ETH zürich

Chair of Systems Design
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Economic Dynamics and Complexity

Lecture 02: Growth, Carrying Capacity, Harvesting

Prof. Frank Schweitzer

Notes:

Outline

Nonlinear Dynamics

Exponential growth

Saturated growth

Harvesting

Growth and dispersal

Notes:

Nonlinear Dynamics

- **Part of Applied Mathematics, Theory of dynamical systems**
  - Coupled differential equations: $\dot{x}_i = f_i(\mathbf{x}, \mathbf{u})$, control parameters $\mathbf{u}$
  - Time evolution, stationary states, in/stability, numerical solvers
  - Classification of solutions dependent on $\mathbf{u}$ ("catastrophe theory")
- **Part of System Dynamics**
- Macro variable $\mathbf{x}(t)$ represents the state of a *system*
  - Coupled sub-systems $i = 1, ..., N$ (with N *small*)
- **Focus:** *Non-linear feedback processes* between subsystems
  - **If** linear superposition (e.g. $\alpha x_1 + \beta x_2$) decomposition possible
- **Focus:** *Different time scales* $(1/\tau_i)$ in subsystems
  - Relaxation of fast and slow variables, "enslaving principle"

Examples:

- "Predator-prey"
  $\dot{x}_1 = f_1(x_1, x_2, \mathbf{u}) = \alpha x_1 - \beta x_1 x_2$
  $\dot{x}_2 = f_2(x_1, x_2, \mathbf{u}) = \gamma x_1 x_2 - \delta x_2$
  $\Rightarrow$ Goodwin model: employment ratio, workers share of output
- Mathematical pendulum
  $(1/\tau_1)\dot{x}_1 = x_2$
  $(1/\tau_2)\dot{x}_2 = -(g/l)\sin x_1$

Notes:

- Steven H. Strogatz: *Nonlinear Dynamics and Chaos*, Westview Press, 2000
- **The importance of being nonlinear:** "Linear systems can be broken down into parts. Each part can be solved separately and finally summed up to get the solution. Whenever parts of a system interfere, cooperate or compete, there are nonlinear interactions going on. The latter systems show behavior that is more than the mere sum of its parts." (Strogatz)

Nonlinear - The new normal

- **Before 1900** (and long after ...)
  - Primary interest in "ideal" systems
    - *Paradigm:* Celestial mechanics
  - Nonlinearity seen as a "perturbation":
    - *Perturbation theory*: Correcting ideal state
    - Example: Cluster expansion of gas theory (Mayer)
- **After 1970** (and also before ...)
  - Nonlinearity as the "normal" case
    - Pattern formation, instabilities
    - Chaos as a real-world phenomenon
  - Shift of *aesthetic* perspectives (in science)
    - *Self-organization*, creative nature, emergence

Dynamics — A Capsule History

1666	Newton	Invention of calculus, explanation of planetary motion
1700s		Flowering of calculus and classical mechanics
1800s		Analytical studies of planetary motion
1890s	Poincaré	Geometric approach, nightmares of chaos
1920–1950		Nonlinear oscillators in physics and engineering, invention of radio, radar, laser
1920–1960	Birkhoff	Complex behavior in Hamiltonian mechanics
	Kolmogorov	
	Arnol'd	
	Moser	
1963	Lorenz	Strange attractor in simple model of convection
1970s	Ruelle & Takens	Turbulence and chaos
	May	Chaos in logistic map
	Feigenbaum	Universality and renormalization, connection between chaos and phase transitions
		Experimental studies of chaos
	Winfree	Nonlinear oscillators in biology
	Mandelbrot	Fractals
1980s		Widespread interest in chaos, fractals, oscillators, and their applications

Notes:

- Source: Strogatz (2000), p.5
- Other famous books: From Being to Becoming (Prigogine/Stengers), The Self-Organizing Universe (Jantsch), The Beauty of Fractals (Peitgen/Richter)

Deterministic vs. stochastic dynamics

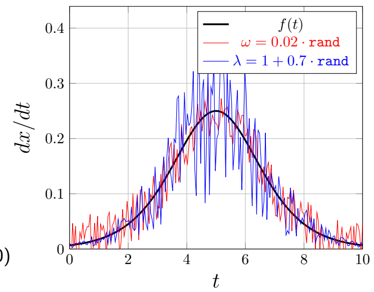
- 1-dimensional system: state variable $x(t)$, continuous time t
  - $x(t)$: Trajectory in (multi-dimensional) phase space $\mathrm{S} \in \Re^n$
- **General form of dynamics:**

$$\frac{dx(t)}{dt} = \dot{x}(t) = \lambda(t) f(x, \mathbf{u}, t) + \omega(t)$$

- $f(x, u_1, .., u_k, t)$: nonlinear function $\Rightarrow$ **deterministic**
  - $\mathbf{u} = \{u_1, ..., u_k\}$: control parameters
  - Unique, regular or chaotic trajectory $x(t)$ ($\rightarrow$ Theorem)
- $\lambda(t)$, $\omega(t)$: perturbations, fluctuations $\Rightarrow$ **stochastic**
  - **Additive** stochastic force $\omega(t) \rightarrow$ Brownian motion
  - **Multiplicative** stochastic force $\lambda(t) \rightarrow$ geometric Brownian motion
    Stochastic influences assumed to be small: $\dot{x} = \lambda(t)x$
    Application: Firm growth (Gibrat, 1931), stock market (Bachelier, 1900)
  - THIS course is about deterministic dynamics

$$f(x) = ax - bx^2$$

$$\mathtt{rand} \sim U(-1, 1)$$



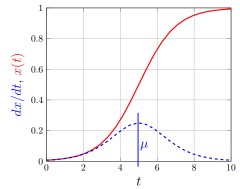
Notes:

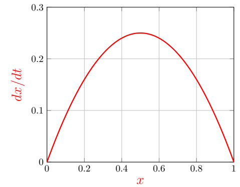
- **Existence and uniqueness theorem:** Consider the initial value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \mathbf{x}(0) = \mathbf{x}_0$. Suppose $\mathbf{f}$ is continuous and all its partial derivatives are continuous for $\mathbf{x}$ in some open set $\mathrm{D} \in \Re^n$, i.e. $\mathbf{f} \in \mathrm{C}^1$. Then for $\mathbf{x} \in \mathrm{D}$, the initial value problem has a solution $\mathbf{x}(t)$ on some time interval $(-\tau, \tau)$ about $t = 0$, and the solution is unique.
- **Implication:** Different trajectories never intersect. If two trajectories intersect, there would be two solutions starting from the same point and this would violate the uniqueness theorem.

Analyzing nonlinear dynamics

$$\frac{dx(t)}{dt} = \beta\, x(t)\, [1 - x(t)]$$

$$x(t) = \frac{1}{1 + e^{-\beta(t-\mu)}}$$





1. **Analytic integration**: Explicit solution $x(t, \mathbf{u})$
2. **Numeric integration**: Plot of $x(t, \mathbf{u})$ over t
3. **Graphic analysis**: Plot of dx/dt over x
   - **Roots** ($dx/dt = 0$) refer to *stationary solutions* $x^\star$
   - **Gradient** $dx/dt|_{x^\star} < 0$ informs about stability

- **Autonomous system**: Plot of *phase space*
  - Non-autonomous system: explicit time dependency $f[\mathbf{x}(t), t]$
  - Autonomous system: no explicit time dependency $f[\mathbf{x}(t)]$
  - Introduce additional variable $x_{n+1} := t$ and new time τ with $d\tau/dt = 1$: $f(\mathbf{x}(t), t) \rightarrow f(x_1(\tau), ..., x_{n+1}(\tau))$
  - Advantage: Trajectories in phase space are "frozen"

Notes:

Outline

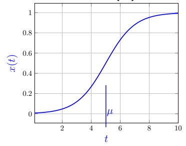
The meaning of growth

- **Dynamics:** $f(x, \mathbf{u}, t)$
  - $x(t)$: variable of interest, e.g. GDP, sales ....
  - $\mathbf{u} = \{u_1, u_2, \ldots\}$: control parameters, e.g. inflation rate
  - $f(\cdot)$: *causes for change*
- **Absolute growth:** $\frac{dx}{dt} = f(x, \mathbf{u}, t)$
  - Discrete time: $\Delta x(t) = x(t) - x(t-1)$
  - Unit: change (*number*) *per time unit*, e.g. (#/day)
- **Relative growth:**

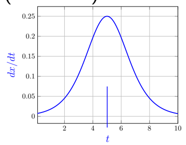
$$\frac{1}{x}\frac{dx}{dt} = \frac{d\ \ln(x)}{dt} = \frac{1}{x} f(x, \mathbf{u}, t)$$

  - Unit: *percentage, fraction*, e.g. 3.5%, one-third, ...
  - "log growth rate", "relative growth rate" → *rate*
  - Still: *per time unit*, e.g. quarterly, annual ...
  - **Note:** size-dependent time scale $(x\ dt)$ (slow down with age)

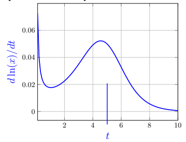
- Dynamics $x(t)$



- (Absolute) Growth dx/dt



- (Relative) Growth rate $d\ln(x)/dt$



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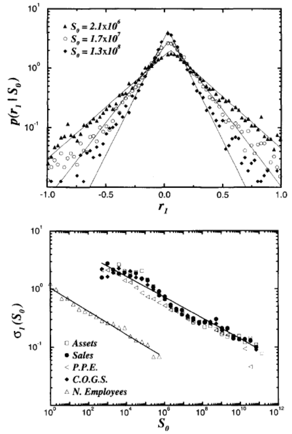
The meaning of (relative) growth rates α

$$\frac{dx}{dt} = \alpha x \Rightarrow \frac{d\ln(x)}{dt} = \alpha$$

- **Growth rates depend on firm size** $\Rightarrow \alpha(x)$
  - Imagine: Firm A sells 500 units, B sells 500.000 units
    A grows by 10%: 50, B grows by 0.1%: 500
  - Percentages do not allow to compare firms of different size
  - Revenues, profits, taxes depend on *absolute* numbers
  - Variance of growth rates *decreases* with size
- **Growth rates are broadly distributed** $\Rightarrow \sigma(x) \sim x^{-0.2}$
  - Imagine: Number of employees constantly grows by 5%
    $t_0 = 100$, $t_{10} = 163$: After 10 years increase by 63%
  - Continuous growth is not realistic/not sustainable
  - Negative growth rates have to be expected (at times)
  - Large variance of growth rates: "fat tails"



Source: Amaral *et al.* (1997)

Exponential growth

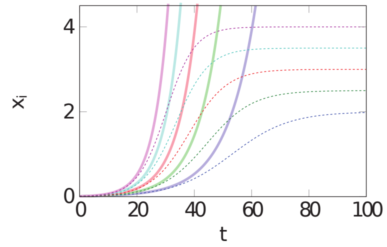
- Quantity x: *continuous*, *positive*, i.e. $x(t) \geq 0$.
- **Law of proportionate growth**

$$\frac{dx(t)}{dt} = \alpha x(t)\ ; \quad x(t) = x(0)\, e^{\alpha t}$$

  - $\alpha > 0$: exponential growth
  - $\alpha < 0$: exponential decay of $x(t)$
- Discrete time $t, t+\Delta t, t+2\Delta t, \dots$ ($\Delta t = 1$)

$$x(t+1) = x(t)\,[1+\alpha]$$

- **Self-reinforcing process**: positive feedback loop
  - always amplifies the direction of change



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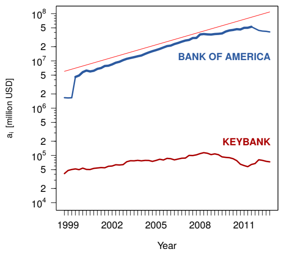
Literature:

- Amaral, L. A. N., Buldyrev, S. V., Havlin, S., Salinger, M. A., Stanley, H. E., and Stanley, M. H. (1997). Scaling behavior in economics: The problem of quantifying company growth. *Physica A*, 244:1–24.

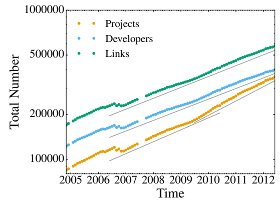
  Data: all publicly traded US manufacturing companies (SIC 2000-3999) for each of the years 1974-1993. All sales values were adjust to 1987 dollars by the GNP price deflator.
- LAN Amaral, SV Buldyrev, S Havlin, H Leschhorn, P Maass, MA Salinger, HE Stanley and MHR Stanley, (1997), Scaling Behavior in Economics: I. Empirical Results for Company Growth. *Journal de Physique I France*, 7:621–633.
- Bottazzi, G., and Secchi, A. (2006). Explaining the Distribution of Firms Growth Rates. *The Rand Journal of Economics*, 37 (2006) 235-256

Notes:

Exponential growth in real systems?



- ***Long*** **time scales**
  - $\alpha > 0$: **unrealistic**: $x \to \infty$
  - $\alpha < 0$: **uninteresting**: $x \to 0$
- ***Intermediate*** **time scales**:
  - exponential growth/decay **possible**, examples:
    (a) Value of OTC derivatives of the Bank of America
    (b) Number of developers registered on `Sourceforge`

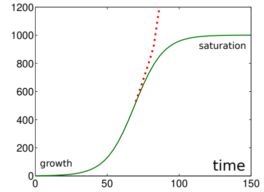


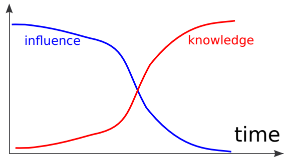
Exponential growth and our influence

- **Early** dynamics: always **linear** $\Rightarrow$ nothing happens

$$x(t) \propto e^{\alpha t} = 1 + \alpha t \ldots$$

- **Late** dynamics: always **underestimated**
  - *A water plant **grows half its size** every day. After 20 days it covers the whole pond. After how many days has it covered half of the pond?*
  - *A water plant **doubles its size** every day. After 20 days it covers the whole pond. After how many days has it covered half of the pond?*
- **We never see it coming** $\Rightarrow$ **The collapse is already there**
  - **Early:** We have all the resources to influence the system, but we do not understand the system.
  - **Late:** We understand the system, but our resources to influence the system are depleted or no longer sufficient.
  - **Good news:** The pond example is misleading. Most growth processes slow down and often saturate.





Notes:

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$$x(t) = x(0)[1+\alpha]^t\,; \quad x(0) = 1\,; \quad \alpha = 1/2$$
$$x(20) = x(0)[3/2]^{20} = N$$
$$20 \ln(3/2) = 20 \cdot 0.405 = \ln N$$
$$x(\hat{t}) = x(0)[3/2]^{\hat{t}} = N/2$$
$$\hat{t} \ln(3/2) = \ln N - \ln 2 = 8.1 - 0.693$$
$$\hat{t} = 18.28$$

- You can compare THIS answer with an answer from ChatGPT (found on quora.com, no date given): "This is a classic example of an exponential growth problem. The formula for exponential growth is: $S = S0 * (1+r)^t$ Where S is the final size, S0 is the initial size, r is the growth rate (in decimal form), and t is the number of time periods.In this case, the initial size is 1, the final size is the whole lake, the growth rate is 0.5 (or 50%), and t is 20 days. We can use this formula to solve for t when S is 1/2 of the lake: $(1/2) = 1 * (1+0.5)^t$ To solve for t, we can take the natural logarithm of both sides: $ln(1/2) = ln(1+0.5)^t = ln(1/2)/ln(1+0.5)t = 6.93147180559945$. So the lotus would cover half the lake in approximately 6.93 days."
- Do you find the mistake?

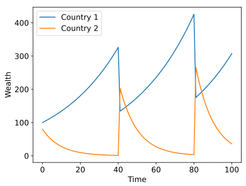
$$x(t) = x(0)[1+\alpha]^t\,; \quad x(0) = 1\,; \quad \alpha = 1$$
$$x(20) = x(0)[2]^{20} = N$$
$$20 \ln(2) = 20 \cdot 0.693 = \ln N$$
$$x(\hat{t}) = x(0)[2]^{\hat{t}} = N/2$$
$$\hat{t} \ln(2) = \ln N - \ln 2 = 13.862 - 0.693$$
$$\hat{t} = 19$$

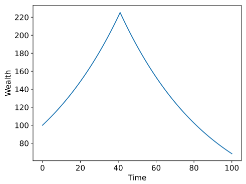
- So if our carbon footprint increases only by 50% instead of 100%, we have twice as much time?? Think again!
- Further, the famous question whether the glass is half full or half empty becomes completely irrelevant if you have no time left to enjoy the remainder.

Break exponential growth?

$$x(t) = x(0)e^{\alpha\ t}$$

- **Law of proportionate growth**: "The rich get richer!"
  - What means "rich?" $\alpha > 0$, or $\alpha_1 > \alpha_2$?
  - Confusion about absolute and relative wealth
- **Wealth redistribution**: "Take from the rich, give to the poor"
  - **Taxation** every 10 time steps:
  - Collect βx from the rich, add it to the wealth of the poor
- **Conclusion:** Redistribution cures no problem
  - To break the dynamics requires $\alpha < 0$





Outline

Nonlinear Dynamics

Exponential growth

Saturated growth

Harvesting

Growth and dispersal

Notes:

Notes:

Saturated growth

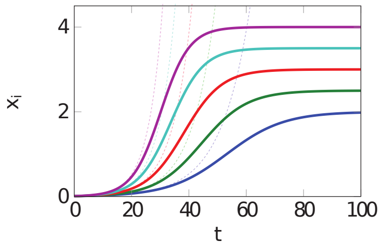
- **Size dependent growth factor:** $\alpha(x) = a - bx$
  - b is assumed to be small
- **Logistic equation:** (Verhulst, 1838)

$$\frac{dx(t)}{dt} = \alpha(x)x = ax - bx^2 \; ; \quad x(t \to \infty) = \frac{a}{b}$$

- **Discete dynamics:** $t \to n$: $x(t) \to x_n$, $x_n \to z_n$

$$\begin{aligned} x_{n+1} &= x_n + a\,x_n - b\,x_n^2 \\ z_{n+1} &= \frac{b}{1+a} x_{n+1} = \frac{b}{1+a}\left[(1+a)x_n - b\,x_n^2\right] \\ \mathbf{z_{n+1}} &= \mathbf{q\,z_n(1-z_n)} \; ; \quad (\mathbf{q = 1 + a}) \end{aligned}$$

  - **Logistic map:** *deterministic chaos* for $3.57 < q < 4$



Economic application: Harvesting

- **Growing natural resources**
  - Saturated growth because of limited carrying capacity
  - Growth rate **decreases** over time
  - Idea: **Harvest** before saturation is reached
- **Example: Fisheries**
  - Regrowing natural resource
  - Optimal harvesting function to maximize yield
  - **Risk:** Overfishing, extinction of population

Source: pixabay.com

Notes:

- The logistic equation was rediscovered by R. Pearl in 1920 and by A. Lotka in 1925.

Notes:

Limited carrying capacity

- (fish) *population* x: has to be sustained by natural resources
- **exponential growth**: $\frac{dx(t)}{dt} = \alpha\, x(t)$ requires $\alpha \neq$ **const.**
  - net growth rate $\alpha = r_b - r_d \lesseqgtr 0$ reflects saturation
- **assumption**: birth/death rates r_b, r_d depend on $x(t)$
  - birth rate decreases: $r_b = b - k_b x(t)$,
  - death rate increases: $r_d = d + k_d x(t)$
  - $r = b - d$: density independent net rate

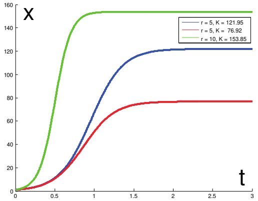
$$\frac{dx(t)}{dt} = \alpha\, x(t) = \Big[r_b - r_d\Big] x(t) = \Big[b - d\Big] x(t) - \Big[k_b + k_d\Big] x^2(t)$$

- define **carrying capacity**: $K = r/(k_b + k_d)$ with $r = b - d$

$$\frac{dx(t)}{dt} = r\, x(t) \left(1 - \frac{x(t)}{K}\right)$$

Solution:

$$x(t) = \frac{K}{(1 + \eta e^{-rt})}\ ;\ \ \eta = \frac{K}{f(0)} - 1$$



- logistic growth with *two different parameters*: r, K (additional influences)

Growth and carrying capacity

- Growth $g(x,t)$: quadratic in x,

$$g(x,t) = \frac{dx}{dt} = r\,x\left(1 - \frac{x}{K}\right) = rx - \frac{r}{K}x^2$$

- **Maximum growth:** $dg(x,t)/dx = 0$

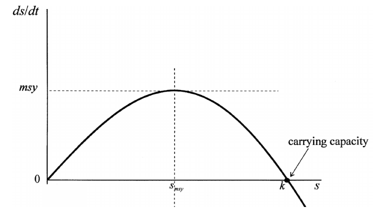
$$x^\star = \frac{K}{2}\ ;\quad g(x^\star) = \frac{rK}{4}$$

- **Equilibria:** $g(x,t) = dx/dt = 0$:

$$x_1^{\mathrm{eq}} = 0\ (\text{instable}),\quad x_2^{\mathrm{eq}} = K\ (\text{stable})$$

- **Stability**: Is system driven away/towards equilibrium state?

$$\text{instable}: \left.\frac{dg}{dx}\right|_{x_1^{\mathrm{eq}}} > 0\ ;\quad \text{stable}: \left.\frac{dg}{dx}\right|_{x_1^{\mathrm{eq}}} < 0$$



Source: Shone, p. 639

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SS02: Saturated growth and carrying capacity

- Explore logistic growth interactively.
- Understand and manipulate key parameters:
  r (Growth rate), K (Carrying capacity), and P_0 (Initial population).
  - Use Python to obtain numerical solutions.
  - Utilize interactive plots to explore parameter roles and effects.

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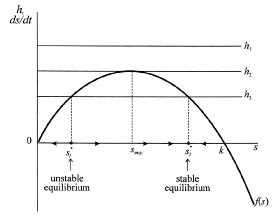
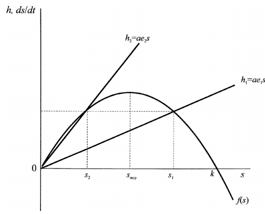
Growth rate and harvesting function

- **Harvesting function:** $h(x, e)$
  - e : effort (manpower, trawlers, ...)
  - "outflow" of resources: $\Delta x / \Delta t$
  - **modified dynamics:** $\hat{g}(x, t) = g(x, t) - h(x, e)$

1. $h(x, e) = \mathrm{const.}$
   - $h_1 > x^\star$: overfishing $\Rightarrow$ extinction
   - $h_2 = x^\star$: maximum yield, but maximum risk
   - $h_3 < x^\star$: outcome depends on $x(t = 0) = x_0$
     - $x_0 < x_1^{\mathrm{eq}}$: extinction
     - $x_0 > x_1^{\mathrm{eq}}$: growth despite harvesting
       $x_2^{\mathrm{eq}} < K$ because of constant harvesting
2. $h(x, e) = a \cdot x \cdot e$: increase with efficiency a (technology)
   - Instable equilibrium at $x_1^{\mathrm{eq}} = 0$
   - With equal a, higher e may **not** result in larger yield (!)
   - Why and when should we spend higher effort?





Source: Shone, pp. 640, 645

Harvesting, supply and demand

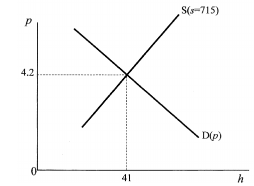
- $h = h(x)$, but determined by supply and demand
  - Demand: $q^d = c_0 - c_1 p$: linear decreasing with price p
  - Supply: $q^s = b_1 p + b_2 x$: linear increasing with price p, size x
  - Market clearing determines price:
    supply=demand: $q^d = q^s = h^{\mathrm{eq}}(x)$

$$c_0 - c_1 p = b_1 p + b_2 x \;\;\Rightarrow\;\; p = \frac{c_0 - b_2 x}{c_1 + b_1}$$

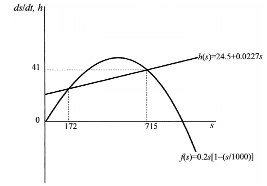
- Price determines harvest: $q^d = q^s = h^{\mathrm{eq}}(x)$

$$h^{\mathrm{eq}}(x) = \frac{c_0 b_1 + c_1 b_2\, x}{c_1 + b_1}$$

- **Problem:** efficiency, profitability not taken into account



$c_0 = 45$, $c_1 = 1$, $b_1 = 1.2$, $b_2 = 0.05$, $r = 0.2$, $K = 10^3$



Source: Shone, p. 644

Crutchfield & Zellner, 1962

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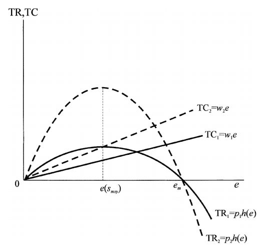
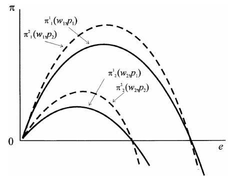
Harvesting and profits

- **Profit = Revenue - Cost:** $\pi = TR - TC$
  - Cost: "Wage" for effort: $TC = we$
  - Revenue depends on price and yield: $TR = ph$, $h = aex$
  - Profit: $\boxed{\pi(e) = e(pa\,x - w)}$
- **Equilibrium:** $g(x,t) - h = 0 \Rightarrow x = K[1 - (ae/r)]$

$$TR(e) = ph = p\,aex = p\,a\frac{K}{r}\,[r - ae]\,e$$

  - $TR = 0$ if $e = 0$ or $e = r/a$, $TR = TR^{\max}$ if $e^{\star} = r/(2a)$

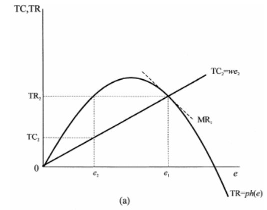


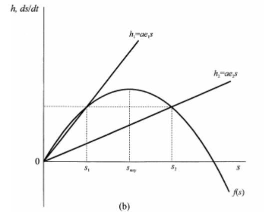


Source: Shone, p. 649

Profits and market entry

- **Scenarios:**
  - market entry as long as $\pi(e) > 0 \Rightarrow e$ increases
  - $e = e_2$: $TR(e_2) > TC(e_2)$: excess profit, entry
  - $e_2 < e < e^{\star}$: $d\pi(e)/de > 0$: profit further increases
  - $e^{\star} < e < e_1$: $d\pi(e)/de < 0$, still entry because of $\pi(e) > 0$
  - $e = e_1$: $TR(e_1) = TC(e_1)$: no profit, no entry
- **Result:** *inefficient equilibrium*
  - negative growth of revenue: $MR = dTR/de < MC = dTC/de$
  - **Unintended consequences**: worse off
  - same yield: $h(e_2) = h(e_1)$, **but** higher effort $e_1 > e_2$
  - ecological drawback: smaller population $x_1 < x_2$
  - *Reason:* "open access" $\Rightarrow$ "Tragedy of the Commons" (Hardin, 1968)





Source: Shone, p. 650

Notes:

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Outline

Nonlinear Dynamics

Exponential growth

Saturated growth

Harvesting

Growth and dispersal

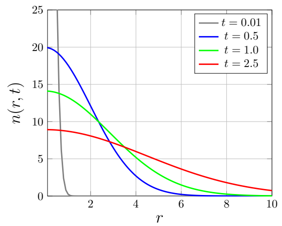
Spatial dispersal

- **Spatial diffusion:** Conserved quantity N_0
  - Example: Dispersal of goods, migration
  - Local density: $n(r,t)$, boundary condition: $N_0 = \int n(r,t)dr$
  - Initial condition: $n(r_0 = 0, 0) = N_0$, $n(r > r_0, 0) = 0$

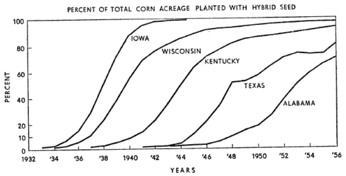
$$\frac{\partial n(r,t)}{\partial t} = D\frac{\partial^2 n(r,t)}{\partial r^2} = D\Delta n(r,t)$$

$$\frac{n(r_i, t+1) - n(r_i, t)}{\Delta t} \approx D\frac{n(r_i+1, t) + n(r_i-1, t) - 2n(r_i, t)}{(\Delta r)^2}$$

Solution: $$\boxed{n(r,t) = \frac{N_0}{2\sqrt{\pi D t}} \exp\left\{-\frac{r^2}{4Dt}\right\}}$$



- **Spatial adoption:** No conserved quantity
  - Example: Spread of knowledge, adoption of technology
    Adoption of "hybrid corn" in USA (1934-1956)
  - Nonlinear combination of two dynamics: (a) adoption, (b) diffusion
    $\Rightarrow$ **Reaction-diffusion equation**



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- Source: Zvi Griliches: Hybrid corn: An exploration in the economics of technological change, *Econometrica* vol 25, no 4 (1957) 501-522

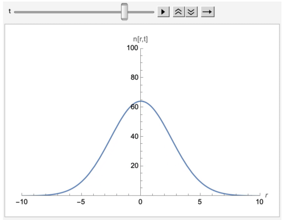
Growth and spread combined

- **Initial condition:** $n(r_0 = 0, 0) = N_0$, $n(r > r_0, 0) = 0$
  - **No** spatial uniform distribution
- **Exponential Growth:** $\partial n(r,t)/\partial t = \alpha n(r,t)$
  - Malthusian population: unlimited reproduction
  - Growth rate = adoption rate $\Rightarrow$ everyone adopts instantaneously
- **Spread:** Diffusion equation $\partial n(r,t)/\partial t = D \Delta n(r,t)$
- **Reaction-diffusion equation**:
  - Dimensionless variables: $\tau = \alpha t$, $\rho = r\sqrt{\alpha/D}$

$$\frac{\partial n}{\partial t} = \alpha n + \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n}{\partial r}\right) \Rightarrow \frac{\partial n}{\partial \tau} = n + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial n}{\partial \rho}\right)$$

Solution:

$$\boxed{n(\rho,\tau) = \frac{N_0}{4\pi\tau}\exp\left\{\tau - \frac{\rho^2}{4\tau}\right\}}$$



Spatial concentration of adopters

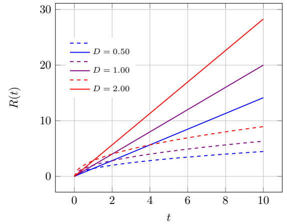
- **Total number of adopters**: $N(t) = N_0 \exp\{\alpha t\}$
- **Adoption area at time t:** Approximated by $\exp\{(0)\}$

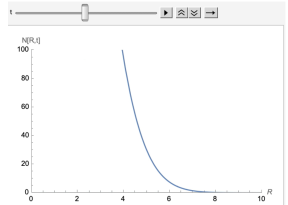
$$R^2(t) = 4\alpha D t^2$$

  - **Important:** $R \propto t$, but for simple diffusion $R \propto \sqrt{t}$
- **Adopters outside:** at distance $r > R(t)$:

$$N(R,t) = 2\pi \int_r^\infty n\left(r\sqrt{\frac{a}{D}}, at\right) rdr$$
$$= N_0 \exp\left(at - \frac{R^2}{4Dt}\right) \approx N_0$$

  - $N(R,t) = N_0 \ll N(t)$, but growth and dispersal will continue





Notes:

- Thomas Robert Malthus, *An Essay on the Principle of Population* (1798)
- Malthus made the famous prediction that population would outrun food supply, leading to a decrease in food per person.

  This Principle of Population was based on the idea that population if unchecked increases at a geometric rate (i.e. 2, 4, 8, 16, 32, 64, 128, etc.) whereas the food supply grows at an arithmetic rate (i.e. 1, 2, 3, 4, 5, 6, 7, 8, etc.).

  Only natural causes (eg. accidents and old age), misery (war, pestilence, and above all famine), moral restraint and vice (which for Malthus included infanticide, murder, contraception and homosexuality) could check excessive population growth $\rightarrow$ Malthusian catastrophe.
- Example from Nico Boccara, *Modeling Complex Systems*, New York: Springer, 2004

Notes:

- Approximate boundary of habitat occupied by invading species $R \propto t$ is in agreement with data on the spread of the muskrat (*Ondatra zibethica*), an american rodent, introduced into Central Europe in 1905.
- For simple diffusion, remember also **Brownian motion** where the mean square displacement in 2-dimensions is: $R^2 = 4Dt$

Questions

1. Remember 3 examples of nonlinear equations from your discipline. How did you solve them?
2. What is the value of plotting a dynamics in phase space?
3. Why looks relative growth so different from absolute growth? What is the problem comparing growth rates?
4. Derive the saturated growth dynamics using the expression for carrying capacity K.
5. Why may higher efficiency in harvesting not result in larger yield?
6. Explain the meaning of "inefficient equilibrium" for the harvesting function. (How) Can we avoid it?
7. Consider the combined dynamics of growth and dispersal. How will the results change if instead of a exponential growth αn, a saturated growth $\alpha n(1 - n)$ is considered?

Notes: