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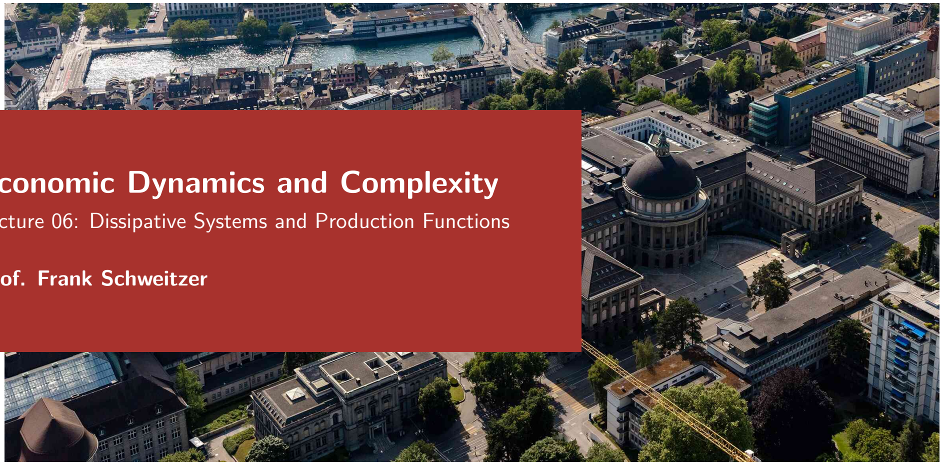


# Economic Dynamics and Complexity

Lecture 06: Dissipative Systems and Production Functions

**Prof. Frank Schweitzer**

Notes:

## Outline

**Dissipative systems**

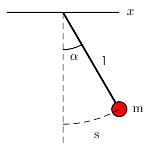
Production functions

Energy and capital input



Notes:

# Idealized vs. real systems: The pendulum



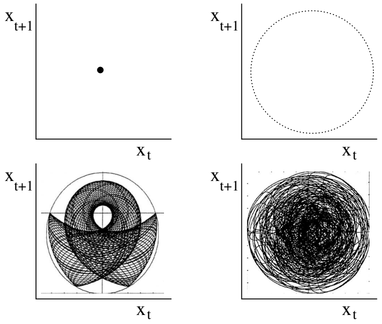

- **Ideal system:** Mathematical pendulum

$$\ddot{\alpha} = -\omega_0^2 \sin\alpha \; ; \quad \omega_0^2 = g/l$$

  - Linearization: $\alpha \approx x/l$, **no friction**
  - Possible solutions: at rest, periodic oscillations
    - depends only on *initial condition*
- **Real system:** Driven pendulum

$$\ddot{\alpha} = -\omega_0^2 \sin\alpha - \gamma\dot{\alpha} + A\sin(\omega_A t)$$

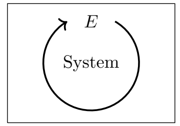
  - **Friction**: $-\gamma\dot{\alpha}$ is compensated by a periodic force
  - System is constantly pumped with energy
  - Possible solutions:
    - Regular motion with higher periodicity
    - Chaotic motion $\rightarrow$ **unpredictable**
- **Simple system with complex dynamics**
  - Precondition: Dissipation, energy import



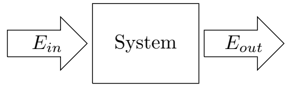



# Conservative vs dissipative systems

- **Conservative system:** Conserved quantity $E(\mathbf{x})$
  - Hamiltonian dynamics: $E = 1/2m\dot{x}^2 + V(x)$
    - *Example: Frictionless pendulum*
  - **Phase space:**
    - Stays constant $\rightarrow$ constant density
    - Existence of a real-valued continuous function that is constant on trajectories, $dE/dt = 0$
    - Stable/unstable nodes, also saddles and centers




- **Dissipative System:** Energy flows in and out
  - **Stationary Non-Equilibrium:** $E_{in} = E_{out}$
    - *Example: Driven pendulum*
  - **Phase space:**
    - Contracting or expanding $\rightarrow$ Inhomogeneous density
    - Sources (repellors) and sinks (attractors)



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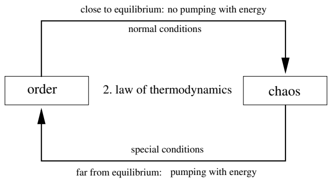
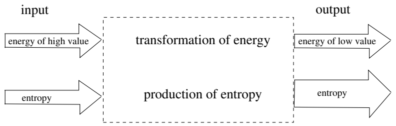
## The bigger picture: Entropy import and export

- **Living systems:** Ecology, Society, Economy, ...
  - ... Transportation, metabolic systems, infrastructure, ...
- **ALL** living systems require **import of energy**
  - 1. Law of Thermodynamics: Conservation of energy
  - Consequence: Energy can be only transformed
  - Import in form of **high valued energy** ("free energy")
  - Transformation produces **entropy**
- **ALL** living systems depend on **entropy export**
  - Export in form of **devalued energy**
  - Examples: Excretion, waste, heat
- **Result:** Order, Structure, Organization
  - **Non-equilibrium**: requires **maintenance**
  - Without Energy import/Entropy export: **Order** → **Chaos**



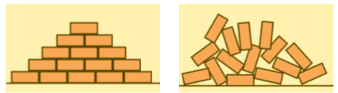





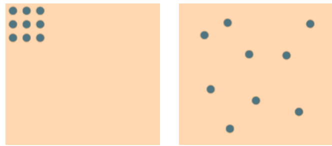

## The meaning of entropy: Measuring order and value

- **Entropy measures order**
  - Ordered states dissolve spontaneously → Increase of entropy
  - *Irreversibility:* no spontaneous reversal
  - Maintaining "order" requires high value energy
  - Exploiting low value energy requires more energy
- **Entropy measures value**
  - Example: Health insurance pays back
    $1 \times 500.000.000\ \mathrm{EUR} = 50.000.000 \times 10\ \mathrm{EUR}$
    What will you do with your 10 EUR?
    ... Increase of pensions/social welfare by 10 EUR?

**"Less is more"**: The lower the entropy, the better



**low entropy** **high entropy**





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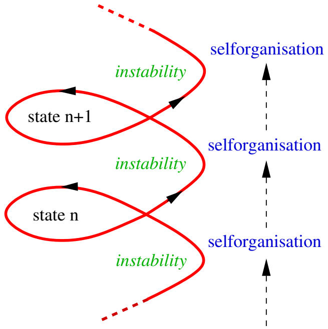
## The value of instability: Evolution

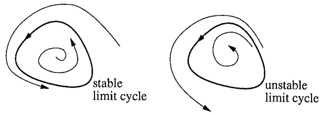
- **Mechanical systems:** Equilibrium = death
- **Living systems** (Ecology, economy, ...)
  - Stationary non-equilibrium: keeps the status quo
  - Stationary states can be stable/unstable
  - Random shocks continuously test stability
  - Resilience: Ability to return to a previous stable state
  - Importance of fast and slow processes (time scales)
- **Evolution**
  - Innovations challenge status quo ("creative destruction")
  - Feedback between system and environmental changes
  - Adaptation, learning and novelty as response
  - Self-organization plays major role (decentralized processes)
  - Open-ended: sequence of system states

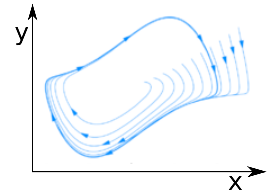





## Limit Cycles

- **Conservative systems:** center ("curl")
  - variety of closed trajectories → undamped oscillations,
  - amplitude depends on initial condtions
  - stable, but not asymptotically stable
- **Dissipative systems:** limit cycle
  - isolated closed trajectory → self-sustained oscillations
  - *neighboring* trajectories are not closed: either attracted or repelled by the limit cycle
  - periodic solution, can be stable or unstable
  - does not depend on initial conditions
  - does not exist in *gradient systems* (1-d)




**Limit cycle:** *No point* where $(\dot{x}, \dot{y}) = (0, 0)$



**Notes:**

Fast and slow processes: Remember the fairy tale "Der Hase und der Igel" (english version: 'The Tortoise and the Hare' - i.e. reverse order).
The consequence for modeling: We focus on the **slow** time scale that determines the system dynamics. Processes on the **fast** time scale have long reached their (quasi) equilibrium.

**Notes:**

## Hopf Bifurcation

- **Limit cycle on either side of the bifurcation point**
  - **2-dim example:** Oscillations with frequency $\omega$

$$\dot{r} = \mu r - r^3 \; ; \quad \dot{\theta} = \omega + br^2$$

- Control parameter $\mu < 0$: Stable spiral with $r^* = 0$
  - Exponentially damped oscillations $\rightarrow$ Equilibrium
- Control parameter $\mu > 0$: Decay changes into growth
  - **Supercritical** Hopf bifurcation
  - Unstable spiral at $r = 0$, Bifurcation point: $r = \sqrt{\mu}$
  - Emergence of a *stable limit cycle*

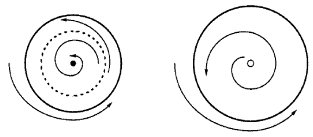



## Subcritical Hopf bifurcation

- After bifurcation trajectories jump to a **distant** attractor
  - Possible attractors: Fixed point, another limit cycle, infinity
  - 3-dim or higher: Chaotic attractor possible
- **Remark:** Analytical criteria difficult
  - Linearization cannot distinguish between sub- or supercritical hopf bifurcation
  - Quick and dirty approach: numerical solution, watch dynamics for different values of $\mu$

**2d Example:**

$$\dot{r} = \mu r + r^3 - r^5 \; ; \quad \dot{\theta} = \omega + br^2$$




(Strogatz, p. 252)

**Degenerate Hopf bifurcation**

- Non-conservative $\rightarrow$ conservative system at the bifurcation point
- No limit cycles on either side of the bifurcation points
- Fixed point becomes nonlinear center instead of weak spiral



**Notes:**

- Strogatz (2000), p. 250

**Notes:**

- limit cycle is always an attractor
- $\mu < 0$: two stable attractors: limit cycle, fixed point at the origin – between them is an unstable cycle (shown as dashed curve)
- as $\mu$ increases, the unstable cycle shrinks (for $\mu = 0$ to zero amplitude) $\rightarrow$ engulfs origin, which becomes unstable
- $\mu = 0$: subcritical bifurcation $\rightarrow$ unstable cycle shrinks to zero amplitude
- $\mu > 0$: one stable attractor: limit cycle, i.e. even trajectories used to remain near the origin have to "jump" to the distant attractor now (Strogatz (2000), p. 252)

## Outline

Dissipative systems

**Production functions**

Energy and capital input



## Economic theory of production

- Production: **Output** $\Leftarrow$ **Input**
  - *process*, that *combines* and *transforms* inputs into outputs
  - production technology: relationship between inputs and outputs
- **Production function**: $Y = \mathcal{F}(x_1, x_2, \ldots, x_n)$
  - Inputs: factors for production $x_1, x_2, \ldots, x_n$, output: $Y$
  - Key inputs: Capital ($K$) and Labor ($L$) $\Rightarrow Y = \mathcal{F}(K, L)$
  - **Input of high valued energy:** Economy as a **dissipative system**
- Important: $\mathcal{F}$ describes only technology, not economic behavior
  - substitute technology: $Y = a + bx_1 + cx_2 + \cdots$
    - *substitute* between different inputs (none really essential)
  - complementary technology: $Y = \min\{x_1, x_2, \ldots\}$
    - complementary inputs (all are essential)
  - Question: Is $\mathcal{F}$ different for substitute and complementary technologies?

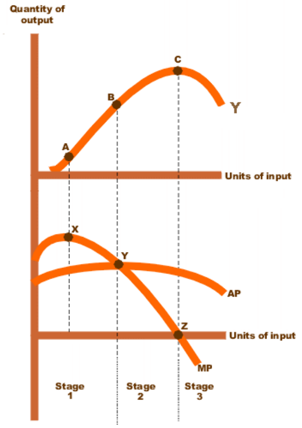


Notes:

Notes:

- This is not too far from the alchemistic idea of transforming lead into gold (using the 'philosopher's stone').
- parameters $a$, $b$, $c$, ... are to be determined empirically
- $a$ is the *fixed input*, $b$, $c$, ... are *variable input coefficients*

# Average and marginal products




- **Average Product**: $AP_{x_i} = Y/x_i$
  - average amount produced by each input unit
  - different for each input factor $\rightarrow i$
- **Marginal Product**: $MP_{x_i} = \Delta Y/\Delta x_i$
  - slope of the production function wrt a given input factor $x_i$
  - additional output produced by one more unit of input
  - all other inputs kept constant



# Returns to Scale

- given production function $Y = \mathcal{F}(K, L)$, e.g. $L = 5$, $K = 3 \Rightarrow Y = \mathcal{F}(5, 3) = 20$
  - How does output $Y$ changes if input is increased by a factor $\lambda$ (e.g. 20% $\rightarrow \lambda = 1.2$)

1. *constant returns to scale* (CRS): $\mathcal{F}_1(\lambda K, \lambda L) = \lambda \mathcal{F}_1(K, L)$
   - example: $\mathcal{F}_1(1.2 \times K, 1.2 \times L) = \mathcal{F}_1(6, 3.6) = \mathbf{24} = 1.2 \times \mathcal{F}_1(5, 3) = 24$
   - if all inputs are increased by $\lambda = 1.2$, output is increased exactly by $\lambda = 1.2$
2. *increasing returns to scale* (IRS): $\mathcal{F}_2(\lambda K, \lambda L) > \lambda \mathcal{F}_2(K, L)$
   - example: $\mathcal{F}_2(1.2 \times K, 1.2 \times L) = \mathcal{F}_2(6, 3.6) = \mathbf{33} > 1.2 \times \mathcal{F}_2(5, 3) = 24$
   - if all inputs are increased by $\lambda = 1.2$, output is increased **more** than $\lambda Y_2$
3. *decreasing returns to scale* (DRS): $\mathcal{F}_3(\lambda K, \lambda L) < \lambda \mathcal{F}_3(K, L)$
   - example: $\mathcal{F}_3(1.2 \times K, 1.2 \times L) = \mathcal{F}_3(6, 3.6) = \mathbf{22} < 1.2 \times \mathcal{F}_3(5, 3) = 24$
   - if all inputs are increased by $\lambda = 1.2$, output is increased **less** than $\lambda Y_3$



**Notes:**

- This plot of the production function assumes (i) an inflection point (A) and (ii) a decrease after C. Most production functions, in particular those discussed in **this** lecture, do not assume that.
  - inflection point at A: marginal product reaches a maximum
  - B: steepest possible line through the origin touches the production function curve, corresponds to the maximum average product
  - beyond C: the production function begins to decline
  - companies ideally want to operate in between points B and C (stage 2)
- Detailed explanation of the three stages in production function:
  - Stage 1: In Stage 1 (from the origin to point B) the variable input is being used with increasing efficiency, reaching a maximum at point B. Because the efficiency of both fixed and variable inputs is improving throughout stage 1, a firm will always try to operate beyond this stage. In stage 1, fixed inputs are underutilized.
  - Stage 2: Output increases at a decreasing rate, and the average and marginal products are declining. In this stage, the employment of additional variable inputs decreases the efficiency of variable inputs.
  - Stage 3: Too much variable input is being used relative to the available fixed input: variable inputs are overutilized. Both the efficiency of variable and fixed inputs decline throughout this stage.
- Source: `http://en.wikipedia.org/wiki/Production_function`

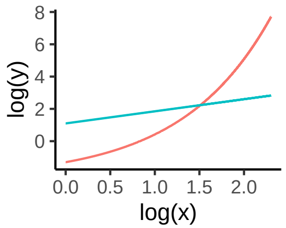
**Notes:**

- Returns to Scale: Explains the behavior of the rate of increase in output (production) relative to the associated increase in all the inputs (factors of production) in the long run.
- Source: `https://en.wikipedia.org/wiki/Returns_to_scale`
- IRS is of course what every firm wants. Can you think of conditions to get IRS, as opposed to DRS?

# Elasticity

- $y = f(x)$: relationship between output $y$ and input $x$
  - how *'elastic'* is $y$ if $x$ is changed? $\Rightarrow$ *elasticity*
  - remember: price elasticity of supply and demand (!)

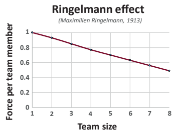
$$\eta = \frac{dy/y}{dx/x} = \frac{x}{y}\frac{dy}{dx} = \frac{d(\ln y)}{d(\ln x)}$$




- **Law of diminishing returns**
  - Short run: $MP = dy/dx$ declines with additional increase in input
    - difficult for firms to increase output if closer to maximum capacity of production
  - Relation to elasticity: $\eta = MP/AP$



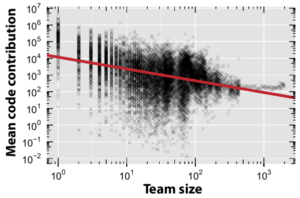
# Decreasing returns to scale in productivity




**Ringelmann effect (*social psychology*)**

- **larger** teams are **less** productive
- declining **motivation**, "social loafing"
- **overhead of coordination**

image: CC-by-SA Bart Derksen




- **Teams of OSS developers**
  - **robust log-linear regression:** $\alpha = 0.86 \pm 0.02$
  - $X \times 2.0 \Rightarrow Y \times 1.1$ (**OSS** projects!)
  - large variation across projects



**Notes:**

- $f(x)$ is a differentiable function of variable $x$
- note: $\frac{dx}{x} = d(\ln x)$
- Short run refers to an increased input given the same facilities, where long run would also include investments in production facilities.

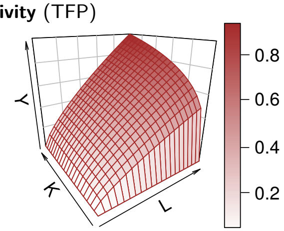
**Notes:**

- I. Scholtes, P. Mavrodiev, F.S.: From Aristotle to Ringelmann: A large-scale analysis of team productivity and coordination in Open Source Software projects, Empirical Software Engineering **21** (2016) 642-683

## Cobb-Douglas production function

- Specific form of a production function $Y = F(K, L)$: $\boxed{Y = A\, L^{\alpha} K^{\beta}}$
  - $A$: level of technology (assumed constant), aka **total factor productivity** (TFP)
  - two input variables: capital $K$, labor $L$
  - two exponents: $\alpha \to L$, $\beta \to K$ $\Rightarrow$ **elasticities**

$$\eta_L = \frac{L}{Y}\frac{dY}{dL} = A\alpha L^{\alpha-1}K^{\beta}\frac{L}{Y} = \alpha$$
$$\eta_K = \frac{K}{Y}\frac{dY}{dK} = A\beta L^{\alpha}K^{\beta-1}\frac{K}{Y} = \beta$$




  - $\boxed{\alpha + \beta = 1}$ $\Rightarrow$ Constant Return to Scale
    - empirics (Douglas) + mathematics (Cobb): $\alpha = 3/4$, $\beta = 1/4$
    - remember demand elasticity: $|E_d| < 1 \to$ **inelastic** (output less elastic wrt capital)



**Notes:**

- Total-factor productivity: In economics, total-factor productivity (TFP) is a variable which accounts for effects in total output not caused by inputs. For example, a year with unusually good weather will tend to have higher output, because bad weather hinders agricultural output. A variable like weather does not directly relate to unit inputs, so weather is considered a total-factor productivity variable.
- original Cobb-Douglas production function: $Y = AL^{3/4}K^{1/4}$
  - **History:** friendship between *Paul Douglas* and *Charles Cobb*
  - Douglas: PD in 1920s: working on the problem of input–output relations
  - survey by NBER: during the decade 1909-1918, the share of output associated with labor was fairly constant at about 74% ($\to 3/4$)
  - Cobb: which production function might account for this?
  - NBER: National Bureau of Economic Research, a very important economic institution in the US $\Rightarrow$ NBER Working paper series

## Empirics

- mostly $\alpha + \beta = 1$, but $\alpha \approx 2/3 \neq 3/4$, $\beta \approx 1/3 \neq 1/4$

PRODUCTION FUNCTION FOR AUSTRALIA, SELECTED FISCAL YEARS

| Cross-Section Studies and Fiscal Year | Observations ($N$) | Values of $k$ | SE of $k$ | Values of $j$ | SE of $j$ | $k + j$ |
|---|---|---|---|---|---|---|
| Australia: | | | | | | |
| 1913 ............. | 85 | .52 | .05 | .47 | .05 | .99 |
| 1923 ............. | 87 | .53 | .05 | .49 | .05 | 1.02 |
| 1927 ............. | 85 | .59 | .05 | .34 | .04 | .93 |
| 1935 ............. | 138 | .64 | .04 | .36 | .04 | 1.00 |
| 1937 ............. | 87 | .49 | .04 | .49 | .04 | .98 |
| Victoria: | | | | | | |
| 1911 ............. | 34 | .74 | .08 | .25 | .11 | .99 |
| 1924 ............. | 38 | .62 | .08 | .31 | .10 | .93 |
| 1928 ............. | 35 | .59 | .07 | .27 | .09 | .86 |
| New South Wales: | | | | | | |
| 1934 ............. | 125 | .64 | .04 | .34 | .03 | .99 |
| Average of all commonwealth and state studies . | 714 | .60 | .06 | .37 | .06 | .97 |
| Average of commonwealth studies only ..... | 482 | .55 | .04 | .43 | .04 | .98 |
| Average of state studies only ..... | 232 | .65 | .07 | .29 | .08 | .94 |



**Notes:**

- Here the production function is of the form: $P = bL^k K^j$
- The sum of exponents for labour and capital is approximately 1. This is close to constant return.
- The average exponent for labour, namely k, is 0.6.
- Douglas, Paul H. "The Cobb-Douglas Production Function Once Again: Its History, Its Testing, and Some New Empirical Values." Journal of Political Economy 84, no. 5 (1976): 903-15. `http://www.jstor.org/stable/1830435`.

## Properties of Cobb-Douglas function

❶ **Constant returns to scale:** $\alpha+\beta=1 \Rightarrow \boxed{F(K,L)=A\,L^{\alpha}K^{1-\alpha}}$

$$F(\lambda\,K,\lambda\,L)=A\,(\lambda\,L)^{\alpha}(\lambda\,K)^{1-\alpha}=A\,\lambda^{\alpha}\,\lambda^{1-\alpha}\,L^{\alpha}K^{1-\alpha}=\lambda\,F(K,L)$$

❷ Marginal products are **positive**:

$$(MP_K)=\frac{\partial F}{\partial K}=A\beta L^{\alpha}K^{\beta-1}=A(1-\alpha)\left(\frac{L}{K}\right)^{\alpha}>0$$
$$(MP_L)=\frac{\partial F}{\partial L}=A\alpha L^{\alpha-1}K^{\beta}=A\alpha\left(\frac{L}{K}\right)^{\alpha-1}>0$$

❸ Marginal products are **decreasing with inputs**:

$$\frac{\partial}{\partial K}(MP_K)=\frac{\partial^2 F}{\partial K^2}=-\alpha(1-\alpha)AL^{\alpha}K^{-\alpha-1}<0$$
$$\frac{\partial}{\partial L}(MP_L)=\frac{\partial^2 F}{\partial L^2}=\alpha(\alpha-1)AL^{\alpha-2}K^{1-\alpha}<0$$

❹ very common **logarithmic** form: $\boxed{\ln Y=\ln A+\alpha\ln L+\beta\ln K}$

- direct relation to empirics $\Rightarrow$ **linear regression**



## Outline

Dissipative systems

Production functions

**Energy and capital input**



**Notes:**

- If $\beta=1-\alpha$, then we can get last part of the marginal products with $\frac{L}{K}$, which is called *organic component*.
- But if the Cobb-Douglas production function has its general form

$$F(K,L)=A\,L^{\alpha}K^{\beta}$$

with $0<\beta<1$, then there are increasing returns if $\alpha+\beta>1$ but decreasing returns if $\alpha+\beta<1$, since

$$F(\lambda\,K,\lambda\,L)=A\,(\lambda\,L)^{\alpha}(\lambda\,K)^{\beta}=A\,\lambda^{\alpha}\,\lambda^{\beta}\,L^{\alpha}K^{\beta}=\lambda^{\alpha+\beta}\,A\,L^{\alpha}K^{\beta}=\lambda^{\alpha+\beta}\,F(K,L),$$

which is greater than or less than $\lambda\,F(K,L)$ as $\alpha+\beta$ is greater or less than one.

**Notes:**

## Deriving the dynamics for production

- **Production function** depends on inputs: $\hat{Y}[X_1(t), X_2(t), ....]$
  - Normalization $Y_0$ =const., i.e. $Y = \hat{Y} - Y_0$
  - Common inputs $X_j(t)$: Capital $K(t)$, labor $L(t)$,
  - New input: "energy" $E(t)$: general form of resources
- **General dynamics:**

$$\frac{dY}{dt} = \sum_j \frac{dY_{X_j}}{dt} \; ; \quad \frac{dY_{X_j}}{dt} = \frac{\partial Y}{\partial X_j}\frac{dX_j}{dt}$$

  - $dY_{X_j}/dt$: growth contributions from inputs $X_j$
  - Assumptions for $Y(X_j)$ and $dX_j/dt$ needed
- Cobb-Douglas production function:

$$Y[X_1(t), X_2(t), ...] = A\, X_1^{\alpha_1}(t)\, X_2^{(\alpha_2)}(t)....$$

  - **Elasticities:** $0 < \alpha_j < 1$: diminishing returns to scale
  - **Total factor productivity:** $A$ (production efficiency)



Notes:

## Dynamics of energy I

- **Power series:** $\dfrac{dE}{dt} = \dot{E} = \sum_{k=0}^{n} a_k E^k = a_0 + a_1 E + a_2 E^2 + \ldots$
  - Need to consider **sources** (inflow) and **sinks** (outflow) of energy
- **Saturation dynamics:** (1) logistic equation: $\dot{E} = f(E, \underline{a}) = a_1 E + a_2 E^2$
  - Stationary solutions $\dot{E} = 0$: $E_1^\star = 0$, $E_2^\star = -a_1/a_2$
  - For meaningful $E$: 1. $a_1 > 0$, 2. $a_2 < 0$ (saturation dynamics), 3. $|a_2| \ll |a_1|$
- **Stability analysis:** Gradient method: $f(x, \underline{a}) = -\dfrac{dV(x, \underline{a})}{dx}$

$$\frac{d^2 V(E, \underline{a})}{dE^2} = -a_1 - 2a_2 E \;\Rightarrow\; \left.\frac{d^2 V(E, \underline{a})}{dE^2}\right|_{r_1^\star} = -a_1 \; ; \quad \left.\frac{d^2 V(E, \underline{a})}{dE^2}\right|_{r_2^\star} = +a_1$$

  - $E_2^\star$ is stable, stability is determined by $a_1$ (comparably large)
  - What dynamics shall we expect from a very stable equilibrium?
    **Conclusion**: $a_2 = 0$ (!)



Notes:

## Dynamics of energy II

- **Saturation dynamics:** (2) linear equation: $\dot{E} = a_0 + a_1 E$
  - stationary solution $E^\star = -a_0/a_1$
  - for meaningful $E$: 1. **either** $a_1 < 0, a_0 > 0$ **or** $a_1 > 0, a_0 < 0$, 2. $|a_1| \ll |a_0|$
  - $a_1 > 0$: exponential growth, so $a_1 < 0$ (exponential decay) is the only option
  - $a_0 > 0$: makes sense, source term for energy
- **Stability analysis:** Method of small perturbations: $E(t) = E^\star + \delta E$

$$\underbrace{\frac{dE^\star}{dt}}_{=0} + \frac{d\delta E}{dt} = \underbrace{a_0 + a_1 r^\star}_{=0} + a_1 \delta E$$

$$\delta E \sim e^{\lambda t} \;\Rightarrow\; \frac{d\delta E}{dt} = \lambda \delta E = a_1 \delta E \;\Rightarrow\; \lambda = a_1$$

  - $E^\star$ is (again) stable, stability is determined by $a_1$ (comparably **small**)
  - System is not unstable, but can change more easily



## Dynamics of energy III

- **Isolated case:** Saturation dynamics → Equilibrium
  - Energy provided at a constant rate $q_0$ (e.g sun radiation)
  - No self-replication, exponential decay (dissipation)

$$\frac{dE(t)}{dt} = q_0 - cE(t)\;; \quad E(t) = \frac{q_0}{c}\left[1 - e^{-ct}\right]\;; \quad E^\star = \frac{q_0}{c}$$

- **Coupling with production:** Use of energy for increasing production:

$$\frac{dE(t)}{dt} = q_0 - cE(t) - \frac{\partial E(t)}{\partial Y}\frac{dY_E}{dt}$$

- **2 Assumptions**: $Y(E)$ and $dY_E/dt$ needed

$$Y(E) = A_E\, E^{1/2}(t) \;\rightarrow\; E = \frac{1}{A_E^2} Y^2 \;\rightarrow\; \frac{\partial E}{\partial Y} = \frac{2}{A_E^2} Y\;; \quad \frac{dY_E}{dt} \propto E(t)Y$$

- **Dynamics of energy resource:** efficiency $d_2 = 2/A_E^2$

$$\frac{dE(t)}{dt} = q_0 - cE(t) - \frac{2}{A_E^2} Y^2 E(t) = q_0 - \left[c + d_2 Y^2\right] E(t)$$



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## Dynamics of capital

- We need assumptions for $Y(K)$ and $dY_K/dt$
- Assumption for $Y(K)$ from Cobb-Douglas production function:

$$Y(K) = A_K \sqrt{K^{1/2}(t)} \rightarrow K = \frac{1}{A_K^2} Y^2 \rightarrow \frac{\partial K}{\partial Y} = \frac{2}{A_K^2} Y = \frac{2K}{Y}$$

- **Assumptions for** $dY_K/dt$: **Saturation dynamics** $\rightarrow$ Growth via investments
  - Savings rate $0 < s < 1$, $\gamma$: Decay of $Y$ without input (defines relaxation time scale)

$$\textbf{Variant 1:} \quad \frac{dY_K}{dt} = s - \gamma Y \qquad \left(\text{Variant 2: } \frac{dY_K}{dt} = \mu s Y - \mu \gamma Y^2\right)$$

- **Dynamics of capital:** Remember: $K = Y^2/A_k^2$
  - Efficiency $\beta = 2/A_K^2$, depreciation $\kappa = 2\gamma$

$$\frac{dK}{dt} = \frac{\partial K}{\partial Y} \frac{dY_k}{dt} = \frac{2K}{Y} [s - \gamma Y] = \beta s Y - \kappa K$$



## Dynamics of production

- Considering $Y(E, K)$

$$\frac{dY}{dt} = \frac{dY_E}{dt} + \frac{dY_K}{dt}$$
$$= s - \gamma Y + d_2 E(t) Y = s - [\gamma - d_2 E(t)]\, Y$$

- **Coupling with dynamics of energy and capital**

$$\frac{dE(t)}{dt} = q_0 - \left[c + d_2 Y^2\right] E(t)$$
$$\frac{dK(t)}{dt} = \beta s Y - \kappa K(t)$$

- **Assumptions**: $Y(E, K)$ and $E(t)$ reach equilibrium fast $\rightarrow$ stationary solution

$$Y_\star = \frac{s}{\gamma - d_2 E_\star} \; ; \quad E_\star = \frac{q_0}{c + d_2 Y_\star^2}$$
$$\gamma d_2 Y_\star^3 - s d_2 Y_\star^2 - [q_0 d_2 - \gamma c]\, Y_\star = cs$$



Notes:

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# Bifurcation analysis

$$\gamma d_2 Y_\star^3 - s d_2 Y_\star^2 - [q_0 d_2 - \gamma c]\, Y_\star = cs$$

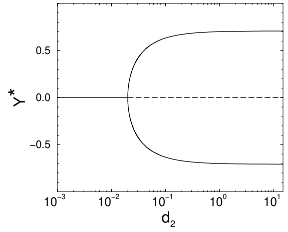
- Assume $s = 0$, i.e. *no feedback* between $Y$, $K$

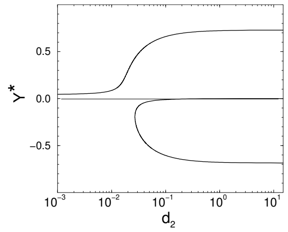
$$\left[\gamma d_2 Y_\star^2 - (q_0 d_2 - \gamma c)\right] Y_\star = 0$$

$$Y_\star^{(1)} = 0\,; \quad Y_\star^{(2,3)} = \pm\sqrt{\frac{q_0}{\gamma} - \frac{c}{d_2}}\,; \quad d_2^{\mathrm{bif}} = \frac{c\gamma}{q_0}$$

  - Positive solution only if: $q_0 d_2 > c\gamma$ ($c\gamma$: dissipation)
  - Negative solution? $Y = \hat{Y} - Y_0$, i.e. $Y < Y_0$
- Assume $0 < s < 1$ and small $cs$ $\rightarrow$ **Bias** toward large positive $Y$

$$Y_\star^{(1)} = 0\,; \quad Y_\star^{(2,3)} = \frac{c}{2\gamma} \pm \sqrt{\frac{c^2}{4\gamma^2} + \left(\frac{q_0}{\gamma} - \frac{c}{d_2}\right)}$$






# Stability analysis

- **Small perturbations of equilibrium solutions**

$$Y = Y^\star + \delta Y\,; \quad E = E^\star + \delta E\,; \quad \left|\frac{\delta Y}{Y^\star}\right| \sim \left|\frac{\delta E}{E^\star}\right| \ll 1$$

- **Dynamics of small perturbations**

$$\begin{bmatrix} \delta\dot{Y} \\ \delta\dot{E} \end{bmatrix} = \begin{bmatrix} -\gamma_0 + d_2 E^\star & d_2 Y^\star \\ -2 d_2 E^\star Y^\star & -c - d_2 Y_\star^2 \end{bmatrix} \begin{bmatrix} \delta Y \\ \delta E \end{bmatrix}$$

- **Ansatz:** $\delta Y \sim e^{\lambda t}$, $\delta E \sim e^{\lambda t}$ $\rightarrow$ **Eigenvalues** $\lambda$:

$$\begin{aligned} \lambda^{(1,2)} &= -\frac{1}{2}\left(\gamma + c + d_2 Y_\star^2 - d_2 E^\star\right) \\ &\pm \sqrt{\frac{1}{4}\left(\gamma + c + d_2 Y_\star^2 - d_2 E^\star\right)^2 - c(\gamma - d_2 E^\star) - d_2 Y_\star^2(\gamma + d_2 E^\star)} \end{aligned}$$

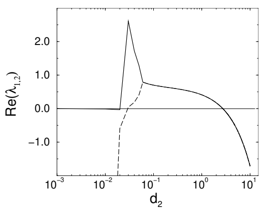


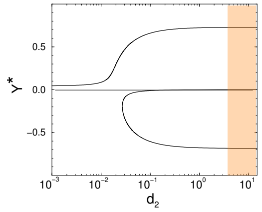
**Notes:**

**Notes:**

## Stability analysis: Results

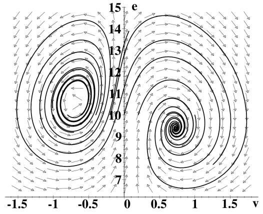
| | | | |
|---|---|---|---|
| 0 | $< d_2 <$ | 0.027 | : 1 stable node |
| 0.027 | $< d_2 <$ | 0.046 | : 1 stable node, 1 instable node, 1 saddle point |
| 0.046 | $< d_2 <$ | 2.720 | : 1 stable focal point, 1 instable focal point, 1 saddle point |
| 2.720 | $< d_2$ | | : 2 stable focal points, 1 saddle point |



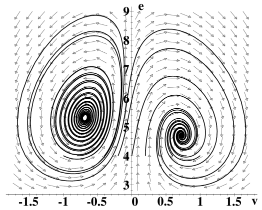




Increased efficiency in resource use (i) **increases production**, but (ii) can **stabilize the drop down** of production

$d_2 = 2.0$




$d_2 = 4.0$






### SS06: Review of Coupled Dynamics and Stability

- Revisit and deepen your understanding of coupled dynamics, focusing on mastering the use of Python and interpretation.
- Extend an existing model and investigate with these tools.
  - Master the identification of fixed points with sympy.
  - Understand phase portraits and how to plot them.
  - Apply these insights to a new model.



Notes:

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## Questions

1. Explain the differences between conservative and dissipative systems. What are the practical consequences? Give examples.
2. Our economy constantly produces electrical energy, using nuclear power plants, photovoltaic systems or wind turbines. How does this fit the 1st and 2nd law of thermodynamics?
3. Compare the limit cycle and the center solution. How can you decide between them?
4. Explain the structure and the meaning of a production function.
5. How can you calculate the marginal product, increasing and decreasing returns to scale?
6. What is special about the Cobb-Douglas production function. Why should $\alpha$, $\beta$ be smaller than 1?
7. Think about the two saturation dynamics for $E(t)$. What are the arguments in favor and against?
8. Shows the degenerated pitch bifurcation diagram a saddle-node bifurcation?



Notes: