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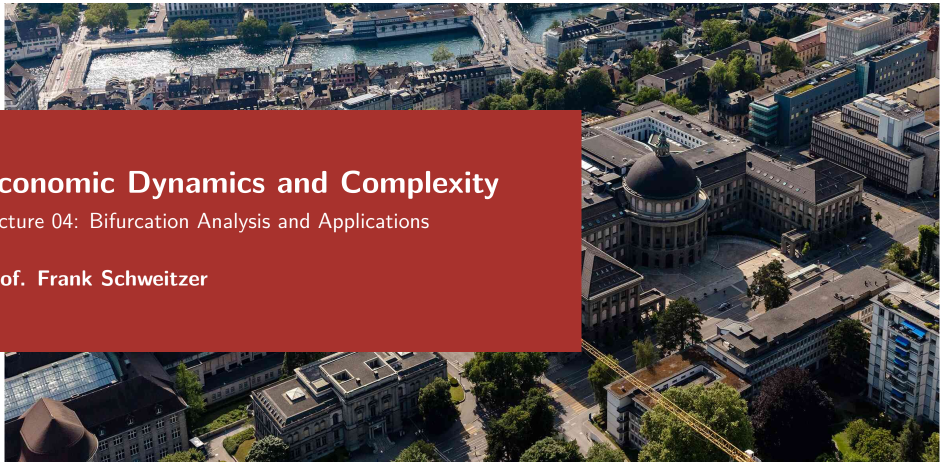


Economic Dynamics and Complexity

Lecture 04: Bifurcation Analysis and Applications

Prof. Frank Schweitzer

Outline

Fixed points and stability

Bifurcations

Applications

Notes:

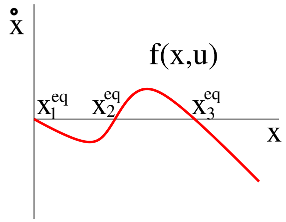
Notes:

Role of control parameters

- 1–dimensional dynamical system
  - one **state variable** $\rightarrow x$
  - set of **control parameters** $\mathbf{u} = \{u_1, u_2, \ldots\}$
  - nonlinear right-hand side: $f(x, \mathbf{u})$
- Dynamical System:

$$\dot{x} = \frac{dx}{dt} = f(x, \mathbf{u})$$

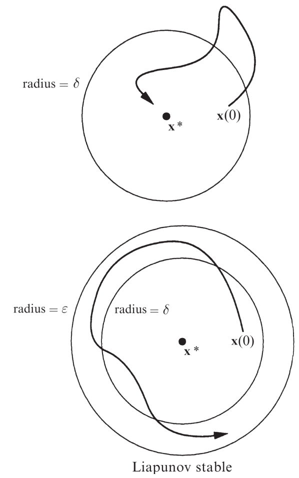
  - **Fixed points:** $\dot{x} = 0 \rightarrow x_1^\star, x_2^\star, \ldots$
    Also called: Equilibria, stationary solutions, attractors
    - Stability?
    - Dependency on $\mathbf{u}$?



Stability of fixed points

Consider a fixed point $x^\star$ of the system $\dot{x} = f(x, u)$.

- **Attracting fixed point:**
  - $x^\star$ is attracting if there is a $\delta > 0$ s.t. $\lim_{t\to\infty} x(t) = x^\star$, whenever $||x(0) - x^\star|| < \delta$
  - Any trajectory starting within a distance δ of $x^\star$ converges to $x^\star$.
- **Liapunov Stability:**
  - $x^\star$ is Liapunov stable, if for every $\varepsilon > 0$, there is a $\delta > 0$ s.t. $||x(t) - x^\star|| < \varepsilon$ whenever $t > 0$ and $||x(0) - x^\star|| < \delta$
  - Trajectories starting within δ of $x^\star$ remain within ε of $x^\star$ for all times
- **Asymptotic Stability:**
  - $x^\star$ is called asymptotically stable if it is both attracting and Liapunov stable.



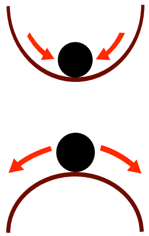
Source: Strogatz, p. 135

Notes:

- Note that we can already deduce from the slope of $f(x)$ whether the equilibria are stable or unstable states: $df/dx < 0$ means stable, $df/dx > 0$ means unstable. Verify this by thinking of driving forces in the neighborhood of the stationary states.

Notes:

Gradient system



- Motion along the gradient of a "landscape"
  $\Rightarrow$ potential $V(x,u)$ where u represents control parameters

$$\dot{x} = \frac{dx}{dt} = f(x,u) = -\frac{dV(x,u)}{dx}$$

- **Equilibria:** $dV/dx = 0 \iff$ critical points (extrema) of $V(x,u)$
  - Note the **minus** sign of the gradient: $-dV/dx$
  - **Minimum** $\rightarrow$ stability: $-d^2V/dx^2 < 0$
  - **Maximum** $\rightarrow$ instability: $-d^2V/dx^2 > 0$
- Interesting: **inflexion points:** $dV/dx = d^2V/dx^2 = 0$
  - depends on control parameters u_1, u_2

Linear stability analysis

- 1-dim dynamics: $\dot{x} = f(x,u)$
- **Fixed points:** $\dot{x} = f(x,u) = 0 \rightarrow x_1^\star, x_1^\star, \dots$
- For each $x_i^\star$: compute **linearization** of $\dot{x} = f(x,u)$:
  - **SMALL perturbation:** $x(t) = x^\star + \delta x(t)$, $\dot{x}^\star = 0$

$$\frac{dx}{dt} = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x^\star} [x - x^\star] + \dots.$$

$$\frac{dx^\star}{dt} + \frac{\delta x}{dt} = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x^\star} \delta x + \dots$$

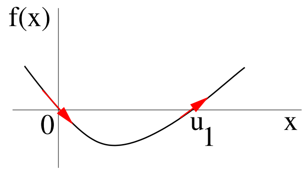
$$\frac{\delta x}{dt} = \lambda\, \delta x \; ; \quad \lambda = \left.\frac{\partial f(x)}{\partial x}\right|_{x^\star}$$

- **Closed form solution:** $\delta x(t) = \delta x(0) e^{\lambda t}$
  - Stability: $\delta x(t) \rightarrow 0$, if $\lambda < 0$
  - Instability: $\delta x(t) \rightarrow \infty$, if $\lambda > 0$
  - Task: Determine λ

Example: $\dot{x} = f(x, u_1) = x^2 - u_1 x$

$x_1^\star = 0,\ x_2^\star = u_1 \; ; \quad \lambda = 2x - u_1$

- $x_1^\star = 0 : \lambda = -u_1 < 0 \Rightarrow$ stable
- $x_2^\star = u_1 : \lambda = u_1 > 0 \Rightarrow$ unstable



Notes:

- Note that $-d^2V/dx^2 < 0$ means $df/dx < 0$ (stability) and $-d^2V/dx^2 > 0$ means $df/dx > 0$ (instability).
- The minus sign in front of dV/dx is a mere convention, to construct an analogy to a mechanical potential where the movement can be easily envisioned.
- *catastrophe theory* (R. Thom, 1969): Generation and destruction of structurally stable attractors

Notes:

Linear stability of multi-dimensional systems

- State **vector:** $\mathbf{x} = \{x_1, x_2, ...\}$
- **Dynamics:** $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$
- **Fixed points:** $\mathbf{x}^\star = \{x_1^\star, x_2^\star, ...\}$
- **Linearization:** $\mathbf{x}(t) = \mathbf{x}^\star + \delta\mathbf{x}$
- **Ansatz:** $\delta\mathbf{x}(t) = e^{\mathbf{J}t}\delta\mathbf{x}(0)$
- Eigenvalues λ_j of $\mathbf{J}$ determine stability type
  1. **All** eigenvalues satisfy $\mathrm{Re}\lambda_j \leq 0$
     $\Rightarrow$ Asymptotic stability
  2. **One** eigenvalue $\mathrm{Re}\lambda_i > 0 \Rightarrow$ Instability

- $e^{\mathbf{J}}$: **Matrix exponential** of $\mathbf{J}$
  - Jacobian Matrix $\mathbf{J}$: $n \times n$ matrix
  - Elements $J_{ij} = \partial f_i(x)/\partial x_j\big|_{x^\star}$
  - Complex eigenvalues: $\lambda_j = a_j - \mathrm{i} b_j$
  - **Real:** $\mathrm{Re}\lambda_j = a_j$, **Imaginary:** $\mathrm{Im}\lambda_j = b_j$
  - **Hyperbolic fixed point:** $x^\star$:
    all λ_j have non-zero real part $\mathrm{Re}\lambda_j$.

- **Limitations:**
  - If $\mathrm{Re}\lambda_j = 0$ (λ_i on the imaginary axis), **nonlinear** system may be **unstable** even if linearization is stable.
  - Therefore: **Linear stability analysis** can only be applied to **hyperbolic fixed points**

Outline

Fixed points and stability

Bifurcations

Applications

Notes:

- Waiting for the multi-dimensional application? Stay tuned for Lecture 05!

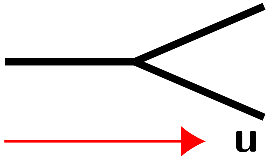
Notes:

Bifurcation

- Branching of *one* stable, real solution in *two* new stable, real solutions dependent on *control parameters* **u**
  - critical values of u_1, u_2: " bifurcation point"
- Why are bifurcations called **catastrophes**? (Thom, 1968)

Fixed points can be created or destroyed, or their stability can be changed.

  - small (*continuous*) changes in u lead to big (*discrete*) changes in x
  - *Gradient dynamics:* small perturbations $V + \delta V$ lead to drastic changes in the number and nature of the critical points



- **Bifurcation diagram**: plot of stable/unstable solutions $x^\star$ against control parameter u

Stable states in gradient systems

- **Dynamics: 2 control parameters**

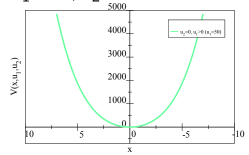
$$\frac{dx}{dt} = f(x, u_1, u_2) = -\frac{dV}{dx}$$

- Potential of 4th order:

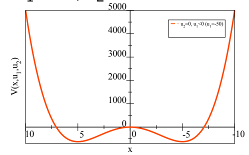
$$V(x, u_1, u_2) = x^4 + u_1 x^2 + u_2 x$$
$$-\frac{dV}{dx} = -4x^3 - 2u_1 x - u_2$$

  - stability: $-d^2V/dx^2|_{x^\star} < 0$
  - **monostability:** $u_1 > 0$
  - **bistability:** $u_1 < 0$
  - **symmetric potential:** $u_2 = 0$
  - **asymmetric potential:** $u_2 \neq 0$

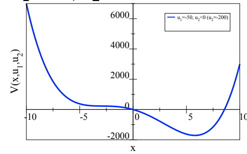
$u_1 > 0, u_2 = 0$



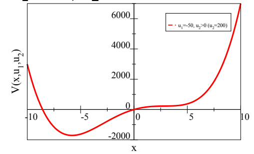
$u_1 < 0, u_2 = 0$



$u_1 < 0, u_2 < 0$



$u_1 < 0, u_2 > 0$



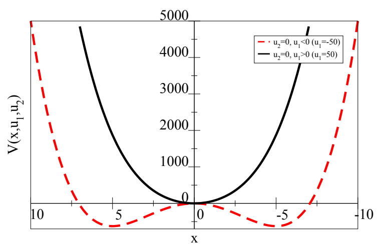
Notes:

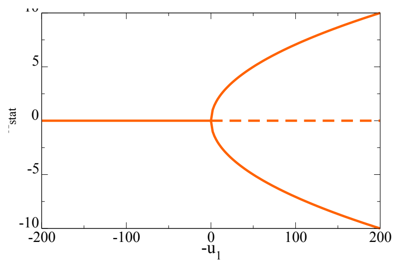
- This means, by changing the control parameter we can control the number of possible solutions. This is important if we want a system (problem) to have more than one solution.
- Terms in Taylor expansion provide a classification of bifurcations in "normal forms".
- In the following example the fixed point at the origin goes from being stable to unstable as u_1 (the bifurcation parameter) goes from positive to negative.
- As a result, very small changes in u_1 can have a dramatic impact on the behavior of the system.
- In addition to the change in stability of the fixed point at the origin, two new (stable) fixed points are created.

Notes:

Pitchfork bifurcation in gradient systems

$u_2 = 0 \to$ symmetric potential $\to$ let us vary u_1





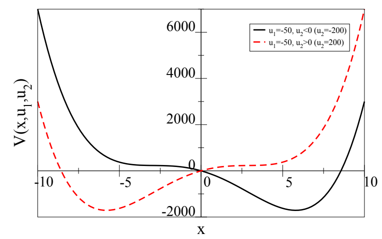
- **Potential:** Transition from **monostability** to **bistability**
  - Control parameter u_1 changes sign
- **Bifurcation diagram:** $x^\star(u_1, u_2)$
  - **Pitchfork–bifurcation** occurs at $u_1 = 0$
  - Application to decision processes

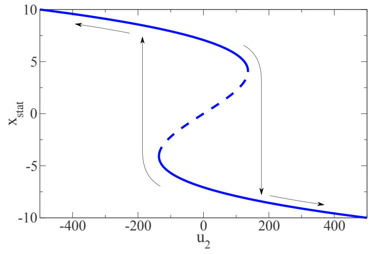
Notes:

- Find the animation on Moodle (literature section)
- The diagram on the left shows the transition of V as u_1 goes from a negative to a positive value.
- The diagram on the right is the associated ***bifurcation*** diagram, which shows the fixed points of the system with respect to the *bifurcation parameter*, u_1 (N.B. in the diagram x_{stat} is plotted against $-u_1$).
- Stable fixed points are represented by solid lines, while unstable fixed points are represented by dashed lines.
- Notice how for $u_1 > 0$ there is only one fixed point which is also stable, while for $u_1 < 0$ there are three fixed points of which two are stable and one is unstable. Compare this with the extrema you see in the left figure.
- Obviously, there are always three equilibrium solutions for the cubic equation. However, given some conditions, only *one* of them is a *real* solution, the two others are *complex solutions*.

Cusp bifurcation in gradient systems

$u_1 < 0$, kept const., $u_2 \neq 0 \to$ asymmetric potential $\to$ let us vary u_2





- **Potential:** Switch between **monostability** and **bistability**
  - Control parameter u_2 changes sign
- **Bifurcation diagram:** $x^\star(u_1, u_2)$
  - **Cusp bifurcation** occurs at $u_1 \neq 0$
  - Application: "unsustainable state" (Jantsch: "hawk/dove")

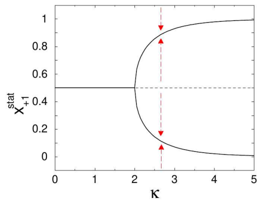
Notes:

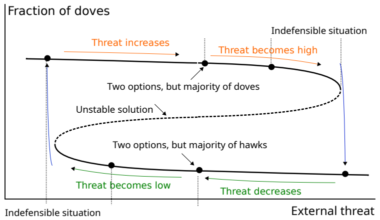
- Find the animation on Moodle (literature section)
- Once again, the left diagram shows the transition of V as u_2 changes from a negative to a positive value.
- The diagram on the right is the associated *bifurcation diagram* which shows the the fixed points of the system along with their stabilities (stable fixed points shown with a solid line, unstable with a dashed line).
- What happens when u_2 slowly grows from 0 to 200 in the right figure? Notice how the stable point 'jumps'; What happens when u_2 then decreases again from 200 to 0? And then from 0 to -200?

- Erich Jantsch, *The Self-Organizing Universe: Scientific and Human Implications of the Emerging Paradigm of Evolution* (1980)

Two illustrative examples

- **Majority/minority in large societies**
  - Two opinions $\{-1,+1\}$, fraction x_{+1}
  - **Control parameter:** $\kappa(s,N)$ with $\kappa^{\text{crit}}=2$
  - Critical population size: $N_c \propto 1/s$
  - Equal distribution of opinions becomes instable
- **Hysteresis of majority/minority**
  - Two opinions $\{-1,+1\}$ (hawks/doves), fraction x_{+1}
  - **Control parameter:** External threat
  - Majority opinion depends on "history"



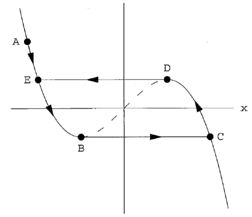


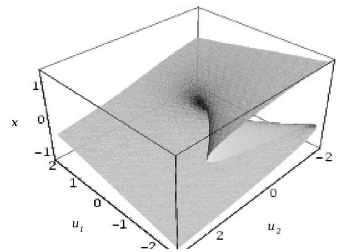
Notes:

Cusp catastrophe and hysteresis

- **Hysteresis:** System jumps at two different values of u_2
  - depends on the **direction** in which u_2 is varied
  - **Hysteresis loop:** Closed path *BCDEB*
  - **Note:** $u_2(x^\star)$ (inverse curve)
- **Cusp "catastrophe"**: Big jump
  - *Smooth surface* shows all fixed points of $V(x,u_1,u_2)$
  - Assume small variations of u_1 and/or u_2
  - Result: Motion between equilibria along different paths
  - Some motion paths remain **continuous**: Avoid catastrophy
  - Some motion paths become **discontinous**: Big jumps

1. **Problem: Optimized trajectory:** Control over $u_1(t)$, $u_2(t)$
2. **Problem:** Influence of noise near critical points

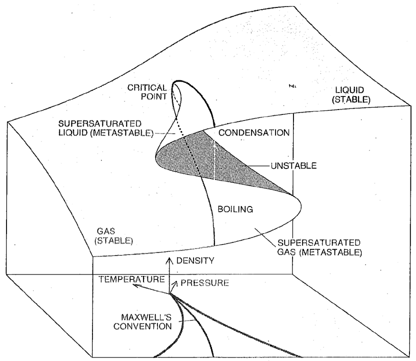




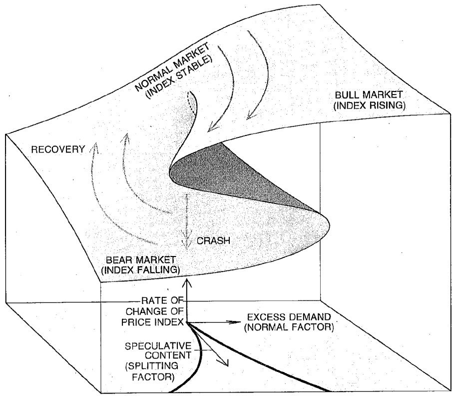
Notes:

- **2d Illustration:** Suppose the value of r is fixed. If $h > h_c$ there exists only one asymptotically stable fixed point A. As h decreases, the point representing the equilibrium passes E to finally reach B. Further decreasing h, the state of the system jumps to the value of the $x-$coordinate of C and follows the solid line below C for even smaller values of h. If now we start to increase h we go back to C and proceed to D , where if we further increase h jump to E and move along the solid line towards A (Source: Boccara (2004), p. 87)
- **3d Illustration:** Each point on this sheet is a fixed point. However, contrary to the above figure, now we have two control parameters, since we don't keep u_1 constant anymore!

Early illustrations (Zeeman, 1976)



PHASE TRANSITIONS between the liquid and the gaseous state of matter conform to a modified cusp model in which temperature and pressure are the control factors. Ordinarily both boiling and condensation take place at the same values of temperature and pressure. Thus there are catastrophic changes, but there is no hysteresis. Under special circumstances, however, a vapor can be cooled below its dew point and a liquid can be heated above its boiling point, so that the behavior surface is followed all the way to the fold curve. The critical point, where liquid and gas exist simultaneously, is represented by the singularity where the pleat disappears.



BEHAVIOR OF THE STOCK MARKET is described by a model in which the controlling parameters are excess demand for stock and the proportion of the market held by speculators as opposed to that held by investors. The behavior itself is measured by the rate at which the index of stock prices is rising or falling. The control factors are oriented not as conflicting factors but as normal and splitting factors. A fall from the top sheet to the bottom represents a crash; the slow recovery is effected through feedback of the price index on the control parameters.

A didactic example

- 1-dim **gradient system**:

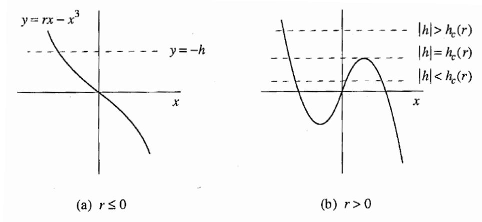
$$V(x) = \frac{1}{4}x^4 - \frac{1}{2}rx^2 - hx$$

$$\frac{dx}{dt} = h + rx - x^3$$

- Two **control parameters** $u_1 = -r$, $u_2 = -h$
  - plot $y = rx - x^3$ and $y = -h \to$ intersections
- **Critical case:** horizontal line $y = -h$ hits $y = rx - x^3$

$$x_{max} = \sqrt{\frac{r}{3}}$$

$$rx_{max} - (x^3_{max}) = \frac{2r}{3}\sqrt{\frac{r}{3}}$$



(Figure 3.6.1 from Strogatz, p. 70)

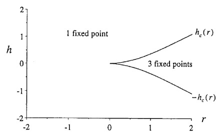
- *Bifurcations* for $h = \pm h_c(r)$

$$h_c(r) = \frac{2r}{3}\sqrt{\frac{r}{3}}$$

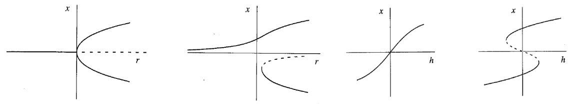
Notes:

Notes:

Imperfect pitchfork bifurcation



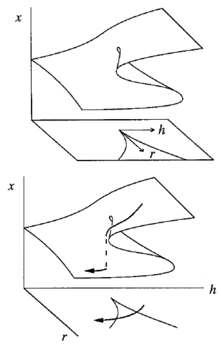
- **Pitchfork bifurcation:**
  - 1 fixed point $\Leftrightarrow$ 3 fixed points
  - $h = 0$: perfect symmetry between x and $-x$
    supercritical (normal) pitchfork bifurcation
  - $h \neq 0$: imperfect bifurcation, symmetry broken



(a) $h=0$ (b) $h\neq 0$ (a) $r \leq 0$ (b) $r > 0$

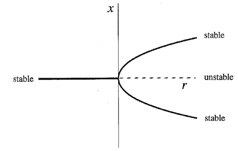
- *Cusp catastrophe surface:* fixed points x^* above (r, h) plane
  - projection onto (r, h) plane $\rightarrow$ **bifurcation curves**

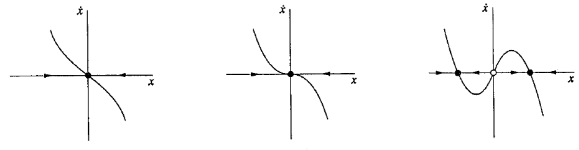
Cusp catastrophe:



More bifurcations: Subcritical pitchfork

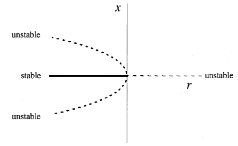
- **Supercritical:** $\dot{x} = rx - x^3$
  - $r \leq 0$: $x^\star = 0 \rightarrow$ 1 stable fixed point $\Rightarrow$ $r > 0$: $r^\star = 0$ unstable
  - 2 new stable fixed points: $x^\star = \pm\sqrt{r}$

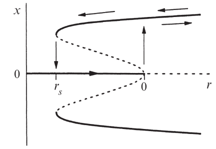




(a) $r<0$ (b) $r=0$ (c) $r>0$

- **Subcritical:** $\dot{x} = rx + x^3$, $x^\star = 0$ stable for $r < 0$
  - Inverted pitchfork, $x^\star = \pm\sqrt{-r}$ are unstable
    Blow-up: $x(t) = \pm\infty$ in finite time if $x_0 \neq 0$
- Stabilizing term: $-x^5$: $\boxed{\dot{x} = rx + x^3 - x^5}$
  - $r_s < r < 0$: 2 large-amplitude fixed points coexist with $x^\star = 0$
    Big jumps, hysteresis possible





Notes:

- Strogatz (2000), pp. 71, 72

Notes:

- Strogatz (2000), p. 56
- example for a 2-dim. supercritical pitchfork system (Strogatz, p. 246)

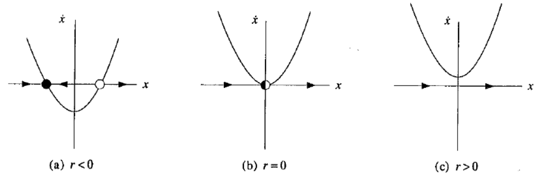
$$\dot{x} = \mu x - x^3$$
$$\dot{y} = -y$$

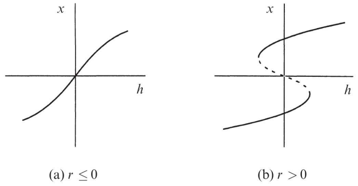
- example for a 2-dim. subcritical pitchfork system

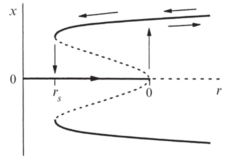
$$\dot{x} = \mu x + x^3$$
$$\dot{y} = -y$$

Saddle-Node Bifurcations

- *"Pairs of stable and unstable fixed points are born or destroyed"*
- **Example 1**: 2 fixed points ⇔ 0 fixed points ($r = 0$)



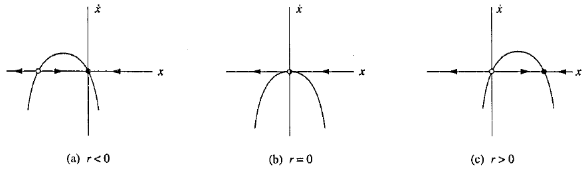




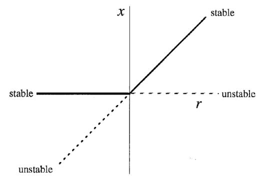
- **Example 2**: 3 fixed points ⇔ 1 fixed point ($r = 0$)
- **Example 3**: 5 fixed points ⇔ 1 fixed point ($r = 0$)
- **Question:**
  Is the supercritical pitchfork bifurcation a saddle-node bifurcation?

Transcritical Bifurcations

- *"Fixed points can never be destroyed, only change their stability"*
- **Example**: $\boxed{\dot{x} = rx - x^2}$



- $x^\star = 0$ exists for all r
- $r < 0$: **unstable** fixed point $x^\star = r$, **stable** fixed point $x^\star = 0$
- $r > 0$: **stable** fixed point $x^\star = r$, **unstable** fixed point $x^\star = 0$



2 fixed points don't disappear at the bifurcation, but switch their stability

Notes:

- Strogatz (2000), p. 45

Notes:

- Strogatz (2000), pp. 50, 51
- An example for a 2-dim. transcritical pitchfork system is

$$\dot{x} = \mu x - x^2$$
$$\dot{y} = -y$$

Outline

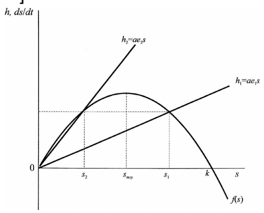
Fixed points and stability

Bifurcations

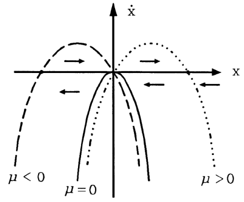
Applications

Economic application: Fisheries

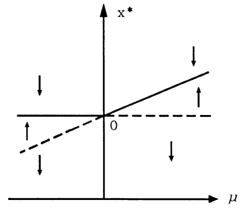
- **Simplified economic models**
  - 1-dimensional, linearization: $\boxed{\dot{x} = g(x, u_1) - h(x, u_2)}$
  - u_1=const., $g(x, u_1) = \sum_k \alpha_k(u_1)x^k$ $(k \leq 3)$
  - u_2 varies, $h(x, u_2) = \hat{h}(x) \cdot u_2$
  - fixed points: $g(x, u_1) = h(x, u_2)$
- **Example:** Fisheries
  - Saturated growth: $g(x, u_1) = [x - u_1 x^2]$
    u_1: carrying capacity
  - Harvesting: $h(x, u_2) = u_2 x$
    $u_2 = e \cdot a$: efficiency times effort
  - Stable fixed point:
    - $x^\star = (1 - u_2)/u_1$ if $u_2 < 1$
    - $x^\star = 0$ if $u_2 > 1$
  - **Transcritical bifurcation**



Plot $\dot{x}$ vs. x: $\dot{x} = [1 - u_2]x - u_1 x^2$



Plot $x^\star$ vs. $\mu = [1 - u_2]$



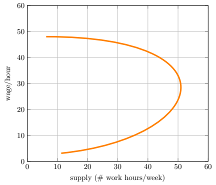
Source: Lorenz 1989, Chapter 3

Notes:

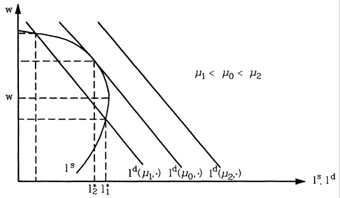
Notes:

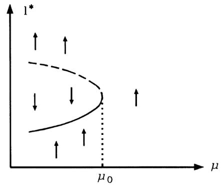
Economic application: Labor market

- **Wage** w dependent on labor supply l^s and demand l^s
  - $\boxed{\dot{w} = \beta[l^d - l^s]}$ wage increases if demand is larger than supply
- **Labor supply/demand** dependent on wage w
  - Demand linearly decreases with wage: $l^d(w) = \mu - \gamma w$
  - Supply (#work hours/week) nonlinearly increases with wage



  - **Reason:** "negative income effect"
    - wage ↑ → income ↑ → demand for leisure ↑ → labor supply ↓
- **Saddle-node bifurcation**

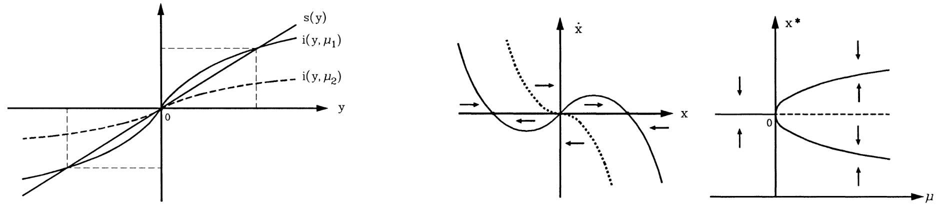




Source: Lorenz 1989, Chapter 3

Economic application: Investment

- **Simplified Kaldor model** of economic output $\boxed{\dot{Y} = \beta[I - S]}$
  - GDP increases if investments I exceed savings S
  - Savings linearly increase with wealth ($\propto$ GDP): $S(Y) = aY$
  - Investments nonlinearly increase with GDP: $I(Y) = \mu[\alpha_1 Y - \alpha_3 Y^3]$
    - **Reason:** Investment opportunities are (comparably) rare in very poor and very rich countries
- **Supercritical pitchfork bifurcation**



Source: Lorenz 1989, Chapter 3

Notes:

Notes:

Catastrophe theory and economics: Critical remarks

- **Economic assumptions made to fit theory**
  - Questionable from economic perspective
  - Example: Zeeman (1974) about bubbles, crashes
- **General critics**
  - Excessive reliance on qualitative methods,
  - Inappropriate quantization in some applications
  - Excessively restrictive mathematical assumptions

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JOURNAL OF Economic Dynamics & Control
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The rise and fall of catastrophe theory applications in economics: Was the baby thrown out with the bathwater?

J. Barkley Rosser Jr.

Pro and Contra

"The only viable applications of catastrophe theory were the qualitative ones, which were ultimately useless, and the ones that attempted to be useful and quantitative were improperly done, at least in the softer social sciences." (Rosser, 2007)

"On the plane of philosophy properly speaking, of metaphysics, catastrophe theory cannot, to be sure, supply any answer to the great problems which torment mankind. But it favors a dialectical, Heraclitean view of the universe, of a world which is the continual theatre of the battle between 'logoi,' between archetypes." (Thom, 1978)

Notes:

- The discussion about questionable assumptions also applies to a model by Hotelling (1921) which combines population growth and diffusion (remember Lecture 02). To make this an economic model, wrong assumptions about the migration of human population had been applied. But it turns out later that the model is well suited to describe biological populations.
- Here are some arguments about "Excessively restrictive mathematical assumptions" (from Rosser, 2007):
  - necessity for a potential function
  - gradient dynamics does not allow the use of time as a control variable
  - elementary catastrophes are only a subset of possible bifurcations

SS04: Understanding Stability and Bifurcation in ODEs

- Explore the stability of ODEs, focusing on an intuitive understanding of stability.
- Apply these concepts to a labor economics model to explore bifurcation and equilibrium stability.
  - Understand what stability means in the context of ODEs and gain intuition behind it.
  - Formulate the labor economics model using a system of equations.
  - Identify fixed points and classify their stability within the labor model.
  - Investigate how equilibrium points change as control parameters are varied.

Notes:

Questions

1. What is the difference between *Liapunov stability* and *Asymptotic stability*?
2. Why can we learn from the sign of df/dx about the stability of a fixed point? Under what conditions?
3. What is the meaning of "bifurcations"? Why are they called "catastrophies"?
4. Explain the impact of the two control parameters u_1, u_2 on the cusp bifurcation.
5. (How) Can we avoid a cusp catastrophy? Give an example.
6. Distinguish sub/supercritical pitchfork, cusp, saddle-node and transcritical bifurcations.
7. Explain the fishery example in terms of bifurcations.
8. What are the advantages and disadvantages of catastrophe theory?

Notes: