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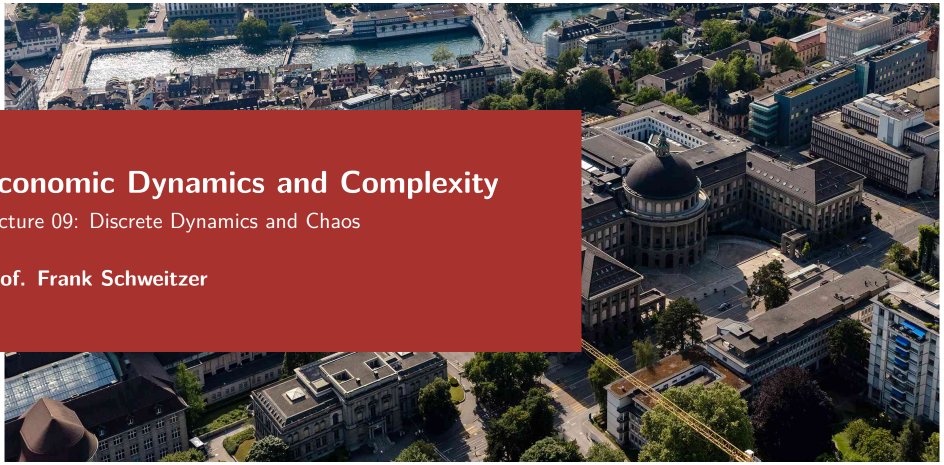


# Economic Dynamics and Complexity

Lecture 09: Discrete Dynamics and Chaos

**Prof. Frank Schweitzer**

Notes:

## Outline

**Discrete time**

Logistic map

Deterministic chaos

Chaos in manufacturing systems

Chaos in time-continuous systems



Notes:

## Continuous vs. discrete time

- **Continuous time:** Mathematical abstraction
  - Time step $\Delta t \to 0$, convenient math treatment
  - **Irreversibility:** The arrow of time (in one direction)
- **Discrete time:** In line with experience, measurements
  - Different units of $\Delta t$ (hour, day, ....)
  - Smallest unit: Planck time $\approx 5 \times 10^{-44} s$
- $\Delta t$ required for **numerical calculations**
  - Introduce *discrete time*: $t_n = t_{n-1} + \Delta t = t_0 + n\Delta t$
  - Discretize dynamic equation: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, Euler Method

$$\mathbf{x}(t_0 + \Delta t) \approx \mathbf{x}_1 = \mathbf{x}_0 + \mathbf{f}(\mathbf{x}_0) \cdot \Delta t$$
$$\mathbf{x}_{n+1} = \mathbf{x}_n + \mathbf{f}(\mathbf{x}_n) \cdot \Delta t$$

  - Correct value at time $t_n$: $x(t_n)$, approximate value $x_n$
  - Error $E := ||\mathbf{x}(t_n) - \mathbf{x}_n||$: $E \propto (\Delta t)$

**What is time?**

*"If no one asks me, I know what it is. If I wish to explain it to him who asks, I do not know."*
Saint Augustine

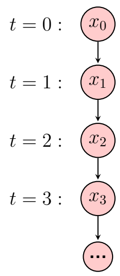
- Improved Euler: $E \propto (\Delta t)^2$
- 4th-order Runge-Kutta: $E \propto (\Delta t)^4$

$$\begin{aligned}
\mathbf{x}_{n+1} &= \mathbf{x}_n + 1/6(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4) \\
\mathbf{k}_1 &= \mathbf{f}(\mathbf{x}_n)\Delta t \\
\mathbf{k}_2 &= \mathbf{f}(\mathbf{x}_n + 1/2\mathbf{k}_1)\Delta t \\
\mathbf{k}_3 &= \mathbf{f}(\mathbf{x}_n + 1/2\mathbf{k}_2)\Delta t \\
\mathbf{k}_4 &= \mathbf{f}(\mathbf{x}_n + \mathbf{k}_3)\Delta t
\end{aligned}$$

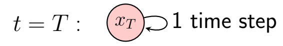


## Discrete case: Long term solutions

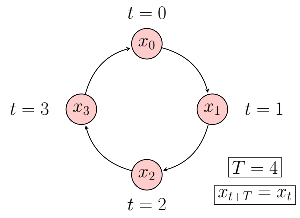
- Discrete time: $t = 0,\ t = 1,\ \dots t = n$
  - Time steps of size $\Delta t = 1$
  - Discrete orbit $\mathcal{X}(t) = \{X(t)\} = X_t$: Set of all values $X$ takes on over time $t$
- Discretization $\to$ **New dynamical regimes**
  - Impacts **number**/ **stability** of stationary solutions
  - Fixed Points, $\mathcal{X} = \{\bar{X}\}$
    - $X_t = X_{t+\Delta t} = \bar{X}$ for all $t$
  - Periodic Orbits, e.g. $\mathcal{X} = \{X_0, X_1, X_2, X_3\}$
    - $X_t = X_{t+T}$ for all $t$, where $T$ is the period
    - Solution encompasses a finite number of states




**Fixed points:**




**Periodic orbits:**






**Notes:**

**Remark:**

- Smaller time steps seem to be more accurate, but require more computation.
- Each computation involves a *round-off error*, that accumulates in a serious way, if $\Delta t$ is too small!

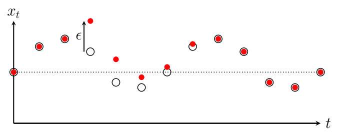
**Notes:**

- In the two cases above we are assuming that the starting point is in the particular orbit chosen, i.e. $X_0 \in \mathcal{X}$
- The problem: the periodic orbit can become very large. How can we still know that it is periodic, i.e. that the sequence of states will repeat some day?

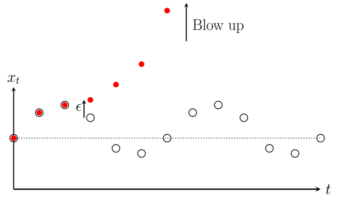
## Discrete case: Stability

- **Stability** of fixed point or periodic orbit:
  For small perturbations the new orbit stays 'close' to the original orbit
- Example: Small perturbation $\varepsilon$ of a periodic orbit
  - Stability: Orbit is regained
  - Instability: Orbit is lost

**Stable periodic orbit:**




**Unstable periodic orbit:**






## Outline

Discrete time

**Logistic map**

Deterministic chaos

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Notes:

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## Remember: Saturated growth

- **Logistic equation:** (Verhulst, 1838)
  - **Size dependent growth factor:** $\alpha(x) = a - bx$
  - $b$ is assumed to be small
  - **Continuous** dynamics, $x(t) > 0$

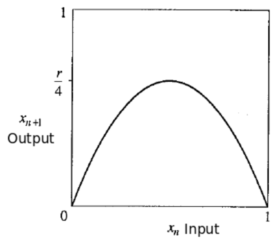
$$\frac{dx(t)}{dt} = \alpha(x)x = ax - bx^2 \; ; \quad x(t \to \infty) = \frac{a}{b}$$

- **Discretization, transformation: Logistic map**
  - $t \to n$: $x(t) \to x_n$, $x_n \to z_n$

$$\begin{aligned} x_{n+1} &= x_n + a\,x_n - b\,x_n^2 \\ z_{n+1} &= \frac{b}{1+a} x_{n+1} = \frac{b}{1+a}\left[(1+a)x_n - b\,x_n^2\right] \\ \mathbf{z_{n+1}} &= \mathbf{r\,z_n(1 - z_n)} \; ; \quad (\mathbf{r = 1 + a}) \end{aligned}$$

  - **Discrete dynamics:** $0 \le z_n \le 1$

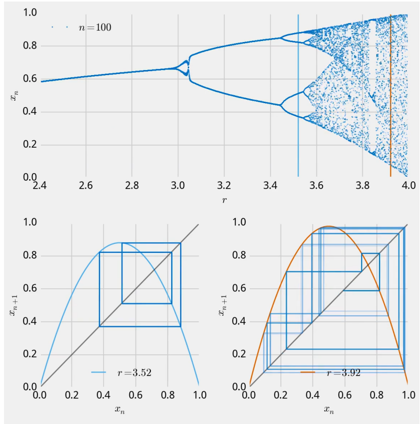
$$\boxed{z_{n+1} = r\, z_n(1 - z_n)}$$

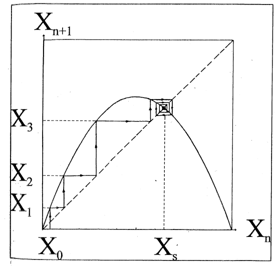



- Control parameter $r > 0$
- Initial condition: $0 < z_0 < 1$



## Iterating the logistic map




- $z_n$ is now called: $x_n$
- **Role of** $r$
  - Scales the time step: $r \times \Delta t$
  - Determines the max of the parabula
  - if $0 \le x_n \le 1$, then $0 \le r \le 4$



**Notes:**

- The logistic equation was rediscovered by R. Pearl in 1920 and by A. Lotka in 1925.

**Notes:**

- Understand that the role of the diagonal in the recurrence plot is to map back $x_{n+1}$ on the $x_n$ axis, to become the starting point for the next iteration.
- Can $x_0$ be zero? Explain!
- The video shows the iterated $x$ values for 2 different r values ($r = 3.52$ and $r = 3.92$). For $r = 3.52$, $x$ settles and oscillate between 2 values, where as for $r = 3.92$, the $x$ value is chaotic, with no observable pattern.

# Role of control parameter $r$

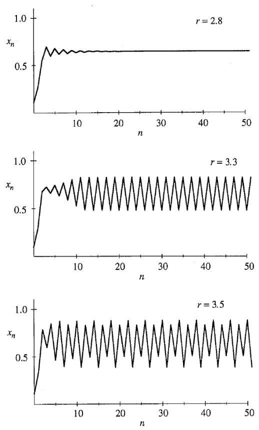
$0 < r < 1$: $x^{stat} \to 0$ (recall: $r = a + 1$, i.e. $a < 0$)

$1 < r < 3$: $x^{stat} = 1 - (1/r) \Rightarrow$ **stationary regime**

- Because $x_{n+1} = x_n \Rightarrow 1 = r(1 - x)$
- Remember: $x = a/b$ is the same as $x = 1 - (1/r)$

$3 < r < 4$: **Oscillations** $\to$ period doubling

- $r = 3.3$: 2 solutions (period-2 cycle, stable)
- $r = 3.5$: 4 solutions (period-4 cycle, stable)
- period-doublings: $8, 16, 32, \ldots$ as $r$ increases






# Period doubling

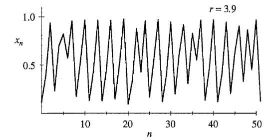
- $r_n$: value of $r$ where a $2^n$–cycle first appears

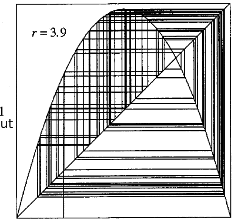
$$\begin{array}{lll} r_1 = 3 & \text{period} & 2 \text{ is born} \\ r_2 = 3.449 & & 4 \\ r_3 = 3.544 & & 8 \\ \vdots & & \\ r_\infty = 3.569 & & \infty \end{array}$$

- Sequence of $r_n$ converges to $r_\infty = 3.56994$
- Universal Feigenbaum constant

$$F = \lim_{n \to \infty} \frac{r_{n+1} - r_n}{r_{n+2} - r_{n+1}} = 4.6692$$

- **Scaling** of $r_n$: $r_n \approx r_\infty - cF^{-n}$ ; $c = 2.632$




**Chaos at** $r_\infty$**:** Sequence of $\{x_n\}$ never settles down to a fixed point or a periodic orbit



**Notes:**

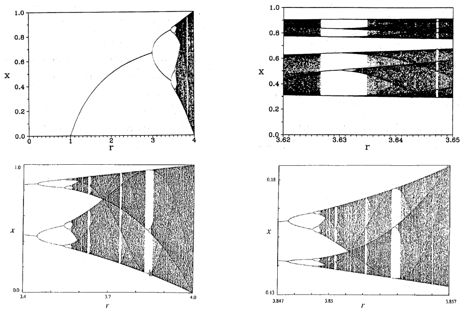
- Pictures: Strogatz (2000), p. 354
- Important note: for the continuous time dynamics, there is ONLY the solution $x = a/b$, i.e. $x = 1 - (1/r)$. But for the discrete dynamics, for larger $r$ solutions are found that do not exist for the continuous time dynamics.
- This becomes clear if you think of $r$ as the scaling parameter of the time step. For large enough $r$, the "integration" with large time steps becomes numerically unstable.
- Remember that $r = 1 + a$. For a "realistic" growth system we would expect $r$ to be not much bigger than 1. Refering back to the model of saturated growth, you can see why a large $r$ (and thus a large $a$) would lead to instability: With a large $a$, the growth rate $\alpha(x)$ becomes large for small $x$, and small for large $x$ (remember: $\alpha = a - b\,x$). The system grows a lot in one timestep, and then decreases a lot in the next, possibly never becoming stable.

**Notes:**

- Strogatz (2000), pp. 355, 356
- Chaos means here that the sequence of $x$ to be followed becomes infinitely long. Does it also mean that all $x$ between 0 and 1 are involved? Obviously not. You can get an infinite number of points also in a finite domain (recall the meaning of irrational numbers)

## Orbit Diagrams: “Chaos” with “windows”

- $r_\infty = 3.569$: 1st appearance of chaos $\to$ attractor changes from finite to infinite set of points
- $r_\infty \leq r \leq 4$: **Mixture** of **order** and **chaos**
  - Windows appear periodically
  - **Example:** large window near $r \sim 3.83$ with stable period-3 cycle
  - **Self-similarity**: pattern repeats at different scales
- $r = 4$: fully developed chaos





## SS09: Logistic Map Exploration in Python

- Exploring the Logistic Map with Python.
- **Learning Objectives:**
  - Contrast the dynamics of the logistic equation in continuous time with the logistic map in discrete time.
  - Examine the effects of the control parameter on the behavior of the logistic map.
  - Use a Cobweb diagram to visualize and understand the behavior of the Logistic map.
  - Study the progression to chaos through period-doubling.
  - Construct and analyze the bifurcation diagram.



**Notes:**

- R. Mahnke, *Nichtlineare Physik in Aufgaben*, Stuttgart: Teubner, 1994, pp. 169, 171

- Sometimes also called the "bifurcation diagram".
- The vertical axis of the diagram shows the possible values $x_n$ can take after many iterations. The role of $r$ is clearly visible in this diagram: For $r < 1$, $x_n$ will always approach zero for large $n$. For $1 < r < 3$ there is only one possible value for $x_n$, at the stable point. For $r > 3$ the number of possible values grows, and for $r \to 4$, $x_n$ can take on any value, if $n$ is large enough.
- The right figure shows the "windows", where there is a finite amount of possible values for $x_n$.

**Notes:**

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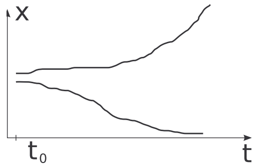
## Chaos: A working definition

- *On one hand:* **Deterministic** dynamics
  - initial condition $x_0$ fixed, coupling constant $r$ fixed
  - each step determined: $x_n \to x_{n+1}$
- *On the other hand:* Long-term **forecast impossible** $x_n$ ($n \to \infty$)
  - Reason: **Exponential divergence** of trajectories
  - Transition: **Order** $\to$ **Chaos** (dependent on $r$)




- **Chaos**: **Aperiodic long-term behaviour** in a **deterministic** system that exhibits **sensitive dependence** on initial conditions.

1. **aperiodic long term behavior**: Trajectories which do *not* settle down to fixed points, periodic orbits or quasiperiodic orbits as $t \to \infty$.
2. **deterministic**: System has no random or noisy inputs or parameters $\Rightarrow$ irregular behavior arises from the system's nonlinearity
3. **sensitivity to initial conditions**: Nearby trajectories separate *exponentially fast* $\Rightarrow$ system has a *positive Liapunov exponent* $\lambda$



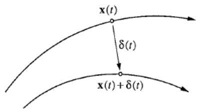
**Notes:**

**Notes:**

- Quasiperiodic Orbit: The type of motion executed by a dynamical system containing two incommensurate frequencies.
- The system $\dot{x} = x$ is deterministic and shows exponential divergence of nearby trajectories. Should we call this system chaotic?
  - Answer: No. Trajectories are repelled to infinity and never return. So infinity acts like an attracting *fixed point*.
- *Chaotic* behavior should be *aperiodic* $\Rightarrow$ excludes fixed points as well as periodic behavior. (Strogatz, 2000, p. 324)

# Calculating the Liapunov Exponent

- Consider initial condition $x_0$ and a nearby point $x_0 + \delta_0$
- Distance $\delta_n$ between trajectories after $n$ iterations:




$$x_{n+1} = f(x_n) \;\rightarrow\; x_{n+1} + \delta_{n+1} = f(x_n + \delta_n)$$
$$\frac{\delta_{n+1}}{\delta_n} = \frac{f(x_n + \delta_n) - f(x_n)}{\delta_n} = f'(x_n)$$

- Calculate distance $|\delta_n| = |\delta_0| e^{n\lambda}$

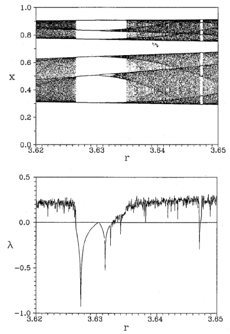
$$\frac{\delta_n}{\delta_0} = \frac{\delta_n}{\delta_{n-1}} \cdot \frac{\delta_{n-1}}{\delta_{n-2}} \cdot \ldots \cdot \frac{\delta_1}{\delta_0}$$
$$\ln\left|\frac{\delta_n}{\delta_0}\right| = \ln\left|\frac{\delta_n}{\delta_{n-1}}\right| + \ln\left|\frac{\delta_{n-1}}{\delta_{n-2}}\right| + \ldots + \ln\left|\frac{\delta_1}{\delta_0}\right| = \sum_{i=0}^{n-1} \ln|f'(x_i)|$$

- **Liapunov exponent** $\lambda$ for $\delta_0 \to 0$:

$$\lambda = \frac{1}{n} \ln\left|\frac{\delta_n}{\delta_0}\right| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)|$$

  - $\lambda > 0$: **necessary but not sufficient** for chaos

**Example: Logistic map**

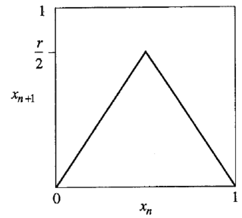





# Example: Tent map

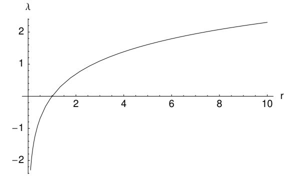
- **Piecewise linear function:**

$$f(x) = \begin{cases} rx & 0 \le x \le 0.5 \\ r - rx & 0.5 < x \le 1 \end{cases}$$




- **Tent map** $x_{n+1} = f(x_n)$:
  - Chaotic behavior for $r > 1$, i.e. $\ln(r) > 0$
  - Calculation of Lyapunov exponent: Since for all $x$: $f'(x) = \pm r$

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln\left|f'(x_i)\right|$$
$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln r = \ln(r)$$




  - **Control parameter** $r = 1$ determines transition to chaos



**Notes:**

- A positive maximal Liapunov exponent is usually taken as an *indication* that the system is chaotic. But note that this describes a necessary, but not a sufficent condition. Remember that other conditions have to be met, as well.
- Top Figure: Strogatz, p. 321, Bottom Figure: Mahnke (1994), p. 171

**Exponential divergence of trajectories:**
**Question:** If $x$ is bound to values between 0 and 1, the maximum difference $\delta_n$ can be not larger than 1. But shouldn't exponential divergence imply that it becomes infinity?
**Answer:** We are considering the divergence of trajectories starting from two *very* close $x_0$, e.g. 0.5 and 0.50000001. So, $\delta_0 = 10^{-8}$. Hence, for a typical $\lambda$ of 0.1, it would take $n \ge 180$ iterations, before $\delta_n \to 1$, which is quite close to $n \to \infty$. If you want larger $n$, just choose $\delta_0$ much smaller!
After all, what matters is according to what rule $\delta_n$ grows over time, and this is indeed *exponentially* (independent of whether it can reach infinity).

**Notes:**

# Example: Logistic map

- **Fixed points:** $x_{n+1} = x_n \to x = rx(1-x)$

$$rx^2 + x(1-r) = 0 \;\to\; x_1^\star = 0\,; \quad x_2^\star = 1 - \frac{1}{r}$$

  - 1st derivative: $\lambda = \ln|f'(x)| = \ln|r(1-2x)|$
    $\lambda_1 = \ln r\,; \quad \lambda_2 = \ln|2-r|$
  - Stability analysis $\to$ **bifurcations:** $r_1^{\rm cr} = 1$, $r_2^{\rm cr} = 3$
- **Periodic orbits!** 2-cycle $\to x = f(f(x)) \equiv f^2(x)$

$$x = r^2 x(1-x)\left[1 - rx(1-x)\right]$$

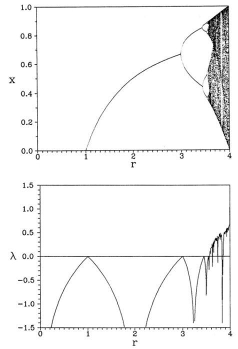
  - 4 Solutions: $x_1^\star = 0$, $x_2^\star = 1 - (1/r)$

$$x_{3,4} = \frac{1}{2}\left(1 + \frac{1}{r}\right) \pm \frac{1}{2r}\sqrt{r^2 - 2r - 3}$$

  - Stability for $3 < r < 3.449$

$$-1 < f'(x_1^\star) f'(x_2^\star) < 1$$
$$-1 < 4 + 2r - r^2 < 1$$

**Dynamics:** $x_{n+1} = r x_n(1 - x_n)$






**Notes:**

- Source: Mahnke (1994)
- Remember that chaos does not mean that all $x$ between 0 and 1 have to be involved. The Lyapunov exponent shows that chaos also exist in a finite subset of possible $x$.

# Deterministic chaos: Unpredictable dynamics, but exactly solvable

- **Chaotic regime:** $r = 4 \to x_{n+1} = 4\,x_n(1 - x_n)$
- Variable transformation: $x_n = \frac{1}{2}\left[1 - \cos(2\pi y_n)\right]$
- Recursive equation:

$$\frac{1}{2}\left[1 - \cos(2\pi y_{n+1})\right] = \frac{1}{2}\left[1 - \cos(4\pi y_n)\right]$$

- Result: $y_{n+1} = 2y_n \mod 1 = 2^n y_0 \mod 1$
- **Closed-form solution** dependent on initial condition

$$x_n = \sin^2\left(2^n \arcsin\sqrt{x_0}\right)$$

  - **rational** $x_0 < 1 \to$ fixed points, periodic cycles
    e.g. $x_0 = 1/5 \to x = (5 \pm \sqrt{5})/8$ (period-2 cycle)
  - **irrational** $x_0 \to$ chaos, Application: stability of planetary system



**Notes:**

- Mahnke (1994), p. 176

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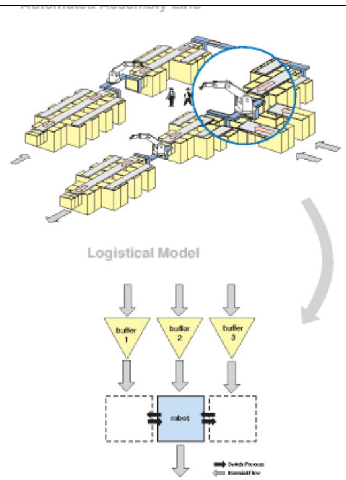
## Chaos in manufacturing sytems




- **Example: car assembly line**
  - focus on parallel buffers (tanks) with limited capacity
  - dynamic system: influx/outflux for each buffer
  - optimal strategy: prevent *overfilling* or *vacancies*
- what is the optimal schedule for filling/emptying buffers?



Notes:

Notes:

- A system without irregularities can behave chaotically. Here we consider manufacturing systems without any irregularities, such as breakdown of parts or irregular influx/outflux.

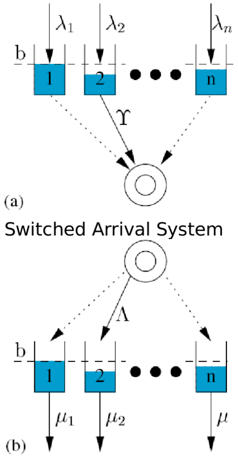
## Server and arrival systems

- Model with $i = 1, \dots, N$ parallel buffers, and one server
  - buffer content $x_i$, maximum capacities $b_i \equiv b$
  - **Scheduling task:** NO buffer should be empty $x_i \leq 0$ or full $x_i \geq b$

(a) **Switched Server System:**
  - server has to **empty** all buffers with rate $\Upsilon$
  - buffers are filled continuously with rates $\lambda_i$

(b) **Switched Arrival System:**
  - server has to **fill** all buffers with rate $\Lambda$
  - buffers are emptied continuously with rates $\mu_i$

Switched Server System






## Modeling the server system

1. **System dynamics implies conservation of mass**

$$\text{inflow rate} \sum_{m=1}^{N} \lambda_m = 1 = \text{outflow rate } \Upsilon$$

$$\sum \frac{dx_i}{dt} = inflow(t) - outflow(t) = 0$$

2. **Dynamics of buffer** $x_i$ for $t \leq s \leq t + \tau$:
   - Server switches at time $t$ and again at time $t + \tau$
   - $q(t) = j$: position of server at time $t$

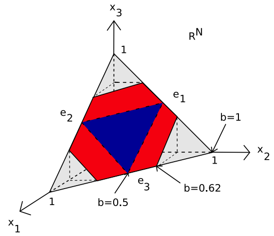
$$x_i(s) = \begin{cases} x_i(t) + \lambda_i \cdot (s - t) & \text{if } i \neq j \\ x_i(t) - (\Upsilon - \lambda_i) \cdot (s - t) & \text{if } i = j \end{cases}$$

3. **Switching rules:**
   - Buffer $x_i$ is full: Server instantaneously switches to $i$
   - If $i$ is empty, server switches to the next buffer

- Each buffer can only be filled up to $b \leq C$: $x_i(t) \leq b, \forall i$
- Total content is constant:

$$x_1(t) + x_2(t) + x_3(t) = C = 1$$

- Feasible region: Hexagon Simplex of $(C, C, C)$






**Notes:**

- What is a **rate**? It is some quantity *per time unit*, i.e. it is a flow variable (either inflow or outflow).
- It is also possible to model systems with different buffer sizes per tank, i.e. $b_i$ instead of just $b$.


K. Peters, J.Worbs, U. Parlitz, and H.P. Wiendahl.
Manufacturing Systems with Restricted Buffer Sizes.
Nonlinear Dynamics of Production Systems, 2004.

Peters, K. and Parlitz, U.
Hybrid systems forming strange billiards
International Journal of Bifurcation and Chaos, 2003
volume 19, number 9, pages 2575-2588


**Notes:**

- Note: the server is at position $j$
- all buffers $i \neq j$ just fill automatically at their individual rates $\lambda_i$ for the time interval $s - t$
- the one buffer with the server also fills automatically at rate $\lambda_i$, but gets emptied at the same time at rate $\Upsilon = 1$, hence $(\Upsilon - \lambda_i) \times [s - t]$, or $(1 - \lambda_i) \times [s - t]$
- the next buffer that becomes full $b = x_i$ is the one that will be served. This cannot be $j$ because the server is already at position $j$.
- for a system with three buffers (buffers), the state space corresponds to a plane in 3 dimensional space
- the state space is constrained to this plane because the total content within all three buffers must remain constant (inflow = outflow)
- because buffers cannot have a negative content, this further restricts the state space to the strictly positive portion of the plane
- the control parameter, b, also imposes a constraint on the state space, by defining the maximum possible values of the state
- the hexagon forms because when a buffer $x_1$ is empty there can be multiple possible states of the other two buffers such that

$$x_2 + x_3 = C \,\&\, x_2 \leq b \,\&\, x_3 \leq b$$

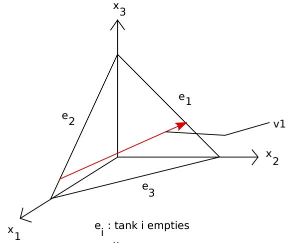
- See also: Chase et al (1993)

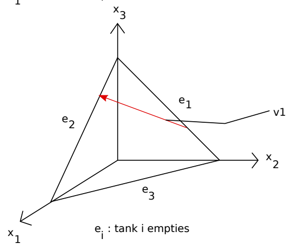
# State progression for the switched server system

- **Dynamics** with $\Upsilon = 1$:

$$\dot{x}_i = \begin{cases} \lambda_i - 1, & \text{if } i = q \text{ (location of server)} \\ \lambda_i, & \text{if } i \neq q \end{cases}$$

- **Example** $i = 1$: $\lambda_1 = 0.1,\ \lambda_2 = 0.5,\ \lambda_3 = 0.4$
  - Dynamics $\dot{x} = [\,-0.9,\ 0.5,\ 0.4]^T \to x_1$: emptied , $x_2$ and $x_3$ filled
  - Continues until $x_1$ becomes empty, or $x_2$ or $x_3$ become full
- **Example** $j = 2$: Dynamics now $\dot{x} = [0.1,\ -0.5,\ 0.4]^T$
  - $q$ changes from $i = 1$ to $j = 2$ (depending on initial conditions, also change to $j = 3$ possible
  - Server now empties buffer $j = 2$ at the rate $\dot{x}_2 = \lambda_2 - 1$.

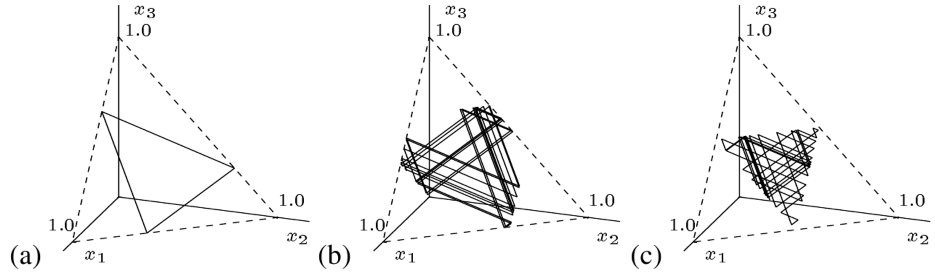





# Occurrence of chaos dependent on $b$

- System dynamics reveals the structure of a strange (chaotic) billiard
  - Current state moves uniformly and linearly inside the bounded region and gets reflected at the boundary.
  - Chaos becomes apparent as the buffer size, b, becomes too small




b=1.0 b = 0.62 b = 0.5



**Notes:**

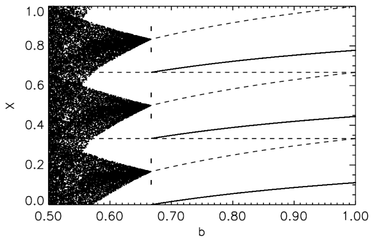
- The red line (v1) shows the direction in which the volumes of the buffers are being modified.
- Example: the server switches from buffer 1 to buffer 2. Therefore all the red lines move toward the boundary marked between $x_1$ and $x_3$.
- Precisely, the values of $x_2$ become smaller (thanks to the server), while all other values ($x_3$, $x_1$) can only increase (thanks to $\lambda_3$, $\lambda_1$).
- If $x_2 = 0$, the server has to move again. Whether it moves to buffer 1 or 3 depends on the initial condition.

**Notes:**

- The figures highlight the fact that with smaller b the server switches more often as one buffer is filled before the one that is served has been emptied
- This is reflected by the increasing number of switching points in the interior of the triangle (along the perimeter of the hexagon described)

## Bifurcation Diagram

- Switched server system with 3 buffers where $\lambda_1 = 1/3$, $\lambda_2 = 1/3$, $\lambda_3 = 1/3$
- For $b < \frac{2}{3}$ chaotic behaviour occurs with period doubling scenario




- **Chaotic regime**: dynamics of buffer contents is **not predictable** in the long run
  - Server switching times cannot be planned $\to$ inefficiency in the production process
- Chaos can be **mitigated** or avoided:
  - Increase buffer capacity $b$ (may be costly)
  - Modify the filling rates $\lambda_i$.
  - Chaos control methods (e.g. OGY): Chaos $\to$ breaks down and becomes periodic motion



## Outline

Discrete time

Logistic map

Deterministic chaos

Chaos in manufacturing systems

**Chaos in time-continuous systems**



Notes:

- We do not aim to understand the complex bifurcation diagram completely
- Meaning of parameter $X$: Normalized 1-dimensional representation of 3-dimensional simplex
- Again, note how the system enters chaotic regime as the buffer capacity, b, becomes too small
- Since all $\lambda_i$ have the same value, bifurcation diagram is symmetric
- Penetration of straight dashed lines corresponds to a reflection point at a boundary in the simplex.
- OGY-technique: see Fritz Colonius, Lars Grüne: Dynamics Bifurcations and Control, Springer 2002.
  - The disadvantage of having chaos in the system is the difficulty of prediction. However chaos is not inherently bad.
  - In the case of b = 0.5, the system is too close to breakdown, any disturbance that pushes b to be less than 0.5 will collapse the system
  - In the case of b = 0.62, as long as the hardware and software can handle frequent switching, it is acceptable.

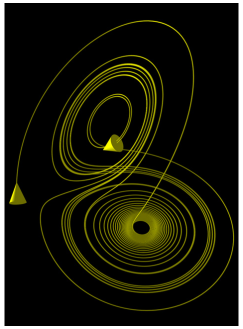
Notes:

## Is there chaos in the system?

1. **Discrete dynamical systems**: Possible in any dimension
   - Discretize simple continuous dynamics (e.g. population growth)
   - Chaos obtained only in a subset of the parameter space
2. **Continuous dynamical systems**: 3 or more dimensions
   - Example: **Lorenz equations** (1963) ("butterfly effect")
     - Developed for weather forecast → convection rolls in the atmosphere
     - also holds for laser, dynamos, waterwheel, ...

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= x(\rho-z)-y\\ \dot{z} &= xy-\beta z\end{aligned}$$

     - *1. Nonlinearity:* quadratic terms $xy$ and $xz$
     - *2. Symmetry:* Same equations if $(x,y) \to (-x,-y)$
   - *Chaotic* attractors also occur in real, continuous systems!

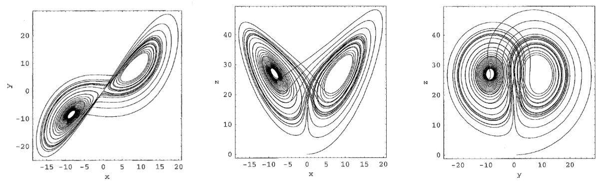




**Notes:**

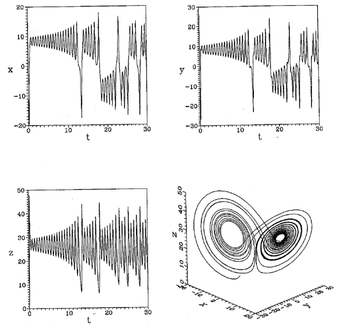
- If you want to know if there is chaos in the system, keep in mind the **3 point working definition**.
- From Wikipedia: The term butterfly effect is related to the work of Lorenz, who in a 1963 paper for the New York Academy of Sciences noted that "One meteorologist remarked that if the theory were correct, one flap of a seagull's wings could change the course of weather forever." Later speeches and papers by Lorenz used the more poetic butterfly. According to Lorenz, upon failing to provide a title for a talk he was to present at the 139th meeting of the AAAS in 1972, Philip Merilees concocted Does the flap of a butterfly?s wings in Brazil set off a tornado in Texas? as a title.
- *Chaotic waterwheel*: Mechanical model of Lorenz equations invented at MIT in 1970s (for construction see: Strogatz, 2000, p 302)

## Chaotic dynamics in continuous time



Source: Boccara

- **Irregular oscillations persist as $t \to \infty$**
  - **Aperiodic**, no self-intersection, no repetition
  - Number of circuits on either side is **unpredictable**
- **"Strange attractor":** Zero volume, but infinite surface area
  - Dimension: 2.05 (numerical experiments → fractals)



Source: Mahnke (1994), p. 134



**Notes:**

- (right) Source: Mahnke (1994), p. 134
- Note: the term *strange attractor* was coined by Ruelle and Takens (1971).
- (left) Source: Boccara, pp. 174/175
- Trajectory appears to cross itself repeatedly → artefact of projecting 3-dim trajectory into 2-dim plane

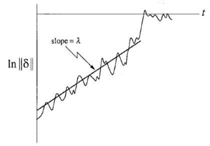
## Liapunov exponent for the Lorenz model

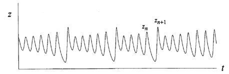
- Motion **on** the attractor **after** transients decayed
  - very sensitive dependence on initial conditions
  - two close trajectories rapidly diverge from each other

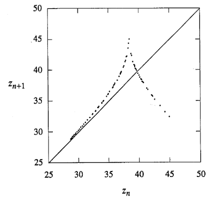
$$\mathbf{x}(t) + \delta(t) \quad \rightarrow \quad ||\delta(t)|| \sim ||\delta_0||e^{\lambda t}$$

  - Numerical studies of the Lorenz attractor: $\lambda = 0.9$

- **Lorenz Map:** Trick to extract order from chaos
  - Lorenz' procedure: Numerical integration of $\dot{z} = xy - bz$
  - Plot maxima of $z(t)$: $z_{n+1}$ vs $z_n$ → **chaotic time series**
  - Lorenz map: $z_{n+1} = f(z_n)$ with $|f'(z) > 1|$






## Questions

1. What are periodic orbits? Why do they appear in discrete dynamics?
2. Derive the logistic map equation from the saturated growth dynamics.
3. Explain the logistic map and its control parameter. What is the meaning of a period-doubling scenario?
4. What is meant by deterministic chaos and how does it arise?
5. What is the Liapunov exponent and how can it be calculated?
6. What is the difference between the switched server system and the switched arrival system?
7. Explain the role of the maximum capacity, $b$, for the dynamics. What is the meaning of "chaos" in such a production system?
8. Is chaos an artefact or ar real world phenomenon? Does it occur in continuous time dynamics?



**Notes:**

- Strogatz (2000), p.321
- Note (1): the curve ln $||\delta||$ vs. $t$ is never a straight line, as the strength of the exponential divergence varies along the attractor.
- Note (2): Exponential divergence must stop when the separation is comparable to the "diameter" of the attractor ⇒ "saturation effect"

**Notes:**