^{-3}$ Webers placed 1 meter from another pole $m_2 = 0.25 \times 10^{-3}$ Webers, in air.

Solution

In MKS

$$F = \frac{m_1 m_2}{4\pi\mu r^2} = \frac{8 \times 0.25 \times 10^{-6}}{16\pi^2 \times 10^{-7} \times 1} \text{ N} = 0.125 \text{ Newton's}$$

Here permeability of medium $\mu = \mu_o \mu_r$, where μ_o is the permeability of a vacuum $\mu_o = 4\pi \times 10^{-7}$ Henry per metre. π_r is the relative permeability. $\mu = 1$

TAKE NOTE

Another method of deriving Equation 3.6 ($\tau = MH\sin\theta$):

Torque$(\tau) = \tau_N + \tau_S$

τ = Force × perpendicular distance between the midpoint (P) and the respective force

Using the Figure above:

- $\tau = [\overrightarrow{F_N} \times \overrightarrow{NC}] + [\overrightarrow{F_S} \times \overrightarrow{PB}]$

where $\sin\theta = \frac{NC}{PN} \Rightarrow NC = PN\sin\theta \Rightarrow NC = l\sin\theta$ and

$$\sin\theta = \frac{PB}{PS} \Rightarrow PB = PS\sin\theta \Rightarrow PB = l\sin\theta$$

- $\tau = \tau_N + \tau_S$:

$\Rightarrow \tau = [\overrightarrow{F_N} \times l\sin\theta] + [\overrightarrow{F_S} \times l\sin\theta]$, but $|\overrightarrow{F_N}| = |-\overrightarrow{F_N}|$:

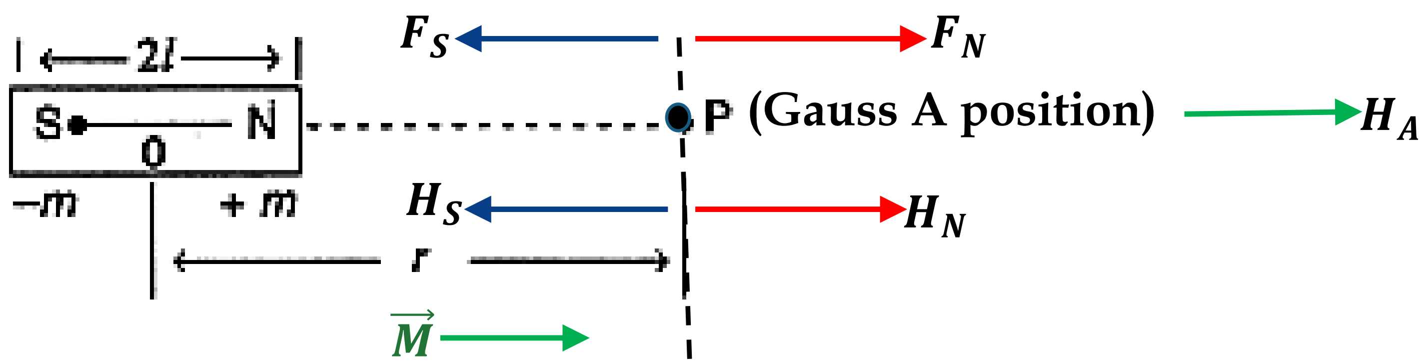
$\Rightarrow \tau = [mH \times l\sin\theta] + [mH \times l\sin\theta] = 2mlH\sin\theta = MH\sin\theta$

3.4 The magnetizing force produced by a bar magnet

Intensity of magnetic field due to a magnetic dipole
(or due to a small bar magnet)

(a) In end-on position or longitudinal position or Gauss A position (H_A):

In this case point lies on the axis of the magnet.



Manetic field at P due to N pole of magnet is given by (direction N pole-P):

$$H_N = \frac{\mu_o m}{4\pi}\frac{1}{(r-l)^2}$$

Magnetic field at P due to S pole of magnet is given by (direction P-S pole):

$$H_S = \frac{\mu_o m}{4\pi}\frac{1}{(r+l)^2}$$

Resultant magnetic field at P is: $\Rightarrow H_A = H_N + H_S$

$$\Rightarrow H_A = \left[\frac{\mu_o m}{4\pi}\frac{1}{(r-1)^2}\right] + \left[\frac{\mu_o(-m)}{4\pi}\frac{1}{(r+l)^2}\right] = \frac{\mu_o m}{4\pi}\left\{\frac{1}{(r-1)^2} - \frac{1}{(r+l)^2}\right\}$$

$$\Rightarrow H_A = km\left\{\frac{1}{(r-1)^2} - \frac{1}{(r+l)^2}\right\}$$

$$H_A = k\ m\,\frac{(r^2 + 2r\ell + \ell^2 - r^2 + 2r\ell - \ell^2)}{(r-\ell)^2\ (r+\ell)^2}$$

$$= k\ \frac{4mr\ell}{(r^2-\ell^2)^2} \quad = \quad k\ \frac{2Mr}{(r^2-\ell^2)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 3.12a$$

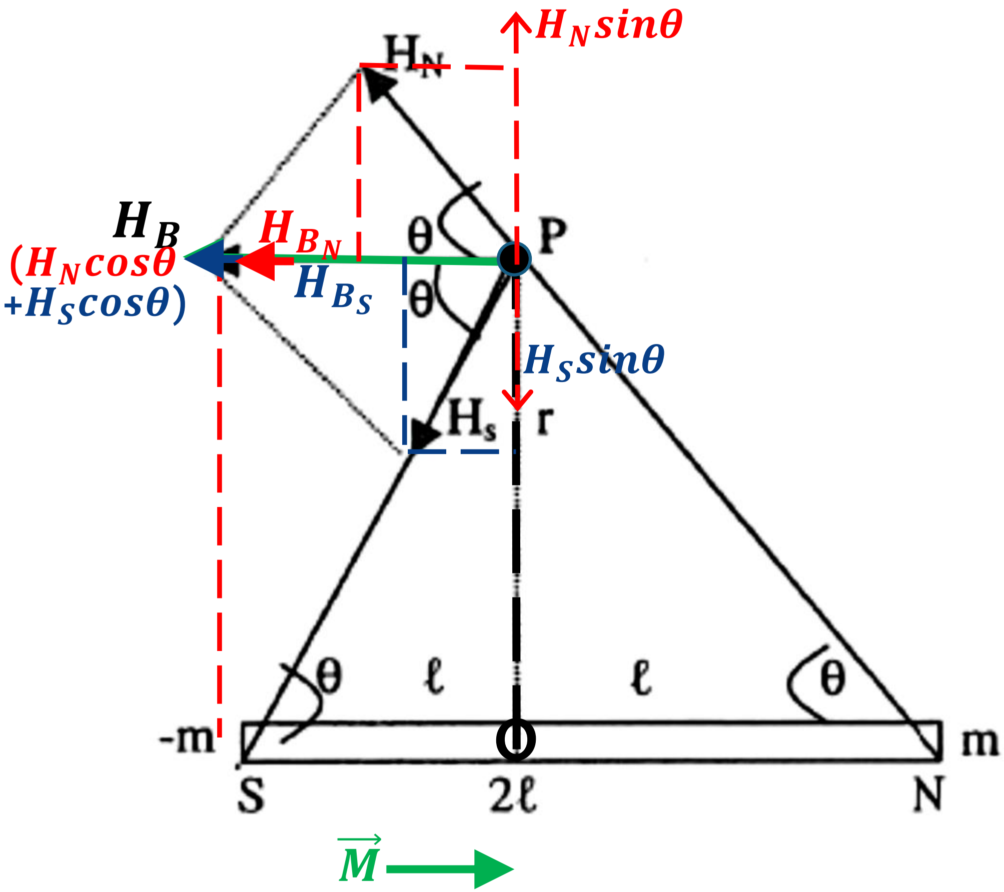
If ℓ is small compared to r, r^2 can be written instead of $(r-\ell)^2$. Hence for a short bar magnet at a considerable distance along its axis, we have

$$H_A = k\ \frac{2M}{r^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 3.12b$$

In cgs $k = 1$ Jn MKS $k = \frac{1}{4\pi\mu}$

b) The broadside or Gauss B position (H_B):

Another position at which we may calculate the Magnetic field strength is the Broadside position of Gauss. This is a point on the perpendicular bisector of a bar magnet.



Manetic field at P due to N pole of magnet is given by (direction N pole-P):

$$H_N = \frac{\mu_o m}{4\pi}\frac{1}{\left((r^2+l^2)^{1/2}\right)^2}$$

Magnetic field at P due to S pole of magnet is given by (direction P-S pole):

$$H_S = \frac{\mu_o m}{4\pi}\frac{1}{\left((r^2+l^2)^{1/2}\right)^2}$$

Thus, $H_N = H_S$

$$\Rightarrow H_N \text{ or } H_S = \frac{\mu_o m}{4\pi}\frac{1}{(r^2+l^2)} = km\frac{1}{(r^2+l^2)} \quad \ldots\ldots\ldots\ldots\ 3.13$$

The resultant force H_B is parallel to magnet and is given by determining the sum of the vector components of H_N and H_S along the H_B (with guide of the dashed red lines in the above figure) or sum of horizontal components of H_N and H_S :

$$cos\theta = \frac{H_{B_N}}{H_N} \qquad \Rightarrow H_{B_N} = H_N\, cos\theta \text{ and}$$

$$cos\theta = \frac{H_{B_S}}{H_S} \qquad \Rightarrow H_{B_S} = H_S \cos\theta$$

- $\therefore H_B = H_{B_N} + H_{B_S} = H_N \cos\theta + H_S \cos\theta$, substitute using Egn.3.13:
- $\Rightarrow H_B = H_S\, cos\theta + H_S\, cos\theta \ \Rightarrow H_B = 2H_S\, cos\theta$
- $\Rightarrow H_B = 2km\frac{1}{(r^2+l^2)}\, cos\theta$; but $cos\theta = \frac{ON}{NP} = \frac{l}{\sqrt{(r^2+l^2)}}$

$$\Rightarrow H_B = 2km \frac{1}{(r^2+l^2)} \frac{l}{(r^2+l^2)^{1/2}}$$

$$= k \frac{2ml}{(r^2+l^2)^{3/2}} = k \frac{M}{(r^2+l^2)^{3/2}} \quad \ldots\ldots\ldots\ldots\ldots \quad 3.14a$$

If ℓ is small compared to r,

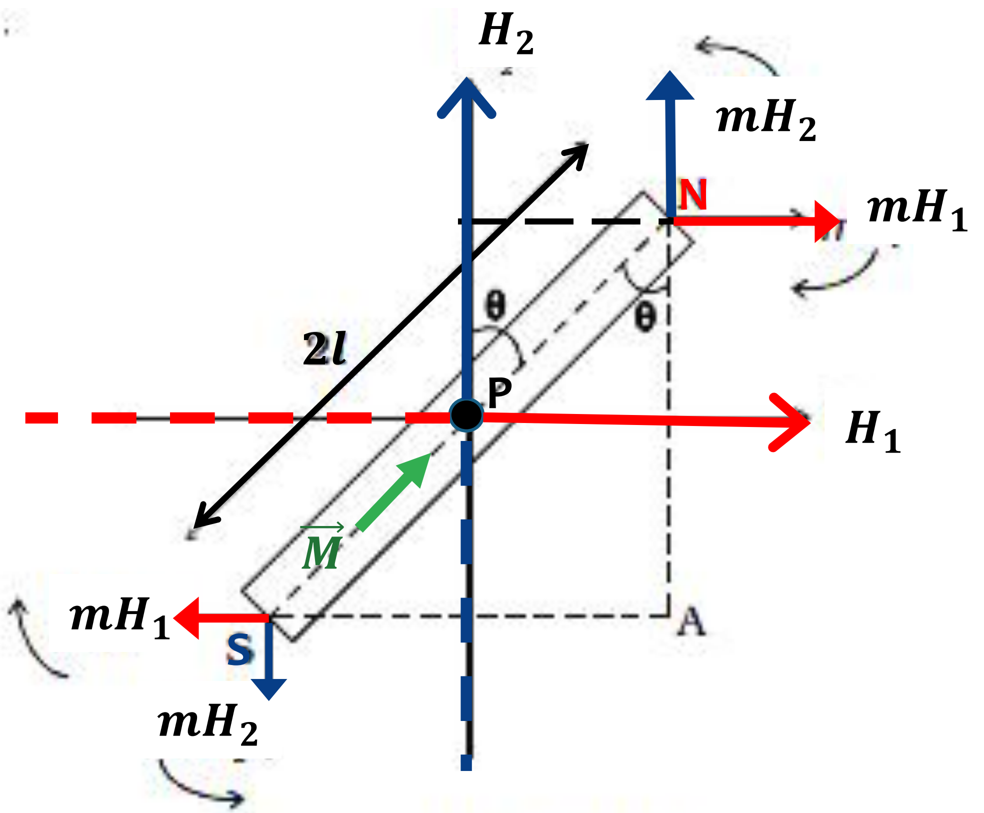
$$H_B = k \frac{M}{r^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 3.14b$$

in CGS knits, $k = 1$

In MKS units $k = \frac{1}{4\pi\mu}$

c) Magnet suspended in crossed fields (Tangent law)

A magnetic needle suspended, at a point (P) where there are two crossed magnetic fields (H_1 and H_2) acting at right angles to each other, will come to rest in the direction of the resultant of the two fields.



Thus these magnetic needle is subjected to two torques tending to rotate the magnet in opposite directions.

- **The torque τ_1 due to the two equal and opposite parallel forces mH_1 and mH_1 tend to set the magnet parallel to H_1.**
- **Similarly the torque τ_2 due to the two equal and opposite parallel forces mH_2 and mH_2 tends to set the magnet parallel to H_2. In a position where the torques balance each other, the magnet comes to rest. Now the magnet makes an angle θ with H_2 as shown in the Figure above.**

- **Torque (τ) = Force $\times$ perpendicular distance between the forces**

Thus, $\tau_1 = (mH_1 \times \text{NA})$ and $\tau_2 = (mH_2 \times \text{SA})$:

The *deflecting torque* due to the forces mH_1 and mH_1 :

- $\Rightarrow \tau_1 = mH_1 \times \mathrm{NA} \Rightarrow \tau_1 = mH_1 \times \mathrm{NS}\cos\theta$
- $\Rightarrow \tau_1 = mH_1 \times 2l\cos\theta \Rightarrow \tau_1 = 2lmH_1\cos\theta$
- $\Rightarrow \tau_1 = MH_1\cos\theta$ 3.15a

Similarly the *restoring torque* due to the forces mH_2 and mH_2:

- $\Rightarrow \tau_2 = mH_2 \times \mathrm{SA} \Rightarrow \tau_2 = mH_2 \times 2l\sin\theta$
- $\Rightarrow \tau_2 = 2lmH_2\sin\theta$
- $\Rightarrow \tau_2 = MH_2\sin\theta$ 3.15b

At *equilibrium*, $\tau_1 = \tau_2: \Rightarrow MH_1\cos\theta = MH_2\sin\theta$

- $\Rightarrow H_1\cos\theta = H_2\sin\theta$
- $\Rightarrow H_1 = H_2\frac{\sin\theta}{\cos\theta} \Rightarrow H_1 = H_2\tan\theta$ 3.16

and this is called Tangent law.

TAKE NOTE

Invariably, in the *applications of tangent law*, the restoring magnetic field H_2 is the horizontal component H_h of Earth's magnetic field H_E.

3.5 The deflection magnetometer

- Its working is based on the principle of tangent law.
- It consist of a small compass magnetic needle and aluminium pointer, pivoted at the centre of a circular box. The box is kept in a wooden frame having two meter scale fitted on its two arms. Reading of the meter scale at any point directly gives the distance of that point from the centre of compass needle.

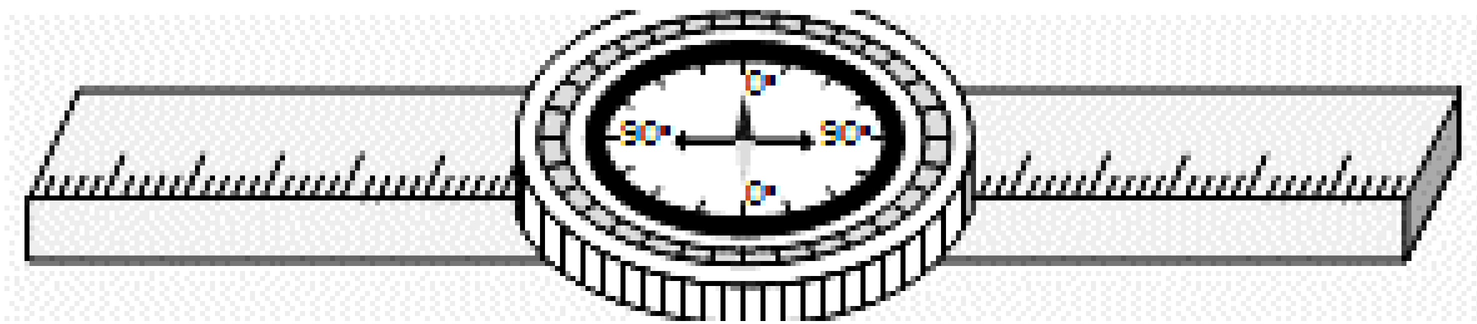


Fig. 1

- Is an instrument used to measure the magnetic moment (M)and polar strength (H) of a bar magnet.
- It is also used to compare the pole strengths of two bar magnets.

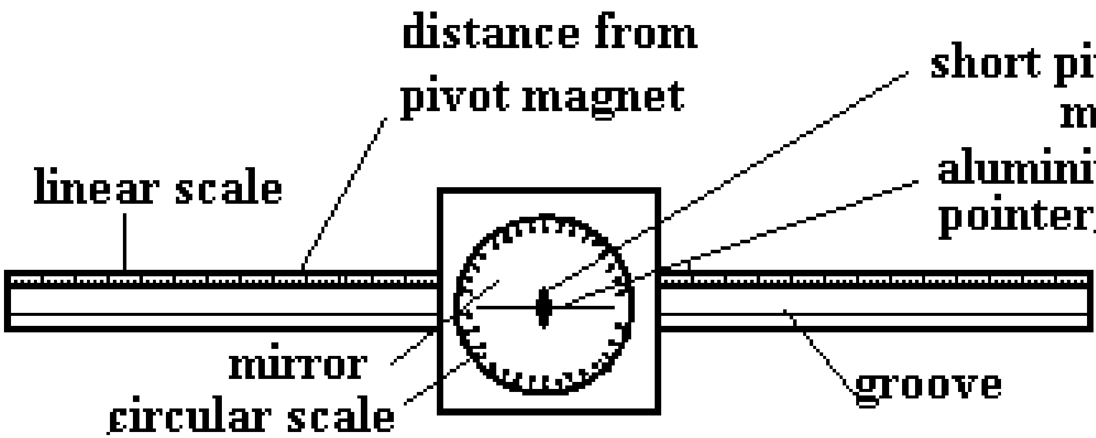

distance from
pivot magnet
short pivot
magnet
aluminium
pointer
linear scale
mirror
circular scale
groove

Fig. 2

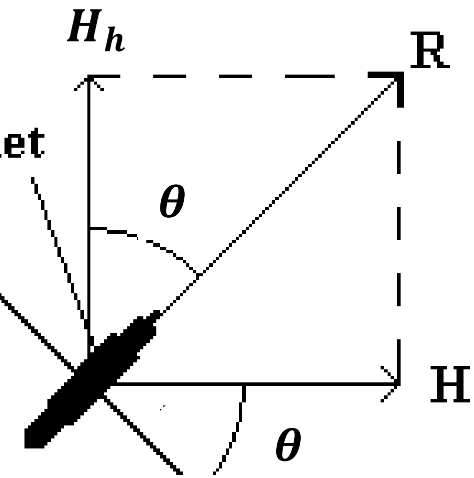

H_h
R
short pivot
magnet
aluminium
pointer
θ
H
θ

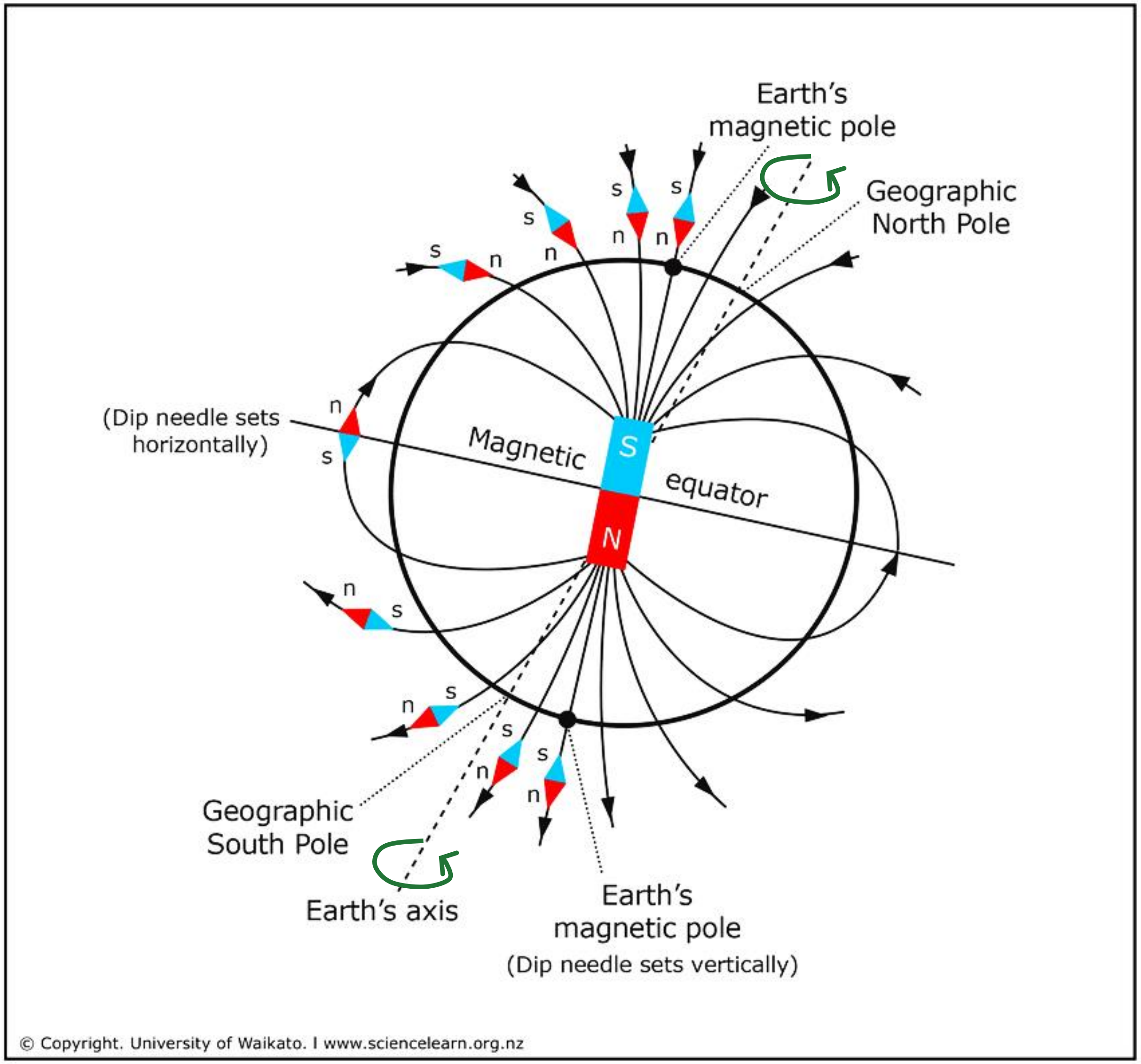
Fig. 3

Near the geographic North Pole is what is called the magnetic North Pole.

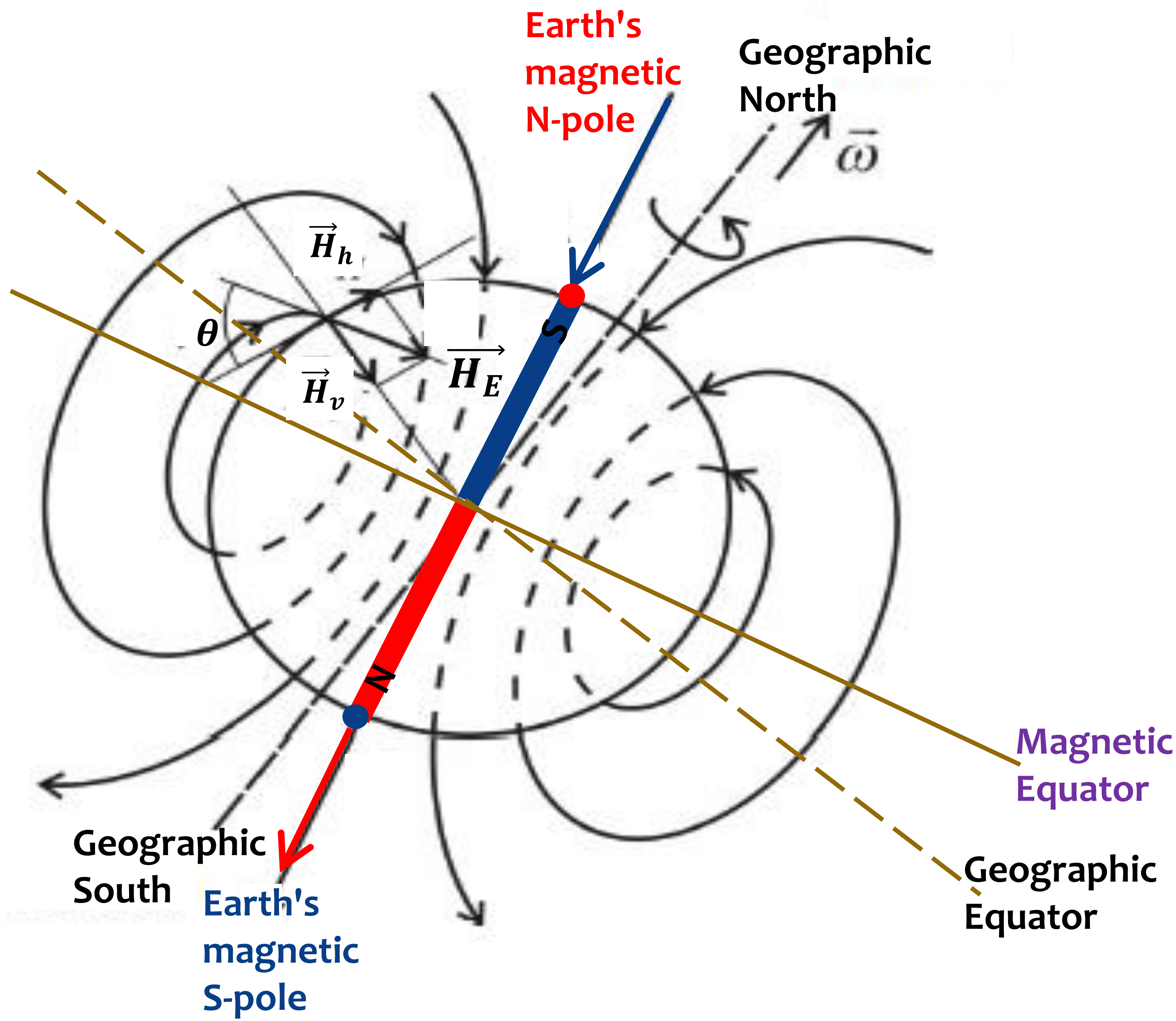
The pole on a bar magnet (compass) which points towards the North is called the north (seeking) pole and are labelled *n* in the diagram. It is that pole which you call the north pole of a magnet.

By convention the direction of magnetic field lines is from the north pole of a bar magnet towards the south pole of bar magnet.

TAKE NOTE SOME DEFINITIONS



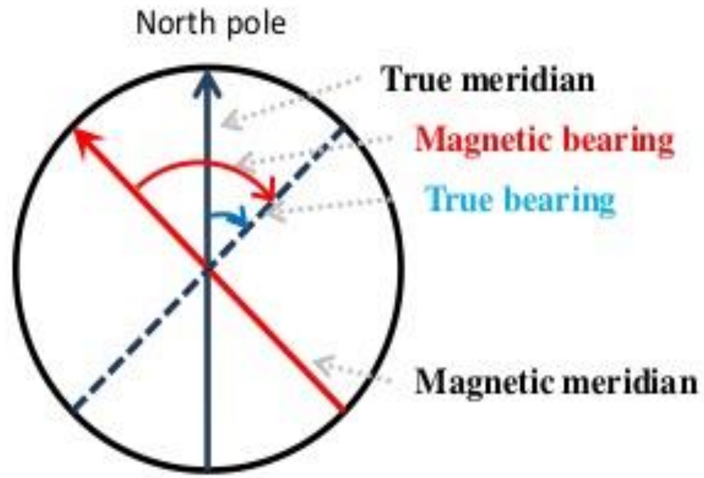
conti. TAKE NOTE



conti. TAKE NOTE

True meridian:

Line or plane passing through geographical north pole and geographical south pole



Magnetic meridian:

The direction indicated by a freely suspended and properly balanced magnetic needle is known as *Magnetic Meridian*. The angle between the magnetic meridian and a line is known as magnetic bearing or simple bearing of the line.

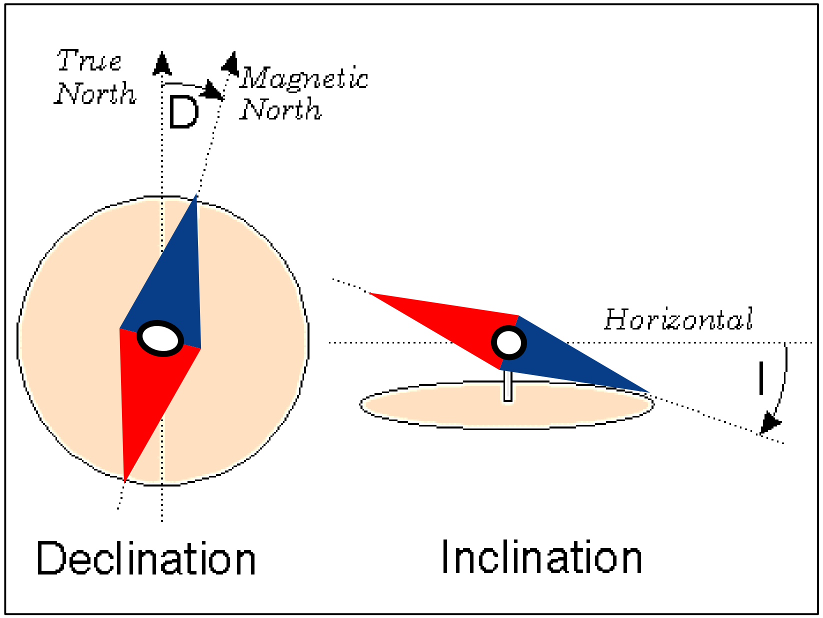
conti. TAKE NOTE

The Earth's magnetic field (H_E) at any point on the Earth can be defined in terms of certain quantities called the magnetic elements of the Earth:

(i) Declination (D) or the magnetic variation θ.

(ii) Dip or inclination (I) δ, and

(iii) The horizontal component of the Earth's magnetic field H_h



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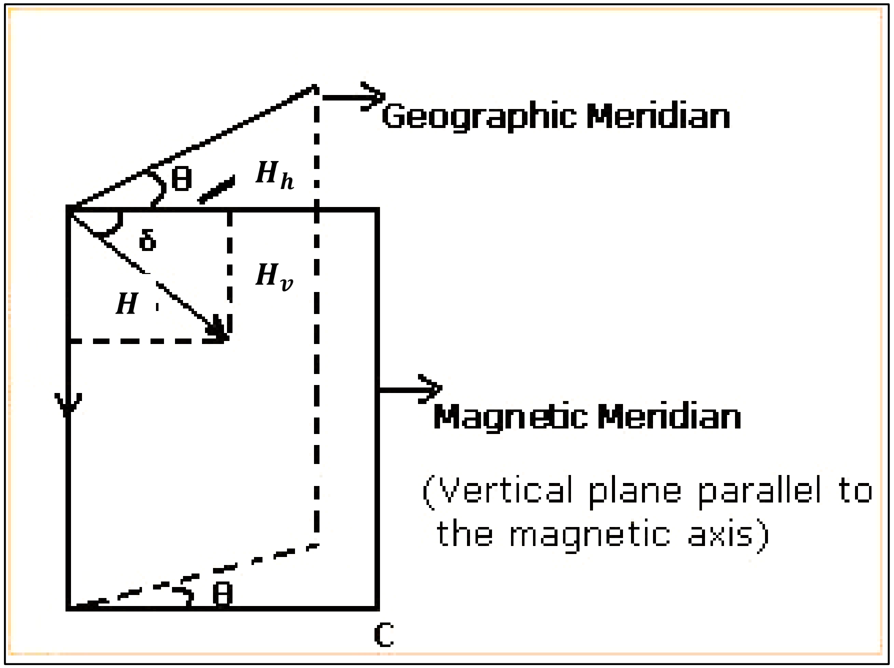


conti. TAKE NOTE

Causes of the Earth's magnetism:

(i) Magnetic masses in the Earth.

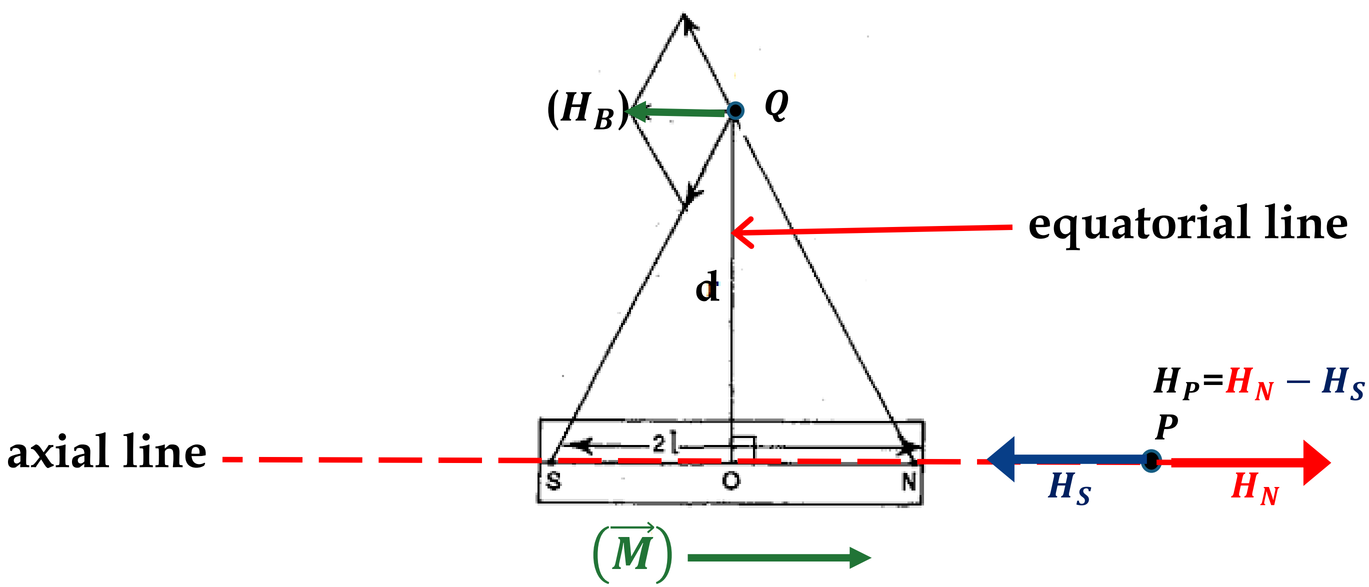
(ii) Electric currents in the Earth.

(iii) Electric currents in the upper regions of the atmosphere.

(iv) Radiations from the Sun.

(v) Action of moon, etc.

Conti. TAKE NOTE



- Magnetic Field $(H_P, H_N$ and $H_S)$ at a point on the *axial line* acts along the dipole moment vector $(\vec{M})$.
- Magnetic Field (H_B) at a point (Q) on the *equatorial line* acts opposite to the dipole moment vector$(\vec{M})$.
- Magnetic length $(2l)$ is the distance between two pole ($P$and Q) $\approx 0.83 \times$ geometrical length

3.5.1 Uses of deflection magnetometer

Deflection magnetometer may be used according to two following positions:

a) **End-on (or) Tan A position- Deflection magnetometer**

b) **Broadside-on (or) Tan B position - Deflection magnetometer**

End-on (or) tan A position- deflection magnetometer

The arms of the deflection magnetometer are placed along East-West direction perpendicular to the true magnetic meridian, so that the aluminium pointer reads 0^o or $zero$ on the scale before placing the bar magnet. The bar magnet is then placed along East - West direction (i.e) parallel to the arms and deflection occurs.

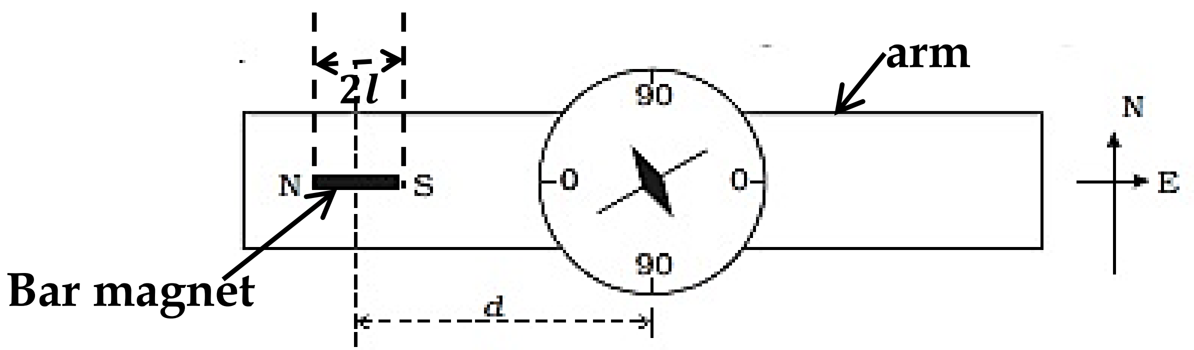


Fig. 4 End-on (or) Tan A position

The magnetic field (H) at a point along the axial line of a bar magnet is perpendicular to the horizontal component (H_h) of Earth's magnetic field (H_E). If a magnetometer and a bar magnet are placed in such way that this condition is satisfied then this arrangement is called Tan A position. That is the compass needle is in end-on position of magnet.

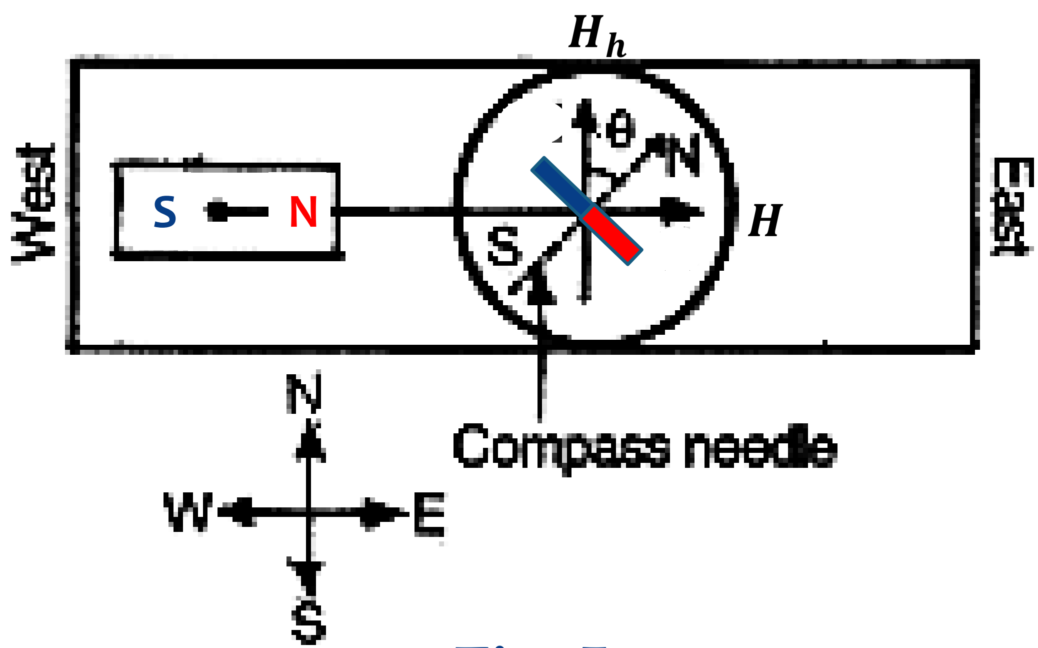


Fig. 5

When a bar magnet of magnetic moment (M) and length ($2l$) is placed at a distance (d) from the center of the magnetic needle, the needle gets deflected through an angle (θ) due to the action of two magnetic fields (H and H_h, as shown in the previous Figures 3, 4, and 5):

- (i) the magnetic field H due to the bar magnet acting along its axis and
- (ii) the horizontal component (H_h) of Earth's magnetic field H_E.

The magnetic field at a distance d acting along the axial line of the bar magnet, $H = \frac{1}{4\pi\mu_O} \frac{2Md}{(d^2-l^2)^2}$.

- According to Tangent law, $H = H_h \tan\theta$ (see Eqn.3.16):

$$\frac{1}{4\pi\mu_O} \frac{2Md}{(d^2-l^2)^2} = H_h \tan\theta \quad \ldots\ldots\ldots\ldots\ 3.17 \text{ (same as Eqn. 3.12a)}$$

Broadside on (or) Tan B position - Deflection magnetometer

Arms of a magnetometer are placed along N-S direction (i.e along the true magnetic meridian) such that are parallel to magnetic needle. so that the pointer reads 90^o on the scale before placing the bar magnet. The bar magnet is then placed along East - West direction (i.e parallel to the aluminum pointer).

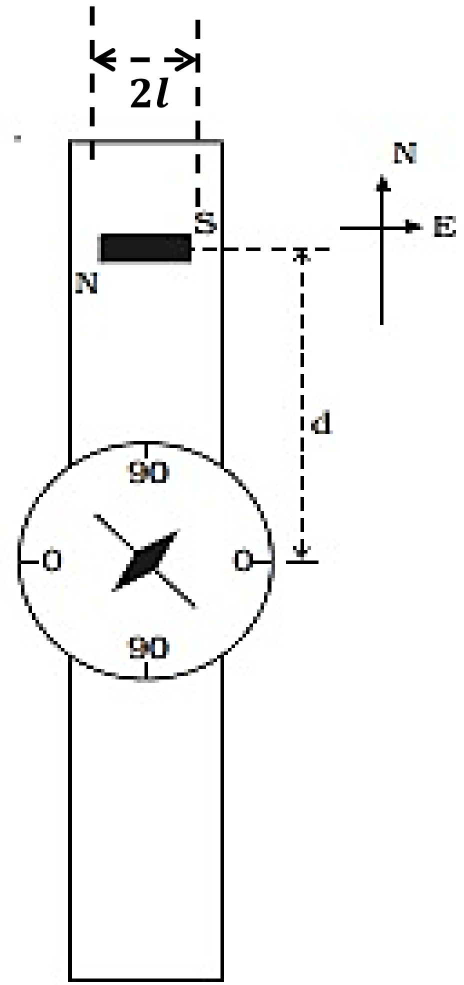
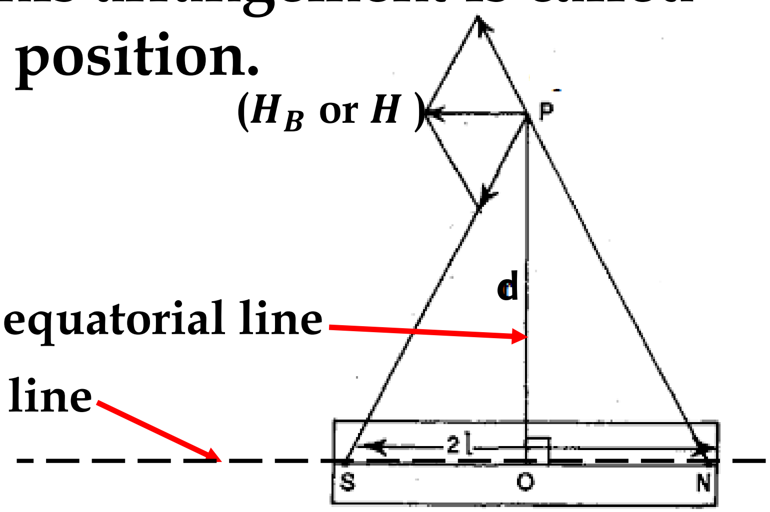
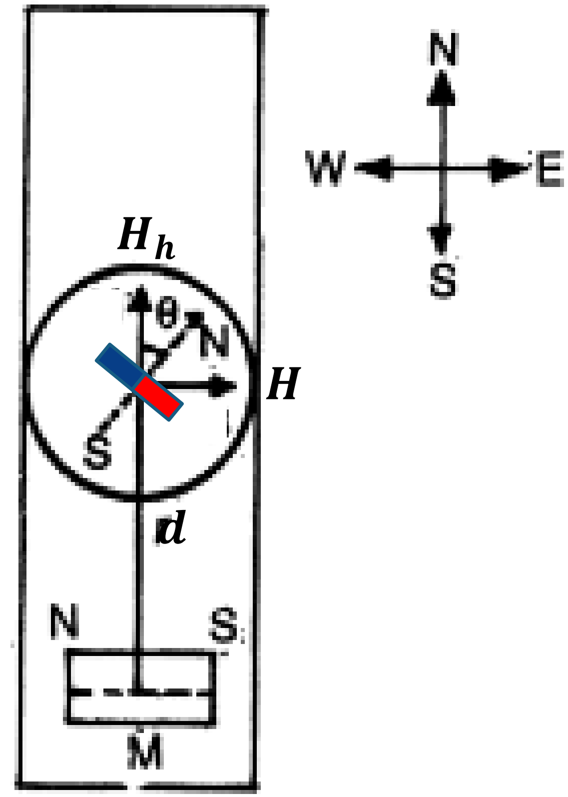


Fig. 6

- The magnetic field (H) at a point along the equatorial line of a bar magnet is perpendicular to the horizontal component (H_h) of Earth's magnetic field(H_E).

If the magnetometer and a bar magnet are placed in such way that this condition is satisfied, then this arrangement is called Tan B position.





- **When a bar magnet of magnetic moment M and length $2l$ is placed at a distance d from the center of the magnetic needle, the needle gets deflected through an angle θ due to the action of the following two magnetic fields(H and H_h):**
- **(i) The field H due to the bar magnet along its equatorial line.**
- **(ii)The horizontal component H_h of Earth's magnetic field H_E.**

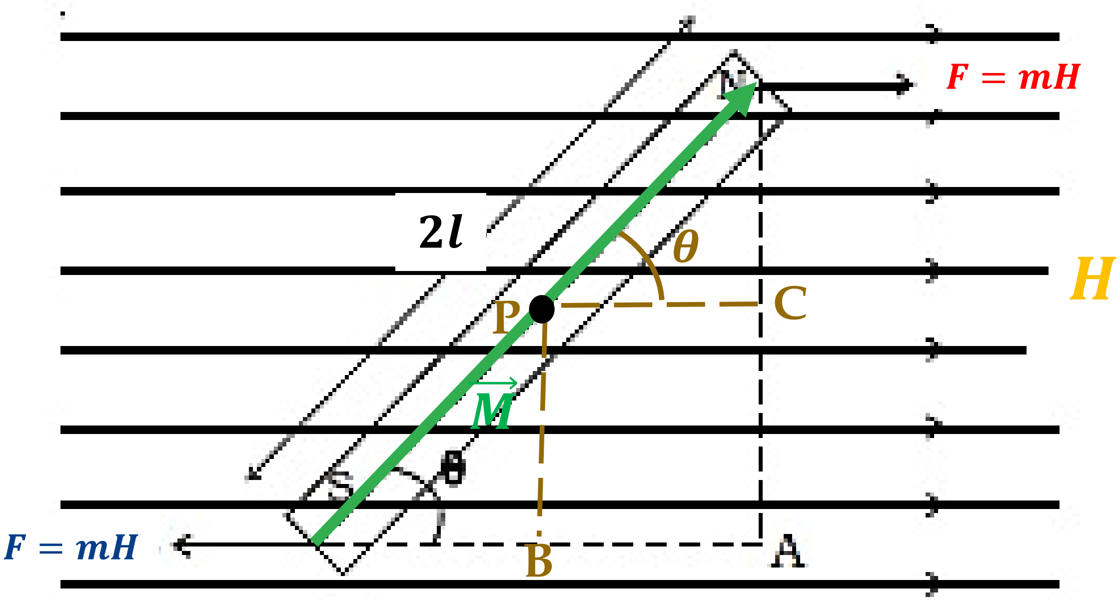
The magnetic field at a distance d along the equatorial line of the bar magnet, $H = \frac{1}{4\pi\mu_0}\frac{M}{(d^2-l^2)^{3/2}}$.

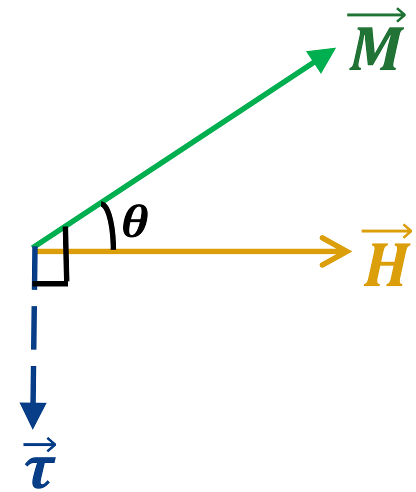
- **According to tangent law: $H = H_h \tan\theta$**
- $\frac{1}{4\pi\mu_0}\frac{M}{(d^2-l^2)^{3/2}} = H_h \tan\theta$
- **If the magnet is short, l is small compared to d and hence l^2 is neglected.** $\frac{1}{4\pi\mu_0}\frac{M}{(d^2-0)^{3/2}} = H_h \tan\theta$
- $\Rightarrow H_h \tan\theta = \frac{1}{4\pi\mu_0}\frac{M}{d^3}$ ………………… 3.18 **(same as Eqn. 3.14a)**

3.6 The vibration of a magnet in a magnetic field

Determination of magnetic moment (M) of a magnet

We now suspend a magnet in a magnetic field. Suppose we turn it slightly from the direction of the field and release it. It will begin to vibrate like a pendulum oscillating in a gravitational field.





From Eqn. 3.6 in section 3.3: $\tau = MH\ sin\theta \Rightarrow \tau$(restoring torque) $= -MHsin\theta$

If the angle θ is small, $\sin\theta \approx \theta$

$\therefore$ Restoring couple $= MH\theta \Rightarrow \vec{\tau} = \vec{M} \times \vec{H}$ 3.19

As the magnetic needle is displaced from the equilibrium position, the torque will try to bring it back in equilibrium position. Hence, acceleration will be related with negative of angular displacement. When compass needle of magnetic moment *M* and moment of inertia *I* is slightly disturbed by an angle *θ* from the mean position of equilibrium, then restoring torque begins to act on the needle which in turn, try to bring the needle back to its mean position.

Since, θ is small.

So, $\sin\theta \approx \theta$

$\therefore$ $\tau = -MH\theta$

But $\tau = I\alpha$ *i.e., torque = (moment of inertia)(angular acceleration)* in rotational motion

where, α = angular acceleration

M = magnetic moment of dipole. I is the moment of inertia

$$\Rightarrow \quad I\alpha = -MH\theta \Rightarrow \alpha = -\left(\frac{MH}{I}\right)\theta \Rightarrow \alpha \propto -\theta$$

$\Rightarrow$ Angular acceleration $\propto$ – Angular displacement

$\Rightarrow$ Therefore, the needle executes SHM. Since $\alpha = -\omega^2\theta \Rightarrow -\omega^2\theta = -\left(\frac{MH}{I}\right)\theta$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{MH}{I}}} \Rightarrow T = 2\pi\sqrt{\frac{I}{MH}} \Rightarrow M = \frac{4\pi^2}{T^2}\frac{I}{H} \quad \text{.........} \quad 3.20$$

TAKE NOTE

What is simple harmonic motion (SHM)?

Is a special type of periodic motion or oscillation where the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement.

What are the properties of SHM?

Is a motion in which,

- **the acceleration (α) is always opposite to displacement(θ),**

$\alpha = -\left(\frac{MH}{I}\right)\theta$, **see the sketch next slide, and**

- **acceleration(α) is directly proportional to displacement(θ).**

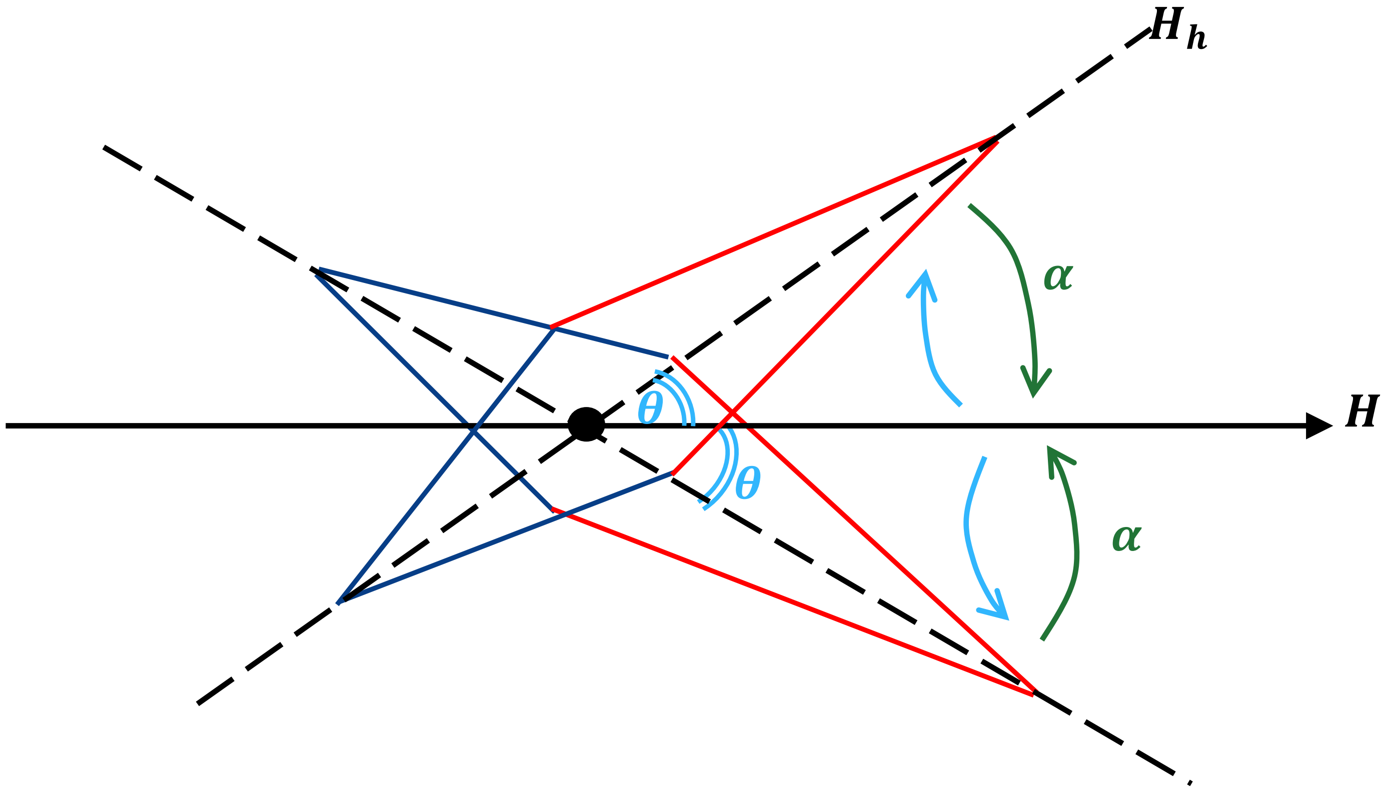
$$\alpha = -\left(\frac{MH}{I}\right)\theta \quad \Rightarrow \alpha \propto \theta$$

Equation for time period (T)in SHM:

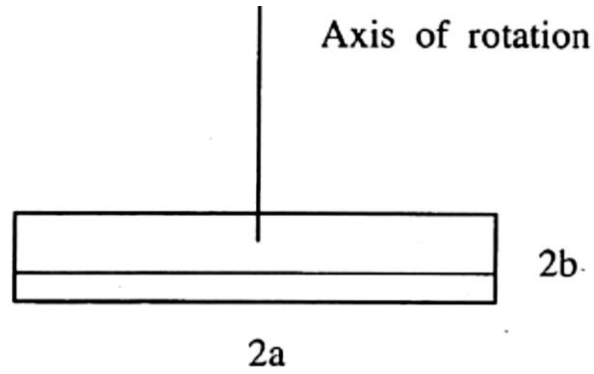
- $T = 2\pi\sqrt{\frac{\text{angular displacement}\,(\theta)}{\text{angular acceleration}\,(\alpha)}} \Rightarrow T = 2\pi\sqrt{\frac{I}{MH}}$
- $\Rightarrow H = \frac{4\pi^2 I}{MT^2}$ **is the Fundamental formula**

conti. TAKE NOTE

To show that in SHM, the acceleration (α) is always opposite to displacement(θ): $\alpha = -\left(\frac{MH}{I}\right)\theta$.



In the case a rectangular bar magnet of mass *m grams* and geometrical length $2a\ cm$ and breath $2b\ cm$, oscillating about an axis, its moment of inertia is (for rectangular cross-section):



$$I = 1/12\ \text{mass}\ [(\text{length})^2 + (\text{breadth})^2] = m\,\frac{(2a)^2 + (2b)^2}{12} \qquad \text{............ 3.21a}$$

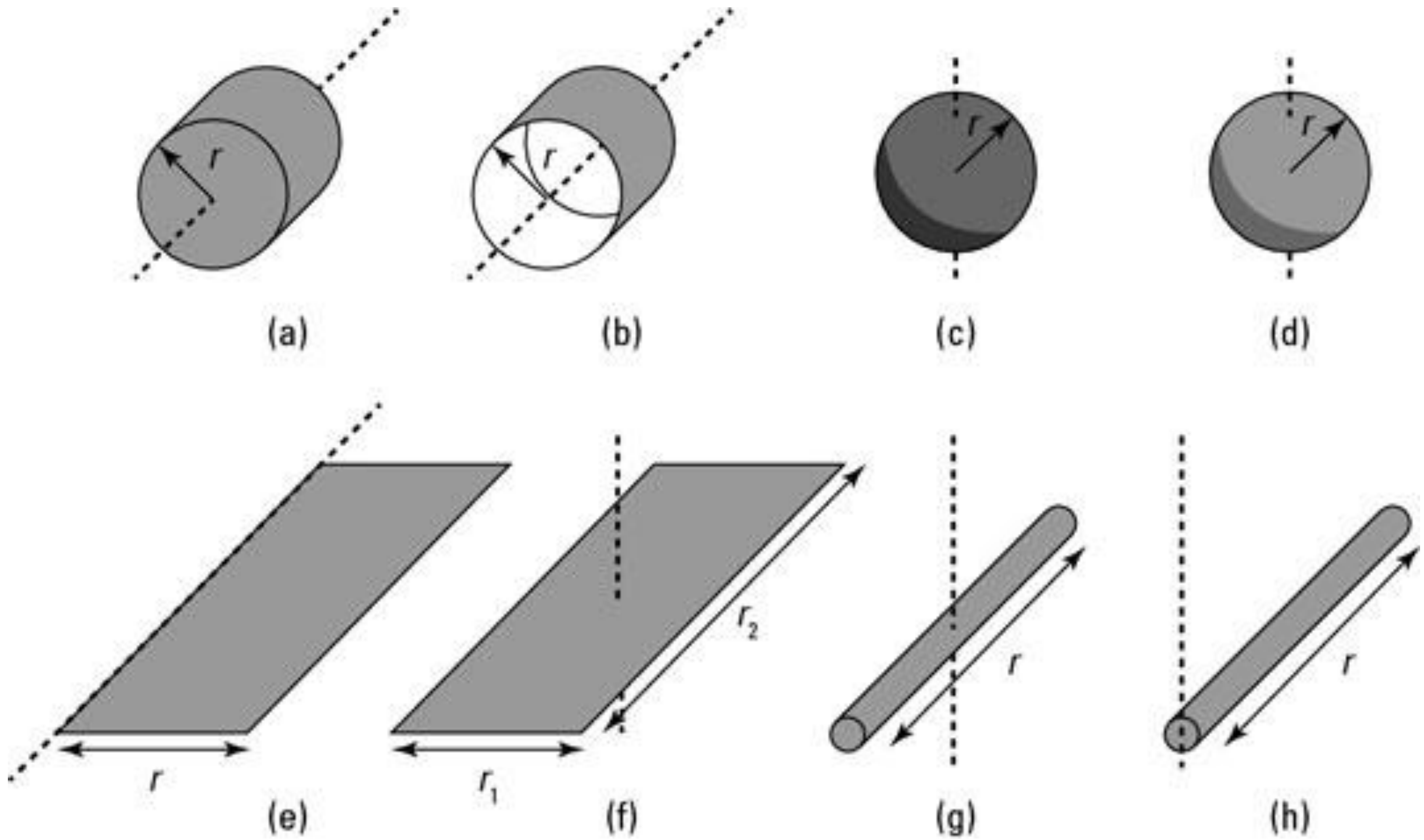
$$I = m\,\frac{a^2 + b^2}{3}\ \text{c.g.s units} \qquad \text{.............................. 3.21b}$$

For a bar magnet of cylindrical cross section

$$I = \text{mass} \times \left(\frac{(\text{length})^2}{12} + \frac{(\text{radius})^2}{4}\right) \qquad \text{.............................. 3.22}$$

TAKE NOTE

The figures below depict the shapes that the moments of inertia equations on next slide correspond to



conti. TAKE NOTE

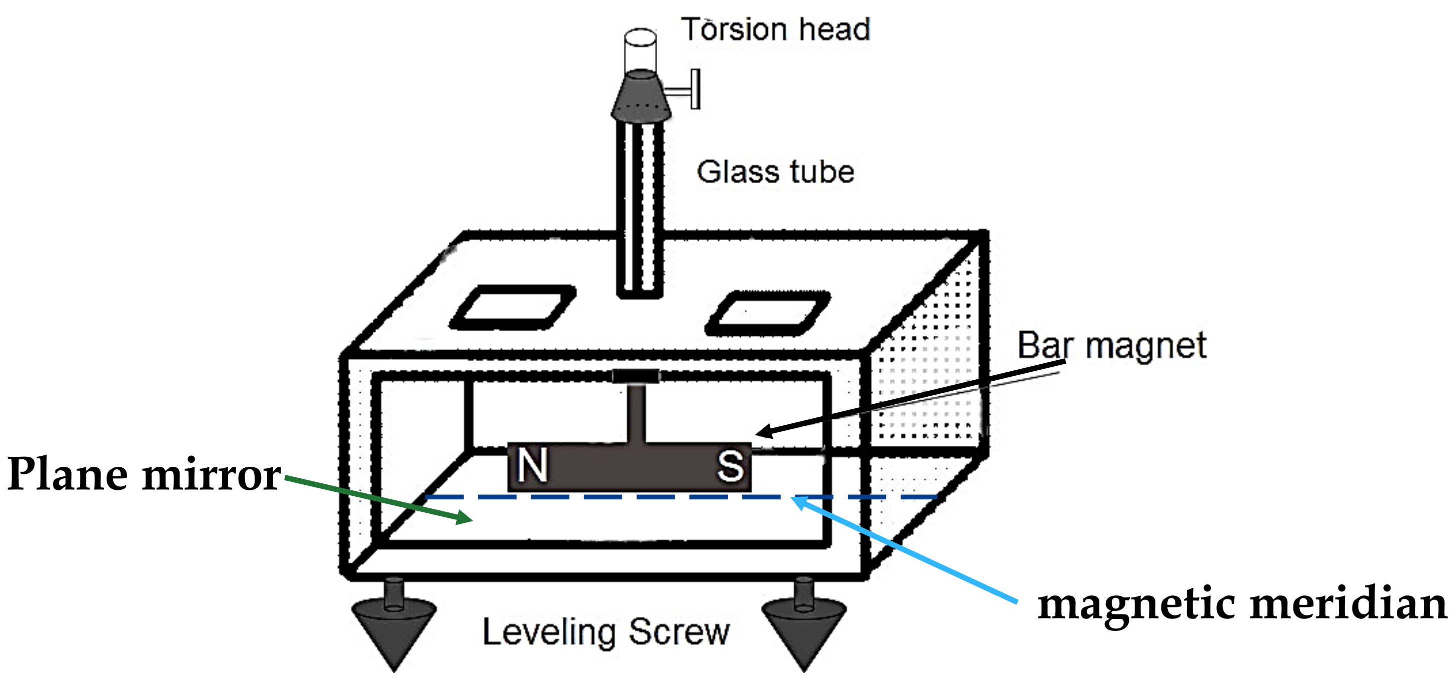
Moments of Inertia for Various Shapes and Solids

Shape	Moment of Inertia
(a) Solid cylinder or disk of radius r	$I = \frac{1}{2}mr^2$
(b) Hollow cylinder of radius r	$I = mr^2$
(c) Solid sphere of radius r	$I = \frac{2}{5}mr^2$
(d) Hollow sphere of radius r	$I = \frac{2}{3}mr^2$
(e) Rectangle rotating around an axis along one edge, where the other edge has length r	$I = \frac{1}{3}mr^2$
(f) Rectangle with sides r_1 and r_2 rotating around a perpendicular axis through the center	$I = \left(\frac{1}{12}\right)m\left(r_1^2 + r_2^2\right)$
(g) Thin rod of length r rotating about its middle	$I = \frac{1}{12}mr^2$
(h) Thin rod of length r rotating about one end	$I = \frac{1}{3}mr^2$

where $r = 2l$

3.7 The vibration Magnetometer

It is an instrument that measures magnetic properties. It consists of a polished wooden box filled with sliding glass door. The top of the case is slotted and carries a glass tube fitted with torsion head to support by means of silk fiber, a brass stirrup to carry magnets and 2 bar magnets. The base of box is fitted with mirror strip, with a longitudinal index line.

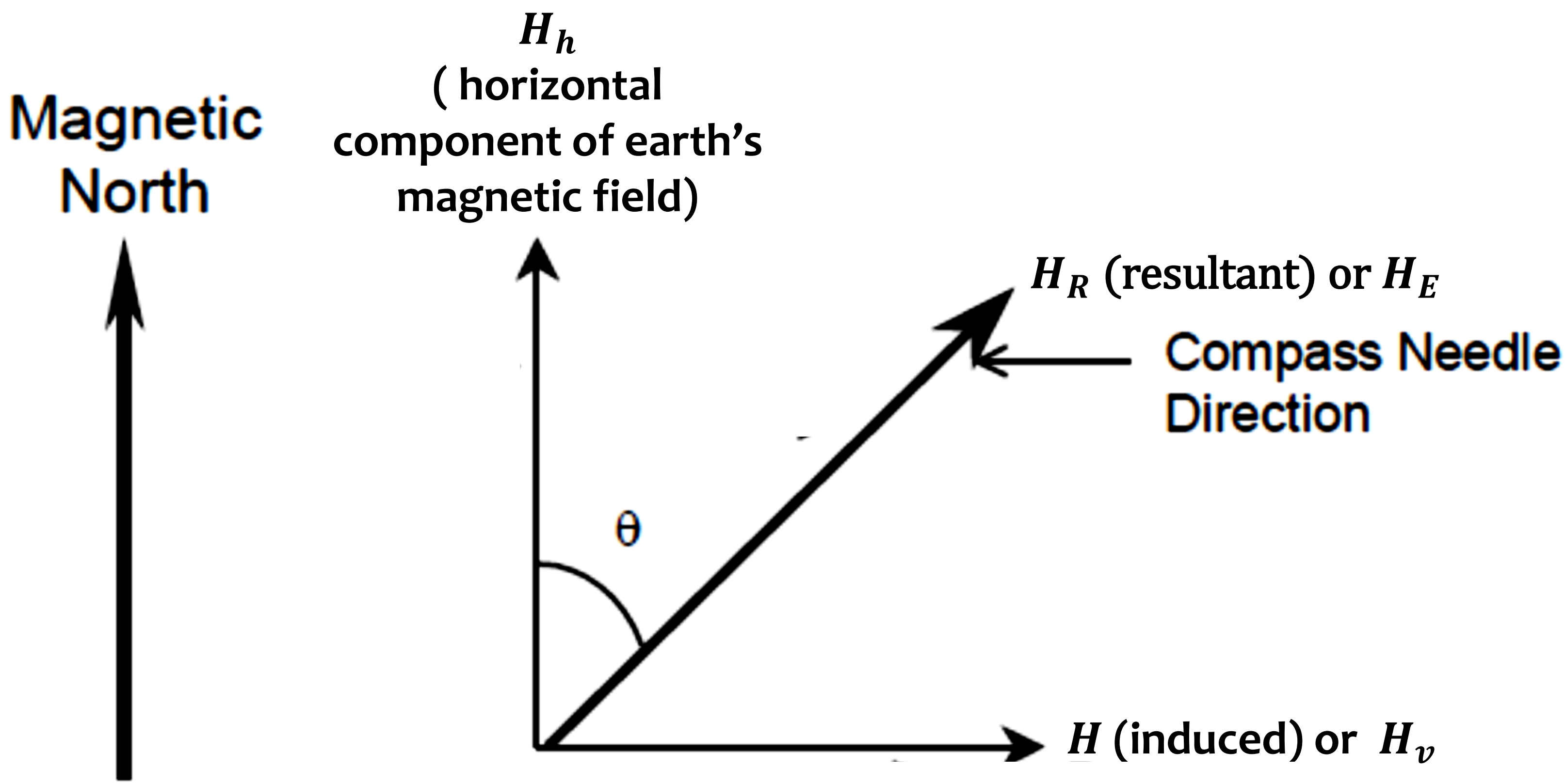


Working principle

A freely suspended magnet in a uniform magnetic field (H)aligns in the direction of the field. When it is displaced through a small angle (θ) in a horizontal plane and released it vibrates in the horizontal plane as it experiences a restoring couple. This keeps the magnet in SHM.

Applications of vibration magnetometer

- **To compare the horizontal components$(H_h)$of the earth's magnetic field (H_E) at different places (in this course we will look at two places).**
- **To determine the magnetic moment $(M)$of a magnet and the horizontal component (H_h) of the earth's magnetic field (H_E) at a given place, when used along with deflection magnetometer.**



Magnetic North
H_h (horizontal component of earth's magnetic field)
H_R (resultant) or H_E
Compass Needle Direction
θ
H (induced) or H_v

To compare the horizontal components H_1 and H_2 of the earth's magnetic field (H_E) at different at two places

- **Since I and M of the magnets are constant, using Equation 3.20:**

$$T^2 = \left(\frac{4\pi^2 I}{M}\right)\frac{1}{H} \Rightarrow T^2 = \left(\frac{4\pi^2 I}{M}\right)\frac{1}{H_h}$$

$$\Rightarrow {T_1}^2 = \left(\frac{4\pi^2 I}{M}\right)\frac{1}{H_1} \text{ and } \Rightarrow {T_2}^2 = \left(\frac{4\pi^2 I}{M}\right)\frac{1}{H_2}$$

$$\therefore \frac{{T_1}^2}{{T_2}^2} = \left[\left(\frac{4\pi^2 I}{M}\right)\frac{1}{H_1}\right] \div \left(\frac{4\pi^2 I}{M}\right)\frac{1}{H_2}\Big] = \frac{H_2}{H_1} \quad \ldots\ldots\ldots\ldots\ldots \quad 3.23$$

$$\frac{T_1^2}{T_2^2} = \frac{H_2}{H_1} \qquad \text{..} \quad 3.23$$

This is a convenient method of determining how the earth's field varies at different points in the laboratory.

We must make an allowance for the effect of the earth's magnetic fields by finding the time of vibration in the earth's field alone, T_e. The magnetic field to be measured is reinforced by the earths magnetic field. If the fields due to the magnets alone is H_1 and H_2 respectively,

$$H_1 + H_e \propto \frac{1}{T_1^2}$$

$$H_2 + H_e \propto \frac{1}{T_2^2}$$

$$H_e \propto \frac{1}{T_e^2}$$

$$\therefore \frac{H_1}{H_2} = \frac{\frac{1}{T_1^2} - \frac{1}{T_e^2}}{\frac{1}{T_2^2} - \frac{1}{T_e^2}} \qquad \text{....................................} \quad 3.24$$

Comparison of magnetic moments $(M_1$ and $M_2)$ of two magnets of the same size and mass

- Since I and H are constant, using Equation 3.20:

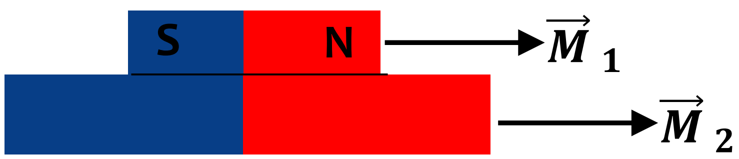
$$T^2 = \left(\frac{4\pi^2 I}{M}\right)\frac{1}{H}$$

- $\Rightarrow {T_1}^2 = \left(\frac{4\pi^2 I}{M}\right)\frac{1}{H_1}$ and $\Rightarrow {T_2}^2 = \left(\frac{4\pi^2 I}{M}\right)\frac{1}{H_2}$

- $\therefore \frac{{T_1}^2}{{T_2}^2} = \left[\left(\frac{4\pi^2 I}{H}\right)\frac{1}{M_1}\right] \div \left(\frac{4\pi^2 I}{H}\right)\frac{1}{M_2}\right] = \frac{M_2}{M_1}$ ………….. 3.25

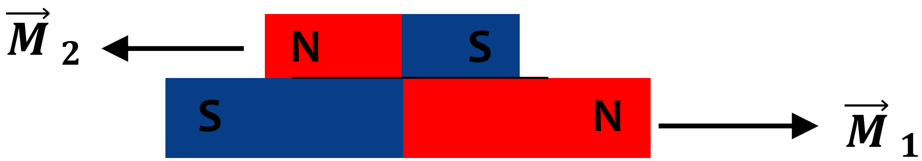
Comparison of magnetic moments (M_1 and M_2) of two magnets by sum method and difference method

i) Sum position:



- Net magnetic moment $(M_s) = M_1 + M_2$
- Net moment of inertia $(I_s) = I_1 + I_2$
- Time period of oscillation of this pair in earth's magnetic field: $T_s = 2\pi\sqrt{\frac{I_s}{M_s H}} = 2\pi\sqrt{\frac{(I_1+I_2)}{(M_1+M_2)\,H}}$ ……….3.26
- and for frequency $(\nu_s) = \frac{1}{2\pi}\sqrt{\frac{(M_1+M_2)\,H}{(I_1+I_2)}}$

ii) Difference position:



- Net magnetic moment $(M_d) = M_1 + M_2$
- Net moment of inertia$(I_d) = I_1 + I_2$
- Time period of oscillation of this pair in earth's magnetic field: $T_d = 2\pi\sqrt{\frac{I_d}{M_d H}} = 2\pi\sqrt{\frac{(I_1+I_2)}{(M_1-M_2)\,H}}$ ………..3.27

  and for frequency $(v_d) = \frac{1}{2\pi}\sqrt{\frac{(M_1-M_2)\,H}{(I_1+I_2)}}$

From Equations 3.26 and 3.27:

- $\frac{T_s}{T_d} = \sqrt{\frac{M_1-M_2}{M_1+M_2}} \Rightarrow \frac{M_1}{M_2} = \frac{{T_d}^2+{T_s}^2}{{T_d}^2-{T_s}^2} = \frac{{v_d}^2+{v_s}^2}{{v_s}^2-{v_d}^2}$ ………..3.28

To determine the magnetic moment (M) of a magnet and the horizontal component (H_h) of the earth's magnetic field (H_E) at a given place, when used along with deflection magnetometer.

A magnet is suspended by means of a long thin copper wire from the ceiling from a vibration magnetometer. The object of the long wire is to reduce torsion to a minimum. The periodic time, T, for small vibrations is

(**NOTE:** for small vibration $H_h \cong H_e$)

$$T = 2\pi\sqrt{\frac{I}{MH_e}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 3.29$$

We now have two unknowns, M and H_e, the moment of the magnet and the earth's horizontal component. The moment of inertia is calculated as given by the formula above.

We now place a deflection magnetometer with its center below the copper wire suspension. This is the place where the value of H_e is required. We now determine the mean deflection, θ, for a suitable distance r of the magnet using, say the end-on position. We have

$$\frac{2Mr}{4\pi\mu_o(r^2-\ell^2)^2} = H_e \tan\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 3.30$$

Squaring equation 3.29 and rearranging it, we have:-

$$M = \frac{4\pi^2 I}{T^2 H_e} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 3.31$$

Substituting for M in (3.30) gives

$$\frac{2\pi Ir}{\mu_o (r^2-\ell^2)^2 \quad T^2 H_e} = H_e \tan\theta \quad \text{or}$$

$$H_e^2 = \frac{2\pi Ir}{\mu_o (r^2-\ell^2)^2 \; T^2 \tan\theta} \quad \ldots\ldots\ldots \quad 3.32$$

Hence H_e is found.

The same experiment yields a result for the moment of the magnet. Indeed, from equation 3.31

$$H_e = \frac{4\pi^2 I}{T^2 M} \quad \text{.................} \quad 3.33$$

Substituting this value in equation 3.32 leads to

$$M^2 = \frac{8\pi^2 \mu_o I \,(r^2-\ell^2)^2 \tan\theta}{rT^2} \quad \text{........................} \quad 3.34$$

The chief uncertainty here is the value of ℓ the "magnetic length" of the magnet. This is avoided if a very short magnet is used so that a simplified equation given as

$$H_e = \frac{2M}{4\pi\mu_o r^3} \quad \text{........................} \quad 3.35$$

is used. This is magnetising force at a distance r from its mid-point.

EXAMPLE 1

1. **On what factors does the period of oscillation of a bar magnet in a uniform magnetic field depend?**

Ans: The time period of oscillation of bar magnet in uniform magnetic field depends on (i) Moment of inertia (I) of the magnet (ii) Moment of the magnet (M)and (iii) the magnetic induction(B). $T = 2\pi\sqrt{\frac{I}{MB}}$

EXAMPLE 2

Two magnets are held together in a vibration magnetometer and are allowed to oscillate in the earth's magnetic field. With like poles together 12 oscillations per minute are made but for unlike poles together only 4 oscillations per minute are executed.

Determine the ratio of their magnetic moments.

SOLUTION

- $T_d = \frac{1}{f_d} = \frac{\text{60 seconds}}{\text{4 oscillations}} = \text{15 seconds/oscillation,}$

- $T_s = \frac{1}{f_s} = \frac{\text{60 seconds}}{\text{12 oscillations}} = \text{5 seconds/oscillation}$

$\therefore \frac{M_1}{M_2} = \frac{{T_d}^2+{T_s}^2}{{T_d}^2-{T_s}^2} = \frac{15^2+5^2}{15^2-5^2} = \frac{5}{4},$ **so the ratio is 5:4**

EXAMPLE 3

A magnet freely suspended in a vibration magnetometer makes 10 oscillations per minute at a place A and 40 oscillations per minute at a place B. If the horizontal component of earth's magnetic field at A is $36 \times 10^{-6} T$, then its value at B is?

SOLUTION

$$T = 2\pi \frac{1}{MH_H} \Rightarrow T \propto \frac{1}{H_H} \Rightarrow \frac{T_A}{T_B} = \frac{(H_H)_B}{(H_H)_A}$$

$$\Rightarrow \frac{60 \text{ seconds}/10 \text{ oscillations}}{60 \text{ seconds}/20 \text{ oscillations}} = \frac{(H_H)_B}{36 \times 10^{-6} T}$$

$$\Rightarrow (H_H)_B = 144 \times 10^{-6} T$$

LECTURE 4: GAUSS LAW

4.1 Introduction

4.2 Field lines and lines of forces

4.3 Flux through a surface (the surface integral)

4.4 Flux of an electric field

4.5 Gauss Law

4.6 Gauss Law and Coulomb's Law

4.7 Gauss Law in differential form

4.8 Application of Gauss Law

4.9 Capacitors and dielectric

4.1 Introduction

In the previous lecture, we replaced calculations of forces with calculations of field intensity and those of potential energy by potential. This made it possible to avoid the serious complications related to the large number of charges involved.

A distribution of charges has a degree of symmetry that provides an additional simplification in calculations. For the case of spherical symmetry of the charge distribution, the field intensity should be the same at all points that are at same distance radially from the surface of the sphere. Gauss's law is a method by which the field intensity is calculated for symmetric charge distributions.

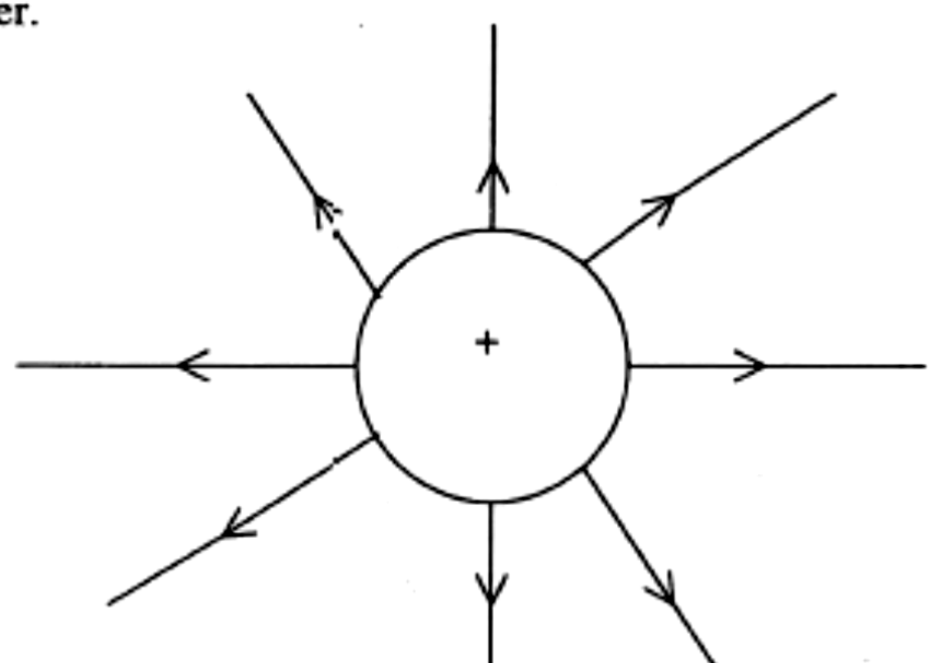
We start our discussion by defining field intensity as line of force per unit area before deriving Gauss law. We finally consider some applications of Gauss law.

The Gaussian surfaces do not have to be spheres.

4.2 Field lines and lines of forces

Electric fields can be mapped out by electrostatic line of force. The electric lines of force may be defined as a line such that the tangent to it is the direction of the force on a small positive charge at that point. Arrows on the lines as shown in figure 4.1show the direction of the force on a positive charge.

The lines of force due to a charged disc or sphere diverge and become farther apart as the field becomes weaker.



The number of lines of force per unit area can be used to represent numerically the strength of the field:

One line per square centimeter represents a field of unit strength.

- Let the source/point/arbitrary charge be Q and the test/unit charge be q the formula for electric force:

$$F = \frac{k \cdot q \cdot Q}{d^2}$$

where $k = 9.0 \cdot 10^9 \ N \cdot m^2/C^2$

d = separation distance between charges (meters)

- If the expression for electric force as given by Coulomb's law is substituted for force, then the electric field ($E = \frac{F}{q}$):

$$E = \frac{F}{q} = \frac{k \cdot \cancel{q} \cdot Q / d^2}{\cancel{q}} = \frac{k \cdot Q}{d^2}$$

$$\boxed{E = \frac{k \cdot Q}{d^2}}$$, it is the Eqn. of point charge Q

- We can define the electric field of an arbitrary charge Q as the force experienced by a unit charge q due to Q, i.e., $E = \frac{F}{q}$.
- The above formula for electric field strength shows that it is dependent upon the quantity of charge on the source charge (Q) and the distance of separation (d) from the source charge.

Consider a sphere of radius 1cm described round a unit point charge. Area of the surface of a sphere is 4π sq.cm.

Strength of the field at surface $= k\,{}^{q}\!/_{r^2} = \frac{1\times1}{1^2} = 1$ **e.s.u of fields strength**

Taking 1 line of force per sq.cm of the surface to represent the unit field, it now means that there must be 4π lines of force emerging from a unit charge. We have

$$\text{Field x Area} = \text{Lines of force}$$
$$\text{E x Area} = 4\pi \text{ lines of force} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 4.1$$

This is Gauss theorem. It follows from this theorem that:

"Electric field very near to the surface of a conductor is $4\pi\delta$, where δ is the surface charge density."

Lines of force do not intersect one another since the direction of the field cannot have two values at one point.

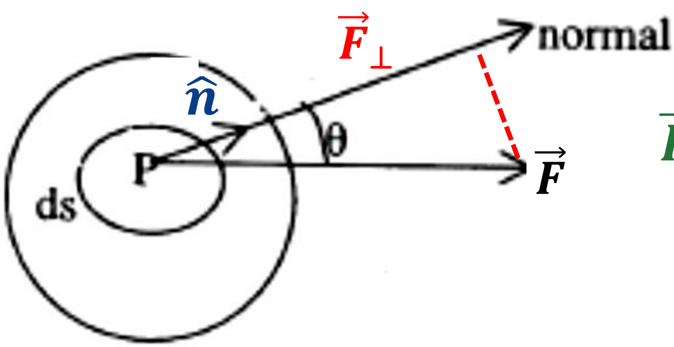
We adopt the convention that the number of lines per unit area over a surface perpendicular to the direction of the lines is proportional to the field strength.

4.3 Flux through a surface (the surface integral)

Let us imagine a smooth surface in the vector field and then consider a continuously varying $\vec{F}$ at a point P inside a small surface element of area ds.

Draw the positive norma $\hat{n}$ of unit length on the element and let ϑ be the angle between $\hat{n}$ and $\vec{F}$.

$\hat{\mathbf{n}}$ - unit vector that points perpendicular (normal) to surface.
- two possible directions for most surfaces; ambiguity.
- for Gaussian surfaces, **n** points out, by convention.

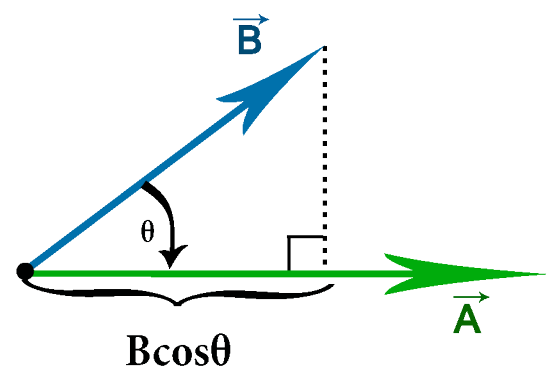


Component of $\vec{F}$ perpendicular to the ds is

$$\vec{F}.\hat{n} = \vec{F}_{\perp}\hat{n} = F\,cos\theta\,\hat{n}$$
$$= F\,\hat{n}cos\theta = F\,cos\theta \,\ldots\ldots\ldots 4.2$$

Fig 4.2 a closed surface and a normal to the surface for a field F

Recall dot product for vectors: The result of the dot product operation will be a single scalar value that is equal to the magnitude of the first vector, times the magnitude of the second vector, times the cosine of the angle between the two vectors



$$\vec{A}\,.\,\vec{B} = |\vec{A}|.|\vec{B}|.\cos\theta$$

Let us consider the flow of fluid. Net amount of fluid going out through the surface is the **flux of velocity** through the surface.

For any arbitrary closed surface.

Net outward flow or **flux** = Average outward normal component of the velocity x area of the surface.

Flux = (Average normal component) (surface area)

$$= \vec{F}.\hat{n}\, ds = F\, \hat{n}\, cos\theta\, ds = F\, cos\theta\, ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 4.2a$$

The integral of this taken over the entire surface is called total flux or surface integral of F through the whole surface, i.e.

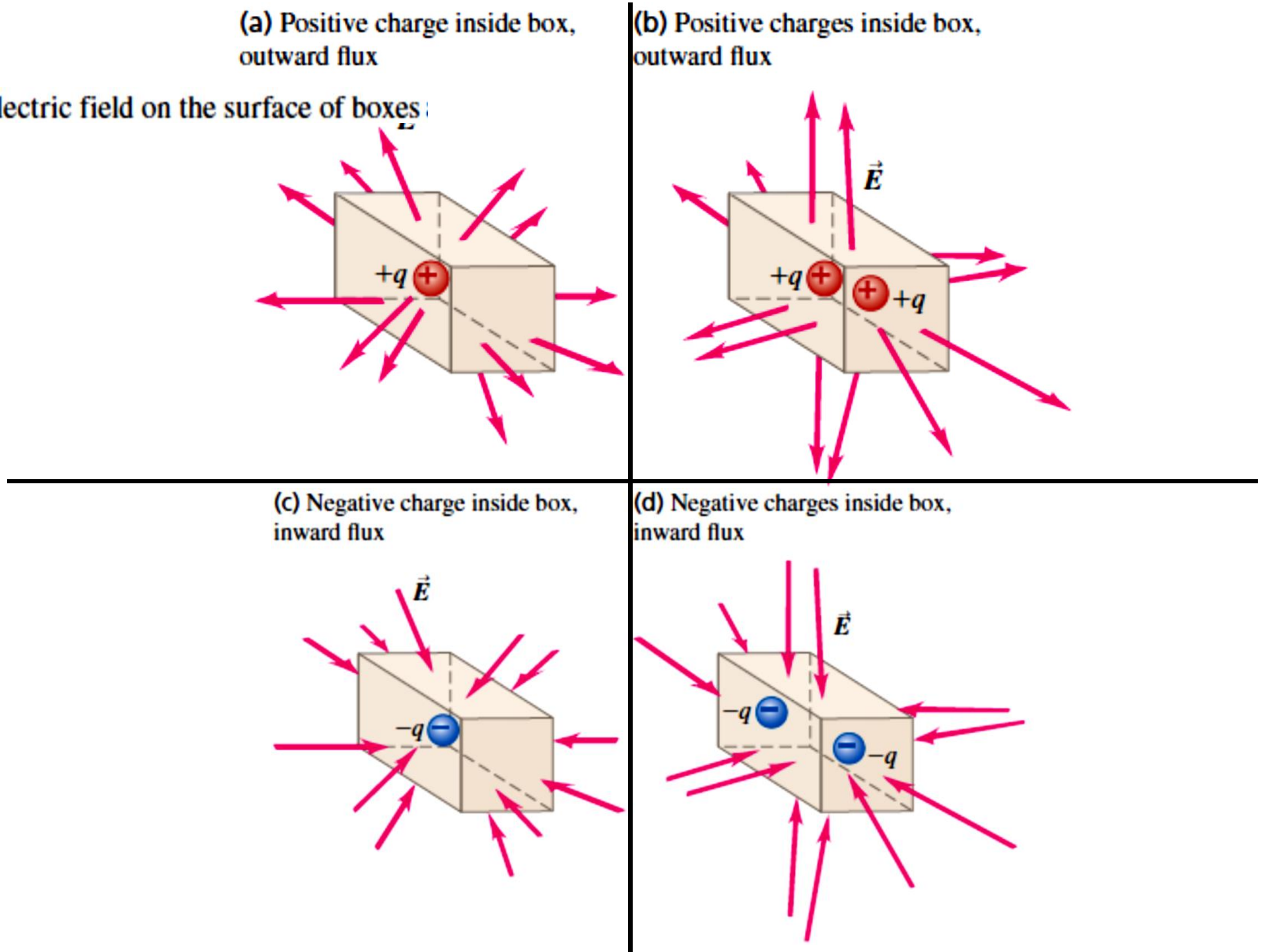
$$\iint_s \vec{F}.\, \hat{n}\, ds = \iint_s F\, \hat{n}\, cos\theta\, ds = \iint_s F\, cos\theta\, ds$$

$$= \textbf{Total fluid flowing outward through } S \quad \ldots\ldots\ldots\ldots 4.2b$$

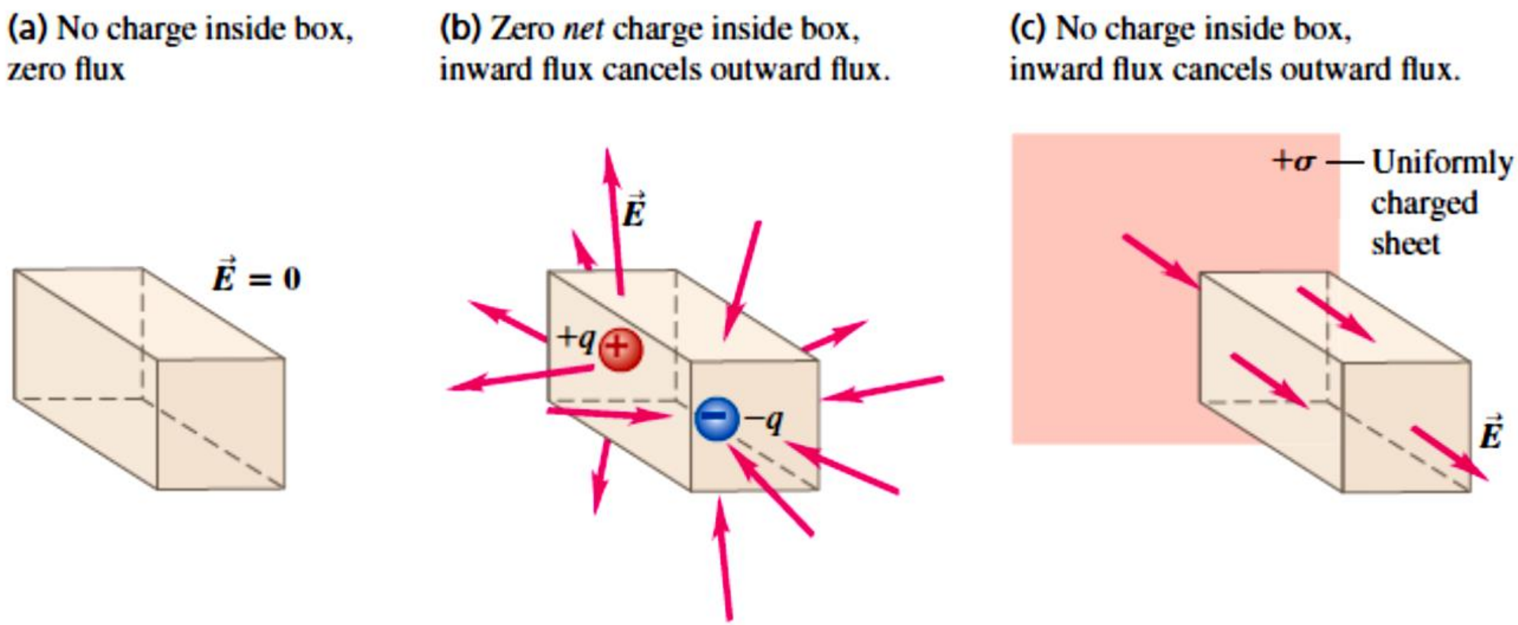
4.4 Flux of an electric field

The word "flux" comes from the Latin word "fluxus" which means to flow. So electric flux means "Flow of electric lines of force from one point to another point."

The electric field on the surface of boxes :



Three cases in which there is zero *net* charge inside a box and no net electric flux through the surface of the box. (a) An empty box with $\vec{E} = 0$. (b) A box containing one positive and one equal-magnitude negative point charge. (c) An empty box immersed in a uniform electric field.



Electric flux Φ_E through a surface:

(component of E-field ⊥ surface) X (surface area) $\Rightarrow \emptyset_E = E_\perp \times A = E\cos\emptyset\, A = E\, A\cos\emptyset = \vec{E}.\vec{A}$

There are two main properties of a vector field, namely, **flux** and **circulation** or **flow** and **rotation**.

We replace $\vec{F}$ with $\vec{E}$ in the formula for flux. We then have an expression for flux of an electric field

$$d\emptyset_E = \vec{E}.d\vec{s} = E_\perp\hat{n}.d\vec{s} = E\cos\alpha\,\hat{n}\,ds = \vec{E}.\vec{n}ds = \vec{E}.d\vec{A}$$

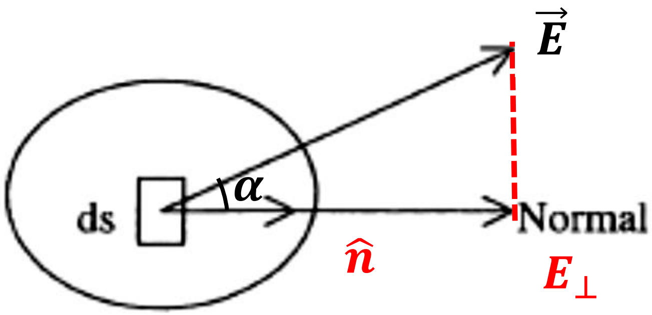


Fig 4.3 A closed surface and normal to the surface for an electric field $\vec{E}$

where $\vec{n}ds = d\vec{A}$ a vector area. Thus $\emptyset_E = \oint_s d\emptyset_E = \oint_s \vec{E}.d\vec{A}$ ………… 4.3a

$$\Phi_E = \int E\cos\phi\, dA = \int E_\perp\, dA = \int \vec{E}\cdot d\vec{A}$$

Electric flux through a surface

Magnitude of electric field $\vec{E}$

Component of $\vec{E}$ perpendicular to surface

Angle between $\vec{E}$ and normal to surface

Element of surface area

Vector element of surface area

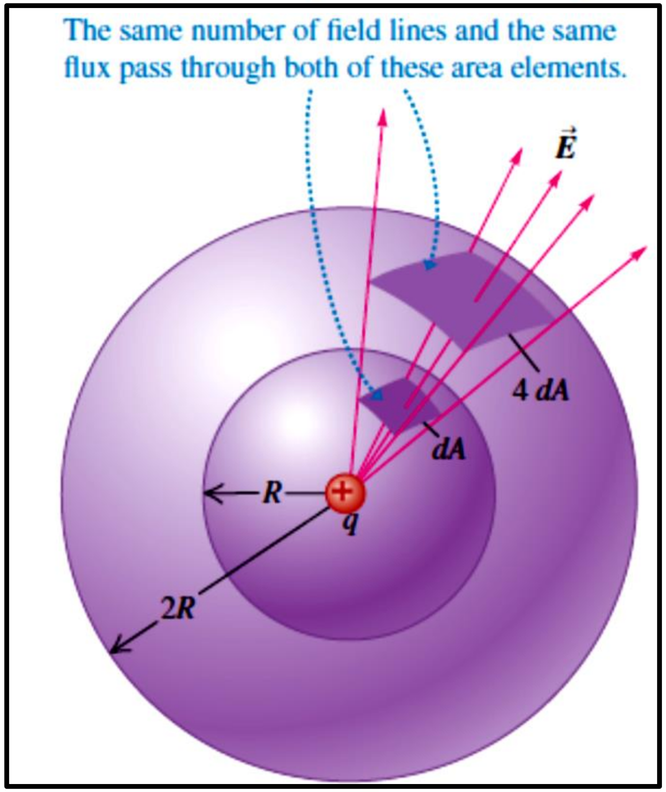
TAKE NOTE

$$\Rightarrow \emptyset_E = E_{\perp} \times \mathbf{A} = E\, cos\emptyset\ \mathbf{A} = E\ \mathbf{A}\cos\emptyset = \vec{E}.\vec{A}$$

For the sphere: $\Rightarrow \emptyset_E = E_{\perp} \times \mathbf{A} = E\, cos\emptyset\ \mathbf{A} = E\ \mathbf{A}$

$$\Rightarrow \Phi_E = EA = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}(4\pi R^2) = \frac{q}{\epsilon_0}$$

The flux is independent of the radius R of the sphere. It depends on only the charge q enclosed by the sphere.



For the irregular shape:

$$\Rightarrow \Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

4.4.1 Unit of flux

From above Equation: $\emptyset_E = \oint_s \ d\emptyset_E = \oint_s \ \vec{E}.d\vec{A}$

$$\Rightarrow \emptyset_E = \oint_s \ \frac{\vec{F}}{Q}.d\vec{A}$$

the unit of flux is e.s.u. cm^2 or $Nm^2/C = \dfrac{Newton\ metre^2}{Coulomb}$

EXAMPLE

Consider a hypothetical cylinder of radius R immersed in a fluid in a uniform electric field $\vec{E}$, the cylinder axis being parallel to the field. What is the flux of this closed surface?

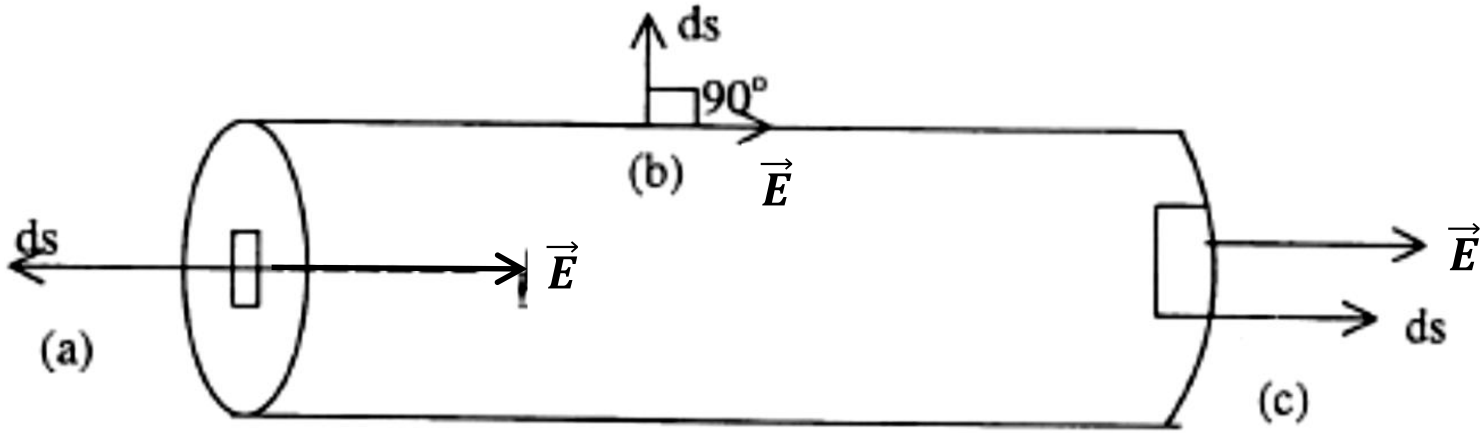


Fig 4.4 Cylindrical surface immersed in a uniform field $\vec{E}$ parallel to its axis

SOLUTION

$\phi_E = \oint E \cdot ds = \int_a E \cdot ds + \int_b E \cdot ds + \int_c E \cdot ds$, **but** $\emptyset_E = \oint_s \vec{E}.d\vec{A} = E \oint_s \cos\alpha \, d\vec{s}$

Recall: $\emptyset_E = \oint_s \vec{E}.d\vec{A} = E\oint_s \cos\alpha\, d\vec{s}$

(a) For the left cap, angle between $\vec{E}$ and ds is 180° for all points, i.e. $\vec{E}$ and ds are oppositely directed.

$$\therefore \int_a E\cdot ds = \int E\cos 180^\circ ds = -E\int ds = -Es$$

where $S = \pi R^2$ is the cap area.

(c) Similarly for the right cap, angle 0°.

$$\int_C E\cdot ds = \int Eds = ES$$

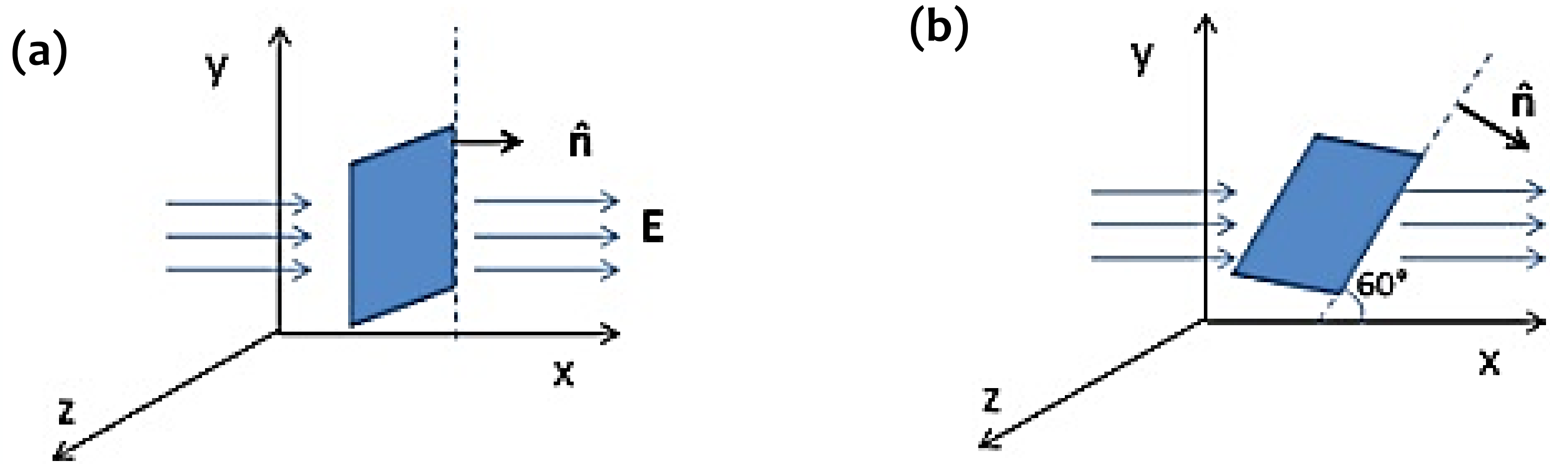
(b) For cylindrical wall, angle 90°

$$\int_b E\cdot ds = \int_C E\cos 90^\circ\, ds = 0$$

$$\therefore Total\ flux = -ES + ES + 0 = 0$$

EXAMPLE

Determine their flux.



SOLUTION

For a uniform electric field **E** = $3 \times 10^3\ \hat{i}$ N/C, the flux through a square sheet of 10 cm side parallel to y-z plane will be:

but $\emptyset_E = E\ A\ cos\theta \quad \Rightarrow \quad \phi = |\mathbf{E}| A \cos\theta$

(a) $\phi = 3 \times 10^3 \times (10^{-1} \times 10^{-1}) \times \cos 0$

$\phi = 30\ Nm^2/C$

(b) When the plane makes an angle of 60° with x-axis, the flux will be:

$\phi = 3 \times 10^3 \times (10^{-1} \times 10^{-1}) \times \cos 60$

$\phi = 15\ Nm^2/C$

4.4.2 Electric Flux density ($\vec{D}$)

Michael Faraday (1791-1867) an English physicist from his famous experiment on pair of concentric metallic spheres, as shown in the Fig. 2.15 concluded that the charge on the outer sphere is always equal in magnitude to the original charge Q placed on inner sphere but is of opposite sign. In other words, some thing was coming out of the inner sphere and reaching the outer sphere in all cases and that some thing depends only on the magnitude of the original charge placed in the inner sphere. It does not depend on the size of the sphere and on the dielectric material between the two spheres. Faraday called this 'something' coming out of the inner sphere and reaching outer sphere as electric displacement or electric flux.

One coulomb of electric charge gives one coulomb of flux $\emptyset = Q$ coulomb, and this electric displacement or electric flux is a scalar quantity. The number of lines or flux crossing unit area of a surface normal to the lines is called electric flux density. It is denoted by $\bar{D}$ and is a vector quantity. The concept of flux is shown in the Fig. 2.16. The number of lines gives a sample of field lines as the total number would be infinite. The direction of $\bar{D}$ at a point is the direction of flux lines at that point and number of lines gives the magnitude.

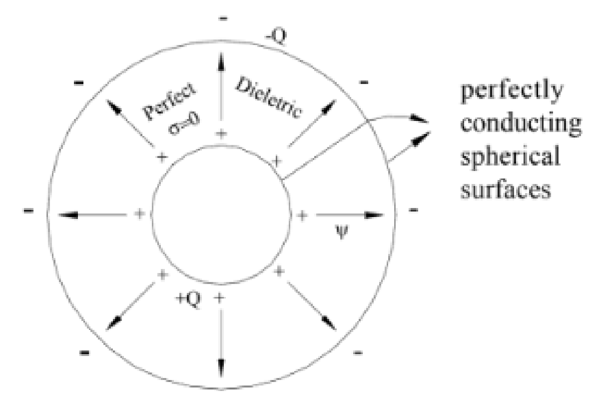


Fig. 2.15 : The electric flux in the region between a pair of charged concentric spheres (Faraday's experiment).

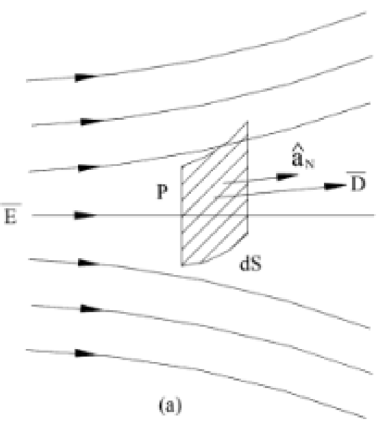


Fig. 2.16: Concept of flux and flux density.

A point charge is surrounded by an electric field. The electric intensity is given by

$$\vec{E} = \left(\frac{K}{\varepsilon}\right)\left(\frac{q}{r^2}\right)\hat{r}\,, \quad \text{where } K = \frac{1}{4\pi} \quad \ldots\ldots\ldots\ldots \quad 4.4.2.1$$

With $\hat{r}$ the unit vector in the radial direction. We may rewrite this equation as

$$\varepsilon\vec{E} = (K)\left(\frac{q}{r^2}\right)\hat{r} = (4\pi K)\left(\frac{q}{4\pi r^2}\right)\hat{r} \quad \ldots\ldots\ldots\ldots \quad 4.4.2.2$$

The dimensions on right hand side are

[q]	=	Statcoulomb
$[r^2]$		Centimetre square

we denote this quantity by D and call it **electric flux density**.

D = ε E ……… 4.4.2.3: where $\varepsilon = \varepsilon_0\varepsilon_r$, ε_o is the permittivity of the free space and ε_r is the relative permittivity.

The electric flux density and the field intensity are vectors in the same direction. This is only true for isotropic media, the media whose properties do not depend on direction.

Flux of D over surface S = SD = Φ_E

$$\therefore \quad D = \frac{\varnothing_E}{S} = \frac{\text{Flux}}{\text{Area}} = \text{flux density} \qquad 4.4.2.4$$

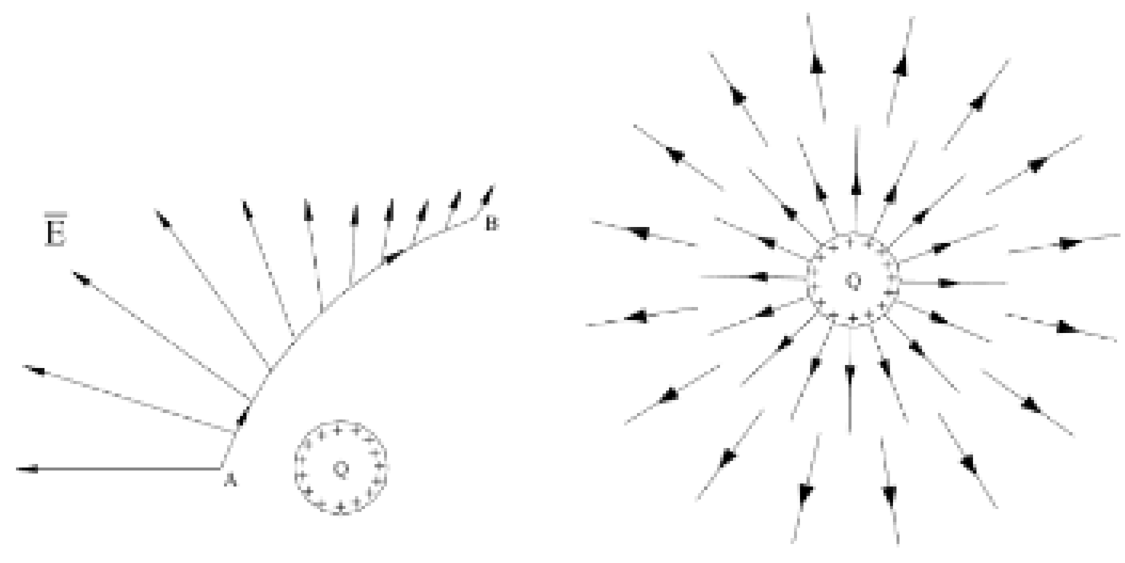
Flux density depends only on the charge and is independent of the permittivity of the medium. it is also called electric displacement vector.

TAKE NOTE: $\Rightarrow \vec{D} = \varepsilon\vec{E} = (K)(\frac{q}{r^2})\hat{r} = (4\pi K)(\frac{q}{4\pi r^2})\hat{r}$ **Eqn. 4.4.2.2**

Electric field intensity $\overline{E}$ and flux density $\overline{D}$ have same form. But the $\overline{D}$ is not a function of ε whereas $\overline{E}$ is a function of ε. The unit of flux density is C/m^2.

An important difference between field lines and lines of force :

The electric field varies in a perfectly continuous way along any path in the field as shown in the Fig. 2.17 (a). However, this type of physical continuity is different from the continuity of the lines of force, which has no physical meaning. The lines of force are as shown in the Fig. 2.17 (b). They satisfy the following conditions : (i) the tangent to a line of force at any point gives the direction of $\overline{E}$ at that point, (ii) number of lines per unit cross sectional area is proportional to the magnitude of $\overline{E}$. But flux lines are not continuous. The physical meaning of continuity of $\overline{E}$ means that the field is continuous, it has different values at different points.

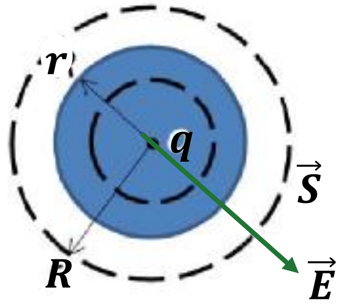


(a) Field lines. **(b) Lines of force.**

Fig. 2.17 : The difference between the field lines and line of force.

EXAMPLE

Calculate the flux of a single positive charge q as shown in the sketch below:



SOLUTION

We take a simple case. The surface is a sphere of radius r centred on the positive charge q. The magnitude of E is the same at every point on the surface and is given by $\Rightarrow E = K\dfrac{q}{r^2}$

$$E = (1/4\pi\varepsilon_0)(q/r^2)$$

$$\text{Flux} = \emptyset_E = \int E.ds = \oint (1/4\pi\varepsilon_0)(q/r^2)ds$$

Angle θ = O since E and $\vec{s}$ are in same direction

$$\phi_E = (1/4\pi\varepsilon_0)(q/r^2)\oint ds = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}4\pi r^2 = \frac{q}{\varepsilon_0}$$

4.5 Gauss Law

Gauss's law (Johann Friederich Karl Gauss, 1777-1855) is an important law and in fact it is converse of Coulomb's law. By Coulomb's law we can calculate electric field intensity for a given charge, but the Gauss's law enables us to determine the charge if $\overline{E}$ or $\overline{D}$ is known. This law says that the total flux coming out of a closed surface is equal to the net charge in the closed surface.

$$\emptyset_E = \oint d\emptyset_E = \oint \vec{D} d\vec{S} = Q_{\text{enclosed}} \quad \ldots\ldots\ldots\ldots 4.3a$$

$$\text{where } \vec{D} = \frac{\emptyset_E}{\text{surface area}} = \frac{\vec{E}.\vec{A}}{\vec{A}} = \vec{E}$$

In general terms, the flux of a vector quantity over any arbitrary closed surface is equal to the strength of the enclosed source of the vector.

Def *: The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.*

In other words, the surface integral of the normal component of electric flux density over any closed surface is equal to the charge enclosed.

The mathematical statement $\oint_S \overline{D}_s . d\overline{S} = Q_{\text{enclosed}} = \int dQ = \int_v \rho_v dv \quad \ldots\ldots\ldots\ldots 4.3b$

it is Gauss's Law in integral form/Maxwell's 1st Equation in integral form

To derive the Gauss Law let us take an arbitrary surface S which encloses a point charge $+Q$. Let us focus our attention on a differential element of area ds of this surface as shown in figure 4.5a below.

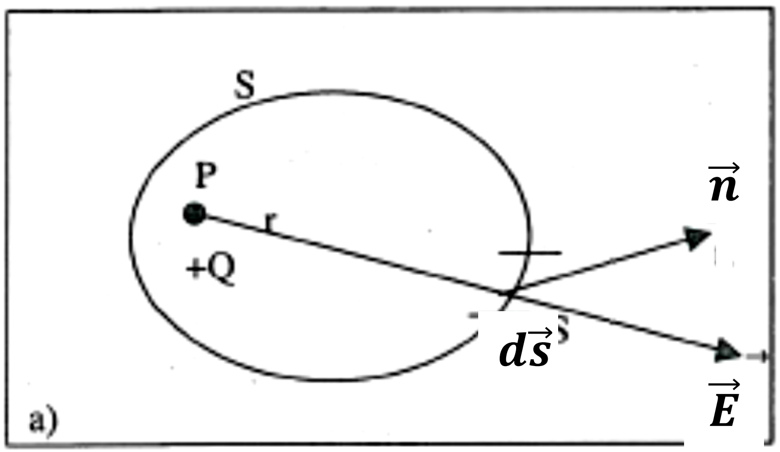


Fig 4.5(a) Gaussian surface surrounding a point charge

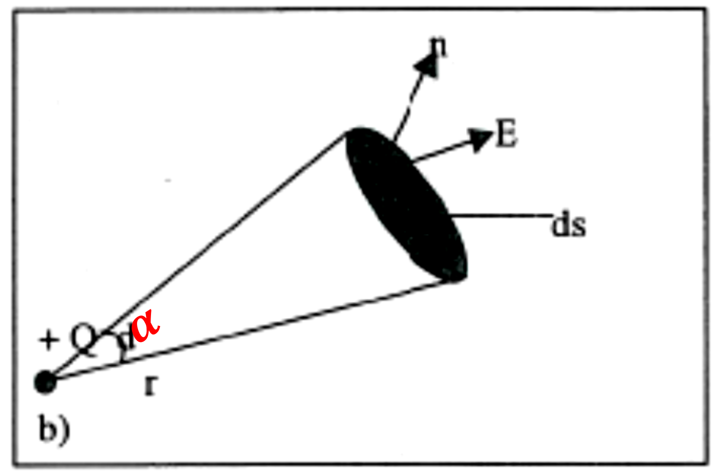


Fig 4.5 (b) Illustrating Gauss Law

At the surface the intensity of the electric field will have the value E and will be directed radially outwards from Q. It will not in general be normal to the surface, but will make an angle α with $\vec{n}$ is a unit vector normal to ds. We wish to evaluate the surface integral of the normal component of E, that is

$$\int_S \vec{E}\,.\vec{n}ds = \int_S E\,cos\alpha\,\vec{n}ds \quad \text{..............4.4a}$$

and $E = k\dfrac{Q}{r^2}$

The surface integral become:

$$\int_S E.\,cos\alpha\,\vec{n}ds = kQ\int_S \frac{cos\alpha\,dA}{r^2}$$

$$= kQ\frac{cos\alpha\,\mathrm{A}}{r^2} = kQ\frac{(cos0^o)\,(4\pi r^2)}{r^2} = 4\pi kQ \quad \text{.....4.4b}$$

Gauss Theorem or law may now be stated as follows:

"The surface integral of the normal component of the electric intensity, taken over any closed surface that encloses a point charge Q, is equal to 4 π KQ "

If we designate the normal component of E by E_n, we may express this theorem as follows:

$$\int_S \vec{E}\,.\vec{n}ds = \int_S E_n\,\vec{n}ds = E_n\int_S \vec{n}ds = E_n\int_S dA = k\frac{Q}{r^2}4\pi r^2 = 4\pi kQ$$

Gauss theorem is a very convenient aid in determining the electric intensity in those cases where the enclosing surface can be chosen of such a shape that symmetry consideration simplify the evaluation of the surface integral. The closed surface near the charge distribution is called a **Gaussian Surface**. It can have any shape, since the magnitude of the field and angle between $\vec{E}$ and normal to this surface have a **constant value**.

Since the above treatment is valid for all pints inside the enclosed surface S, the charge Q need not be a point charge, but represents, in general the total charge enclosed by the surface.

Since the total charge is a sum of a number of point charges, we can write Gauss Law in the form

$$\oint_S \vec{E}\,.\vec{n}ds = 4\pi k\sum Q \quad \text{...} \quad 4.5$$

where the summation is taken only over charges lying within the closed surface.

We now state Gauss law:

The surface integral of the normal component of $\vec{E}$ over any closed surface in an electrostatic field equals $\frac{4\pi K}{}\sum Q$, where $\sum Q$ is the net charge inside the surface.

From Equation 4.4: $\int_S \vec{E}\,.\vec{n}ds = 4\pi kQ$

$$\oint \vec{E}\,.\,d\vec{A} = 4\pi kQ_{\text{enclosed}}$$

$$\Rightarrow \oint \vec{E}\,.\,d\vec{A} = 4\pi\frac{1}{4\pi\varepsilon_0}Q_{\text{enclosed}} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

$$\Rightarrow \int_S \vec{E}\,.\vec{n}ds = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$ ....4.6 ; is Gauss's Law

The charge enclosed may be of different types such as several point charges $Q = \sum Q_n$ or line charge $Q = \int \rho_L\, dL$ or a surface charge $Q = \int \rho_s\, dS$ or a volume charge distribution $Q = \int \rho_v\, dv$.

Generally $\oint_S \vec{D}\,.\,d\vec{S} = Q = \int_V \rho_V\, dV$ ....4.7 (as 4.3b)

TAKE NOTE

Various forms of Gauss's Law:

Magnitude of electric field $\vec{E}$

Component of $\vec{E}$ perpendicular to surface

Total charge enclosed by surface

$$\Phi_E = \oint E\cos\theta \, dA = \oint E_\perp \, dA = \oint \vec{E}\cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$

Electric flux through a closed surface

Angle between $\vec{E}$ and normal to surface

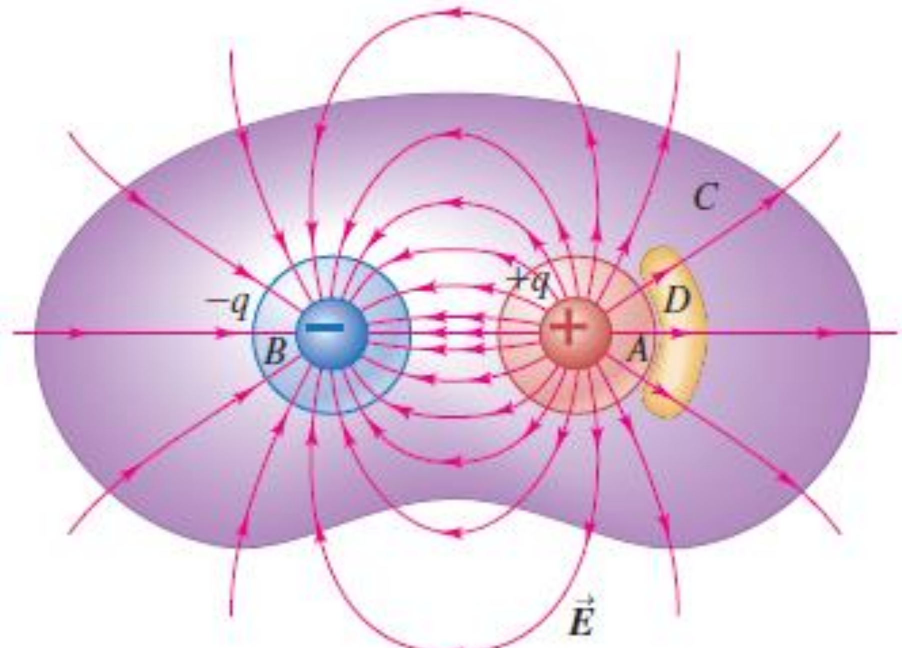
Element of surface area

Vector element of surface area

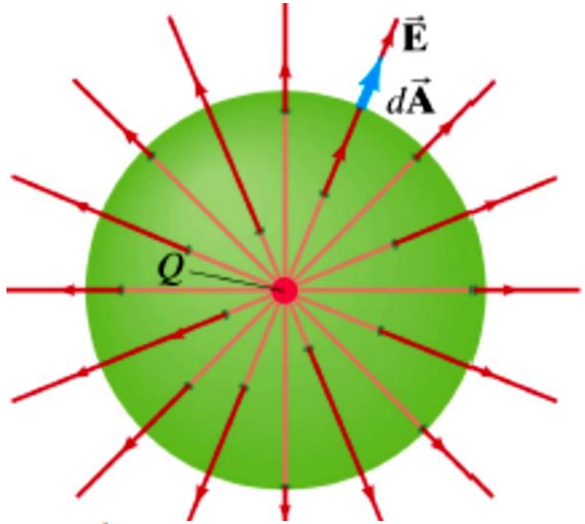
Electric constant

<u>Closed surfaces A, B, C, and D:</u>

- $\emptyset_{E_A} = \frac{+q}{\varepsilon_0}$,
- $\emptyset_{E_B} = \frac{-q}{\varepsilon_0}$,
- $\emptyset_{E_C} = \emptyset_{E_A} + \emptyset_{E_B}$
  $= \frac{+q}{\varepsilon_0} + \frac{-q}{\varepsilon_0} = 0$
- $\emptyset_{E_D} = 0$



4.6 Gauss Law and Coulomb's Law



At each point on the surface, $\vec{E}$ is perpendicular to the surface, and its magnitude is the same at every point.

Suppose we have a point-like charge Q. Let's compute the electric field at a distance, r, from the charge:

1. Define sphere of radius r around point-like charge Q
2. Apply Gauss's Law

$$\oint \vec{E} d\vec{A} = \frac{q_{encl}}{\varepsilon_0} = \left(\frac{Q}{\varepsilon_0}\right) \text{.....Gauss's Law}$$

$$\text{but} \oint \vec{E} d\vec{A} = E \oint dA = E 4\pi r^2$$

$$\Rightarrow E 4\pi r^2 = \left(\frac{Q}{\varepsilon_0}\right)$$

$$\Rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2} = k\frac{Q}{r^2} \quad \Rightarrow \quad \vec{E} = k\frac{Q}{r^2}\hat{r}$$

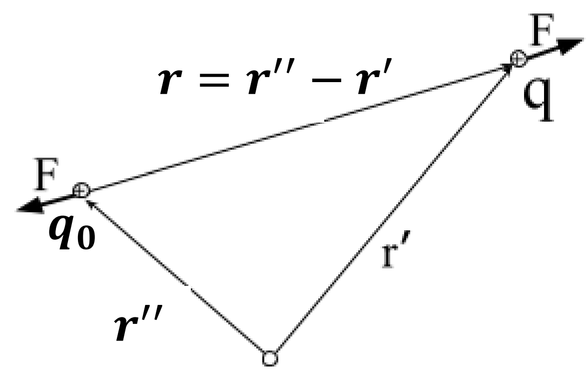
Since $\vec{E}$ is radial, on the surface we have

$$\oint_S \vec{E}\,.\vec{n}ds = \oint_S E\,.\cos 0^o\,\vec{n}ds = \oint_S E\,dA = (k\frac{q}{r^2})\,4\pi r^2 = 4\pi kq$$

$$\therefore E\oint_S ds = 4\pi kq$$

$$E\cdot 4\pi r^2 = 4\pi kq$$

$$E = k\frac{q}{r^2} \quad \text{.......................} \quad 4.8a$$



hence $\vec{E} = k\frac{q}{r^2}\hat{r}$ 4.8b

Let us put a second point charge q_0 at the point where $\vec{E}$ is calculated. The force that acts on q_0 is

$$\vec{F} = \vec{E}q_0 = k\frac{q\,q_0}{r^2}\hat{r} \quad \text{.......................} \quad \text{4.9, is Coulomb's Law}$$

Coulomb's law states that the force between two static point electric charges is proportional to the inverse square of the distance between them, acting in the direction of a line connecting them (see sketch above).

TAKE NOTE

The Gauss's Law is the opposite of or alternative for Coulomb's Law. Gauss's law and Coulomb's law are equivalent - meaning that they are one and the same thing. Either one of them can be derived from the other.

- **Consider a point charge q. As per Coulomb's law, the electric field produced by it is given by: $E = k\frac{q}{r^2}$ or $\vec{F} = \vec{E}q_0 = k\frac{q\,q_0}{r^2}\hat{r}$**
- **Consider a sphere of radius r centered on charge q. So, for the surface S of this sphere you have:**

$\int_S \vec{E}\,.\vec{n}ds = E_n \int_S dA = k\frac{q}{r^2}\; 4\pi r^2 = 4\pi k q = 4\pi\frac{1}{4\pi\varepsilon_0}q = \frac{q}{\varepsilon_0}$, which is Gauss' law and it will hold for a surface of any shape or size, provided that it is a closed surface enclosing the charge q.

<u>Note that</u> if the r^2 in the expression for the surface area of the sphere in the numerator did not exactly cancel out the r^2 in the denominator of Coulomb's law, the surface integral would actually depend on r. Hence you would not have the result that the surface integral is independent of the area of the surface, which is what is implied by Gauss' law.

<u>Note that</u> by performing these steps in reverse, you can also derive Coulomb's law from Gauss' law, thus demonstrating that they are equivalent.

4.7 Gauss Law in differential form

When a charge is distributed over a volume such that ρ is the density, then the charge enclosed by the surface enclosing the volume is given by

$$q = \int_V \rho \, dv \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 4.10a$$

substituting this in Gauss Law we get {Gauss's Law: $\oint \vec{E} \,.\, d\vec{A} = \frac{q_{enclosed}}{\varepsilon_0}$, and $k = \frac{1}{4\pi\varepsilon_0}$}

$$\int_S \vec{E} \cdot d\vec{s} = 4\pi \, k \int_V \rho \, dv \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 4.10b$$

According to Gauss Divergence theorem: "Flux across S is equal to Volume integral of the divergence of the function $\vec{E}$ over the volume v enclosed by the surface."

$$\text{Flux} = \oint_S \vec{E} \cdot d\vec{s} = \oint \vec{E} \cdot \hat{n} \, ds = \iiint_V div\, E \, dv$$, compare with Eqn. 4.10b:

$$\therefore \; div\; E = 4\pi k \; \rho \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 4.11$$

In a vacuum, $\varepsilon_0 = 1$, $k = 1$ in cgs units. In SI units $k = \frac{1}{4\pi\varepsilon_0}$

$$div\; E = 4\pi \rho \qquad \text{c.gs} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 4.12a$$

$$div\; E = \rho \qquad \text{SI units} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 4.12b$$

Equation 4.12 are Gauss's Law in differential form.

4.8 Application of Gauss Law

4.8.1 Introduction

The usefulness of Gauss Law depends on our ability to evaluate the integral

$$\oint_S \vec{E} \cdot d\vec{S}$$

Gaussian surface is a closed surface in three-dimensional space through which the flux of a vector field is calculated (see Fig. below)

For this we construct a closed surface near the charge distribution over which the magnitude $\vec{E}$ and the angle between $\vec{E}$ and the normal to this surface have a constant value. This surface is called Gaussian surface. That is,

(a) The magnitude of $\vec{E}$ is constant for uniform charge distributions

(b) The angle θ is constant for symmetrical charge distributions and symmetrical surfaces

Thus, the necessary condition for the application of Gauss Law is a uniform symmetrical charge distribution and symmetrical Gaussian surface.

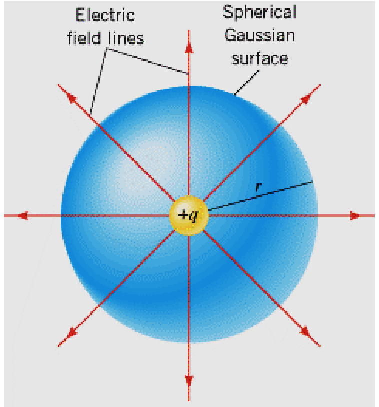
We have three main kinds of symmetry. These are:

If the charge distribution has

(a) Spherical symmetry – Gaussian surface is a concentric sphere

(b) Cylindrical symmetry – Gaussian surface is a Coaxial cylinder

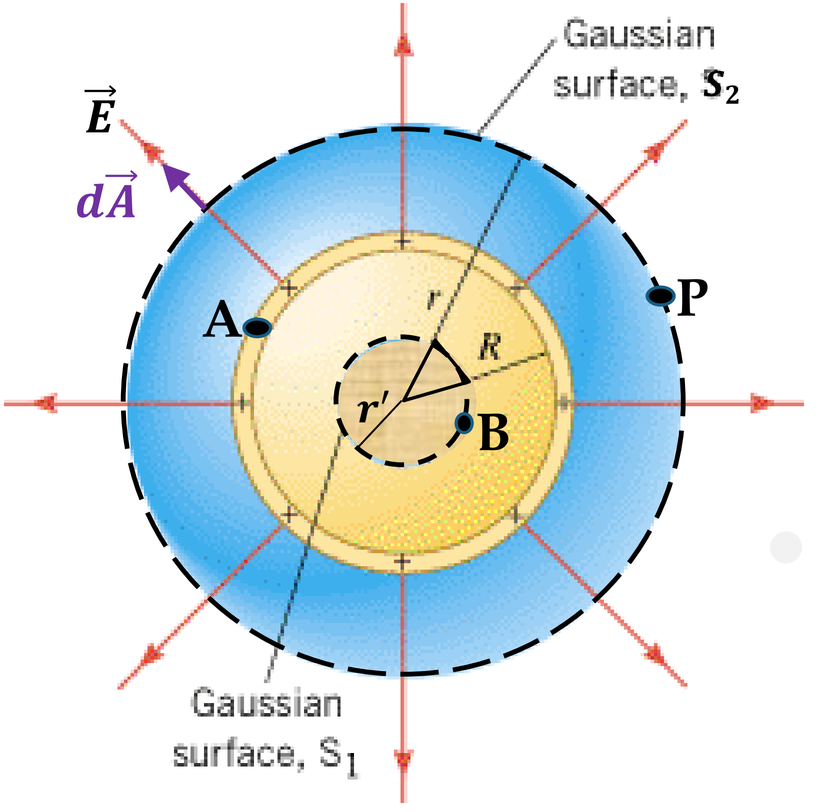
(c) Plane symmetry – Gaussian surface is **a pill box**



4.8.2 Electric Field due to a uniformly charged sphere

Suppose a charge q is uniformly distributed over a solid sphere of radius *R* as shown below:

- **For spherical charge distribution, the Gaussian surfaces (S_1 and S_2) are spheres concentric with the charged sphere.**
- **Gaussian surface containing *P* encloses all charges in the sphere.**
- **Gaussian surface containing *B* encloses a smaller sphere than the sphere (*R*) of charge distribution.**



We now consider the electric field due to a uniformly charged sphere at three different points.

i) At a point P outside the charged sphere

From Gauss's law,

$$\Phi_E = \oint_S \vec{E}\,.\,\vec{dS} = \frac{q}{\varepsilon_0} \quad \text{.........} \; 4.13$$

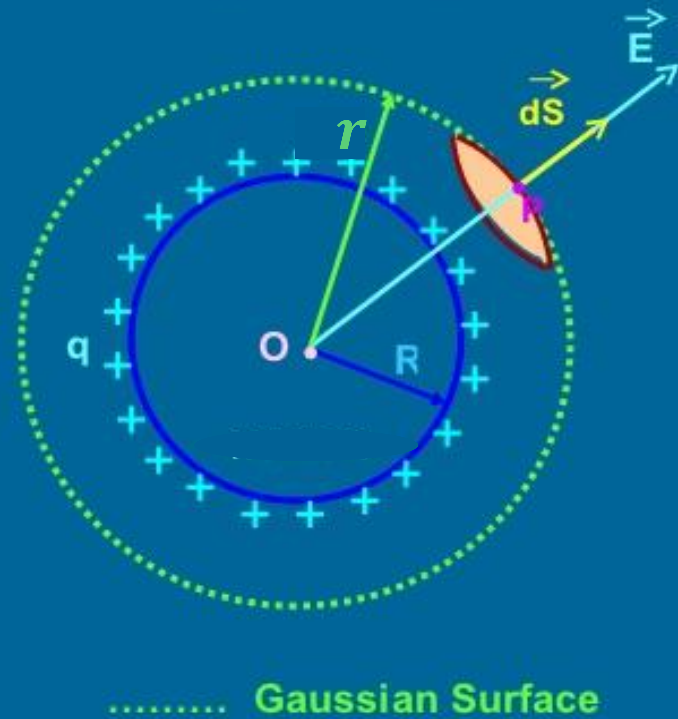
Since $\vec{E}$ and $\vec{dS}$ are in the same direction,
$\theta = 0^o$

$$\therefore \quad \Phi_E = \oint_S E \; dS = \frac{q}{\varepsilon_0}$$

$$\text{or} \quad \Phi_E = E \oint_S dS = \frac{q}{\varepsilon_0}$$

$$E \times 4\pi r^2 = \frac{q}{\varepsilon_0} \quad \text{or} \quad \boxed{E = \frac{q}{4\pi\varepsilon_0 r^2}} \; \text{...} \; 4.14$$

Since $q = \sigma \times 4\pi R^2$, $\quad \therefore \quad \boxed{E = \dfrac{\sigma \; R^2}{\varepsilon_0 \;\; r^2}}$



Eqn. 4.14 is similar to the electric field due to a point charge. Thus outside the sphere, the electric field behaves as though it is due to a point charge (carrying all the charge of the shell) at the center of the shell.

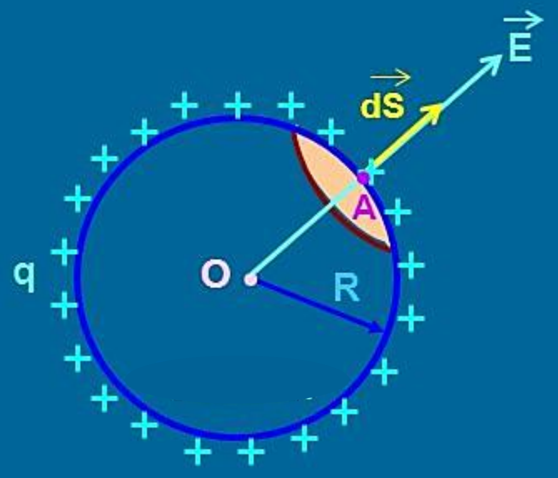
where σ is surface charge density $= \dfrac{q}{\text{surface area}}$

$$= \frac{\text{volume enclosed by surface area} \times \text{volume charge density } (\rho)}{\text{surface area}}$$

ii) At a point A on the surface of the charged sphere

From Gauss's law,

$$\Phi_E = \oint_S \vec{E} \cdot \vec{dS} = \frac{q}{\varepsilon_0}$$



Since $\vec{E}$ and $\vec{dS}$ are in the same direction,

$\theta = 0^o$

$$\therefore \quad \Phi_E = \oint_S E \; dS = \frac{q}{\varepsilon_0}$$

$$\text{or} \quad \Phi_E = E \oint_S dS = \frac{q}{\varepsilon_0}$$

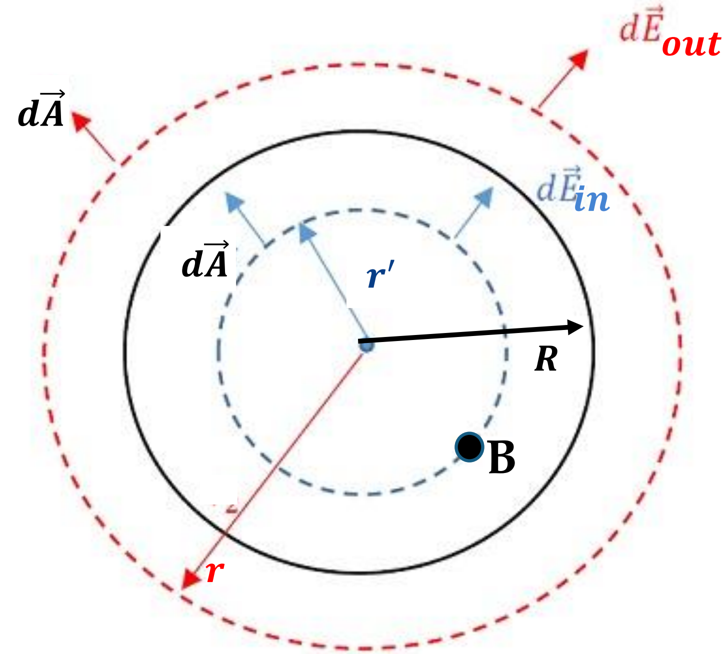
$$E \times 4\pi R^2 = \frac{q}{\varepsilon_0} \quad \text{or} \quad \boxed{E = \frac{q}{4\pi\varepsilon_0 R^2}} \quad \dots\dots\dots \; 4.15$$

Since $q = \sigma \times 4\pi R^2$,

$$\therefore \quad \boxed{E = \frac{\sigma}{\varepsilon_0}}$$

Electric field due to a uniformly charged thin spherical shell at a point on the surface of the shell is maximum.

iii) At a point B inside the charged sphere



TAKE NOTE

If the volume charge density is ρ, then the total charge enclosed by the sphere is

$$q = (\tfrac{4}{3})\rho\pi r'^3.$$

If you spread that charge uniformly around the conducting surface of the sphere (surface area $4\pi r'^2$), you find that the surface charge density $(\sigma) = (\tfrac{1}{3})\rho r$. That is

$$\Rightarrow \sigma \text{ is surface charge density} = \frac{q_{enclosed\ by\ Gaussian\ surface}}{surface\ area}$$

$$\Rightarrow \sigma = \frac{(volume\ enclosed\ by\ surface\ area)\times(volume\ charge\ density\ (\rho))}{surface\ area}$$

$$\Rightarrow \sigma = \frac{(\frac{4}{3})\rho\pi r'^3}{4\pi r'^2} = \frac{1}{3}\rho r'$$

The outward flux through the surface of a sphere is given by

$$\oint \vec{E}\cdot d\vec{s} = E\cdot 4\pi r'^2 = 4\pi k q \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 4.15a$$

total charge enclosed by the Gaussian surface $(q_{encl.})$

= volume enclosed by surface x charge density

$$\Rightarrow q_{encl.} = \left(\tfrac{4}{3}\right)\rho\pi r'^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 4.15b$$

Re-arrange 4.15a: $\quad E\,(\text{inside the sphere}) = \dfrac{4\pi k q}{4\pi r'^2}$

$$\Rightarrow E = \frac{4\pi k}{4\pi r'^2} \times \frac{4}{3}\rho\pi r'^3 = 4\pi k\frac{1}{3}\rho r' \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 4.15c$$

<u>at the surface of the sphere</u> $r' = R: \Rightarrow E = 4\pi k\frac{1}{3}\rho R \quad \ldots\ldots\ldots \quad 4.15d$

Re-writing Eqn. 4.15c: $E = 4\pi k\frac{1}{3}\left[\dfrac{q}{\frac{4}{3}\pi R^3}\right] r' = kq\dfrac{r'}{R^3} \quad \ldots\ldots\ldots \quad 4.15e$

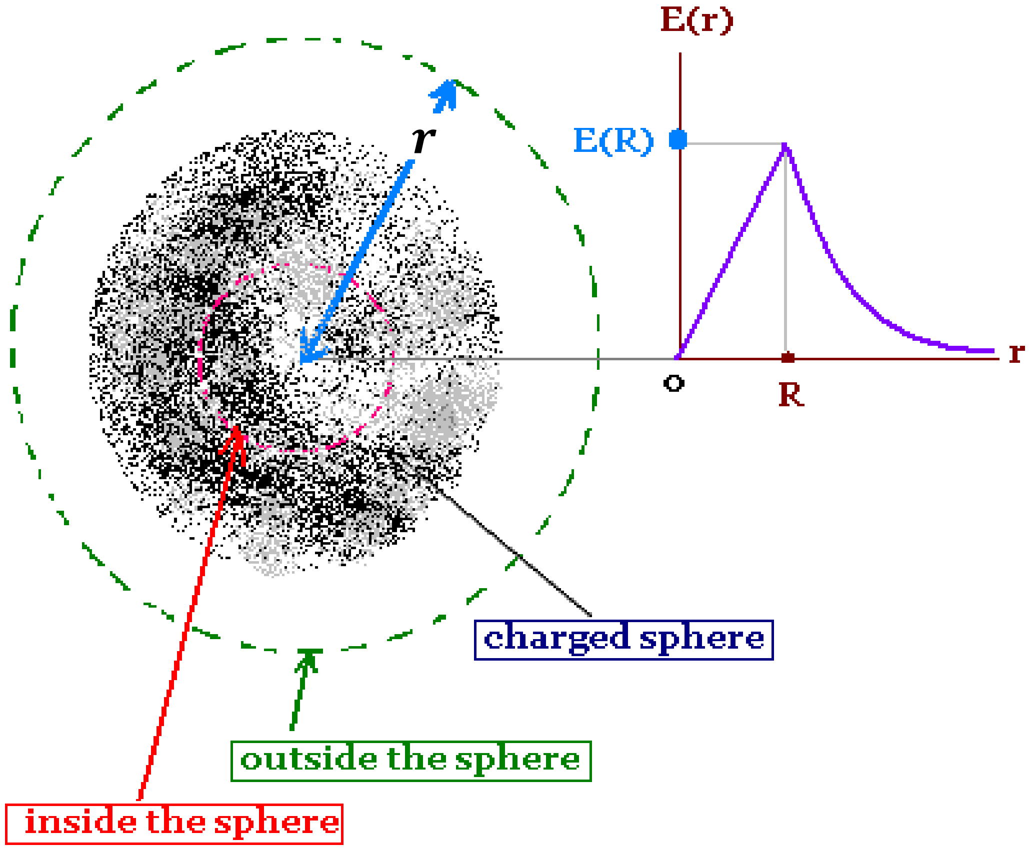
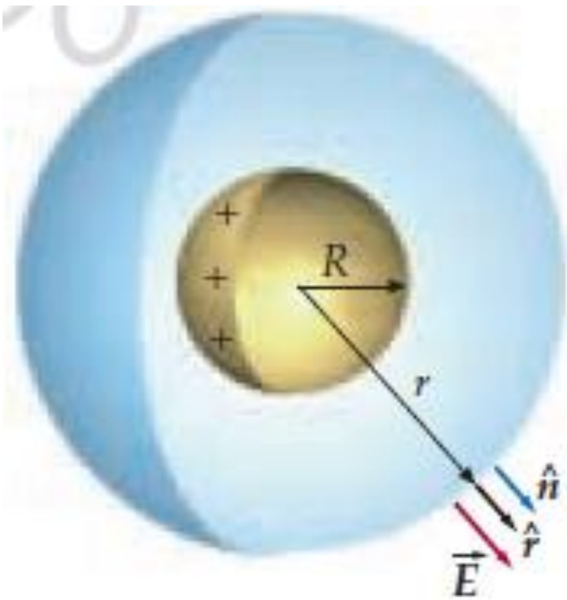


Fig.4.8 Electric field due to uniformly charged sphere

4.8.3 Electric field due to a charged spherical shell

A spherical shell is a hollow sphere of small thickness. Let the conductor have a radius R. We construct a Gaussian surface of radius r at a point p as shown in figure 4.9:-



$\Rightarrow$

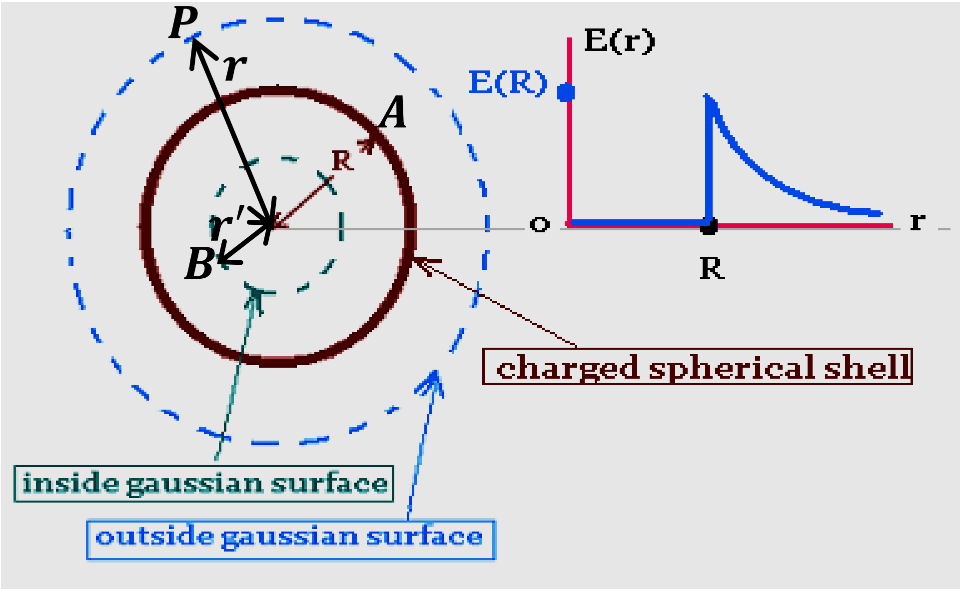


Fig.4.9 Gaussian surface for a charged hollow spherical shell

The charge distribution is spherical, then E is radial, and its magnitude depends only on the distance r from the center of the sphere.
Therefore the Gaussian surface is the spherical surface that has the same center as the spherical shell of charge, of radius $r' < R$ inside and $r > R$ outside.

i) At a point P outside the shell:

From Gauss's law,

$$\Phi_E = \oint_S \vec{E} \cdot \vec{dS} = \frac{q}{\varepsilon_0} \quad \ldots\ldots\ldots\ 4.16$$

Since $\vec{E}$ and $\vec{dS}$ are in the same direction, $\theta = 0^o$

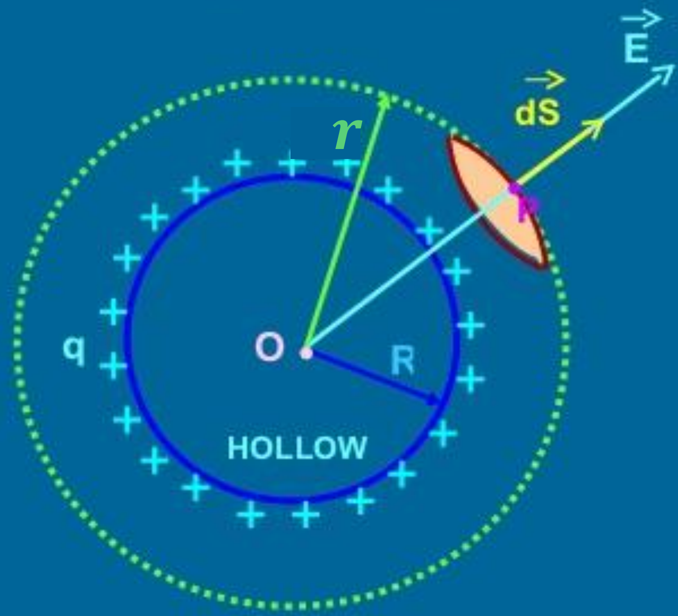
$$\therefore \quad \Phi_E = \oint_S E \, dS = \frac{q}{\varepsilon_0}$$

$$\text{or} \quad \Phi_E = E \oint_S dS = \frac{q}{\varepsilon_0}$$

$$E \times 4\pi r^2 = \frac{q}{\varepsilon_0} \quad \text{or} \quad \boxed{E = \frac{q}{4\pi\varepsilon_0 r^2}} \ldots 4.17$$

Since $q = \sigma \times 4\pi R^2$,

$$\therefore \quad \boxed{E = \frac{\sigma R^2}{\varepsilon_0 r^2}}$$



……… Gaussian Surface

Eqn. 4.17 is similar to the electric field due to a point charge. Thus outside the sphere, the electric field behaves as though it is due to a point charge (carrying all the charge of the shell) at the center of the shell.

ii) At a point A on the surface of the shell:

From Gauss's law,

$$\Phi_E = \oint_S \vec{E} \cdot \vec{dS} = \frac{q}{\varepsilon_0}$$

Since $\vec{E}$ and $\vec{dS}$ are in the same direction,

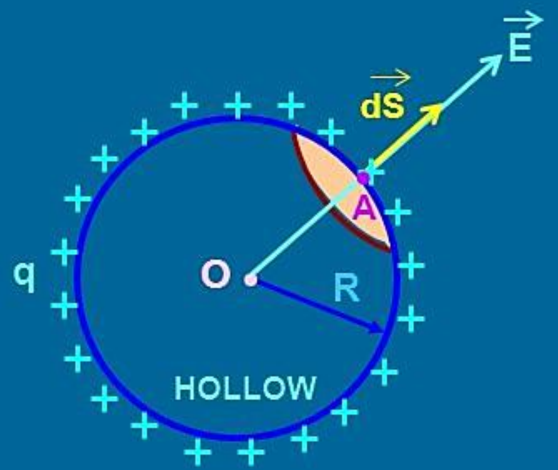
$\theta = 0^o$

$$\therefore \quad \Phi_E = \oint_S E \; dS = \frac{q}{\varepsilon_0}$$

$$\text{or} \quad \Phi_E = E \oint_S dS = \frac{q}{\varepsilon_0}$$

$$E \times 4\pi R^2 = \frac{q}{\varepsilon_0} \quad \text{or} \quad \boxed{E = \frac{q}{4\pi\varepsilon_0 R^2}} \quad \text{.........} \; 4.18$$

Since $q = \sigma \times 4\pi R^2$, $\quad \therefore \boxed{E = \frac{\sigma}{\varepsilon_0}}$



Electric field due to a uniformly charged thin spherical shell at a point on the surface of the shell is maximum.

iii) At a point B inside the shell:

From Gauss's law,

$$\Phi_E = \oint_S \vec{E} \cdot \vec{dS} = \frac{q}{\varepsilon_0}$$

Since $\vec{E}$ and $\vec{dS}$ are in the same direction,
$\theta = 0^o$

$$\therefore \quad \Phi_E = \oint_S E \, dS = \frac{q}{\varepsilon_0}$$

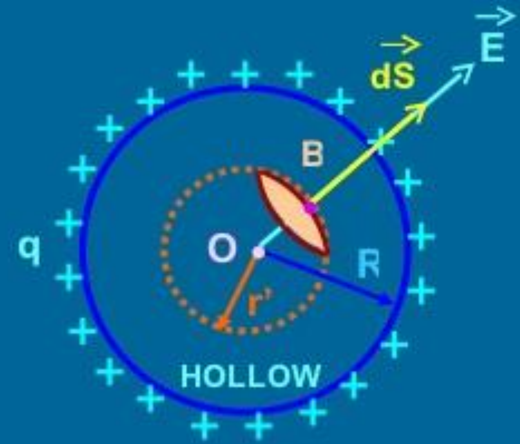
$$\text{or} \quad \Phi_E = E \oint_S dS = \frac{q}{\varepsilon_0}$$

$$E \times 4\pi r'^2 = \frac{q}{\varepsilon_0} \quad \text{or} \quad \boxed{E = \frac{0}{4\pi\varepsilon_0 \, r'^2}} \quad \ldots 4.19$$

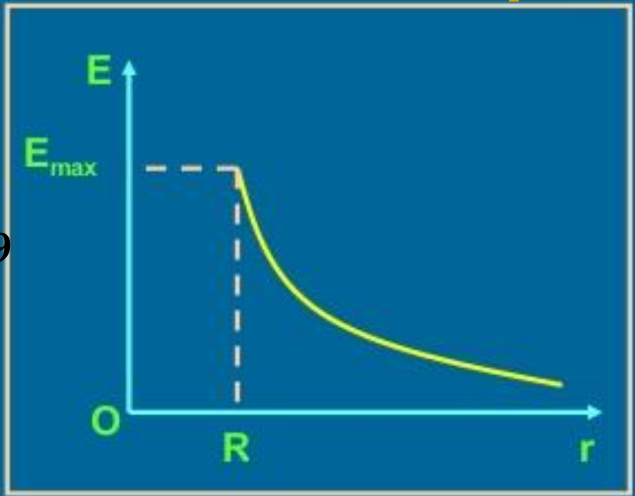
(since q = 0 inside the Gaussian surface)

$$\therefore \quad \boxed{E = 0} \quad \ldots\ldots\ldots 4.20$$

This property E = 0 inside a cavity is used for electrostatic shielding.



Gaussian surface inside the shell : $q = 0C$



$E = 0$, for both hollow and solid spherical conductor..

The electric field inside a conducting sphere is zero, so the potential remains constant at the value it reaches at the surface.

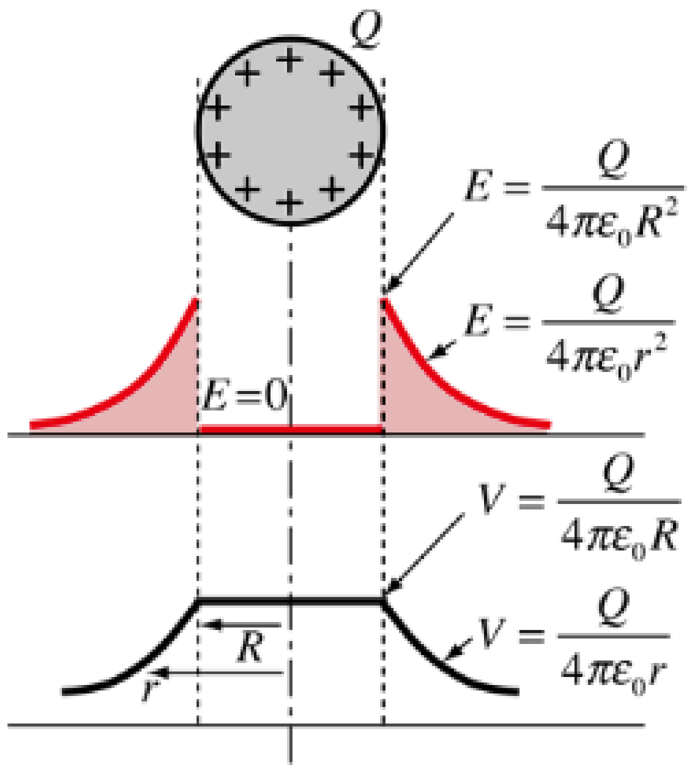


Fig.4.10 Electric field due to charged spherical shell

4.8.4 Field due to two concentric spherical conductors

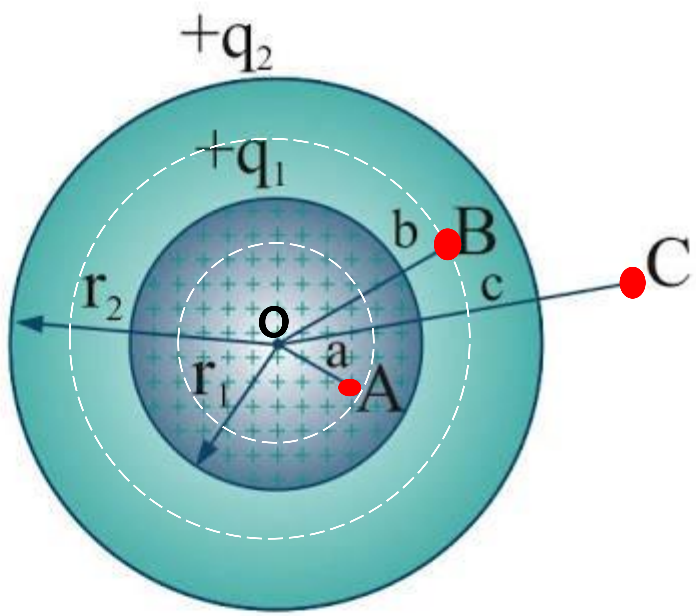


Fig. 4.11 Field due to two concentric spherical conductors/shells and their Gaussian surfaces (are in broken white lines)

So we find the electric fields at A, B and C, where $a = r_a, b = r_b$and $c = r_c$

Consider charges $+q_1$ and $+q_2$ uniformly distributed over the surfaces of two thin concentric metallic spherical shells of radii r_1 and r_2 respectively. Determine the electric field E at a points A, B, and C (i.e., E_A, E_B, and E_C).

Due to symmetric distribution of charge, the magnitude of E at all points on the Gaussian surface will be same and radially outward in direction. Thus, for any area element dS taken on the Gaussian surface, the field vector $\vec{E}$ and area vector $d\vec{S}$ both are parallel, therefore, $\vec{E}.d\vec{S} = E\, dS \cos 00 \; = \; E\, dS$

Hence, the flux through the entire Gaussian surface will be

$\emptyset_E = \int \vec{E}.d\vec{S} = \int E dS = E \int dS$, but $\int dS = 4\pi r^2$

$\emptyset_E = E(4\pi r^2)$ ……………………………… 4.21a

According to Gauss's Law of electrostatics: $\Phi = q/\varepsilon_0$ ……. 4.21b

By comparing Eqns. 4.21a and 4.21b:

$E(4\pi r^2) = \; q/\varepsilon_0 \Rightarrow E = \frac{q}{\varepsilon_0(4\pi r^2)}$ ……. 4.21c. Three cases arises here:

(i) Electric Field (E_A) at a Point A inside the inner shell of two thin concentric spherical shells $r_a \leq r_1$:

In this case the surface does not enclose any charge ($q_1 = q_2 = 0$), meaning there would be no charge within the Gaussian surface due to both conductors. Therefore, according to Gauss's law $q = 0$. Thus, $E = 0$ at a point inside the inner shell.

(ii) Electric Field (E_B) at a point B between the two shells of two thin concentric spherical shells $r_1 < r_b \leq r_2$:

In this case, consider that the point B at which the electric field is to be determined lies between r_1 and r_2. Therefore, the net charge enclosed by the Gaussian sphere is only the charge residing on inner shell, which is q_1 alone. Therefore, from Eqn. 4.21c:

$$E = \frac{q_1}{\varepsilon_0 (4\pi {r_b}^2)} \quad \text{..........................} \quad 4.22$$

(iii) Electric Field (E_c) at a point C outside the outer shell of two thin concentric spherical shells:

In this situation, consider that the point C lies outside the outer shell at a distance r_c from the centre O, such that $r_c > r_2$. In this case the net charges enclosed by the Gaussian surface will be the sum of charges on both shells, that is, $Q = q_1 + q_2$. Thus from Eqn. 4.21c,

$$E = E_1 + E_2 = \frac{q_1}{\varepsilon_0(4\pi r_c{}^2)} + \frac{q_2}{\varepsilon_0(4\pi r_c{}^2)}$$

$$= \frac{q_1+q_2}{\varepsilon_0(4\pi r_c{}^2)} = k\frac{q_1+q_2}{r_c{}^2} \quad \text{.................} \quad 4.23$$

TAKE NOTE

If two spherical shells have charges equal and opposite that is, if one shell has $+q$ charge and the other has $-q$. In this situation, inside the inner spherical shell (point A) and outside the outer spherical shell (point C) the electric field is zero.

4.8.5 Field inside a uniformly charged wire (line charge)

We construct a Gaussian surface in the form of a circular cylinder of radius r and length l with the wire at its axis. E is constant over the cylindrical surface by Gauss Law as figure 4.12 shows:-

$$\oint \vec{E} \cdot \vec{ds} = \oint E\, ds = E \oint ds = E.2\pi r l$$

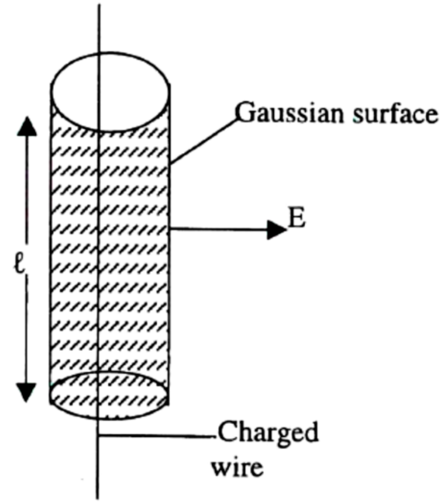


Fig 4.12 Gaussian surface for a uniformly charged wire

If the thickness is neglected, then density of charge is linear, λ.

There is no flux through the two end faces or circular caps because E is tangential to the faces, and no component is perpendicular to the faces.

$$\therefore \vec{E}.\vec{ds} = 0.$$

$$\text{total outward flux} = E \cdot 2\pi r l = \frac{4\pi k}{} \cdot \lambda l$$

$$\therefore E = k\frac{2\lambda l}{rl} = 2k\frac{\lambda}{r} \quad \text{.......................... 2.24}$$

4.8.6 Field of charged conducting plate

An infinite conducting plate is one having thickness that allows the charge to migrate to separate sides of the plate in response to the repulsive electrostatic forces between them. If the charge density on each side of the conducting plate is the same as the charge density of the infinite sheet, then the total charge enclosed would be $2\rho A$.

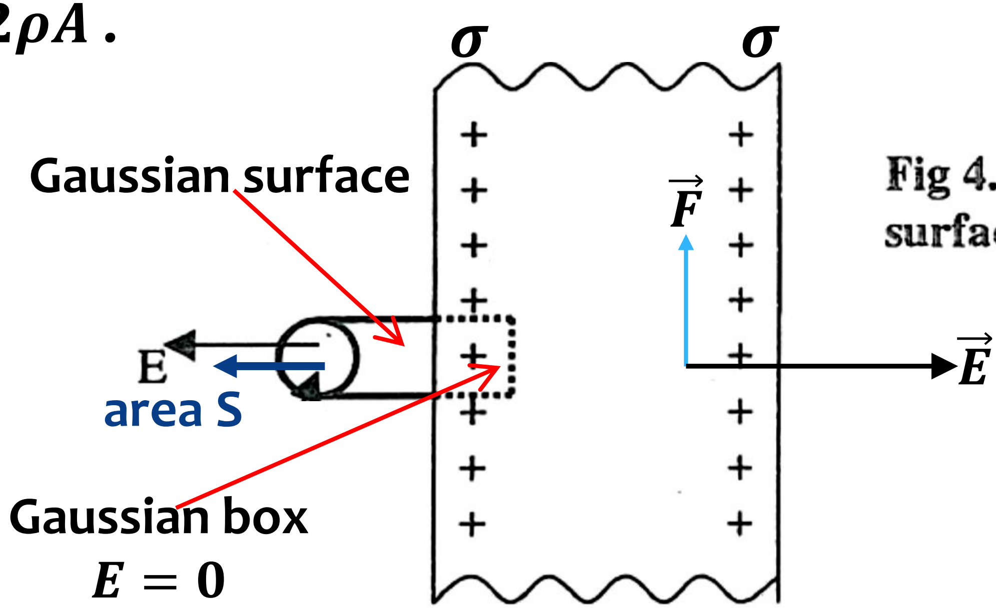


Fig 4.13 cylindrical Gaussian surface in place conducting plate.

No lines of force cut the side walls of cylinder since they are parallel to the side walls, so

$E = 0$ (perp to these walls) **i.e., $\theta = 90^0$. Same as inside the conductor.**

Flux through the cylinder = E x A **Recall:** $\oint_s \vec{E} \cdot d\vec{S} = EAcos0^0 = EA$

$$\emptyset_E = \oint_s \vec{E}\,.\,d\vec{S} = EAcos0^0 = EA, \text{ and}$$

by Gauss's Law: $\emptyset_E = \frac{q}{\varepsilon_0} = 4\pi kq$

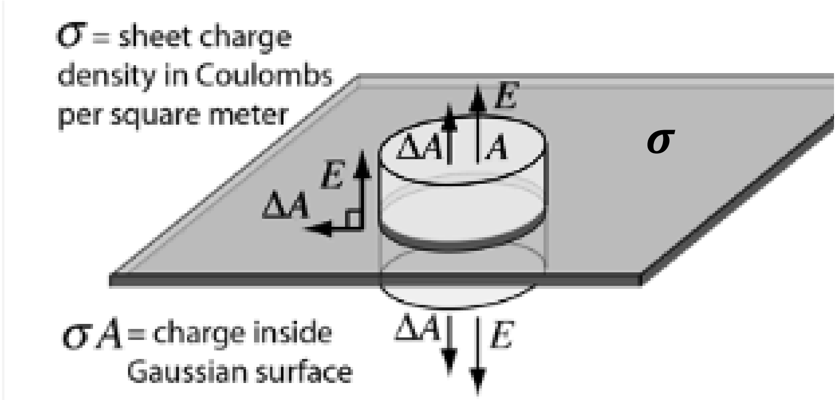
$$\Rightarrow EA = 4\pi kq = 4\pi k\sigma A$$

$$\therefore E = 4\pi k\sigma = \frac{\sigma}{\varepsilon_0} \qquad \cdots\cdots\cdots\cdots\cdots\cdots \quad 4.25$$

where

$$k = \begin{cases} 1 & \text{cgs units} \\ \dfrac{1}{4\pi\varepsilon_0} & \text{MKS units} \end{cases}$$

4.8.7 Field of an infinite plane sheet of charge



Recall: $\emptyset_E = EA$ and $\emptyset_E = q/\varepsilon_0$

$\Rightarrow \emptyset_E = E2A$ and $\emptyset_E = \sigma A/\varepsilon_0$

$\Rightarrow \emptyset_E = E2A = \frac{\sigma A}{\varepsilon_0}$

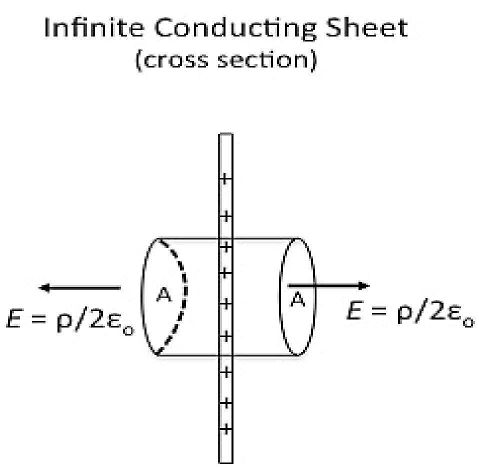
$\Rightarrow E = \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{2} 4\pi K = 2\pi K\sigma \ldots 4.26$

Fig. 4.14 Gaussian surface for an infinite plane sheet of charge

Consider an infinite sheet of charge. The charge is assumed to be uniformly distributed over the plane with a surface charge density of σ as shown above. The electric field will be perpendicular to the surface. Therefore only the ends of a cylindrical Gaussian surface will contribute to the electric flux.

In this case a cylindrical Gaussian surface perpendicular to the charge sheet is used. The resulting field is half that of a conductor at equilibrium with this surface charge density.

TAKE NOTE



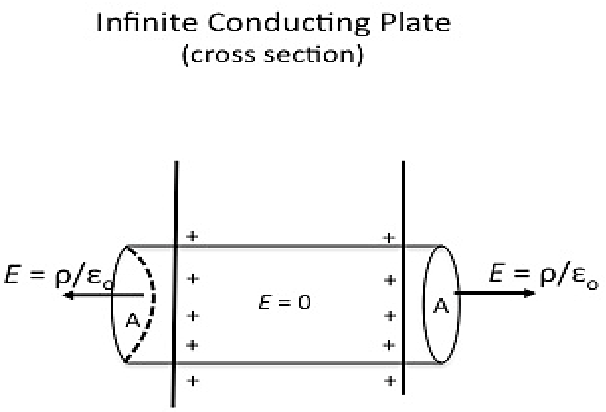
ρ = charge density

Total enclosed charge = ρA

Total flux $\Phi = 2EA = \rho A/\varepsilon_0$

Electric field $E = \rho/2\varepsilon_o$

Fig. 4.14



ρ = charge density

Total enclosed charge $= 2\rho A$

Total flux $\Phi = 2EA = 2\rho A/\varepsilon_0$

Electric field $E = \rho/\varepsilon_o$

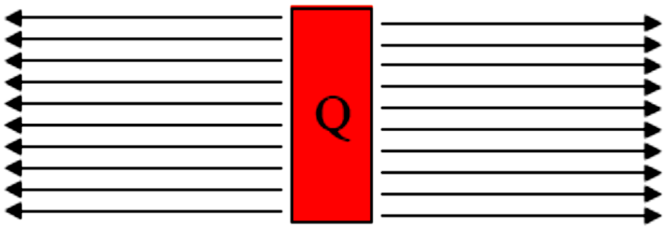
Fig. 4.13

An infinite plane/sheet of charge is as being one atom/molecule thick. Thus there are not two separate sides of charge. The total enclosed charge is ρA.

An infinite conducting plate is one having thickness that allows the charge to migrate to separate sides of the plate in response to the repulsive electrostatic forces between them. If the charge density on each side of the conducting plate is the same as the charge density of the infinite sheet, then the total charge enclosed would be $2\rho A$.

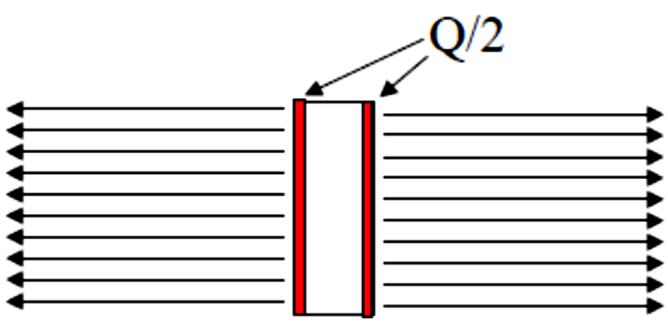
On the other hand, if the same quantity of charge on the infinite sheet Fig. 4.14 were placed on the conducting plate Fig.413, the charge would split up making the density on each side of the plate $(\rho/2)$ and the total enclosed charge ρA, giving the same result as the infinite sheet of charge.

conti. TAKE NOTE



$$E = \frac{\sigma}{2\varepsilon_0} = \frac{Q}{2A\varepsilon_0}$$

Insulating plate: charge distributed homogeneously.



$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{2A\varepsilon_0}$$

Conducting plate: charge distributed on the outer surfaces.

4.8.8 Field due to two infinite parallel plane sheets

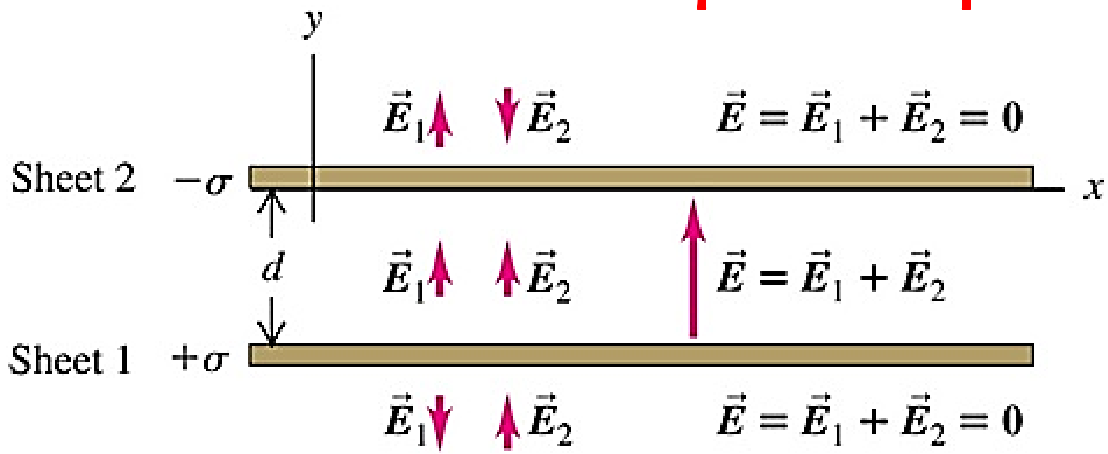


Fig. 4.15 Two infinite plane sheets are placed parallel to each other, separated by a distance d. It's an important case of parallel plate capacitor

The lower sheet has a uniform positive surface charge density $+\sigma$, and the upper sheet has a uniform negative surface charge density $-\sigma$ with the same magnitude (i.e., $E_1 = E_2$).

Recall electric field of an infinite sheet Eqn.4.26: $E = \dfrac{\sigma}{2\varepsilon_0} = 2\pi K\sigma$

Total electric field between the two sheets are added together to give:

$$E = E_1 + E_2 = \left[\frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0}\right] = [2\pi K\sigma + 2\pi K\sigma]$$

$\Rightarrow E = \dfrac{\sigma}{\varepsilon_0} = 4\pi K\sigma$ .... **4.27 (TAKE NOTE: same as Eqn. 4.25)**

The electric field at the sides cancels ($E = \mp E_1 + \pm E_2 = 0$).

Example 4.1

A hollow metallic spherical shell 1m radius is charged uniformly with a positive charge of 4 micro-coulombs.

a) What is the electric field intensity at the center of the sphere?
b) What is the electric potential at the center of the sphere
c) What is the magnitude of the electric field intensity at a point 2m from the center of the center of the sphere?
e) What is the electric potential at a point 2m from the center of the sphere?

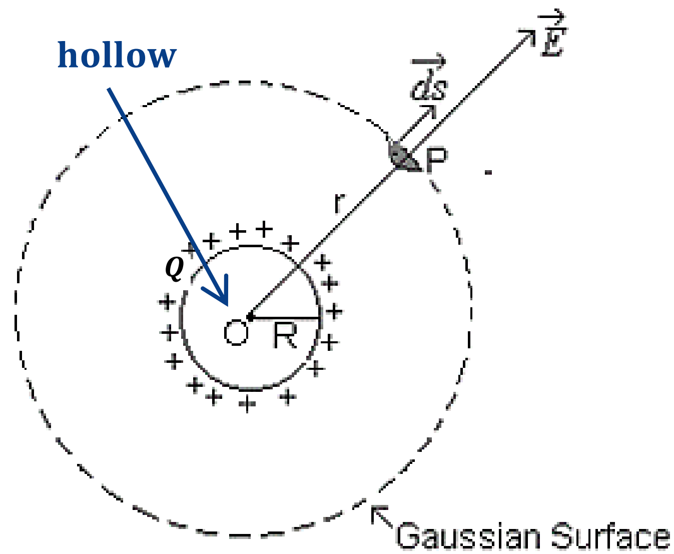
Solution

$R = 1m, Q = 4\mu C, E = ?, V = ?$

a) E=0 at the center of the sphere, since there is no charge in the space inside the sphere. i.e., at $r < R, Q = 0$
b) The electric potential at the center of the sphere is the same as at its surface. It is constant inside the sphere.

$$V = E.R = ER = \frac{Q}{4\pi R^2}\, R = \frac{Q}{4\pi R}$$

$$= 9x10^9 \frac{Nm^2}{C^2} \; \frac{4x10^{-6}C}{1} = 36x\ 10 \text{volts}$$



c) $r > R$

$$E = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2} = 9x10^9 \frac{Nm^2}{C^2}\; \frac{4\,x10^{-6}C}{2^2m^2} = 9x10^3\ \text{volts}/_{m}$$

$$\text{d) } V = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r} = 9X10^9 \frac{Nm^2}{C^2} \cdot \frac{4x10^{-6}C}{2m}$$

$$= 18x10^3 \text{ Volts}$$

Example 4.2

Two large metal plates of area $0.5m^2$ face each other and are separated by 10cm and carry equal and opposite charges on their inner surfaces. If E between the plates is $60 ^N/_C$, find the charge on the plates

Solution

Area of plate = $1m^2$

$$E = E_1 + E_2 = \frac{\sigma}{\varepsilon_0}$$

which is independent of separation, and hence constant for the distance 10cm.

$\sigma = \frac{\text{charge}}{\text{Area}} = \frac{q}{A}$ **, substitute in the above equation.**

$$\Rightarrow E = \frac{q}{A\varepsilon_o}$$

$$\therefore 60 = \frac{q}{0.5x8.85x10^{-12}} \quad \Rightarrow q = 0.5x60x8.85x10^{-12}C$$

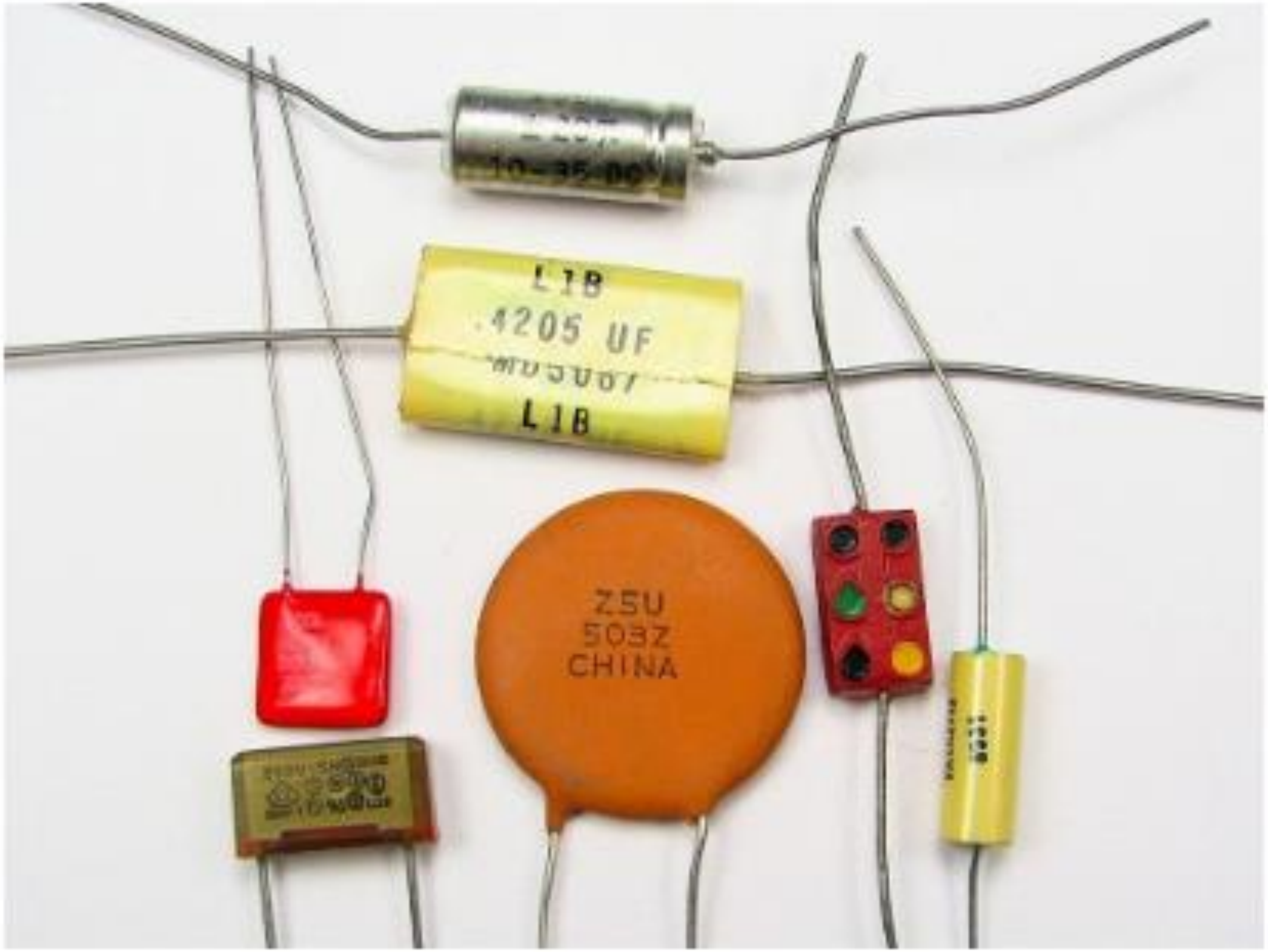
$$q = 265.5x10^{-12}C = 2.655x10^{-12}C$$

4.9 Capacitors and dielectric

4.9.1 Introduction

Capacitance is the ability of a system to store an electric charge/electrical energy. Yourself you can store some electrical energy and therefore have capacitance. The devices called capacitors are designed to store large amounts of electrical energy and are primarily made of ceramic, glass, or plastic, depending upon purpose and size. It consists of two conducting surfaces (usually metal plates) separated by an insulating/dielectric material like air, rubber, or paper. Capacitors In this section we will discuss various designs of capacitors, the spherical, cylindrical and parallel-plate types, since capacitance depends entirely on the geometry of the capacitor. Derive their capacitance, energy, polarization and susceptibility equations.

conti. Introduction

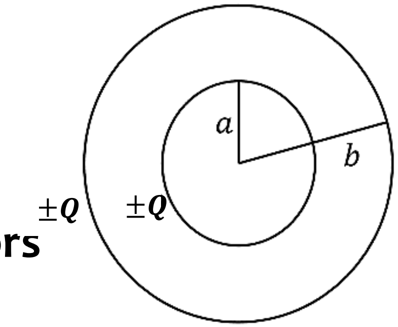


Types of capacitors according to shape

We can design the capacitor of different geometry. The more commonly used capacitors are:

a) Spherical capacitor
b) Cylindrical capacitor
c) Parallel-plate capacitor

SPHERICAL CAPACITOR



i) Spherical capacitor is formed from two concentric spherical conducting shells separated by a vacuum. The capacitance can be obtained by evaluating the voltage difference between the conductors for a given charge on each.

By applying Gauss's law to an charged conducting sphere:

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} \quad \cdots\cdots \quad 4.28$$

The voltage between the spheres can be found by integrating the electric field along a radial line , so the angle between $\vec{E}$ and $\vec{r}$ is zero:

$$\Delta V = \frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{1}{r^2}dr = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right] \Rightarrow \Delta V = Q\,\frac{[\frac{1}{a} - \frac{1}{b}]}{4\pi\varepsilon_0} = Q\,\frac{1}{C} \Rightarrow C = \frac{Q}{\Delta V} \quad \cdots\cdots \quad 4.29a$$

As the spheres are brought very close together, the charges on them remain constant and potential difference is considerable reduced: $\Rightarrow C = \frac{Q}{V}$ ……. 4.29b

$$\Rightarrow CV = Q \Rightarrow Q \propto V$$

Eqn. 4.29a ($\Delta V = Q\,\frac{1\left[\frac{1}{a}-\frac{1}{b}\right]}{4\pi\varepsilon_0}$) can be rewritten as $\Rightarrow \Delta V = kQ\left[\frac{1}{a}-\frac{1}{b}\right]$

$$\therefore C = \frac{Q}{V} = \frac{Q}{kQ\left[\frac{1}{a}-\frac{1}{b}\right]} = \frac{1}{k}\frac{1}{\left[\frac{b-a}{ab}\right]} = \frac{1}{k}\frac{ab}{b-a} \quad \ldots\ldots\ldots\ldots\ldots\ldots 4.30$$

Hence the capacitance C depends only on their radii.

The SI units for capacitance $\Rightarrow C = \frac{Q}{V}$

unit of $C = \frac{\text{columb(c)}}{\text{volt(V)}} = Farad(F)$

Hence, the capacitance of a capacitor is numerically equal with the charge raise the potential difference between the plates of the capacitor by 1 volt.

One Farad Capacitance

We have, $q = CV \quad \Rightarrow \quad C = \frac{q}{V}$

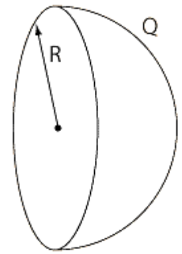
$$\text{for}, q = 1C, V = 1volt$$

$$C = \frac{1C}{1V} = 1Farad$$

Hence, the capacitance of a capacitor is said to be 1 Farad

Hence, the capacitance of a capacitor is said to be 1 Farad if 1 Columb charge is required to increase the potential difference between its plates by 1 volt.

ii) Isolated spherical capacitor



An isolated charged conducting sphere has capacitance. Applications for such a capacitor may not be immediately evident, but it does illustrate that a charged sphere has stored some energy as a result of being charged. Taking the concentric sphere capacitance expression of Eqn. 4.29a: $C = \dfrac{4\pi\varepsilon_0}{\left[\dfrac{1}{a} - \dfrac{1}{b}\right]}$

and taking the limits $a \rightarrow R$ and $b \rightarrow \infty$ gives $C = 4\pi\varepsilon_0 R$ ……………………………… 4.31a

TAKE NOTE

The above equation can be confirmed by examining the potential of a charged conducting sphere: $V = \dfrac{Q}{4\pi\varepsilon_0 r}$ so at the surface $C = \dfrac{Q}{V} = 4\pi\varepsilon_0 R$ ……………………………… 4.31b

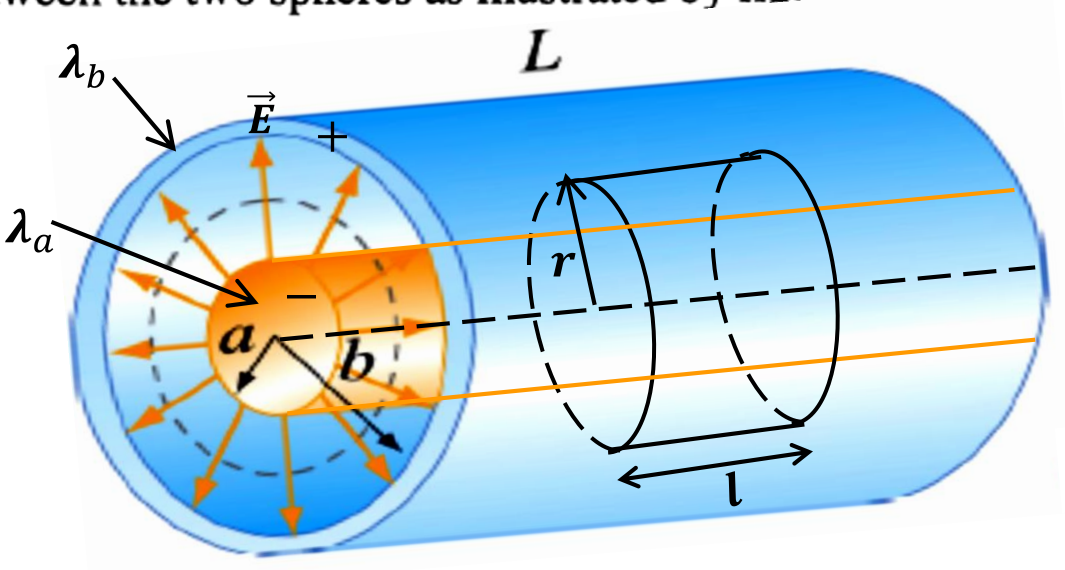
In CGS-system, $4\pi\epsilon_0 = 1$ and $C = R$

So, in CGS_system, the capacitance of the isolated charged sphere is numerically equal to its radius.

If we have an isolated charged sphere of radius a, its capacitance is $C' = 4\pi\epsilon_0 a$. Since $\frac{ab}{b-a} > a$, it follows that C>C'. Therefore, the arrangement of the two spherical shells leads to increase the capacitance of a spherical conductor.

Cylindrical capacitor

Such a capacitor consists of two coaxial cylinders of radius a and b and length L. To calculate the capacitance, we construct a Gaussian surface shown by dotted line having a radius r, between the two spheres as illustrated by figure 4.20



The electric field is non-vanishing only in the region a $< r < b$ and λ is linear charge density.

To calculate the capacitance, we first compute the electric field everywhere. Due to the cylindrical symmetry of the system, we choose our Gaussian surface to be a coaxial cylinder with length $\ell < L$ and radius r where $a < r < b$. Using Gauss's law, we have

$$\oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = E(2\pi r\ell) = \frac{\lambda\ell}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{\lambda}{2\pi\varepsilon_0 r} \quad \text{...............} \quad 4.32$$

where $\lambda = Q/L$ is the charge per unit length.

Notice that the electric field is non-vanishing only in the region $a < r < b$. For $r < a$, the enclosed charge is $q_{\text{enc}} = 0$ since any net charge in a conductor must reside on its surface. Similarly, for $r > b$, the enclosed charge is $q_{\text{enc}} = \lambda\ell - \lambda\ell = 0$ since the Gaussian surface encloses equal but opposite charges from both conductors.

The potential difference is given by

$$\Delta V = V_b - V_a = -\int_a^b E_r\, dr = -\frac{\lambda}{2\pi\varepsilon_0}\int_a^b \frac{dr}{r} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right) \quad \cdots\cdots\cdots\cdots \quad 4.33$$

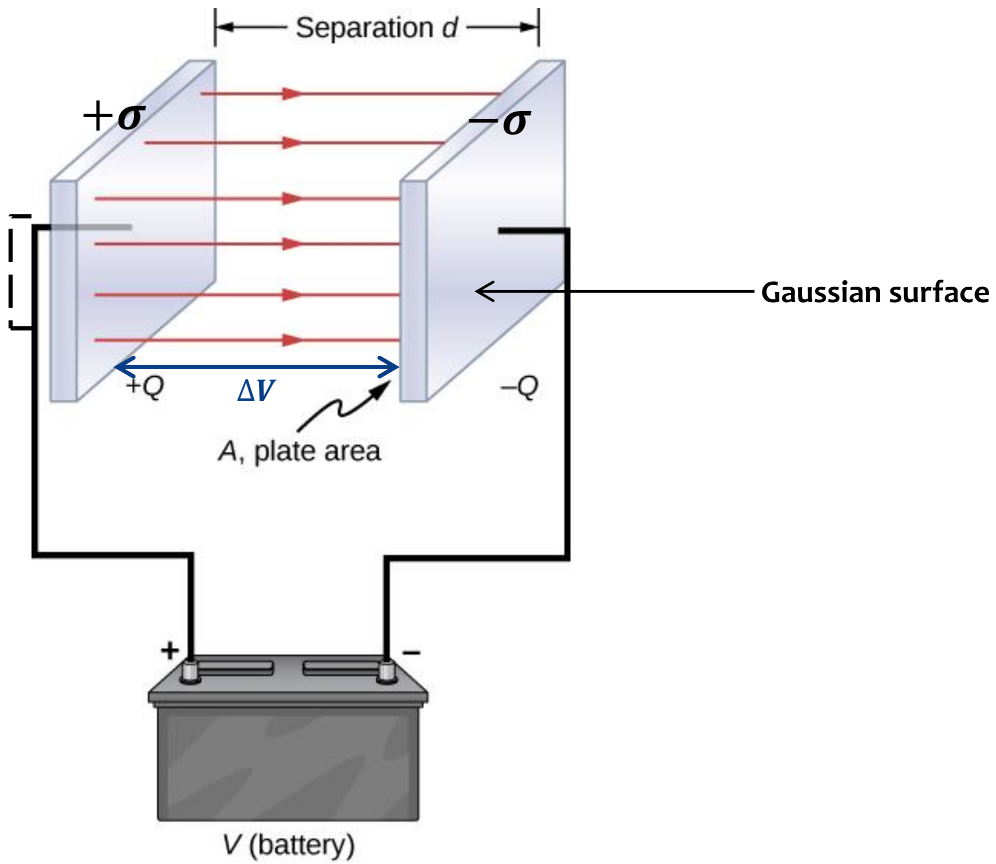
where we have chosen the integration path to be along the direction of the electric field lines. As expected, the outer conductor with negative charge has a lower potential. This gives

$$\boxed{C = \frac{Q}{|\Delta V|} = \frac{\lambda L}{\lambda \ln(b/a)/2\pi\varepsilon_0} = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}} \quad \cdots\cdots\cdots\cdots\cdots \quad 4.34$$

Once again, we see that the capacitance C depends only on the geometrical factors, L, a and b.

Parallel-plate capacitor

Is a system composed of two identical parallel-conducting plates separated by a distance.



The electric field will be uniform and perpendicular to the plates everywhere except near the edges. We can neglect the edges. If σ is the surface charge density, we construct a Gaussian surface as shown in figure 4.21. We found that the field intensity is given as:

Surface charge density σ on the plates: $\sigma = Q/A$ …………………… 4.35

When d is small, the electrical field between the plates is fairly uniform and that its magnitude: $E = \frac{\sigma}{\varepsilon_0}$ ………………………….4.36

Since the electrical field $\vec{E}$ between the plates is uniform, the potential difference between the plates can be expressed in terms of the work done on a positive test charge $+Q$ when it moves from the positive to the negative plate: $V = \text{work done/charge} = \frac{Fd}{Q} \Rightarrow V = Ed = \frac{\sigma}{\varepsilon_0} d = \frac{Q}{A\varepsilon_0} d$ ….. 4.37

Therefore $C = \frac{Q}{V} = \frac{Q}{\frac{Q}{A\varepsilon_0} d} = \frac{QA\varepsilon_0}{Qd} = \varepsilon_0 \frac{A}{d} = \frac{A}{4\pi k d}$ ………….4.38

From the above equation that capacitance is a function *only of the* geometry and what material fills the space between the plates (in this case, vacuum) of this capacitor. This is true for all capacitors: The capacitance is independent of Q or V. If the charge changes, the potential changes correspondingly so that Q/V remains constant. Also $C \propto \frac{1}{E}$.

EXAMPLE

a. What is the capacitance of an empty parallel-plate capacitor with metal plates that each have an area of $1.00\,m^2$, separated by 1.00 mm?
b. How much charge is stored in this capacitor if a voltage of $3.00 \times 10^3 V$ is applied to it?

SOLUTION

a) $$C = \epsilon_0 \frac{A}{d} = \left(8.85 \times 10^{-12} \frac{F}{m}\right) \frac{1.00\,m^2}{1.00 \times 10^{-3} m} = 8.85 \times 10^{-9} F = 8.85\,nF.$$

b) $$Q = CV = (8.85 \times 10^{-9} F)(3.00 \times 10^3 V) = 26.6\,\mu C.$$

EXAMPLE

Suppose you wish to construct a parallel-plate capacitor with a capacitance of 1.0 F. What area must you use for each plate if the plates are separated by 1.0 mm?

SOLUTION

$$A = \frac{Cd}{\epsilon_0} = \frac{(1.0\,F)(1.0 \times 10^{-3} m)}{8.85 \times 10^{-12} F/m} = 1.1 \times 10^8\,m^2.$$

4.9.3 The energy of a charged capacitor

Derivation: Energy stored in a capacitor

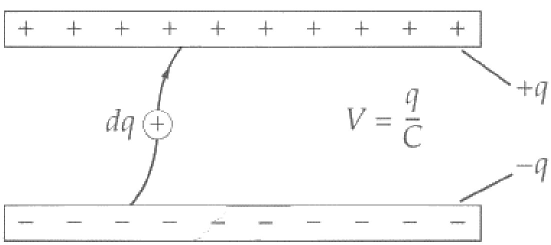
$$dU = V\,dq = \frac{q}{C}\,dq$$

$$W = \int_0^Q dq\,|\Delta V| = \int_0^Q dq\,\frac{q}{C} = \frac{1}{2}\frac{Q^2}{C} \quad \text{.... 4.39b}$$

$$U = \int dU = \int_0^Q \frac{q}{C}\,dq = \frac{1}{2}\frac{Q^2}{C}$$

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2 \quad \text{.... 4.39a}$$

or $\mathbf{U} = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q\,Q}{C} = \frac{1}{2}\frac{QQ}{(Q/\Delta V)} = \frac{1}{2}Q\Delta V$



Total work (W) needed to charge a capacitor is the electrical potential energy (U) stored in it, or $U = W$, (see Eqns.4.39).
When the charge is expressed in coulombs, potential is expressed in volts, and the capacitance is expressed in farads, this relation gives the energy in joules.

The energy supplied to a capacitor in charging it is **stored** by the capacitor and released when it discharges. It reasonable to assume that the energy stored by a capacitor is stored in the electric field since the electric field increases as Q or Δ V increase

This work done is stored up in the form of potential energy and supplies the energy necessary to drive the current when the capacitor is discharged.

Since energy stored in a capacitor is U or U_E, energy density (u_E) stored in a vacuum between the plates of a charged parallel-plate capacitor can be determined.

We just have to divide U by the volume Ad of space between its plates and take into account that for a parallel-plate capacitor, we have $E = \frac{\sigma}{\varepsilon_0}$ and $\mathrm{C} = \frac{\varepsilon_0 A}{d}$: $\boxed{u_E = \frac{U_E}{\text{Volume}} = \frac{1}{2}\,\varepsilon_0 E^2}$

$$\Rightarrow \; u_E = \frac{U}{Ad} = \frac{1}{2}\frac{Q^2}{C}\frac{1}{Ad} = \frac{1}{2}\frac{Q^2}{\epsilon_0 A/d}\frac{1}{Ad}$$

$$= \frac{1}{2}\frac{1}{\epsilon_0}\left(\frac{Q}{A}\right)^2 = \frac{\sigma^2}{2\epsilon_0} = \frac{(E\epsilon_0)^2}{2\epsilon_0} = \frac{\epsilon_0}{2}E^2 \quad \text{.................} \quad 4.40$$

4.9.4 Combinations of capacitances

Capacitors may be connected by conducting wires into a network such as the parallel and series combination.

a) Capacitor In Parallel

Combination of capacitors are used in practice to obtain a variety of capacitance values from a small number of capacitors. If we want the combination to have a total capacitance greater than the individual values, the capacitors are connected in parallel as shown in Fig. 4.22

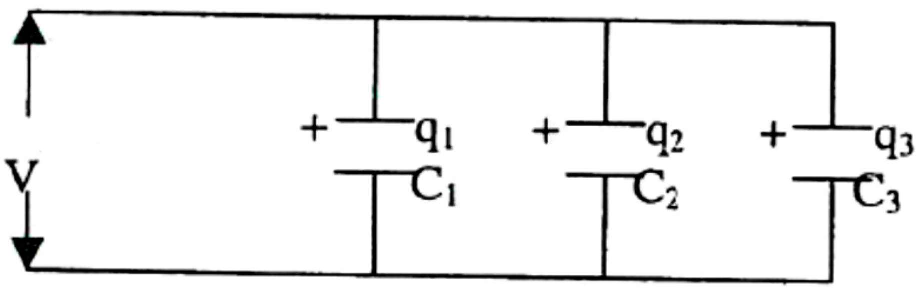


Fig 4.22 Capacitors in parallel combination

The three capacitors C_1, C_2, C_3, are connected in parallel across a common potential difference V. Applying the relation $q = CV$ to each capacitor yields. The following:

$q_1 = C_1V \qquad q_2 = C_2V \qquad q_3 = C_3V$

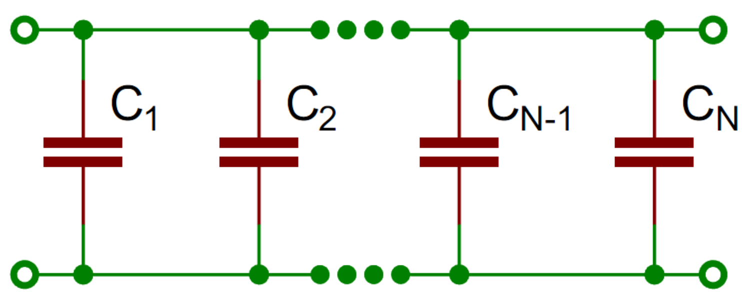
Total charge q on the combination is

$$q = q_1 + q_2 + q_3 = (C_1 + C_4 + C_3)\,V \quad \ldots\ldots\ldots\ldots\ldots \quad 4.41$$

The equivalent capacitance C is therefore $$C = \frac{q}{V} = C_1 + C_2 + C_3 + \ldots \quad \ldots\ldots\ldots \quad 4.42$$

The result can be extended to any number of parallel – connected capacitors. **See Fig. below**

Total capacitance = sum of individual capacitances



$$C_{Tot} = C_1 + C_2 + \ldots + C_{N-1} + C_N$$

b) Capacitors In Series

It may be desired to have a capacitor combination with a total capacitance less than the individual capacitances. The capacitors may be then be connected in series as shown in figure 4.25 below.

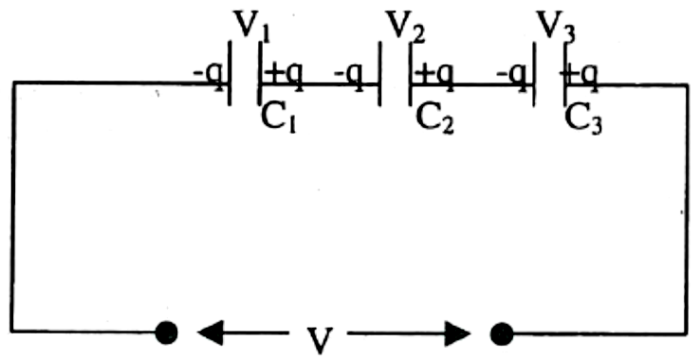


Fig 4.23 Capacitors in series combination

The magnitude q is the charge on each plate and must be the same. This is true because the net charge on the part of the circuit enclosed by + q and –q must be zero. Hence, charge on the combination is equal to the charge on each capacitor as given below:-

$$q = q_1 = q_2 = q_3$$

The potential difference across the individual capacitors are

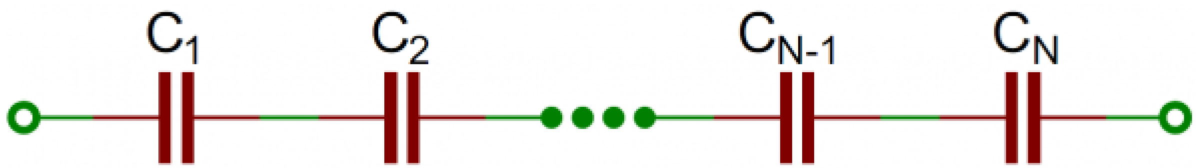
$$V_1 = \frac{q}{C_1} \qquad V_2 = \frac{q}{C_2} \qquad V_3 = \frac{q}{C_3}$$

Thus, the total potential difference is

$$V = V_1 + V_2 + V_3 = \frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3} = \frac{q}{C} \quad \ldots\ldots\ldots \; 4.43$$

$$\therefore \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \quad \ldots\ldots \; 4.44$$

Where C is the effective capacitance of the series combination



$$\frac{1}{C_{Tot}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_{N-1}} + \frac{1}{C_N}$$

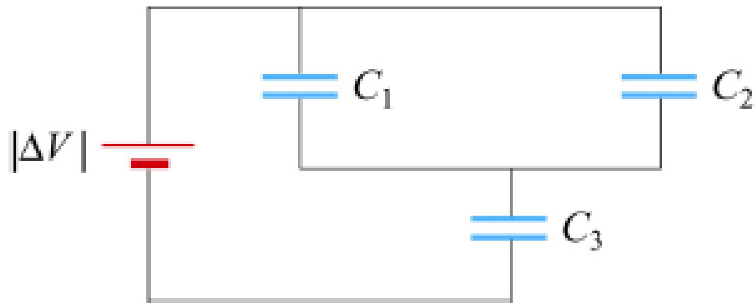
The total capacitance of *N* capacitors in series is the inverse of the sum of all inverse capacitances.

If you only have two capacitors in series, you can use the "product-over-sum" method to calculate the total capacitance:

$$C_{Tot} = \frac{C_1 C_2}{C_1 + C_2}$$

EXAMPLE

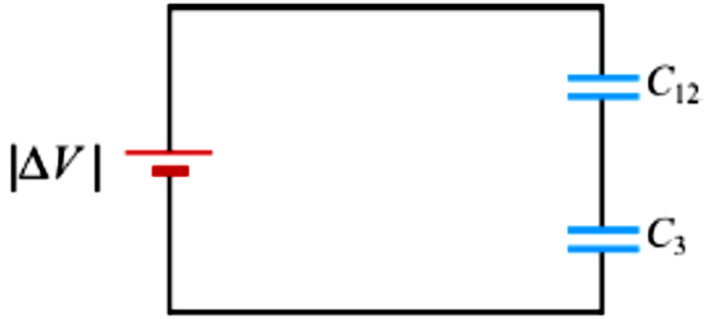
Find the equivalent capacitance for the combination of capacitors shown in Figure

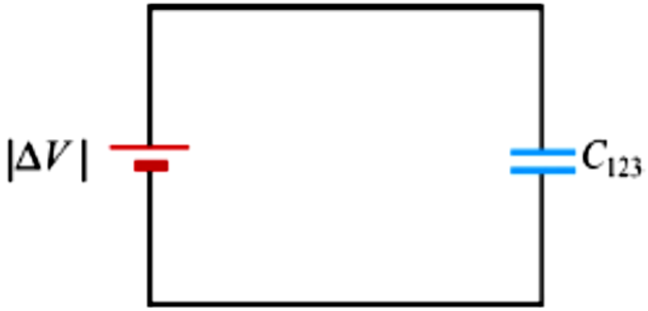


Solution:

Since C_1 and C_2 are connected in parallel, their equivalent capacitance C_{12} is given by

$$C_{12} = C_1 + C_2$$





$$\frac{1}{C_{123}} = \frac{1}{C_{12}} + \frac{1}{C_3}$$

OR

$$\boxed{C_{123} = \frac{C_{12}C_3}{C_{12} + C_3} = \frac{(C_1 + C_2)C_3}{C_1 + C_2 + C_3}}$$

4.9.5 Gauss Law and Dielectrics

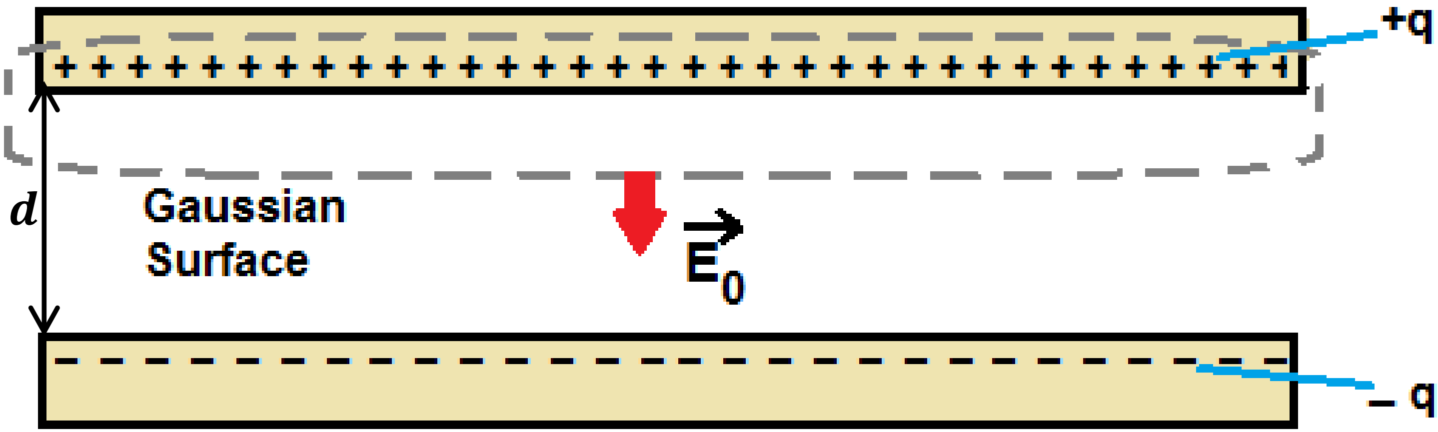
There is difficulty of storing a large amount of charge ($Q = C.V$) in capacitors. If d is made smaller to produce a larger capacitance ($C = \varepsilon_0 . A/d$), then the maximum voltage must be reduced proportionally ($Q(\text{doesn't change}) = \uparrow C. \downarrow V$) to avoid breakdown (since$E = V/d \Rightarrow E = \downarrow V/\downarrow d$). An important solution to this difficulty is to put an insulating material, called a *dielectric*, between the plates of a capacitor and allow d to be as small as possible. Not only does the smaller d make the capacitance greater, but many insulators can withstand greater electric fields than air before breaking down.

To see how this happens, suppose a capacitor has a capacitance C_0 when there is no material between the plates. When a dielectric material is inserted to completely fill the space between the plates, the capacitance increases to

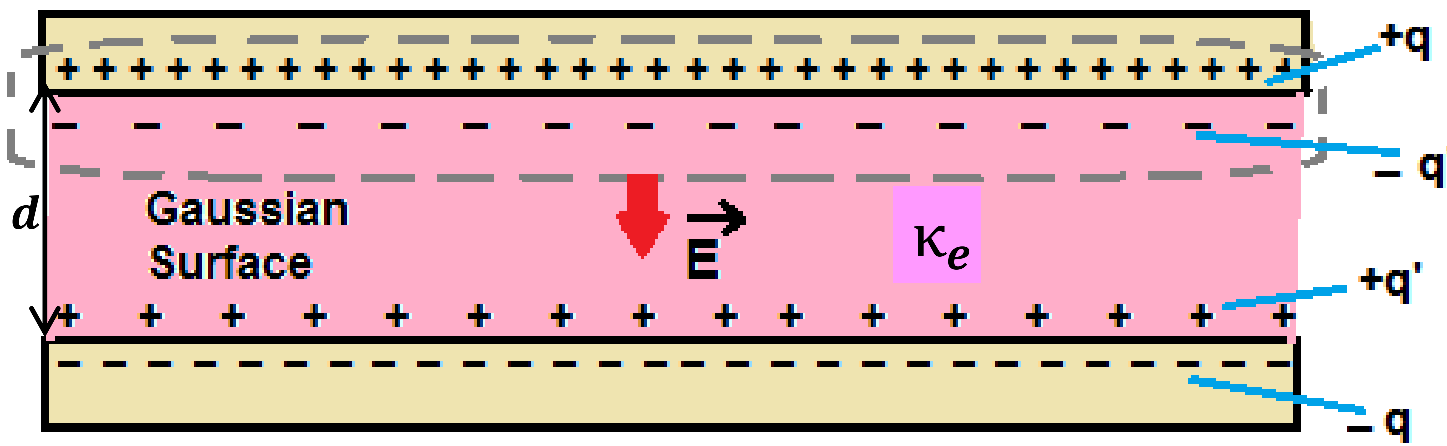
$$C = \kappa_e C_0 \quad \cdots\cdots\cdots\cdots\cdots \quad 4.45$$

where κ_e is called the dielectric constant $\Rightarrow \kappa_e = \dfrac{E_0\ (\text{in vacuum})}{E\ (\text{in dielectric})}$; $C = \dfrac{Q}{d} . \dfrac{1}{E} \quad \Rightarrow C \propto \dfrac{1}{E}$

Let us apply Gauss law to a parallel-plate capacitor with and without a dielectric, as shown in the figure below. The Gaussian surfaces are constructed as shown.



(a) without dielectric



(b) with a dielectric

Fig 4.24 Parallel plate capacitor with and without dielectric gives.

a) without dielectric

Electric field $\vec{E}_0$ is given by Gauss's Law: $\oint \vec{E} \,.\, d\vec{A} = \frac{q}{\varepsilon_0}$ 4.46a

But $\vec{E} = \vec{E}_0$, so Eqn. 4.46a becomes:

$$\Rightarrow \varepsilon_0 \oint \vec{E}_0 \,.\, d\vec{A} = q \text{ .... } 4.47$$

$$\Rightarrow \varepsilon_0 \vec{E}_0 . \hat{n}\vec{A} = q$$

$$\Rightarrow \varepsilon_0 E_0 A = q$$

$$\Rightarrow E_0 = \frac{q}{\varepsilon_0 A} \text{ ....} 4.48$$

b) with a dielectric

Electric field $\vec{E}$ is given by Gauss's Law: $\oint \vec{E} \,.\, d\vec{A} = \frac{q-q'}{\varepsilon_0}$ 4.49

$$\Rightarrow \varepsilon_0 \oint \vec{E} \,.\, d\vec{A} = q - q' \Rightarrow \varepsilon_0 \vec{E} . \hat{n}\vec{A} = q - q'$$

$$\Rightarrow \varepsilon_0 E A = q - q' \qquad \Rightarrow E = \frac{q-q'}{\varepsilon_0 A} \text{ ....} 4.50$$

The effect of the dielectric is to weaken the original field $\vec{E}_0$ by a factor of κ_e. So that $E = \frac{E_0}{\kappa_e}$, $\Rightarrow E = \frac{q}{\varepsilon_0 A \kappa_e}$ ……4.51

Eqn. 4.50 is equal to Eqn. 4.51:

$$\therefore \frac{q-q'}{\varepsilon_0 A} = \frac{q}{\varepsilon_0 A \kappa_e} \Rightarrow \varepsilon_0 A . \frac{(q-q')}{\varepsilon_0 A} = \frac{q}{\kappa_e} \Rightarrow (q - q') = \frac{q}{\kappa_e} \quad \text{….4.52}$$

Above equation shows that the magnitude of q' of an induced surface charge is less than that of the free charge q, and is zero if no dielectric is present.

Substituting $(q - q')$ in Eqn. 4.49 , Gauss's Law becomes:

$$\oint \vec{E} . d\vec{A} = \frac{q}{\kappa_e \varepsilon_0}$$

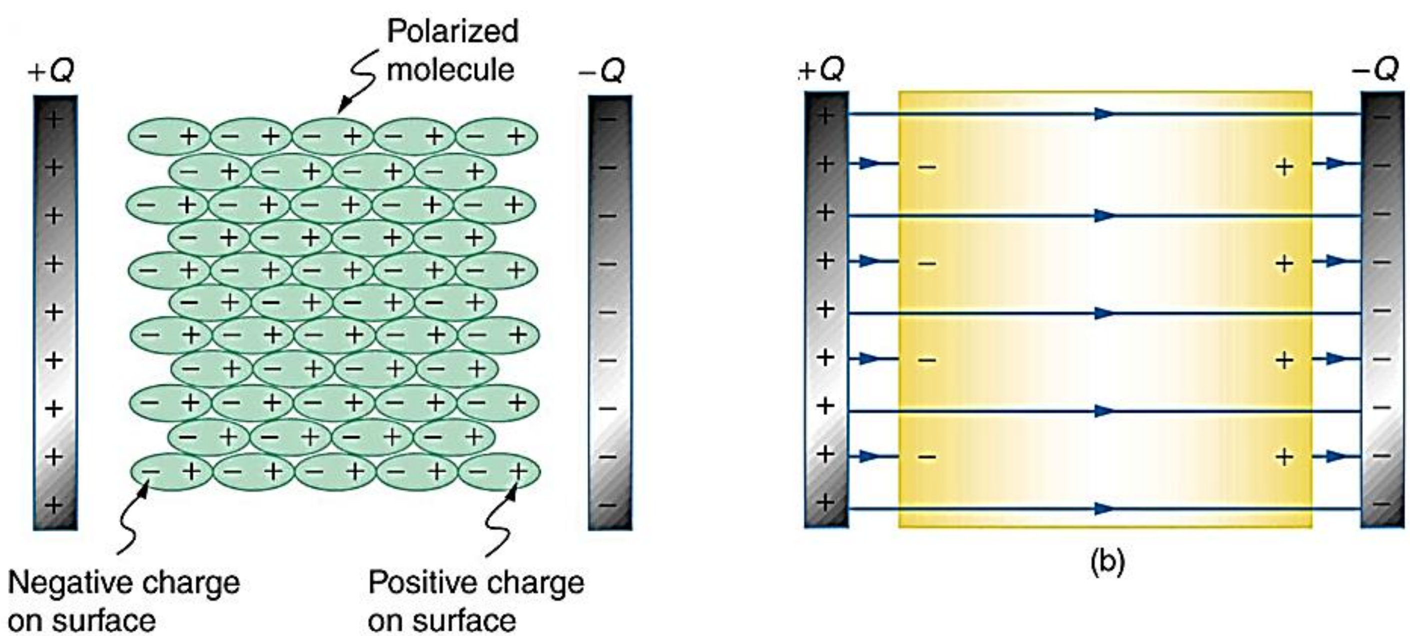
$$\Rightarrow \varepsilon_0 \oint \kappa_e \vec{E} . d\vec{A} = q$$ [Gauss's Law with dielectrics] …. 4.46b

The following should be taken into account:

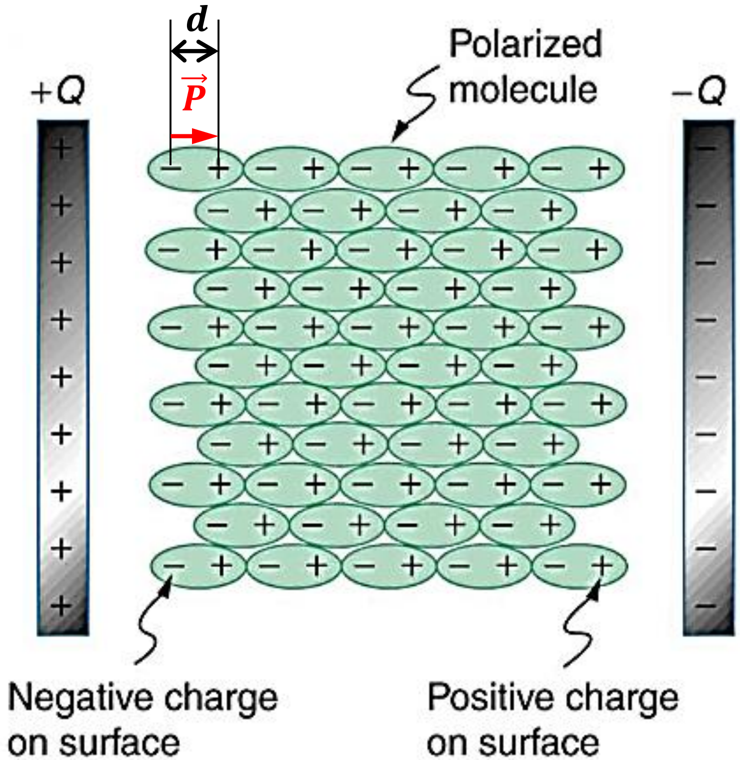
1. The flux integral now involves $\kappa_e E$, not just E. (The vector $\varepsilon_0 \kappa_e E$ is sometimes called the electric displacement D, so that Eqn. 4.46b ($\varepsilon_0 \oint \kappa_e \vec{E}\,.\,d\vec{A} = q$) can be written in the form $< j\, D\, .\, iX > = \ q$).
2. The charge q enclosed by the Gaussian surface is now taken to be the free charge only. The induced surface charge is deliberately ignored on the right side of Eqn. 4.46b, having been taken fully into account by introducing the dielectric constant κ_e on the left side: $\oint \vec{E}\,.\,d\vec{A} = \frac{q-q'}{\varepsilon_0} \Rightarrow \varepsilon_0 \oint \kappa_e \vec{E}\,.\,d\vec{A} = q$
3. Eqn. 4.46b differs from Eqn. 4.46a , our original statement of Gauss' law, only in that ε_0 in the latter equation has been replaced by $\varepsilon_0 \kappa_e$. We keep κ_e inside the integral of Eqn. 4.46b to allow for cases in which κ_e is not constant over the entire Gaussian surface.

4.9.6 A dielectric and polarization of dielectric

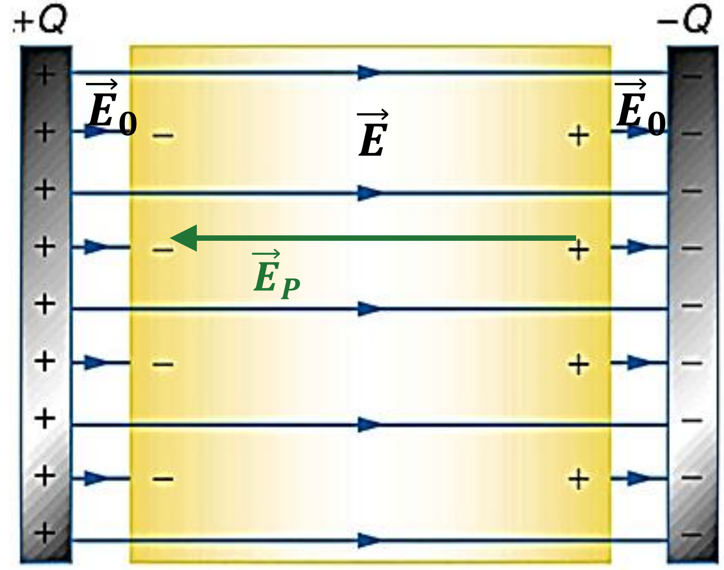
Microscopically, a dielectric increases capacitance by polarization of the insulator. The more easily it is polarized, the greater its dielectric constant (k_e). The effect of polarization can be best explained in terms of the characteristics of the Coulomb force. The Figures below show the separation of charge schematically in the molecules of a dielectric material placed between the charged plates of a capacitor. The Coulomb force between the closest ends of the molecules and the charge on the plates is attractive and very strong, since they are very close together. This attracts more charge onto the plates than if the space were empty and the opposite charges were a distance d away.



(a) (b)



(a)



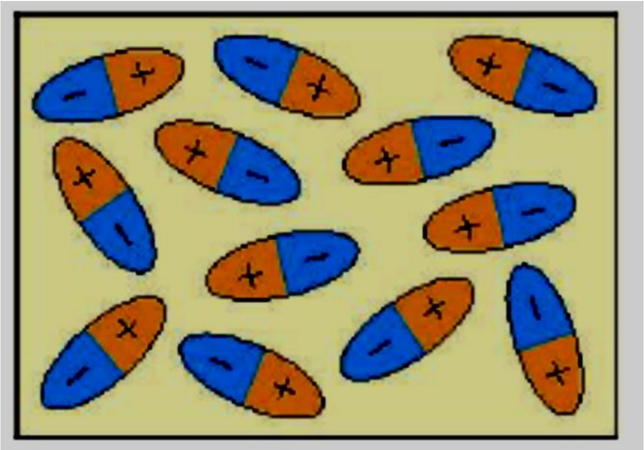
(b)

The molecules in the insulating material between the plates of a capacitor are polarized by the charged plates. This produces a layer of opposite charge on the surface of the dielectric that attracts more charge onto the plate, increasing its capacitance. P is the electric dipole moment ($P = Qd$).

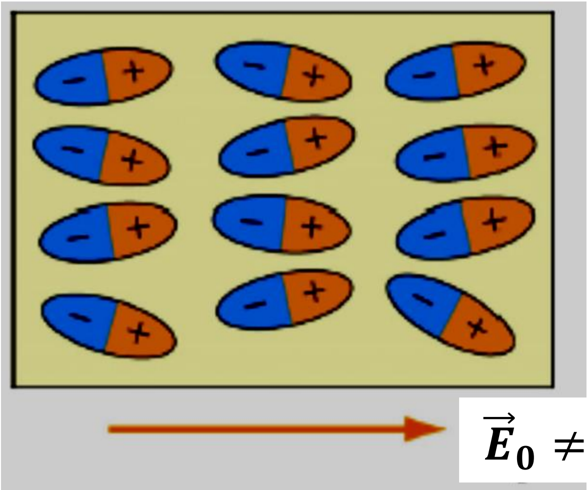
The dielectric reduces the electric field strength ($\downarrow E = \downarrow V/d$) inside the capacitor, resulting in a smaller voltage between the plates for the same charge ($Q = \uparrow C . \downarrow V$). The capacitor stores the same charge for a smaller voltage, implying that it has a larger capacitance because of the dielectric.

There are two types of dielectrics, polar and non-polar:

There are two types of dielectrics. The first type is polar dielectrics, which are dielectrics that have permanent electric dipole moments. An example of this type of dielectric is water.



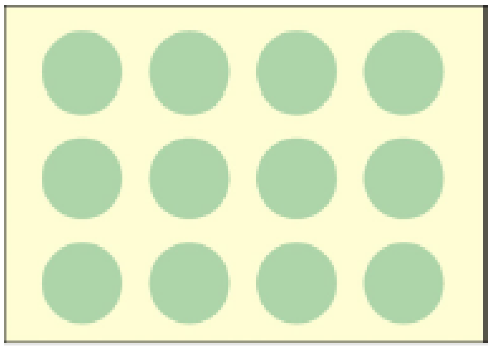
$\vec{E}_0 = 0$



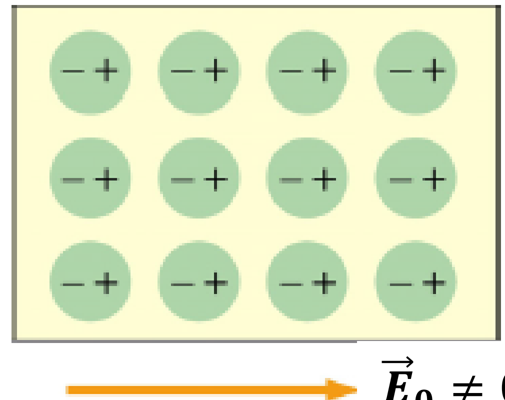
$\vec{E}_0 \neq 0$

When an external electric field $\vec{E}_0$ is present, a torque is set up and causes the molecules to align with $\vec{E}_0$. However, the alignment is not complete due to random thermal motion. The aligned molecules then generate an electric field that is opposite to the applied field but smaller in magnitude.

The second type of dielectrics is the non-polar dielectrics, which are dielectrics that do not possess permanent electric dipole moment. Electric dipole moments can be induced by placing the materials in an externally applied electric field.



$\vec{E}_0 = 0$

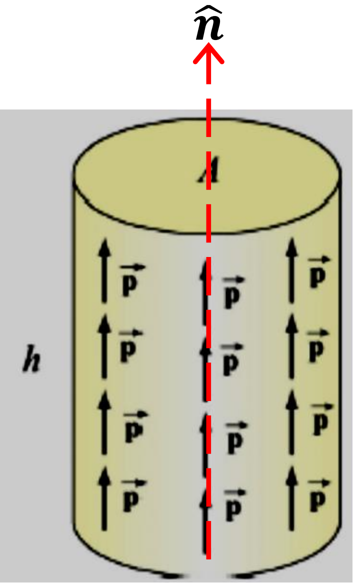


$\vec{E}_0 \neq 0$

The induced surface charges on the faces produces an electric field $\vec{\mathbf{E}}_P$ in the direction opposite to $\vec{\mathbf{E}}_0$, leading to $\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 + \vec{\mathbf{E}}_P$, with $|\vec{\mathbf{E}}| < |\vec{\mathbf{E}}_0|$.

POLARIZATION

One of the concepts crucial to the understanding of dielectric materials is the average electric field produced by many little electric dipoles which are all aligned. Suppose we have a piece of material in the form of a cylinder with area A and height h, and that it consists of N electric dipoles, each with electric dipole moment $\vec{\mathbf{p}}$ spread uniformly throughout the volume of the cylinder.



polarization vector $\vec{\mathbf{P}}$ is the net electric dipole moment vector per unit volume

$$\vec{\mathbf{P}} = \frac{1}{\text{Volume}} \sum_{i=1}^{N} \vec{\mathbf{p}}_i \quad \cdots\cdots\cdots\cdots \; 4.53$$

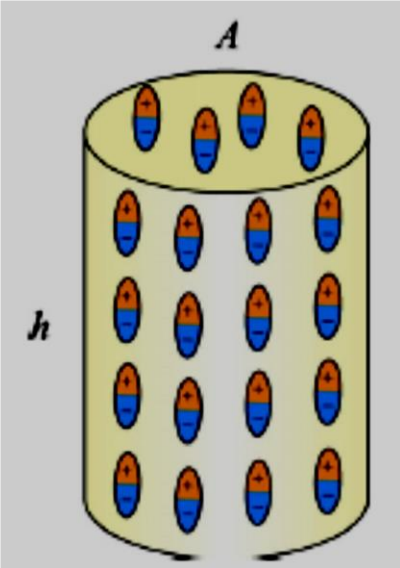
assume for the moment that all of the electric dipole moments $\vec{\mathbf{p}}$ are aligned with the axis of the cylinder. Since each electric dipole has its own electric field associated with it, in the absence of any external electric field, if we average over all the individual fields produced by the dipole, what is the average electric field just due to the presence of the aligned dipoles?

For the above cylinder, the magnitude of $\vec{P}$:

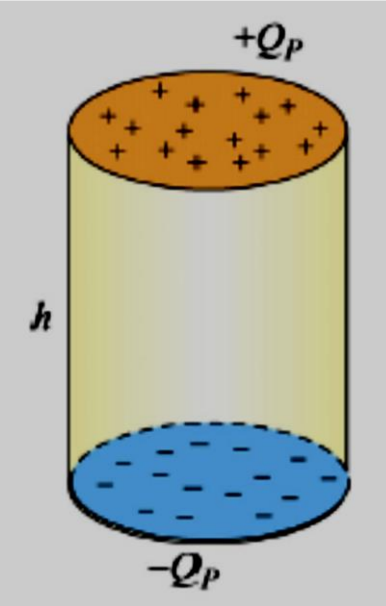
$$P = \frac{Np}{Ah} \qquad \text{……………………… 4.54}$$

and the direction of $\vec{\mathbf{P}}$ is parallel to the aligned dipoles.

Now all the little ± charges associated with the electric dipoles in the interior of the cylinder are replaced with two equivalent charges, $\pm Q_P$, on the top and bottom of the cylinder, respectively.



uniform dipole distribution.



Equivalent charge distribution.

The equivalence can be seen by noting that in the interior of the cylinder, positive charge at the top of any one of the electric dipoles is *canceled* on average by the negative charge of the dipole just above it. The only place where cancellation does not take place is for electric dipoles at the top of the cylinder, since there are no adjacent dipoles further up. Thus the interior of the cylinder appears uncharged in an average sense (averaging over many dipoles), whereas the top surface of the cylinder appears to carry a net positive charge. Similarly, the bottom surface of the cylinder will appear to carry a net negative charge.

the electric dipole moment Q_P produces, $Q_P h$, is equal to the total electric dipole moment of all the little electric dipoles.

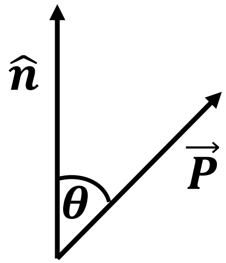
$$Q_P h = Np \text{, or } Q_P = \frac{Np}{h} \quad \text{.....................} \ 4.55$$

To compute the electric field produced by Q_P, we note that the equivalent charge distribution resembles that of a parallel-plate capacitor, with an equivalent surface charge density σ_P that is equal to the magnitude of the polarization:

$$\sigma_P = \frac{Q_P}{A} = \frac{Np}{Ah} = P \quad \text{.....................} \ 4.56$$

SI units of P are $(\mathrm{C \cdot m})/\mathrm{m}^3$, or $\mathrm{C/m^2}$ same as the surface charge density.

In general if the polarization vector makes an angle θ with $\hat{\mathbf{n}}$, the outward normal vector of the surface, the surface charge density would be



$$\sigma_P = \vec{\mathbf{P}} \cdot \hat{\mathbf{n}} = P\cos\theta \quad \text{............. 4.57}$$

But $\hat{n}$ and $\vec{P}$ are always in the same direction, $\theta = 0^o \Rightarrow \sigma_P = P$,

so average electric field of magnitude: $E_P = P/\varepsilon_0$..........4.58a

Since the direction of this electric field is *opposite* to the direction of $\vec{\mathbf{P}}$, in vector notation $\vec{\mathbf{E}}_P = -\vec{\mathbf{P}}/\varepsilon_0$ 4.58b

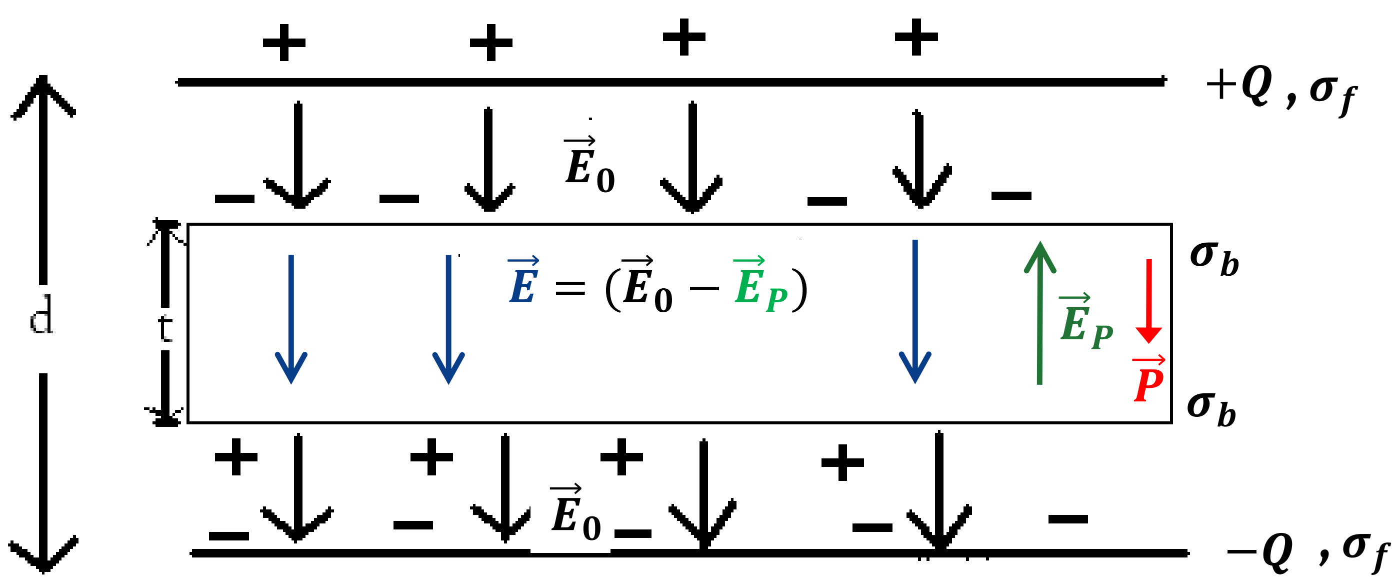
dipoles are randomly oriented, then the polarization $\vec{\mathbf{P}} = \mathbf{0}$.

Examine the effects of introducing dielectric material into a system.

first assume that the atoms or molecules comprising the dielectric material have a *permanent* electric dipole moment. If left to themselves, these permanent electric dipoles in a dielectric material never line up spontaneously, so that in the absence of any applied external electric field, $\vec{\mathbf{P}} = \vec{\mathbf{0}}$ due to the random alignment of dipoles, and the average electric field $\vec{\mathbf{E}}_P$ is zero as well.

However, when we place the dielectric material in an external field $\vec{\mathbf{E}}_0$, the dipoles will experience a torque $\vec{\tau} = \vec{\mathbf{p}} \times \vec{\mathbf{E}}_0$ that tends to align the dipole vectors $\vec{\mathbf{p}}$ with $\vec{\mathbf{E}}_0$. The effect is a net polarization $\vec{\mathbf{P}}$ parallel to $\vec{\mathbf{E}}_0$, and therefore an average electric field of the dipoles $\vec{\mathbf{E}}_P$ *anti-parallel* to $\vec{\mathbf{E}}_0$, i.e., that will tend to *reduce* the total electric field strength below $\vec{\mathbf{E}}_0$. The total electric field $\vec{\mathbf{E}}$ is the sum of these two fields: See the Figure below: $\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 + \vec{\mathbf{E}}_P = \vec{\mathbf{E}}_0 - (\vec{\mathbf{P}} / \varepsilon_0)$ 4.59

4.9.7 The three electric vectors



It is possible to discuss the consequences of polarization without consideration to the atomic processes involved. The polarization p depends on the nature of the substance, and for most dielectrics, it is proportional to the electric field. $\vec{E}$. It is also parallel to the field $\vec{E}$. We define a property of the dielectric, called **susceptibility** X_e by the equation.

$$p = X_e \varepsilon_0 E \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 4.60a$$

Since only a material substance can be polarized, the susceptibility of a vacuum is zero.

The surface density of a bound charge at any point on the surface of a dielectric is equal to the normal component of p at the surface. For a special case of a surface perpendicular to E,

$$\sigma_b = p = X_e \varepsilon_0 \vec{E} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 4.60b$$

region between the plates and the dielectric, we encounter two different kinds of charge. On the surface of the dielectric, there is a concentration of the bound charge per unit area σ_b. equal to p. On the surface of a conductor, there is a concentration of free charge per unit area σ_f from an external source. This is equal to $\varepsilon_o E_f$.

$$\sigma_f = \varepsilon_0 \vec{E}_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 4.60c$$

In most cases, the polarization $\vec{\mathbf{P}}$ is not only in the same direction as $\vec{\mathbf{E}}_0$, but also linearly proportional to $\vec{\mathbf{E}}_0$ (and hence $\vec{\mathbf{E}}$.) $\Rightarrow$ $\vec{\mathbf{P}} = \varepsilon_0 \chi_e \vec{\mathbf{E}}$ 4.60d

where χ_e is the *electric susceptibility* *4.61a*

Materials that obey this relationship are *linear dielectrics.*

Re-arrange Eqn. 4.59 then substitute Eqn. 4.60 in it: $\vec{E} = \vec{E}_0 - \frac{\vec{P}}{\varepsilon_0} \Rightarrow \vec{E}_0 = \vec{E} + \frac{\vec{P}}{\varepsilon_0}$

$$\Rightarrow \vec{E}_0 = \vec{E} + \frac{\varepsilon_0 \chi_e \vec{E}}{\varepsilon_0} \Rightarrow \vec{E}_0 = (1 + \chi_e)\,\vec{E} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 4.62$$

where $\kappa_e = (1 + \chi_e)$ is the *dielectric constant* *4.61b*

Then multiply ε_0 both sides of Eqn. 4.62, for it to be analogous to Eqn. 4.60:

$$\varepsilon_0 \vec{E}_0 = \varepsilon_0\,(1 + \chi_e)\,\vec{E} \Rightarrow \varepsilon_0 \vec{E}_0 = \varepsilon \vec{E} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 4.63$$

where $\varepsilon = \varepsilon_0\,(1 + \chi_e)$ is the *permittivity of the dielectric* 4.61c

Since permittivity of a dielectric is always greater than the permittivity of a vacuum, it is convenient to have a ratio of the two. The ratio of permittivity of the dielectric to that of a vacuum is called relative permittivity or dielectric constant.

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}$$

It is a pure dimensionless number. It is equal to unity for a vacuum and is greater than unity for a material substance

The dielectric constant κ_e is always greater than one since $\chi_e > 0$.

$$\Rightarrow \quad E = \frac{E_0}{\kappa_e} < E_0 \quad \text{.....................} \quad 4.63$$

Thus, we see that the effect of dielectric materials is always to decrease the electric field below what it would otherwise be.

In the case of dielectric material where there are no permanent electric dipoles, a similar effect is observed because the presence of an external field $\vec{\mathbf{E}}_0$ induces electric dipole moments in the atoms or molecules. These induced electric dipoles are parallel to $\vec{\mathbf{E}}_0$, again leading to a polarization $\vec{\mathbf{P}}$ parallel to $\vec{\mathbf{E}}_0$, and a reduction of the total electric field strength.

The *electric Displacement* $(\boldsymbol{D})$

$$\Rightarrow \boldsymbol{D} = \boldsymbol{\varepsilon_0} \kappa_{\boldsymbol{e}} \vec{\boldsymbol{E}} \text{ (see slide 284)} \Rightarrow \boldsymbol{D} = \boldsymbol{\varepsilon_0}(\mathbf{1} + \chi_{\boldsymbol{e}})\vec{\boldsymbol{E}}$$

$$\Rightarrow \boldsymbol{D} = \boldsymbol{\varepsilon_0}\vec{\boldsymbol{E}} + \boldsymbol{\varepsilon_0}\chi_{\boldsymbol{e}}\vec{\boldsymbol{E}} \Rightarrow \boldsymbol{D} = \boldsymbol{\varepsilon_0}\vec{\boldsymbol{E}} + \vec{\boldsymbol{P}} \Rightarrow \boldsymbol{D} = \vec{\boldsymbol{\sigma}}_{\boldsymbol{f}} \quad \ldots\ldots \quad 4.64\text{a}$$

OR

$$\Rightarrow \boldsymbol{D} = \boldsymbol{\varepsilon_0}(\mathbf{1} + \chi_{\boldsymbol{e}})\vec{\boldsymbol{E}} \Rightarrow \boldsymbol{D} = \boldsymbol{\varepsilon}\vec{\boldsymbol{E}} \Rightarrow \boldsymbol{D} = \vec{\boldsymbol{\sigma}}_{\boldsymbol{f}} \quad \ldots\ldots\ldots\ldots\ldots \quad 4.64\text{b}$$

where $\boldsymbol{\sigma_f}$ is the concentration of free charge per unit area on the surface of a conductor/capacitor.

In free space, the energy per unit volume stored in the field equals $\frac{1}{2}\varepsilon_o E^2$. For a dielectric material, we replace ε_o by ε, which leads us to the expression of the energy stored per unit volume in a dielectric

Recall Eqn. 4.40: $u_E = \frac{1}{2}\varepsilon_0 E^2$ **(in free space)**

$\Rightarrow u_E = \frac{1}{2}\varepsilon E^2$ **(in dielectric)**

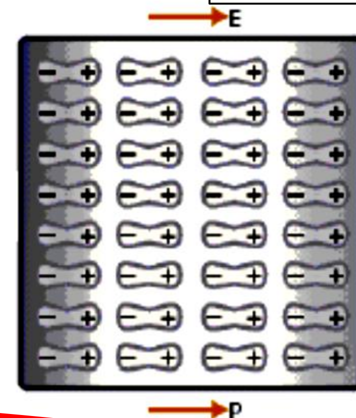
So energy per unit volume (u_E):

$$u_E = \left(\frac{W}{v}\right) = \frac{1}{2}\varepsilon E.E = \frac{1}{2}D.E = \frac{1}{2}D.\frac{D}{\varepsilon} = \frac{D^2}{\varepsilon} \quad \dots\dots\dots\dots \quad 4.65$$

We can readily see that the energy stored per unit volume with a material dielectric is greater than that for free space. This is due to the extra work that must be done to polarize the dielectric.

SUMMARY

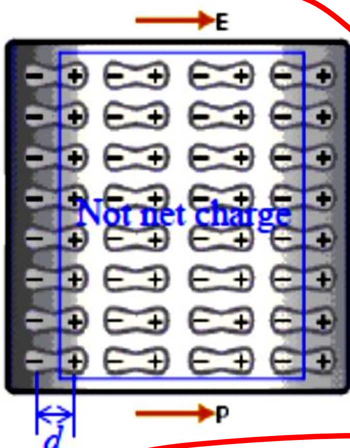
TAKE NOTE:
where ρ_{SP} is σ_P



In all cases of polarization, no matter what the mechanism is (electronic, ionic, orientational), **E** induces **P**.

For materials without spontaneous polarization and for not too strong **E**, **P** $\propto$ **E**.

$$\boxed{\mathbf{P} = \chi\varepsilon_0\mathbf{E}}$$



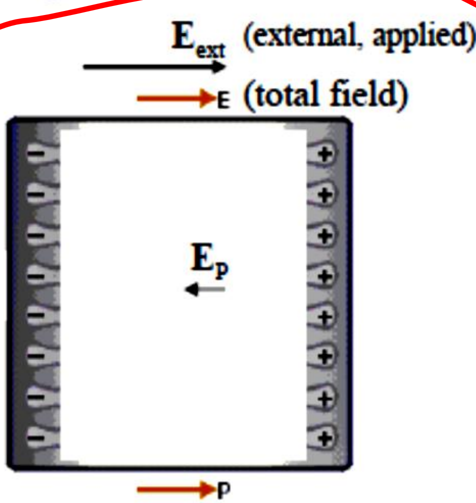
No net charge in the interior. Two sheet charges at surfaces.

$$(|\rho_{sP}|A)d = P(Ad) \Longrightarrow |\rho_{sP}| = P$$

The polarization charge density, or "internal" charge density

As any P is from – to + while any **E** is from + to –, we write:

$$\boxed{\rho_{sP} = -P}$$



The polarization (or "internal") charge leads to a polarization (or "internal") field

$$E_P = \rho_{sP}/\varepsilon_0 \Longrightarrow \varepsilon_0 E_P = \rho_{sP} = -P \Longrightarrow E_P = -P/\varepsilon_0$$

More generally, in the vector form: $\boxed{\varepsilon_0\mathbf{E}_P = -\mathbf{P}}$

Notice that the polarization (internal) field is always in opposite direction to the external, applied field $\mathbf{E}_{ext}$.

Example 4.4

A 4 microfarad capacitor C_1, charged to a potential of 200 volts is suddenly connected to and shares its charge with an uncharged 2 microfarad capacitor C_2. What is the loss of energy in this process?

Solution

In order for C_1 to share charges with C_2, they must be connected in parallel.

$$\mathbf{Q = C_1\, DV_1 = 4 \times 10^{-6} \times 200V = 800 \times 10^{-6}\, C}$$

$$\mathbf{Q_1 + Q_2 = 8 \times 10^{-4}\, C = C_1\, \Delta V + C_2\, \Delta V}$$
$$\mathbf{= (4 \times 10^{-6}\, far + 2 \times 10^{-6}\, far)\, \Delta V}$$

$$\therefore \Delta v = \frac{8 \times 10^{-6} Coul}{6 \times 10^{-6}\ Farads} = 1.33 \times 10^{2}\ volts$$

Work done before connecting

$$W_{before} = \tfrac{1}{2}\, G\, (\Delta V)^2\ \tfrac{1}{2}\, 4 \times 10^{-6} farads \times (200V)^2$$

$$= 8 \times 10^{-2} joules$$

$$W_{after} = \tfrac{1}{2}\, C\, \Delta V^2 = \tfrac{1}{2}\ [(4 + 2)\ X\ 10^{-6} farads \times (1.33 \times 10^{2})^2$$

$$= 1.34 \times 10^{-2} joules$$

$$\Delta W = (8 - 1.34) \times 10^{-2} joules$$

$$= 7.66 \times 10^{-2} joules$$