Chair of Systems Design
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Economic Dynamics and Complexity

Lecture 12: Productivity and Resilience

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Outline

Notes:

Notes:

General derivation of production function

- Consider two Inputs X_1, X_2, Output $Y(X_1, X_2)$
- **Homogeneity** of degree n: $Y(\alpha X_1, \alpha X_2) = \alpha^n Y(X_1, X_2)$
  - Remember: Returns to scale. Linear homogeneity: $n = 1$

$$1 \cdot Y(X_1, X_2) = X_1 \frac{\partial Y(X_1, X_2)}{\partial X_1} + X_2 \frac{\partial Y(X_1, X_2)}{\partial X_2}$$

- **Diminishing returns to scale:**

$$\frac{\partial^2 Y(X_1, X_2)}{\partial X_1^2} < 0 \; ; \; \frac{\partial^2 Y(X_1, X_2)}{\partial X_2^2} < 0$$

- **Separation of variables**: $Y(X_1, X_2) = G(X_1)H(X_2)$

$$G(X_1)H(X_2) = X_1 \frac{dG(X_1)}{dX_1} H(X_2) + X_2 \frac{dH(X_2)}{dX_2} G(X_1)$$
$$1 = X_1 \frac{dG(X_1)/dX_1}{G(X_1)} + X_2 \frac{dH(X_2)/dX_2}{H(X_2)}$$

- Can only hold if

$$X_1 \frac{dG(X_1)/dX_1}{G(X_1)} = \alpha$$
$$X_2 \frac{dH(X_2)/dX_2}{H(X_2)} = \beta$$
$$\alpha + \beta = 1$$

- Integration:

$$\int \frac{dG(X_1)/dX_1}{G(X_1)} dX_1 = \int \frac{\alpha}{X_1} dX_1 = \alpha \ln X_1 + C_1$$

- Result with $Y(1,1) = A = e^{(C_1 + C_2)}$

$$\boxed{Y(X_1, X_2) = A X_1^{\alpha} X_2^{\beta}}$$

Alternative derivation: Profit maximization

- **Production function**: $Y(K, L)$, K: capital, L: labor
- **Profit**: price $p\times$ output minus cost for capital, labor

$$\pi = pY(K, L) - rK - wL$$

  - **Profit maximization:** $\partial\pi/\partial K = \partial\pi/\partial L = 0$

$$p\frac{\partial Y(K, L)}{\partial L} - w = 0 \; ; \; p\frac{\partial Y(K, L)}{\partial K} - r = 0$$

  - Determines $L^*(r, w, p)$, $K^*(r, w, p)$
- Fractions α, β used to pay labor, capital

$$\alpha p Y(K, L) = wL^* \; ; \; \beta p Y(K, L) = rK^*$$
$$\frac{[\partial Y(K, L)/\partial L]_{K^*,L^*}}{Y(K^*, L^*)} = \frac{\alpha}{L^*} \; ; \; \frac{[\partial Y(K, L)/\partial L]_{K^*,L^*}}{Y(K^*, L^*)} = \frac{\beta}{K^*}$$

$$\frac{\partial}{\partial L} \ln Y(K, L)_{K^*,L^*} = \frac{\alpha}{L^*}$$
$$\frac{\partial}{\partial K} \ln Y(K, L)_{K^*,L^*} = \frac{\beta}{K^*}$$

$$\ln Y(K, L) = \alpha \ln L + H(K) + C_L$$
$$\ln Y(K, L) = \beta \ln K + G(L) + C_K$$
$$H(K) = \beta \ln K + C_K$$
$$G(L) = \alpha \ln L + C_L$$
$$Y(K, L) = AK^{\beta} L^{\alpha} \; ; \; A = ^{C_K + C_L}$$

Constant returns to scale: $\alpha + \beta = 1$

$$Y(\lambda K, \lambda L) = A(\lambda L)^{\alpha}(\lambda K)^{\beta} = \lambda^{\alpha+\beta} A L^{\alpha} K^{\beta} = \lambda^{\alpha+\beta} Y(K, L)$$

Notes:

Rewriting $Y(X_1, X_2)$ in terms of partial derivatives follows the Euler Theorem.

Notes:

- r: capital rental rate (i.e. interest rate), w: wage rate, p: price level
- $G(L)$: a constant of integration that may depend on L, here it is $\alpha \ln L$
- $H(K)$: a constant of integration that may depend on K, here it is $\beta \ln K$
- More information: Cobb, C. W. and P. H. Douglas (1928), A theory of production. American Economic Review 18(1):139-165.
- A: Total factor productivity (TFP): Accounts for effects in total output not caused by inputs. For example, a year with unusually good weather will tend to have higher output, because bad weather hinders agricultural output. A variable like weather does not directly relate to unit inputs, so weather is considered a total-factor productivity variable.

Outline

Derivation of production function

Continuous models of business cycles

Resilience and performance

Formalizing the Kaldor model

- Remember Kaldor's model: Dynamics of production:

$$\dot{Y}(I,S) = F(Y,K) = \alpha\,[I(Y,K) - S(Y,K)]$$
$$\dot{K} = G(Y,K) = I(Y,K) - \kappa K$$

  - Nonlinear functions for investment $I(Y,K)$, savings $S(Y,K)$
  - Capital K accumulates from investment, κ: depreciation rate
- Equilibrium: $\dot{Y} = \dot{K} = 0 \to (Y^*, K^*)$, $\kappa = 0$
- Taylor expansion gives the following Jacobian matrix:

$$J = \begin{bmatrix} \frac{\partial F}{\partial Y} & \frac{\partial G}{\partial Y} \\ \frac{\partial F}{\partial K} & \frac{\partial G}{\partial K} \end{bmatrix} = \begin{bmatrix} F_Y & G_Y \\ F_K & G_K \end{bmatrix} = \begin{bmatrix} \alpha(I_Y - S_Y) & \alpha(I_K - S_K) \\ I_Y & I_K \end{bmatrix}$$

$$\tau := \mathrm{tr}(\mathbf{J}) = (F_Y + G_K) = \alpha(I_Y - S_Y) + I_K$$
$$\Delta := \det(\mathbf{J}) = (F_Y G_K - F_K G_Y)$$
$$= \alpha(I_Y - S_Y)I_K - \alpha I_Y(I_K - S_K)$$
$$= \alpha(I_Y S_K - I_K S_Y) > 0$$

- Roots of the characteristic polynomial

$$\det(\mathbf{J} - \lambda\mathbf{I}) = \lambda^2 - \tau\lambda + \Delta$$

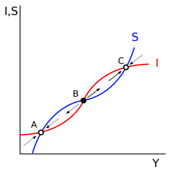
- **Solution**

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$

- Stability for real negative parts

$$\tau < 0 \to \alpha(I_Y - S_Y) + I_K < 0$$

- When is $(I_Y - S_Y) < 0$??? (A,C)



Notes:

Notes:

- κ: constant depreciation rate
- α: adjustment speed in the market of goods
- K: capital stock

Example from Gabisch/Lorenz p 158

Transitional dynamics

- **Curve:** $\dot{K} = 0 \rightarrow G_Y dY + G_K dK = 0$

$$\left.\frac{dK}{dY}\right|_{\dot{K}=0} = -\frac{G_Y}{G_K} = -\frac{I_Y}{I_K} > 0$$

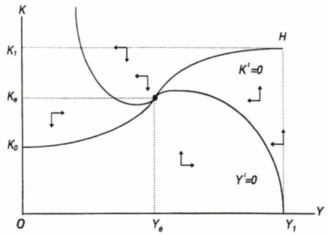
  - $\dot{K} = 0$: Upward sloping curve
  - Above: $I_K < 0$, therefore $\dot{K} < 0$
  - Below: $I_K > 0$, therefore $\dot{K} > 0$
- **Curve:** $\dot{Y} = 0 \rightarrow F_Y dY + F_K dK = 0$

$$\left.\frac{dK}{dY}\right|_{\dot{Y}=0} = -\frac{F_Y}{F_K} = -\frac{I_Y - S_Y}{I_K - S_K} \lesseqgtr 0$$

  - Sign/slope of $\dot{Y} = 0$ depends on S_Y and I_Y
  - Negative: low and high Y, positive: near Y^*
  - $d\dot{Y}/dK = \alpha(I_K - S_K) > 0$ below $\dot{Y} = 0 \rightarrow Y$ increases
  - $d\dot{Y}/dK = \alpha(I_K - S_K) < 0$ above $\dot{Y} = 0 \rightarrow Y$ decreases

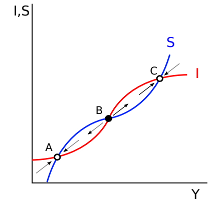
Important assumptions:

$$\frac{dI}{dY} = I_Y > 0 \; ; \quad \frac{dS}{dY} = S_Y > 0$$
$$\frac{dI}{dK} = I_K < 0 \; ; \quad \frac{dS}{dK} = S_K > 0$$

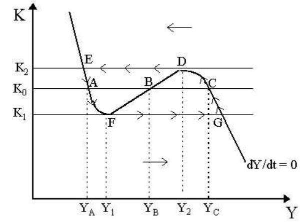


Result: Limit Cycle

Hysteresis



- **Focus on equilibrium points** $I(Y, K) = S(Y, K)$
  - Assume $\kappa = 0$ to simplify equations
  - $\Delta > 0$ because of $I_K < 0$, $S_K, S_Y, I_Y > 0$
  - $\tau < 0$ at stable points (A), (C): $I_Y < S_Y$
  - $\tau > 0$ at instable point (B): $I_Y > S_Y$
- **Hysteresis**
  - (C)$\rightarrow$(D): K rises from K_0 to K_2
  - (D)$\rightarrow$(E): Y_C and Y_B merge in (D) $\rightarrow$ unstable $\rightarrow$ jump
  - (E)$\rightarrow$(F): K declines to K_1, Y_A and Y_B merge in (F), etc.



Notes:

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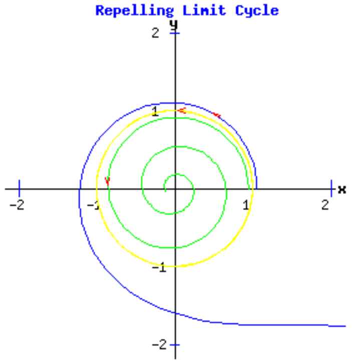
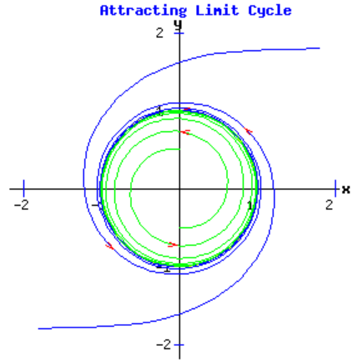
Limit cycle





Van-der-Pol oscillator

- Differentiate Kaldor's nonlinear function:

$$\dot{Y} = \alpha \left[I(Y,K) - S(Y,K) \right]$$
$$\ddot{Y} = \alpha \left[I_Y \dot{Y} + I_K \dot{K} - S_Y \dot{Y} - S_K \dot{K} \right]$$

- Substitute function $\dot{K} = 0 = I(Y,K)$:

$$\ddot{Y} - \alpha [I_Y - S_Y] \dot{Y} - \alpha [I_K - S_K] I(Y,K) = 0$$

- **Assumption 1:** Capital determined by savings

$$\dot{Y} \approx 0 \rightarrow I \approx S \rightarrow \dot{K} = S$$

- **Assumption 2:** $(I_K - S_K)$ independent of capital stock

$$I_k[1 - (S_K/I_K)] \rightarrow I_K[1 + c] \text{ if } S(K) \propto K \; ; \; I(K) \propto -K$$
$$\ddot{Y} - \alpha \left[I_Y - S_Y \right] \dot{Y} - \alpha I_K S(Y) = 0$$

- **Assumption 3:** Symmetric shapes of $I(Y,K)$, $S(Y,K)$

$$S(Y) = a - bY + cY^3$$
$$S_Y = -b + c_1 Y^2$$
$$I(Y) = -a + bY - cY^3$$
$$I_Y = b - c_1 Y^2$$

- Parabolic function for $[S_Y - I_Y]$

$$\ddot{x} + A(x)\dot{x} + B(x) = 0$$
$$A(x) = \alpha[S_Y - I_Y] = \mu(x^2 - 1)$$
$$B(x) = x$$

- Van-der-Pol equation:

$$\boxed{\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0}$$

Notes:

Notes:

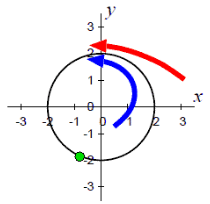
- $A(x)$: spring force
- $A(x)\dot{x}$: damping factor
- μ: related to adjustment coefficient α of the damping term

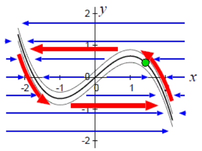
This discussion follows: A. Chian, *Complex Systems Approach to Economic Dynamics*, Springer 2007, pp. 13-14. Unfortunately, the book does not explain the model better than the slide.

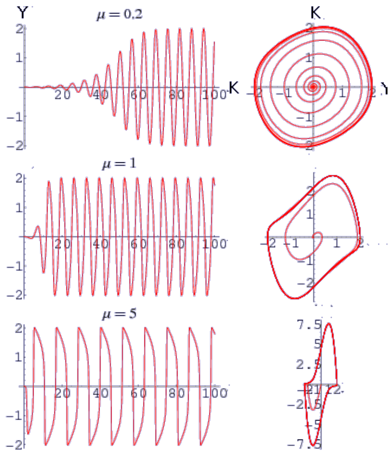
Van-der-Pol oscillator: Dynamics

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

- control parameter $\mu < 0$: decaying spiral
- $\mu > 0$: oscillations of different size/shape



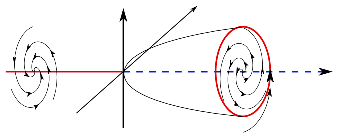




Hopf bifurcation

- **Non-linear damping term:** $\mu(x^2 - 1)\dot{x}$
  - Positive for $|x| > 1$: large amplitude oscillations decay
  - Negative for $|x| < 1$: oscillations grow if become too small
- **Control parameter** μ
  - Hopf bifurcation at $\mu = 0$
  - Stable limit cycle for $\mu > 0$
  - Varying cycle shape and size dependent on μ
- **Hopf bifurcation**
  - Eigenvalues are complex conjugates
  - Pass through the imaginary axis at bifurcation
  - Analytical criteria difficult
  - **Example:** 2d oscillations with frequency ω

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$



$$\dot{r} = \mu r - r^3 \; ; \quad \dot{\theta} = \omega + br^2$$

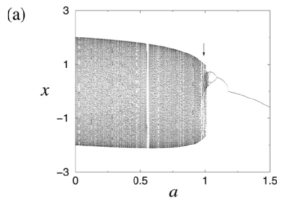
Notes:

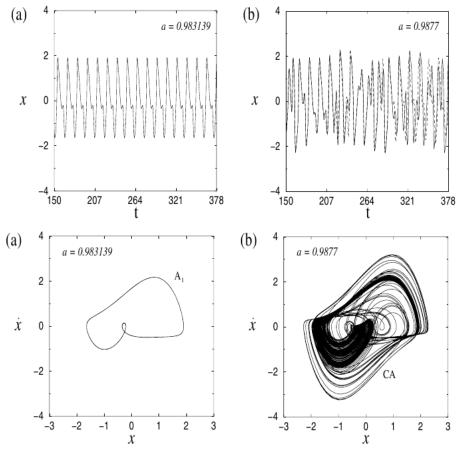
Notes:

Forced Van-der-Pol oscillator

$$\boxed{\ddot{x} + \mu(x^2 - 1)\dot{x} + x = I(t)}$$

- $I(t) \rightarrow$ **autonomous investment**
- **Exogenous force:** $I(t) = a \sin(\omega t)$
- Amplitude a: Control parameter
- **Result:** Period doubling scenario





Source: A. Chian (2007), p. 17

From linear to nonlinear accelerator-multipliers

- **Model**
  - **Capital** K: Fixed fraction of income $\rightarrow K = \nu Y$
  - **Investment** I: Change of capital $\rightarrow I = \dot{K} = \nu \dot{Y}$
  - **Savings**: Fixed fraction of income $\rightarrow S = sY$
- **Dynamics:**
  - **Assumption 1:** $\dot{Y} \propto (I - S) \rightarrow$ Remember Kaldor
  - **Assumption 2:** $\dot{I} \propto (\dot{K} - I) = (\nu \dot{Y} - I) \rightarrow$ delayed
  - **Unify adjustment speeds** (suitable time unit)

$$\dot{Y} = I - sY \ ; \quad \dot{I} = \nu \dot{Y} - I$$

  - Differentiate, using : $I = \dot{Y} + S = \dot{Y} + sY$

$$\ddot{Y} = \dot{I} - s\dot{Y} = (\nu - s)\dot{Y} - I = (\nu - s)\dot{Y} - \dot{Y} - sY$$

$$\ddot{Y} - (\nu - 1 - s)\dot{Y} + sY = 0$$

- Replace linear accelerator

$$\nu \dot{Y} \rightarrow \nu(\dot{Y} - \frac{1}{3}\dot{Y}^3)$$

- Make $I(t)$ exogenous:
  $\rightarrow$ **autonomous investment**

$$\ddot{Y} - (\nu - 1 - s)\dot{Y} + \frac{\nu}{3}\dot{Y}^3 + sY = I(t)$$

- Existence of **limit cycles**

Notes:

The pictures are taken from A. Chian (2007), p. 17, 19
The bifurcation diagram does not allow us to conclude a chaotic dynamics. In fact, the maximum Liapunov exponent is zero for $0 < a < 1$ and negative for $a > 1$. Only at about $a = 1$ it becomes slightly larger than zero.
Lorenz (1987) and Lorenz and Nusse (2002) have considered the generalization of this oscillator equation in an economic context.

- Periodic and chaotic time series. (a) A periodic time series x(t) for $a = 0.983139$, (b) two chaotic time series for $a = 0.9877$ with slightly different initial conditions:$x = 0.2108$ and $\dot{x} = 0.0187$ for the solid line $x = 0.2100$ and $\dot{x} = 0.0187$ for the dashed line

Notes:

The adjustment equations assume different speeds for the two adaptive processes. Samuelson and Hicks assumed identical speed for all kind of adjustments $\rightarrow$ Unify.
The basic equation is capable of producing damped or explosive oscillations, depending on the sign of $(\nu - 1 - s)$, but nothing else. The boundary case $\nu = (1 + s)$ produces a standing cycle, but its probability is small, and it is structurally unstable.
Source: Puu(1997), Nonlinear economic dynamics.

Two coupled economies

- **Dynamics:** $\dot{Y}_i = I_i - S_i \; (i = 1, 2)$
- **Import:** $M_i = m_i Y_i \rightarrow$ Increase of S_i
- **Export:** $X_i = m_j Y_j \rightarrow$ Increase of I_j
- Linear multiplier-accelerator model

$$\dot{Y}_i = I_i - s_i Y_i$$

- **Savings:** $S_i = s_i Y_i$, **investments** I_i
- Savings growth: $\dot{S}_i = s_i \dot{Y}_i + m_i Y_i$
- Investment dynamics: $\dot{I}_i = \nu_i \dot{Y}_i - I_i + m_j Y_j$
- **nonlinear replacement**

$$\nu \dot{Y} \rightarrow \nu(\dot{Y} - \frac{1}{3}\dot{Y}^3)$$

- Differentiate, using $I_i = \dot{Y}_i + S_i = \dot{Y}_i + s_i Y_i$:

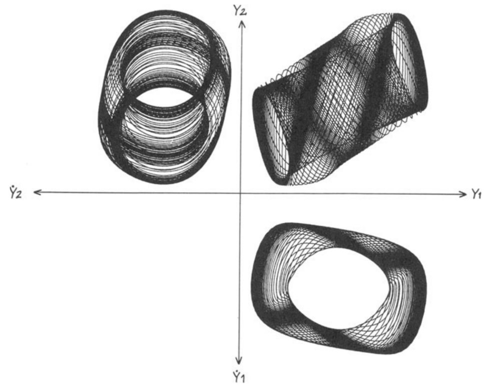
$$\ddot{Y}_i + (s_i + m_i)Y_i - m_j Y_j = (\nu_i - 1 - s_i)\dot{Y}_i - \frac{\nu_i}{3}\dot{Y}_i^3$$

Coupled oscillations

- **Dynamics**: Periodic solutions of a common period ω

$$Y_i = A_i(t)\cos\omega t + B_i(t)\sin\omega t$$

- A_i and B_i:slowly varying functions of time $\rightarrow$ **Quasiperiodic solution**
- **NO cycle is repeated**



Notes:

Notes:

- The phase diagrams for both regions are displayed in the second and fourth quadrants with Lissajou figures in the first and third showing the covariation of the two oscillators

Driven oscillations of a small economy

- System 1: **Small economy**, System 2: **Rest of the world**
  - World economy oscillates at its own amplitude, frequency
  - System 2 $\to$ **driving force** with amplitude K/ω

$$\ddot{Y}_1 + (s_1 + m_1)Y_1 - m_2 Y_2 = (\nu_1 - 1 - s_1)\dot{Y}_1 - \frac{\nu_1}{3}\dot{Y}_1^3$$

  - Set phase lead equal to zero, rescale

$$\ddot{Y}_1 + Y_1 = 5\dot{Y}_1 - \frac{5}{3}\dot{Y}_1^3 + \frac{K}{\omega}\cos\omega t$$

  - Differentiate:

$$\dddot{Y}_1 + \dot{Y}_1 = 5(1 - \dot{Y}_1^2)\ddot{Y}_1 - K\sin\omega t$$

- Define $X = \dot{Y}_1 \to$ **Forced Van-der-Pol oscillator**:

$$\ddot{X} + X = 5(1 - X^2)\dot{X} - K\sin\omega t$$

  - Period doubling scenario $\to$ **Chaos** for $K = 5$, $\omega \in (2.457, 2.463)$

Notes:

Kapitaniak, following Parlitz and Lauterborn, has demonstrated that the above equation has solutions that go through a period doubling cascade to chaos when $k = 5$ and ω takes values in the interval from 2.457 to 2.463.

Questions

1. What are the three main assumptions to derive a Cobb-Douglas production function?
2. Recall the conditions to obtain stable/unstable oscillations and fixed points.
3. Why do we observe a hysteresis in the Kaldor model?
4. What is a van-der-Pol oscillator? Why is it relevant for economic dynamics?
5. Why is the existence of limit cycles a sign of business cycles?
6. Recall the definition of deterministic chaos. Should we always expect chaotic business cycle dynamics?
7. Describe the relation between models of coupled systems and and models of forced oscillations.

Notes:

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Derivation of production function

Continuous models of business cycles

Resilience and performance

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